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## A NOTE ON WALLACE'S THEOREM

## 0. BOTTEMA and J.T. GROENMAN

If $A_{4}$ is a point on the circumcircle $\Omega$ of triangle $A_{1} A_{2} A_{3}$, its projections on the sides $A_{2} A_{3}, A_{3} A_{1}, A_{1} A_{2}$ are on a straight line $L_{4}$, the Wallace line of $A_{4}$ with respect to the triangle. (This line is still too often called the Simson line, after Robert Simson (1687-1768), although it has long been known that the theorem was first discovered by William Wallace in 1797, long after Simson's death [1].) If $\mathrm{H}_{4}$ is the orthocenter of $A_{1} A_{2} A_{3}$, then $\tau_{4}$ passes through the midpoint $M$ of $A_{4} H_{4}$ [2]. Another classical result is that $M$ occupies a "symmetrical" position with respect to the four points $A_{i}$ :

THEOREM. Let $A_{i}, i=1,2,3,4$, be four points on a circle $\Omega$. For $i=1,2,3,4$ and $\{j, k, 2\}=\{1,2,3,4\}-\{i\}$, the Wallace Zines $l_{i}$ of $A_{i}$ with respect to triangle $A_{j} A_{k} A_{l}$ are all concurrent in a point which is the midpoint of $A_{i} H_{i}$, where $H_{i}$ is the orthocenter of $A_{j} A_{k} A_{l}$.

The point of concurrency of the $\tau_{i}$ will be denoted by $W_{4}$ and called the Wallace point of the four concyclic points $A_{i}$. We give here a short proof of this classical theorem and then show that it can be generalized to a theorem about $n$ points on a circle.

Proof. With respect to a Cartesian frame, let $\Omega$ be the unit circle with center at the origin 0 . The points $A_{i}$ on $\Omega$ are given by $\left(\cos \phi_{i}, \sin \phi_{i}\right)$, or $\left(c_{i}, s_{i}\right)$ for short. The centroid of triangle $A_{1} A_{2} A_{3}$ is

$$
G=\left(\frac{1}{3} \sum_{i=1}^{3} c_{i}, \frac{1}{3} \sum_{i=1}^{3} s_{i}\right) ;
$$

the three points $0, G$, and the orthocenter $H_{4}$ are on the Euler line of the triangle, and $\mathrm{OH}_{4}=30 \mathrm{G}$. Hence

$$
H_{4}=\left(\sum_{i=1}^{3} c_{i}, \sum_{i=1}^{3} s_{i}\right),
$$

and the midpoint of $A_{4} H_{4}$, which lies on $\tau_{4}$, is

$$
\begin{equation*}
\left(\frac{1}{2} \sum_{i=1}^{4} c_{i}, \frac{1}{2} \sum_{i=1}^{4} s_{i}\right) \tag{1}
\end{equation*}
$$

Since (1) is invariant under permutations of ( $1,2,3,4$ ), it is the required Wallace point $W_{4}$, the point of concurrency of the four Wallace lines $\tau_{i}$ which bisects each segment $A_{i} H_{i}$. The centroid of the four points $A_{i}$ is

$$
G_{4}=\left(\frac{1}{4} \sum_{i=1}^{4} c_{i}, \frac{1}{4} \sum_{i=1}^{4} s_{i}\right)
$$

hence the points $0, G_{4}, W_{4}$ are collinear, and we have $0 W_{4}=20 G_{4}$.
We now consider five points $A_{i}$ on $\Omega$, and we denote the Wallace point of $\left\{A_{1}, A_{2}, A_{3}, A_{4}\right\}$ by $W_{4,5}$, etc. Then $W_{4,5}$ is given by (1) and, as $A_{5}=\left(c_{5}, s_{5}\right)$,

$$
\begin{equation*}
\frac{2}{3} W_{4,5}+\frac{1}{3} A_{5}=\left(\frac{1}{3} \sum_{i=1}^{5} c_{i}, \frac{1}{3} \sum_{i=1}^{5} s_{i}\right), \tag{2}
\end{equation*}
$$

which is invariant under permutations of $(1,2, \ldots, 5)$. Thus the five lines joining an arbitrary point of the quintuple to the Wallace point of the remaining four all pass through one point $W_{5}$, given by (2), the Wallace point of the quintuple. On each join, we have

$$
A_{i} W_{5}: W_{5} W_{4, i}=2: 1 .
$$

The centroid $G_{5}$ of the quintuple is

$$
\left(\frac{1}{5} \sum_{i=1}^{5} c_{i}, \frac{1}{5} \sum_{i=1}^{5} s_{i}\right) ;
$$

hence $0, G_{5}, W_{5}$ are collinear, and we have

$$
O G_{5}: O W_{5}=3: 5 .
$$

Obviously, we may derive a chain of theorems defining by induction the Wallace point $W_{n}$ of an $n$-tuple of points on the circle $\Omega$. The chain starts with $n=3$, the point $W_{3}$ being the orthocenter of $A_{1} A_{2} A_{3}$. Let

$$
W_{n-1}=\left(\frac{1}{n-3} \sum_{i=1}^{n-1} c_{i}, \frac{1}{n-3} \sum_{i=1}^{n-1} s_{i}\right), \quad n>3 .
$$

As $A_{n}=\left(c_{n}, s_{n}\right)$, the point

$$
W_{n}=\frac{n-3}{n-2} W_{n-1}+\frac{1}{n-2} A_{n}=\left(\frac{1}{n-2} \sum_{i=1}^{n} c_{i}, \frac{1}{n-2} \sum_{i=1}^{n} s_{i}\right)
$$

is invariant under permutations of $(1,2, \ldots, n)$, and its coordinates follow from those of $W_{n-1}$ when $n-1$ is replaced by $n$.

The $n$ joins $A_{i} W_{n-1, i}$ are concurrent at $W_{n}$, and on each join we have

$$
A_{i} W_{n}: W_{n} W_{n-1, i}=n-3: 1, \quad n>3 .
$$

The points $0, W_{n}$, and the centroid

$$
\mathrm{G}_{n}=\left(\frac{1}{n} \sum_{i=1}^{n} c_{i}, \frac{1}{n} \sum_{i=1}^{n} s_{i}\right)
$$

are collinear and

$$
O G_{n}: O W_{n}=n-2: n, \quad n \geq 3 .
$$

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2. H.S.M. Coxeter and S.L. Greitzer, Geometry Revisited, Random House, New York, 1967 (now available from the M.A.A), p. 45.

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MATHEMATICAL CLERIHEWS
Karl Friedrich Gauss
Did not carouse,
But worked till weary
On number theory.

Pierre de Fermat
Asserted, "C'est ça!" -
And wrote too large in
A certain margin.

René Descartes
Devised the art
Of function graphs.
His face caused laughs.

Isaac Newton,
Highfalutin',
Stirred up ructions
With his fluxions.

# DIGITAL SUMS, NIVEN NUMBERS, AND NATURAL DENSITY 

## ROBERT E. KENNEDY

At a recent conference on number theory [1], Ivan Niven mentioned a question which appeared in the children's pages of a certain newspaper. It was: Find a whole number which is twice the sum of its digits. One possible activity associated with this question would be to investigate the set of positive integers which are divisible by their digital sum. Even though this idea is easily understood, surprisingly little regarding it can be found in the literature. To facilitate the discussion, we make the following

Definition. A positive integer is called a Niven number if it is divisible by its digital sum.

Some examples of Niven numbers are: 8, 12, 180, and 4050. That there are infinitely many Niven numbers is easily seen, since (for example) each positive integral power of 10 is a Niven number.

In [2], the notion of a Niven number was first introduced in order to show how mathematics can be developed: how we can make conjectures based on a fundamental idea, seek examples and counterexamples, and use ideas from various areas of mathematics. In this article, the emphasis will be on the "natural density" of the set of Niven numbers. Thus, if $A(N)$ denotes the number of Niven numbers not exceeding $N$, we are interested in the status of

$$
\lim _{N \rightarrow \infty} \frac{A(N)}{N} .
$$

In investigating this open question, various approaches may be used. For example, it would be enlightening to find either
(1) a set, $S_{1}$, of Niven numbers with a nonzero density; or
(2) a set, $S_{2}$, of non-Niven numbers with a nonzero density.

If, for $i=1,2$, we denote by $S_{i}(N)$ the number of members of $S_{i}$ not exceeding $N$, then we have

$$
S_{1}(N) \leq A(N) \leq N-S_{2}(N) .
$$

Thus

$$
\frac{S_{1}(N)}{N} \leq \frac{A(N)}{N} \leq 1-\frac{S_{2}(N)}{N},
$$

and so the density of the set of Niven numbers (if it exists) lies between the density of $S_{1}$ and the density of the complement of $S_{2}$.

A candidate for $S_{1}$ would be any infinite arithmetic progression of Niven numbers.

Thus, if there exist integers $a \neq 0$ and $b$ such that $a n+b$ is Niven for each positive integer $n$, then

$$
S_{1}=\{a n+b: n=1,2,3, \ldots\}
$$

would be a set of Niven numbers with a nonzero density since

$$
\lim _{N \rightarrow \infty} \frac{S_{1}(N)}{N}=\frac{1}{a} .
$$

This would be sufficient to show that the density of the set of Niven numbers could not possibly be zero. Whether or not such integers $a$ and $b$ exist is still an unanswered question. However, there does not exist an integer a such that an is Niven for all $n$. This is shown by the following theorem, which gives a method of constructing, for any given integer $a$, a multipie an which is not a Niven number.

THEOREM 1. Given a positive integer $a$, let $2^{s}$ be the highest power of 2 which is $a$ factor of $a$. Then, if $m$ is the number of digits of $a$,

$$
N=a^{2^{s+1}} \sum_{k=0}^{-1} 10^{k m}
$$

is a multiple of a which is not Niven.
Proof. Let $D$ be the digital sum of $N$. Then $D$ is the product of $2^{s+1}$ and the digital sum of $a$. Thus $2^{s+1}$ is a factor of $D$, but not a factor of $N$ by the maximality of $s$. Hence $D$ is not a factor of $N$, and $N$ is not a Niven number.

To exemplify this theorem, let $a=12$. Then $s=2, m=2$, and we have

$$
N=12\left(10^{0}+10^{2}+10^{4}+10^{6}+10^{8}+10^{10}+10^{12}+10^{14}\right),
$$

which can be written as $N=1212121212121212$. Note that $12 \mid N$, but $N$ is not Niven since 8 divides the digital sum of $N$ but does not divide $N$. Thus $N$ is a multiple of 12 which is not Niven. Essentially, the theorem states that $N$ is constructed by concatenating $a$ to itself $2^{s+1}-1$ times.

The above theorem, in the author's opinion, makes it highly probable that there is no infinite arithmetic progression consisting only of Niven numbers. So, if a set such as $S_{1}$ exists, it will probably not be an arithmetic progression. In fact, all subsets of the set of Niven numbers found to date have zero density. This would be the case if the natural density of the set of Niven numbers were zero.

For example, let $S$ be the set of all positive integers whose digital sum is 3 . Then $S$ is a zero density subset of the set of Niven numbers. The proof of this is not too involved, but it does point out a class of sets of integers each of which has zero density. The next theorem gives such a class.

THEOREM 2. Let $S_{t}(N)$ be the number of positive integers, not exceeding $N$, which have digital sum $t$. Then

$$
\lim _{N \rightarrow \infty} \frac{S_{t}(N)}{N}=0
$$

Proof. Each integer (not exceeding $N$ ) with digital sum $t$ may be written in the form

$$
10^{n_{1}}+10^{n_{2}}+\ldots+10^{n^{2}},
$$

where

$$
0 \leq n_{i} \leq\left[\log _{10} N\right]+1, \quad i=1,2, \ldots, t
$$

where, as usual, the square brackets denote the integral part operator. Thus there are at most $\left.\left(\Gamma \log _{10} N\right]+2\right)^{t}$ integers with digital sum $t$. Since

$$
\lim _{N \rightarrow \infty} \frac{\left(\left[\log _{10} N\right]+2\right)^{t}}{N}=0,
$$

it follows that

$$
\lim _{N \rightarrow \infty} \frac{S_{t}(N)}{N}=0
$$

Note that this result states that the set of positive integers with digital sum $t$ has a natural density of zero.

A collection, $S_{2}$, of non-Niven numbers with a nonzero density can be found. In particular, let $S_{2}$ be the set of all odd integers with an even number of odd digits. Since such an integer will have an even digital sum, it cannot be a Niven number. It is perhaps intuitively obvious that this set has a natural density of $\frac{1}{4}$, but with the next theorem we can show that this is indeed the case.

THEOREM 3. Let $T(N)$ be the number of positive odd integers, not exceeding $N$, which have an even number of odd digits. Then $T(100 n)=25 n$ for each positive integer $n$.

Outline of proof. The proof follows by mathematical induction on $n$, together with the observation that, from 100 n to $100 n+100$, exactly 25 odd integers with an even number of odd digits occur.

Now, given $N$, there is an $n$ such that

$$
100 n<N \leq 100(n+1) .
$$

Then, by Theorem 3,

$$
25 n<T(N) \leq 25(n+1),
$$

and so

$$
\frac{1}{4}\left(\frac{n}{n+1}\right)<\frac{T(N)}{N} \leq \frac{1}{4}\left(\frac{n+1}{n}\right) .
$$

Thus, the natural density of the set of odd integers with an even number of odd digits is $\frac{7}{4}$ since

$$
\lim _{n \rightarrow \infty} \frac{1}{4}\left(\frac{n}{n+1}\right)=\lim _{n \rightarrow \infty} \frac{1}{4}\left(\frac{n+1}{n}\right)=\frac{1}{4} .
$$

From this and the fact that $A(N) \leq N-T(N)$, we have

$$
\lim _{N \rightarrow \infty} \frac{A(N)}{N} \leq \frac{3}{4},
$$

provided the limit exists. This is not a very refined result, but it does give some information about the ratio $A(N) / N$. Another indication of how of ten Niven numbers occur is given by the following theorem and its corollary.

THEOREM 4. For any positive integer $N$, at least one of

$$
10 N+1, \quad 10 N+11, \quad 10 N+21
$$

is not a Niven number.
Outline of proof. Let ( ) ${ }_{s}$ denote the digital sum of the number inside the parentheses. Since at least two of $N, N+1, N+2$ do not end in 9, it follows that at least one pair of $(N)_{s},(N+1)_{s},(N+2)_{s}$ differ by 1 . Hence at least one of these digital sums is odd. Noting that $(10 K)_{s}=(K)_{s}$ for all $K$, at least one of

$$
(10 N)_{s}, \quad(10 N+10)_{s}, \quad(10 N+20)_{s}
$$

is odd. Since each of $10 \mathrm{~N}, 10 \mathrm{~N}+10$, and $10 \mathrm{~N}+20$ ends in 0 , at least one of

$$
(10 N+1)_{s}, \quad(10 N+11)_{s}, \quad(10 N+21)_{s}
$$

is even. Therefore one of the odd integers $10 N+1,10 N+11,10 N+21$ has an even digital sum and thus cannot be a Niven number.

Corollary. There cannot exist more than 21 consecutive integers each of which is a Niven number.

Even though a sequence of 21 consecutive Niven numbers has not been discovered, Theorem 4 and its proof can be used to show that such a sequence would have to start with a number of the form $100 \mathrm{~N}+90$.

In conclusion, I would like to emphasize the open questions which have been mentioned.
(a) Does the set of Niven numbers have a natural density? If it does, what is it? (It has been shown by computer that $A(90000000)=6347468$, which gives a partial density of 0.07052742. )
(b) Can an infinite arithmetic progression be composed entirely of Niven numbers?
(c) Does there exist a sequence of 21 consecutive Niven numbers? If not, what is the longest possible sequence of consecutive Niven numbers?

When you investigate the above questions, you will discover, as I have, that even the simplest sounding idea can be a vehicle for one's mathematical growth.

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※ $\quad$ *
*
THE OLYMPIAD CORNER: 35
M.S. KLAMKIN

I present this month the problems of the 1982 Australian and British Mathematical Olympiads, through the courtesy of Jim Williams and R.C. Lyness, respectively. I welcome solutions to all of these problems. I expect to give in coming issues the problems of the 1982 Canadian, U.S.A., and International Mathematical 0lympiads.

1982 AUSTRALIAN MATHEMATICAL OLYMPIAD
Paper I, 16 March 1982. Time: 4 hours

1. If $A$ and $B$ toss $n+1$ and $n$ fair coins, respectively, what is the probability $P_{n}$ that $A$ gets more heads than $B$ ?
2. The fractional part $\{x\}$ of $x$ is defined as the smallest nonnegative number such that $x-\{x\}$ is an integer. For example, $\{1.6\}=0.6$ and $\{\pi\}=\pi-3$. Show that

$$
\lim _{n \rightarrow \infty}\left\{(2+\sqrt{3})^{n}\right\}=1
$$

3. Let $A B C$ be a triangle, and let the internal bisector of the angle $A$ meet the circumcircle again at $P$. Define $Q$ and $R$ similarly. Prove that

$$
A P+B Q+C R>B C+C A+A B
$$

Paper II, 17 March 1982. Time: 4 hours
4. Find what real numbers $d$ have the following property:

If $f(x)$ is a continuous function for $0 \leq x \leq 1$ and if $f(0)=f(1)$, there exists $t$ such that $0 \leq t<t+d \leq 1$ and $f(t)=f(t+d)$. Equivalently, every continuous graph from $(0,0)$ to $(1,0)$ has a horizontal chord of length $d$.
5. The sequence $p_{1}, p_{2}, \ldots$ is defined as follows:
$p_{1}=2 ; p_{n}=$ the largest prime divior of $p_{1} p_{2} \ldots p_{n-1}+1, n \geq 2$.
Prove that 5 is not a member of this sequence.
6. A number is written into each square of an $n \times n$ table. We know that any two rows of the table are different. Prove that the table contains a column such that, on omitting it, the remaining table has no equal rows either.
(Note. The rows $1,1,2,7,5$ and $1,1,7,2,5$ are different, that is, not equal.)

## 1982 BRITISH MATHEMATICAL OLYMPIAD <br> 18 March 1982. Time: $3 \frac{1}{2}$ hours

1. $P Q R S$ is a quadrilateral of area $A .0$ is a point inside it. Prove that if

$$
2 A=O P^{2}+O Q^{2}+O R^{2}+O S^{2}
$$

then $P Q R S$ is a square and 0 is its centre.
2. A multiple of 17 when written in the scale of 2 contains exactly three
digits 1 . Prove that it contains at least six digits 0 , and that if it contains exactly seven digits 0 , then it is even.
3. If

$$
s_{n}=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}, \quad n>2,
$$

prove that

$$
n(n+1)^{a}-n<s_{n}<n-(n-1) n^{b},
$$

where $a$ and $b$ are given in terms of $n$ by $a n=1$ and $b(n-1)=-1$.
4. A sequence of real numbers $u_{1}, u_{2}, u_{3}, \ldots$ is given by $u_{1}$ and the recurrence relation

$$
u_{n}^{3}=u_{n-1}+\frac{15}{64}, \quad n \geq 2
$$

By considering the curve

$$
x^{3}=y+\frac{15}{64},
$$

or otherwise, describe with proof the behaviour of $u_{n}$ as $n$ tends to infinity.
5. A right circular cone stands on a horizontal base of radius $r$. Its vertex $V$ is at a distance 1 from every point on the perimeter of the base. A plane section of the cone is an ellipse whose lowest point is $L$ and whose highest point is H. On the curved surface of the cone, to one side of the plane VLH, two routes from L to H are marked. Route $R_{1}$ is along the semiperimeter of the ellipse and $R_{2}$ is the route of shortest length.

Find the condition that $R_{1}$ and $R_{2}$ intersect between $L$ and $H$.
6. Prove that the number of sequences $a_{1} \alpha_{2} \ldots a_{n}$ with each of their $n$ terms $a_{i}=0$ or 1 and containing exactly $m$ occurrences of 01 is

$$
\binom{n+1}{2 m+1} .
$$

Editor's Note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2Gl.
$\therefore$
*
$\%$

## PROBLEMS--PROBLEMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before December 1, 1982, although solutions received after that date will also be considered until the time when a solution is published.
741. Proposed by Allan Wm. Johnson Jr., Washington, D.C.

Solve the doubly true addition

$$
7 \cdot \text { THREE }+5 \cdot \text { FIVE }+4 \cdot \text { ELEVEN }=\text { NINETY. }
$$

742. Proposed by Charles W. Trigg, San Diego, California.

Find three three-digit primes that are composed of the nine nonzero digits and have a sum that is a triangular number.
743. Proposed ky George Tsintsifas, Thessaloniki, Greece.

Let $A B C$ be a triangle with centroid $G$ inscribed in a circle with center
0 . A point $M$ lies on the disk $\omega$ with diameter $O G$. The lines $A M, B M, C M$ meet the
circle again in $A^{\prime}, B^{\prime}, C^{\prime}$, respectively, and $G^{\prime}$ is the centroid of triangle $A^{\prime} B^{\prime} C^{\prime}$ 。 Prove that
(a) $M$ does not lie in the interior of the disk $\omega^{\prime}$ with diameter $O G^{\prime}$;
(b) $[A B C] \leq\left[A^{\prime} B^{\prime} C^{\prime}\right]$, where the brackets denote area.

744, Proposed by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.
(a) Prove that, for all nonnegative integers $n$,

$$
\begin{array}{l|l}
5 & \mid 2^{2 n+1}+3^{2 n+1}, \\
7 & \mid 2^{n+2}+3^{2 n+1}, \\
11 & \mid 2^{8 n+3}+3^{n+1}, \\
13 & \mid 2^{4 n+2}+3^{n+2}, \\
17 & \mid 2^{6 n+3}+3^{4 n+2}, \\
19 & \mid 2^{3 n+4}+3^{3 n+1}, \\
29 & \mid 2^{5 n+1}+3^{n+3}, \\
31 & \mid 2^{4 n+1}+3^{6 n+9} .
\end{array}
$$

(b) Of the first eleven primes, only 23 has not figured in part (a). Prove that there do not exist polynomials $f$ and $g$ such that

$$
23 \mid 2^{f(n)}+3^{g(n)}
$$

for all positive integers $n$.
745, Proposed by Roger Izard, Dallas, Texas.
In the adjoined figure, triangles $A B C$ and DEF are both equilateral, and angles BAD, CBE, and $A C F$ are all equal. Prove that triangles $A B C$ and DEF have the same center.
746. Proposed by Jack Garfunkel, Flushing, N.Y. Given are two concentric circles and a triangle $A B C$ inscribed in the outer circle. A tangent to the outer circle at $A$ is rotated about $A$
 in the counterclockwise sense until it first touches the inner circle, say at $P$. The procedure is repeated at $B$ and $C$, resulting in points $Q$ and $R$, respectively, on the inner circle. Prove that triangle $P Q R$ is directly similar to triangle $A B C$.
747. Proposed by J.T. Groenman, Arnhem, The Netherlands.

Let $A B C$ be a triangle (with sides $a, b, c$ in the usual order) inscribed in a circle with center 0 . The segments $B C, C A, A B$ are divided internally in the same ratio by the points $A_{1}, B_{1}, C_{1}$, respectively, so that

$$
\overline{\mathrm{BA}}_{1}: \overline{\mathrm{A}_{1} \mathrm{C}}=\overline{\mathrm{CB}}_{1}: \overline{\mathrm{B}_{1} \mathrm{~A}}=\overline{\mathrm{AC}}_{1}: \overline{\mathrm{C}_{1} \mathrm{~B}}=\lambda: \mu,
$$

where $\lambda+\mu=1$. A line through $A_{1}$ perpendicular to $O A$ meets the circle in two points, one of which, $P_{a}$, lies on the arc CAB; and points $P_{b}, P_{c}$ are determined analogously by lines through $B_{1}, C_{1}$ perpendicular to $O B, O C$. Prove that

$$
\overline{\mathrm{AP}}_{a}^{2}+\overline{\mathrm{BP}}_{b}^{2}+\overline{\mathrm{CP}}_{c}^{2}=a^{2}+b^{2}+c^{2}
$$

independently of $\lambda$ and $\mu$.
Investigate the situation if the word "internally" is replaced by "externaliy".
748. Proposed by H. Kestelman, University College, London, England. Let $a_{1}, a_{2}, \ldots, a_{n}$ be distinct complex numbers. For the Vandermonde matrix

$$
M=\left(\begin{array}{ccccc}
1 & a_{1} & a_{1}^{2} & \cdots & a_{1}^{n-1} \\
1 & a_{2} & a_{2}^{2} & \cdots & a_{2}^{n-1} \\
\vdots & \vdots & \vdots & & \vdots \\
1 & a_{n} & a_{n}^{2} & \cdots & a_{n}^{n-1}
\end{array}\right)
$$

show that the elements of the $j$ th column of $M^{-1}$ are the coefficients in the polynomial $f_{j}$, of degree $n-1$, given by

$$
f_{j}(t)=\prod_{r \neq j} \frac{t-a_{r}}{a_{j}-a_{r}}
$$

749: Proposed by Ram Rekha Tiwari, Radhaur, Bihar, India.
Solve the system

$$
\begin{aligned}
& \frac{y z(x+y+z)(y+z-x)}{(y+z)^{2}}=a^{2} \\
& \frac{z x(x+y+z)(z+x-y)}{(z+x)^{2}}=b^{2} \\
& \frac{x y(x+y+z)(x+y-z)}{(x+y)^{2}}=c^{2} .
\end{aligned}
$$

750*. Proposed by Anders Lönnberg, Mockfj̈ard, Sweden. For which real $a$ does the graph of

$$
y=x^{x^{x^{\alpha}}}, x>0
$$

have a point of inflection with horizontal tangent (that is, where $y^{\prime}=0$ )?

## SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.
613. [1981: 79; 1982: 55, 67] Proposed by Jack Garfunkel, Flushing, N.Y. If $A+B+C=180^{\circ}$, prove that

$$
\cos \frac{1}{2}(B-C)+\cos \frac{1}{2}(C-A)+\cos \frac{1}{2}(A-B) \geq \frac{2}{\sqrt{3}}(\sin A+\sin B+\sin C) .
$$

(Here $A, B, C$ are not necessarily the angles of a triangle, but you may assume that they are if it is helpful to achieve a proof without calculus.)
II. Solution by George Tsintsifas, Thessaloniki, Greece.

We assume that $A, B, C$ are the angles of a triangle labelled in such a way that $B$ and $C$ are "on the same side" of $60^{\circ}$, that is,

$$
A \leq 60^{\circ} \leq B, C \quad \text { or } \quad A \geq 60^{\circ} \geq B, C .
$$

In either case, we have

$$
\left(\cos \frac{A}{2}-\frac{\sqrt{3}}{2}\right)\left(\cos \frac{B}{2}-\frac{\sqrt{3}}{2}\right) \leq 0
$$

and

$$
\left(\cos \frac{A}{2}-\frac{\sqrt{3}}{2}\right)\left(\cos \frac{C}{2}-\frac{\sqrt{3}}{2}\right) \leq 0,
$$

from which

$$
4 \pi \cos \frac{A}{2} \leq 2 \sqrt{3}\left(\cos \frac{A}{2}+\cos \frac{B}{2}\right) \cos \frac{C}{2}-3 \cos \frac{C}{2}
$$

and

$$
4 \pi \cos \frac{A}{2} \leq 2 \sqrt{3}\left(\cos \frac{A}{2}+\cos \frac{C}{2}\right) \cos \frac{B}{2}-3 \cos \frac{B}{2} .
$$

Adding the last two inequalities, we have

Since

$$
8 \pi \cos \frac{A}{2} \leq 2 \sqrt{3}\left(\Sigma \cos \frac{B}{2} \cos \frac{C}{2}\right)+2 \sqrt{3} \cos \frac{B}{2} \cos \frac{C}{2}-3\left(\cos \frac{B}{2}+\cos \frac{C}{2}\right) .
$$

$4 \pi \cos \frac{A}{2}=\Sigma \sin A$ and $2 \cos \frac{B}{2} \cos \frac{C}{2}=\cos \frac{B-C}{2}+\cos \frac{B+C}{2} \leq 1+\sin \frac{A}{2}$,
we have so far shown that

$$
2 \Sigma \sin A \leq 2 \sqrt{3}\left(\Sigma \cos \frac{B}{2} \cos \frac{C}{2}\right)+\sqrt{3}\left(1+\sin \frac{A}{2}\right)-3\left(\cos \frac{B}{2}+\cos \frac{C}{2}\right),
$$

and the desired inequality will follow from

$$
\begin{equation*}
2 \sqrt{3}\left(\Sigma \cos \frac{B}{2} \cos \frac{C}{2}\right)+\sqrt{3}\left(1+\sin \frac{A}{2}\right)-3\left(\cos \frac{B}{2}+\cos \frac{C}{2}\right) \leq \sqrt{3}\left(\Sigma \cos \frac{B-C}{2}\right), \tag{1}
\end{equation*}
$$

which we proceed to establish.
Since

$$
\cos \frac{B-C}{2}=2 \cos \frac{B}{2} \cos \frac{C}{2}-\sin \frac{A}{2}, \text { etc. }
$$

so that

$$
\Sigma \cos \frac{B-C}{2}=2\left(\Sigma \cos \frac{B}{2} \cos \frac{C}{2}\right)-\Sigma \sin \frac{A}{2},
$$

(1) is equivalent to each of the following four inequalities

$$
\begin{gather*}
1+2 \sin \frac{A}{2} \leq \sqrt{3}\left(\cos \frac{B}{2}+\cos \frac{C}{2}\right)-\left(\sin \frac{B}{2}+\sin \frac{C}{2}\right) \\
\cos 60^{\circ}+\cos \frac{B+C}{2} \leq\left(\cos 30^{\circ} \cos \frac{B}{2}-\sin 30^{\circ} \sin \frac{B}{2}\right)+\left(\cos 30^{\circ} \cos \frac{C}{2}-\sin 30^{\circ} \sin \frac{C}{2}\right) \\
2 \cos \left(30^{\circ}+\frac{B+C}{4}\right) \cos \left(30^{\circ}-\frac{B+C}{4}\right) \leq \cos \left(30^{\circ}+\frac{B}{2}\right)+\cos \left(30^{\circ}+\frac{C}{2}\right) \\
\cos \left(30^{\circ}-\frac{B+C}{4}\right) \leq \cos \frac{B-C}{4} \tag{2}
\end{gather*}
$$

and it is sufficient to establish (2). Now

$$
60^{\circ} \geq B \Rightarrow 120^{\circ} \geq 2 B \Rightarrow 120^{\circ}-(B+C) \geq B-C \Rightarrow 30^{\circ}-\frac{B+C}{4} \geq \frac{B-C}{4}
$$

and

$$
60^{\circ} \leq B \Rightarrow 2 B-120^{\circ} \geq 0 \Rightarrow B+C-120^{\circ} \geq C-B \Rightarrow \frac{B+C}{4}-30^{\circ} \geq \frac{C-B}{4} .
$$

In either case (2) holds, and the proof is complete.
Editor's conment.
Solution I of this inequality appears in an article by V.N. Murty [1982: 67].
635. [1981: 146] Proposed by Dan Sokolowsky, [now at] Califormia State University at Los Angeles.
In Figure 1 below, 0 is the circumcenter of triangle $A B C$, and $P Q R \perp O A, P S T \perp O B$. Prove that

$$
\begin{equation*}
P Q=Q R \Leftrightarrow P S=S T . \tag{1}
\end{equation*}
$$



Figure 1


Figure 2
I. Solution by the proposer.

We will show that

$$
\begin{equation*}
P Q \cdot P S=Q R \cdot S T \text {, } \tag{2}
\end{equation*}
$$

from which (1) follows. Let $P Q R$ and PST meet the circle again at $M$ and $N$, respectively, as shown in Figure 2. Then

$$
\underline{\angle A Q M}=\frac{1}{2}(\operatorname{arc} B P+\operatorname{arc} A M)=\frac{1}{2}(\operatorname{arc} N B+\operatorname{arc} P A)=\underline{ } B S T
$$

and

$$
\underline{\angle A R P}=\frac{1}{2}(\operatorname{arc} P A+\operatorname{arc} M C)=\frac{1}{2}(\operatorname{arc} A M+\operatorname{arc} M C)=\underline{S B} T ;
$$

hence $\triangle A Q R \sim \triangle T S B$ and

$$
\begin{equation*}
O R \cdot S T=A O \cdot B S . \tag{3}
\end{equation*}
$$

Also,

$$
\underline{\angle P A B}=\frac{1}{2} \operatorname{arc} B P=\frac{1}{2} \operatorname{arc} N B=\underline{B P S}
$$

and

$$
\underline{I A P M}=\frac{1}{2} \operatorname{arc} A M=\frac{1}{2} \operatorname{arc} P A=\underline{/ P B S} ;
$$

hence $\triangle A P Q \sim \triangle P B S$ and

$$
\begin{equation*}
P Q \cdot P S=A Q \cdot B S \tag{4}
\end{equation*}
$$

Now (2) follows from (3) and (4), $\square$
The point $P$ such that $P Q=Q R$ is easily characterized and constructed. It
follows from the above solution that $\triangle P A B \sim \triangle Q A P$, so

$$
\begin{equation*}
\frac{P A}{P \bar{B}}=\frac{Q A}{Q \bar{P}} \tag{5}
\end{equation*}
$$

and $\triangle A B C \sim \triangle A R Q$, so

$$
\begin{equation*}
\frac{C A}{C B}=\frac{O A}{\bar{Q} \bar{R}^{\circ}} \tag{6}
\end{equation*}
$$

It now follows form (5) and (6) that $P Q=Q R$ if and only if

$$
\begin{equation*}
\frac{P A}{P B}=\frac{C A}{C B}, \tag{7}
\end{equation*}
$$

and this is the required characterization. To construct the point $P$ satisfying (7), let $X$ be the midpoint of arc $A C B$, and let the bisector of angle $C$ of the triangle meet $A B$ in $Y$. Then line $X Y$ will meet the circle again in the required point $P$, for

$$
\frac{P A}{P B}=\frac{Y A}{\overline{Y B}}=\frac{C A}{C B} .
$$

II. Solution by Dan Pedoe, University of Minnesota.

Using the theory of projective ranges, pencils, and projective ranges on a conic, a proof of this interesting problem is rapidly obtained, and also the strong implication that a similar theorem holds for a central conic, and finally that the theorem can be stated for any conic, and that the point 0 in the problem is irrelevant.

We use cross ratios and the notation $\infty$ for the point at infinity on the line under consideration (see [1] for all concepts used). Suppose $P Q=Q R$. Then

$$
\left\{P R, Q_{\infty}\right\}=-1,
$$

and therefore

$$
A\left\{P R, Q_{\infty}\right\}=-1
$$

Since $A$ lies on the circle, we have

$$
\{P C, B A\}=-1,
$$

the join $A A$ being the tangent at $A$, and this tangent is parallel to PQR. Hence

$$
B\{P C, B A\}=-1,
$$

and taking the join $B B$ as the tangent at $B$, which is parallel to PST, we get

$$
\{P T, \infty S\}=-1,
$$

from which $P S=S T$ follows. The converse also clearly holds.
The first generalization is to a central conic, with 0 as centre, and PQR, PST parallel to the tangents to the conic at $A, B$, respectively. This indicates that the
point 0 is irrelevant, and the fact that the points $A, B$ and $P, C$ form a harmonic range on the conic leads to the following theorem, which is illustrated in Figure 3 :

If $A B C$ is a triangle inscribed in a conic, and if PQR and PST are parallel to the tangents at $A$ and $B$, respectively, then

$$
P Q=Q R \Leftrightarrow P S=S T .
$$



Furthermore, if $\mathrm{PQ}=\mathrm{QR}$, then the tangents to the conic at $A$ and $B$ intersect on $C P$, so that $P$ is uniquely determined, and the tangents to the conic at P and C intersect on AB .

Also solved by ALFRED AEPPLI, University of Minnesota; JORDI DOU, Barcelona, Spain; J.T. GROENMAN, Arnhem, The Netherlands; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and GEORGE TSINTSIFAS, Thessaloniki, Greece.

## REFERENCE

1. D. Pedoe, A Course of Geometry for Colleges and Universities, Cambridge University Press, New York and Cambridge, England, 1970.
$\% \quad \% \quad$ \%
2. [1981: 146] Proposed by Ferrell Wheeler, student, Texas A \& M University, College Station, Texas.
For an n-point planar graph $G_{n}$ consisting of the points $A_{1}, A_{2}, \ldots, A_{n}$, let $N(i)$ be the number of points $A_{j}, j \neq i$, such that the line $A_{i} A_{j}$ divides the plane into two half-planes both containing the same number of points of $G_{n}$. The spread of $G_{n}$ is then defined by

$$
\sigma\left(G_{n}\right) \equiv \sum_{i=1}^{n} N(i)
$$

(a) For what values of $k$ does there exist an $n$-point graph $G_{n}$ such that $\sigma\left(G_{n}\right)=k$ ?
(b) In particular, prove or disprove that $\sigma\left(G_{6}\right) \neq 8$ for any 6-point graph $G_{6}$.

Solution by Jordi Dou, BarceZona, Spain.
(a) Such a graph exists if and only if $k$ is even. For suppose $k$ is even, and let $G_{k}$ be the graph consisting of the vertices of a convex $k$-gon. It is clear that $N(i)=1$ for $i=1,2, \ldots, k$ and that $\sigma\left(G_{k}\right)=k$. Conversely, let $G_{n}$ be any $n$-point graph. The lines $A_{i} A_{j}$ which satisfy the condition are counted

once in $N(i)$ and once in $N(j)$, and so $\sigma\left(G_{n}\right)$ is even.
(b) There are 6-point graphs $G_{6}$ such that $\sigma\left(G_{6}\right)=8$. For example, as shown in the figure, let $A_{1} A_{2} A_{3} A_{4} A_{5}$ be a convex pentagon, and let $A_{6}$ be an interior point of triangle $A_{1} A_{3} A_{4}$ above line $A_{2} A_{5}$. Then the points $A_{i}$ form a graph $G_{6}$ for which

$$
N(1)=N(2)=N(3)=N(4)=N(5)=1 \text { and } N(6)=3 \text {, }
$$

and so $\sigma\left(G_{6}\right)=8$.
A partial solution was received from BIKASH K. GHOSH, Bombay, India.

*     *         * 

637. [1981: 146] Proposed by Jayanta Bhattacharya, Midnapur, West Bengal, India. Given $a, b, c>0,0<A, B, C<\pi$, and

$$
\begin{aligned}
& a=b \cos C+c \cos B, \\
& b=c \cos A+a \cos C, \\
& c=a \cos B+b \cos A,
\end{aligned}
$$

prove that

$$
\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C} \text { and } A+B+C=\pi
$$

Solution by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.

With the given $a, b, c>0$, the system

$$
\left\{\begin{array}{l}
a=c y+b z  \tag{1}\\
b=c x+a z \\
c=b x+a y
\end{array}\right.
$$

has a unique solution $(x, y, z)$, since the determinant of the coefficients is $2 a b c \neq 0$; and the hypothesis of the problem states that this solution is

$$
\begin{equation*}
(x, y, z)=(\cos A, \cos B, \cos C) \tag{2}
\end{equation*}
$$

Since $\cos \theta<1$ for all $\theta$, the given system yields

$$
a<b+c, \quad b<c+a, \quad c<a+b
$$

hence there is a triangle, $T$, with sides $\alpha, b, c$. Let $T=A^{\prime} B^{\prime} C^{\prime}$, with $B^{\prime} C^{\prime}=\alpha$, etc. It is known from elementary trigonometry that

$$
\left(\cos A^{\prime}, \cos B^{\prime}, \cos C^{\prime}\right)
$$

is a solution of (1); hence, from (2),

$$
\cos A=\cos A^{\prime}, \quad \cos B=\cos B^{\prime}, \quad \cos C=\cos C^{\prime},
$$

and now $0<A, B, C<\pi$ and $0<A^{\prime}, B^{\prime}, C^{\prime}<\pi$ imply that

$$
A=A^{\prime}, \quad B=B^{\prime}, \quad C=C^{\prime} .
$$

Finally, the desired conclusion follows from the law of sines and the angle sum theorem applied to triangle $T$.

Also solved by JORDI DOU, Barcelona, Spain; BIKASH KUMAR GHOSH, Bombay, India; G.C. GIRI, Midnapore College, West Bengal, India; J.T. GROENMAN, Arnhem, The Netherlands; M.S. KLAMKIN, University of Alberta; L.F. MEYERS, The Ohio State University; V.N. MURTY, Pennsylvania State University, Capitol Campus; DEBASIS SANYAL, Jalpaiguri Government Engineering College, West Bengal, India; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; TAN DINH NGO, J.F. Kennedy H.S., Bronx, N.Y.; KENNETH S. WILLIAMS, Carleton University, Ottawa; J.A. WINTERINK, Albuquerque Technical Vocational Institute, New Mexico; and the proposer.

Editor's comment.
The solution we have featured would seem to be the "natural" one for this problem ("one from the book", as Paul Erdös would say [1]). It shows that the hypothesis of our problem implies that the quantities $a, b, c, A, B, C$ satisfy $A+B+C=\pi$ and not only the law of sines but also every known or yet to be discovered identity or inequality involving the sides and angles of a triangle. Except for Klamkin, whose solution approximated the one given here, all other solvers kept their eyes firmly fixed on the law of sines, resulting in longer (in some cases much longer) calculations. Some of these solutions had a few weak spots to which we complaisantly closed our eyes.

## REFERENCE

1. G.L. Alexanderson, "An Interview with Paul Erdös", Two-Year College Mathematics Journat, 12 (September 1981) 249-259, esp. p. 254.

*     * 
* 

638. [1981: 146] Proposed by S.C. Chan, Singapore.

A fly moves in a straight line on a coordinate axis. Starting at the origin, during each one-second interval it moves either a unit distance in the positive direction or, with equal probability, a unit distance in the negative direction.
(a) Obtain the mean and variance of its distance from the origin after $t$ seconds.
(b) The fly is trapped if it reaches a point 6 units from the origin in the positive direction. What is the probability that it will be trapped within 8 seconds?

Solution by Jordi Dou, Barcelona, Spain.
(a) The distance from the origin after $t$ seconds is given by the random variable $X=H-T$, where $H$ is the number of heads (a move in the positive direction) and $T$ is the number of tails (a move in the negative direction) in tosses of a fair
coin, and we have

$$
X=H-(t-H)=2 H-t_{0}
$$

Since the mean of $H$ is $t / 2$ and its variance $t / 4$, the mean of $X$ will be 0 and its variance $t$.
(b) For $t \leq 8$, let $P_{t}$ be the probability that the fly will be trapped in exactly $t$ seconds. Then we have

$$
P_{t}=0 \text { for } t=1,2,3,4,5,7, \quad P_{6}=\left(\frac{1}{2}\right)^{6}=\frac{1}{64},
$$

and

$$
P_{8}=P(X=4 \text { when } t=6) \cdot \frac{1}{2} \cdot \frac{1}{2}=P(H=5) \cdot \frac{1}{4}=\left(\frac{1}{2}\right)^{6}\left(\frac{6}{5}\right) \cdot \frac{1}{4}=\frac{3}{128} .
$$

Hence the probability that the fly will be trapped within 8 seconds is

$$
\sum_{t=1}^{8} P_{t}=\frac{1}{64}+\frac{3}{128}=\frac{5}{128} .
$$

Partially solved (part(b) only) by MAHESH KUMAR SANGANERIA, student, Midnapore College, West Bengal, India. Two incorrect solutions were received.
$\% \quad \% \quad \%$
639. [1981: 146] Proposed by Hayo Ahlburg, Benidorm, Alicante, Spain.

If $x+y+z=0$, prove that

$$
\frac{x^{5}+y^{5}+z^{5}}{5}=\frac{x^{3}+y^{3}+z^{3}}{3} \cdot \frac{x^{2}+y^{2}+z^{2}}{2}
$$

I. Solution by J.A. McCallum, Medicine Hat, Alberta.

This problem can be solved in several fancy ways, including by using symmetric functions and (good heavens!) by taking logarithms [1, p. 442], but probably no solution is more immediate than the following straightforward one, which uses only the most elementary algebra and can be understood by any Grade 10 student.

With $z=-(x+y)$, we have

$$
\frac{x^{3}+y^{3}+z^{3}}{3}=-x y(x+y), \quad \frac{x^{2}+y^{2}+z^{2}}{2}=x^{2}+x y+y^{2}
$$

and

$$
\frac{x^{5}+y^{5}+z^{5}}{5}=-x y(x+y)\left(x^{2}+x y+y^{2}\right)
$$

from which the desired result follows.
II. Solution by M.S. Klamkin, University of Alberta.

Let $S_{r}=x^{r}+y^{r}+z^{r}$ and suppose $S_{1}=0$; then $S_{3} / 3=x y z$ and we have, from

Crux 514 [1981: 56],

$$
\begin{equation*}
S_{n+3}=\frac{1}{3} S_{3} S_{n}+\frac{1}{2} S_{2} S_{n+1}, \quad n=0,1,2, \ldots \tag{1}
\end{equation*}
$$

With $n=2$ in (1), we obtain

$$
\begin{equation*}
\frac{S_{5}}{5}=\frac{S_{3}}{3} \cdot \frac{S_{2}}{2}, \tag{2}
\end{equation*}
$$

and this is the desired result, which is already known 「1, p. 442].
With $n=1$ in (1), we have $S_{4}=S_{2}^{2} / 2$. With this, and (2), and (1) with $n=4$, we obtain

$$
\begin{equation*}
\frac{S_{7}}{7}=\frac{S_{5}}{5} \cdot \frac{S_{2}}{2}, \tag{3}
\end{equation*}
$$

a result which is also known [1, p. 443].
Inspired by this problem, I proposed the following problem which was included in the recent 1982 U.S.A. Mathematical Olympiad:

Let $S_{r}=x^{r}+y^{r}+z^{r}$. Find all integer pairs $\{m, n\}$ such that

$$
\begin{equation*}
\frac{S_{m+n}}{m+n}=\frac{S_{m}}{m} \cdot \frac{S_{n}}{n} \tag{4}
\end{equation*}
$$

is meaningful and true for all real triples $(x, y, z)$ with $x+y+z=0$, having given that two solutions are $\{m, n\}=\{3,2\}$ and $\{5,2\}$.

It turns out that the two given solutions, established above in (2) and (3), are the only ones, as we now show.

Solution. We must obviously have $m n \neq 0$ and $m+n \neq 0$. Noreover, the requirement that ( $x, y, z$ ) must satisfy (4) even when $0 \in\{x, y, z\}$ shows that we must have $m, n>0$. We now consider three cases.

If $m$ and $n$ are both odd, then $(x, y, z)=(1,-1,0)$ does not satisfy (4), so this case produces no solution.

If $m$ and $n$ are both even, then substituting $(x, y, z)=(1,-1,0)$ in (4) yields

$$
\frac{2}{m+n}=\frac{2}{m} \cdot \frac{2}{n} \quad \text { or } \quad(m-2)(n-2)=4
$$

so $m=n=4$. But then $(x, y, z)=(1,1,-2)$ does not satisfy (4). Again, no solution.
Finally, suppose one of the pair $\{m, n\}$ is odd and the other even. Without loss of generality, we may assume that $m=2 q+1$ and $m+n=2 p+1$, so that $n=2(p-q)$ with $p>q$. Substituting $(x, y, z)=(t, 1,-t-1)$ in (4) yields

$$
\begin{equation*}
\frac{t^{2 p+1}+1-(t+1)^{2 p+1}}{2 p+1}=\frac{t^{2 q+1}+1-(t+1)^{2 q+1}}{2 q+1} \cdot \frac{t^{2(p-q)}+1+(t+1)^{2(p-q)}}{2(p-q)} \tag{5}
\end{equation*}
$$

Since (5) is to hold for all $t>0$, we divide the two sides of (5) by

$$
t^{2 p}=t^{2 q} \cdot t^{2(p-q)}
$$

and then let $t \rightarrow \infty$. The result is

$$
-1=-1 \cdot \frac{2}{2(p-q)}, \quad \text { or } \quad p-q=1
$$

Thus we must have $n=2$ in any solution. Since $p=q+1$, setting $t=1$ in (5) yields

$$
\frac{2^{2 q+3}-2}{2 q+3}=\frac{2^{2 q+1}-2}{2 q+1} \cdot \frac{2^{2}+2}{2}
$$

which is equivalent to

$$
(2 q-5) \cdot 2^{2 q+1}+8 q+16=0
$$

for which the only solutions are $q=1$ and 2 . Thus $m=3$ or 5 , and the only solutions are the two given ones.

The problem is more complicated if we add the requirement that $x y z \neq 0$, because then negative values of $m$ and $n$ must be considered. The solutions $\{m, n\}=\{3,2\}$ and $\{5,2\}$ are still valid, and an analysis similar to the above (but much longer) shows that the only other possible solution of (4) is $\{m, n\}=\{3,-1\}$, that is, that

$$
\begin{equation*}
\frac{x^{2}+y^{2}+z^{2}}{2}=\frac{x^{3}+y^{3}+z^{3}}{3} \cdot \frac{x^{-1}+y^{-1}+z^{-1}}{-1} \tag{6}
\end{equation*}
$$

holds whenever $x+y+z=0$ and $x y z \neq 0$. To verify that (6) is indeed true, we merely note that

$$
\Sigma x^{2}=-2 \Sigma y z, \quad \Sigma x^{3}=3 x y z, \quad \Sigma x^{-1}=\Sigma y z / x y z,
$$

and (6) follows.
Also solved by JAYANTA BHATTACHARYA, Midnapur, West Bengal, India; W.J. BLUNDON, Memorial University of Newfoundland; S.C. CHAN, Singapore; CHRIS DEARLING, Queen Elizabeth Park School, Oakville, Ontario: JORDI DOU, Barcelona, Spain; ROLAND H. EDDY, Memorial University of Newfoundland; BENJI FISHER, Bronx H.S. of Science, Bronx, N.Y.; JACK GARFUNKEL, Flushing, N.Y.; BIKASH K. GHOSH, Bombay, India; G.C. GIRI, Midnapore College, West Bengal, India; J.T. GROENMAN, Arnhem, The Netherlands; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; LEROY F. MEYERS, The Ohio State University; V.N. MURTY, Pennsylvania State University, Capitol Campus; BOB PRIELIPP, University of Wisconsin-Oshkosh; STANLEY RABINOWITZ, Digital Equipment Corp., Merrimack, New Hampshire; DEBASIS SANYAL, Jalpaiguri Government Engineering College, West Bengal, India; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; TAN DINH NGO, student, J.F. Kennedy H.S., Bronx, N.Y.; GEORGE TSINTSIFAS, Thessaloniki, Greece; KENNETH M. WILKE, Topeka, Kansas; KENNETH S. WILLIAMS, Carleton University, Ottawa; and the proposer.

## Editor's conment

It happens occasionally that a proposer unwittingly submits, and we publish, a problem that turns out to be already known. We are seldom sorry when this happens, because our readers usually come up with an improved solution, or with an interesting extension, or generalization, or new related problem. Before his problem was published, our proposer, in looking through the back volumes of this journal, came across the following problem [1976: 180]: If $x+y+z+u=0$, then

$$
\begin{equation*}
\frac{x^{5}+y^{5}+z^{5}+u^{5}}{5}=\frac{x^{3}+y^{3}+z^{3}+u^{3}}{3} \cdot \frac{x^{2}+y^{2}+z^{2}+u^{2}}{2} \tag{7}
\end{equation*}
$$

He saw that his problem followed by setting $u=0$ and asked to have it withdrawn. But we were aware of the results (2) and (3), and were curious to know if there were other solutions to (4). So we published the problem in deadpan fashion, hoping that some reader would answer our unspoken question, and promised our proposer that when the time came to publish a solution we would inform our readers of his reluctance to have the problem published. Once again, our readers came through: solution II completely illuminates the question. Now who will do for (7) what Klamkin did for our present problem?

## REFERENCE

1. H.S. Hall and S.R. Knight, Higher Algebra, Macmillan, London, 1887 (first of many editions).
$*$
2. [1981: 147] Proposed by George Tsintsifas, Thessaloniki, Greece. Let 0 be an interior point of an oval (compact convex set) in the real Euclidean plane $R^{2}$, and let $\chi(\phi)$ be the length of the "chord" of the oval through 0 making an angle $\phi$ with some fixed ray $0 x$. If $K$ is the area of the oval, prove that

$$
K \geq \frac{1}{2} \int_{0}^{\pi / 2} \times(\phi) \times(\phi+\pi / 2) d \phi .
$$

Solution by J. Chris Fisher, University of Regina.
We can relax the given conditions somewhat. Suppose $C$ is a simple closed curve, enclosing a region of area $K$, such that, for each $\phi, 0 \leq \phi \leq 2 \pi$, there corresponds a chord of $C$, of length $X(\phi)$, with endpoints $P=P(\phi)$ and $Q=Q(\phi)$, making an angle $\phi$ with a fixed reference direction. Assume furthermore that $P$ and $Q$ both traverse $C$ in the counterclockwise sense as $\phi$ increases. (Note that the chords PQ do not necessarily have a common point 0 for all $\phi$.$) Let a=\chi(\phi)$ and $b=\chi(\phi+\theta)$ for some fixed
angle $\theta, 0 \leq \theta \leq 2 \pi$, and let

$$
I=\int_{0}^{2 \pi} a b d \phi
$$

Even if the curve $C$ is not convex, the desired inequality will follow from the following

THEOREM. If there is a point 0 such that $0 \in \mathrm{PQ}$ for all $\phi$, then $K \geq I / 8$, with equality (at least) if $C$ is a circle with center 0.

Proof. For all $\phi$, let $a_{1}, a_{2}, b_{1}, b_{2}$ be the lengths of segments $\mathrm{OP}(\phi), \mathrm{OQ}(\phi)$, $O P(\phi+\theta), O Q(\phi+\theta)$, respectively, as shown in the figure. We note that


$$
K=\frac{1}{2} \int_{0}^{2 \pi} a_{1}^{2} d \phi=\frac{1}{2} \int_{0}^{2 \pi} a_{2}^{2} d \phi=\frac{1}{2} \int_{0}^{2 \pi} b_{1}^{2} d \phi=\frac{1}{2} \int_{0}^{2 \pi} b_{2}^{2} d \phi
$$

Since $a_{i}^{2}+b_{j}^{2} \geq 2 a_{i} b_{j}(i, j=1,2)$, it follows that

$$
\begin{aligned}
8 K & =\int_{0}^{2 \pi}\left(a_{1}^{2}+a_{2}^{2}+b_{1}^{2}+b_{2}^{2}\right) d \phi \\
& \geq \int_{0}^{2 \pi}\left(a_{1} b_{1}+a_{1} b_{2}+a_{2} b_{1}+a_{2} b_{2}\right) d \phi \\
& =\int_{0}^{2 \pi}\left(a_{1}+a_{2}\right)\left(b_{1}+b_{2}\right) d \phi \\
& =\int_{0}^{2 \pi} a b d \phi \\
& =I,
\end{aligned}
$$

and the desired inequality follows. Equality holds if and only if

$$
\begin{equation*}
a_{i}(\phi)=b_{j}(\phi), \quad 0 \leq \phi \leq 2 \pi, \quad i, j=1,2, \tag{1}
\end{equation*}
$$

that is, if and only if the curve $C$ is invariant under a rotation of $\theta$ about 0 . So equality certainly holds if $C$ is a circle with center 0 ; furthermore, if (1) holds for all $\theta, 0 \leq \theta \leq 2 \pi$ (as well as all $\phi$ ), then $C$ must be a circle with center 0 . This completes the proof of the theorem.

We apply this to our problem (ignoring the unnecessary convexity requirement). With $\theta=\pi / 2$, we have

$$
K \geq \frac{I}{8}=\frac{1}{8} \int_{0}^{2 \pi} a b d \phi=\frac{1}{2} \int_{0}^{\pi / 2} a b d \phi=\frac{1}{2} \int_{0}^{\pi / 2} x(\phi) x(\phi+\pi / 2) d \phi,
$$

as required. Equality holds if and only if the curve $C$ is invariant under a rotation of $\pi / 2$ about 0 . Thus, equality holds if $C$ is a circle, a square, or, in general, a regular $4 n$-gon (convex or star-shaped), all with center 0 .

There has been a recent interest in the quantity $I$ when $\theta=0$, that is, in

$$
I=\int_{0}^{2 \pi} x^{2}(\phi) d \phi
$$

Curiously, if $C$ is convex and $P Q$ is either (i) the area bisector for all $\phi$, or (ii) the perimeter bisector for all $\phi$, then the inequality is reversed: $K \leq I / 8$. This was conjectured by Erwin Lutwak and proved by Goodey [1]. The proof is vastly more complicated than what was required for the present problem.

Also solved by ALfRED AEPPLI, University of Minnesota; JORDI DOU, Barcelona, Spain; M.S. KLAMKIN, University of Alberta; BENGT MÅNSSON, Lund, Sweden; LEROY F. MEYERS, The Ohio State University; and the proposer.

## REFERENCE

1. Paul R. Goodey, "Mean Square Inequalities...", Preprint, Royal Holloway College, Egham, Surrey, England.

* 
* 

$\%$
641. [1981: 147] Proposed by AlZan Wm. Johnson Jr., Washington, D.C. Deduce three different solutions of the decimal multiplication

$$
\begin{array}{r}
\text { TWO } \\
\text { SIX } \\
\hline \text { TWELVE }
\end{array}
$$

from these doubly-true clues:
(a) In the first solution, the sum TWO + SIX is a cube and both SIX -3 and SIX +5 are prime numbers.
(b) In the second solution, TWO +9 is a prime number and both TWO +7 and 4-SIX + TWO/ 2 are square numbers.
(c) In the third solution, $3 \cdot$ SIX $+2 \cdot T W 0+3$ is a square number and SIX equals a power of 2 multiplied by a power of 3 .

Solution by Gali Salvatore, Perkins, Québec.
Clearly $S=9$. Since TWELVE/TWO $=$ SIX $<1000$, we must have TWO > TWE and $8 \geq$ TWO - TWE. Observing that TWO•SIX = TWELVE $>1000 \cdot$ TWE, we have

```
8000 \geq 1000•TWO - 1000•TWE > 1000•TWO - TWO•SIX,
```

and so

$$
\begin{equation*}
\text { TWO }<\frac{8000}{1000-\text { SIX }} . \tag{1}
\end{equation*}
$$

(a) There are three pairs of three-digit primes beginning with 9 and differing by 8 , namely:

$$
911 \text { and 919, } 929 \text { and 937, } 983 \text { and 991; }
$$

and corresponding to these as potential values of SIX are

$$
\begin{equation*}
\text { 914, 932, } 986 \tag{2}
\end{equation*}
$$

Bearing in mind that TWO < 8000/(1000-986) < 572 from (1) and (2), we have

$$
1117=203+914 \leq \text { TWO }+ \text { SIX }<572+986=1558,
$$

and the cube must be 1331. Now 1331 - SIX = TWO leads to digit duplications for the first two values of SIX in (2), so we must have SIX $=986$, TWO $=345$, and the unique reconstruction is

$$
345 \cdot 986=340170
$$

(b) We seek a prime that is 2 more than a three-digit square. The only such primes are 227 and 443, and the corresponding values of TWO are 218 and 434, the last of which has duplicated digits. So TWO $=218$. From this and (1), we have 963 < SIX $\leq 976$, and so

$$
\begin{equation*}
3961<4 \cdot \text { SIX }+109 \leq 4013 . \tag{3}
\end{equation*}
$$

The only square satisfying (3) is 3969 , from which SIX $=965$, and the unique reconstruction is

$$
218 \cdot 965=210370
$$

(c) The double inequality $900<\operatorname{SIX}=2^{m_{3} n}<1000$ is possible only for SIX $=2^{2} \cdot 3^{5}=972$. If

$$
3 \cdot 972+2 \cdot \text { TWO }+3=a^{2},
$$

then $a^{2}$ is odd and, from this and (1),

$$
103<\text { TWO }=\frac{a^{2}-2919}{2}<286
$$

so that $3125<a^{2}<3491$ and $a^{2}=3249$ or 3481. If $a^{2}=3481$, then TWO $=281$ and $2=X=T$. So $a^{2}=3249$, from which TWO $=165$, and we have the unique reconstruction

```
                                    165.972 = 160380.
```

Also solved by N. ESWARAN, student, Indian Institute of Technology, Kharagpur, West Bengal, India; BIKASH K. GHOSH, Bombay, India; J.A. McCALLUM, Medicine Hat, Alberta; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec (parts (a) and (b) only); CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.
McCallum noted that the three solutions given above are the only decimal solutions of
TWO•SIX = TWELVE.

We'11 take his word for it.
*
*
*
642. [1981: 147] Proposed by Charles W. Trigg, San Diego, California.

In the convenient notation $a_{4}=a \alpha a \alpha$, the subscript indicates the number of consecutive like digits. Thus $22_{2} 6_{0} 9^{4}=229994$.

The base ten number $10_{k} 30_{k}$ is known to be a palindromic prime for $k=0,1,2,3$. Is it composite for any $k>3$, and if so what are its factors?
I. Report from CRAY-1, S/N 22 computer, Lawrence Livermore Laboratory, University of California (transmitted by Harry L. Nelson).

```
                    131 = PRIME
                    10301 = PRIME
                    1003001 = PRIME
                    100030001 = PRIME
                    10000300001 = 19.526331579
                    1000003000001 = 29.34482862069
                    100000030000001 = 139•636761•1129819
                    10000000300000001 = 61.163934431147541
                    1000000003000000001 = 59.281.60317269015019
                    100000000030000000001 = 26176969.3805613959129
                    10000000000300000000001 = PRIME
                    1000000000003000000000001=661•4752481.318330426589661
100000000000030000000000001=19•11059•64019•7433984563573699
1000000000000030000000000... SORRY, GOT TO GET BACK TO WORK NOW.
```

II. Solution by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hompshire.

Let $N(k)=10_{k} 30_{k} 1, k=0,1,2, \ldots$. It is easy to show that there are infinitely many composite numbers $N(k)$, as a consequence of the following

THEOREM. Suppose $p \mid N$, where $p$ is a prime and $N$ a positive integer in decimal notation. If $N^{\prime}$ is the number obtained by inserting $p-1$ consecutive zeros between two consecutive digits (or at the end) of $N$, then $p \mid N^{\prime}$.

For example, 7|42 and 7|40000002.

Proof. The theorem clearly holds for $p=2,3,5$, so we assume $p>5$. The desired result then follows from the fact that, by Fermat's Theorem,

$$
n \cdot 10^{p-1} \equiv n(\bmod p), \quad n=0,1,2, \ldots
$$

We apply this to our problem. It is known that $N(k)$ is prime for $k=0,1,2,3$, as stated in the proposal, and a smidgin of compute power shows that

$$
19|N(4), \quad 19| N(12), \quad 29|N(5), \quad 29| N(21), \quad 59|N(8), \quad 59| N(48) .
$$

Hence, for $i=0,1,2, \ldots$, we have

$$
\begin{array}{rc}
131 \mid N(130 i), & 10301 \mid N(1+10300 i), \text { etc., } \\
19 \mid N(4+18 i), & 19 \mid N(12+18 i), \\
29 \mid N(5+28 i), & 29 \mid N(21+28 i), \\
59 \mid N(8+58 i), & 59 \mid N(48+58 i) .
\end{array}
$$

Also solved by E.C. BUISSANT DES AMORIE, Amstelveen, The Netherlands; CLAYTON W. DODGE, University of Maine at Orono; BIKASH K. GHOSH, Bombay, India; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; L.F. MEYERS, The Ohio State University; JACQUES SAUVE, University of Waterloo (now at the University of Campina Grande, Brazil); and KENNETH M. WILKE, Topeka, Kansas.

Editor's comment.
The number $N(7433984563573710)$ would be approximately 30 billion kilometres long if typed in full, enough to give even a CRAY-1 computer indigestion. Yet it follows from our solutions I and II that this number is divisible by the prime 7433984563573699. Check that on your abacus!
*
*
643. [1981: 147] Proposed by J.T. Groenman, Arnhem, The Netherlands.

For $i=1,2,3$, a given triangle has vertices $A_{i}$, interior angles $\alpha_{i}$, and sides $\alpha_{i}$. Segment $A_{i} D_{i}$, which terminates in $\alpha_{i}$, bisects angle $\alpha_{i} ; m_{i}$ is the perpendicular bisector of $A_{i} D_{i}$; and $E_{i}=\alpha_{i} \cap m_{i}$. Prove that
(a) the three points $E_{i}$ are collinear;
(b) the three segments $E_{i} A_{i}$ are tangent to circumcircle of the triangle;
(c) if $p_{i}$ is the length of $E_{i} A_{i}$, and if $a_{1} \leq a_{2} \leq a_{3}$, then $\left(1 / p_{3}\right)+\left(1 / p_{1}\right)=1 / p_{2}$.

Solution by George Tsintsifas, Thessaloniki, Greece.
Obviously, the $E_{i}$ are the centers and $E_{i} A_{i}$ the radii of the three Apollonian circles of triangle $A_{1} A_{2} A_{3}$. The circumcircle of the triangle is orthogonal to each of the Apollonian circles, and the latter have two common points, J and $\mathrm{J} '$, the isodynamic points of the triangle 「1, pp. 260-262].
(a) The points $E_{i}$ are collinear on the Lemoine axis of the triangle [1, p. 253]. (This is obvious from the fact that the lines $E_{2} E_{3}, E_{3} E_{1}, E_{1} E_{2}$ are all perpendicular to the common chord $\mathrm{JJ}^{\prime}$.)
(b) This part follows immediately from the orthogonality property mentioned above.
(c) It is easy to show (or see [1, p. 156, Art. 319]) that $\bar{E}_{1} \bar{A}_{2} / \bar{E}_{1} \bar{A}_{3}=a_{3}^{2} / a_{2}^{2}$, from which, if $\alpha_{2} \neq \alpha_{3}$,

Thus

$$
p_{1}^{2}=\bar{E}_{1} A_{1}^{2}=\overline{\mathrm{E}}_{1} \overline{\mathrm{~A}}_{2} \cdot \overline{\mathrm{E}} 1 \mathrm{~A}_{3}=\frac{\left(\alpha_{1} \alpha_{2} \alpha_{3}\right)^{2}}{\left(a_{2}^{2}-a_{3}^{2}\right)^{2}},
$$

and so

$$
p_{1}=\frac{a_{1} a_{2} a_{3}}{\left|a_{2}^{2}-a_{3}^{2}\right|}
$$

with similar formulas for $p_{2}$ and $p_{3}$. Hence, if $a_{1}<a_{2}<\alpha_{3}$, we have

$$
\frac{1}{p_{3}}+\frac{1}{p_{1}}-\frac{1}{p_{2}}=\frac{\left(a_{2}^{2}-a_{1}^{2}\right)+\left(a_{3}^{2}-a_{2}^{2}\right)-\left(a_{3}^{2}-a_{1}^{2}\right)}{a_{1} a_{2} a_{3}}=0
$$

as required.
We have so far assumed that the $a_{i}$ are all distinct. If, for example, $a_{2}=a_{3}$, then $E_{1}$ is a point at infinity, we replace $1 / p_{1}$ by 0 , and the desired result still holds.

Also solved by O. BOTTEMA, Delft, The Netherlands; JORDI DOU, Barcelona, Spain; ROLAND H. EDDY, Memorial University of Newfoundland; BENJI FISHER, student, Bronx H.S. of Science; BIKASH K. GHOSH, Bombay, India; STANLEY RABINOWITZ, Digital Equipment Corp., Merrimack, New Hampshire; KeSIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer.

## REFERENCE

1. Nathan Altshiller Court, College Geometry, Barnes and Noble, New York, 1952.

*     *         * 

644, [1981: 147] Proposed by Jack Garfunkel, Flushing, N.Y.
If I is the incenter of triangle $A B C$ and lines $A I, B I$, CI meet the circumcircle of the triangle again in $D, E, F$, respectively, prove that

$$
\frac{A I}{I D}+\frac{B I}{I E}+\frac{C I}{I F} \geq 3 .
$$

I. Solution by S.C. Chan, Singapore.

We adopt the usual meanings for $R, r, s, a, b, c$. Referring to the figure, we have

$$
\begin{aligned}
\angle D I B & =\frac{1}{2}(\operatorname{arc} B D+\operatorname{arc} E A) \\
& =\frac{1}{2} \operatorname{arc} D E \\
& =\underline{I D B I},
\end{aligned}
$$

so

$$
I D=D B=2 R \sin \frac{A}{2} ;
$$

and it is easily seen that

$$
A I=r \csc \frac{A}{2}=4 R \sin \frac{B}{2} \sin \frac{C}{2}
$$

Thus, from the half-angle formulas we have

$$
\frac{A I}{I D}=\frac{2 \sin \frac{B}{2} \sin \frac{C}{2}}{\sin \frac{A}{2}}=\frac{2(s-a)}{a}
$$



With this and two similar results, we get

$$
\begin{aligned}
\frac{\mathrm{AI}}{\mathrm{ID}}+\frac{\mathrm{BI}}{\mathrm{IE}}+\frac{\mathrm{CI}}{\mathrm{IF}} & =\frac{2(s-a)}{a}+\frac{2(s-b)}{b}+\frac{2(s-c)}{c} \\
& =\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{c}{a}+\frac{a}{c}\right)+\left(\frac{a}{b}+\frac{b}{a}\right)-3 \\
& \geq 3
\end{aligned}
$$

since $(x / y)+(y / x) \geq 2$ whenever $x, y>0$. Equality clearly holds just when $a=b=c$.
II. Solution by Roland $H$. Eddy, Memorial University of Newfoundland.

We will sharpen the inequality and also find an upper bound. With

$$
A I=r \csc \frac{A}{2}, \quad I D=2 R \sin \frac{A}{2}, \quad \sin ^{2} \frac{A}{2}=\frac{(s-b)(s-c)}{b c}, \text { etc., }
$$

we obtain

$$
\begin{equation*}
\frac{A I}{I D}+\frac{B I}{I E}+\frac{C I}{I F}=\frac{r}{2 R}\left\{\frac{s(b c+c a+a b)-3 a b c}{(s-a)(s-b)(s-c)}\right\} . \tag{1}
\end{equation*}
$$

With the known relations

$$
b c+c a+a b=r^{2}+4 R r+s^{2}, \quad a b c=4 R r s, \quad r^{2} s=(s-a)(s-b)(s-c),
$$

the right member of (1) becomes

$$
\begin{equation*}
\frac{r^{2}-8 R r+s^{2}}{2 R r} \tag{2}
\end{equation*}
$$

With the help of (1) and (2), the Steinig-Blundon inequalities [1, pp. 50-51],

$$
\begin{gathered}
-156- \\
16 R r-5 r^{2} \leq s^{2} \leq 4 R^{2}+4 R r+3 r^{2},
\end{gathered}
$$

are easily shown to be equivalent to

$$
\begin{equation*}
2\left(2-\frac{r}{R}\right) \leq \frac{\mathrm{AI}}{\mathrm{ID}}+\frac{\mathrm{BI}}{\mathrm{IE}}+\frac{\mathrm{CI}}{\mathrm{IF}} \leq 2\left(\frac{R}{r}-1+\frac{r}{R}\right) . \tag{3}
\end{equation*}
$$

In a certain sense (explained in the reference), the upper and lower bounds in (3) are the best possible. We now use the familiar $R \geq 2 r$ to obtain our final result:

$$
\begin{equation*}
3 \leq 2\left(2-\frac{r}{R}\right) \leq \frac{\mathrm{AI}}{\mathrm{ID}}+\frac{\mathrm{BI}}{\mathrm{IE}}+\frac{\mathrm{CI}}{\mathrm{IF}} \leq 2\left(\frac{R}{r}-1+\frac{r}{R}\right) . \tag{4}
\end{equation*}
$$

The lower bound is attained just when $R=2 r$, that is, just when the triangle is equilateral, and then equality holds throughout in (4).
III. Comment by M.S. Klamkin, University of Alberta.

With $A I=r \csc (A / 2)$ and $I D=2 R \sin (A / 2)$, etc., the proposed inequality becomes

$$
\frac{r}{2 R} \Sigma \csc ^{2} \frac{A}{2} \geq 3
$$

[Having established this], we therefore get, with $R \geq 2 r$,

$$
\Sigma \csc ^{2} \frac{A}{2} \geq \frac{6 R}{r} \geq 12
$$

which sharpens the known inequality $\Sigma \csc ^{2}(\mathrm{~A} / 2) \geq 12$ [1, p.32].
It may be of interest to note also at this time that

$$
\begin{equation*}
\frac{A I}{I D} \cdot \frac{B I}{I E} \cdot \frac{C I}{I F} \leq 1 \tag{5}
\end{equation*}
$$

This follows immediately from

$$
\frac{A I}{I D}=\frac{2 \sin \frac{B}{2} \sin \frac{C}{2}}{\sin \frac{A}{2}}, \text { etc., } \quad r=4 R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2},
$$

and $R \geq 2 r$.
IV. Conment by Jordi Dou, Barcelona, Spain.

The inequality in this problem is interesting, and surprising in view of (5) and of the known result [2]

$$
\frac{A I+B I+C I}{I D+I E+I F} \leq 1
$$

Also solved by JAYANTA BHATTACHARYA, Midnapur, West Bengal, India; W.J. BLUNDON, Memorial University of Newfoundland; O. BOTTEMA, Delft, The Netherlands; JORDI DOU, Barcelona, Spain; BIKASH K. GHOSH, Bombay, India; G.C. GIRI, Midnapore College, West Bengal, India; J.T. GROENMAN, Arnhem, The Netherlands; M.S. KLAMKIN, Univer-
sity of Alberta; V.N. MURTY, Pennsylvania State University, Capitol Campus; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; TAN DINH NGO, student, J.F. Kennedy H.S., Bronx, N.Y.; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer.

Editor's comment.
Klamkin stated without proof that the following three inequalities are each equivalent to that of our problem:

$$
\begin{gathered}
2 \Sigma \sin ^{2} \frac{B}{2} \sin ^{2} \frac{C}{2} \geq 3 \pi \sin \frac{A}{2}, \\
9+2 \Sigma \cos B \cos C \geq 7 \Sigma \cos A, \\
s^{2} \geq 14 R r-r^{2} .
\end{gathered}
$$

Two recent problems [3, 4] are of related interest. The first replaces the incentre $I$ by the centroid $G$ and states that

$$
\frac{A G}{G D}+\frac{B G}{G E}+\frac{C G}{G F}=3
$$

holds for every triangle. The second replaces $I$ by an arbitrary point $P$ inside the circumcircle and asks for the set of points $P$ such that

$$
\frac{A P}{P D}+\frac{B P}{P E}+\frac{C P}{P F} \leq 3 .
$$

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1. 0. Bottema et al., Geometric Inequalities, Wolters-Noordhoff, Groningen, 1969.
1. O.P. Lossers, Solution of Problem S23 (proposed by Leon Bankoff and Jack Garfunkel), American Mathematical Monthly, 88 (1981) 536-537.
2. Thu Pham, Solution of Problem 1119 (proposed by K.R.S. Sastry), Mathematics Magazine, 55 (1982) 180-182.
3. J.C. Binz, Solution of Problem 1120 (proposed by Peter Ørno), ibid., 55 (1982) 182.
4. [1981: 177] Proposed by Dmitry P. Mav20, Moscow, U.S.S.R.

Can the product of four successive terms of an arithmetic progression with rational terms be an exact fourth power?

Solution by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.

The answer is, in general, NO. For let

$$
\begin{equation*}
(a-3 d)(a-d)(a+d)(a+3 d)=y^{4}, \tag{1}
\end{equation*}
$$

where we assume, without loss of generality, that ( $a, d, y$ ) is an integer triple. (If $a=p / q$ and $d=r / s$, we can always multiply both sides of (1) by $q^{4} s^{4}$.)
Equation (1) is equivalent to

$$
(2 d)^{4}+y^{4}=\left(a^{2}-5 d^{2}\right)^{2}
$$

This equation has no integer solutions if $d y \neq 0$, for Euler showed [1] that the equation $x^{4}+y^{4}=z^{2}$ has no integer solutions if $x y \neq 0$. There are, trivially, infinitely many solutions if $d y=0$.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; M.S. KLAMKIN, University of Alberta; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

REFERENCE

1. Leonard Eugene Dickson, History of the Theory of Numbers, Chelsea, New York, 1952, Vol. II, p. 618.

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## THE PUZZLE CORNER

Puzzle No. 16: Alphametic
SOLVE
PUZZLE my friend, there are six ways to do, Only two of which here are taken as true:

Trial and error, hard work, intelligence rare, Even SLEEP is required for the first of the pair. Now answer me true: Is ZERO a square? For the second solution-you'll get it in time, Roman $V$ is quite odd, by itself not a crime, But what might be implied is a ZERO that's prime!

Puzzle No. 17: Alphametic
Both ONE and EIGHT are cubes this time.
The old queen of math said: "That suits me just fine,"
With wavering voice, a grimace upon her,
"If only ELEVEN hadn't been prime
Then THIRTEEN would surely have taken the honour."
HANS HAVERMANN
Weston, Ontario
Answer to Puzzle No. 15 「1982: 96]: No table; not able; notable.

