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CONTENTS

THE GAUSSIAN "30"

MICHAEL DALEZMAN

It seems reasonable to speculate that every natural number is special in some nontrivial way (and if one were not, *that* would make it special). The number 30 has a well-known property that distinguishes it from all other natural numbers. *It is the largest integer n with the following property:*

$$1 < p < n \text{ and } (p,n) = 1 \implies p \text{ is a prime.}$$

According to Dickson [2], this was first proved in 1893 by S. Schatunowsky with the help of Bertrand's Postulate (1845, proved in 1850 by Chebyshev): *If $n > 1$, there is at least one prime between n and $2n$.* Schatunowsky's proof was apparently not well publicized, for the question appeared again in 1899 in a French mathematical journal [8]. Of the solutions later published, E. Maillet [4], P. Wolfskehl [4, 10], and E. Landau [5] used the following route:

First, with the help of Bertrand's Postulate they proved what we now call Bonse's Inequality [9]: *If $p_1, p_2, p_3, \ldots$ is the sequence of rational primes and $i \geq 4$, then $p_1 p_2 \ldots p_i > p_{i+1}^2$.* This result was then used to prove the property of number 30. Landau [5] also gave a direct proof. It was based on the evaluation of certain infinite series and used neither Bertrand's Postulate nor Bonse's Inequality. In 1907 Bonse gave an elementary proof of the inequality named after him and then used it to prove the property of the number 30 [1, 7, 9].

In this article, we will prove that $5+5i$ is the Gaussian integer corresponding to the rational integer 30, that is, $5+5i$ *is (to within a unit factor) the Gaussian integer α of largest possible norm $N(\alpha)$ with the following property:*

$$1 < N(\pi) \leq N(\alpha) \text{ and } (\pi,\alpha) = 1 \implies \pi \text{ is a Gaussian prime.}$$

Our approach will follow that of Maillet and Wolfskehl. After establishing Bertrand's Postulate for Gaussian integers, we will prove Bonse's Inequality and use it to establish the property of $5+5i$. In the sequel, p will denote a rational prime; Gaussian integers will be represented by Greek letters and, in particular, π and ρ will always denote Gaussian primes; and $N(\alpha)$ or $N\alpha$ will be used indifferently for the norm of α.

It is known [6] that the norm N is a multiplicative function, that is, that $N(\alpha\beta) = N(\alpha)N(\beta)$ and that the Gaussian primes consist of

(a) $1+i$, of norm 2;

(b) $a \pm bi$, where the norm a^2+b^2 is a rational prime congruent to 1 modulo 4;

(c) p, of norm p^2, where p is a rational prime congruent to 3 modulo 4.

Let π_1, π_2, π_3, ... be the sequence of Gaussian primes ordered according to nondecreasing norms. The first six Gaussian primes in the sequence are

$$\pi_1 = 1 + i \quad \text{with norm } 2,$$

$$\pi_2, \pi_3 = 2 \pm i \quad \text{with norm } 5,$$

$$\pi_4 = 3 \quad \text{with norm } 9,$$

$$\pi_5, \pi_6 = 2 \pm 3i \quad \text{with norm } 13.$$

Erdös proved [3] a form of Bertrand's Postulate for primes of the form $4n+1$: if $x \geq 6.5$, then there exists a prime $p \equiv 1 \pmod 4$ such that $x < p \leq 2x$. We can use this to prove

BERTRAND'S POSTULATE FOR GAUSSIAN INTEGERS. If $x \geq 2.5$, then there exists a Gaussian prime ρ such that $x < N\rho \leq 2x$.

Proof. If $x \geq 6.5$, then the p which is provided by Erdös's result gives rise to a Gaussian prime ρ (in fact to two conjugate primes) such that $x < N\rho = p \leq 2x$. If $2.5 \leq x < 6.5$, then π_2 or π_4 will have norm in $(x, 2x]$. □

The following results are therefore valid for $i \geq 2$: there exists a Gaussian prime ρ such that $N\pi_i < N\rho \leq 2N\pi_i$; hence

$$N\pi_{i+1} \leq 2N\pi_i \quad \text{and} \quad N\pi_{i+1}^2 \leq 4N\pi_i^2 \leq N\pi_i^3.$$

We can now prove

BONSE'S INEQUALITY FOR GAUSSIAN PRIMES. If $i \geq 5$, then

$$N\pi_1 N\pi_2 \ldots N\pi_{i-1} > N\pi_i^2. \tag{1}$$

Proof. The inequality is easily verified for $i = 5$. Proceeding inductively and assuming (1), we get

$$N\pi_1 N\pi_2 \ldots N\pi_{i-1} N\pi_i > N\pi_i^3 \geq N\pi_{i+1}^2. \quad \square$$

We are now ready to prove that $5+5i = \pi_1\pi_2\pi_3$ is the Gaussian "30". We first show that

$$1 < N\beta \leq N(5+5i) = 50 \text{ and } (\beta, 5+5i) = 1 \implies \beta \text{ is a prime}. \tag{2}$$

Suppose that β is a Gaussian integer satisfying the hypothesis of (2). If β is not a prime, then there are primes ρ and π with, say, $N\rho \leq N\pi$ such that $\rho\pi | \beta$. Then

$$N\rho^2 \leq N\rho N\pi \leq N\beta \leq 50 \quad \text{and} \quad N\rho \leq \sqrt{50}.$$

Thus $N\rho \leq 7$ and $\rho = \pi_1$, π_2, or π_3. Since each of these primes divides $5+5i$, we have the contradiction $(\beta, 5+5i) \neq 1$, and thus (2) is established.

To prove that $5+5i$ is the Gaussian integer of largest possible norm having property (2), let α be a Gaussian integer having this property and suppose that $N\alpha \geq 50$. For $i = 1,2,3$, we have

$$N\pi_i^2 \leq 25 < 50 \leq N\alpha.$$

Thus π_1^2, π_2^2, and π_3^2 are nonprimes none of which, by (2), can be relatively prime to α. Hence $\pi_1\pi_2\pi_3|\alpha$ and $\alpha = \gamma\pi_1\pi_2\pi_3$. We will now assume that $N\alpha \geq 81$ and get a contradiction. First observe that $N\pi_4^2 = 81 \leq N\alpha$, and $\pi_4|\alpha$ follows from (2). Now let j be such that

$$\pi_1\pi_2\ldots\pi_{j-1}|\alpha \quad \text{and} \quad \pi_j \nmid \alpha.$$

Then $j \geq 5$ and, by Bonse's Inequality,

$$N\alpha \geq N\pi_1 N\pi_2 \ldots N\pi_{j-1} > N\pi_j^2.$$

Thus π_j^2 satisfies the hypothesis of (2) and so must be a prime. This contradiction shows that we must have $N\alpha < 81$. Now we have

$$50 \leq N\alpha = N(\gamma)N(\pi_1\pi_2\pi_3) = N(\gamma)\cdot 50 < 81.$$

We conclude that $N\gamma = 1$, so γ is a unit and, to within a unit factor,

$$\alpha = \pi_1\pi_2\pi_3 = 5+5i. \quad \square$$

Bonse [1] also proved that $p_1p_2\ldots p_i > p_{i+1}^3$ for all $i \geq 5$ and used this inequality to prove that 1260 is the largest integer n with the property

$$1 < m < n \text{ and } (m,n) = 1 \implies m \text{ is a product of at most two primes.}$$

These results can also be extended to Gaussian integers. The Gaussian "1260" can be shown to be $30+30i$.

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* * *

THE PUZZLE CORNER

Puzzle No. 8: Homonym (7; 4 3)

Since two and seven both can be
Enumeration bases, and confuse,
It ONEs my students mightily
To TWO's the only base they'll use.

Puzzle No. 9: Rebus (1 5 12)

144 = 10000

He said, "This night
Is dark as pitch, by Phoebus!"
But there was light—
So what he said was REBUS.

ALAN WAYNE, Holiday, Florida

Answer to Puzzle No. 5 [1981: 294]: FOUR = 3025, FIVE = 3917, SIX = 496, SEVEN + NINE = 47178 + 8987 = 56165.

Answer to Puzzle No. 6 [1981: 300]: Farey sequence (Far EY).

Answer to Puzzle No. 7 [1981: 300]: Horatio.

* * *

A 1982 POTPOURRI

CHARLES W. TRIGG

As a result of the gallimaufry (ragout or hash) I served in these pages in January 1981, the editor has demoted me from Prince to Count of Digit Delvers [1981: 6]. This year's "rotten pot" will probably result in my further demotion to the role of a mere Court Jester (still an honorable position in the court of the Queen of Mathematics). There are seventeen ingredients in the pot, identified by the seventeen letters in NINETEEN EIGHTY-TWO (and a soupçon of garlic indicated by the hyphen). The ingredients were allowed to stand in the California sunshine until they had acquired a sufficient degree of ripeness to justify the title. The fulsome details follow.

N

$1982 = 2 \cdot 991$. Thus 1982 and its two factors are each divisible by their respective digital roots. 991 is the 167th prime, and 167 is itself a prime. The reverse of the year, $2891 = 7^2 \cdot 59$; and 7 is the 2^2th prime while 59 is the 17th prime, whereas 2 and 17 themselves are primes. The overlapping meld of the year and its reverse, 1982891, is a palindromic prime and it occupies a prime place in the sequence of primes, being the 147743rd prime. (Unfortunately, 147743 itself does not occupy a prime place in the sequence of primes, being the 13656th prime. Too much California sunshine, probably.)

I

$1 \cdot 9 = 9$ and $8 \cdot 2 = 16$, consecutive squares; $19+82 = 101$, a prime; $1+\sqrt{9} = 8/2$; $1 = -\sqrt{9}+\sqrt{8 \cdot 2}$; and $\sqrt[\sqrt{1 \cdot 9}]{8} = 2$.

N

$(1982)^2 = 3928324$, in which the difference of the sums of the alternate digits is $19 - 12 = 7$, a prime. Also, the sums of the digits equidistant from the center are 5, 11, and 7, all primes.

E

The number formed by the first seven significant digits of $1/1982$, 5045408, has the same digital root as the 1982nd prime, 17207. The integer formed by the first seven significant digits of $\log_{10} 1982$, 3297103, is a prime, and its rank in the sequence of primes, 236720, has the same digit sum as 1982.

T

$$1982 = 19 \cdot 82 + 198 \cdot 2 + 19 + 8 + 2 + 1 \cdot 9 - 8 - 2$$
$$= 1 \cdot 982 + 19 - 8 - 2 + 1 \cdot 982 + 19 - 8 - 2$$
$$= (198+2)(1 \cdot 9) + 82 + 19 + 82 + \sqrt{1 \cdot 9} - \sqrt{8 \cdot 2}$$
$$= (1 \cdot \sqrt{9}+8 \cdot 2)(1 \cdot 98+2) + (1+9) \cdot 8 + 2$$
$$= (1 \cdot 9+8+2)(1+98+2) + (1+9)(8-2) + 1 \cdot 9 - 8 + 2$$
$$= [1+\sqrt{9}(8-2)](1+9)(8+2) + (1+9 \cdot 8/2)(-1+9-8+2) + 1+ \sqrt{9} + 8/2.$$

E

With nine nonzero digits, $1982 = 45 - 36 + 2 \cdot 987 - 1$
$= 57 - 48 + 6 \cdot 329 - 1$
$= 3 + 4 + 5 \cdot 6 - 8 + 9 \cdot 217.$

E

In this, the main ingredient in the pot, the symbol $!x$ represents "subfactorial x" so that $!3 = 2$ and $!2 = 1$. The symbol $\|$ indicates concatenation; for example, $12\|47 = 1247$.

$0 = 1+9-8-2$
$1 = -1 \cdot 9+8+2$
$2 = -1+9-8+2$
$3 = 1 \cdot 9-8+2$
$4 = 1+9-8+2$
$5 = 1 \cdot 9-8/2$
$6 = 1+9-8/2$
$7 = 1 \cdot \sqrt{9}+8/2$
$8 = 1+\sqrt{9}+8/2$
$9 = 19-8-2$
$10 = -1+9+\sqrt{8/2}$
$11 = 1 \cdot 9+\sqrt{8/2}$
$12 = 1+9+\sqrt{8/2}$
$13 = 19-8+2$
$14 = 1+\sqrt{9}+8+2$
$15 = 19-8/2$
$16 = (1+\sqrt{9})(8/2)$
$17 = 19-\sqrt{8/2}$
$18 = 1 \cdot \sqrt{9}(8-2)$
$19 = 1 \cdot \sqrt{9}+8 \cdot 2$
$20 = (-1+\sqrt{9})(8+2)$
$21 = 19+\sqrt{8/2}$
$22 = 1 \cdot \sqrt{9} \cdot 8-2$
$23 = 19+8/2$
$24 = (1+\sqrt{9})(8-2)$
$25 = 1 \cdot 9+8 \cdot 2$
$26 = 1 \cdot \sqrt{9} \cdot 8+2$
$27 = 1+\sqrt{9} \cdot 8+2$
$28 = (1+\sqrt{9})!+8/2$
$29 = -1+\sqrt{9}(8+2)$
$30 = 1 \cdot \sqrt{9}(8+2)$
$31 = 1+\sqrt{9}(8+2)$
$32 = (-1+\sqrt{9}) \cdot 8 \cdot 2$
$33 = -1+(9+8) \cdot 2$
$34 = (1+\sqrt{9}) \cdot 8+2$
$35 = 19+8 \cdot 2$
$36 = 1 \cdot 9 \cdot 8/2$
$37 = 1+9 \cdot 8/2$
$38 = 19\sqrt{8/2}$
$39 = [-1+(\sqrt{9})!] \cdot 8-!2$
$40 = (1+\sqrt{9})(8+2)$
$41 = [-1+(\sqrt{9})!] \cdot 8+!2$
$42 = [-1+(\sqrt{9})!] \cdot 8+2$
$43 = 1+(\sqrt{9})! \cdot (8-!2)$
$44 = -1+\{[(\sqrt{9})!]!\}/(8 \cdot 2)$
$45 = -1+(\sqrt{9})! \cdot 8-2$
$46 = 1 \cdot (\sqrt{9})! \cdot 8-2$
$47 = -1+\sqrt{9} \cdot 8 \cdot 2$
$48 = -1+98/2$
$49 = 1 \cdot 98/2$
$50 = 1+98/2$
$51 = 1+(\sqrt{9})! \cdot 8+2$
$52 = (1\|\sqrt{9}) \cdot 8/2$
$53 = -1+9(8-2)$
$54 = 1 \cdot 9(8-2)$
$55 = 1+9(8-2)$
$56 = [1+(\sqrt{9})!] \cdot 8 \cdot !2$
$57 = [1+(\sqrt{9})!] \cdot 8+!2$
$58 = -(1+\sqrt{9})!+82$
$59 = -1+(\sqrt{9})!(8+2)$
$60 = (1+9)(8-2)$
$61 = 1+(\sqrt{9})!(8+2)$
$62 = (-1+9) \cdot 8-2$

$63 = -19+82$	$76 = -1\cdot(\sqrt{9})!+82$	$89 = 1+(\sqrt{9})!+82$
$64 = (1+\sqrt{9})\cdot 8\cdot 2$	$77 = 1-(\sqrt{9})!+82$	$90 = -1+9+82$
$65 = (-1+9)\cdot 8+!2$	$78 = -1-\sqrt{9}+82$	$91 = 1\cdot 9+82$
$66 = (-1+9)\cdot 8+2$	$79 = -1\cdot\sqrt{9}+82$	$92 = 1+9+82$
$67 = 1\cdot\sqrt{9}+8^2$	$80 = 1-\sqrt{9}+82$	$93 = 1+[(\sqrt{9})!]!/8+2$
$68 = 1+\sqrt{9}+8^2$	$81 = (1+9)\cdot 8+!2$	$94 = \{-1+[(\sqrt{9})!]\cdot 8\}\cdot 2$
$69 = -1+9\cdot 8-2$	$82 = (1+9)\cdot 8+2$	$95 = -1+98-2$
$70 = 1\cdot 9\cdot 8-2$	$83 = -1+!(\sqrt{9})+82$	$96 = 1\cdot 98-2$
$71 = 1+9\cdot 8-2$	$84 = -1+\sqrt{9}+82$	$97 = 1+98-2$
$72 = -1-9+82$	$85 = 1\cdot\sqrt{9}+82$	$98 = 1+98-!2$
$73 = -1+9\cdot 8+2$	$86 = 1+\sqrt{9}+82$	$99 = -1+98+2$
$74 = 1\cdot 9\cdot 8+2$	$87 = -1+(\sqrt{9})!+82$	$100 = (1+9)(8+2)$
$75 = 1+9\cdot 8+2$	$88 = 1\cdot(\sqrt{9})!+82$	$101 = 1+98+2$

N

Since 82 - 19 = 63, both 19 and 82 are members of five arithmetic progressions with common differences of 1, 3, 7, 9, and 21, respectively. Of interest is the progression

19, 28, 37, 46, 55, 64, 73, 82

in which the only internal prime terms are mutual reverses.

E

When the reiterative routine of adding an integer to its reverse (resulting in a *versum*) is started with 1982, the first, second, and fifth versums are composed of distinct digits, and the fourth versum is a palindrome formed from consecutive even digits:

1982	4873	8657	16225	68486
2891	3784	7568	52261	68486.
4873	8657	16225	68486	136972

No other versum before the 400th one is palindromic. The 26th versum, 15741916202375, contains every decimal digit except 8, its digital root and that of its rank 26.

I

Repeatedly add the squares of the digits of the integers, starting with

$$1^2 + 9^2 + 8^2 + 2^2 = 150.$$

Subsequent terms in the sequence are

26, 40, *16, 37, 58, 89, 145, 42, 20, 4, 16, ...

Four operations to enter the eight-term regenerative loop.

G

Repeatedly add the squares of the odd digits to the sum of the even digits of the integers: $1^2 + 9^2 + 8 + 2 = 92$, then 83, 17, 50, 25, 27, 51, 26, 8, the last being a self-replicating digit.

H

Repeatedly add the squares of the even digits to the sum of the odd digits of the integers: $1 + 9 + 8^2 + 2^2 = 78$, then 71, 8, 64, 52, and the self-replicating 9.

T

Beginning with the tens' digit, repeatedly add the squares of the alternate digits to the sum of the other digits of the integers: $1^2 + 9 + 8^2 + 2 = 76$, and then 55, 30, 9.

Y

Beginning with the units' digit, repeatedly add the squares of the alternate digits to the sum of the other digits of the integers: $1 + 9^2 + 8 + 2^2 = 94$, and then 25, 27, 51, 6, 36, 39, 84, 24, 18, 65, 31, 4, 16, 37, 52, *9, 81, 9, ... Seventeen operations to enter the two-term regenerative loop.

$1 + 9 - 8 = 2$

T

$$\begin{vmatrix} 1 & 8 \\ 2 & 9 \end{vmatrix} \cdot \begin{vmatrix} 9 & 1 \\ 8 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 9 \\ 8 & 2 \end{vmatrix}.$$

W

$$1982_{\text{ten}} = 2642_{\text{nine}} = 5531_{\text{seven}} = 13102_{\text{six}} = 30412_{\text{five}}$$

$$= 132332_{\text{four}} = 2201102_{\text{three}} = 11110111110_{\text{two}}.$$

The digits of the year in base nine are all even, and those in base seven are all odd. In each base except ten, the distinct digits are in arithmetic progression. In base six, the product of the nonzero digits equals the base. In each of the bases five, three, and two, all digits in that base appear in the year. All nonzero digits in base four appear in the year.

0

Using heptagonal numbers, which are of the form $H(n) = n(5n-3)/2$, we have

$$H(2816) = \underline{1982}0416;$$
$$H(8905) = \underline{1982}34205,\ H(8906) = \underline{1982}78731;$$
$$H(28158) = \underline{1982}140173,\ \ldots,\ H(28164) = \underline{1982}984994.$$

Finally, if k is even and $\|$ is the concatenation symbol, we have

$$H(k\|1227) = \ldots 1982 \quad \text{and} \quad H(k\|1852) = \ldots 1982.$$

And how was *your* year?

2404 Loring Street, San Diego, California 92109.

* * *

NOTES ON NOTATION: V

LEROY F. MEYERS

The subject of this note is what Victor Borge would call a flyspeck.

The mean value theorem states that if f is a function which is continuous on the nondegenerate closed interval $[a,b]$ and differentiable at least on its interior (a,b), then there is a number c in (a,b) such that $f(b)-f(a) = f'(c)(b-a)$. The following proof, to be analyzed in the course of this note, is standard.

Let f be continuous on $[a,b]$ and differentiable on (a,b). Define

$$F(x) = (b-a)\big(f(x)-f(a)\big) - (x-a)\big(f(b)-f(a)\big),\quad a \le x \le b. \tag{1}$$

Then F is continuous on $[a,b]$ and differentiable on (a,b). Furthermore, $F(a) = F(b) = 0$. Then Rolle's theorem applied to F on $[a,b]$ yields

$$F'(c) = (b-a)f'(c) - 1\cdot\big(f(b)-f(a)\big) = 0,\quad a < c < b. \tag{2}$$

Therefore

$$f(b) - f(a) = f'(c)(b-a),\quad a < c < b. \quad \square \tag{3}$$

A person coming across this proof for the first time may well have difficulty understanding it, not only because of the magically defined function F, but also because of three flyspecks: the commas in the numbered lines.

Each of these lines contains two mathematical sentences: an equation and a double inequality; these are separated by a comma. Since there are no explicit conjunctions in the surrounding text, these mathematical sentences are joined by what teachers of English call a *comma fault* (or *comma splice*) and teachers of classical Greek call *asyndeton*. In ordinary English, a run-on sentence usually indicates the absence of a conjunction like "and". However, in *none* of the numbered lines will insertion of "and" give the correct mathematical meaning. Sometimes the comma fault is avoided by the use of the preposition "for" in the sense "for the situation in which" in place of the comma. However, this does not remedy the situation in this example.

The difficulty is that the comma is used in two different ways in lines (1) and (2). (Line (3) is like line (2).) The distinction becomes obvious when the lines are written out in full:

$$F(x) = (b-a)(f(x)-f(a)) - (x-a)(f(b)-f(a)) \text{ for all } x \text{ such that } a \le x \le b. \qquad (1')$$

$$F'(c) = (b-a)f'(c) - 1\cdot(f(b)-f(a)) \text{ for some } c \text{ such that } a < c < b. \qquad (2')$$

The distinction is thus between a *universal quantifier* "for all x" and an *existential quantifier* "for some c".

A second example is copied from an unnamed book:

$$\text{Define: } \phi(a) = p^{-n},\ a \ne 0,\ \phi(0) = 0. \qquad (4)$$

(Here p is a prime and a is a rational number written in the form $p^n \cdot r/s$, where n is an integer and neither r nor s is divisible by p.) The two commas have different meanings. In full:

$$\text{Define: } \phi(a) = p^{-n} \text{ if } a \ne 0; \text{ and } \phi(0) = 0. \qquad (4')$$

(Here, as in many other cases, no ambiguity is caused by contracting the universal quantifier "for all a such that a ..." to "if a ...".) The use of commas in the definition of a function by cases, as in (4), is unfortunately extremely common and forces the reader to guess the author's intention.

Moral: Don't be afraid to use a few more words (as in (4')) to clarify a statement.

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* * *

CROOKS[1] WHAT?

For the benefit of recent subscribers, we reprint the following from an earlier issue [1980: 332]:

Crux Mathematicorum is an idiomatic Latin phrase meaning: a puzzle or problem for mathematicians. The phrase appears in the Foreign Words and Phrases Supplement of *The New Century Dictionary*, D. Appleton - Century Co., 1946, Vol. 2, p. 2438. It also appears in *Websters New International Dictionary*, Second Unabridged Edition, G. & C. Merriam Co., 1959, Vol. 1, p. 637.

* * *

[1]To paraphrase What's-his-name, I am not a crooks. (*Crux*)

THE OLYMPIAD CORNER: 31

M.S. KLAMKIN

I present this month the problems proposed in two recent competitions. As usual, I invite readers to send me elegant solutions for possible later publication in this column.

I am grateful to Bernhard Leeb, who was one of the West German team members in the 1981 International Mathematical Olympiad, for the English translation of the two rounds of the 1981 West German Olympiad.

WEST GERMAN MATHEMATICAL OLYMPIAD 1981

First Round

1. Let a and n be positive integers and $s = a+a^2+\ldots+a^n$. Prove that $s \equiv 1 \pmod{10}$ if and only if $a \equiv 1 \pmod{10}$ and $n \equiv 1 \pmod{10}$.

2. Prove that if a,b,c are the lengths of the sides of a scalene triangle and $a+b = 2c$, then the line joining the incentre and the circumcentre of the triangle is parallel to one side of the triangle.

3. A cell is removed from a $2^n \times 2^n$ checkerboard. Prove that the remaining surface can be tiled with L-shaped trominoes.

4. Prove that if p is a prime, then $2^p + 3^p$ is not of the form n^k, where n and k are integers greater than 1.

Second Round

1. The sequence $\{a_1, a_2, a_3, \ldots\}$ is defined as follows: a_1 is an arbitrary positive integer and, for $n > 1$, $a_n = [3a_{n-1}/2]+1$. Is it possible to choose a_1 such that a_{100001} is odd and a_n is even for all $n \le 100000$?

2. A bijective projection of the plane onto itself projects every circle onto a circle. Prove that it projects every straight line onto a straight line.

3. Let n be a positive power of 2. Prove that from any set of $2n-1$ positive integers one can choose a subset of n integers such that their sum is divisible by n.

4. Let M be a nonempty set of positive integers such that $4x$ and $[\sqrt{x}]$ both belong to M whenever x does. Prove that M is the set of all positive integers.

"The Eötvös Competition was organized in Hungary in 1894 and played a remarkable role in the development of mathematics in that small country. (Before the first world war Hungary had 19 million inhabitants; at present [1961] it has about 10 million.)" (Gábor Szegö in [1, p. 5].) See [1] for more historical information and for the problems and solutions of the Eötvös Competitions from 1894 to 1928. This reference is the American edition of *Problems of the Mathematics Contests*, by József Kürschák, first published in 1929 and revised in 1955 by György Hájos, Gyula Neukomm, and János Surányi. In honor of the author of this book, the Eötvös Competition is now called the József Kürschák Competition. It is open to all students who have not yet entered university. Thanks to the courtesy of László Csirmaz, I am able to present below an English translation of the latest one, which took place on 24 October 1981.

THE 1981 JÓZSEF KÜRSCHÁK COMPETITION

Time: 4 hours

1. The points A,B,C,P,Q,R lie in a plane. Prove that

$$AB + PQ + QR + RP \le AP + AQ + AR + BP + BQ + BR,$$

where XY denotes the distance between points X and Y.

2. Let $n > 2$ be an even number. The squares of an $n\times n$ chessboard are colored with $n^2/2$ colors in such a way that every color was used for coloring exactly two of the squares. Prove that one can place n rooks on squares of different colors in such a way that no two of the rooks can capture each other.

3. For a natural number n, $r(n)$ denotes the sum of the remainders of the divisions

$$n\div 1,\quad n\div 2,\quad n\div 3,\quad \dots,\quad n\div n.$$

Prove that $r(k) = r(k-1)$ for infinitely many natural numbers k.

REFERENCE

1. Elvira Rapaport (translator), *Hungarian Problem Book I, II*, New Mathematical Library Nos. 11 and 12, Random House, New York, 1963 (now available from the M.A.A., Washington, D.C.).

Editor's Note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

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PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before June 1, 1982, although solutions received after that date will also be considered until the time when a solution is published.

701. *Proposed by Alan Wayne, Holiday, Florida.*

Ever since her X-rated contribution to the solution of Crux 411 [1979: 299] shocked the old lady from Dubuque, EDITH ORR (long may she swing!) has maintained a low profile. Yet we all know that she is always THERE at the editor's elbow, like as not blowing in his ear. So we should all be able to solve the subtraction

$$\begin{array}{r} \text{EDITH} \\ -\ \ \text{ORR} \\ \hline \text{THERE} \end{array}.$$

702. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Given are three noncollinear points A_1, B_1, C_1 and three positive real numbers l, m, n whose sum is 1. Show how to determine a point M inside the circumcircle of triangle $A_1B_1C_1$ such that, if A_1M, B_1M, C_1M meet the circumcircle again in A,B,C, respectively, then we have

$$\frac{[BMC]}{l} = \frac{[CMA]}{m} = \frac{[AMB]}{n} = [ABC],$$

where the brackets denote the area of a triangle.

703. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

A right triangle ABC has legs AB = 3 and AC = 4. A circle γ with center G is drawn tangent to the two legs and tangent internally to the circumcircle of the triangle, touching that circumcircle in H. Find the radius of γ and prove that GH is parallel to AB.

704. *Proposed by H. Kestelman, University College, London, England.*

Suppose $A_1, A_2, \ldots, A_j$ are square matrices of any orders and no two have a common eigenvalue. If $f_1,f_2,\ldots,f_j$ are any given polynomials, then there is a polynomial F such that $F(A_r) = f_r(A_r)$ for all r.

705. *Proposed by Andy Liu, University of Alberta.*

On page 315 of U.I. Lydna's definitive treatise *Medieval Religion* (published in 1947) is a little piece of irrelevancy entitled "Pagan Island". This island was of circular shape and had along its coast exactly 26 villages whose names, in cyclic order, had the initials A, B, ..., Z. At various times in its history, the island was visited by exactly 26 missionaries whose names all had different initial letters. Each missionary first went to the village which matched his initial. While more than one missionary might be on the island at the same time, no two ever appeared (or disappeared) in the same village at the same time. As the name of the island implies, all the villages were initially pagan. When a missionary arrived at an unconverted village, he converted it and then moved on to the next village along the coast. When a missionary arrived at a converted village, he was promptly devoured and the village became unconverted again.

While the missionaries' fate was never in doubt, how many villages remained converted after their combined effort?

706. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let $F(x) = 7x^{11} + 11x^7 + 10ax$, where x ranges over the set of all integers. Find the smallest positive integer a such that $77|F(x)$ for every x.

707. *Proposed by Charles W. Trigg, San Diego, California.*

In the decimal system, how many eight-digit palindromes are the products of two consecutive integers?

708. *Proposed by Vedula N. Murty, Pennsylvania State University, Capitol Campus.*

A triangle has sides $a,b,c,$ semiperimeter s, inradius r, and circumradius R.

(a) Prove that

$$(2a-s)(b-c)^2 + (2b-s)(c-a)^2 + (2c-s)(a-b)^2 \geq 0,$$

with equality just when the triangle is equilateral.

(b) Prove that the inequality in (a) is equivalent to each of the following:

$$3(a^3+b^3+c^3+3abc) \leq 4s(a^2+b^2+c^2),$$

$$s^2 \geq 16Rr-5r^2.$$

709. *Proposed by F.G.B. Maskell, Algonquin College, Ottawa, Ontario.*

ABC is a triangle with incentre I, and DEF is the pedal triangle of the point I with respect to triangle ABC. Show that it is always possible to construct with straightedge and compass four circles each of which is tangent to each of the circumcircles of triangles ABC, EIF, FID, and DIE, provided that triangle ABC is not equilateral.

710. *Proposed by Gali Salvatore, Perkins, Québec.*

Let z' and z'' be the roots of the equation

$$z + \frac{1}{z} = 2(\cos\phi + i\sin\phi),$$

where $0 < \phi < \pi$.

(a) Show that $z'+i$, $z''+i$ have the same argument, and that $z'-i$, $z''-i$ have the same modulus.

(b) Find the locus of the roots z',z'' in the complex plane when ϕ varies from 0 to π.

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SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

584. [1980: 283; 1981: 290] *Proposed by F.G.B. Maskell, Algonquin College, Ottawa.*

If a triangle is isosceles, then its centroid, circumcentre, and the centre of an escribed circle are collinear. Prove the converse.

II. *Solution (one of two) by Leon Bankoff, Los Angeles, California.*

Suppose the centroid G, the circumcentre O, and the excentre I_a of triangle ABC are collinear. Since G, O, and the orthocentre H are collinear (on the Euler line of the triangle), it follows that H, O, and I_a are collinear. Now it is well known [1] that the lines AH and AO are isogonal conjugates, that is, they make equal angles with the internal bisector AI_a. But the collinearity of H, O, and I_a implies that $\angle HAI_a = \angle OAI_a = 0$, so the altitude AD falls along AI_a. Hence AD is the perpendicular bisector of side BC, from which AB = AC follows.

REFERENCE

1. Nathan Altshiller Court, *College Geometry*, Barnes & Noble, New York, 1952, p. 59, Theorem 73(b).

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601. [1981: 19] *Proposed by J.A.H. Hunter, Toronto, Ontario.*

Even amateur gardeners know the value of the maxim *weed'em and reap*. But first you must find the WEEDS. Find them in

$$\begin{array}{r} \text{GEE} \\ \text{TAM} \\ \text{SEE} \\ \underline{\text{THEM}} \\ \text{WEEDS} \end{array}.$$

Solution by Kenneth M. Wilke, Topeka, Kansas.

It is easy to see that W = 1 and (T,E) = (7,0), (8,0), (9,0), or (9,2). The possibility (T,E) = (9,2) can be eliminated because it requires {G,S,H} = {8,7,6} and a carry of 2 from the tens' column, which is impossible with E = 2. Also, (T,E) = (9,0) requires the impossible G+S+H = 1; and (T,E) = (7,0) requires {G,S,H} = {9,8,6}, resulting in M < 5 and the impossible A = D. Hence we have (T,E) = (8,0) and {G,S,H} = {3,4,5}, {2,3,7}, or {2,4,6} with S even. Of these three possibilities, only {G,S,H} = {2,3,7} permits the necessary A+1 = D. Thus S = 2, {G,H} = {3,7}, and the remaining M = 6, A = 4, D = 5 fall out easily. There is only one WEEDS, but the interchangeability of G and H shows that there are two solutions:

$$\begin{array}{r} 300 \\ 846 \\ 200 \\ \underline{8706} \\ 10052 \end{array} \quad \text{and} \quad \begin{array}{r} 700 \\ 846 \\ 200 \\ \underline{8306} \\ 10052 \end{array}.$$

Also solved by JAMES BOWE, Erskine College, Due West, South Carolina; CLAYTON W. DODGE, University of Maine at Orono; BIKASH K. GHOSH, Bombay, India; W.C. IGIPS, Danbury, Connecticut; ALLAN WM. JOHNSON JR., Washington, D.C.; EDGAR LACHANCE,Ottawa, Ontario; J.A. McCALLUM, Medicine Hat, Alberta; HERMAN NYON, Paramaribo, Surinam; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; CHARLES W. TRIGG, San Diego, California; and the proposer

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602. [1981: 19] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Given are twenty natural numbers a_i such that

$$0 < a_1 < a_2 < \ldots < a_{20} < 70.$$

Show that at least one of the differences $a_i - a_j$, $i > j$, occurs at least four **times**.

(A student proposed this problem to me. I don't know the source.)

I. *Essence of the solutions submitted by nearly all the solvers identified below.*

We prove the following stronger result: If twenty natural numbers a_i satisfy $0 < a_1 < a_2 < \ldots < a_{20} \le 70$, then at least one of the nineteen differences $\Delta_j = a_{j+1} - a_j$ occurs at least four times.

First observe that $\Sigma\Delta_j = a_{20} - a_1 \le 70-1 = 69$. If each difference occurs at most three times, then

$$\Sigma\Delta_j \ge 3(1+2+3+4+5+6) + 7 = 70,$$

a contradiction. Hence at least one of the differences Δ_j occurs at least four times.

II. *Comment by Joe Konhauser, Macalester College, Saint Paul, Minnesota.*

This problem is a variant of the first part of the following known two-part problem [1]:

(1) Given eight positive integers $a_1 < a_2 < \ldots < a_8 \le 16$. Prove that there exists a k such that $a_i - a_j = k$ has at least three solutions.

(2) Find a set $a_1, \ldots, a_8$ for which $a_i - a_j = k$ has at most three solutions for any k.

Solutions were received from JORDI DOU, Barcelona, Spain; BIKASH K. GHOSH, Bombay, India; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; ANDY LIU, University of Alberta; LEROY F. MEYERS, The Ohio State University; DAN SOKOLOWSKY, California State University at Los Angeles; ROBERT A. STUMP, Hopewell, Virginia; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

This problem raises more questions than it answers:

(a) If *all* the differences $\Delta_{ij} = a_i - a_j$, $i > j$, are considered, and $a_{20} \le 70$, must at least one difference occur at least k times for some $k > 4$?

(b) If all the difference Δ_{ij} are considered but the restriction $a_{20} \le 70$ is removed, what is the maximal value of a_{20} for which it is true that at least one difference must occur at least four times?

(c) What if there are n numbers a_i instead of twenty?

REFERENCE

1. Leo and William Moser (proposers), Problem E 1359, *American Mathematical Monthly* 66 (1959) 234; solution by J.L. Pietenpol, *ibid.*, p. 815.

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603. [1981: 19] *Proposed by Hayo Ahlburg, Benidorm, Alicante, Spain.*

If k is a positive integer, show that $n^5 + 1$ is a factor of

$$(n^4 - 1)(n^3 - n^2 + n - 1)^k + (n+1)n^{4k-1}.$$

I. *Solution by Kenneth S. Williams, Carleton University, Ottawa, Ontario.*

We will consider the problem over the complex field and show that, for any odd prime p and positive integer k, $z^p + 1$ is a factor of

$$\begin{aligned} F(z) &\equiv (z^{p-1}-1)(z^{p-2}-z^{p-3}+\ldots+z-1)^k + (z+1)z^{(p-1)k-1} \\ &= (z+1)(z^{p-2}-z^{p-3}+\ldots+z-1)^{k+1} + (z+1)z^{(p-1)k-1} \\ &= (z+1)\{(z^{p-2}-z^{p-3}+\ldots+z-1)^{k+1} + z^{(p-1)k-1}\} \\ &\equiv (z+1)G(z). \end{aligned} \tag{1}$$

Our problem will then correspond to the special case $p = 5$.

If the pth roots of -1 are

$$-1,\ \omega_1,\ \omega_2,\ \ldots,\ \omega_{p-1}, \tag{2}$$

then

$$\begin{aligned} z^p+1 &= (z+1)(z-\omega_1)(z-\omega_2)\ldots(z-\omega_{p-1}) \\ &= (z+1)(z^{p-1}-z^{p-2}+\ldots-z+1). \end{aligned} \tag{3}$$

It then suffices to show that $F(-1) = 0$, which is obvious from (1), and that $G(\omega) = 0$, where ω is any one of the imaginary roots in (2). To prove the latter, we note from (3) that

$$\omega^{p-2}-\omega^{p-3}+\ldots+\omega-1 = \omega^{p-1},$$

and so

$$G(\omega) = \omega^{(p-1)(k+1)} + \omega^{(p-1)k-1} = \omega^{(p-1)k-1}(\omega^p+1) = 0.$$

II. *Solution by Jordi Dou, Barcelona, Spain.*

For any integer n, let

$$a = n^3-n^2+n-1 \quad \text{and} \quad d = n^4-n^3+n^2-n+1.$$

Using for convenience the notation $M(x)$ to represent any integral multiple of the integer x, we will show that, for any positive integer k, $n^5+1 = (n+1)d$ is a factor of

$$(n^4-1)a^k + (n+1)n^{4k-1} = (n+1)(a^{k+1} + n^{4k-1}),$$

or, equivalently, that $a^{k+1} + n^{4k-1} = M(d)$. Since $a = n^4-d$, we have

$$a^{k+1} = (n^4-d)^{k+1} = n^{4k+4} + M(d);$$

hence

$$\begin{aligned} a^{k+1} + n^{4k-1} &= n^{4k+4} + M(d) + n^{4k-1} \\ &= n^{4k-1}(n^5+1) + M(d) \\ &= M(d). \quad \square \end{aligned}$$

If we put successively $n = 2, 3, 4, \ldots$ in the last equation, we obtain the interesting relations

$$5^{k+1} + 2^{4k-1} = M(11),$$

$$20^{k+1} + 3^{4k-1} = M(61),$$

$$51^{k+1} + 4^{4k-1} = M(205),$$

etc., each of which holds for $k = 1,2,3,\ldots$.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; CLAYTON W. DODGE, University of Maine at Orono; BIKASH K. GHOSH, Bombay, India; G.C. GIRI, Midnapore College, West Bengal, India; J.T. GROENMAN, Arnhem, The Netherlands; W.C. IGIPS, Danbury, Connecticut; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; ANDY LIU, University of Alberta; LEROY F. MEYERS, The Ohio State University; BOB PRIELIPP, University of Wisconsin-Oshkosh; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, California State University at Los Angeles; ROBERT A. STUMP, Hopewell, Virginia; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

Six solvers painstakingly threaded their way through a proof by mathematical induction. O.K., so it gets you there. But the scenery along the road is uninspiring.

Kierstead commented: "A most enjoyable problem, especially since it looks impossible at first glance. The author is to be congratulated."

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604. [1981: 19] *Proposed by Stanley Wagon and Joan P. Hutchinson, Smith College, Northampton, Massachusetts.*

Show that one can determine with only $n(n-1)$ additions whether a real $n\times n$ matrix is such that the sum of the elements of D is constant whenever D is an n-element set having exactly one element from each row and one from each column. (The most naïve approach requires $n!(n-1)$ additions.)

I. *Solution by Euclide Paracelso Bombasto Umbugio, University of Guayazuela.*

No additions are needed! Just take the exponentials of the n^2 entries in the matrix and compute (naïvely) the $n!(n-1)$ *products*.

[This method will not work if complex entries are allowed, since several distinct complex numbers can have the same exponential. Fortunately for the good professor, the proposal (needlessly) restricted the entries to real numbers. (Editor)]

II. *Solution by the proposers.*

Let A be a real [or complex] $n\times n$ matrix and consider the following four statements (in which the term "constant vector" means a vector whose components are all equal):

(1) A has the property (call it P) mentioned in the proposal.

(2) The difference between any two rows of A is a constant vector.

(3) The difference between any two columns of A is a constant vector.

(4) There are matrices B with constant row vectors, and C with constant column vectors, such that $A = B + C$.

The desired result clearly follows from the equivalence (1) $\iff$ (2). We show more: that, in fact, the four statements (1)-(4) are all equivalent.

Since property P is valid for matrices with constant rows and for those with constant columns, and is preserved under addition, it follows that (4) implies (1). It is easy to see that (2) and (3) are equivalent. To show that (3) implies (4), let $A = (a_{ij})$ and let $d_2,\ldots,d_n$ be such that $\bar{C}_j - \bar{C}_1 = (d_j,\ldots,d_j)$, where $\bar{C}_j$ denotes the jth column. Then

$$A = \begin{bmatrix} a_{11} & a_{11} & \cdots & a_{11} \\ a_{21} & a_{21} & \cdots & a_{21} \\ & & \vdots & \\ a_{n1} & a_{n1} & \cdots & a_{n1} \end{bmatrix} + \begin{bmatrix} 0 & d_2 & \cdots & d_n \\ 0 & d_2 & \cdots & d_n \\ & & \vdots & \\ 0 & d_2 & \cdots & d_n \end{bmatrix}.$$

Finally, to prove that (1) implies (3), observe that if A has property P, then so does any 2×2 minor of A. Now, given i,j, let $d = a_{1j} - a_{1i}$. Then $a_{1i}+a_{kj} = a_{1j}+a_{ki}$ for any k, whence $d = a_{kj} - a_{ki}$ and $\bar{C}_j - \bar{C}_i = (d,\ldots,d)$.

Also solved by LEROY F. MEYERS, The Ohio State University. A partial solution (discussion of some special cases) was received from BIKASH K. GHOSH, Bombay, India.

Editor's comment.

If a square matrix has property P, the constant sum of the set D mentioned in the proposal has been called the *conjuring number* of the matrix by Howard Eves. This was reported in an editor's comment [1981: 64], where Eves was credited with stating (without proof) that if the first n^2 natural numbers are written in order in an $n\times n$ array increasing from left to right and from top to bottom, then the resulting matrix has property P and its conjuring number is $\frac{1}{2}n(n^2+1)$. These statements are easy consequences of our present solution II. Eves had earlier given a 6×6 array with property P and conjuring number 1980. (See [1980: 8] and correction in [1980: 96].) It is now clear from our solution II how such arrays can be constructed with any desired conjuring number.

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605, [1981: 48] *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Can every fourth-order pandiagonal magic square (magic also along the broken diagonals) be written in the form

$F+y$	$G+x$	$G-x$	$F-y$
$G-w$	$F+z$	$F-z$	$G+w$
$F+x$	$G+y$	$G-y$	$F-x$
$G+z$	$F-w$	$F+w$	$G-z$

where $w = x + y + z$?

I. *Solution by Kenneth M. Wilke, Topeka, Kansas.*

The answer is yes. Kraitchik [1] shows that every fourth-order pandiagonal magic square can be written in the form

a	b	c	$S-(a+b+c)$
e	$S-(a+b+e)$	$a-c+e$	$b+c-e$
$\frac{S}{2}-c$	$(a+b+c)-\frac{S}{2}$	$\frac{S}{2}-a$	$\frac{S}{2}-b$
$\frac{S}{2}-(a-c+e)$	$\frac{S}{2}-(b+c-e)$	$\frac{S}{2}-e$	$(a+b+e)-\frac{S}{2}$

where S is the magic constant. The proposer's form can be obtained from Kraitchik's by setting $a = F+y$, $b = G+x$, $c = G-x$, $e = G-w$, and $S = 2(F+G)$, where $w = x+y+z$ as in the proposal.

II. *Comment by the proposer.*

I show that the cells of every fourth-order pandiagonal magic square can be rearranged to form a fourth-order symmetrical magic square. (I gave another proof in [2].) The 16 numbers of the square in the proposal,

$$\begin{matrix} F+w & F+x & F+y & F+z & F-z & F-y & F-x & F-w \\ G-w & G-x & G-y & G-z & G+z & G+y & G+x & G+w, \end{matrix} \qquad (1)$$

form 8 disjoint pairs each of which sums to $F+G$, which is half the magic sum

$2(F+G)$ and equals the symmetry constant of the following symmetrical magic square whose entries consist of the numbers (1):

$F+w$	$G-x$	$G-y$	$F-z$
$F-w$	$G+x$	$G+y$	$F+z$
$G-z$	$F-y$	$F-x$	$G+w$
$G+z$	$F+y$	$F+x$	$G-w$

It also turns out that the cells of every fourth-order symmetrical magic square can be rearranged to form a fourth-order pandiagonal magic square. My proof of this statement is given in [2].

Every third-order magic square is symmetrical and its cells rearrange into the 3×3 array shown below on the left. The cells of every fifth-order pandiagonal magic square rearrange into the 5×5 array shown below on the right.

p	$p+c$	$p+2c$
$p+d$	$p+c+d$	$p+2c+d$
$p+2d$	$p+c+2d$	$p+2c+2d$

p	$p+c_1$	$p+c_2$	$p+c_3$	$p+c_4$
$p+d_1$	$p+c_1+d_1$	$p+c_2+d_1$	$p+c_3+d_1$	$p+c_4+d_1$
$p+d_2$	$p+c_1+d_2$	$p+c_2+d_2$	$p+c_3+d_2$	$p+c_4+d_2$
$p+d_3$	$p+c_1+d_3$	$p+c_2+d_3$	$p+c_3+d_3$	$p+c_4+d_3$
$p+d_4$	$p+c_1+d_4$	$p+c_2+d_4$	$p+c_3+d_4$	$p+c_4+d_4$

The orderliness of (1) suggests that it is the fourth-order analogue of these results.

As in the case of the 3×3 or 5×5 array, the orderliness of (1) facilitates the manual construction of pandiagonal or symmetrical fourth-order magic squares whose cells are to contain integers having a predetermined characteristic. Thus, if prime magic squares are desired, then visual examination of a table of primes discloses many examples like

$$(w,x,y,z,F,G) = (32,22,20,-10,51,261) \text{ and } (39,45,-21,15,58,152).$$

Also solved by the proposer.

REFERENCES

1. Maurice Kraitchik, *Mathematical Recreations,* Second Revised Edition, Dover Publications, New York, 1953, pp. 186-187.

2. Allan Wm. Johnson Jr., Letter to the Editor, *Journal of Recreational Mathematics,* 12 (1979-80) 207-209.

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606. [1981: 48] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $\sigma_n = A_0A_1\ldots A_n$ be an n-simplex in Euclidean space R^n and let $\sigma'_n = A'_0A'_1\ldots A'_n$ be an n-simplex similar to and inscribed in σ_n, and labeled in such a way that

$$A'_i \in \sigma_{n-1} = A_0A_1\ldots A_{i-1}A_{i+1}\ldots A_n, \qquad i = 0,1,\ldots,n.$$

Prove that the ratio of similarity

$$\lambda \equiv A'_iA'_j/A_iA_j \geq 1/n.$$

[If no proof of the general case is forthcoming, the editor hopes to receive a proof at least for the special case $n = 2$.]

Partial solution by Gali Salvatore, Perkins, Québec.

We will consider here only the special case $n = 2$, for which the intricate notation of the proposal is unnecessary. We will assume that triangle A'B'C' is directly similar to triangle ABC, with A',B',C' on the *lines* (not necessarily on the segments, which is unduly restrictive) BC,CA,AB, respectively.

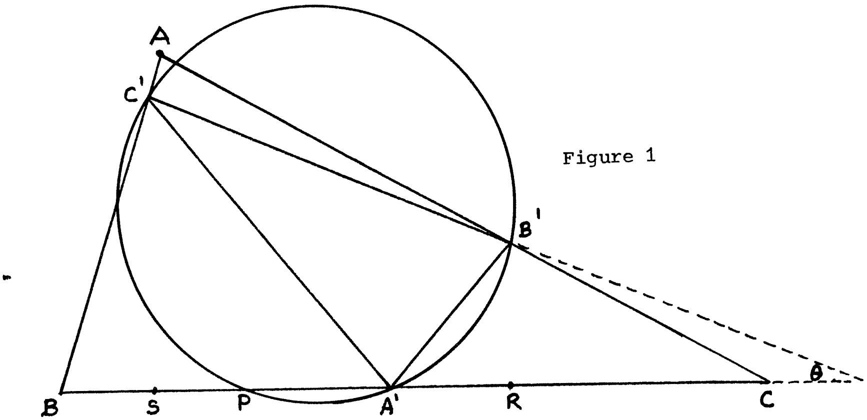

Figure 1

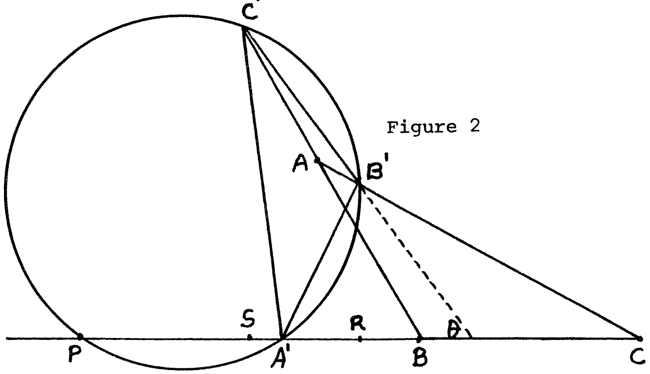

Figure 2

If the lines BC and B'C' are parallel, it is easy to show that A'B'C' is the medial triangle of ABC, and so $\lambda = B'C'/BC = \frac{1}{2}$. Suppose now that the lines BC and B'C' are not parallel but meet at an acute angle θ, as shown in the figures. Let the circumcircle of A'B'C' meet line BC again at P. Observe that triangle C'BP is isosceles, since its angles at B and P are both equal to (as in Figure 1) or both supplementary to (as in Figure 2) $\angle A'B'C'$; and similarly triangle B'PC is isosceles with equal angles at P and C. Thus, if RS is the orthogonal projection of segment B'C' upon line BC, then R and S are the midpoints of segments PC and PB, respectively, whence $RS = \frac{1}{2}BC$. Since $B'C' = RS\sec\theta$, it follows that

$$\lambda = B'C'/BC = \tfrac{1}{2}\sec\theta > \tfrac{1}{2},$$

and $\lambda \geq \frac{1}{2}$ holds in every case.

Partial solutions (for the special case $n = 2$) were also submitted by LEROY F. MEYERS, The Ohio State University; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer (six solutions).

Editor's comment.

The result $\lambda \geq 1/n$ remains open for general $n > 2$. If true, it will require an analytic solution, but a neat synthetic solution may still be possible for the special case $n = 3$.

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607. [1981: 49] *Proposed by Gali Salvatore, Perkins, Québec.*

Let S be the set of all positive integers that are, in base ten, palindromes with fewer than 10 digits. S contains an abundance of multiples of 3, of 9, and of 27, but it contains only *one* multiple of 81. It is the largest number in the set:

$$999999999 = 81 \cdot 12345679.$$

What is there in the structure of palindromes that explains this strange behaviour?

Solution by Leroy F. Meyers, The Ohio State University.

The "strange behaviour" is a consequence of the fact, proved below, that *a palindrome (in base ten) is divisible by* 81 *if and only if the sum of its digits is divisible by* 81. Since 999 999 999 is the smallest positive integer with a digit sum of 81, and is a palindrome, it must be the smallest (positive) palindrome which is divisible by 81.

To prove the italicized criterion, we will need the fact, easily established by induction, that

$$10^k \equiv 9k + 1 \pmod{81}$$

for every nonnegative integer k. Let P be an $(n+1)$-digit palindrome, say

$$P = \sum_{k=0}^{n} a_k \cdot 10^k,$$

where $0 \le a_k \le 9$ and $a_{n-k} = a_k$ for $0 \le k \le n$. Then

$$2P = \sum_{k=0}^{n} a_k(10^k+10^{n-k}) \equiv \sum_{k=0}^{n} a_k\{9k+1+9(n-k)+1\}$$

$$\equiv (9n+2)\sum_{k=0}^{n} a_k \pmod{81}.$$

Since 2 and $9n+2$ are both relatively prime to 81, P is divisible by 81 if and only if Σa_k is divisible by 81. □

The well-known criteria for divisibility by 3 and 9 show that the italicized statement remains true if 81 is replaced by 3 or by 9; and every step of the above proof remains valid if 81 is replaced by 27. So it is tempting to conjecture that a palindrome (in base ten) is divisible by 3^m if and only if the sum of its digits is divisible by 3^m. Unfortunately, the conjecture fails when $m = 5$: the eleven-digit palindrome

$$29\,799\,999\,792 = 3^5 \cdot 122\,633\,744\,,$$

for example, is divisible by 3^5 but its digit sum is only 3^4.

According to the criterion established above, if a palindrome has an even number of digits and is divisible be 81, then its digit sum must be even and divisible by 81. The smallest such palindrome is clearly the eighteen-digit

$$999\,999\,999\,999\,999\,999,$$

with a digit sum of 162.

Also solved by CLAYTON W. DODGE, University of Maine at Orono; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; and KENNETH M. WILKE, Topeka, Kansas. A comment was submitted by BOB PRIELIPP, University of Wisconsin-Oshkosh.

Editor's comment.

One solver (not named above) apparently misunderstood the problem and sent the proof of a related result. Another reader (who shall also be nameless) offered to send a 7-page partial solution. He refrained from sending it after the editor begged for mercy.

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608. [1981: 49] *Proposed by Ngo Tan, student, J.F. Kennedy H.S., Bronx, N.Y.*

ABC is a triangle with sides of lengths a, b, c and semiperimeter s. Prove that

$$\cos^4 \tfrac{1}{2}A + \cos^4 \tfrac{1}{2}B + \cos^4 \tfrac{1}{2}C \le \tfrac{1}{2}\frac{s^3}{abc},$$

with equality if and only if the triangle is equilateral.

Solution de Hippolyte Charles, Waterloo, Québec.

Les formules bien connues $\cos \tfrac{1}{2}A = \sqrt{s(s-a)/bc}$, etc., donnent

$$\Sigma\cos^4 \tfrac{1}{2}A = \frac{s^2\Sigma a^2(s-a)^2}{a^2b^2c^2}.$$

L'inégalité proposée est donc équivalente à l'inégalité

$$\Sigma 2a^2(s-a)^2 \le abcs \qquad (1)$$

déjà établie par Blundon [1981: 218] comme étant équivalente à celle de Crux 570. Comme dans (1), l'égalité a lieu seulement pour le triangle équilatéral.

Also solved by JAYANTA BHATTACHARYA, Midnapur, West Bengal, India; W.J. BLUNDON, Memorial University of Newfoundland; JACK GARFUNKEL, Flushing, N.Y.; BIKASH K. GHOSH, Bombay, India; J.T. GROENMAN, Arnhem, The Netherlands (two solutions); GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer.

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609*. [1981: 49] *Proposed by Jack Garfunkel, Flushing, N.Y.*

$A_1B_1C_1D_1$ is a convex quadrilateral inscribed in a circle and M_1, N_1, P_1, Q_1 are the midpoints of sides B_1C_1, C_1D_1, D_1A_1, A_1B_1, respectively. The chords A_1M_1, B_1N_1, C_1P_1, D_1Q_1 meet the circle again in A_2, B_2, C_2, D_2, respectively. Quadrilateral $A_3B_3C_3D_3$ is formed from $A_2B_2C_2D_2$ as the latter was formed from $A_1B_1C_1D_1$, and the procedure is repeated indefinitely. Prove that quadrilateral $A_nB_nC_nD_n$ "tends to" a square as $n \to \infty$.

What happens if $A_1B_1C_1D_1$ is not convex?

Editor's comment.

It is the editor's duty to report on solutions submitted by readers, to publish the best, and, occasionally, to make comments (meant to be helpful) pointing out some mistakes made by solvers.

For this problem, only one solution(?) was received, which would occupy 6 or 7 pages if printed here. It was largely incomprehensible to this poor benighted editor, who did not have the time or the patience to pore through the mass of calculations it contained, especially as it appeared to be based on the preposterous notion that in a circle the length of a chord is proportional to the length of the subtended arc.

The conclusion reached, if anyone cares to know, was that, yes indeed, $A_nB_nC_nD_n$ does "tend to" a square as $n\to\infty$, and that whether or not $A_1B_1C_1D_1$ is convex. But our solver's interpretation of "nonconvex quadrilateral" was "a [convex, according to his figure] quadrilateral lying entirely on one side of a diameter of the circle". Anyway, Mr. X, thanks for trying.

The problem is interesting, and knowledgeable readers are urged to try and solve it. Possibly the class of quadrilaterals for which the conclusion of the problem is expected to hold should be more sharply defined. The conclusion does not hold, for example, for some degenerate (and therefore convex) quadrilaterals, say when $B_1 = C_1$ and $A_1 = D_1$.

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610. [1981: 49] *Proposed by Hayo Ahlburg, Benidorm, Alicante, Spain.*

If m is odd and $(m,5) = 1$, prove that $m^{21}-m \equiv 0 \pmod{13200}$.

Solution by F.G.B. Maskell, Algonquin College, Ottawa, Ontario.

Since $13200 = 2^4\cdot 3\cdot 5^2\cdot 11$ and the four indicated factors are relatively prime in pairs, the desired result follows from the congruences (1)-(4) established below with the help of the following factorizations, in which $P_k(m)$ denotes a polynomial in m of degree k:

$$\begin{aligned} m^{21} - m &= (m-1)(m+1)(m^{10}+1)P_9(m) \\ &= (m-1)m(m+1)P_{18}(m) \\ &= (m^{20}-1)m \\ &= (m^{11}-m)P_{10}(m). \end{aligned}$$

Since m is odd, $8|(m-1)(m+1)$ and $2|m^{10}+1$, so

$$(m-1)(m+1)(m^{10}+1) \equiv 0 \pmod{16}. \tag{1}$$

One of three consecutive integers must be divisible by 3, so

$$(m-1)m(m+1) \equiv 0 \pmod{3}. \tag{2}$$

Since $(m,5) = 1$, the Euler-Fermat Theorem gives

$$m^{20} - 1 = m^{\phi(25)} - 1 \equiv 0 \pmod{25}. \tag{3}$$

Finally, since 11 is a prime, Fermat's Theorem gives

$$m^{11} - m \equiv 0 \pmod{11}. \tag{4}$$

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; JAMES BOWE, Erskine College, Due West, South Carolina; CLAYTON W. DODGE, University of Maine at Orono; JORDI DOU, Barcelona, Spain; BIKASH K. GHOSH, Bombay, India (partial solution); J.T. GROENMAN, Arnhem, The Netherlands; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; LEROY F. MEYERS, The Ohio State University; BOB PRIELIPP, University of Wisconsin-Oshkosh; LAWRENCE SOMER, Washington, D.C.; ROBERT A. STUMP, Hopewell, Virginia; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

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611. [1981: 49] *Proposed by Alan Wayne, Pasco-Hernando Community College, New Port Richey, Florida.*

This decimal cryptarithmic addition is dedicated to the memory of the late Sidney Penner, formerly Problem Editor of the *New York State Mathematics Teachers' Journal:*

$$\begin{array}{r} \text{SIDNEY} \\ \text{EDITOR} \\ \underline{\text{SIDNEY}} \\ \text{PENNER} \end{array}.$$

Editor Sidney Penner was unique, and so is EDITOR SIDNEY PENNER.

Solution by Charles W. Trigg, San Diego, California.

The columns of this cryptarithm, proceeding from the right, determine the following equations, with $(Y,k) = (0,0)$ or $(5,1)$ in the first:

$$2Y = 10k, \qquad (1)$$

$$E + O + k = 10, \qquad (2)$$

$$N + T + 1 = 10, \qquad (3)$$

$$2D + I + 1 = N + 10p, \qquad (5)$$

$$2I + D + p = E + 10q, \qquad (6)$$

$$2S + E + q = P. \qquad (7)$$

Clearly, $0 < S \le 4$, $0 < E \le 7$, and $P \ge 4$. With these restrictions in mind, starting with equation (5) and the possible combinations of D and I producing N, and working through the equations (5), (6), (7), (3) in that order, when duplicate digits appeared or when both 0 and 5 appeared in a set, the set was discarded. The surviving digit sets are shown in the left part of the table on the following page.

When (1) and (2) were applied in that order to the surviving sets, the right part of the table shows that values of O were produced that duplicated members of the set, except in the two cases indicated by the assignment of the tenth digit to R. Thus the two solutions are as shown below the table on the following page.

D	I	N	[p]	E	[q]	S	P	T	Y	[k]	O	R
0	7	8	[0]	4	[1]	2	9	1	5	[1]	5	-
1	6	9	[0]	3	[1]	2	8	0	5	[1]	6	-
4	8	7	[1]	1	[2]	3	9	2	0	[0]	9	-
4	8	7	[1]	1	[2]	3	9	2	5	[1]	8	-
6	2	5	[1]	1	[1]	3	8	4	0	[0]	9	7
6	4	7	[1]	5	[1]	1	8	2	0	[0]	5	-
7	3	8	[1]	4	[1]	2	9	1	0	[0]	6	5
7	3	8	[1]	4	[1]	2	9	1	5	[1]	5	-
7	8	3	[2]	5	[2]	1	9	6	0	[0]	5	-
8	2	9	[1]	3	[1]	1	6	0	5	[1]	6	-
9	6	5	[2]	3	[2]	1	7	4	0	[0]	7	-

$$\begin{array}{r} 326510 \\ 162497 \\ \underline{326510} \\ 815517 \end{array} \quad \text{and} \quad \begin{array}{r} 237840 \\ 473165 \\ \underline{237840} \\ 948845 \end{array}. \quad \square$$

As he grew as an **ED**ITOR, so did PENNER, retaining his near palindromic nature with P = E + R. The total man, SIDNEY PENNER, grew from 1142027 to 1186685, a growth of

$$44658 = 1\cdot 2\cdot 3\cdot 9\cdot 827,$$

wherein all the nonzero digits appear and only the powers of 2 appear twice.

Also solved by J.A.H. HUNTER, Toronto, Ontario; ALLAN WM. JOHNSON JR., Washington, D.C., J.A. McCALLUM, Medicine Hat, Alberta; and KENNETH M. WILKE, Topeka, Kansas. Partial solutions (one answer only) were received from BIKASH K. GHOSH, Bombay, India; EDGAR LACHANCE, Ottawa, Ontario; HERMAN NYON, Paramaribo, Surinam; and the proposer.

Editor's comment.

The proposer (and the three solvers who believed him) will please note that, though Editor Sidney Penner may have been unique, EDITOR SIDNEY PENNER has the unique distinction of being unique and uniquer. This led our featured solver to state, in a letter accompanying his solution, that "Alan must be on the wane". Wayne would probably call this "a Triggonometric utterance".

Judging from the numbering of his equations, our featured solver apparently believes that 3 + 1 = 5. Could Trigg too be on the wane?

* * *

CORRECTION. In Problem 695 [1981: 302], the second sentence should read: For $j,k \neq i$, M_i are the midpoints of B_jB_k; and m_i are the lines through M_i perpendicular to a_i.