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CONTENTS

A SIMPLE PROOF OF THE BUTTERFLY PROBLEM

KESIRAJU SATYANARAYANA

The Butterfly Problem has been attracting attention at least since 1815, and the number of known proofs is quite large (see the extensive list of references in [1], also [2] and [3]). Assuming that mathematical lepidopterists will always be happy to meet a new specimen, we present here a simple elementary analytical proof which we believe to be new.

In its simplest form, the problem may be stated as follows (see Figure 1):

THE BUTTERFLY PROBLEM. Through the midpoint M *of a chord* AB *of a circle, two other chords,* CD *and* EF, *are drawn.* ED *and* CF *intersect* AB *in* P *and* Q, *respectively. Prove that* PM = MQ.

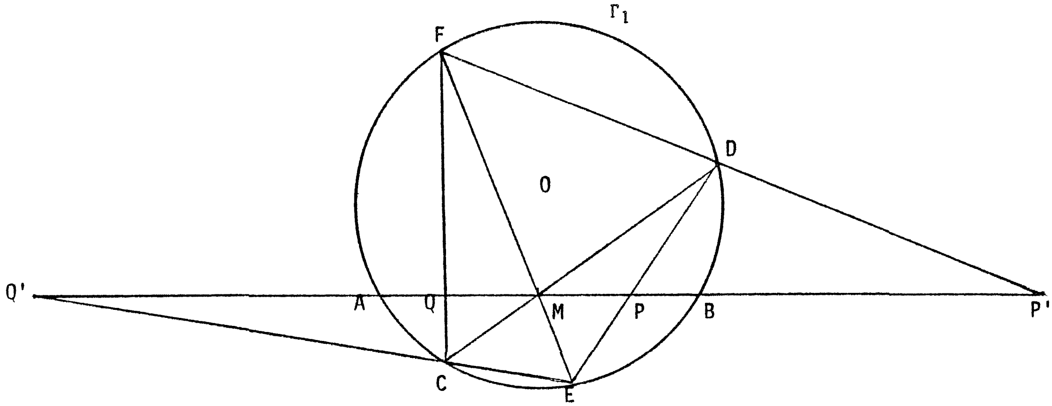

Figure 1

Proof. Let Γ_1 be the given circle. We introduce a rectangular coordinate system with origin M, x-axis AB, and y-axis MO, where $O(0,d)$ is the centre of the circle. If the circle has radius r, its equation is

$$\Sigma_1 \equiv x^2 + (y-d)^2 - r^2 = 0.$$

As the lines CD and EF pass through the origin, they form a degenerate conic Γ_2 whose equation is of the form

$$\Sigma_2 \equiv ax^2 + 2hxy + by^2 = 0.$$

Now, for any k, l,

$$\Sigma \equiv k\Sigma_1 + l\Sigma_2 = 0$$

represents a conic Γ through the points common to Γ_1 and Γ_2, that is, through C,D,E,F; and every conic through C,D,E,F is representable in this form.

Suppose the conic $\Sigma = 0$ intersects AB in V and W. The equation of AB is $y = 0$, and

$$\Sigma_1(x,0) = x^2 + d^2 - r^2, \qquad \Sigma_2(x,0) = ax^2;$$

hence the abscissas of V and W are the roots of $\Sigma(x,0) = 0$, that is, of

$$k(x^2 + d^2 - r^2) + lax^2 = 0.$$

Since this equation has no first-degree term, the sum of its roots is zero, so $\overline{MV} + \overline{MW} = 0$, and

$$VM = MW. \tag{1}$$

Now (1) holds for all conics through C,D,E,F, and the pair of lines ED,CF is such a conic, so PM = MQ follows from (1). □

The pair of lines CE,DF is also a conic through C,D,E,F. If these lines intersect AB in P' and Q', as shown in Figure 1, then P'M = MQ' also follows from (1).

Suppose that Γ_1 is, instead of a circle, an arbitrary *proper* conic of equation

$$\Sigma_1 \equiv Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0.$$

With the rest of the notation as before, we have

$$\Sigma(x,0) = k(Ax^2+2Gx+C) + lax^2.$$

If the coordinates of A and B are $(-\alpha,0)$ and $(\alpha,0)$, respectively, then $\Sigma_1(-\alpha,0) = \Sigma_1(\alpha,0) = 0$ implies that $G = 0$; so the equation $\Sigma(x,0) = 0$ has no first-degree term, and the rest of the proof is as before. We have thus proved

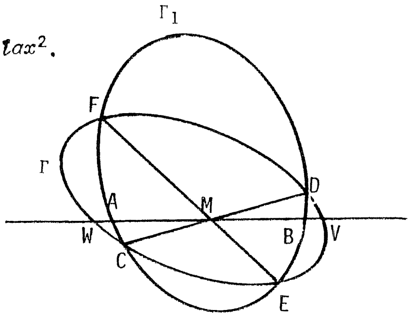

Figure 2

THE GENERALIZED BUTTERFLY PROBLEM.

Through the midpoint M *of a chord* AB *of a proper conic* Γ_1, *two other chords,* CD *and* EF, *are drawn. A conic* Γ *through* C,D,E,F *intersects* AB *in* V *and* W. *Prove that* VM = MW.

This problem is illustrated in Figure 2. A proof by projective geometry can be found in Eves [4].

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4. Howard Eves, *A Survey of Geometry*, Revised Edition, Allyn and Bacon, Boston, 1972, pp. 255-256.

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* * *

BIOGRAPHICAL NOTE

Professor Satyanarayana, who is 84 years young, has been since 1978 one of the most prolific contributors of solutions to *Crux* problems, mostly, but not exclusively, in the field of geometry. He has contributed problems and solutions to other journals as well; one of his problems, in fact, was recently published in the *American Mathematical Monthly* (Problem E 2873 in the March 1981 issue). He has also had several articles published in *The Mathematics Student*.

He was born on October 6, 1897 in Malakapalli, West Godavari District, India. He holds First Class Honours B.A. and M.A. Degrees in Mathematics from Madras University. He lectured for 34 years to B.A., B.Sc., and B.Ed. classes of the Academic and Teachers' Training Colleges at Rajahmundry, and retired in 1953 as Principal of the Teachers' Training College. Since his retirement, he has published three research-oriented books on geometry:

1. Angles and In- and Ex-Elements of Triangles and Tetrahedra (1962). Rs 10/-.

2. Poristic Theory, Pedal Lines, Kantor Lines, Kantor Points and Allied Topics (1976). Rs 30/-.

3. Dihedral Angles and In- and Ex-Elements of n-Space Simplexes (1979). Rs 10/-.

These books are available from Visalaandhra Publishing House, Vijayawada 520 004, Andhra Pradesh, India.

* * *

THE PUZZLE CORNER

Puzzle No. 5: Alphametic

FOUR is a square and FIVE is a prime,
SIX has been perfect for quite a long time.
Can you confirm when you've finished this poem
That SEVEN + NINE is a palindroem?

HANS HAVERMANN, Weston, Ontario

* * *

MORE NINE-DIGIT PATTERNED PALINDROMIC PRIMES

CHARLES W. TRIGG

Delving once more (see [1]) into the set of 5172 nine-digit palindromic primes, a list of which was prepared by Jacques Sauvé on a PDP-11/45 at the University of Waterloo, we find a number of subsets of these primes with the same characteristics or patterns. Some selected subsets are given below. But first we do a bit of juggling with the digits of the cardinal number of the set, 5172.

According to some points of view, all the digits of 5172 are primes, and $-(5+1)+7+2$ supplies the missing prime digit, 3. Furthermore,

$$5 \cdot 1 - 7 + 2 = 0 = 5 - 1 \cdot 7 + 2$$

and

$$-5 + 1 + 7 + 2 = 5,$$
$$5 + 1 - 7 + 2 = 1,$$
$$5 - 1 + \sqrt{7 + 2} = 7,$$
$$5 - \sqrt{1 \cdot 7 + 2} = 2.$$

Now to our self-appointed task. Three of the primes have eight like digits, namely:

111181111, 111191111, and 777767777.

If a nine-digit palindrome contains seven like digits, the other two digits must be like. The twenty-two such palindromic primes having nine digits fall into four different patterns. They are listed separately below according to the positions of the two like digits.

188888881	121111121	110111011	111010111
199999991	131111131	112111211	111515111
322222223	181111181	113111311	111616111
355555553	323333323	115111511	333434333
722222227		331333133	333535333
		335333533	
		338333833	
		991999199	

If a nine-digit palindrome contains six like digits and the other three digits are like, then the palindrome is divisible by 3. So there are no palindromic primes of this type.

The thirty-three palindromic primes with four like digits and five like digits include the seven smoothly undulating primes [2]

323232323, 383838383, 727272727, 919191919, 929292929, 979797979, 989898989.

The other twenty-six primes fall into the five patterns separately exhibited below:

331111133	100111001	112212211	181888181	322323223
772222277	133111331	118818811	313111313	355353553
779999977	377333773	338838833	323222323	722727227
992222299	766777667	994494499	383888383	911919119
995555599	944999449	998898899	959555959	
	977999779			
	988999889			

Thus all nine-digit palindromic primes composed of just two distinct digits are accounted for.

There are thirty-seven palindromic primes which are permutations of five consecutive digits. Each digit except the central one appears twice in the prime. These primes are assembled below into columns according to their digit sets.

102343201	125343521	345262543	345676543	745686547	759686957
312040213	134525431	346525643	354767453	746858647	957686759
320141023	142353241		356474653	756848657	967585769
321404123	153424351		357646753	786545687	976858679
324101423	312545213		736545637		978656879
	315424513		745363547		986757689
	351242153		745636547		
	352141253		753646357		
	352414253		756343657		
			764535467		
			765343567		

The first prime in the fourth column, 345676543, is the only peak [2] nine-digit palindromic prime composed of consecutive digits [3].

The thirteen palindromic primes composed of five consecutive odd digits are:

135979531	319575913	719535917	913575319
153979351	371595173	759313957	971535179
157393751	395717593	791535197	973515379
157939751			

As a central digit, 3 and 7 each appear four times, 1 appears three times, 9 appears twice, and 5 not at all.

There are forty-two nine-digit palindromic primes composed of digits that are powers of 2. The three composed of 1's and 2's, and the five composed of 1's and 8's, have been mentioned previously. The twenty-nine primes composed of three distinct powers, and the five containing the four digit-powers are:

121414121	121282121	111484111	144818441*e*	122484221
122444221	121818121	111848111	144848441*f*	128444821
141242141	128121821	114484411*a*	148414841*e*	142888241
144212441	128181821	114818411*b*	148444841*f*	184212481
	128282821	114848411*c*	148818841*g*	184222481
		118414811*b*	148888841	
		118848811*d*	184414481*e*	
		141484141*a*	184818481*g*	
		141848141*c*	188141881*d*	
		144484441	188414881*g*	

Primes followed by the same lower-case letter are permutations of the same digit set.

There are twenty-eight nine-digit palindromic primes in which every digit is a power of 3. The six primes composed of 1's and 3's, and the five primes composed of 1's and 9's have been previously mentioned. The other seventeen are:

113939311*a*	191939191	319131913*a*	319999913*b*	391999193*b*
139131931*a*	193191391*c*	319191913*c*	331999133*d*	399191993*b*
139999931*b*	199393991*b*	319393913*d*	391333193	913939319
191313191	313999313*d*			

Primes followed by the same lower-case letter are permutations of the same digit set.

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1. Charles W. Trigg, "Nine-Digit Patterned Palindromic Primes", this journal, 7 (June-July 1981) 168-170.

2. ________, "Special Palindromic Primes", *Journal of Recreational Mathematics*, 4 (July 1971) 169-170.

3. Léo Sauvé, Editor's comment, this journal, 6 (November 1980) 289-290.

2404 Loring Street, San Diego, California 92109.

* * *

MAMA-THEMATICS

Frau Gauss, speaking of her son Carl Friedrich: "He likes to get down to the roots of things."

ALAN WAYNE, Holiday, Florida

THE OLYMPIAD CORNER: 30

M.S. KLAMKIN

The following problems, for which readers are invited to send me elegant solutions, are from *Középiskolai Matematikai Lapok*, 62 (May 1981) 208-209. They are labeled "Olympiad Preparatory Problems" and were collected by József Szikszai, Miskolc. I am grateful to Frank Papp for supplying the English versions of these problems.

1. Which of the following two numbers is larger:

$$\sqrt[7]{7+\sqrt[7]{7}}-\sqrt[7]{7},\qquad \sqrt[7]{7}-\sqrt[7]{7-\sqrt[7]{7}}\quad ?$$

2. Justify the following assertion: If the positive numbers $x_1,x_2,\ldots,x_n$ have product 1, then

$$\sum_{i=1}^{n} x_i^n \le \sum_{i=1}^{n} x_i^{n+1}.$$

3. Determine the pairs (m,n) of natural numbers for which the equation

$$\frac{1-\sin^2 nx}{1-\sin^2 mx}=\sin nx$$

has real solutions.

4. Show that

$$\sum_{k=1}^{n-1}\cot(k\pi/n)\cdot\cos^2(k\pi/n)=0.$$

5. If n is a given natural number, solve the equation

$$(2x-1)^n+(1-x)^n=x^n.$$

6. If n is a given natural number, determine the largest and least values of the expression

$$\prod_{k=1}^{n}(2-\cos^2\alpha_k)+\prod_{k=1}^{n}\cos^2\alpha_k.$$

7. Show that, for nonnegative numbers a,b,c,d,

$$(a+c)(b+d)(2a+c+d)(2b+c+d)\ge 4cd(2a+c)(2b+d).$$

8. Let G denote the geometric mean of the n positive numbers a_i and, for natural numbers k, let P_k denote the kth power mean, i.e.,

$$P_k=\left(\sum_{i=1}^{n}a_i^k/n\right)^{1/k}.$$

Show that

$$(n-1)G^n \le nP_{n-1}^{n-1}P_1 - P_n^n.$$

9. Show that, for an arbitrary pair (n,k) of natural numbers, there is a unique natural number $f(n,k)$ which satisfies the relation

$$(\sqrt{n+1} + \sqrt{n})^k = \sqrt{f(n,k)+1} + \sqrt{f(n,k)}.$$

10. For which real numbers x,y is the following inequality satisfied?

$$\sqrt[3]{\frac{x^3+y^3}{2}} \ge \sqrt{\frac{x^2+y^2}{2}}\ .$$

A large number of problems published earlier in this column are still awaiting a published solution. Space permitting, we would like to publish elegant solutions to as many of them as possible, and readers are invited to collaborate in this project by submitting their solutions to me. The following solution to one of the backlog problems is not particularly elegant. Readers are urged to find a better one.

J-33. [1981:144] A straight line CD and two points A and B not on the line are given. Locate the point M on this line such that $\angle AMC = 2\angle BMD$.

Solution.

We begin by showing that there is on CD a unique point M such that $\angle AMC = k\angle BMD$ for any $k > 0$. We may assume that A and B are on the same side of CD, for otherwise we could replace one point by its mirror-image across CD.

When $k = 1$, the construction of the point M is well known. It occupies the position M' shown in Figure 1. As M moves to the left on CD, the ratio $\rho = \angle AMC/\angle BMD$ increases monotonically and becomes unbounded; and as M moves to the right on CD, ρ decreases monotonically to zero. Hence, by continuity, for every $k > 0$ there is a unique point M on CD for which $\rho = k$. This point M is to the left or to the right of M' according as $k > 1$ or $k < 1$.

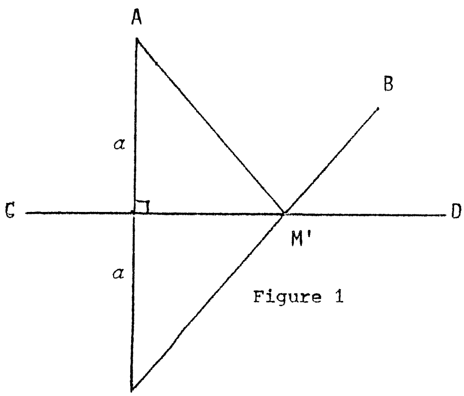

Figure 1

We show that when $k = 2$, as in our problem, the point M can be constructed with straightedge and compass. Let A' and B' be the feet of the perpendiculars from A and B, respectively, upon CD, and set AA' $= a$ and BB' $= b$, as shown in Figure 2.

We orient line CD so that $\overline{A'B'} = c$ is positive or negative according as B is to the right or to the left of A. In any case we have $a \cot 2\theta + b \cot \theta = c$. Using a familiar trigonometric identity for $\cot 2\theta$, this equation is easily shown to be equivalent to

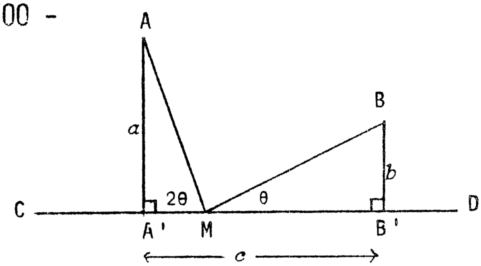

Figure 2

$$(a+2b)\cot^2\theta - 2c\cot\theta - a = 0, \qquad (1)$$

from which we get

$$MB' = b\cot\theta = \frac{b\{c+\sqrt{c^2+a(a+2b)}\}}{a+2b}, \qquad (2)$$

which shows that the point M is constructible with straightedge and compass. Note that we have used the positive root of (1) for $\cot\theta$ because $\angle AMC = 2\theta < 180^\circ$ implies that $\angle BMD = \theta < 90^\circ$. □

There may be an elegant construction that avoids the straightforward but tedious Euclidean construction of (2). Readers are invited to find one and send it to me.

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

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THE PUZZLE CORNER

Puzzle No. 6: Rebus (*5 8)

EY, EY, EY, ...

The ALL we mathematicians may define
As certain fractions ordered in a line.

Puzzle No. 7: Rebus (*7)

H/O

Not everything, you see,
In ALL's philosophy.

ALAN WAYNE, Holiday, Florida

*　　　*　　　*

Readers are urged to verify on the front page of this issue that the addresses of the editor (Léo Sauvé) and managing editor (F.G.B. Maskell) are different, being on different campuses of Algonquin College. The appropriate address should be used in each case to ensure safe and prompt arrival of readers' communications.

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before May 1, 1982, although solutions received after that date will also be considered until the time when a solution is published.

691. *Proposed by J.A. McCallum, Medicine Hat, Alberta.*

Here is an alphametic about the man who coined the word "alphametic", the well-known author of the syndicated column *Fun With Figures*, J.A.H. Hunter:

```
      H E'S
    T H A T
      F U N
  M A T H S.
 -----------
H U N T E R
```

The apostrophe has no mathematical significance and the answer, like the man himself, is unique.

692. *Proposed by Dan Sokolowsky, California State University at Los Angeles.*

S_n is a set of n distinct objects. For a fixed $k \geq 1$, $2k$ subsets of S_n are denoted by A_i, B_i, $i = 1,\ldots,k$. Find the largest possible value of n for which the following conditions (a)-(d) can hold simultaneously for $i = 1,\ldots,k$.

(a) $A_i \cup B_i = S_n$.

(b) $A_i \cap B_i = \phi$.

(c) For each pair of distinct elements of S_n, there exists an i such that the two elements are either both in A_i or both in B_i.

(d) For each pair of distinct elements of S_n, there exists an i such that one of the two elements is in A_i and the other is in B_i.

693*. *Proposed by Ferrell Wheeler, student, Texas A & M University.*

On a 4×4 tick-tack-toe board, a winning path consists of four squares in a row, column, or diagonal. In how many ways can three X's be placed on the board, not all on the same winning path, so that if a game is played on this partly-filled board, X going first, then X can absolutely force a win?

694*. *Proposed by Jack Garfunkel, Flushing, N.Y.*

Three congruent circles with radical center R lie inside a given triangle with incenter I and circumcenter O. Each circle touches a pair of sides of the triangle. Prove that O, R, and I are collinear.

(This generalizes Problem 5 of the 1981 International Mathematical Olympiad [1981: 223], where it was specified that the three circles had a common point.)

695. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

For $i = 1,2,3$, A_i are the vertices of a triangle with sides a_i and excircles with centers I_i touching a_i in B_i. For $j,k \neq i$, M_i are the midpoints and m_i the right bisectors of B_jB_k. Prove that the m_i are concurrent.

696. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be a triangle; a,b,c its sides; and s,r,R its semiperimeter, inradius and circumradius. Prove that, with sums cyclic over A,B,C,

(a) $\frac{3}{4} + \frac{1}{4}\Sigma\cos\frac{1}{2}(B-C) \geq \Sigma\cos A$;

(b) $\Sigma a\cos\frac{1}{2}(B-C) \geq s(1 + 2r/R)$.

697. *Proposed by G.C. Giri, Midnapore College, West Bengal, India.*

Let

$$a = \tan\theta + \tan\phi,$$
$$b = \sec\theta + \sec\phi,$$
$$c = \csc\theta + \csc\phi.$$

If the angles θ and ϕ are such that the requisite functions are defined and $bc \neq 0$, show that $2a/bc < 1$.

698*. *Proposé par Hippolyte Charles, Waterloo, Québec.*

Les sommes partielles de la série harmonique (laquelle, on le sait bien, est divergente) sont définies par

$$s_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}.$$

La série $\sum_{n=1}^{\infty} \frac{1}{s_n}$ est-elle convergente ou divergente?

699. *Proposed by Charles W. Trigg, San Diego, California.*

A quadrilateral is inscribed in a circle. One side is a diameter of the circle and the other sides have lengths of 3, 4, and 5. What is the length of the diameter of the circle?

700. *Proposed by Jordi Dou, Barcelona, Spain.*

Construct the centre of the ellipse of minimum eccentricity circumscribed to a given convex quadrilateral.

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

585. [1980: 284] *Proposed by Jack Garfunkel, Flushing, New York.*

Consider the following three inequalities for the angles A,B,C of a triangle:

$$\cos\frac{B-C}{2}\cos\frac{C-A}{2}\cos\frac{A-B}{2} \ge 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}, \tag{1}$$

$$\csc\frac{A}{2}\cos\frac{B-C}{2} + \csc\frac{B}{2}\cos\frac{C-A}{2} + \csc\frac{C}{2}\cos\frac{A-B}{2} \ge 6, \tag{2}$$

$$\csc\frac{A}{2} + \csc\frac{B}{2} + \csc\frac{C}{2} \ge 6. \tag{3}$$

Inequality (3) is well-known (*American Mathematical Monthly* 66 (1959) 916) and it is trivially implied by (2). Prove (1) and show that (1) implies (2).

Solution by M.S. Klamkin, University of Alberta.

We will show that inequalities (1) and (2) are just disguised forms of the two well-known elementary inequalities

$$(b+c)(c+a)(a+b) \ge 8abc \tag{1'}$$

and

$$bc(b+c) + ca(c+a) + ab(a+b) \ge 6abc, \tag{2'}$$

which are valid for arbitrary nonnegative real values of a,b,c. These are easily established by the arithmetic-geometric mean inequality (or see Bottema et al. [1]). If a,b,c are the sides of a triangle, then (1') and (2') are equivalent to

$$\frac{b+c}{a}\cdot\frac{c+a}{b}\cdot\frac{a+b}{c} \ge 8 \tag{1''}$$

and

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c} \ge 6. \tag{2''}$$

Now, by the law of sines,

$$\frac{b+c}{a} = \frac{\sin B+\sin C}{\sin A} = \frac{2\cos(A/2)\cos(B-C)/2}{2\sin(A/2)\cos(A/2)} = \frac{\cos(B-C)/2}{\sin(A/2)}. \tag{4}$$

If we substitute this and two similar expressions in (1") and (2"), we obtain (1) and (2).

To show that (1') implies (2'), and hence that (1) implies (2), we apply the arithmetic-geometric inequality: with sum and product cyclic over a,b,c, we have

$$\Sigma bc(b+c) \ge 3\{\Pi bc(b+c)\}^{1/3} \ge 6abc. \quad \square$$

For the geometrical significance of (1), we refer to an exercise in Todhunter [2] and an inequality of Gridasov [3]. In the exercise, one is to show that if the bisectors of angles A,B,C of a triangle meet the opposite sides in D,E,F, respectively, then

$$\frac{[DEF]}{[ABC]} = 2\Pi\{\sin(A/2)\}/\{\cos(B-C)/2\}, \tag{5}$$

where [DEF] denotes the area of triangle DEF, etc. Gridasov's inequality is that $4[DEF] \le [ABC]$, which follows from (1) and (5). This inequality has been extended by this author to arbitrary concurrent cevians for simplexes [4]. (See also this journal [1978: 255-256].)

To establish (5), we use an elementary result given in [4], viz.,

$$\frac{[DEF]}{[ABC]} = \frac{2xyz}{(y+z)(z+x)(x+y)}, \tag{6}$$

where x,y,z are the barycentric coordinates of the point P of concurrency of the three cevians. Here we have

$$\vec{P} = x\vec{A} + y\vec{B} + z\vec{C}, \qquad x,y,z \ge 0,\ x+y+z = 1,$$

where the vectors to P and the vertices are taken from an origin outside the plane of the triangle. If the cevians are angle bisectors, then $x = a$, $y = b$, $z = c$ (assuming that the side lengths have been normalized so that $a+b+c = 1$), and (5) follows from (6) and (4).

Also solved by W.J. BLUNDON, Memorial University of Newfoundland (two solutions); J.T. GROENMAN, Arnhem, The Netherlands; V.N. MURTY, Pennsylvania State University, Capitol Campus (two solutions); NGO TAN, student, J.F. Kennedy H.S., Bronx, N.Y.; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer.

Editor's comment.

The proposer gave without proof the following geometric equivalent of (1): *Let* I *be the incenter of triangle* ABC, *and let the bisectors of angles* A,B,C *meet the circumcircle again in* R,S,T, *respectively. Then*

$$AR\cdot BS\cdot CT \ge 8\cdot IR\cdot IS\cdot IT.$$

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1. O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969, pp. 12, 13.

2. I. Todhunter, *Plane Trigonometry*, Macmillan, London, 1880, p. 309.

3. V. Gridasov, *Matematika i fizika*, Sofia, 6 (1965) 52-53.

4. M.S. Klamkin, "A Volume Inequality for Simplexes", *Publ. Fac. D'Elektrotehn.*, Univ. of Belgrade, No. 357 - No. 380 (1971) 3-5.

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586. [1980: 284] *Proposed by Ngo Tan, student, J.F. Kennedy H.S., Bronx, N.Y.*

(a) Given a natural number n, show that the equation

$$9n^3 = 6abn + ab(a+b) \qquad (1)$$

has no solution in natural numbers a and b.

(b) Using (a), or otherwise, show that none of the following expressions is a perfect square for any natural number n:

$$36n^3 + 36n^2 + 12n + 1,$$
$$12n^3 + 36n^2 + 36n + 9,$$
$$4n^3 + 36n^2 + 108n + 81.$$

Solution by the proposer.

(a) Equation (1) is equivalent to

$$(3n+a)^3 + (3n+b)^3 = (3n+a+b)^3$$

which, by Fermat's Last Theorem, has no solution in natural numbers (or even in positive rationals). Hence, for any given n, equation (1) has no solution in natural numbers a and b.

(b) Solving (1) for a, we get

$$a = \frac{-b(b+6n) + \sqrt{\Delta}}{2b}, \qquad (2)$$

where

$$\Delta = b^2(b+6n)^2 + 36bn^3.$$

A necessary condition for the natural number b to be part of a solution (a,b) of (1) is that $b|9n^3$. For such a natural number b, Δ cannot be a perfect square; otherwise (a,b), with a given by (2), would be a positive rational solution of (1), which is impossible. In particular, for $b = 1, 3, 9$, the values of Δ, $\Delta/9$, and $\Delta/81$, respectively, viz.,

$$36n^3 + 36n^2 + 12n + 1,$$
$$12n^3 + 36n^2 + 36n + 9,$$
$$4n^3 + 36n^2 + 108n + 81,$$

are not perfect squares for any natural number n.

Also solved by KENNETH M. WILKE, Topeka, Kansas.

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587. [1980: 317] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $\sigma = A_0A_1\ldots A_n$ be an n-simplex in R^n. A straight line cuts the $(n-1)$-dimensional faces

$$\sigma_i \equiv A_0A_1\ldots A_{i-1}A_{i+1}\ldots A_n, \qquad i = 0,1,\ldots n$$

in the points B_i. If M_i is the midpoint of the straight line segment A_iB_i, show that all the points M_i lie in the same $(n-1)$-dimensional plane.

I. *Comment by M.S. Klamkin, University of Alberta.*

Coincidentally, I had proposed the same problem in *Elemente der Mathematik*, and a simple solution by I. Paasche was published in that journal [31 (1976) 14-15]. This problem extends the known results for $n = 2, 3$ for which the midpoints are collinear and coplanar, respectively.

II. *Comment by Hessel Pot, Woerden, The Netherlands.*

The special case $n = 2$ brings a question to mind. Starting with a triangle and a line, the three midpoints are again on a line, so the process can be repeated with this new line. Is there any sort of convergence or regularity when the process is repeated indefinitely?

Solutions were received from M.S. KLAMKIN, University of Alberta; LEROY F. MEYERS, The Ohio State University; HESSEL POT, Woerden, The Netherlands; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer.

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588. [1980: 317] *Proposed by Jack Garfunkel, Flushing, N.Y.*

Given is a triangle ABC with internal angle bisectors t_a, t_b, t_c and medians m_a, m_b, m_c to sides a, b, c, respectively. If

$$m_a \cap t_b = P,\ m_b \cap t_c = Q,\ m_c \cap t_a = R,$$

and L, M, N are the midpoints of the sides a, b, c, respectively, prove that

$$\frac{AP}{PL}\cdot\frac{BQ}{QM}\cdot\frac{CR}{RN} = 8.$$

Solution by Roland H. Eddy, Memorial University of Newfoundland.

Since BP bisects angle B in triangle ABL, we have AP/PL $= c/(a/2) = 2c/a$. With this and two similar results, we have

$$\frac{AP}{PL}\cdot\frac{BQ}{QM}\cdot\frac{CR}{RN} = \frac{2c}{a}\cdot\frac{2a}{b}\cdot\frac{2b}{c} = 8. \qquad \square$$

We show that if we replace the medians by the altitudes (when the triangle is acute-angled), the Gergonne cevians, or the Nagel cevians, we obtain

$$\Pi \equiv \frac{AP}{PL}\cdot\frac{BQ}{QM}\cdot\frac{CR}{RN} \geq 8.$$

For the altitudes we have AP/PL $= c/c\cos B = \sec B$ and two similar results, from which

$$\Pi = \sec A \sec B \sec C \geq 8. \qquad (1)$$

The Gergonne cevians join the vertices to the points of contact of the incircle with the opposite sides. They are concurrent in the *Gergonne point* of the triangle. The Nagel cevians join the vertices to the points of contact with the opposite sides of the excircles relative to those sides. They are concurrent in the *Nagel point* of the triangle. In both the Gergonne and the Nagel cases, we find AP/PL = $c/(s-c)$ and two similar results, from which

$$\Pi = \frac{abc}{(s-a)(s-b)(s-c)} \geq 8. \tag{2}$$

Inequalities (1) and (2) can be found in O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1968, pp. 25, 12.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; CLAYTON W. DODGE, University of Maine at Orono; JORDI DOU, Barcelona, Spain; G.C. GIRI, Midnapore College, West Bengal, India; J.T GROENMAN, Arnhem, The Netherlands; FRED A. MILLER, Elkins, West Virginia; NGO TAN, student, J.F Kennedy H.S., Bronx, N.Y.; HESSEL POT, Woerden, The Netherlands; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; MALCOLM A. SMITH, Georgia Southern College, Statesboro, Georgia; DAN SOKOLOWSKY, California State University at Los Angeles; ROBERT A. STUMP, Hopewell, Virginia; and the proposer.

Editor's comment.

See Crux 685 [1981: 275] for a related problem.

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589. [1980: 317] *Proposed by Ngo Tan, student, J.F. Kennedy H.S., Bronx, N.Y.*

In a triangle ABC with semiperimeter s, sides of lengths a, b, c, and medians of lengths m_a, m_b, m_c, prove that:

(a) There exists a triangle with sides of lengths $a(s-a)$, $b(s-b)$, $c(s-c)$.

(b) $(m_a/a)^2 + (m_b/b)^2 + (m_c/c)^2 \geq 9/4$, with equality if and only if the triangle is equilateral.

Solution by M.S. Klamkin, University of Alberta.

(a) The desired result follows immediately if we set

$$x = s-a > 0, \qquad y = s-b > 0, \qquad z = s-c > 0,$$

for then the triangle inequality

$$b(s-b) + c(s-c) > a(s-a),$$

for example, becones $y(z+x) + z(x+y) > x(y+z)$, which is equivalent to $2yz > 0$.

(b) With $4m_a^2 = 2b^2+2c^2-a^2$, etc., the required inequality is easily found to be equivalent to

$$(a^2/b^2+b^2/c^2+c^2/a^2) + (b^2/a^2+c^2/b^2+a^2/c^2) \geq 6,$$

which is true since each quantity in parentheses is at least 3 by the A.M.-G.M. inequality, with equality if and only if $a = b = c$. $\square$

A dual inequality to (b) is

$$(a/m_a)^2 + (b/m_b)^2 + (c/m_c)^2 \geq 4.$$

More generally, for any triangle inequality

$$I(a,b,c,m_a,m_b,m_c) \geq 0$$

we have the dual inequality

$$I(m_a,m_b,m_c,\tfrac{3}{4}a,\tfrac{3}{4}b,\tfrac{3}{4}c) \geq 0,$$

because the three medians of a triangle also form a triangle whose medians are $\frac{3}{4}$ the respective sides of the original triangle (see Nathan Altshiller Court, *College Geometry*, Barnes and Noble, New York, 1952, p.66).

Also solved by W.J.BLUNDON, Memorial University of Newfoundland; S.C. CHAN, Singapore; JORDI DOU, Barcelona, Spain; ROLAND H. EDDY, Memorial University of Newfoundland; JACK GARFUNKEL, Flushing, N.Y.; G.C. GIRI, Midnapore College, West Bengal, India; J.T. GROENMAN, Arnhem, The Netherlands; LEROY F. MEYERS, The Ohio State University; BOB PRIELIPP, University of Wisconsin-Oshkosh; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, Califormia State University at Los Angeles; ROBERT A. STUMP, Hopewell, Virginia; and the proposer.

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590. [1980: 317] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Find all real solutions of the equation $[x^3] - 3[x^2] + 2[x] = 0$, where the brackets denote the greatest integer function.

Solution by Hessel Pot, Woerden, The Netherlands.

The step function defined by $f(x) = [x^3] - 3[x^2] + 2[x]$ for all real x obviously satisfies $f(x) < 0$ for $x < 0$. If $x \geq 3$, then $[x^3] \geq [3x^2] \geq 3[x^2]$ and $f(x) \geq 2[x] > 0$. So the solutions of the equation $f(x) = 0$ all lie in the interval $[0,3)$, to which the subsequent discussion is restricted.

The step function is continuous (from the right) at $x = 0$, continuous at $x = 1$ and $x = 2$ since, for example,

$$f(1+\varepsilon) = f(1) = f(1-\varepsilon)+1-3+2 = f(1-\varepsilon)$$

when ε is a small positive number, and continuous from the right at all points of discontinuity, which are the square roots s and the cube roots c of integers in the intervals $(1,2)$ and $(2,3)$. As x increases from 1 to 3, $f(x)$ decreases by 3 when $x = s$ (i.e., when x^6 is a cube not a square), and it increases by 1 when $x = c$ (i.e., when x^6 is a square not a cube). It is now easy to evaluate mentally $f(x)$ when x^6 ranges along the combined ascending sequence of squares and cubes less than 3^6. Part of this sequence, with the corresponding values of $f(x)$, is tabulated at the top of the next page.

x^6 =	0	1^6	2^2	2^3	3^2	4^2	5^2	3^3	6^2	7^2
$f(x)$ =	0	0	1	-2	-1	0	1	-2	-1	0
x^6 =	2^6	9^2	10^2	11^2	5^3	12^2	13^2	14^2	6^3	15^2
$f(x)$ =	0	1	2	3	0	1	2	3	0	1

The rest of the sequence, $16^2, 17^2, \ldots, 26^2$, produces no more zeros for f because it contains at least three f-increasing squares between two succeeding f-decreasing cubes.

The above tabulation shows that the required solution set is

$$[0,\sqrt[3]{2}) \cup [\sqrt[3]{4},\sqrt[3]{5}) \cup [\sqrt[3]{7},\sqrt[3]{9}) \cup [\sqrt{5},\sqrt[3]{12}) \cup [\sqrt{6},\sqrt[3]{15}).$$

Also solved by JORDI DOU, Barcelona, Spain; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; LEROY F. MEYERS, The Ohio State University; HARRY L. NELSON, Livermore, California; DAN SOKOLOWSKY, California State University at Los Angeles; ROBERT A. STUMP, Hopewell, Virginia; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

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591. [1980: 318] *Proposed by Charles W. Trigg, San Diego, California.*

December is a good month to solve the cryptarithm

$$\text{SEER} + \text{SEER} + \text{SEER} = \text{MAGI},$$

which memorializes the start of a historical event. The three wise men, Pythagoreans all, insisted that AG be twice the square of a prime. Find the unique solution.

Solution by Clayton W. Dodge, University of Maine at Orono.

Assume the addition is written in column form. Since $2p^2$ = 08, 18, 50, and 98 for primes p with $2p^2 < 100$, and since A and G can differ by only the carries into their columns, we can have AG = 08 or 98 only. Thus G = 8, so (G,E,A) = (8,2,6), (8,9,9), or (8,6,9). Only the third case is permissible since AG ≠ 68 and A ≠ E. Now R ≠ 2 since I ≠ 6, and R ≠ 3 since I ≠ 9. Hence R = 1 and I = 3. Then S = 2 and M = 7. The unique solution is

$$2661 + 2661 + 2661 = 7983.$$

Also solved by J.A.H. HUNTER, Toronto, Ontario; ALLAN WM. JOHNSON JR., Washington, D.C; HERMAN NYON, Paramaribo, Surinam; BOB PRIELIPP, University of Wisconsin-Oshkosh (two solutions); DONVAL R. SIMPSON, Fairbanks, Alaska; RAM REKHA TIWARI, Radhaur, Bihar, India; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

One solver wondered why December was a good month to publish this cryptarithm. The answer is, obviously, because the December 1980 issue of *Crux* was delivered to readers in early January 1981, just around the Feast of Epiphany.

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592. [1980: 318] *Proposed by Leroy F. Meyers, The Ohio State University.*

(a) Given a segment AB of length l, and a rusty compass of fixed opening r, show how to find a point C such that the length of AC is the mean proportional between r and l, by use of the rusty compass only, if $\frac{1}{4}l \le r \le l$ but $r \ne \frac{1}{2}l$.

(b) Show that the construction is impossible if $r = \frac{1}{2}l$.

(c)* Is the construction possible if $r < \frac{1}{4}l$ or $r > l$?

(This problem was inspired by Dan Pedoe's Problem 492.)

Solution of parts (a) and (b) by the proposer.

Since only one radius is possible, it will be unambiguous and convenient to denote by (P) a circle with center P and radius r.

(a) The circles (A) and (B) intersect the segment AB in unique points A' and B', respectively. Since $\frac{1}{4}l \le r \le l$ and $r \ne \frac{1}{2}l$, we have $0 < A'B' \le 2r$, and so the circles (A') and (B') intersect. Let C be one of the points of intersection, and let D be the midpoint of AB. Then CD $\perp$ AB and

$$AC^2 = AD^2 + CD^2 = AD^2 + A'C^2 - A'D^2$$

$$= (\tfrac{1}{2}l)^2 + r^2 - |\tfrac{1}{2}l-r|^2 = lr.$$

Hence AC is the mean proportional between l and r. The rusty compass was used exactly four times. Note that if $r = \frac{1}{4}l$, then the triangle ACD is degenerate, but the calculation goes through; if $r = l$, then A' = B and B' = A, so that C is a point of intersection of the first two circles drawn.

(b) If $r = \frac{1}{2}l$, then the circles (A) and (B) are tangent at the midpoint D of AB. In the above notation, A' = B' = D, so that the circles (A') and (B') are not distinct, and the construction, if possible, must be continued in a different way. However, the only points which can be obtained successively as intersections of circles with centers already determined are those of the triangular lattice of side length r. The three smallest distances between any two of these points are r, $r\sqrt{3}$, and $2r$,none of which is the mean proportional $r\sqrt{2}$. Hence the mean proportional cannot be determined by rusty compass alone.

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593. [1980: 318] *Proposed by Andy Liu, University of Alberta.*

Grandpa is 100 years old and his memory is fading. He remembers that last year — or was it the year before that? — there was a big birthday party in his honor, each guest giving him a number of beads equal to his age. The total number of beads was a five-digit number, $x67y2$, but to his chagrin he cannot recall what x and y stand for. How many guests were at the party?

Solution by J.T. Groenman, Arnhem, The Netherlands.

Suppose the big party occurred last year, when Grandpa was 99. Then, with $0 < x \leq 9$ and $0 \leq y \leq 9$, we have

$$x67y2 \equiv x + 10y + 69 \equiv x + 10y - 30 \equiv 0 \quad \pmod{99},$$

which implies that $x+10y = 30$ and $10|x$. So there is no solution and the big party must have occurred the year before, when Grandpa was 98. Now we have

$$x67y2 \equiv 4x + 10y + 38 \equiv 4x + 10y - 60 \equiv 0 \quad \pmod{98}.$$

The acceptable values of x and y must therefore satisfy $2x+5y = 30$, and the only solution is $x = 5$, $y = 4$.

The number of guests was thus $56742/98 = 579$.

Also solved by CLAYTON W. DODGE, University of Maine at Orono; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; G.C. GIRI, Midnapore College, West Bengal, India; HANS HAVERMANN, Weston, Ontario; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; LEROY F. MEYERS, The Ohio Sate University; HERMAN NYON, Paramaribo, Surinam; BOB PRIELIPP, University of Wisconsin-Oshkosh; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, California State University at Los Angeles; ROBERT A. STUMP, Hopewell, Virginia; RAM REKHA TIWARI, Radhaur,Bihar, India; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec (deux solutions); CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

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594. [1980: 318] *Proposed by John Veness, Cremorne, N.S.W., Australia.*

Let N be a natural number which is not a perfect cube. Investigate the existence, nature, and number of solutions of either or both of the Diophantine equations

$$x^3 - Ny^3 = \pm 1$$

in positive integers x and y.

Comment by Kenneth M. Wilke, Topeka, Kansas.

We will let x and y range over the set Z of all rational integers. The solutions of the two equations in the proposal, if any, are then included among those of the Diophantine equation

$$x^3 + dy^3 = 1, \tag{1}$$

which has been extensively studied, especially by B. Delaunay, whose methods were later refined and extended by T. Nagell.

We quote from Mordell [1]. "The integer solutions [of (1)] are trivial when d is a perfect cube. Then if $|d| > 1$, the only solution is $x = 1$, $y = 0$, and when $|d| = 1$, there is another solution $x = 0$, $dy = 1$. We may suppose now that $d > 1$ and is free from cubed factors since these can be absorbed in y^3. We consider the cubic field $K = Q(\sqrt[3]{d})$. The integers in K of the form $x + y\sqrt[3]{d} + z\sqrt[3]{d}^2$, where x,y,z are rational integers, form a ring $Z[\sqrt[3]{d}]$, the units in which are those integers η whose norms $N(\eta) = \pm 1$. Let ε be the fundamental unit in the ring chosen so that $0 < \varepsilon < 1$. Then all the units in $Z[\sqrt[3]{d}]$ are given by $\eta = \pm\varepsilon^n$, where n takes all integer values." Mordell then goes on to discuss and prove Delaunay's result:

The equation $x^3 + dy^3 = 1$ $(d > 1)$ *has at most one integer solution with* $xy \neq 0$. *This is given by the fundamental unit in the ring when it is a binomial unit, i.e.,* ε *takes the form* $\varepsilon = x + y\sqrt[3]{d}$.

Mordell then gives Nagell's more comprehensive result as it applies to the more general equation $ax^3 + by^3 = c$. The Delaunay-Nagell Theorem as it applies to equation (1) is given by LeVeque [2] as follows (adjusted only for notation):

The equation $x^3 + dy^3 = 1$ *has at most one solution in integers* $x,y \neq 0$. *If* (x_1,y_1) *is a solution, the number* $x_1 + y_1\sqrt[3]{d}$ *is either the fundamental unit of* $K = Q(\sqrt[3]{d})$ *or its square; the latter can happen for only finitely many values of* d.

See Cohn [3] for a discussion of values of d for which (1) has no nontrivial solution.

A comment was also received from HERMAN NYON, Paramaribo, Surinam.

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1. L.J. Mordell, *Diophantine Equations*, Academic Press, New York, 1969, pp. 203-205, 219-225.

2. William Judson LeVeque, *Topics in Number Theory*, Addison-Wesley, Reading, Massachusetts, 1956, Vol.II, pp. 112-120.

3. J.H.E. Cohn, "The Diophantine Equation $x^3 = dy^3 + 1$", *Journal London Math. Soc.*, 42 (1967) 750-752.

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595. [1980: 318] *Proposed by G.C. Giri, Midnapore College, West Bengal, India.*

Let $f(x,y) = a^2\cos x\cos y + a(\sin x + \sin y) + 1$. Prove that

$$f(\beta,\gamma) = 0 \text{ and } f(\gamma,\alpha) = 0 \implies f(\alpha,\beta) = 0.$$

Solution by the proposer.

It follows from the hypothesis that $\theta = \alpha$ and $\theta = \beta$ are solutions of the equation

$$a^2\cos\gamma\cos\theta + a(\sin\gamma + \sin\theta) + 1 = 0$$

and hence of

$$a^4\cos^2\gamma\cos^2\theta = \{a(\sin\gamma + \sin\theta) + 1\}^2 \tag{1}$$

as well as of

$$a^2\sin^2\theta = \{a^2\cos\gamma\cos\theta + a\sin\gamma + 1\}^2. \qquad (2)$$

With $\cos^2\theta = 1 - \sin^2\theta$, (1) is equivalent to a quadratic in $\sin\theta$ for which the sum of the roots is

$$\sin\alpha + \sin\beta = -\frac{2(1 + a\sin\gamma)}{a(1 + a^2\cos^2\gamma)}. \qquad (3)$$

With $\sin^2\theta = 1 - \cos^2\theta$, (2) is equivalent to a quadratic in $\cos\theta$ for which the product of the roots is

$$\cos\alpha\cos\beta = \frac{(1 + a\sin\gamma)^2 - a^2}{a^2(1 + a^2\cos^2\gamma)}. \qquad (4)$$

Now, from (3) and (4),

$$a^2\cos\alpha\cos\beta + a(\sin\alpha + \sin\beta) + 1 = 0,$$

that is, $f(\alpha,\beta) = 0$.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; and KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India.

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596. [1981: 18] *Proposed by Leroy F. Meyers, The Ohio State University.*

Automorphic numbers were discussed in my comment II to Crux 321 [1978: 252]. An *automorphic number* (in base ten) is a positive integer k whose square ends in k. (Initial zeros are permitted.) Not counting the trivial solutions 1, 01, 001, ..., there are exactly two n-digit automorphic numbers for each positive integer n. Examples are

5, 6, 25, 76, 376, 625, 0625, 9376.

For an arbitrary positive integer n, find explicit formulas for the two nontrivial n-digit automorphic numbers.

Solution by Robert A. Stump, Hopewell, Virginia (revised by the editor).

If $k > 1$ is an n-digit automorphic number, then, by definition,

$$k^2 - k = k(k-1) \equiv 0 \pmod{10^n}; \qquad (1)$$

so, since $(k,k-1) = 1$ and $10 \nmid k$, either

$$5^n \mid k \quad\text{and}\quad 2^n \mid k-1 \qquad (2)$$

or

$$2^n \mid k \quad\text{and}\quad 5^n \mid k-1. \qquad (3)$$

There is at most one n-digit number k satisfying (2); for if k' is also such a number, then $k-k' \equiv 0 \pmod{10^n}$, and $k = k'$ since $|k-k'| < 10^n$. This number, if it exists, will be denoted by a_n. Similarly, there is at most one n-digit number b_n satisfying (3).

Having shown their uniqueness, we now show that a_n and b_n exist for every positive integer n. In fact, we show that, explicitly,

$$a_n = \text{the number formed by the last } n \text{ digits of } 5^{2^{n-1}} \tag{4}$$

and

$$b_n = \text{the number formed by the last } n \text{ digits of } 2^{4\cdot 5^{n-1}}. \tag{5}$$

First, observe that if a positive integer k satisfies (2) or (3) for some positive integer n, then k has at least n digits, it satisfies (1), and the number formed by its last n digits is automorphic. Now $2^{n-1} \geq n$ for every n and, from Euler's generalization of Fermat's Theorem,

$$5^{2^{n-1}} = 5^{\phi(2^n)} \equiv 1 \pmod{2^n};$$

so (2) holds for $k = 5^{2^{n-1}}$ and (4) is established. The proof of (5) is similar. It is based on

$$2^{4\cdot 5^{n-1}} = 2^{\phi(5^n)} \equiv 1 \pmod{5^n}$$

and on the fact that $4\cdot 5^{n-1} \geq n$ for every n. □

It follows from (2) and (3) that

$$10^n | a_n b_n \qquad \text{and} \qquad 10^n | (a_n - 1)(b_n - 1);$$

hence

$$a_n b_n - (a_n - 1)(b_n - 1) + 1 = a_n + b_n \equiv 1 \pmod{10^n};$$

and since $1 < a_n < 10^n$ and $1 < b_n < 10^n$, so that $2 < a_n + b_n < 2\cdot 10^n$, we conclude that

$$a_n + b_n = 10^n + 1. \tag{6}$$

So when one of the two n-digit automorphic numbers has been calculated, the other can be found more easily from (6).

If the positive integer x has at least n digits, the number formed by its last n digits is

$$x - 10^n[x/10^n]. \tag{7}$$

So if a more mathematically explicit formulation is required for (4) and (5), e.g., for a computer who does not "speak English", one can always substitute $x = 5^{2^{n-1}}$ or $2^{4\cdot 5^{n-1}}$ in (7) to obtain a_n or b_n.

Also solved by the proposer. Comments were received from HAYO AHLBURG, Benidorm, Alicante, Spain; BIKASH K. GHOSH, Bombay, India; J.T. GROENMAN, Arnhem, The Netherlands; ANDY LIU, University of Alberta; HERMAN NYON, Paramaribo, Surinam; and KENNETH M. WILKE, Topeka, Kansas.

Editor's comment.

The proposal was quite explicit in asking for explicit formulas for a_n and b_n. Yet only our featured solver and the proposer addressed themselves specifically to that question. Of the other "comments" received, some gave references (the most important of which had already appeared in this journal [1978: 254]) where limited lists of automorphic numbers can be found, and others showed how to calculate a few automorphic numbers (or even infinitely many, by recurrence). Strictly speaking, all these discussions are beside the point here if they don't (and they didn't) lead to explicit formulas for a_n and b_n.

* * *

597. [1981: 18] *Proposed by Roland H. Eddy, Memorial University of Newfoundland.*

Consider the equalities

$$\sqrt{2\frac{2}{3}} = 2\sqrt{\frac{2}{3}} \quad\text{and}\quad \sqrt{a\frac{b}{c}} = a\sqrt{\frac{b}{c}}.$$

The first occurs in W. Knight's item "...But Don't Tell Your Students" [1980: 240], which inspired this problem. Find all positive integer triples (a,b,c), with b and c square-free and $(b,c) = 1$, that satisfy the second.

Solution by Leroy F. Meyers, The Ohio State University.

Suppose that a,b,c are positive integers such that b and c are square-free and relatively prime, and

$$\sqrt{a + \frac{b}{c}} = a\sqrt{\frac{b}{c}}. \tag{1}$$

Then $ac = b(a^2-1)$. Since $(a,a^2-1) = (b,c) = 1$, we must have $a|b$ and $b|a$, so $a = b$ and $c = a^2-1$. Hence a, like b, is square-free, and so are $a+1$ and $a-1$ since their product c is square-free. Thus we have the necessary conditions:

$$a-1,\ a,\ a+1 \textit{ are all square-free; } b = a;\ c = a^2-1. \tag{2}$$

These conditions are also sufficient. For suppose a,b,c are positive integers satisfying (2). Then (1) holds, b is square-free, $(b,c) = (a,a^2-1) = 1$, and we have only left to show that c is square-free. Observe that the square-free numbers $a+1$ and $a-1$ must both be odd (otherwise one would be divisible by 4), so their gcd must be odd. Since this gcd divides their difference 2, it must be 1. The square-free numbers $a+1$ and $a-1$ are therefore relatively prime, and their product c is square-free.

We conclude that the triple (a,b,c) is a solution to our problem if and only if it satisfies (2). There are infinitely many solutions, for Sierpiński [1] affirms: "One can prove that there exist infinitely many triples of consecutive natural numbers such that each of the numbers is square-free." The first few values of a leading to solutions are: 2, 6, 14, 22, 30, 34, 38, 42, 58, 66, 70, 78, 86, 94, 102, 106, 110, 114, 130, 138, 142, 158, 166, 178, 182, 186, 194.

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; LEON BANKOFF, Los Angeles, California; JAMES BOWE, Erskine College, Due West, South Carolina; CLAYTON W. DODGE, University of Maine at Orono; BIKASH K. GHOSH, Bombay, India; J.T. GROENMAN, Arnhem, The Netherlands; J.A.H. HUNTER, Toronto, Ontario; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; ANDY LIU, University of Alberta; J.A. McCALLUM, Medicine Hat, Alberta; HERMAN NYON, Paramaribo, Surinam; BOB PRIELIPP, University of Wisconsin-Oshkosh; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, California State University at Los Angeles; ROBERT A. STUMP, Hopewell, Virginia; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; KENNETH S. WILLIAMS, Carleton University, Ottawa, Ontario; and the proposer.

Editor's Comment.

The Sierpiński reference in the above solution was added for completeness by the editor, who took it from the solution of Bob Prielipp.

Most of the other solvers arrived at the necessary conditions (2). But also, the editor regrets to report, most did not seem to be aware that a proof of sufficiency was also required for completeness. Even a bit of feeble arm waving in that direction would have been welcome. They thus appeared to take an attitude towards mathematical proofs that they would not tolerate in their students.

Ahlburg and Nyon hinted at the following more general result:

Let $n \geq 2$. If a,b,c are positive integers such that b and c are nth-power-free and relatively prime, and

$$\sqrt[n]{a+\frac{b}{c}} = a\sqrt[n]{\frac{b}{c}},$$

then

$$a-1,\ a,\ a^{n-1}+\ldots+a+1 \textit{ are all nth-power-free};\ b = a;\ c = a^n-1. \qquad (3)$$

The proof that conditions (3) are necessary follows the same pattern as in the case $n = 2$, but the proof of sufficiency breaks down when we try to show that c is nth-power-free. For example, when $n = 3$ and $a = 10$, then 9, 10, 111 are all cube-free, but $c = 999 = 3^3\cdot 37$ is not. Thus conditions (3) would have to be strengthened to make them sufficient. In any case, this emphasizes the fact that a proof of sufficiency was absolutely essential in the case $n = 2$.

REFERENCE

1. Wacław Sierpiński, *Elementary Theory of Numbers*, Warszawa, 1964, p.34.

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598. [1981: 18] *Proposed by Jack Garfunkel, Flushing, N.Y.*

Given a triangle ABC and a segment PQ on side BC, find, by Euclidean construction, segments RS on side CA and TU on side AB such that, if equilateral triangles PQJ, RSK, and TUL are drawn outside the given triangle, then JKL is an equilateral triangle.

Solutions or comments were received from JORDI DOU, Barcelona, Spain; BIKASH K. GHOSH, Bombay, India; J.T. GROENMAN, Arnhem, The Netherlands; ANDY LIU, University of Alberta; LEROY F. MEYERS, The Ohio State University; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer.

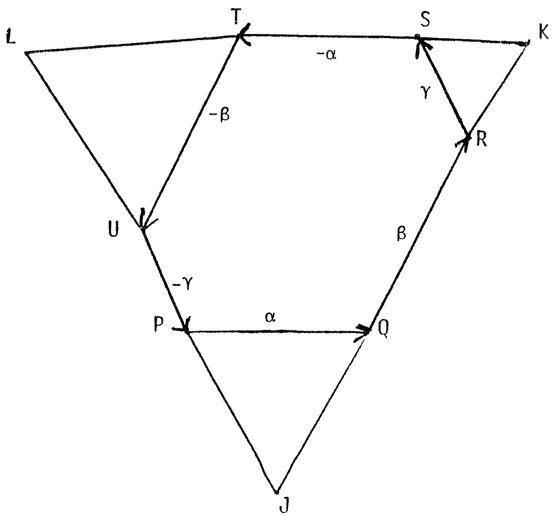
Editor's comment.

As several readers pointed out, this problem is ill-posed and additional conditions would have to be imposed to make it viable. As it stands now, with the point J known, if *any* equilateral triangle JKL is drawn with K on the opposite side of CA from B and L on the opposite side of AB from C, and then, by a trivial construction, equilateral triangles RSK and TUL are drawn with RS on CA and TU on AB, then the segments RS and TU constitute one of infinitely many solutions. Nothing to write home about. This problem, it is clear, should have been diverted to the circular file. But the proposer and the editor were both asleep at the switch.

A viable problem can, however, still be salvaged from the debacle. It is clear from the proposer's solution that the problem he *should* have proposed is the following:

Given a (not necessarily convex) hexagon PQRSTU in which two pairs of opposite sides (hence also the third pair) are equal and parallel, equilateral triangles PQJ, RSK, and TUL are drawn externally. Prove that triangle JKL is equilateral.

Solution adapted from the proposer's.



We represent vectors by complex numbers (denoted by Greek letters). Let $\omega = e^{2\pi i/3}$ (so that $\omega^3 = 1$) and

$$\vec{PQ} = \alpha, \quad \vec{QR} = \beta, \quad \vec{RS} = \gamma,$$

as shown in the figure. Then

$$\vec{JK} = \vec{JQ} + \vec{QR} + \vec{RK}$$

$$= -\alpha\omega^2 + \beta - \gamma\omega$$

and

$$\vec{KL} = \vec{KS} + \vec{ST} + \vec{TL}$$

$$= -\gamma\omega^2 - \alpha + \beta\omega.$$

Thus $\omega\vec{JK} = \vec{KL}$, which shows that triangle JKL is equilateral.

* * *

599. [1981: 18] *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Prove that 36 divides the sum of the 36 integers composing a sixth-order magic square that is *pandiagonal* (magic also along the broken diagonals) or *symmetrical* (pairs symmetrical with respect to the center have a constant sum).

Solution by the proposer.

It suffices to show that 6 divides the magic sum M of every sixth-order magic square that is pandiagonal or symmetrical.

Proof for symmetrical squares.

Let T be the constant sum of pairs symmetrical with respect to the center. If the top row contains the numbers A, B, C, D, E, F, then the bottom row contains the numbers $T-F, T-E, \ldots, T-A$. These twelve numbers form two complete rows whose sum is $6T = 2M$. Hence $T = M/3$, which shows that $3|M$.

To show that also $2|M$, we partition the sixth-order square into four third-order squares, the nine numbers in each third-order square summing to Q_1, Q_2, Q_3, Q_4, repectively, as shown in Figure 1. Because the square is magic, we have

Q_1	Q_2
Q_3	Q_4

Figure 1

$$Q_1 + Q_2 = Q_2 + Q_4 = 3M;$$

and because it is symmetrical, we have

$$Q_1 + Q_4 = 9T = 3M.$$

Hence $Q_1 = Q_4 = 3M/2$, and $2|M$.

Proof for pandiagonal squares.

To show that $2|M$, we consider the square in Figure 2, which is assumed to be magic and pandiagonal.

A_1	B_1	A_2	B_2	A_3	B_3
C_1	D_1	C_2	D_2	C_3	D_3
A_4	B_4	A_5	B_5	A_6	B_6
C_4	D_4	C_5	D_5	C_6	D_6
A_7	B_7	A_8	B_8	A_9	B_9
C_7	D_7	C_8	D_8	C_9	D_9

Figure 2

Let

$$A = \sum_{i=1}^{9} A_i, \qquad B = \sum_{i=1}^{9} B_i, \qquad D = \sum_{i=1}^{9} D_i.$$

Because the square is magic, if we add rows 1,3,5, and then separately add columns 2,4,6, we get

$$A + B = B + D = 3M;$$

and because it is pandiagonal, adding the three northwest-southeast diagonals which begin at A_1, A_2, A_3 gives

$$A + D = 3M.$$

Hence $A = D = 3M/2$, and $2|M$.

To show that $3|M$, we consider the square of Figure 3, which is assumed magic and pandiagonal.

A_1	B_1	C_1	A_2	B_2	C_2
D_1	E_1	F_1	D_2	E_2	F_2
G_1	H_1	I_1	G_2	H_2	I_2
A_3	B_3	C_3	A_4	B_4	C_4
D_3	E_3	F_3	D_4	E_4	F_4
G_3	H_3	I_3	G_4	H_4	I_4

Figure **3**

Here we define

$$A = \sum_{i=1}^{4} A_i, \qquad B = \sum_{i=1}^{4} B_i,$$

with similar definitions for C, D, E, F, G, H, I. Adding separately rows 1,4, then rows 2,5, then rows 3,6, we get

$$A + B + C = 2M, \quad D + E + F = 2M, \quad G + H + I = 2M;$$

and adding serarately columns 1,4, then columns 2,5, then columns 3,6 gives

$$A + D + G = 2M, \quad B + E + H = 2M, \quad C + F + I = 2M.$$

Finally, adding separately the northwest-southeast diagonals beginning at A_1, A_2, then the northeast-southwest diagonals beginning at C_1, C_2, we get

$$A + E + I = 2M, \qquad C + E + G = 2M.$$

If follows from these results that the square in Figure 4 is magic with magic sum $2M$. **Since** every third-order magic square has a magic sum equal to thrice the center number, we conclude that $E = 2M/3$, and so $3|M$. □

A	B	C
D	E	F
G	H	I

Figure 4

By continuing to add broken diagonals in Figure 3, it is easy to show that the square of Figure 4 is also pandiagonal, which is possible only if all its entries are equal. This proves that the entries in a sixth-order pandiagonal magic square can be partitioned into nine disjoint quartets each of which sums to $2M/3$.

The 36 consecutive integers m, $m+1$, ..., $m+35$ add up to $18(2m+35)$, so a magic square composed of these numbers must have magic sum $M = 3(2m+35)$. Because this magic sum is odd, parity prevents this magic square from being pandiagonal or symmetrical. That 36 (more generally, $(4p+2)^2$) consecutive integers cannot be arranged into a magic square that is pandiagonal or symmetrical was first proved over 60 years ago by Planck [1].

A nearly complete solution was submitted by KENNETH M. WILKE, Topeka, Kansas; and a somewhat inconclusive argument dealing with special cases was submitted by BIKASH K. GHOSH, Bombay, India.

REFERENCE

1. C. Planck, "Pandiagonal Magics of Orders 6 and 10 With Minimal Numbers", *The Monist*, 29 (1919) 307-316.

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THE DOT POLKA

The editor has received, through the courtesy of Leon Bankoff, a generous extract from a book entitled *One Million*, by Hendrik Hertzberg, published in 1970 by Simon and Schuster. The book contains one million dots.

We quote from the introduction. "There are 5000 dots to a page—10000 on each double-page spread. ...Notes are scattered—like mileposts—here and there in the inside margins. Each note corresponds to a number, and the dot signifying that number is readily identifiable." The notes, of which there are several hundreds, range from 2 (population of the Garden of Eden), through 32500 (number of laps in Lapland) and 155024 (number of breasts in Brest), to 1000000 (number of dots in the book). One of the most stirring passages in the book is reproduced below.

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If the reader stares fixedly at the above for a few minutes, the dots will soon begin to dance before his eyes. They are dancing, of course, a polka.

INDEX TO VOLUME 7, 1981

AUTHORS OF ARTICLES AND NOTES

TITLES OF ARTICLES AND NOTES

PROBLEMS AND SOLUTIONS

The numbers refer to the pages in which the corresponding name appears with a problem proposal, a solution, or a comment.