# Crux Mathematicorum

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# Crux Mathematicorum

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## GENERAL INFORMATION

Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practise or teach mathematics. Its purpose is primarily educational, but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publication should be sent to the Editor:

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# THE OLYMPIAD CORNER
## No. 97
### R.E. WOODROW

*All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.*

The first item is the 29th I.M.O. (Sydney & Canberra). I would like to thank Bruce Shawyer and Richard Nowakowski for collecting information and sharing it with me.

This year a record 268 students from 49 countries participated in the contest written July 9–21 at Sydney and Canberra, Australia. The maximum team size for each country was again six students, the same as for the last five years.

The six problems of the competition were assigned equal weights of seven points each (the same as in the last seven I.M.O.'s) for a maximum possible individual score of 42 (and a maximum possible team score of 252). For comparison see the last seven I.M.O. reports in [1981: 220], [1982: 223], [1983: 205], [1984: 249], [1985: 202], [1986: 169] and [1987: 207].

This year first place (gold) medals were awarded to students with scores from 32 to a perfect 42. There were 5 perfect papers (compared to 22 in 1987 at Havana) and 4 more students scored 40 or 41. In all 17 gold medals were awarded. The second place (silver) medals were awarded to the 48 students with scores in the range 23–31. There were 66 third place (bronze) medals awarded to students with scores in the interval 14–22. In addition honourable mention was given any student receiving full marks on at least one problem. Congratulations to the following seventeen students who received a gold medal.

| Name | Country | Score |
|---|---|---|
| Nicusor Dan | Romania | 42 |
| Nicolai Filonov | U.S.S.R. | 42 |
| Hong Yu He | China | 42 |
| Bao Chau Ngo | Vietnam | 42 |
| Adrian Vasiu | Romania | 42 |
| Xi Chen | China | 41 |
| Sergei Ivanov | U.S.S.R. | 41 |
| Julien Cassaigne | France | 40 |
| Ravi Vakil | Canada | 40 |
| Dimitri Tuliakov | U.S.S.R. | 37 |
| Dimitri Ivanov | U.S.S.R. | 36 |
| Thorsten Kleinjung | W. Germany | 35 |
| Terence Tao | Australia | 34 |
| Andreas Siebert | E. Germany | 33 |
| Wolfgang Stoecher | Austria | 33 |
| Shoni Dar | Israel | 32 |
| Mats Persson | Sweden | 32 |

The international jury (comprised of the team leaders from the participating countries) set out to give a paper that was somewhat harder than the one given last year in Havana. An indication of their success is the width of the gold medal band this year, and the rather astonishing absence of the U.S.A. from the ranks of gold medalists! Nevertheless the jury awarded medals to 49% of the participants, down only slightly from the 50.6% in 1987. Terence Tao of Australia celebrated his thirteenth birthday during the contest and was paraded around the cafeteria by his teammates. He adds a gold this year with a score of 32 to the silver (Havana) and bronze (Warsaw) he won previously. While there were no female gold medalists this year, Zvesdelina Stankova (Bulgaria) and Jianmei Wang (China) were the highest placed females with scores of 29. Also, Stankova was awarded a Special Prize (a John Conway wooden puzzle) for her solution to question 6.

As the I.M.O. is officially an individual event, the compilation and comparison of team scores is unofficial, if inevitable. These team scores were compiled by adding up the individual scores of the team members. The totals are given in the following table. Congratulations to the winning team from the U.S.S.R. and to the teams from Romania and China who tied for second place not far behind.

| Rank | Country | Score (Max 252) | Prizes | | | Total Prizes |
|---|---|---|---|---|---|---|
| | | | 1st | 2nd | 3rd | |
| 1. | U.S.S.R. | 217 | 4 | 2 | – | 6 |
| 2.–3. | China | 201 | 2 | 4 | – | 6 |
| 2.–3. | Romania | 201 | 2 | 4 | – | 6 |
| 4. | W. Germany | 174 | 1 | 4 | 1 | 6 |
| 5. | Vietnam | 166 | 1 | 4 | – | 5 |
| 6. | U.S.A. | 153 | – | 5 | 1 | 6 |
| 7. | E. Germany | 145 | 1 | 4 | – | 5 (Team of 5) |
| 8. | Bulgaria | 144 | – | 4 | 2 | 6 |
| 9. | France | 128 | 1 | 1 | 3 | 5 |
| 10. | Canada | 124 | 1 | 1 | 2 | 4 |
| 11. | U.K. | 121 | – | 3 | 2 | 5 |
| 12. | Czechoslovakia | 120 | – | 2 | 2 | 4 |
| 13.–14. | Israel | 115 | 1 | – | 4 | 5 |
| 13.–14. | Sweden | 115 | 1 | – | 4 | 5 |
| 15. | Austria | 110 | 1 | 1 | 1 | 3 |
| 16. | Hungary | 109 | – | 2 | 2 | 4 |
| 17. | Australia | 100 | 1 | – | 1 | 2 |
| 18. | Singapore | 96 | – | 2 | 2 | 4 |
| 19. | Yugoslavia | 92 | – | – | 4 | 4 |
| 20. | Iran | 86 | – | 1 | 3 | 4 |
| 21. | Netherlands | 85 | – | – | 3 | 3 |
| 22. | Republic of Korea | 79 | – | – | 3 | 3 |
| 23. | Belgium | 76 | – | – | 3 | 3 |
| 24. | Hong Kong | 68 | – | – | 2 | 2 |
| 25. | Tunisia | 67 | – | – | 3 | 3 (Team of 4) |
| 26. | Colombia | 66 | – | – | 3 | 3 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 27.–28.–29. | Finland | 65 | – | – | 2 | 2 |
| 27.–28.–29. | Greece | 65 | – | – | 1 | 1 |
| 27.–28.–29. | Turkey | 65 | – | – | 3 | 3 |
| 30. | Luxembourg | 64 | – | 1 | 2 | 3 (Team of 3) |
| 31. | Morocco | 62 | – | – | 2 | 2 |
| 32. | Peru | 55 | – | – | 1 | 1 |
| 33. | Poland | 54 | – | 1 | – | 1 (Team of 3) |
| 34. | New Zealand | 47 | – | 1 | – | 1 |
| 35. | Italy | 44 | – | – | 1 | 1 (Team of 4) |
| 36. | Algeria | 42 | – | 1 | – | 1 (Team of 5) |
| 37. | Mexico | 40 | – | – | 1 | 1 |
| 38. | Brazil | 39 | – | – | – | 0 |
| 39. | Iceland | 37 | – | – | 1 | 1 (Team of 4) |
| 40. | Cuba | 35 | – | – | – | 0 |
| 41. | Spain | 34 | – | – | – | 0 |
| 42. | Norway | 33 | – | – | – | 0 |
| 43. | Ireland | 30 | – | – | – | 0 |
| 44. | Philippines | 29 | – | – | – | 0 (Team of 5) |
| 45.–46. | Argentina | 23 | – | – | – | 0 (Team of 3) |
| 45.–46. | Kuwait | 23 | – | – | – | 0 |
| 47. | Cyprus | 21 | – | – | – | 0 |
| 48. | Indonesia | 6 | – | – | – | 0 (Team of 3) |
| 49. | Ecuador | 1 | – | – | – | 0 (Team of 1) |

This year the Canadian team moved up to tenth place. The team members, scores and the leaders were as follows:

| | | |
|---|---|---|
| Ravi Vakil | 40 | (gold medal) |
| Patrick Surry | 25 | (silver medal) |
| Colin Springer | 22 | (bronze medal) |
| David McKinnon | 15 | (bronze medal) |
| Gurraj Sangha | 11 | (honourable mention) |
| Philip Jong | 11 | |

Leaders: Bruce Shawyer, Memorial University of Newfoundland
Ron Scoins, Waterloo

Observer: Richard Nowakowski, Dalhousie.

The U.S.A. team slipped to sixth place, but members put in a solid performance. The team members were:

| | | |
|---|---|---|
| Jordan Ellenberg | 31 | (silver medal) |
| John Woo | 31 | (silver medal) |
| Samuel Kutin | 26 | (silver medal) |
| Tal Kubo | 24 | (silver medal) |
| Eric Wepsic | 23 | (silver medal) |
| Hubert Bray | 18 | (bronze medal) |

The leaders of the American team were:

Gerald Heuer, Concordia College
Gregg Patruno, Columbia University.

The next few Olympiads are:

| | |
|---|---|
| 1989 | Braunschweig, West Germany |
| 1990 | China |
| 1991 | Sweden |
| 1992 | East Germany |
| 1993 | Turkey. |

Canada has tentatively been awarded the honour for 1995.

*

We next give the problems of this year's I.M.O. competition. Solutions to these problems, along with those of the 1988 U.S.A. Mathematical Olympiad, will appear in a booklet entitled *Mathematical Olympiads 1988* which may be obtained for a small charge from:

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Executive Director
M.A.A. Committee on H.S. Contests
917 Oldfather Hall
University of Nebraska
Lincoln, Nebraska, U.S.A. 68588

## THE 29TH INTERNATIONAL MATHEMATICAL OLYMPIAD

Canberra, Australia

First Day

July 15, 1988

Time: 4 1/2 hours

1. Consider two coplanar circles of radii $R$ and $r$ $(R > r)$ with the same centre. Let $P$ be a fixed point on the smaller circle and $B$ a variable point on the larger circle. The line $BP$ meets the larger circle again at $C$. The perpendicular $l$ to $BP$ at $P$ meets the smaller circle again at $A$ (if $l$ is tangent to the circle at $P$ then $A = P$).

(i) Find the set of values of $BC^2 + CA^2 + AB^2$.

(ii) Find the locus of the midpoint of $AB$.

2. Let $n$ be a positive integer and let $A_1,A_2,\dots,A_{2n+1}$ be subsets of a set $B$. Suppose that

(a) Each $A_i$ has exactly $2n$ elements,

(b) each $A_i \cap A_j$ $(1 \le i < j \le 2n + 1)$ contains exactly one element, and

(c) every element of $B$ belongs to at least two of the $A_i$.

For which values of $n$ can one assign to every element of $B$ one of the numbers 0 and 1 in such a way that each $A_i$ has 0 assigned to exactly $n$ of its elements?

3. A function $f$ is defined on the positive integers by

$$f(1) = 1, \ f(3) = 3,$$
$$f(2n) = f(n),$$
$$f(4n + 1) = 2f(2n + 1) - f(n),$$
$$f(4n + 3) = 3f(2n + 1) - 2f(n),$$

for all positive integers $n$. Determine the number of positive integers $n$, less than or equal to 1988, for which $f(n) = n$.

Second Day
July 16, 1988
Time: 4 1/2 hours

4. Show that the set of real numbers $x$ which satisfy the inequality

$$\sum_{k=1}^{70} \frac{k}{x - k} \geq \frac{5}{4}$$

is a union of disjoint intervals, the sum of whose lengths is 1988.

5. $ABC$ is a triangle right–angled at $A$, and $D$ is the foot of the altitude from $A$. The straight line joining the incentres of the triangles $ABD$, $ACD$ intersects the sides $AB$, $AC$ at the points $K$, $L$ respectively. $S$ and $T$ denote the areas of the triangles $ABC$ and $AKL$ respectively. Show that $S \geq 2T$.

6. Let $a$ and $b$ be positive integers such that $ab + 1$ divides $a^2 + b^2$. Show that

$$\frac{a^2 + b^2}{ab + 1}$$

is the square of an integer.

* * *

We next give solutions for the problems of the 20th *Canadian Mathematics Olympiad* (1988) posed in the last issue of the Corner [1988: 163]. The solutions come from R. Nowakowski, Dalhousie University, who is chairman of the Canadian Mathematics Olympiad Committee of the Canadian Mathematical Society.

1. For what values of $b$ do the equations $1988x^2 + bx + 8891 = 0$ and $8891x^2 + bx + 1988 = 0$ have a common root?

*Solution.*

From the equations we see that

$$b = \frac{-8891 - 1988x^2}{x} \text{ and } b = \frac{-1988 - 8891x^2}{x}$$

respectively. Putting these two equal we find $x = \pm 1$. If $x = 1$ is the common root then $b = -10879$, if $x = -1$ is the common root then $b = 10879$.

2. A house is in the shape of a triangle, perimeter $P$ metres and area $A$ square metres. The garden consists of all the land within 5 metres of the house. How much land do the garden and house together occupy?

*Solution.*

The garden consists of 3 rectangular pieces and three sectors of a circle. The rectangular pieces all have width 5 metres and their total length is $P$ metres. Their combined area is therefore $5P$ square metres. At a corner of the house, with interior angle $x$, the angle within the sector is $360^\circ - 180^\circ - x = 180^\circ - x$. The sum of the angles in all three sectors is $3(180^\circ) - (\text{sum of interior angles}) = 360^\circ$. Therefore the sectors fit together to form a circle of radius 5. Their combined area is $25\pi$. The total area of house and garden is thus $A + 25\pi + 5P$ square metres.

3. Suppose that $S$ is a finite set of points in the plane where some are coloured red, the others are coloured blue. No subset of three or more similarly coloured points is collinear. Show that there is a triangle

(i) whose vertices are all the same colour; and such that

(ii) at least one side of the triangle does not contain a point of the opposite colour.

*Solution.*

Consider the set of triangles whose vertices are in the set $S$. Call a triangle *monochromatic* if all its vertices are the same colour. Let $T$ be a monochromatic triangle of least nonzero area. If every side of $T$ contains a vertex of the other colour then the triangle formed by choosing such a vertex along each side of $T$ is monochromatic and has smaller nonzero area, contrary to the choice of $T$.

4. Let

$$x_{n+1} = 4x_n - x_{n-1},\ x_0 = 0,\ x_1 = 1,$$

and

$$y_{n+1} = 4y_n - y_{n-1},\ y_0 = 1,\ y_1 = 2.$$

Show for all $n \geq 0$ that $y_n^2 = 3x_n^2 + 1$.

*Solution.*

The result is proved by simultaneous induction on the two statements

(a) $y_n^2 = 3x_n^2 + 1$

and

(b) $y_n y_{n-1} = 3x_n x_{n-1} + 2$.

Both statements are true for $n = 1$.

(i) $$\begin{aligned} y_{n+1}^2 &= (4y_n - y_{n-1})^2 = 16y_n^2 - 8y_ny_{n-1} + y_{n-1}^2 \quad \text{(by definition of } y_{n+1}) \\ &= 48x_n^2 + 16 - 8y_ny_{n-1} + 3x_{n-1}^2 + 1 \quad \text{(by induction and (a))} \\ &= 48x_n^2 + 16 - 8(2 + 3x_nx_{n-1}) + 3x_{n-1}^2 + 1 \quad \text{(by induction and (b))} \\ &= 48x_n^2 - 24x_nx_{n-1} + 3x_{n-1}^2 + 1 = 3(4x_n - x_{n-1})^2 + 1 \\ &= 3x_{n+1}^2 + 1. \quad \text{(by definition of } x_{n+1}) \end{aligned}$$

(ii) $$\begin{aligned} y_{n+1}y_n &= (4y_n - y_{n-1})y_n = 4y_n^2 - y_ny_{n-1} \quad \text{(by definition of } y_{n+1}) \\ &= 4(3x_n^2 + 1) - (3x_nx_{n-1} + 2) \quad \text{(by induction, (a) and (b))} \\ &= 3x_n(4x_n - x_{n-1}) + 2 \\ &= 3x_nx_{n+1} + 2. \quad \text{(by definition of } x_{n+1}) \end{aligned}$$

5. Let $S = \{a_1, a_2, \ldots, a_r\}$ denote a set of integers where $r$ is greater than 1. For each non–empty subset $A$ of $S$, we define $p(A)$ to be the product of all the integers contained in $A$. Let $m(S)$ be the arithmetic average of $p(A)$ over all non–empty subsets $A$ of $S$. If $m(S) = 13$ and if $m(S \cup \{a_{r+1}\}) = 49$ for some positive integer $a_{r+1}$, determine the values of $a_1, a_2, \ldots, a_r$ and $a_{r+1}$.

*Solution.*

For any $n$ and $A = \{a_1, a_2, \ldots, a_n\}$ note that

$$(1 + a_1)(1 + a_2)\ldots(1 + a_n) = (2^n - 1)m(A) + 1.$$

It follows that

$$\begin{aligned} (1 + a_1)(1 + a_2)\ldots(1 + a_{r+1}) &= (2^{r+1} - 1)m(S \cup \{a_{r+1}\}) + 1 \\ &= [(2^r - 1)m(S) + 1](1 + a_{r+1}). \end{aligned}$$

Thus

$$[13(2^r - 1) + 1](1 + a_{r+1}) = (2^{r+1} - 1)49 + 1.$$

Solving for $2^r$ (and using $2^{r+1} = 2 \cdot 2^r$),

$$2^r = \frac{12(a_{r+1} - 3)}{13a_{r+1} - 85}. \tag{1}$$

Now the right side of (1) is a decreasing function of $a_{r+1}$. Since $a_{r+1} = 1$ gives $2^r < 1$, no integers less than $85/13$ need be considered as possible values for $a_{r+1}$, i.e. $a_{r+1} \geq 7$. Since $r \geq 2$ we also require

$$\frac{12(a_{r+1} - 3)}{13a_{r+1} - 85} \geq 4,$$

which works out to $a_{r+1} \leq 38/5$. Thus $a_{r+1} = 7$, and we get $r = 3$ and

$$(1 + a_1)(1 + a_2)(1 + a_3) = (2^3 - 1)13 + 1 = 92 = 2 \cdot 2 \cdot 23.$$

Therefore the only solution in positive integers (up to rearrangement) is $1 + a_1 = 2$, $1 + a_2 = 2$, $1 + a_3 = 23$, i.e.

$$a_1 = 1, \; a_2 = 1, \; a_3 = 22.$$

The other 13 essentially different integral solutions are left for the reader.

*

The results of the 1988 Canadian Mathematics Olympiad are as follows:

| | |
|---|---|
| Gurraj Sangha | 1st Prize |
| David McKinnon | 2nd Prize |
| Philip Jong | 3rd Prize |
| Peter Copeland | 4th Prize |
| Graham Denham | 4th Prize |
| Samuel Maltby | 4th Prize |
| Phil Reiss | 4th Prize |
| Patrick Surry | 4th Prize |

It is worth noting that the highest scores on the contest were given to Ravi Vakil followed by Colin Springer, but neither person was eligible for an official prize. They were able to represent Canada at the I.M.O. because of the different eligibility criteria.

There were some ambiguities in the format and wording of the 1988 C.M.O. For instance, I received some comments on the subscripts and superscripts in problem 4 which, from the way they were printed on the exam sheet, offered potential for confusion. (Reportedly, none of the top students were fazed by this.) Readers who tried the problems printed in last month's Corner may have noticed other difficulties. For example, "set of integers" in problem 5 should likely have been "set of positive integers", and the question of rearrangements more clearly specified to limit the number of solutions to 1 instead of 14 (if nonpositive integers are allowed) or 72 (if also permutations are allowed). E.T.H. Wang pointed out that at least five points are needed in problem 3. A more serious possible confusion in problem 3 was conveyed to me by Alan Mekler of Simon Fraser University who points out that "side of the triangle" could also be taken to mean the line determined by two of the vertices. With this interpretation the problem is a good bit more challenging. What about the good contestant who reads the more difficult interpretation, but does not arrive at the solution? I reproduce below the elegant solution of Alistair Lachlan of Simon Fraser University for the problem as it was relayed to me by Alan Mekler.

**3**. [1988: 163] *1988 Canadian Mathematics Olympiad.*

Suppose that $S$ is a finite set of points in the plane where some are coloured red, the others blue. No subset of three or more similarly coloured points is collinear. Show that there is a triangle

(i) whose vertices are all the same colour; and such that

(ii) at least one side of the triangle does not contain a point of the opposite colour.

*Solution by A. Lachlan and A. Mekler, Simon Fraser University.*

Note that $S$ must contain at least 5 points, otherwise there are trivial counter-examples with no monochromatic triangles. We prove the result with "side of the triangle" interpreted to mean the entire line containing the side.

Suppose we have a counterexample with $b$ blue points and $r$ red points, and $b + r \geq 5$. If either $b$ or $r$ equals 3 or less, then it is easy to see that there is a triangle of the required type; thus $b > 3$ and $r > 3$. Call any line containing 2 red points a "red line". There are exactly $\binom{r}{2}$ red lines, and, since we have a counterexample, each red line must have a blue point on it. However, any blue point can be on at most $r/2$ red lines. Thus

$$b \geq \binom{r}{2}/(\tfrac{r}{2}) = r - 1.$$

Similarly $r \geq b - 1$, hence $|r - b| \leq 1$. In particular, as $b,r > 3$, $\binom{b}{2} > r$ and $\binom{r}{2} > b$.

Now, since every red line contains a blue point and $\binom{r}{2} > b$, there is a blue point $B_1$ which lies on two distinct red lines, say the lines containing (distinct) red points $Q_1$, $Q_2$ and $Q_3$, $Q_4$, respectively. Let $B_2,\ldots,B_b$ enumerate the rest of the blue points. Choose red points $R_2,\ldots,R_b$ so that $R_i$ lies on the line $B_1B_i$. Notice that these points exist and are distinct. As well, at most one of $Q_1$, $Q_2$ is among $R_2,\ldots,R_k$ and similarly for $Q_3$, $Q_4$. Counting the points we have

$$r \geq |\{Q_1,Q_2,Q_3,Q_4\} \cup \{R_2,\ldots,R_b\}| \geq b + 1$$

so that $r > b$. Similarly $b > r$, a contradiction.

*  *  *

# PROBLEMS

*Problem proposals and solutions should be sent to the editor, whose address appears on the inside front cover of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.*

*Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.*

*To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before* **April 1, 1989,** *although solutions received after that date will also be considered until the time when a solution is published.*

**1361**. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let $ABC$ be a triangle with sides $a$, $b$, $c$ and angles $\alpha$, $\beta$, $\gamma$, and let its circumcenter lie on the escribed circle to the side $a$.

(i) Prove that $-\cos\alpha + \cos\beta + \cos\gamma = \sqrt{2}$.

(ii) Find the range of $\alpha$.

**1362**. *Proposed by M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

Determine the sum

$$\sum_{j=0}^{n}\sum_{k=0}^{n}\begin{bmatrix}n\\ j+k\end{bmatrix}\begin{bmatrix}n\\ j\end{bmatrix}\begin{bmatrix}n\\ k\end{bmatrix}\omega^{-j-2k}$$

where $\omega$ is a primitive cube root of unity.

**1363***. *Proposed by P. Erdos, Hungarian Academy of Sciences.*

Let there be given $n$ points in the plane, no three on a line and no four on a circle. Is it true that these points must determine at least $n$ distinct distances, if $n$ is large enough? I offer \$25 U.S. for the first proof of this.

**1364**. *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.*

Let $a$ and $b$ be integers. Find a polynomial with integer coefficients that has $\sqrt[3]{a}+\sqrt[3]{b}$ as a root.

**1365**. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Prove that

$$\frac{3}{\pi}<\frac{\sin A}{\pi-A}+\frac{\sin B}{\pi-B}+\frac{\sin C}{\pi-C}<\frac{3\sqrt{3}}{\pi}$$

where $A$, $B$, $C$ are the angles (in radians) of an acute triangle.

**1366***. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

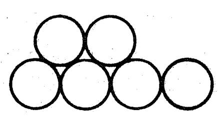
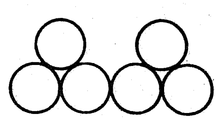
Prove or disprove that

$$\frac{x}{\sqrt{x+y}}+\frac{y}{\sqrt{y+z}}+\frac{z}{\sqrt{z+x}}\geq\frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{\sqrt{2}}$$

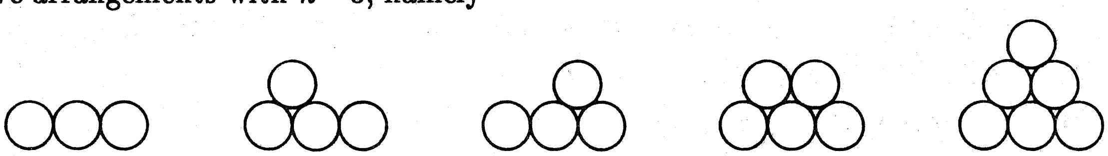
for all positive real numbers $x$, $y$, $z$.

**1367**. *Proposed by Richard K. Guy, University of Calgary.*

Consider arrangements of pennies in rows in which the pennies in any row are contiguous, and each penny not in the bottom row touches two pennies in the row below.

For example,  is allowed, but  isn't.

How many arrangements are there with $n$ pennies in the bottom row? To illustrate, there are five arrangements with $n = 3$, namely

 .

**1368**. *Proposed by Florentin Smarandache, Craiova, Romania.*

Let $ABCD$ be a tetrahedron and $A_1 \in CD$, $A_2 \in CB$, $C_1 \in AD$, $C_2 \in AB$ be four coplanar points. Let $E = BC_1 \cap DC_2$ and $F = BA_1 \cap DA_2$. Prove that the lines $AE$ and $CF$ intersect.

**1369**. *Proposed by G.R. Veldkamp, De Bilt, The Netherlands.*

The perimeter of a triangle is 24 cm and its area is 24 cm$^2$. Find the maximal length of a side and write it in a simple form.

**1370**. *Proposed by Peter Watson–Hurthig, Columbia College, Burnaby, British Columbia.*

Let $L(n)$ be the number of steps required to go from $n$ to 1 in the Collatz sequence

$$C_1(n) = n, \qquad C_{k+1}(n) = \begin{cases} 3\,C_k(n) + 1 & \text{if } C_k(n) \text{ is odd}, \\ C_k(n)/2 & \text{if } C_k(n) \text{ is even}. \end{cases}$$

It is notoriously unknown whether $L(n)$ exists for all positive integers $n$. Show that there exist infinitely many $n$ such that

$$L(n) = L(n+1) = L(n+2).$$

*   *   *

# SOLUTIONS

*No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.*

**1067**. [1985: 221; 1987: 27] *Proposed by Jack Garfunkel, Flushing, N.Y.*

(a)$^*$ If $x$, $y$, $z > 0$, prove that

$$\frac{xyz(x + y + z + \sqrt{x^2 + y^2 + z^2})}{(x^2 + y^2 + z^2)(yz + zx + xy)} \le \frac{3 + \sqrt{3}}{9}.$$

II. *Generalization by Murray S. Klamkin, University of Alberta.*

We show more generally that

$$\frac{T_n^{3/n}(T_1 + \sqrt{S_2})}{S_2 T_2} \le \frac{n + \sqrt{n}}{n\binom{n}{2}},$$

where

$$T_1 = x_1 + x_2 + \cdots + x_n,$$

$$T_2 = \sum_{i \neq j} x_i x_j,$$

$$T_n = x_1x_2\dots x_n,$$
$$S_2 = x_1^2 + x_2^2 + \cdots + x_n^2.$$

The given inequality is then the case $n = 3$.

By the Maclaurin and power mean inequalities,

$$T_n^{1/n} \le \sqrt{T_2/\binom{n}{2}} \le T_1/n \le \sqrt{S_2/n}.$$

Thus

$$T_1 + \sqrt{S_2} \le (\sqrt{n} + 1)\sqrt{S_2}$$

and

$$\frac{T_n^{3/n}}{T_2} \le \frac{1}{\binom{n}{2}}\sqrt{T_2/\binom{n}{2}} \le \frac{1}{\binom{n}{2}}\sqrt{S_2/n},$$

which yield the result.

* * *

**1122**. [1986: 50; 1987: 197] *Proposed by Richard K. Guy, University of Calgary, Calgary, Alberta.*

Find a dissection of a $6 \times 6 \times 6$ cube into a small number of connected pieces which can be reassembled to form cubes of sides 3, 4, and 5, thus demonstrating that $3^3 + 4^3 + 5^3 = 6^3$. One could ask this in at least four forms:

(a) the pieces must be bricks, with integer dimensions;

(b) the pieces must be unions of $1 \times 1 \times 1$ cells of the cube;

(c) the pieces must be polyhedral;

(d) no restriction.

*Editor's comment.*

CHARLES H. JEPSEN, Grinnell College, has sent in a 10-brick solution to part (a), improving his 11-brick solution mentioned on [1987: 197]. Here it is in layers, with the 10 bricks labelled $A$ to $J$.

| | | | | | |
|---|---|---|---|---|---|
| A | A | A | A | B | C |
| A | A | A | A | B | C |
| A | A | A | A | B | C |
| A | A | A | A | B | E |
| D | D | D | D | D | E |
| D | D | D | D | D | F |

layer 1

| | | | | | |
|---|---|---|---|---|---|
| A | A | A | A | B | C |
| A | A | A | A | B | C |
| A | A | A | A | B | C |
| A | A | A | A | B | E |
| D | D | D | D | D | E |
| D | D | D | D | D | F |

layer 2

| | | | | | |
|---|---|---|---|---|---|
| A | A | A | A | B | C |
| A | A | A | A | B | C |
| A | A | A | A | B | C |
| A | A | A | A | B | E |
| D | D | D | D | D | E |
| D | D | D | D | D | F |

layer 3

| | | | | | |
|---|---|---|---|---|---|
| *A* | *A* | *A* | *A* | *B* | *G* |
| *A* | *A* | *A* | *A* | *B* | *G* |
| *A* | *A* | *A* | *A* | *B* | *G* |
| *A* | *A* | *A* | *A* | *B* | *G* |
| *D* | *D* | *D* | *D* | *D* | *G* |
| *D* | *D* | *D* | *D* | *D* | *F* |

layer 4

| | | | | | |
|---|---|---|---|---|---|
| *H* | *H* | *H* | *H* | *H* | *G* |
| *H* | *H* | *H* | *H* | *H* | *G* |
| *H* | *H* | *H* | *H* | *H* | *G* |
| *H* | *H* | *H* | *H* | *H* | *G* |
| *H* | *H* | *H* | *H* | *H* | *G* |
| *I* | *I* | *I* | *J* | *J* | *J* |

layer 5

| | | | | | |
|---|---|---|---|---|---|
| *H* | *H* | *H* | *H* | *H* | *G* |
| *H* | *H* | *H* | *H* | *H* | *G* |
| *H* | *H* | *H* | *H* | *H* | *G* |
| *H* | *H* | *H* | *H* | *H* | *G* |
| *H* | *H* | *H* | *H* | *H* | *G* |
| *I* | *I* | *I* | *J* | *J* | *J* |

layer 6

When reassembled, $A$ forms the 4-cube, the 3-cube is

| | | |
|---|---|---|
| *C* | *C* | *C* |
| *C* | *C* | *C* |
| *C* | *C* | *C* |

layer 1

| | | |
|---|---|---|
| *E* | *E* | *E* |
| *I* | *I* | *I* |
| *J* | *J* | *J* |

layer 2

| | | |
|---|---|---|
| *E* | *E* | *E* |
| *I* | *I* | *I* |
| *J* | *J* | *J* |

layer 3

and the 5–cube is

| | | | | |
|---|---|---|---|---|
| *H* | *H* | *H* | *H* | *H* |
| *H* | *H* | *H* | *H* | *H* |
| *H* | *H* | *H* | *H* | *H* |
| *H* | *H* | *H* | *H* | *H* |
| *H* | *H* | *H* | *H* | *H* |

layer 1

| | | | | |
|---|---|---|---|---|
| *H* | *H* | *H* | *H* | *H* |
| *H* | *H* | *H* | *H* | *H* |
| *H* | *H* | *H* | *H* | *H* |
| *H* | *H* | *H* | *H* | *H* |
| *H* | *H* | *H* | *H* | *H* |

layer 2

| | | | | |
|---|---|---|---|---|
| *D* | *D* | *D* | *D* | *G* |
| *D* | *D* | *D* | *D* | *G* |
| *D* | *D* | *D* | *D* | *G* |
| *D* | *D* | *D* | *D* | *G* |
| *D* | *D* | *D* | *D* | *G* |

layer 3

| | | | | |
|---|---|---|---|---|
| *D* | *D* | *D* | *D* | *G* |
| *D* | *D* | *D* | *D* | *G* |
| *D* | *D* | *D* | *D* | *G* |
| *D* | *D* | *D* | *D* | *G* |
| *D* | *D* | *D* | *D* | *G* |

layer 4

| | | | | |
|---|---|---|---|---|
| *B* | *B* | *B* | *B* | *G* |
| *B* | *B* | *B* | *B* | *G* |
| *B* | *B* | *B* | *B* | *G* |
| *B* | *B* | *B* | *B* | *G* |
| *F* | *F* | *F* | *F* | *G* |

layer 5

This still leaves the possibility of an 8– or 9–brick solution for (a).

*　　　*　　　*

**1215**. [1987: 53; 1988: 119] *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Let $a$, $b$, $c$ be nonnegative real numbers with $a + b + c = 1$. Show that

$$ab + bc + ca \le a^3 + b^3 + c^3 + 6abc \le a^2 + b^2 + c^2 \le 2(a^3 + b^3 + c^3) + 3abc,$$

and for each inequality determine all cases when equality holds.

*Comment by Murray S. Klamkin, University of Alberta.*

It should have been noted with the published solution [1988: 119] that all the given inequalities are known. To see this, we convert the inequalities to homogeneous form by multiplying selectively by $a + b + c$ and using the elementary symmetric functions

$$T_1 = a + b + c, \quad T_2 = bc + ca + ab, \quad T_3 = abc.$$

The given inequalities are then equivalent to

$$T_1T_2 \le (T_1^3 - 3T_1T_2 + 3T_3) + 6T_3 \le T_1(T_1^2 - 2T_2) \le 2(T_1^3 - 3T_1T_2 + 3T_3) + 3T_3,$$

and all of these are known elementary inequalities. The first and third inequalities are both

$$T_1^3 + 9T_3 \ge 4T_1T_2$$

or equivalently

$$\sum a(a-b)(a-c) \ge 0,$$

a special case of the Schur inequality

$$\sum a^n(a-b)(a-c) \ge 0.$$

The middle inequality reduces to the well known Cauchy inequality $T_1T_2 \ge 9T_3$ or

$$(a + b + c)(1/a + 1/b + 1/c) \ge 9.$$

*　　　*　　　*

**1225***. [1987: 86] *Proposed by David Singmaster, The Polytechnic of the South Bank, London, England.*

What convex subset $S$ of a unit cube gives the maximum value for $V/A$, where $V$ is the volume of $S$ and $A$ is its surface area? (For the two-dimensional case, see *Crux* 870 [1986: 180].)

*Editor's comment.*

The best anyone has done with this problem ("anyone" being either RICHARD I. HESS, Rancho Palos Verdes, California; or the proposer) is to consider, analogous to the two-dimensional case, sets $S_r$ obtained by rounding the edges and corners of the unit cube to cylindrical and spherical caps of radius $r$. For these sets the maximum $V/A$ was found numerically to be at $r = 0.25848326$, where $V = 0.851069$, $A = 4.5930139$, and

$$V/A = 0.18529641.$$

Can someone improve on this?

* * *

1249*. [1987: 150] *Proposed by D.S. Mitrinovic and J.E. Pecaric, University of Belgrade, Belgrade, Yugoslavia.*

Prove the triangle inequalities

(a) $$\sum \sin^4 A \le 2 - \frac{1}{2}\left[\frac{r}{R}\right]^2 - 3\left[\frac{r}{R}\right]^4 \le 2 - 5\left[\frac{r}{R}\right]^4$$

(b) $$\sum \sin^2 2A \ge 6\left[\frac{r}{R}\right]^2 + 12\left[\frac{r}{R}\right]^4 \ge 36\left[\frac{r}{R}\right]^4$$

(c) $$\sum \sin 2B \sin 2C \le 5\left[\frac{r}{R}\right]^2 + 8\left[\frac{r}{R}\right]^3 \le 9\left[\frac{r}{R}\right]^2$$

where the sums are cyclic over the angles $A$, $B$, $C$ of a triangle, and $r$, $R$ are the inradius and circumradius respectively.

*Solution by Vedula N. Murty, Pennsylvania State University at Harrisburg.*

The inequality $r/R \le 1/2$ proves the second inequalities of (a), (b), and (c).

For the remaining inequalities we put $x = r/R$, $y = s/R$ where $s$ is the semiperimeter. We claim that the first inequalities of (a), (b), and (c) are equivalent respectively to:

$$L = y^4 - y^2(6x^2 + 8x) + 25x^4 + 8x^3 + 20x^2 - 16 \le 0,$$
$$M = y^4 - y^2(6x^2 + 8x + 4) + 25x^4 + 8x^3 + 32x^2 + 16x \le 0,$$
$$N = y^4 + y^2(2x^2 - 8x - 4) + x^4 - 24x^3 + 16x \le 0,$$

where $0 < x \le 1/2$ and $0 < y$. To prove this we need only note the following identities:

$$\sum \sin^4 A = \left[\left[\sum \sin A\right]^2 - 2\sum \sin B \sin C\right]^2 - 2\left[\sum \sin B \sin C\right]^2 + 4\prod \sin A \sum \sin A;$$

$$\sum \sin^2 2A = \left[\sum \sin 2A\right]^2 - 2\sum \sin 2B \sin 2C;$$

$$\sum \sin 2B \sin 2C = 4\prod \cos^2 A + 4\prod \sin^2 A + 4\prod \cos A;$$

$$\sum \sin 2A = 4 \prod \sin A = 2xy;$$

$$\sum \sin B \sin C = \frac{y^2 + x^2 + 4x}{4};$$

$$\sum \sin A = y;$$

$$\prod \sin A = \frac{xy}{2};$$

$$\prod \cos A = \frac{y^2 - (x + 2)^2}{4}.$$

Substitution of these expressions and some algebraic simplification proves the assertion.

Next note that

$$L - M = 4[y^2 - (3x^2 + 4x + 4)]$$

and

$$M - N = 8x^2[(3x^2 + 4x + 4) - y^2].$$

Steinig (see [1], item 5.8) proved that

$$y^2 \leq 3x^2 + 4x + 4,$$

therefore $L \leq M$ and $N \leq M$. Thus if we prove that $M \leq 0$ we immediately establish $L \leq 0$ and $N \leq 0$.

It remains to prove $M \leq 0$. The equation $M = 0$ is a quadratic in $y^2$ and has two real roots

$$y_1^2 = 3x^2 + 4x + 2 - 2\sqrt{1 - x^2 + 4x^3 - 4x^4}$$

and

$$y_2^2 = 3x^2 + 4x + 2 + 2\sqrt{1 - x^2 + 4x^3 - 4x^4}.$$

Thus

$$M = (y^2 - y_1^2)(y^2 - y_2^2),$$

and to show $M \leq 0$ we must show

$$y_1^2 \leq y^2 \leq y_2^2.$$

For this we note the known inequality

$$2 + 10x - x^2 - 2(1 - 2x)^{3/2} \leq y^2 \leq 2 + 10x - x^2 + 2(1 - 2x)^{3/2}$$

(see [1], item 5.10), and we now verify that

$$y_1^2 \leq 2 + 10x - x^2 - 2(1 - 2x)^{3/2} \tag{1}$$

and

$$y_2^2 \geq 2 + 10x - x^2 + 2(1 - 2x)^{3/2} \tag{2}$$

for $0 < x \leq 1/2$.

(1) is equivalent successively to

$$3x^2 + 4x + 2 - 2\sqrt{1 - x^2 + 4x^3 - 4x^4} \leq 2 + 10x - x^2 - 2(1 - 2x)^{3/2},$$

$$2x^2 - 3x + (1 - 2x)^{3/2} \leq \sqrt{1 - x^2 + 4x^3 - 4x^4},$$

and by squaring and rearranging,

$$8x^4 - 24x^3 + 22x^2 - 6x \le (6x - 4x^2)(1 - 2x)^{3/2},$$
$$2x(x - 1)(2x - 1)(2x - 3) \le 2x(3 - 2x)(1 - 2x)^{3/2},$$

which is true (for $0 < x \le 1/2$) since the left side is negative and the right side positive.

Similarly, (2) is equivalent to

$$3x^2 + 4x + 2 + 2\sqrt{1 - x^2 + 4x^3 - 4x^4} \ge 2 + 10x - x^2 + 2(1 - 2x)^{3/2},$$
$$\sqrt{1 - x^2 + 4x^3 - 4x^4} \ge -2x^2 + 3x + (1 - 2x)^{3/2},$$
$$-8x^4 + 24x^3 - 22x^2 + 6x \ge (6x - 4x^2)(1 - 2x)^{3/2},$$
$$2x(1 - x)(1 - 2x)(3 - 2x) \ge 2x(3 - 2x)(1 - 2x)^{3/2},$$

and finally

$$1 - x \ge \sqrt{1 - 2x},$$

which is clearly true by squaring.

*Reference:*

[1] O. Bottema et al, *Geometric Inequalities.*

*Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria. One other reader noted that the three right–hand inequalities followed easily.*

* * *

<u>1250</u>. [1987: 151] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

We have a regular octahedron with vertices $A_1, A_2, \ldots, A_6$. Let $P$ be a point and let $n_1, n_2, \ldots, n_8$ be the distances from $P$ to the eight faces of the octahedron. Let

$$S_1 = \sum_{i=1}^{6} \overline{PA_i}^2, \quad S_2 = \sum_{j=1}^{8} n_j^2.$$

Prove that $S_1/S_2$ is independent of $P$.

*Solution by Richard I. Hess, Rancho Palos Verdes, California.*

Define the vertices as

$$A_1 = (0,0,1),\ A_2 = (0,0,-1),\ A_3 = (1,0,0),$$
$$A_4 = (-1,0,0),\ A_5 = (0,1,0),\ A_6 = (0,-1,0)$$

and let $P = (x,y,z)$. Then

$$\begin{aligned} S_1 &= x^2 + y^2 + (z-1)^2 + x^2 + y^2 + (z+1)^2 + (x-1)^2 + y^2 + z^2 + (x+1)^2 \\ &\quad + y^2 + z^2 + x^2 + (y-1)^2 + z^2 + x^2 + (y+1)^2 + z^2 \\ &= 6(x^2 + y^2 + z^2 + 1). \end{aligned}$$

The eight faces are all at distance $d = 1/\sqrt{3}$ from the origin with normals $(\pm d, \pm d, \pm d)$, where all eight choices of + or – are taken. For each such normal $\mathbf{n}_i$, $1 \le i \le 8$, $\mathbf{n}_i \cdot (x,y,z)$ gives the distance $q_i = (\pm x \pm y \pm z)/\sqrt{3}$ from $P$ to the plane through the origin with normal $\mathbf{n}_i$. The sum of the squares of the distances from $P$ to the two faces of the octahedron parallel to this plane is then

$$(q_i + d)^2 + (q_i - d)^2 = 2q_i^2 + 2d^2.$$

Thus

$$S_2 = \frac{1}{2}\sum_{i=1}^{8} (2q_i^2 + 2d^2)$$

$$= \frac{2}{3}[(x + y + z)^2 + (x + y - z)^2 + (x - y + z)^2 + (-x + y + z)^2 + 4]$$

$$= \frac{8}{3}(x^2 + y^2 + z^2 + 1).$$

Therefore $S_1/S_2 = 9/4$ irrespective of $P$.

*Also solved by JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ZUN SHAN and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; D.J. SMEENK, Zaltbommel, The Netherlands; C. WILDHAGEN, Tilburg University, Tilburg, The Netherlands; and the proposer.*

*　　　　　*　　　　　*

1251. [1987: 179] *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.* (Dedicated to Léo Sauvé.)

(a) Find all integral $n$ for which there exists a regular $n$–simplex with integer edge and integer volume.

(b)* Which such $n$–simplex has the smallest volume?

*Solution by the proposer.*

(a) If the edge of the $n$–simplex is $a$, the volume is given by

$$\frac{a^n}{n!}\sqrt{\frac{n+1}{2^n}}.$$

This expression will be rational only if $n + 1$ is a square or twice a square. It can then be made integral by choosing $a$ to be large enough. If $n + 1$ is a square, then we must have $2^n$ a square, so $n$ is even. Thus $n + 1$ will be the square of an odd number, so

$$n = 4k^2 + 4k \tag{1}$$

for some positive integer $k$. If $n + 1$ is twice a square, then we have

$$n = 2k^2 - 1 \tag{2}$$

for some positive integer $k > 1$. Equations (1) and (2) give all possible values for $n$.

(b)* I don't have a rigorous solution to this part. It seems clear that the smallest integral volume occurs for the smallest $n$, since $n!$ contains many primes and these can't all be cancelled by the $n + 1$ term in the numerator. Thus these primes must appear in $a$, and the term $a^n$ increases much faster than $n!$ does. So it appears that the smallest volume occurs when $n = 7$ and $a = 2 \cdot 3 \cdot 5 \cdot 7 = 210$. The volume then is

$$2 \cdot 3^5 \cdot 5^6 \cdot 7^6 = 893397093750.$$

*Also solved (part (a)) by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria. There was one partial solution submitted.*

*The proposer is probably correct about part (b), although one reader claims the minimum volume to be* 1, *occurring when* $n = 1 = a$ ! *Okay then,* **assuming $n > 1$**, *can anyone give a simple argument for part (b)?*

* * *

<u>1252</u>. [1987: 179] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $ABC$ be a triangle and $M$ an interior point with barycentric coordinates $\lambda_1, \lambda_2, \lambda_3$. We denote the pedal triangle and the Cevian triangle of $M$ by $DEF$ and $A'B'C'$ respectively. Prove that

$$\frac{[DEF]}{[A'B'C']} \geq 4\lambda_1\lambda_2\lambda_3(s/R)^2,$$

where $s$ is the semiperimeter and $R$ the circumradius of $\Delta ABC$, and $[X]$ denotes the area of figure $X$.

I. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

We will prove the stronger result

$$\frac{[DEF]}{[A'B'C']} \geq \frac{(\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)(\lambda_3 + \lambda_1)}{2}\left[\frac{s}{R}\right]^2,$$

with equality if and only if $M$ is the incenter of $\Delta ABC$. The given inequality then follows via

$$(\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)(\lambda_3 + \lambda_1) \geq 8\lambda_1\lambda_2\lambda_3.$$

Let $F = [ABC]$ and $r_1, r_2, r_3$ the distances from $M$ to the sides $a_1, a_2, a_3$, respectively, of $\Delta ABC$. Then

$$\lambda_1 = \frac{[MBC]}{F} = \frac{a_1r_1}{2F}, \quad \lambda_2 = \frac{a_2r_2}{2F}, \quad \lambda_3 = \frac{a_3r_3}{2F}. \tag{1}$$

Furthermore, since

$$[DEM] = \frac{r_1r_2\sin\ C}{2} = \frac{cr_1r_2}{4R}, \text{ etc.}$$

we have

$$[DEF] = [DEM] + [EFM] + [FDM]$$
$$= \frac{a_1r_2r_3 + a_2r_3r_1 + a_3r_1r_2}{4R}. \tag{2}$$

From p.89 of Bottema et al, *Geometric Inequalities*, we take

$$[A'B'C'] = \frac{2\lambda_1\lambda_2\lambda_3F}{(\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)(\lambda_3 + \lambda_1)}. \tag{3}$$

Finally by the weighted arithmetic–harmonic inequality we get

$$\frac{a_1r_2r_3 + a_2r_3r_1 + a_3r_1r_2}{a_1 + a_2 + a_3} \geq \frac{a_1 + a_2 + a_3}{\dfrac{a_1}{r_2r_3} + \dfrac{a_2}{r_3r_1} + \dfrac{a_3}{r_1r_2}}$$

or

$$a_1r_2r_3 + a_2r_3r_1 + a_3r_1r_2 \geq \frac{4s^2r_1r_2r_3}{a_1r_1 + a_2r_2 + a_3r_3}, \tag{4}$$

with equality if and only if $r_1 = r_2 = r_3$, i.e. $M$ is the incenter of $\Delta ABC$. Now (1), (2), and (4) yield

$$[DEF] \geq \frac{s^2r_1r_2r_3}{R(a_1r_1 + a_2r_2 + a_3r_3)} = \frac{4s^2F^2\lambda_1\lambda_2\lambda_3}{Ra_1a_2a_3(\lambda_1 + \lambda_2 + \lambda_3)}$$

$$= \frac{s^2F\lambda_1\lambda_2\lambda_3}{R^2}, \tag{5}$$

where we used $a_1a_2a_3 = 4RF$ and $\lambda_1 + \lambda_2 + \lambda_3 = 1$. (3) and (5) yield the desired result.

II. *Generalizations by Murray S. Klamkin, University of Alberta.*

[Klamkin also proved (5), having noted that the original inequality then follows from the fact that the maximum of $[A'B'C']$ is $F/4$, occurring when the three cevians are the three medians (see [1978: 256]). In the process he obtained (4), using

$$a_1r_1 + a_2r_2 + a_3r_3 = 2F = r(a_1 + a_2 + a_3),$$

where $r$ is the inradius of $\Delta ABC$, to write it in the form

$$\frac{a_1}{r_1} + \frac{a_2}{r_2} + \frac{a_3}{r_3} \geq \frac{a_1 + a_2 + a_3}{r}. \tag{6}$$

He then went on to say ...]

We now give some generalizations of (6). For $n$, $m$ real, $n \geq 0$, we have by Hölder's inequality that

$$\left[\frac{a_1^m}{r_1^n} + \frac{a_2^m}{r_2^n} + \frac{a_3^m}{r_3^n}\right]^{\frac{1}{n+1}}(a_1r_1 + a_2r_2 + a_3r_3)^{\frac{n}{n+1}}$$

$$\geq \left[\frac{a_1^m}{r_1^n}\right]^{\frac{1}{n+1}}(a_1r_1)^{\frac{n}{n+1}} + \left[\frac{a_2^m}{r_2^n}\right]^{\frac{1}{n+1}}(a_2r_2)^{\frac{n}{n+1}} + \left[\frac{a_3^m}{r_3^n}\right]^{\frac{1}{n+1}}(a_3r_3)^{\frac{n}{n+1}}$$

$$= a_1^{\frac{m+n}{n+1}} + a_2^{\frac{m+n}{n+1}} + a_3^{\frac{m+n}{n+1}},$$

or

$$\left[\frac{a_1^m}{r_1^n} + \frac{a_2^m}{r_2^n} + \frac{a_3^m}{r_3^n}\right]r^n(a_1 + a_2 + a_3)^n \geq \left[a_1^{\frac{m+n}{n+1}} + a_2^{\frac{m+n}{n+1}} + a_3^{\frac{m+n}{n+1}}\right]^{n+1}. \tag{7}$$

If $m = 1$, (7) becomes

$$\frac{a_1}{r_1^n} + \frac{a_2}{r_2^n} + \frac{a_3}{r_3^n} \geq \frac{a_1 + a_2 + a_3}{r^n}.$$

Putting $n = 1$ as well yields (6).

The above inequalities and all others of the type

$$x + y + z \geq w$$

can be extended to

$$F(x) + F(y) + F(z) \geq F(x + y + z) \geq F(w)$$

where $F$ is an increasing concave function with $F(0) = 0$ ($F$ is then subadditive). For example, letting $F(x) = x^\lambda$ where $0 < \lambda < 1$, (6) becomes

$$\left[\frac{a_1}{r_1}\right]^\lambda + \left[\frac{a_2}{r_2}\right]^\lambda + \left[\frac{a_3}{r_3}\right]^\lambda \geq \left[\frac{a_1 + a_2 + a_3}{r}\right]^\lambda .$$

*Also solved by the proposer. There was one partial solution submitted.*

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**1253.** [1987: 179] *Proposed by Richard I. Hess, Rancho Palos Verdes, California.*

Player $A$ starts with \$3 and player $B$ starts with \$10. On each turn a fair coin is tossed, with the outcome that either $B$ pays $A$ \$3 or $A$ pays $B$ \$2. Play continues until one player wins by having won all the other player's money. Which player is more likely to win?

*Solution by the proposer.*

Let $p_i$ by the probability that $A$ wins when $A$ starts with \$$i$ and $B$ starts with \$$13 - i$. We want to find $p_3$. By the conditions of the problem,

$$p_i = \frac{1}{2}(p_{i-2} + p_{i+3}), \quad 1 \leq i \leq 12,$$

where we define

$$p_i = 0 \text{ for } i \leq 0, \quad p_i = 1 \text{ for } i \geq 13.$$

Thus

$$p_3 = \frac{1}{2}p_1 + \frac{1}{2}p_6 = \frac{1}{2}p_4 + \frac{1}{4}p_9 = \frac{1}{4}p_2 + \frac{3}{8}p_7 + \frac{1}{8}p_{12}$$
$$= \frac{5}{16}p_5 + \frac{1}{4}p_{10} + \frac{1}{16} = \frac{5}{32}p_3 + \frac{9}{32}p_8 + \frac{3}{16},$$

so that

$$9p_3 = 3p_8 + 2. \tag{1}$$

Continuing,

$$9p_3 = \frac{3}{2}p_6 + \frac{3}{2}p_{11} + 2 = \frac{3}{4}p_4 + \frac{3}{2}p_9 + \frac{11}{4}$$
$$= \frac{3}{8}p_2 + \frac{9}{8}p_7 + \frac{3}{4}p_{12} + \frac{11}{4} = \frac{3}{4}p_5 + \frac{15}{16}p_{10} + \frac{25}{8}$$
$$= \frac{3}{8}p_3 + \frac{27}{32}p_8 + \frac{115}{32},$$

so that

$$276p_3 = 27p_8 + 115. \tag{2}$$

Now (1) and (2) yield

$$p_3 = \frac{97}{195} < \frac{1}{2},$$

so $B$ is more likely to win.

*Also solved by HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; and R.D. SMALL, University of New*

*Brunswick, Fredericton. None of these solutions was as succinct as the proposer's. There was also one incorrect solution received.*

* * *

**1254**. [1987: 179] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $ABC$ be a triangle and $n \geq 1$ a natural number. Show that

$$\left|\sum \sin n(B-C)\right| \begin{cases} < 1 & \text{if } n = 1, \\ < 3\sqrt{3}/2 & \text{if } n = 2, \\ \leq 3\sqrt{3}/2 & \text{if } n \geq 3, \end{cases}$$

where the sum is cyclic.

*Solution by Murray S. Klamkin, University of Alberta.*

Letting

$$x = n(B-C), \quad y = n(C-A), \quad z = n(A-B), \tag{1}$$

and denoting the given sum by $S_n$, our problem is to find the extrema of

$$|S_n| = |\sin x + \sin y + \sin z|$$

subject to $x + y + z = 0$. Our solution is via Lagrange multipliers. The Lagrangian is

$$\mathscr{L} = \sin x + \sin y + \sin z - \lambda(x + y + z).$$

The critical points will satisfy $\mathscr{L}_x = \mathscr{L}_y = \mathscr{L}_z = 0$ or

$$\cos x = \cos y = \cos z = \lambda. \tag{2}$$

Since $x + y + z = 0$, it follows easily that

$$\cos^2 x + \cos^2 y + \cos^2 z = 1 + 2 \cos x \cos y \cos z.$$

Thus

$$3\lambda^2 = 1 + 2\lambda^3,$$

so that $\lambda = 1$ or $-1/2$. Now by (2), if $\lambda = 1$ then $\sin x = 0$ etc., while if $\lambda = -1/2$ then $\sin x = \pm\sqrt{3}/2$ etc. Thus

$$|S_n| \leq 3\sqrt{3}/2 \tag{3}$$

for all critical points.

Without loss of generality we may assume $A \leq B \leq C$. Then if $n = 1$, (1) and (2) with $\lambda = -1/2$ imply that

$$A - B = -120^\circ = B - C,$$

and so $C - A = 240^\circ$ which is impossible. Thus for $n = 1$ any critical points in the interior of the feasible region will correspond to $\lambda = 1$, in which case

$$|S_n| = 0.$$

For $n = 2$ and $\lambda = -1/2$, (1) and (2) imply

$$\{A - B, B - C\} \subseteq \{-60^\circ, -120^\circ\}$$

and it is easily seen that the only solution is the degenerate triangle $C = 120^\circ$, $B = 60^\circ$,

$A = 0^\circ$. Thus for $n = 2$ strict inequality holds in (3) at critical points. For $n \geq 3$ we have a nondegenerate solution

$$A = 60^\circ - \frac{120^\circ}{n}, \quad B = 60^\circ, \quad C = 60^\circ + \frac{120^\circ}{n}$$

giving equality in (3).

It remains to check that the given inequalities hold on the boundary of the region containing those points $(x,y,z)$ which actually correspond to triangles $ABC$. This boundary contains precisely all degenerate triangles, those in which at least one of $A$, $B$, $C$ is 0. $A = B = 0$ gives

$$|S_n| = 0$$

which satisfies the inequalities for all $n$. For $A = 0$ we get

$$\begin{aligned} |S_n| &= |\sin n(B-C) + \sin nC - \sin nB| \\ &= |\sin(180n - 2nC) + \sin nC - \sin(180n - nC)| \\ &= \begin{cases} |\sin 2nC| & , \ n \text{ odd} \\ |2 \sin \ nC - \sin 2nC|, & n \text{ even.} \end{cases} \end{aligned}$$

For $n$ odd, we have $|S_n| \leq 1$. We have $|S_1| = 1$ for the degenerate triangle $C = 135^\circ$, $B = 45^\circ$. For $n$ even, we let

$$f(C) = 2 \sin nC - \sin 2nC,$$

for which

$$\begin{aligned} f'(C) &= 2n \cos nC - 2n \cos 2nC \\ &= 2n(1 + \cos nC - 2 \cos^2 nC), \end{aligned}$$

so that $f$ takes on its extreme values when $\cos nC = 1$ or $-1/2$. Then we get respectively

$$\sin nC = 0, \ \ f(C) = 0$$

and

$$\sin nC = \pm\sqrt{3}/2, \ \ \sin 2nC = \pm\sqrt{3}/2, \ \ |f(C)| \leq 3\sqrt{3}/2.$$

Thus $|S_n| \leq 3\sqrt{3}/2$ holds on the boundary for all $n > 1$.

In summary, all three given inequalities are correct and best possible, with equality holding for $n = 1$ and 2 only for degenerate triangles.

We now consider the analogous problem with cos instead of sin. Putting

$$C_n = \sum \cos n(B - C),$$

clearly $C_n \leq 3$, with equality for equilateral triangles. To obtain the minimum value for $C_n$, we proceed as before, using Lagrange multipliers. Again with the substitution (1), the Lagrangian is

$$\mathcal{L} = \cos x + \cos y + \cos z - \lambda(x + y + z)$$

and the equations for the critical points are

$$\sin x = \sin y = \sin z = \lambda. \tag{4}$$

The identity we use here (again for $x + y + z = 0$) is

$$\sin 2x + \sin 2y + \sin 2z = -4 \sin x \sin y \sin z. \tag{5}$$

If $\cos x$, $\cos y$, and $\cos z$ are not all the same sign, then from (4)

$$C_n = \sum \cos x \geq -1. \tag{6}$$

We note that when $n = 1$, equality in (6) holds only if two of $\cos(B - C)$, $\cos(C - A)$, $\cos(A - B)$ equal $-1$, i.e. only for degenerate triangles. If $\cos x$, $\cos y$, $\cos z$ all have the same sign, then from (5)

$$3 \sin 2x = -4 \sin^3 x,$$
$$3 \cos x = -2(1 - \cos^2 x),$$

so

$$(2 \cos x + 1)(\cos x - 2) = 0,$$

that is, $\cos x = -1/2$. Thus

$$C_n \geq -3/2.$$

As before, for $n > 2$ equality holds for the nondegenerate triangle

$$A = 60^\circ - \frac{120^\circ}{n}, \quad B = 60^\circ, \quad C = 60^\circ + \frac{120^\circ}{n};$$

for $n = 2$ equality holds only for a degenerate triangle; and for $n = 1$ there are no solutions.

Now we consider the boundary. For $A = B = 0$,

$$C_n = 1 + 2 \cos nC \geq -1,$$

and for $n = 1$ equality holds only for a degenerate triangle. For $A = 0$,

$$\begin{aligned} C_n &= \cos n(B - C) + \cos nC + \cos nB \\ &= \cos(180n - 2nC) + \cos nC + \cos(180n - nC) \\ &= \begin{cases} -\cos 2nC, & n \text{ odd} \\ \cos 2nC + 2\cos nC, & n \text{ even.} \end{cases} \end{aligned}$$

Thus for $n$ odd, $C_n \geq -1$, and for $n$ even,

$$\begin{aligned} C_n &= 2 \cos^2 nC + 2 \cos nC - 1 \\ &= 2(\cos nC + 1/2)^2 - 3/2 \geq -3/2. \end{aligned}$$

Hence, excluding degenerate triangles, we have shown that

$$\sum \cos n(B - C) \begin{cases} > -1 & \text{if } n = 1, \\ > -3/2 & \text{if } n = 2, \\ \geq -3/2 & \text{if } n \geq 3. \end{cases}$$

*Also solved by the proposer. Two partial solutions were received.*

* * *

**1255.** [1987: 180] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

(a) Find all positive integers $n$ such that $2^{13} + 2^{10} + 2^n$ is the square of an integer.

(b)* Find all positive integers $n$ such that $2^{14} + 2^{10} + 2^n$ is the square of an integer.

I. *Solution to (a) by several readers.*

Rewrite

$$2^{13} + 2^{10} + 2^n = y^2$$

as

$$2^{10}(8 + 1) + 2^n = y^2,$$
$$(2^5 \cdot 3)^2 + 2^n = y^2,$$

and finally

$$2^n = y^2 - 96^2 = (y + 96)(y - 96).$$

Thus each of $y + 96$ and $y - 96$ must be a power of 2, and since they differ by 192 we must have

$$y + 96 = 256 = 2^8,$$

so

$$y - 96 = 64 = 2^6,$$

and hence the only solution is $n = 8 + 6 = 14$.

II. *Solution to (b) by Kee–wai Lau, Hong Kong.*

We shall see that

$$2^{14} + 2^{10} + 2^n$$

is the square of an integer if and only if $n = 4, 13, 15, 16$, or $19$.

For $1 \le n \le 10$ it can be checked easily that $2^{14} + 2^{10} + 2^n$ is the square of an integer only for $n = 4$. We now assume $n \ge 11$ and let $k = n - 10 \ge 1$. Then

$$2^{14} + 2^{10} + 2^n = (32)^2(2^k + 17),$$

so we want to know when $2^k + 17$ is the square of an integer. Now it has been proved ([1], [2]) that the only positive integers $k$ for which $y^2 - 17 = 2^k$ is solvable in integers are 3, 5, 6, and 9. Hence our result.

*References:*

[1] F. Beukers, On the generalized Ramanujan–Nagell equation I, *Acta Arithmetica* 38 (1981) 389–410.

[2] N. Tzanakis, On the Diophantine equation $y^2 - D = 2^k$, *J. Number Theory* 17 (1983) 144–164.

III. *Editor's comments.*

The proposer pointed out a similar problem on [1982: 46], with solution as in I above.

Two readers, LEROY F. MEYERS and P. PENNING, independently considered the general Diophantine equation

$$2^a + 2^b + 2^c = x^2 \tag{1}$$

for $a$, $b$, $c$, $x$ nonnegative integers, $a \le b \le c$. They found the "basic" solutions

$$2^1 + 2^1 + 2^5 = 6^2, \tag{2}$$

$$2^0 + 2^{k+1} + 2^{2k} = (1 + 2^k)^2, \ \ k \ge 0, \tag{3}$$

$$2^0 + 2^4 + 2^5 = 7^2, \tag{4}$$

and

$$2^0 + 2^4 + 2^9 = 23^2, \tag{5}$$

from which further solutions can be derived by multiplying both sides by an arbitrary even positive power of 2. Thus, of the five values of $n$ given in II, 15 and 19 come from (4) and (5) respectively (by multiplying by $2^{10}$), and 4, 13 and 16 similarly come from (3). Neither Meyers nor Penning could show that there are no other solutions to (1). Any further "basic" solution, however, will have $a = 0$ and $2 \le b < c$ where $c$ is odd. Does some reader know whether additional solutions exist?

*Part (a) solved by HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE–WAI LAU, Hong Kong; STEWART METCHETTE, Culver City, California; LEROY F. MEYERS, The Ohio State University; P. PENNING, Delft, The Netherlands; ZUN SHAN and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; C. WILDHAGEN, Tilburg University, Tilburg, The Netherlands; and the proposer.*

*Hess, Lau, Meyers, Penning, and the proposer found all five solutions to (b), while Janous, Metchette, and Shan and Wang only missed one.*

* * *

**1256.** [1987: 180] *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

Let $ABC$ be a triangle with sides satisfying $a^3 = b^3 + c^3$. Determine the range of angle $A$.

*Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

We consider the following more general situation. Let $r > 2$ and $a$, $b$, $c$ be positive real numbers such that

$$a^r = b^r + c^r. \tag{1}$$

We shall determine that $a$, $b$, $c$ are the sides of a triangle $ABC$, and that the range of angle $A$ is

$$[\arccos(1 - 2^{(2-r)/r}), \tfrac{\pi}{2}). \tag{2}$$

In particular when $r = 3$ we get

$$A \in [\arccos(1 - 2^{-1/3}), \tfrac{\pi}{2})$$

or approximately

$$78.1^\circ \le A < 90^\circ.$$

Without loss of generality let $a = 1$. Then by (1) $b,c < 1$. We also assume $b \ge c$. Then from (1)

$$c \le 1/2^{1/r}, \quad b = (1 - c^r)^{1/r}.$$

We first claim that $a$, $b$, $c$ form a triangle. Indeed, we have to check $b + c > 1$, i.e.,

$$(1 - c^r)^{1/r} + c > 1.$$

This inequality is either easily verified directly or it follows from Minkowski's inequality.

Letting the triangle be $ABC$, by the law of cosines we have

$$\cos A = \frac{(1 - c^r)^{2/r} + c^2 - 1}{2c(1 - c^r)^{1/r}}$$

$$= \frac{(1 - t)^{2/r} + t^{2/r} - 1}{2t^{1/r}(1 - t)^{1/r}} = f(t), \qquad (3)$$

where we have put $t = c^r$ (and thus $0 < t \le 1/2$). Let's now discuss $f(t)$ on the interval $(0,1/2]$. Differentiation of (3) and a simplification (of medium length) leads to

$$f'(t) = \frac{t^{2/r} - (1 - t)^{2/r} + 1 - 2t}{2r[t(1 - t)]^{1+1/r}}.$$

Thus the sign of $f'(t)$ is equal to the sign of

$$z(t) = 1 - 2t + t^{2/r} - (1 - t)^{2/r}.$$

Now

$$z'(t) = -2 + \frac{2}{r}[t^{(2-r)/r} + (1 - t)^{(2-r)/r}]$$

and

$$z''(t) = \frac{2(2 - r)}{r^2}[t^{(2-2r)/r} - (1 - t)^{(2-2r)/r}].$$

As $r > 2$ and $t \le 1/2$, we infer $z''(t) < 0$, i.e. $z$ is concave. From this we get

$$z(t) \ge \min\{z(0)\ ,\ z(1/2)\} = 0.$$

Consequently $f'(t) \ge 0$, i.e. $f$ increases on $(0,1/2]$. Finally, from (3) we have

$$f(1/2) = 1 - 2^{(2-r)/r}$$

and

$$\lim_{t \to 0} f(t) = 0,$$

and the range (2) follows.

[*Editor's note*: this generalization was also essentially obtained by the proposer. From his solution the range of $A$ comes out to be

$$[2\arcsin(\sqrt[r]{2}/2)\ ,\ \pi/2);$$

same as (2), but a bit simpler-looking.]

*Also solved by SEUNG–JIN BANG, Seoul, Korea; HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; J.T. GROENMAN, Arnhem, The Netherlands;*

*RICHARD I. HESS, Rancho Palos Verdes, California; KEE–WAI LAU, Hong Kong; ZUN SHAN and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; and the proposer.*

* * *

**1257**. [1987: 180] *Proposed by Jordan Stoyanov, Bulgarian Academy of Sciences, Sofia, Bulgaria.*

Find all rational $x$ such that $3x^2 - 5x + 4$ is the square of a rational number.

*Solution by P. Penning, Delft, The Netherlands.*

We consider the more general equation

$$ax^2 + bx + c^2 = y^2,$$

where $a$, $b$, $c$ are rational numbers. This may be rewritten as

$$x(ax + b) = (y + c)(y - c)$$

and then as

$$\frac{y + c}{x} = \frac{ax + b}{y - c} = r, \tag{1}$$

which is equivalent to the system

$$rx - y = c$$
$$ax - ry = -b - rc.$$

Solving for $x$, we get

$$x = \frac{2cr + b}{r^2 - a}$$

where (from (1)) all rational solutions $x$ will be found by allowing $r$ to take on all possible rational values.

In the original problem, $a = 3$, $b = -5$, $c = 2$ so that the solution is

$$x = \frac{4r - 5}{r^2 - 3}, \quad r \text{ rational.}$$

*Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ZUN SHAN and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; C. WILDHAGEN, Tilburg University, Tilburg, The Netherlands; and the proposer. Their solutions, although all correct, varied greatly in appearance with the above and with each other. Two other readers sent in incomplete answers.*

* * *

**1258**. [1987: 180] *Proposed by Ian Witten, University of Calgary, Calgary, Alberta.*

Think of a picture as an $m \times n$ matrix $A$ of real numbers between 0 and 1 inclusive, where $a_{ij}$ represents the brightness of the picture at the point $(i,j)$. To reproduce the picture on a computer we wish to approximate it by an $m \times n$ matrix $B$ of 0's and 1's, such that every "part" of the original picture is "close" to the corresponding part of the reproduction. These are the ideas behind the following definitions.

A *subrectangle* of an $m \times n$ grid is a set of positions of the form

$$\{(i,j) \mid r_1 \le i \le r_2\,,\ s_1 \le j \le s_2\}$$

where $1 \le r_1 \le r_2 \le m$ and $1 \le s_1 \le s_2 \le n$ are constants. For any subrectangle $R$, let

$$d(R) = \left| \sum_{(i,j)\in R} (a_{ij} - b_{ij}) \right|,$$

where $A$ and $B$ are as given above, and define

$$d(A,B) = \max d(R),$$

the maximum taken over all subrectangles $R$.

(a) Show that there exist matrices $A$ such that $d(A,B) > 1$ for every 0–1 matrix $B$ of the same size.

(b)* Is there a constant $c$ such that for every matrix $A$ of any size, there is some 0–1 matrix $B$ of the same size such that $d(A,B) < c$?

I. *Solution to* (*a*) *by C. Wildhagen, Tilburg University, Tilburg, The Netherlands.*

If $R$ is a subrectangle of $[m] \times [n]$, let $A_R$ denote the submatrix of $A$ consisting of those entries of $A$ whose indices belong to $R$, and define

$$w(A_R) = \sum_{(i,j)\in R} a_{ij}.$$

Then we have to show that there exists some $m \times n$ matrix $A$ with entries in the interval $[0,1]$ such that for each $m \times n$ $\{0,1\}$–matrix $B$,

$$|w(A_R) - w(B_R)| > 1 \tag{1}$$

for some subrectangle $R$ of $[m] \times [n]$.

Let $m = 3$ and $n = 11$. Choose a number $\epsilon$ so that

$$1/12 < \epsilon < 1/11.$$

Take for $A$ the $3 \times 11$ matrix with each entry equal to $\epsilon$. Suppose $B$ is some $3 \times 11$ $\{0,1\}$–matrix for which (1) fails for every $R$. Then

(i) *any submatrix $R$ of $B$ of area at most* 11 *contains at most one* 1, for otherwise

$$|w(A_R) - w(B_R)| \ge 2 - 11\epsilon > 2 - 11/11 = 1;$$

(ii) *any submatrix $R$ of $B$ of area at least* 12 *contains at least one* 1, for otherwise

$$|w(A_R) - w(B_R)| \ge 12\epsilon > 1.$$

It follows from (i) that each row of $B$ contains at most one 1. If $B$ has a zero row, then it is easy to see that $B$ must contain either a $2 \times 6$ or a $3 \times 4$ submatrix of 0's, contradicting (ii). Thus $B$ contains exactly three 1's, one in each row. Let them be located in columns $j_1$, $j_2$, $j_3$ from top to bottom. Note that from (i) and (ii),

$$|j_1 - j_2|, |j_2 - j_3| = 5 \text{ or } 6,$$

since a $2 \times 5$ subrectangle must contain at most one 1, and a $2 \times 6$ subrectangle at least one 1. On the other hand, by (i) a $3 \times 3$ rectangle cannot contain two 1's, so $|j_1 - j_3| \geq 3$. Now it is easy to see that without loss of generality we must have $j_1 = 1$, $j_2 = 6$, $j_3 = 11$; but then $B$ has a $3 \times 4$ zero submatrix, contradicting (ii). Thus (1) must hold for some $R$, for each $3 \times 11$ $\{0,1\}$–matrix $B$.

II. *Solution to (a), and comment, by the editor.*

With the choice $\epsilon = 2/23$, the above proof actually shows that the given matrix $A$ has the following stronger property: for every $3 \times 11$ $\{0,1\}$–matrix $B$, there is some subrectangle $R$ such that

$$|w(A_R) - w(B_R)| \geq 24/23.$$

We now give a slightly more complicated argument to raise this bound further to 21/19. This is the best the editor has been able to do, and shows that, if it exists, the constant $c$ referred to in part (b) must be greater than 21/19.

Let $A$ be the $2 \times n$ matrix

$$\begin{bmatrix} 12/19 & 12/19 & \cdots & 12/19 \\ 9/19 & 9/19 & \cdots & 9/19 \end{bmatrix}$$

where $n$ is a sufficiently large positive integer. Suppose that $B$ is a $2 \times n$ $\{0,1\}$–matrix such that (to borrow the notation of the first proof)

$$|w(A_R) - w(B_R)| < 21/19 \tag{2}$$

for all subrectangles $R$. Then it is clear that

(i) no column of $B$ can be zeros;

(ii) the first row of $B$ cannot have three consecutive 1's, for otherwise there is a $1 \times 3$ subrectangle $R$ for which

$$|w(A_R) - w(B_R)| = 3 - 3(12/19) = 21/19;$$

(iii) the first row of $B$ cannot have two consecutive 0's;

(iv) the first row of $B$ cannot contain the submatrix 01010, for otherwise there is a $1 \times 5$ subrectangle $R$ for which

$$|w(A_R) - w(B_R)| = 5(12/19) - 2 > 21/19;$$

(v) the second row of $B$ cannot contain the submatrix 1101 (or 1011), for otherwise there is a $1 \times 4$ subrectangle $R$ for which

$$|w(A_R) - w(B_R)| = 3 - 4(9/19) = 21/19;$$

(vi) the second row of $B$ cannot contain the submatrix 00100, for otherwise there is a $1 \times 5$ subrectangle $R$ for which

$$|w(A_R) - w(B_R)| = 5(9/19) - 1 > 21/19.$$

We further claim that the first row of $B$ cannot contain the submatrix 0110110. Otherwise, by (i) there is a $2 \times 7$ submatrix of $B$ which looks like

$$\begin{matrix} 0&1&1&0&1&1&0 \\ 1&&&1&&&1 \end{matrix}$$

which by (v) must in fact be

$$\begin{matrix} 0&1&1&0&1&1&0 \\ 1&0&0&1&0&0&1 \end{matrix}$$

which is impossible by (vi).

Now, by (ii), (iii), (iv), and this last result, the first row of $B$ must, except for a couple of entries at each end, look like

$$0\,1\,1\,0\,1\,0\,1\,1\,0\,1\,0\,1\,1\,0\,1 \cdots .$$

Since for the submatrix 01101 and its corresponding $1 \times 5$ subrectangle $R$ we have

$$w(A_R) - w(B_R) = 5(12/19) - 3 = 3/19 > 0,$$

if $n$ is large enough there is a long submatrix of the first row for whose corresponding subrectangle $R$ $w(A_R) - w(B_R)$ becomes arbitrarily large, contradicting (2).

Part (b) remains completely open. The editor would be most interested in an answer to this question, or even in an improvement to the bound 21/19. Readers might also like to try to increase the bound 21/19 for $2 \times n$ matrices only, or the bound 24/23 for matrices $A$ with all entries equal.

* * *

<u>1259</u>. [1987: 181] *Proposed by M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

If $x,y,z \geq 0$, disprove the inequality

$$(yz + zx + xy)^2(x + y + z) \geq 9xyz(x^2 + y^2 + z^2).$$

Determine the largest constant one can replace the 9 with to obtain a valid inequality.

*Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Put $y = z = 1$, $x \to \infty$. Then there must hold

$$\frac{(2x + 1)^2(x + 2)}{x(x^2 + 2)} \geq 9$$

as $x \to \infty$. But the left side approaches 4 as $x \to \infty$, which disproves the given inequality and shows in fact that 9 cannot be replaced by any constant larger than 4. We claim that 4 works. Indeed, as the inequality is homogeneous and symmetric we may put $z = 1$, and then we have to show

$$(xy + x + y)^2(x + y + 1) \geq 4xy(x^2 + y^2 + 1). \qquad (1)$$

Multiplying out and collecting terms leads to

$$x^3y^2 + x^2y^3 + x^3 + y^3 + 5x^2y^2 + 5xy^2 + 5x^2y + x^2 + y^2 \geq 2x^3y + 2xy^3 + 2xy,$$

that is,

$$x^3(y-1)^2 + y^3(x-1)^2 + (y-x)^2 + 5xy(xy + x + y) \geq 0.$$

From this obviously true inequality the validity of (1) immediately follows. Furthermore, it also shows that (1) still holds true if we add $5xy(xy + x + y)$ to the right–hand side. For the original inequality this means: *if* $x,y,z \geq 0$ *then*

$$(xy + yz + zx)^2(x + y + z) \geq xyz[4(x^2 + y^2 + z^2) + 5(xy + yz + zx)],$$

with equality if and only if $x = y = z$.

A more general question would be: if $p$, $q$, $r$, $s > 0$ are such that $q \neq s$ and $2p + q = 3r + s$, determine the largest constant $C = C(p,q,r,s)$ such that

$$(xy + yz + zx)^p(x^q + y^q + z^q) \geq Cx^ry^rz^r(x^s + y^s + z^s)$$

holds for all $x$, $y$, $z \geq 0$.

*Also solved (usually by the same method) by HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; KEE–WAI LAU, Hong Kong; GILLIAN NONAY and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; and the proposer. One incomplete and one incorrect solution were also received.*

* * *

We have a little space left over this issue, so here's a "filler" the editor cut out of the Thunder Bay *Chronicle–Journal* around 1978.

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