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CONTENTS

THE OLYMPIAD CORNER: 76

M.S. KLAMKIN

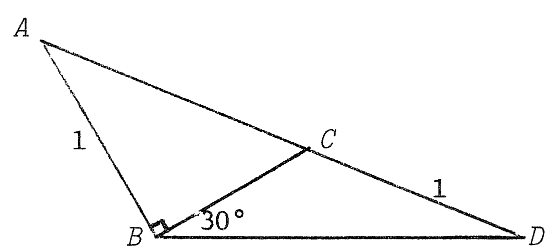
All communications about this column should be sent to M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada, T6G 2G1.

This month I give the problems of the 1986 Canadian Mathematical Olympiad (received through secondary sources) and the 1986 U.S.A. Mathematical Olympiad by courtesy of Walter Mientka. The U.S.A.M.O. was set by J. Konhauser, A. Liu, G. Patruno, and I. Richards (chairman).

EIGHTEENTH CANADIAN MATHEMATICS OLYMPIAD (1986)

Wednesday, May 7, 1986 9:00 a.m. - 12:00 noon

1.



In the diagram AB and CD are of length 1 while angles ABC and CBD are 90° and 30° respectively. Find AC.

2. A Mathlon is a competition in which there are M athletic events. Such a competition was held in which only A, B and C participated. In each event p_1 points were awarded for first place, p_2 for second and p_3 for third where $p_1 > p_2 > p_3 > 0$ and p_1, p_2, p_3 are integers. The final score for A was 22, for B was 9 and for C was also 9. B won the 100 metres. What is the value of M and who was second in the high jump?

3. A chord ST of constant length slides around a semicircle with diameter AB. M is the mid-point of ST and P is the foot of the perpendicular from S to AB. Prove that angle SPM is constant for all positions of ST.

4. For positive integers n and k, define $F(n,k) = \sum_{r=1}^{n} r^{2k-1}$. Prove that $F(n,1)$ divides $F(n,k)$.

5. Let u_1, u_2, $u_3, \ldots$ be a sequence of integers satisfying the recurrence relation $u_{n+2} = u_{n+1}^2 - u_n$. Suppose $u_1 = 39$ and $u_2 = 45$. Prove that 1986 divides infinitely many terms of the sequence.

*

THE FIFTEENTH U.S.A. MATHEMATICAL OLYMPIAD

April 22, 1986 Time: 3 1/2 hours

1. Part a. Do there exist 14 consecutive positive integers each of which is divisible by one or more primes p from the interval $2 \leq p \leq 11$?

Part b. Do there exist 21 consecutive positive integers each of which is divisible by one or more primes p from the interval $2 \leq p \leq 13$?

2. During a certain lecture, each of five mathematicians fell asleep exactly twice. For each pair of these mathematicians, there was some moment when both were sleeping simultaneously. Prove that, at some moment, some three of them were sleeping simultaneously.

3. What is the smallest integer n, greater than one, for which the root-mean-square of the first n positive integers is an integer?

Note: The root-mean-square of n numbers $a_1, \ldots, a_n$ is defined to be $[(a_1^2 + \ldots + a_n^2)/n]^{1/2}$.

4. Two distinct circles K_1 and K_2 are drawn in the plane. They intersect at points A and B, where AB is a diameter of K_1. A point P on K_2 and inside of K_1 is also given.

Using only a "T-square" (i.e. an instrument which can produce the straight line joining two points and the perpendicular to a line through a point on or off the line), find an explicit construction for:

two points C and D on K_1 such that CD is perpendicular to AB and CPD is a right angle.

5. By a *partition* π of an integer $n \geq 1$, we mean a representation of n as a sum of one or more positive integers, where the summands must be put in nondecreasing order. (E.g. if $n = 4$, then the partitions π are $1 + 1 + 1 + 1$, $1 + 1 + 2$, $1 + 3$, $2 + 2$, and 4.)

For any partition π, define $A(\pi)$ to be the number of 1's which appear in π, and define $B(\pi)$ to be the number of distinct integers which appear in π. (E.g. if $n = 13$ and π is the partition $1 + 1 + 2 + 2 + 2 + 5$, then $A(\pi) = 2$ and $B(\pi) = 3$.)

Prove that, for any fixed n, the sum of $A(\pi)$ over all partitions π of n is equal to the sum of $B(\pi)$ over all partitions π of n.

* * *

I now give solutions to some problems from Corner 62 [1985: 36].

1. *Proposed by Australia.*

A total of 1983 cities are served by ten airlines. There is direct service (without stopovers) between any two cities and all airline schedules run both ways. Prove that at least one of the airlines can offer a round trip with an odd number of landings.

Solution.

More generally, suppose there are n airlines $A_1, A_2, \ldots, A_n$ and m cities $C_1, C_2, \ldots, C_m$ with $m > 2^n$. We will show that there is at least one airline A_i having a round trip with an odd number of landings. For $n = 1$, the result is immediate since the one airline must serve at least 3 cities and $C_1C_2C_3C_1$ is a round trip with 3 landings. We now use induction and assume the result is valid for $n - 1$ airlines for $n > 1$. We can assume that all the round trips by A_n consist of an even number of landings, otherwise our proof is done. Now we can separate the cities served by A_n into the two non-empty classes $\{D_1, D_2, \ldots, D_r\}$ and $\{E_1, E_2, \ldots, E_s\}$ where $r + s = m$, so that each flight by A_n flies only between a D-city and an E-city. [For a proof of this, consider any city, say D_1. Call each city linked by A_n to D_1 an E-city, call all the cities linked by A_n to any of these E-cities a D-city, etc. If any cities remain, pick one and call it a D-city. Call each city linked to it by A_n an E-city, etc. Continue until all cities have been considered. No contradiction can arise since all the round trips have an even number of landings.] Since $r + s = m > 2^n$, at least one of r, s, say r, is greater than 2^{n-1}. But the cities $D_1, D_2, \ldots, D_r$ are linked only by the n-1 airlines $A_1, A_2, \ldots, A_{n-1}$, and hence by the inductive hypothesis at least one of them offers a round trip with an odd number of landings. For the original problem, we only have to note that $2^{10} = 1024 < 1983$.

To show that the above general result is sharp, we give a schedule for $m = 2^n$ cities for which there are no round trips with an odd number of landings. Let the cities be F_k where $k = 0, 1, \ldots, 2^n-1$. We now write each such k as an n-digit binary number (possibly starting with a number of zero digits). We link F_i and F_j with A_1 if the first digits of i and j are distinct, with A_2 if the first digits are the same but the second digits are different, ..., with A_n if the first $n - 1$ digits are the same but the last (n^{th}) digits are different. Then for any i, all round trips by A_i have an even number of landings since the i^{th} digit alternates.

2. *Proposed by Australia and the U.S.A. (independently).*

The altitude from a vertex of a given tetrahedron intersects the opposite face in its orthocenter. Prove that all four altitudes of the tetrahedron are concurrent.

Solution.

Let $PABC$ be the tetrahedron and H be the foot of the altitude from P. Since AH and PH are orthogonal to BC, so also is AP. Similarly, $AB \perp PC$ and $AC \perp PB$. A tetrahedron whose opposite sides are orthogonal in pairs is said to be orthocentric. It is a known result (see p.71 of N. Altshiller-Court, *Modern Pure Solid Geometry*, Chelsea, New York, 1964, for a synthetic proof) that the four altitudes of an orthocentric tetrahedron are concurrent.

For practice in vector geometry, give a vector solution of the above problem.

3. *Proposed by Brazil.*

Which of the numbers 1,2,...,1983 have the largest number of positive divisors?

Solution.

The divisors of a number n with prime decomposition

$$p_1^{a_1} p_2^{a_2} \ldots p_m^{a_m}$$

are all the terms of the product expansion of

$$(1 + p_1 + \ldots + p_1^{a_1})(1 + p_2 + \ldots + p_2^{a_2}) \ldots (1 + p_m + \ldots + p_m^{a_m}).$$

Consequently, the number of its positive divisors $N(n)$ is given by $(1 + a_1)(1 + a_2) \ldots (1 + a_m)$. Since this number only depends on the exponents a_i and not the prime factors, it will be maximized for a given m if the primes $p_1,\ldots,p_m$ are chosen to be the first m primes. Thus we need only consider numbers of the form 2^a, 2^a3^b, $2^a3^b5^c$, and $2^a3^b5^c7^d$ since $2\cdot3\cdot5\cdot7\cdot11 > 1983$. Moreover, it is easy to see we may assume $a \geq b \geq c \geq d$. With a little trial and error, we find the maximum $N(n)$ for each of the latter forms are:

$$N(2^{10}) = 11,\ N(2^6\cdot3^3) = 28,\ N(2^3\cdot3^2\cdot5^2) = N(2^5\cdot3^2\cdot5) = 36,$$
$$N(2^4\cdot3\cdot5\cdot7) = 40.$$

Hence $2^4\cdot3\cdot5\cdot7 = 1680$ has the largest number of divisors.

4. *Proposed by Canada.*

Find all possible finite sequences $\{n_0,n_1,\ldots,n_k\}$ of integers such that, for each $i = 0, 1,\ldots, k$, i appears in the sequence n_i times.

Editorial note: Such sequences are called self-descriptive strings and are treated in the following two papers:

[1] M.D. McKay, M.S. Waterman, Self-descriptive strings, *Math. Gazette* 66 (1982) 1-4.

[2] T. Gardiner, Self-descriptive lists - a short investigation, *Math. Gazette* 68 (1984) 5-10.

It is shown in [1] that for $k \geq 6$, a self-descriptive string exists and is unique, and is given by the second row of the table

Number (i)	0	1	2	3	4	...	$k-4$	$k-3$	$k-2$	$k-1$	k
Occurrence (n_i)	$k-3$	2	1	0	0	...	0	1	0	0	0 .

The only self-descriptive strings for $k < 6$ are $\{1,2,1,0\}$, $\{2,0,2,0\}$, and $\{2,1,2,0,0\}$. For a related paper, see L. Sallows and V.L. Eijkhout, Co-descriptive strings, *Math. Gazette* 70 (1986) 1-10.

5. *Proposed by Canada.*

Let $a_0 = 0$ and

$$a_{n+1} = k(a_n+1) + (k+1)a_n + 2\sqrt{k(k+1)a_n(a_n+1)}\ , \quad n = 0,1,2,\ldots,$$

where k is a positive integer. Prove that a_n is a positive integer for $n = 1,2,3,\ldots$.

Solution.

Since $a_0 = 0$, $a_1 = k$. We solve the given recurrence equation for a_n in terms of a_{n+1}:

$$a_{n+1} - (2k+1)a_n - k = 2\sqrt{k(k+1)a_n(a_n+1)}$$
$$a_{n+1}^2 + (2k+1)^2a_n^2 + k^2 - 2(2k+1)a_{n+1}a_n - 2ka_{n+1} + 2k(2k+1)a_n$$
$$= 4k(k+1)(a_n^2+a_n)$$
$$a_n^2 - 2ka_n - 2(2k+1)a_{n+1}a_n + (a_{n+1}-k)^2 = 0$$

so

$$a_n = \frac{2k + 2(2k+1)a_{n+1} \pm \sqrt{4[k+(2k+1)a_{n+1}]^2 - 4(a_{n+1}-k)^2}}{2}$$
$$= k + (2k+1)a_{n+1} \pm \sqrt{(4k^2+4k)a_{n+1}^2 + (4k^2+4k)a_{n+1}}$$
$$= k(a_{n+1}+1) + (k+1)a_{n+1} - 2\sqrt{k(k+1)a_{n+1}(a_{n+1}+1)}$$

since $a_n < a_{n+1}$. We now add this equation to the given equation with n replaced by $n + 1$ to give

$$a_n + a_{n+2} = 2k(a_{n+1} + 1) + 2(k+1)a_{n+1}$$

or

$$a_{n+2} = (4k+2)a_{n+1} - a_n + 2k.$$

It now follows by induction that a_n is a positive integer for $n = 2,3,4,\ldots$.

For extensions of this result see M.S. Klamkin, Perfect squares of the form $(m^2-1)a_n^2 + t$, *Math. Mag.* 42 (1969) 111-113.

<u>6</u>. *Proposed by Cuba.*

Show that there exist infinitely many sets of 1983 consecutive positive integers each of which is divisible by some number of the form a^{1983}, where $a \neq 1$ is a positive integer.

Solution by Curtis Cooper, Central Missouri State University, Warrensburg, Missouri.

The following proof is due to S.M. Akers Jr. (solution III to problem 106, *Math. Mag.* 25 (1952) 222; reprinted in R. Honsberger, *Mathematical Morsels*, M.A.A., 1978, pp.136-137). It shows by induction on n that for any integer $m \geq 1$ there exists a set of n consecutive natural numbers each of which is divisible by a number of the form a^m, where $a \neq 1$ is a positive integer.

(i) For $n = 1$, a^m for any integer $a > 1$ satisfies the requirement.

(ii) Suppose for $n \geq 1$, the n consecutive natural numbers $A_1, A_2, \ldots, A_n$ are each divisible by an m^{th} power > 1. Now we look for $n + 1$ consecutive numbers with the same property. Let

$$a_1^m,\ a_2^m, \ldots,\ a_n^m > 1$$

divide

$$A_1,\ A_2, \ldots,\ A_n,$$

respectively, and let $L = a_1^m a_2^m \ldots a_n^m$. Next let $A_{n+1} = A_n + 1$ and $A = A_{n+1}[(L+1)^m - 1]$. Then

$$A + A_1\ ,\ A + A_2\ , \ldots,\ A + A_{n+1}$$

are $n + 1$ consecutive numbers divisible by

$$a_1^m\ ,\ a_2^m\ , \ldots,\ a_n^m\ ,\ (L+1)^m$$

respectively. Thus the result is true by induction.

7. *Proposed by Finland.*

Let r and s be integers, with $s > 0$. Show that there exists an interval I of length $1/s$ and a polynomial $P(x)$ with integral coefficients such that, for all $x \in I$,

$$|P(x) - r/s| < 1/s^2.$$

Solution.

Let n be a positive integer and $P(x) = r(1 - (sx-1)^{2n})/s$, $\frac{1}{2s} \le x \le \frac{3}{2s}$. Then $|sx-1| \le \frac{1}{2}$. Clearly, $P(x)$ is an integral polynomial and

$$|P(x) - r/s| = \left|\frac{r}{s}(sx-1)^{2n}\right| \le \left|\frac{r}{s}\right| 2^{-2n}.$$

Finally, we can choose n sufficiently large so that $\left|\frac{r}{s}\right| 2^{-2n} < \frac{1}{s^2}$. (Just take $2n > \log_2|rs|$).

* * *

PROBLEMS

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before January 1, 1987, although solutions received after that date will also be considered until the time when a solution is published.

1141. [1986: 106] (Corrected) *Proposed by Hidetosi Fukagawa, Yokosuka High School, Tokai-City, Aichi, Japan.*

Disjoint, non-touching spheres O_1 and O_2 are inside and tangent to a sphere O. Four spheres S_1, S_2, S_3, S_4, each tangent to two of the others as well as to O_1, O_2, and O, are packed in a ring in that order inside O and around O_1 and O_2. Show that

$$\frac{1}{r_1} + \frac{1}{r_3} = \frac{1}{r_2} + \frac{1}{r_4}$$

where r_i is the radius of S_i.

1151*. *Proposed by Jack Garfunkel, Flushing, N.Y.*

Prove (or disprove) that for an obtuse triangle ABC,

$$m_a + m_b + m_c \le s\sqrt{3}$$

where m_a, m_b, m_c denote the medians to sides a, b, c and s denotes the semiperimeter of ΔABC. Equality is attained in the equilateral triangle.

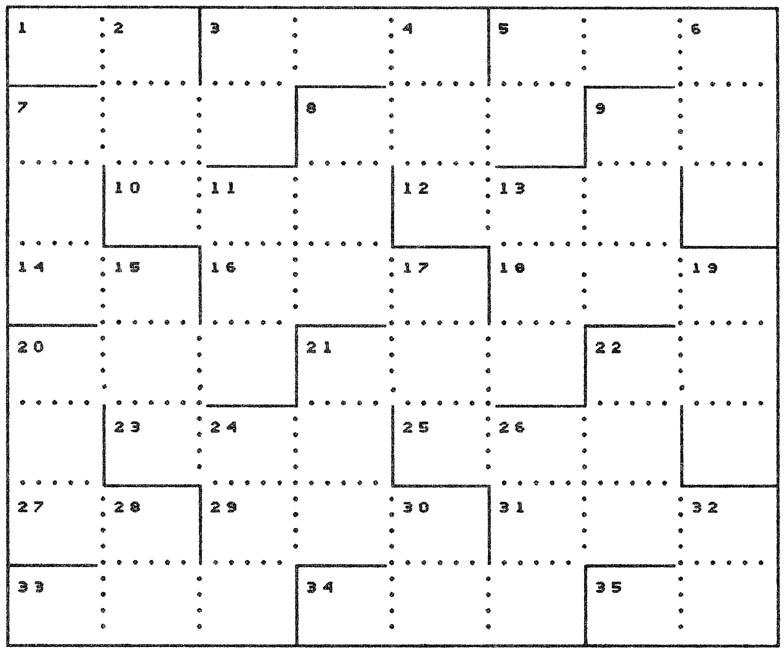
1152. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Prove that

$$\Sigma \cos\frac{\alpha}{2} \le \frac{\sqrt{3}}{2} \Sigma \cos\frac{1}{4}(\beta - \gamma)$$

where α, β, γ are the angles of a triangle and the sums are cyclic over these angles.

<u>1153</u>. *Proposed by Richard K. Guy, University of Calgary, Calgary, Alberta.*



1B	3D	9B	29B	7A	21D	12B	11U	20U
2D	6D	5B	19U	15D	7D	22D	18B	15U
27A	2D	26D	20A	8D	8A	16A	31A	33A
5D	3A	25B	30D	14A	9A	16B	24D	23B
28D	35A	3U	30U	9U	13D	22A	32U	32D
4U	21A	21D	19U	17D	10A	32U	34A	33A

The answers are distinct 2- and 3-digit decimal numbers, none beginning with zero. Each of the above sets of answers is a primitive Pythagorean triple, in increasing size, so that the third member is the hypotenuse. A = across, B = back, D = down, U = up. For example, 1B has its tens & units digits in the squares labelled 2 & 1 respectively; 11U is a 3-digit number with its tens & units digits in squares 16 & 11 respectively.

<u>1154</u>. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let A, B, and C be the angles of an arbitrary triangle. Determine the best lower and upper bounds of the function

$$f(A,B,C) = \Sigma \sin\frac{A}{2} - \Sigma \sin\frac{A}{2}\sin\frac{B}{2}$$

(where the summations are cyclic over A, B, C) and decide whether they are attained.

1155. *Proposed by Roger Izard, Dallas, Texas.*

In triangle ABC cevians AD, BE, and CF meet at point O. Points F, B, C, and E are concyclic. Points A, F, D, and C are also concyclic. Show that AD, BE, and CF are altitudes.

1156. *Proposed by Hidetosi Fukagawa, Yokosuka High School, Aichi, Japan.*

At any point P of an ellipse with semiaxes a and b $(a > b)$, draw a normal line and let Q be the other meeting point. Find the least value of length PQ, in terms of a and b.

1157*. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Find all triples of positive integers (r,s,t), $r \le s, t$, for which $(rs + r + 1)(st + s + 1)(tr + t + 1)$ is divisible by $(rst - 1)^2$. This problem was suggested by Routh's Theorem (see Crux [1981: 199]).

1158. *Proposed by Svetoslav Bilchev, Technical University, Russe, Bulgaria.*

Prove that

$$\sum \frac{1}{(\sqrt{2} + 1)\cos\frac{A}{8} - \sin\frac{A}{8}} \ge \sqrt{6 - 3\sqrt{2}}$$

where the sum is cyclic over the angles A, B, C of a triangle. When does equality occur?

1159. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be a triangle and P some interior point with distances $AP = x_1$, $BP = x_2$, $CP = x_3$. Show that

$$(b + c)x_1 + (c + a)x_2 + (a + b)x_3 \ge 8F ,$$

where a, b, c are the sides of ΔABC and F is its area.

1160. *Proposed by Clark Kimberling, University of Evansville, Evansville, Indiana.*

Let A', B', C' be the first points of intersection of the angle bisectors of a triangle ABC with its incircle Γ. The tangents to Γ at A', B', C' form a triangle $A''B''C''$. Prove that the lines AA'', BB'', CC'' are concurrent.

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SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

<u>897</u>. [1983: 313; 1985: 63; 1985: 123] *Proposed by Vedula N. Murty, Pennsylvania State University, Capitol Campus.*

If $\lambda > \mu$ and $a \geq b \geq c > 0$, prove that

$$b^{2\lambda}c^{2\mu} + c^{2\lambda}a^{2\mu} + a^{2\lambda}b^{2\mu} \geq (bc)^{\lambda+\mu} + (ca)^{\lambda+\mu} + (ab)^{\lambda+\mu}$$

with equality just when $a = b = c$.

Further generalizations by M.S. Klamkin, University of Alberta, Edmonton, Alberta.

I. First we show that if $a_1 \geq a_2 \geq \cdots \geq a_n > 0$ and either $\lambda > \mu > 0$ or $0 > \lambda > \mu$, then

$$a_1^{2\lambda}a_2^{2\mu} + a_2^{2\lambda}a_3^{2\mu} + \cdots + a_n^{2\lambda}a_1^{2\mu} \geq (a_1a_2)^{\lambda+\mu} + (a_2a_3)^{\lambda+\mu} + \cdots + (a_na_1)^{\lambda+\mu} \tag{1}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

From inequality (6) on [1980: 107], we have that if $x_1 \geq x_2 \geq \cdots \geq x_n > 0$ and $m \geq 1$ then

$$x_1^m x_2 + x_2^m x_3 + \cdots + x_n^m x_1 \geq x_1 x_2^m + x_2 x_3^m + \cdots + x_n x_1^m . \tag{2}$$

Suppose $\lambda > \mu > 0$, and put $x_i = a_i^{2\mu}$ and $m = \lambda/\mu$. Then $x_1 \geq x_2 \geq \cdots \geq x_n > 0$ and $m \geq 1$, so (2) becomes

$$\Sigma\, a_1^{2\lambda}a_2^{2\mu} \geq \Sigma\, a_1^{2\mu}a_2^{2\lambda} \tag{3}$$

where the sums here and below are cyclic over $a_1, a_2, \ldots, a_n$. Consequently, by (3) and the A.M.-G.M. inequality,

$$\Sigma\, a_1^{2\lambda}a_2^{2\mu} \geq \sum \frac{a_1^{2\lambda}a_2^{2\mu} + a_1^{2\mu}a_2^{2\lambda}}{2} \geq \Sigma\, (a_1a_2)^{\lambda+\mu} .$$

From the second inequality, equality in (1) implies that $a_1 = a_2 = \cdots = a_n$.

If $0 > \lambda > \mu$ and $a_1 \geq a_2 \geq \cdots \geq a_n > 0$, then $0 < -\lambda < -\mu$ and

$$a_n^{-1} \geq a_{n-1}^{-1} \geq \cdots \geq a_2^{-1} \geq a_1^{-1} > 0 ,$$

so applying (1) we have

$$(a_n^{-1})^{2(-\mu)}(a_{n-1}^{-1})^{2(-\lambda)} + \dots + (a_1^{-1})^{2(-\mu)}(a_n^{-1})^{2(-\lambda)} \geq (a_n^{-1}a_{n-1}^{-1})^{-\mu-\lambda} + \dots + (a_1^{-1}a_n^{-1})^{-\mu-\lambda}$$

or

$$a_n^{2\mu}a_{n-1}^{2\lambda} + \dots + a_1^{2\mu}a_n^{2\lambda} \geq (a_n a_{n-1})^{\mu+\lambda} + \dots + (a_1 a_n)^{\mu+\lambda}$$

which is again (1).

II. Next we prove that if $x_1 \geq x_2 \geq \dots \geq x_n > 0$ and $0 \leq \alpha \leq 1$, then the function

$$F(\alpha,n) = \Sigma\, x_1^{2-\alpha}x_2^{\alpha}$$

is a nonincreasing function of α, that is,

$$F(\alpha_1,n) \geq F(\alpha_2,n) \quad \text{for} \quad 0 \leq \alpha_1 \leq \alpha_2 \leq 1.$$

Note that

$$F(\alpha,n) = x_1^2\left[\frac{x_2}{x_1}\right]^\alpha + x_2^2\left[\frac{x_3}{x_2}\right]^\alpha + \dots + x_n^2\left[\frac{x_1}{x_n}\right]^\alpha ,$$

and thus

$$F'(\alpha,n) = x_1^2\left[\frac{x_2}{x_1}\right]^\alpha \ell n\left[\frac{x_2}{x_1}\right] + x_2^2\left[\frac{x_3}{x_2}\right]^\alpha \ell n\left[\frac{x_3}{x_2}\right] + \dots + x_n^2\left[\frac{x_1}{x_n}\right]^\alpha \ell n\left[\frac{x_1}{x_n}\right]$$

and

$$F''(\alpha,n) = x_1^2\left[\frac{x_2}{x_1}\right]^\alpha \ell n^2\left[\frac{x_2}{x_1}\right] + x_2^2\left[\frac{x_3}{x_2}\right]^\alpha \ell n^2\left[\frac{x_3}{x_2}\right] + \dots + x_n^2\left[\frac{x_1}{x_n}\right]^\alpha \ell n^2\left[\frac{x_1}{x_n}\right] .$$

Since $F''(\alpha,n) \geq 0$, F is convex. Thus we need only show that $F'(1,n) \leq 0$, that is,

$$x_1x_2\ell n\left[\frac{x_2}{x_1}\right] + x_2x_3\ell n\left[\frac{x_3}{x_2}\right] + \dots + x_nx_1\ell n\left[\frac{x_1}{x_n}\right] \leq 0. \tag{4}$$

We do this by induction on n.

When $n = 3$, (4) is equivalent to

$$xy\ell n\left[\frac{y}{x}\right] + yz\ell n\left[\frac{z}{y}\right] + zx\ell n\left[\frac{x}{z}\right] \leq 0 \tag{5}$$

for $x \geq y \geq z > 0$. When $y = z$, the left side of (5) is just

$$xy\ell n\left[\frac{y}{x}\right] + y^2\ell n 1 + yx\ell n\left[\frac{x}{y}\right] = 0,$$

so we may assume $x \geq y > z > 0$. Moreover, since (5) is homogeneous in x, y, z, we can set $z = 1$. Inequality (5) then reduces to

$$xy\ell ny - xy\ell nx - y\ell ny + x\ell nx \leq 0,$$

or

$$\frac{x\ln x}{x-1} \geq \frac{y\ln y}{y-1},$$

and so we wish to prove that the function

$$f(x) = \frac{x\ln x}{x-1}$$

increases for $x > 1$. This can be done analytically. It also follows geometrically by rewriting $f(x)$ as

$$f(x) = \frac{\displaystyle\int_{1/x}^{1} \frac{dt}{t}}{1 - \frac{1}{x}},$$

since the above expression gives the average height under the decreasing curve $y = \frac{1}{t}$ for $\frac{1}{x} \leq t \leq 1$. (A similar method was used in [1980: 75].)

Now, assuming that $F'(1,k) \leq 0$ and noting that $x_1 \geq x_k \geq x_{k+1}$, we have that

$$\begin{aligned} F'(1,k+1) &= F'(1,k) - x_k x_1 \ln\left[\frac{x_1}{x_k}\right] + x_k x_{k+1} \ln\left[\frac{x_{k+1}}{x_k}\right] + x_{k+1} x_1 \ln\left[\frac{x_1}{x_{k+1}}\right] \\ &= F'(1,k) + \left\{x_1 x_k \ln\left[\frac{x_k}{x_1}\right] + x_k x_{k+1} \ln\left[\frac{x_{k+1}}{x_k}\right] + x_{k+1} x_1 \ln\left[\frac{x_1}{x_{k+1}}\right]\right\} \\ &\leq 0 + 0 = 0 \end{aligned}$$

by the induction hypothesis and (5). Thus $F'(1,n) \leq 0$ for all $n \geq 3$, and so $F(\alpha,n)$ is nonincreasing in α. In particular, $F(0,n) \geq F(\alpha,n) \geq F(1,n)$, or

$$\Sigma\, x_1^2 \geq \Sigma\, x_1^{2-\alpha} x_2^{\alpha} \geq \Sigma\, x_1 x_2 \tag{6}$$

for $0 \leq \alpha \leq 1$.

Letting $a_1 \geq a_2 \geq \ldots \geq a_n > 0$ and $\lambda > \mu > 0$, we put

$$x_i = a_i^{\lambda+\mu}$$

and

$$\alpha = \frac{2\mu}{\lambda + \mu};$$

then $x_1 \geq x_2 \geq \ldots \geq x_n > 0$ and $0 < \alpha < 1$, and the second inequality of (6) reduces to (1).

If instead we assume $a_1 \geq a_2 \geq \ldots \geq a_n > 0$ and $0 > \lambda > \mu$, then we put

$$x_i = a_{n+1-i}^{\lambda+\mu}$$

and

$$\alpha = \frac{2\lambda}{\lambda + \mu};$$

then again $x_1 \geq x_2 \geq \cdots \geq x_n > 0$ and $0 < \alpha < 1$, and (6) becomes

$$\sum a_n^{(\lambda+\mu)(2 - \frac{2\lambda}{\lambda+\mu})} a_{n-1}^{(\lambda+\mu)\cdot\frac{2\lambda}{\lambda+\mu}} \geq \Sigma\, a_n^{\lambda+\mu} a_{n-1}^{\lambda+\mu}$$

or

$$\Sigma\, a_n^{2\mu} a_{n-1}^{2\lambda} \geq \Sigma(a_n a_{n-1})^{\lambda+\mu} ,$$

which is (1) in this case as well.

III. For our final generalization we shall prove that if $x \geq y \geq z > 0$ and $t \geq 0$, then

$$x\left[\frac{y}{z}\right]^t + y\left[\frac{z}{x}\right]^t + z\left[\frac{x}{y}\right]^t \geq x + y + z. \tag{7}$$

Letting

$$H(t) = x\left[\frac{y}{z}\right]^t + y\left[\frac{z}{x}\right]^t + z\left[\frac{x}{y}\right]^t - x - y - z ,$$

we have $H(0) = 0$, and so it is enough to prove that $H(t)$ is nondecreasing for $t \geq 0$. Then

$$H'(t) = x\left[\frac{y}{z}\right]^t \ell n\left[\frac{y}{z}\right] + y\left[\frac{z}{x}\right]^t \ell n\left[\frac{z}{x}\right] + z\left[\frac{x}{y}\right]^t \ell n\left[\frac{x}{y}\right]$$

and

$$H''(t) = x\left[\frac{y}{z}\right]^t \ell n^2\left[\frac{y}{z}\right] + y\left[\frac{z}{x}\right]^t \ell n^2\left[\frac{z}{x}\right] + z\left[\frac{x}{y}\right]^t \ell n^2\left[\frac{x}{y}\right] ,$$

and since $H''(t) \geq 0$, H is convex. Thus it suffices to show that $H'(0) \geq 0$, that is,

$$x\ell n\left[\frac{y}{z}\right] + y\ell n\left[\frac{z}{x}\right] + z\ell n\left[\frac{x}{y}\right] \geq 0.$$

As before, we can assume $x \geq y > z = 1$, and so we must prove

$$x\ell ny - y\ell nx + \ell nx - \ell ny \geq 0$$

or

$$\frac{\ell nx}{x - 1} \leq \frac{\ell ny}{y - 1}$$

for $x \geq y > 1$. This can be rewritten as

$$\frac{\int_1^x \frac{1}{t}\,dt}{x - 1} \leq \frac{\int_1^y \frac{1}{t}\,dt}{y - 1}$$

and so follows by another average height argument.

Now assume $a \geq b \geq c > 0$ and $\lambda \geq \mu$, $\lambda + \mu < 0$. Put

$$x = a^{-(\lambda+\mu)} , \quad y = b^{-(\lambda+\mu)} , \quad z = c^{-(\lambda+\mu)}$$

and

$$t = 1 - \frac{2\lambda}{\lambda + \mu} .$$

Then $x \geq y \geq z > 0$ and $t \geq 0$, so (7) implies that

$$a^{-(\lambda+\mu)}\left[\frac{b}{c}\right]^{\lambda-\mu} + b^{-(\lambda+\mu)}\left[\frac{c}{a}\right]^{\lambda-\mu} + c^{-(\lambda+\mu)}\left[\frac{a}{b}\right]^{\lambda-\mu} \geq a^{-(\lambda+\mu)} + b^{-(\lambda+\mu)} + c^{-(\lambda+\mu)}.$$

Multiplying by $(abc)^{\lambda+\mu}$, we obtain Murty's original inequality. If instead we assume $a \geq b \geq c > 0$ and $\lambda \geq \mu$, $\lambda + \mu > 0$, then we put

$$x = c^{-(\lambda+\mu)}, \quad y = b^{-(\lambda+\mu)}, \quad z = a^{-(\lambda+\mu)}$$

and

$$t = 1 - \frac{2\mu}{\lambda + \mu}.$$

Again, $x \geq y \geq z > 0$ and $t \geq 0$, and (7) implies Murty's inequality.

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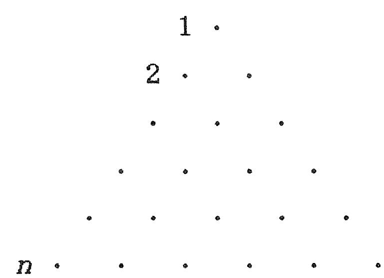
976*. [1984: 262] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

(a) For all possible sets of n distinct points in a plane, let $T(n)$ be the maximum number of equilateral triangles having their vertices among the n points. Evaluate $T(n)$ explicitly in terms of n, or (at least) find a good upper bound for $T(n)$.

(b) If $a_n = T(n)/n$, prove or disprove that the sequence $\{a_n\}$ is monotonically increasing.

(c) Prove or disprove that $\lim_{n\to\infty} a_n = \infty$.

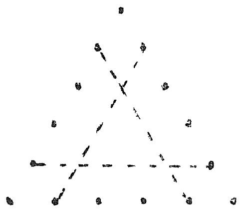
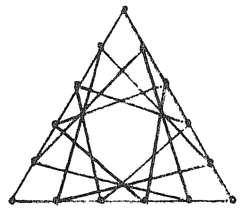
I. *Partial solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*



(a) We limit ourselves to the triangular lattice formed with equilateral triangles, with n rows of points, the i$^{\text{th}}$ row having i points, as shown. Let t_n be the number of all possible equilateral triangles with vertices among these $\binom{n+1}{2}$ lattice points. Then:

$$t_1 = 0, \quad t_2 = 1, \quad t_3 = 5.$$

Furthermore, we have the following recursion formula for t_n:

$\downarrow$ (inclusion-exclusion) $\qquad\qquad \downarrow$

$$t_n = (3t_{n-1} - 3t_{n-2} + t_{n-3}) \qquad + \qquad (n-1)\ , \quad \text{where } n \geq 4.$$

This is equivalent to

$$(t_n - 2t_{n-1} + t_{n-2}) - (t_{n-1} - 2t_{n-2} + t_{n-3}) = n - 1 \tag{1}$$

or

$$b_n - b_{n-1} = n - 1 \tag{2}$$

where $b_n = t_n - 2t_{n-1} + t_{n-2}$. Since $b_3 = t_3 - 2t_2 + t_1 = 3$, (2) implies that

$$b_n = (n-1) + \ldots + 4 + 3 + 3 = \left[{n \atop 2}\right]\ ,$$

and so

$$t_n - 2t_{n-1} + t_{n-2} = \left[{n \atop 2}\right]\ .$$

Now this can be written as

$$c_n - c_{n-1} = \left[{n \atop 2}\right]$$

where $c_n = t_n - t_{n-1}$. Also $c_3 = t_3 - t_2 = 4 = \left[{3 \atop 2}\right] + \left[{2 \atop 2}\right]$, so

$$c_n = \sum_{i=2}^{n} \left[{i \atop 2}\right] = \left[{n+1 \atop 3}\right]\ .$$

Finally, we have

$$t_n - t_{n-1} = \left[{n+1 \atop 3}\right]$$

and $t_3 = 5 = \left[{4 \atop 3}\right] + \left[{3 \atop 3}\right]$, so

$$t_n = \sum_{i=3}^{n+1} \left[{i \atop 3}\right] = \left[{n+2 \atop 4}\right]\ .$$

All the above were for $n \geq 4$, but we see that this final formula holds for $n = 1,2,3$ as well. Thus, if $k = \left[{n+1 \atop 2}\right]$,

$$\frac{T(k)}{k^2} \geq \frac{\left[{n+2 \atop 4}\right]}{\left[{n+1 \atop 2}\right]^2} \sim \frac{1}{6}$$

so $T(k)$ is of order at least $k^2/6$. (In particular, part (c) is true.)

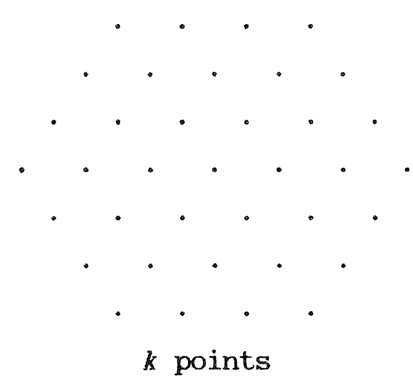
II. *Partial solution by Jordan B. Tabov, Sofia, Bulgaria.*

[Editor's note: Tabov also found the formula $t_n = \binom{n+2}{4}$ above, and thus that $\underline{\lim}\ \dfrac{T(n)}{n^2} \geq \dfrac{1}{6}$.]

We shall now prove that

$$\overline{\lim}\ \frac{T(n)}{n^2} \leq \frac{1}{3} .$$

Given any n points, there are $\binom{n}{2}$ segments connecting these points in pairs. Each of these segments may be a side of 2 equilateral triangles. Therefore $T(n)$ cannot exceed $\frac{2}{3}\binom{n}{2} = \frac{1}{3}n(n-1)$.



A better lower bound for $\underline{\lim}\ \dfrac{T(n)}{n^2}$ may be obtained by considering a *hexagonal* geoboard H formed by equilateral triangles as shown in the figure, with k lattice points on each side (including the endpoints). The total number of lattice points in H is $3k^2 + p(k)$, where $p(k)$ is linear.

Let $h(k)$ be the number of equilateral triangles with vertices among the points of H. The number of such triangles with 2 or 3 vertices on the outer ring of H is a polynomial of degree 2 in k, and the number with no vertices on the outer ring is of course $h(k-1)$. The number of equilateral triangles with exactly one vertex on the outer ring works out to be $7k^3$ + a polynomial of degree 2. Thus

$$h(k) = h(k-1) + 7k^3 + q(k),$$

where $q(k)$ is a polynomial of degree 2, and so

$$h(k) = 7\sum_{i=1}^{k} i^3 \quad + \text{ a polynomial of degree 3}$$

$$= \frac{7}{4}k^4 + \text{ a polynomial of degree 3.}$$

Hence

$$\underline{\lim}\ \frac{T(n)}{n^2} \geq \lim \frac{\frac{7}{4}k^4}{(3k^2)^2} = \frac{7}{36} .$$

This result and the result obtained from triangular geoboards lead me to formulate the following

Conjecture: $\lim \dfrac{T(n)}{n^2} = \dfrac{1}{5}$.

(since $\dfrac{1}{5} = \dfrac{1}{6} + \dfrac{1}{6^2} + \dfrac{1}{6^3} + \ldots$).

III. *Comment by the Editor*.

This problem is by no means answered and quite a few questions remain.

(1) Does $\lim \dfrac{T(n)}{n^2}$ exist? If so, what is it?

(2) What configuration of n points yields the maximum number of equilateral triangles?

(3) What is the answer to part (b) of the original problem?

(4) Two solvers independently found, using different proofs, that the triangular grid with n points on each side has exactly $\binom{n+2}{4}$ equilateral triangles. Is there a natural correspondence between such triangles and the four-element subsets of an $(n + 2)$-element set?

Part (c) *was also answered by FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio.*

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1017. [1985: 51] *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

If the figure on the left is a pandiagonal magic square, then so is the figure on the right.

A	B	C	D
E	F	G	H
I	J	K	L
M	N	O	P

A	B	N	M
E	F	J	I
H	G	K	L
D	C	O	P

Both figures are arrangements of the same 16 arbitrary numbers $A,B,C,\ldots,P$, and both have A in the upper left corner cell. Enumerate all the ways the arbitrary $A,B,C,\ldots,P$ can be arranged to form pandiagonal magic squares in which A is fixed as shown.

Solution by the proposer.

From *Crux* 605 [1982: 22] it follows that every pandiagonal fourth-order magic square is composed of 8 disjoint pairs of numbers, each of which sums to half the magic sum, and that these 8 pairs can be written and situated as follows:

$f + s$	$f - y$	$f + t$	$f - x$
$f - z$	$f + u$	$f + w$	$f + v$
$f - t$	$f + x$	$f - s$	$f + y$
$f - w$	$f - v$	$f + z$	$f - u$

(*)

(Here, subtracting f from each entry results in the same configuration as subtracting $\frac{1}{2}(F + G)$ from each entry of the array in *Crux* 605.) Thus $4f$ will be the magic sum, and

$$s + t - x - y = 0 \tag{1}$$

$$u + v + w - z = 0 \tag{2}$$

$$s - t - w - z = 0 \tag{3}$$

$$u - v + x - y = 0 \,. \tag{4}$$

By (1) and (3),

$$s = \frac{1}{2}(w + x + y + z)$$

which, with (1), gives

$$t = -\frac{1}{2}(w - x - y + z).$$

By (2) and (4),

$$u = -\frac{1}{2}(w + x - y - z)$$

$$v = -\frac{1}{2}(w - x + y - z) \,.$$

These last four equations show that every cell in (*) can be expressed in terms of f, w, x, y, z.

In particular, the cell in the upper left corner equals

$$f + \frac{1}{2}(w + x + y + z)$$

whose value does not change if w, x, y, z are permuted in any of the 24 possible ways. Moreover we find that

$$f + t = f + s - (w + z)$$

$$f - t = f + s - (x + y)$$

$$f + u = f + s - (w + x)$$

$$f - u = f + s - (y + z)$$

$$f + v = f + s - (w + y)$$

$$f - v = f + s - (x + z)$$

and thus permuting w, x, y, z will not change the values of the entries in (*), only their position. Note that $(f + t) + (f - t)$, for example, is unchanged by such a permutation, so the resulting array will still be in the form (*). Therefore there are at least 24 ways to arrange the 16 arbitrary $A,B,C,\ldots,P$ into a pandiagonal fourth-order magic square with A in the upper left corner.

There are in fact exactly 24 such magic squares. This follows from Frénicle's 1693 enumeration of all (essentially different) fourth-order magic squares on the integers $1,2,3,\ldots,16$. He counted only those squares with the properties that

(i) the smallest of the four corner cells occurs in the upper left corner, and

(ii) of the two cells rookwise adjacent to the upper left corner cell, the cell in the top row is the smaller.

These two properties ensure that rotations and reflections are not counted as different. The resulting 880 magic squares are displayed and categorized on pp.188-198 of W.H. Benson and O. Jacoby, *New Recreations with Magic Squares*, Dover, 1976. Of these, 48 turn out to be pandiagonal (type I in Benson and Jacoby's list), with 12 having 1 in the upper left corner. Since we do not wish property (ii) above, we double this figure, arriving at the same 24 magic squares (and only those) found earlier.

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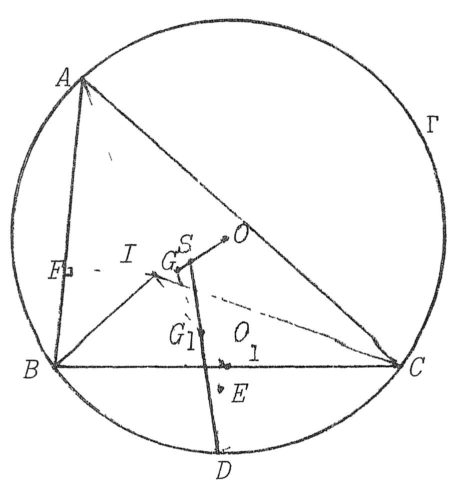
<u>1018</u>. [1985: 51] *Proposed by Kurt Schiffler, Schorndorf, Federal Republic of Germany.*

Let ABC be a triangle with incentre I. Prove that the Euler lines of triangles IBC, ICA, IAB, and ABC are all concurrent.

Solution by G.R. Veldkamp, de Bilt, The Netherlands, and W.A. van der Spek, Leeuwarden, The Netherlands.



Let Γ be the circumcircle of ΔABC, with center O and radius R, and let r be the radius of the incircle of ΔABC. Let G be the median point, so that GO is the Euler line of ΔABC. Let D be the intersection of AI with Γ, that is, the midpoint of arc BC. Then it is well-known (e.g. Theorem 292, page 185 of R.A. Johnson's *Advanced Euclidean Geometry*) that D is the center of a circle passing through B, I, and C. This means that D is the circumcenter of

ΔBIC, so that

$$CD = ID. \tag{1}$$

Let G_1 be the median point of ΔBIC, so that DG_1 is the Euler line of this triangle.

Let O_1 be the midpoint of BC. Then since G and G_1 are median points,

$$\frac{\overline{AG}}{\overline{GO_1}} = \frac{\overline{IG_1}}{\overline{G_1O_1}} = 2.$$

Thus $GG_1 \| AID$, and hence, letting E be the intersection of GG_1 and OD,

$$\frac{\overline{GG_1}}{\overline{G_1E}} = \frac{\overline{AI}}{\overline{ID}} \tag{2}$$

and

$$\overline{DE} = \frac{2}{3}\,\overline{DO_1}\ . \tag{3}$$

It also follows that the Euler line DG_1 of ΔBIC will intersect the Euler line GO of ΔABC between G and O. We let S be the point of intersection.

Let F be the foot of the perpendicular from I to AB. Since $\angle BAD = \angle BCD$, $\Delta AFI \sim \Delta CO_1D$, and hence

$$\frac{\overline{IA}}{\overline{CD}} = \frac{\overline{FI}}{\overline{O_1D}} = \frac{r}{\overline{O_1D}}\ . \tag{4}$$

Applying the theorem of Menelaus to the triangle GOE with the transversal SG_1D, we have

$$\begin{aligned}
1 &= \frac{\overline{GS}}{\overline{OS}}\cdot\frac{\overline{OD}}{\overline{ED}}\cdot\frac{\overline{EG_1}}{\overline{GG_1}} \\
&= \frac{\overline{GS}}{\overline{OS}}\cdot\frac{\overline{OD}}{\overline{ED}}\cdot\frac{\overline{ID}}{\overline{AI}} && \text{by (2)} \\
&= \frac{\overline{GS}}{\overline{OS}}\cdot\frac{\overline{OD}}{\overline{ED}}\cdot\frac{\overline{CD}}{\overline{AI}} && \text{by (1)} \\
&= \frac{\overline{GS}}{\overline{OS}}\cdot\frac{R}{\frac{2}{3}\cdot\overline{DO_1}}\cdot\frac{\overline{CD}}{\overline{AI}} && \text{by (3)} \\
&= \frac{\overline{GS}}{\overline{OS}}\cdot\frac{3R}{2\overline{DO_1}}\cdot\frac{\overline{DO_1}}{r} && \text{by (4)} \\
&= \frac{\overline{GS}}{\overline{OS}}\cdot\frac{3R}{2r}\ ,
\end{aligned}$$

that is,

$$\frac{\overline{GS}}{\overline{OS}} = \frac{2r}{3R}\ .$$

Now if we consider the Euler lines of ΔAIB or ΔCIA rather than ΔBIC, we will arrive at the same ratio; thus the Euler lines of ΔAIB, ΔCIA, and ΔBIC all intersect the Euler line of ΔABC in the same point S (we call this point the *Schiffler point* of ΔABC).

Also solved by D.J. SMEENK, Zaltbommel, The Netherlands.

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1019. [1985: 51] *Proposed by Weixuan Li and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Determine the largest constant k such that the inequality

$$x \le \alpha \sin x + (1 - \alpha)\tan x$$

holds for all $\alpha \le k$ and for all $x \in [0,\pi/2)$.

(The inequality obtained when α is replaced by 2/3 is the Snell-Huygens inequality, which is fully discussed in Problem 115 [1976: 98-99, 111-113, 137-138].)

Solution by Richard I. Hess, Rancho Palos Verdes, California.

Let

$$f(x) = \alpha \sin x + (1 - \alpha)\tan x - x$$

and suppose $\alpha = \frac{2}{3} + \epsilon$. Then for $x << 1$,

$$\sin x = x - \frac{x^3}{6} + O(x^5)$$

$$\tan x = x + \frac{x^3}{3} + O(x^5)$$

and so

$$\begin{aligned} f(x) &= (\frac{2}{3} + \epsilon)(x - \frac{x^3}{6} + O(x^5)) + (\frac{1}{3} - \epsilon)(x + \frac{x^3}{3} + O(x^5)) - x \\ &= - \frac{\epsilon x^3}{6} - \frac{\epsilon x^3}{3} + O(x^5) \\ &= - \frac{\epsilon x^3}{2} + O(x^5). \end{aligned}$$

For suitably small x this must be negative for $\epsilon > 0$, so the given inequality won't hold unless $\epsilon \le 0$. Thus the largest value of k is $\frac{2}{3}$.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; M.S. KLAMKIN, University of Alberta, Edmonton, Alberta; EDWIN M. KLEIN, University of Wisconsin, Whitewater, Wisconsin; KEE-WAI LAU, Hong Kong; JORDAN B. TABOV, Sofia, Bulgaria; A. TAMANAS, Thornton Heath, Surrey, England; and the proposers.

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1020. [1985: 51] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Solve, for $x \in [0,2\pi)$, the equation

$$176\cos x + 64\sin x = 75\cos 2x + 80\sin 2x + 101.$$

Solution by Edwin M. Klein, University of Wisconsin, Whitewater, Wisconsin.

Clearly $x \neq \pi$, so using the substitution $t = \tan(x/2)$ we have

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$$

and thus

$$\sin 2x = 2\sin x \cos x = \frac{4t(1-t^2)}{(1+t^2)^2}$$

$$\cos 2x = 1 - 2\sin^2 x = \frac{1-6t^2+t^4}{(1+t^2)^2}.$$

The given equation then simplifies to

$$176(1-t^4) + 128t(1+t^2) = 75(1-6t^2+t^4) + 320t(1-t^2) + 101(1+t^2)^2$$

or

$$352t^4 - 448t^3 - 248t^2 + 192t = 0$$

or

$$8t(2t-1)(2t-3)(11t+8) = 0.$$

Hence, $\tan(x/2) = 0,\ 1/2,\ 3/2,\ -8/11$ and so the solutions are

$$x = 2\arctan 0 = 0$$
$$x = 2\arctan 1/2 \approx 53.13^\circ$$
$$x = 2\arctan 3/2 \approx 112.62^\circ$$
$$x = 2\pi - 2\arctan 8/11 \approx 287.95^\circ.$$

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; SAM BAETHGE, San Antonio, Texas; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; KEE-WAI LAU, Hong Kong; J.A. McCALLUM, Medicine Hat, Alberta; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer. There were two partial solutions received.

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1021. [1985: 82] *Proposed by Allan Wm. Johnson, Washington, D.C.*

In the etymological decimal addition

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SERGE
  ^DE
NIMES
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DENIM
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maximize $\hat{\text{N}}$IMES (where $\hat{\text{I}} = \text{I}$), the city in southern France that gave its name to denim cloth.

Solution.

$$\begin{array}{r} 16256 \\ 86 \\ \underline{70361} \\ 86703 \end{array} .$$

Found by J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; J.A. McCALLUM, Medicine Hat, Alberta; GLEN E. MILLS, Valencia Community College, Orlando, Florida; and the proposer. There was one incorrect solution submitted. The proposer notes that the word jean, *as in jeans, comes from the word Genoa (Gênes in French) where this fabric was produced and exported. Leroy F. Meyers is reminded of the homophonous verse*

Gal, amant de la reine, alla, tour magnanime,
Galamment de l'Arène à la Tour Magne, à Nîmes.

* * *

<u>1023</u>*. [1985: 82] *From a Trinity College, Cambridge, examination paper dated June 7, 1901.*

Show that, for $n = 1,2,3,\ldots,$

$$\sum_{k=1}^{n} \text{Arctan}\,\frac{2}{k^2} = \frac{3\pi}{4} - \text{Arctan}\,\frac{1}{n} - \text{Arctan}\,\frac{1}{n+1}\,.$$

Solution by M. Parmenter, Memorial University of Newfoundland, St. John's, Newfoundland.

We will prove this by induction.

When $n = 1$, the equation to be proved reads

$$\text{Arctan}2 = \frac{3\pi}{4} - \frac{\pi}{4} - \text{Arctan}\,\frac{1}{2}\,.$$

This is correct since $\text{Arctan}2 + \text{Arctan}\,\frac{1}{2} = \frac{\pi}{2}$.

Assume the equation for $n = s$. Then when $n = s + 1$, the left side of the above equation is

$$\sum_{k=1}^{s+1} \text{Arctan}\,\frac{2}{k^2} = \sum_{k=1}^{s} \text{Arctan}\,\frac{2}{k^2} + \text{Arctan}\,\frac{2}{(s+1)^2}$$
$$= \frac{3\pi}{4} - \text{Arctan}\,\frac{1}{s} - \text{Arctan}\,\frac{1}{s+1} + \text{Arctan}\,\frac{2}{(s+1)^2}\,.$$

We wish to prove this is equal to

$$\frac{3\pi}{4} - \text{Arctan}\,\frac{1}{s+1} - \text{Arctan}\,\frac{1}{s+2}\,,$$

that is, we have to prove

$$\text{Arctan}\,\frac{2}{(s+1)^2} = \text{Arctan}\,\frac{1}{s} - \text{Arctan}\,\frac{1}{s+2}\,.$$

But these are all first quadrant angles and

$$\tan\left[\text{Arctan}\,\frac{1}{s} - \text{Arctan}\,\frac{1}{s+2}\right] = \frac{\frac{1}{s} - \frac{1}{s+2}}{1 + \frac{1}{s(s+2)}} = \frac{2}{s(s+2)+1} = \frac{2}{(s+1)^2}\,,$$

as required.

Also solved by MICHAEL W. ECKER, University of Scranton, Scranton, Pennsylvania; JACK GARFUNKEL, Flushing, N.Y.; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta, Edmonton, Alberta; VEDULA N. MURTY, Penn State University, Capitol Campus; BOB PRIELIPP, University of Wisconsin, Oshkosh, Wisconsin; NATARAJAN SIVAKUMAR, student, University of Alberta, Edmonton, Alberta; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; and WONG NGAI YING, Hong Kong. For more involved Arctangent summations, see M.L. Glasser and M.S. Klamkin, "On some inverse tangent summations", Fibonacci Quart. *14 (1976) 385-388.*

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1024. [1985: 82] *Proposed by William Tunstall Pedoe, student, The High School of Dundee, Scotland.*

Prove that an odd number which is a perfect square cannot be perfect.

I. *Solution by Frank P. Battles and Laura L. Kelleher, Massachusetts Maritime Academy, Buzzards Bay, Massachusetts.*

If a number is perfect the sum of its divisors is twice the number itself and hence this sum is even. Consider a number which is an odd perfect square. Since it is odd, all of its divisors are odd. Since it is a perfect square, it has an odd number of divisors. It follows that the sum of its divisors is odd, and hence this number is not a perfect number.

II. *Solution by Edwin M. Klein, University of Wisconsin, Whitewater, Wisconsin.*

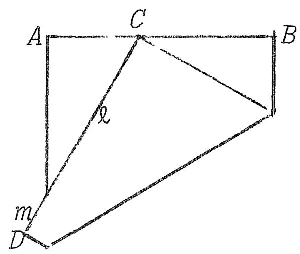
More generally, if n is a perfect square or twice a perfect square, then n is not perfect, because $\sigma(n)$ is odd (cf. David M. Burton, *Elementary Number Theory*, p.118 ex. 7(b) and p.224 ex. 2(b)).

Also solved by SAM BAETHGE, San Antonio, Texas; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; M.S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; LEROY F. MEYERS, The Ohio State University, Columbus, Ohio; BOB PRIELIPP, University of Wisconsin, Oshkosh, Wisconsin; DAN SOKOLOWSKY, Brooklyn, N.Y.; STAN WAGON, Smith College, Northampton, Massachusetts; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; KENNETH M. WILKE, Topeka, Kansas; and the proposer. Prielipp and Wagon used the known result that any odd perfect number is of the form $p^{\alpha}m^2$ *where p is a prime and p and* α *are both congruent to 1 mod 4 (e.g. see p.128 of Beck, Bleicher, and Crowe,* Excursions into Mathematics). *Meyers suspects that no perfect* k^{th} *power* ($k > 1$) *is a perfect number; any comments from out there?*

* * *

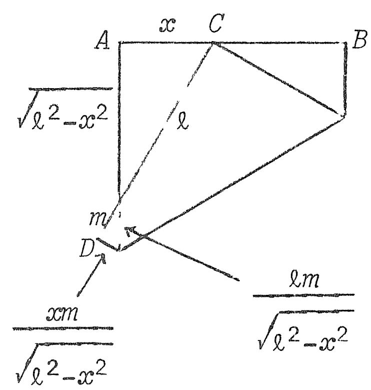
<u>1025</u>. [1985: 83] *Proposed by Peter Messer, M.D., Mequon, Wisconsin.*

A paper square *ABCD* is folded so that vertex *C* falls on *AB* and side *CD* is divided into two segments of lengths ℓ and m, as shown in the figure. Find the minimum value of the ratio ℓ/m.



I. *Solution by Sam Baethge, San Antonio, Texas.*

Without loss of generality let $AB = 1$ and label the figure as shown by using the similar right triangles.



Then

$$\frac{xm}{\sqrt{\ell^2 - x^2}} + \frac{\ell m}{\sqrt{\ell^2 - x^2}} + \sqrt{\ell^2 - x^2} = 1.$$

Since $m + \ell = 1$, ℓ/m is minimized when ℓ is minimized. Thus we substitute for m and solve for ℓ, yielding

$$(x + \ell)(1 - \ell) + (\ell^2 - x^2) = \sqrt{\ell^2 - x^2}$$

$$(\ell + x)(1 - \ell + \ell - x) = \sqrt{\ell^2 - x^2}$$

$$(\ell + x)^2(1 - x)^2 = \ell^2 - x^2$$

$$(\ell + x)(1 - x)^2 = \ell - x$$

$$\ell[(1 - x)^2 - 1] = -x - x(1 - x)^2$$

$$\ell = \frac{x[1 + (1 - x)^2]}{2x - x^2} = \frac{x^2 - 2x + 2}{2 - x} = -x + \frac{2}{2 - x}.$$

Then

$$\frac{d\ell}{dx} = -1 + \frac{2}{(x - 2)^2}.$$

For $\frac{d\ell}{dx} = 0$, $(x - 2)^2 = 2$ or $x = 2 - \sqrt{2}$. Then $\ell = 2\sqrt{2} - 2$ and $m = 3 - 2\sqrt{2}$, and so

$$\ell/m = \frac{2\sqrt{2} - 2}{3 - 2\sqrt{2}} = (2\sqrt{2} - 2)(3 + 2\sqrt{2}) = 2\sqrt{2} + 2.$$

Note that the minimum occurs when the acute angles at C are each 45°.

II. *Solution by Dan Sokolowsky, Brooklyn, N.Y.*

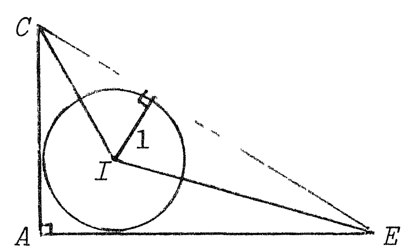
Let E denote the point dividing CD into segments of length ℓ and m. By problem 995 of this journal ([1984: 319], solution [1986: 58]) the inradius of ΔCAE has length $m = DE$, so

$$\frac{\ell}{m} = \frac{\text{hypotenuse of } \Delta CAE}{\text{inradius of } \Delta CAE}.$$

This ratio is the same for similar triangles. We therefore lose no generality in confining ourselves to the set $\mathcal{S}_1$ of all right triangles which have common inradius 1, in which case the stated problem is equivalent to the following:

[1] Minimize the hypotenuse h over all triangles in $\mathcal{S}_1$.

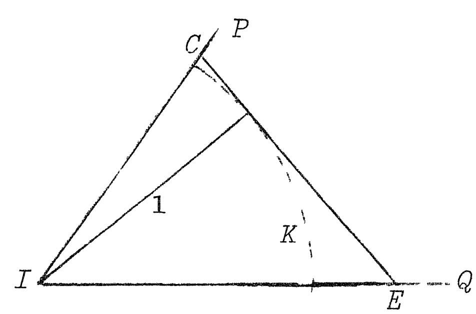
Let $\Delta = \Delta CAE \in \mathcal{S}_1$ and let I denote its incenter. Then



$$\begin{aligned} \angle CIE &= 180^\circ - (\angle ICE + \angle IEC) \\ &= 180^\circ - \tfrac{1}{2}(\angle ACE + \angle AEC) \\ &= 180^\circ - 45^\circ \\ &= 135^\circ, \end{aligned}$$

and thus [1] is equivalent to a special case of

[2] Given $\angle PIQ$ and a circular arc K from PI to QI with radius 1 and center I, let CE be a tangent to K with C on IP and E on IQ. Determine the minimum length of CE.



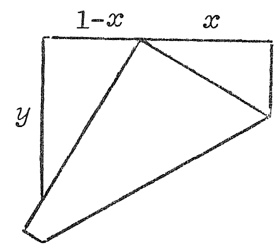
It is well known that CE is minimum when it meets IP and IQ at equal angles $\alpha = (180^\circ - \angle PIQ)/2$, and thus

$$\min CE = 2\cot\alpha.$$

In the special case of our problem, $\alpha = \frac{1}{2}(180^\circ - 135^\circ) = 22\frac{1}{2}^\circ$. Hence

$$\min \ell/m = 2\cot 22\tfrac{1}{2}^\circ = 2(\sqrt{2} + 1).$$

III. *Comment by the proposer.*



Most of my mathematics revolves around my ten-year hobby of origami. A consequence of the above corner-to-edge folding of a square is a method of rapid folding of a square into thirds, fifths, sevenths, and beyond. Labeling the figure as shown, we can obtain

$$y = \frac{2x}{1+x},$$

and therefore, for example:

- if $x = \frac{1}{2}$ (easily found), then $y = \frac{2}{3}$, so y folded in half yields one edge folded into thirds.
- if $x = \frac{1}{4}$ (easily found), then $y = \frac{2}{5}$, which produces an edge folded into fifths.
- if $x = \frac{3}{4}$ then $y = \frac{6}{7}$, which produces an edge folded into sevenths.

This method was discovered by Koji Fushimi and was reported in the *British Origami Magazine* 95 (August 1982) p.20, with the comment that it was the easiest and best method known for achieving thirds and fifths. For another construction, see "Mathematics of origami" by Jacques Justin, in 115 (December 1985) pp.18-20 of the same magazine.

Also solved by LEON BANKOFF, Los Angeles, California; FRANK P. BATTLES and LAURA KELLEHER, Massachusetts Maritime Academy, Buzzards Bay, Massachusetts; JORDI DOU, Barcelona, Spain; JACK GARFUNKEL, Flushing, N.Y.; J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; KEE-WAI LAU, Hong Kong; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer. Numerical solutions of varying accuracy were found by RICHARD I. HESS, Rancho Palos Verdes, California; J.A. McCALLUM, Medicine Hat, Alberta; and LA MOYNE L. PORTER, Shaker Heights, Ohio.

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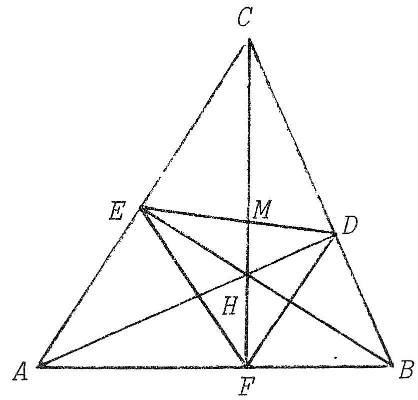
<u>1026</u>. [1985: 83] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

D, E, and F are points on sides BC, CA, and AB, respectively, of triangle ABC, and AD, BE, and CF concur at point H. If H is the incenter of triangle DEF, prove that H is the orthocenter of triangle ABC.

(This is the converse of a well-known property of the orthocenter.)

I. *Solution by Jordi Dou, Barcelona, Spain.*



Let M be the intersection of ED and CH. In the complete quadrilateral with edges CE, EH, HD, and DC, M and F divide CH harmonically, that is, $(CH,MF) = -1$ (e.g. Theorem 2.8.10, page 85 of [1]). Thus (e.g. Theorem 2.5.7, page 75 of [1]), the cross ratio $D(CA,EF)$ must also be -1. That is,

$$\frac{\sin\overline{CDE}/\sin\overline{EDA}}{\sin\overline{CDF}/\sin\overline{FDA}} = -1.$$

But DA bisects $\angle EDF$, so $\sin\overline{FDA} = -\sin\overline{EDA}$. Thus

$$\sin\overline{CDE} = \sin\overline{CDF} = \sin\overline{FDB},$$

which implies that DC is perpendicular to DA. Therefore AD and similarly BE and CF are altitudes of ΔABC, that is, H is the orthocenter of ΔABC.

Reference:

[1] H. Eves, *A Survey of Geometry* (Revised Edition), Allyn and Bacon, 1972.

II. *Solution by J.T. Groenman, Arnhem, The Netherlands.*

We use normal homogeneous trilinear coordinates with respect to ΔDEF. We obtain $D = (1,0,0)$, $E = (0,1,0)$, $F = (0,0,1)$, $H = (1,1,1)$. Thus

HD is the line $y = z$, so $A = (a_1,a_2,a_2)$

HE is the line $x = z$, so $B = (b_2,b_1,b_2)$

HF is the line $x = y$, so $C = (c_2,c_2,c_1)$.

Also, B, D, C are collinear, and thus we get

$$\begin{vmatrix} b_2 & b_1 & b_2 \\ 1 & 0 & 0 \\ c_2 & c_2 & c_1 \end{vmatrix} = 0$$

or

$$b_1c_1 = b_2c_2.$$

In the same way,

$$c_1a_1 = c_2a_2$$

and

$$a_1b_1 = a_2b_2.$$

Hence

$$\frac{b_1}{b_2} = \frac{c_2}{c_1} = \frac{a_1}{a_2} = \frac{b_2}{b_1},$$

whence $b_1{}^2 = b_2{}^2$. Since $b_1 = b_2$ would give $B = (1,1,1) = H$, we must have

$b_1 = -b_2$ and hence $B = (1,-1,1)$. This means that B (and similarly A and C) is an excenter of ΔDEF. It follows that AC, BC, and AB are the exterior angle-bisectors of ΔDEF, and thus are perpendicular to BE, AD, and CF respectively. Therefore AD, BE, CF are the altitudes of ΔABC and H is the orthocenter of this triangle.

Also solved by R.H. EDDY, Memorial University of Newfoundland, St. John's, Newfoundland; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAN SOKOLOWSKY, Brooklyn, N.Y.; and the proposer.

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<u>1027</u>. [1985: 248 (corrected)] *Proposed by M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

Determine all quadruples (a,b,c,d) of nonzero integers satisfying the Diophantine equation

$$abcd\left[\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right]^2 = (a + b + c + d)^2$$

and such that $a^2 + b^2 + c^2 + d^2$ is a prime.

Solution par C. Festraets-Hamoir, Bruxelles, Belgique.

$$abcd\left[\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right]^2 = (a + b + c + d)^2$$

$$\Longleftrightarrow abcd\left[\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{2}{ab} + \frac{2}{ac} + \frac{2}{ad} + \frac{2}{bc} + \frac{2}{bd} + \frac{2}{cd}\right]$$

$$= a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd$$

$$\Longleftrightarrow \frac{bcd}{a} + \frac{acd}{b} + \frac{abd}{c} + \frac{abc}{d} = a^2 + b^2 + c^2 + d^2$$

$$\Longleftrightarrow b^2c^2d^2 + a^2c^2d^2 + a^2b^2d^2 + a^2b^2c^2 = abcd(a^2 + b^2 + c^2 + d^2).$$

Posons $a^2 + b^2 + c^2 + d^2 = p$ (p premier). On a

$$b^2c^2d^2 + a^2c^2d^2 + a^2b^2d^2 + a^2b^2c^2 \equiv 0 \pmod p$$

$$b^2c^2(d^2 + a^2) + a^2d^2(c^2 + b^2) \equiv 0 \pmod p$$

$$b^2c^2(-b^2 - c^2) + a^2d^2(c^2 + b^2) \equiv 0 \pmod p$$

$$(a^2d^2 - b^2c^2)(b^2 + c^2) \equiv 0 \pmod p$$

$$a^2d^2 - b^2c^2 \equiv 0 \pmod p \qquad \text{car } (b^2 + c^2, p) = 1.$$

De même, on a

$$a^2c^2 - b^2d^2 \equiv 0 \pmod p.$$

D'où, par addition

$$a^2d^2 - b^2c^2 + a^2c^2 - b^2d^2 \equiv 0 \pmod p$$

$$(a^2 - b^2)(d^2 + c^2) \equiv 0 \pmod p$$

$$a^2 - b^2 \equiv 0 \pmod p \qquad \text{car } (d^2 + c^2, p) = 1.$$

Par symétrie, on obtient

$$a^2 \equiv b^2 \equiv c^2 \equiv d^2 \pmod{p}$$

$$p = a^2 + b^2 + c^2 + d^2 \equiv 4a^2 \pmod{p},$$

$$a^2 = 0 \quad \text{ou} \quad a^2 = p$$

ce qui est impossible. Donc, il n'existe aucun quadruple d'entiers positifs non nuls (a,b,c,d) satisfaisant les conditions données.

Also solved by FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; and the proposer.

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