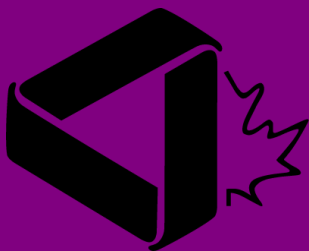


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THE OLYMPIAD CORNER: 62

M.S. KLAMKIN

I give one new problem set this month. It consists of 25 problems proposed (but unused) by various participating countries in past International Mathematics Olympiads. I will extend this list in forthcoming columns. As usual, I solicit from all readers elegant solutions to these problems with, if possible, extensions or generalizations.

1. *Proposed by Australia.*

A total of 1983 cities are served by ten airlines. There is direct service (without stopovers) between any two cities and all airline schedules run both ways. Prove that at least one of the airlines can offer a round trip with an odd number of landings.

2. *Proposed by Australia and the U.S.A. (independently).*

The altitude from a vertex of a given tetrahedron intersects the opposite face in its orthocenter. Prove that all four altitudes of the tetrahedron are concurrent.

3. *Proposed by Brazil.*

Which of the numbers 1, 2, ..., 1983 have the largest number of positive divisors?

4. *Proposed by Canada.*

Find all possible finite sequences $\{n_0, n_1, \dots, n_k\}$ of integers such that, for each $i = 0, 1, \dots, k$, i appears in the sequence n_i times.

5. *Proposed by Canada.*

Let $a_0 = 0$ and

$$a_{n+1} = k(a_n + 1) + (k+1)a_n + 2\sqrt{k(k+1)a_n(a_n + 1)}, \quad n = 0, 1, 2, \dots,$$

where k is a positive integer. Prove that a_n is a positive integer for $n = 1, 2, 3, \dots$.

6. *Proposed by Cuba.*

Show that there exist infinitely many sets of 1983 consecutive positive integers each of which is divisible by some number of the form a^{1983} , where $a \neq 1$ is a positive integer.

7. *Proposed by Finland.*

Let r and s be integers, with $s > 0$. Show that there exists an interval I of length $1/s$ and a polynomial $P(x)$ with integral coefficients such that, for all $x \in I$,

$$|P(x) - \frac{x}{s}| < \frac{1}{s^2}.$$

8, *Proposed by Finland.*

Let $F: [0,1] \rightarrow \mathbb{R}$ be a continuous function satisfying

$$\begin{cases} F(2x) = bF(x), & 0 \leq x \leq \frac{1}{2}, \\ F(x) = b + (1-b)F(2x-1), & \frac{1}{2} \leq x \leq 1, \end{cases}$$

where $b = (1+c)/(2+c)$ and $c > 0$. Prove that $0 < F(x)-x < c$ for all $x \in (0,1)$.

9, *Proposed by the German Democratic Republic.*

Let P_1, P_2, \dots, P_n be n distinct points in a plane. Prove that

$$\max_{1 \leq i < j \leq n} \overline{P_i P_j} > \frac{\sqrt{3}}{2}(n-1) \min_{1 \leq i < j \leq n} \overline{P_i P_j}.$$

10, *Proposed by Great Britain.*

If the sides a, b, c of a triangle satisfy

$$2(bc^2 + ca^2 + ab^2) = b^2c + c^2a + a^2b + 3abc,$$

prove that the triangle is equilateral. Prove also that the equation can be satisfied by positive real numbers that are not the sides of a triangle.

11, *Proposed by Great Britain.*

Prove that there is a unique infinite sequence $\{u_0, u_1, u_2, \dots\}$ of positive integers such that, for all $n \geq 0$,

$$u_n^2 = \sum_{r=0}^n \binom{n+r}{r} u_{n-r}.$$

12, *Proposed by Israel.*

For a given set X of 1983 members there exists a family of subsets $\{S_1, S_2, \dots, S_k\}$ such that

- (i) the union of any three of these subsets is the entire set X , and
- (ii) the union of any two of these subsets contains at most 1979 members.

Determine the largest possible value of k .

13, *Proposed by Israel.*

There are 1983 points on a given circle, and each is given one of the affixes ± 1 . Prove that, if the number of points with the affix $+1$ is greater than 1789, then at least 1207 of the points have the property that the partial sums that can be formed by summing their own affix and those of their consecutive neighbors on the circle up to any other point, in either direction on the circle, are all strictly positive.

14. *Proposed by Mongolia.*

Show that there exist distinct natural numbers n_1, n_2, \dots, n_k such that

$$\pi^{-1984} < 25 - \left(\frac{1}{n_1} + \frac{1}{n_2} + \dots + \frac{1}{n_k} \right) < \pi^{-1960}.$$

15. *Proposed by Mongolia.*

The set $\{1, 2, \dots, 49\}$ is partitioned into three subsets. Show that at least one of the subsets contains three different numbers a, b, c such that $a+b = c$.

16. *Proposed by The Netherlands.*

Prove that in any parallelepiped the sum of the lengths of the edges does not exceed twice the sum of the lengths of the four principal diagonals.

17. *Proposed by Poland.*

Given nonnegative real numbers x_1, x_2, \dots, x_k and positive integers k, m, n such that $km \leq n$, prove that

$$n \left\{ \prod_{i=1}^k x_i^m - 1 \right\} \leq m \sum_{i=1}^k (x_i^n - 1).$$

18. *Proposed by Romania.*

A polynomial $P(x)$ of degree 990 satisfies

$$P(k) = F_k, \quad k = 992, 993, \dots, 1982,$$

where $\{F_k\}$ is the Fibonacci sequence, defined by

$$F_1 = F_2 = 1, \quad F_{n+1} = F_n + F_{n-1}, \quad n = 2, 3, 4, \dots$$

Prove that $P(1983) = F_{1983} - 1$.

19. *Proposed by Sweden.*

Let a and b be integers. Is it possible to find integers p and q such that the integers $p+na$ and $q+nb$ are relatively prime for any integer n ?

20. *Proposed by Sweden.*

AB is the diameter of a circle γ with center O. A segment BD is bisected by the point C on γ , and AC and DO intersect at P. Prove that there is a point E on AB such that P lies on the circle with diameter AE.

21. *Proposed by the U.S.A.*

The sum of all the face angles at all but one of the vertices of a given simple polyhedron is 5160° . Find the sum of all the face angles of the polyhedron.

22. *Proposed by the U.S.A.*

Determine all pairs (a, b) of positive real numbers with $a \neq 1$ such that

$$\log_a b < \log_{a+1}(b+1).$$

23. *Proposed by the U.S.S.R.*

A tetrahedron is inscribed in a unit sphere. The tetrahedron is such that the center of the sphere lies in its interior. Show that the sum of the edge lengths of the tetrahedron exceeds 6.

24. *Proposed by the U.S.S.R.*

The proper divisors of the natural number n are arranged in increasing order, $x_1 < x_2 < \dots < x_k$. Find all numbers n such that

$$x_5^2 + x_6^2 - 1 = n.$$

25. *Proposed by the U.S.S.R.*

A triangle T_1 is constructed with the medians of a right triangle T . If R_1 and R are the circumradii of T_1 and T , respectively, prove that $R_1 > 5R/6$.

*

I now give comments and solutions to various problems from earlier columns.

17. [1981: 17; 1984: 145] *From a 1973 Moscow Olympiad.*

Twelve painters live in 12 houses which are built along a circular street and are painted some white, some blue. Each month one of the painters, taking with him enough white and blue paint, leaves his house and walks along the road in the clockwise sense. On the way, he repaints every house (starting with his own) the opposite colour. He stops work as soon as he repaints some white house blue. In a year, each painter undertakes such a journey exactly once. Show that at the end of a year each house will be painted its original colour, provided that at the beginning of the year at least one house was painted blue.

II. *Comment by Leroy F. Meyers, The Ohio State University.*

This problem is equivalent to Problem 705 [1982: 325]. In particular, my solution to No. 705 showed that there are always exactly two complete circuits, a fact that was not brought out explicitly in the original solution to the present problem.

*

H-2. [1981: 114; 1984: 148] *From Középiskolai Matematikai Lapok 60 (1979) 140.*

Let n be a positive integer. As a first step, we have given the sequence $\{a_1, a_2, \dots, a_k\}$, where $k = 2^n$ and each a_i is 1 or -1. As a second step, we form the new sequence $\{a_1a_2, a_2a_3, \dots, a_ka_1\}$ and continue to repeat this process to generate new sequences. Show that, by at most the 2^n th iterated step, we arrive at a constant sequence with every term equal to 1.

II. *Comment by Leroy F. Meyers, The Ohio State University.*

This problem is really Ducci's problem in disguise. See my note "Ducci's Four-Number Problem: A Short Bibliography" [1982: 262-266]. For a proof that the coefficients of $(x+y)^m$ are all odd when $m = 2^n - 1$, see my solution to Problem 90 [1982: 279] or the earlier solution in [1976: 34].

*

A4. [1983: 138] *From the 1983 Netherlands Invitational Mathematics Examination.*

What is the smallest amount, in cents, that cannot be made up with at most ten of the coins of denominations 1¢, 5¢, 10¢, 25¢, 50¢, and 100¢?

Comment by E. Frederick Lang, M.D., Grosse Pointe, Michigan.

No solution was provided for this problem, but the answer given [1983: 142] was 444. I gave the problem to my grandson Christopher D. Lang, who is 15 years old and attends Athens High School in Troy, Michigan. He found that the correct answer is 394.

*

F.2415. [1983: 237; 1984: 150] *From Középiskolai Matematikai Lapok (March 1983).*

Choose 400 different points inside a unit cube. Show that 4 of these points lie inside some sphere of radius $4/23$.

II. *Solution by Fred Galvin, University of Kansas.*

The elegant solution given earlier showed that 376 points suffice. We show, less elegantly but more efficiently, that 321 points suffice.

Let n and k be positive integers, let r be a positive real number, and suppose there are n points X_1, X_2, \dots, X_n in the unit cube such that no $k+1$ of them belong to a ball of radius r . Let B_i be the ball of radius r centered at X_i and

$$B = B_1 \cup B_2 \cup \dots \cup B_n.$$

No point belongs to $k+1$ of the B_i 's. For if, say, P belonged to $B_1 \cap B_2 \cap \dots \cap B_{k+1}$, then X_1, X_2, \dots, X_{k+1} would all belong to the ball of radius r centered at P , contradicting the hypothesis. Hence, with the bars denoting volume, we have

$$k|B| \geq |B_1| + |B_2| + \dots + |B_n|,$$

that is,

$$|B| \geq \frac{4}{3}\pi r^3 \cdot \frac{n}{k}. \quad (1)$$

On the other hand, since B is contained in the "rounded cube" consisting of all points at distance at most r from the unit cube, we have

$$|B| \leq 1 + 6r + 3\pi r^2 + \frac{4}{3}\pi r^3. \quad (2)$$

Thus, combining (1) and (2) gives

$$n \leq k\left(\frac{3}{4\pi r^3} + \frac{9}{2\pi r^2} + \frac{9}{4r} + 1\right).$$

For $k = 3$ and $r = 0.1739$, we get $n < 320.0988\dots$, so for 321 points inside a unit cube, 4 of them will lie inside some sphere of radius $0.1739 < 4/23$,

*

], [1984: 40] *From the 1983 Brazilian Mathematical Olympiad.*

Show that the equation

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{1983}$$

has a finite number of solutions, where x, y, z are natural numbers.

Solution by K.S. Murray, Brooklyn, N.Y.

We may assume that $x \leq y \leq z$ in any solution (x, y, z) . Thus $3/x \geq 1/1983$, so $x \leq 3 \cdot 1983$ and the number of possibilities for x is finite. Then, for each possible value of x , it is seen from

$$\frac{2}{y} \geq \frac{1}{1983} - \frac{1}{x}, \quad \text{or} \quad y \leq \frac{2}{(1/1983) - (1/x)},$$

that the number of possibilities for y is also finite. Finally, for each possible (x, y) , the equation gives only one possible value of z . \square

The above solution clearly indicates how an inductive proof would go for the following more general theorem:

For every rational w and every positive integer s the equation

$$\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_s} = w$$

has a finite number ≥ 0 of solutions in positive integers x_1, x_2, \dots, x_s .

In fact, this theorem appears, with a complete proof by induction on s , on pages 85-86 of W. Sierpiński's *250 Problems in Elementary Number Theory* (American Elsevier, New York, 1970).

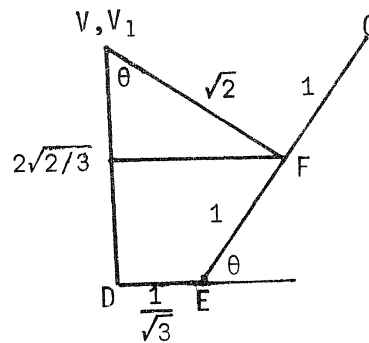
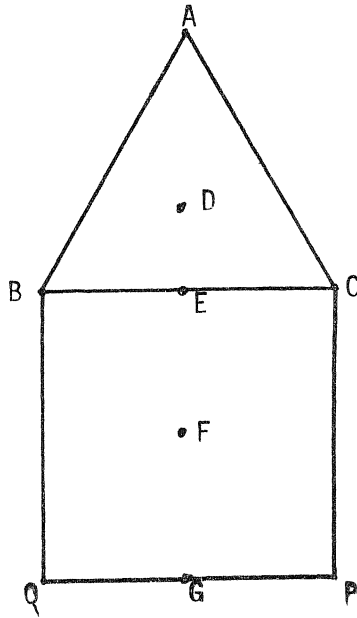
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2. [1984: 40] *From the 1983 Brazilian Mathematical Olympiad.*

Triangle ABC is equilateral and has side a . Squares BCPQ, CAMN, and ABRS are constructed, and they are the bases of three square pyramids with vertices V_1, V_2, V_3 , all the edges being of length a . The pyramids are rotated about BC, CA, and AB until V_1, V_2, V_3 all coincide. Show that, after the rotations, MNPQRS is a regular hexagon.

Solution by K.S. Murray, Brooklyn, N.Y.

It is clear that after the rotation, when V_1, V_2, V_3 occupy the common position V , the points M, N, P, Q, R, S are all coplanar. We show that then the hexagon $MNPQRS$ is parallel to ABC and contains V . For convenience, let $a = 2$. It follows easily that the altitudes of $V-ABC$ and V_1-BCPQ are $2\sqrt{2/3}$ and $\sqrt{2}$, respectively. Now let D, E, F, G be the centroids of ABC , BC , $BCPQ$, and PQ , respectively, and let θ be the angle of rotation of $BCPQ$ about BC that brings V_1 into coincidence with V . We have the following configurations:



After the rotation

In order for V_1 to coincide with V , the following equations must be satisfied:

$$\frac{1}{\sqrt{3}} + \cos \theta = \sqrt{2} \sin \theta, \quad (1)$$

$$\sin \theta + \sqrt{2} \cos \theta = 2\sqrt{2/3}. \quad (2)$$

And if G is to be at the same height as V , we must have

$$2 \sin \theta = 2\sqrt{2/3}. \quad (3)$$

It follows easily that (3) satisfies both (1) and (2).

The rest follows by the symmetry of the configuration $MNPQRS$ with center V .

*

3. [1984: 40] *From the 1983 Brazilian Mathematical Olympiad.*

Show that $1 + \frac{1}{2} + \dots + \frac{1}{n}$ is not an integer for any natural number $n \geq 2$.

Comment by M.S.K.

This is a known problem. For example, see Problem 85, pages 21 and 155, in the

highly recommended book *Selected Problems and Theorems in Elementary Mathematics*, by D.O. Shklyarsky, N.N. Chentsov, and I.M. Yaglom, Mir Publishers, Moscow, 1979. (The book can be obtained (in English) through Imported Publications, Inc., 320 West Ohio Street, Chicago, Illinois 60610.)

For $n \geq 2$, there is always a positive integer m such that $2^m \leq n < 2^{m+1}$. Combining the fractions using the least common denominator, which must be of the form $2^m k$ with k odd, we obtain

$$\text{sum} = \frac{1 + \text{even no.}}{2^m k},$$

which cannot be an integer.

It is also known more generally that if a and d are positive integers, then

$$\frac{1}{a} + \frac{1}{a+d} + \dots + \frac{1}{a+(n-1)d}$$

is never an integer except in the trivial case $a = n = 1$,

*

4, [1984: 40] *From the 1983 Brazilian Mathematical Olympiad.*

Show that all the points of a circle can be coloured, each with one of two colours, in such a way that no inscribed right triangle has its three vertices all of the same colour.

Solution by Paul Wagner, Chicago, Illinois.

Just colour any point and its diametrically opposite point with different colours.

Rider by M.S.K.

Determine whether or not it is possible to two-colour a circle in such a way that the three vertices of any inscribed angle of measure $360^\circ/k$ are all of the same colour, where $k > 2$ is an integer.

*

5, [1984: 40] *From the 1983 Brazilian Mathematical Olympiad.*

(a) Prove that $1 \leq \sqrt[n]{n} \leq 2$ for every natural number $n \geq 1$.

(b) Find the smallest real number k such that $1 \leq \sqrt[n]{n} \leq k$ for every natural number $n \geq 1$.

Solution by Gali Salvatore, Perkins, Québec.

(b) Let $f(x) = \sqrt[x]{x}$. The following facts about this function are known and easy to prove by calculus (see [1], or see the graph of the function in [2]): $f(x)$ is strictly increasing for $0 < x < e$, has an absolute maximum at $x = e$, is strictly decreasing for $x > e$, and approaches 1 as $x \rightarrow \infty$. Since $2 < e < 3$, we therefore have

$1 < \sqrt[n]{n} \leq k$ for $n = 3, 4, 5, \dots$ when $k = \sqrt[3]{3}$, and no smaller k will do. Since also $f(2) = f(4)$ and $f(1) = 1$, it follows that $1 \leq \sqrt[n]{n} \leq k$ holds for every natural number n when $k = \sqrt[3]{3}$, and no smaller k will do.

(a) This follows from $n < n+1 \leq (1+1)^n$, or from part (b) since $\sqrt[3]{3} < 2$.

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1. Problem 55 (proposed by Louis Rotando, solution by Robert Plummer), *The Two-Year College Mathematics Journal*, 8 (1977) 97.
2. R. Arthur Knoebel, "Exponentials Reiterated", *American Mathematical Monthly*, 88 (1981) 235-252, esp. p. 236.

*

6. [1984: 40] *From the 1983 Brazilian Mathematical Olympiad.*

A sphere being given, show that the largest number of spheres congruent to and tangent to the given sphere, no two of which have any interior point in common, is at least 12 and at most 14.

Try to refine this estimate.

Solution by Angelo N. Barone, University of Sao Paulo.

Let A_1, A_2, \dots, A_6 be the consecutive vertices of a regular hexagon with side $2r$ and center O . We look at the regular tetrahedra $OA_1A_2A_7$, $OA_3A_4A_8$, and $OA_5A_6A_9$, where A_7, A_8, A_9 lie on the same side of the plane of the hexagon. We also look at the regular tetrahedra $OA_1A_2A_{10}$, $OA_3A_4A_{11}$, and $OA_5A_6A_{12}$, where A_{10}, A_{11}, A_{12} lie on the other side of the plane of the hexagon. All the edges of the polyhedron $A_1A_2 \dots A_{12}$ have length $2r$ and the distance from O to each of its vertices is $2r$. Therefore the 12 spheres with centers A_i and radius r are tangent to the sphere with center O and radius r . This shows that the number we are looking for is at least 12.

We now consider two externally tangent congruent spheres of radius r and the cone tangent to one of them whose vertex is the centre of the other. The angle at the vertex of an axial section of the cone measures $\pi/3$. The cone determines on the central sphere (i.e., the one with the vertex of the cone at its center) a spherical cap whose

$$\text{height} = r\left(1 - \frac{\sqrt{3}}{2}\right) \quad \text{and} \quad \text{area} = 2\pi r^2\left(1 - \frac{\sqrt{3}}{2}\right).$$

It is easy to verify that 15 times this area is strictly larger than the area of the sphere. Therefore the number we are looking for is at most 14.

Comment by M.S.K.

For a refinement of the estimate "at least 12 and at most 14" and for further comments, see Problem 826 in this issue.

*

Gy. 2142. [1984: 75] *From Középiskolai Matematikai Lapok 67 (1983) 80.*

A 12×12 chessboard has alternating black and white squares. In one operation, every square in a single row (or column) is repainted the opposite colour (white squares repainted black and black ones white). The operation is then repeated on another row (or column). Is it possible that, after a certain number of operations, all the squares on the chessboard are black?

Unsigned solution.

We assume that the rows and columns are numbered consecutively from top to bottom and left to right, respectively, and that the chessboard is oriented so that the top left square is white. Then all the squares will end up black if we first perform the operation on rows 1,3,5,7,9,11 and then on columns 2,4,6,8,10,12.

Rider by M.S.K.

What are all the possible numbers of black squares that one can obtain?

*

Gy. 2143. [1984: 75] *From Középiskolai Matematikai Lapok 67 (1983) 80.*

A *word* is any sequence of letters. Starting with the word AB , new words are formed by the repeated use of the following rules in any order of succession:

- (i) If a word ends in B , add C at the end.
- (ii) If a word begins with A , double the word that follows the initial A (e.g., $ABC \rightarrow ABCBC$).
- (iii) If a word contains three consecutive letters B , replace them by a single C .
- (iv) Omit two consecutive letters C if they occur anywhere in a word.

Consider all the words formed in this way. Does the word AC figure among them?

Comment by John Morvay, Dallas, Texas.

This problem is identical (except for notation) with the *MU*-puzzle presented in Chapter I of Douglas R. Hofstadter's *Gödel, Escher, Bach* (Basic Books, New York, 1979). Later, in Chapter IX, Hofstadter shows that the *B-count* of a word (the number of times B occurs in it) can never be a multiple of 3. In particular, the *B-count* of a word cannot be zero, so AC is not a word in the system. Hofstadter's argument goes as follows:

1. The *B-count* begins at 1 (not a multiple of 3).
2. Rules (i) and (iv) leave the *B-count* of any word unchanged.
3. Rules (ii) and (iii) affect the *B-count* of a word in such a way as never to create a multiple of 3 unless given one initially.

*

Gy. 2144. [1984: 75] *From Középiskolai Matematikai Lapok 67 (1983) 80.*

Determine all natural numbers n such that $2^n - 1$ equals the square or higher integral power of a natural number.

Solution by Rob Frielipp, University of Wisconsin-Oshkosh.

The property clearly holds for $n = 1$. We show that it holds for no other natural number n . Suppose, on the contrary, that

$$2^n - 1 = x^k, \quad (1)$$

where n, x, k are natural numbers and $n \geq 2$, $k \geq 2$. Clearly x is odd. If k is even, then the right side of (1) is congruent to 1 modulo 4 while the left side is congruent to -1 modulo 4: contradiction. For odd k , rewrite (1) in the equivalent form

$$2^n = x^k + 1 = (x + 1)(x^{k-1} - x^{k-2} + \dots - x + 1).$$

The second factor on the right divides 2^n and it is odd since it has an odd number of odd addends. Hence this factor equals 1, so $2^n = x + 1$ and $k = 1$: contradiction.

*

Gy. 2145. [1984: 75] *From Középiskolai Matematikai Lapok 67 (1983) 80.*

Solve the following system of equations:

$$x^3 + y^3 + z^3 = 8,$$

$$x^2 + y^2 + z^2 = 22,$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = -\frac{z}{xy},$$

Solution by Glen E. Mills, Pensacola Junior College, Florida.

The last equation is equivalent to

$$(z + x)(z + y) = 0, \quad xyz \neq 0.$$

We find all the complex solutions for which $z = -y$. The solutions for which $z = -x$ can then be obtained from these by permuting the values of x and y .

If $z = -y$, the first equation gives $x^3 = 8$, so

$$x = 2 \text{ or } 2\omega \text{ or } 2\omega^2,$$

where ω is a primitive cube root of unity. Then the second equation gives correspondingly

$$y^2 = 9 \text{ or } 11 - 2\omega^2 \text{ or } 11 - 2\omega,$$

so

$$y = \pm 3 \text{ or } \pm\sqrt{11 - 2\omega^2} \text{ or } \pm\sqrt{11 - 2\omega}.$$

Thus the possible solutions are

$$(x, y, z) = (2, \pm 3, \mp 3) \text{ or } (2\omega, \pm\sqrt{11-2\omega^2}, \mp\sqrt{11-2\omega^2}) \text{ or } (2\omega^2, \pm\sqrt{11-2\omega}, \mp\sqrt{11-2\omega}),$$

with similarly placed signs corresponding. Conversely, it is easily verified that all of these are in fact solutions of the system.

*

F. 2434. [1984: 75] *From Középiskolai Matematikai Lapok 67 (1983) 80.*

Prove that the equation

$$x^3 + 4x^2 + 6x + c = 0$$

cannot have three distinct real roots for any real number c .

Solution by Glen E. Mills, Pensacola Junior College, Florida.

For any real c , $P(x) \equiv x^3 + 4x^2 + 6x + c$ is a real polynomial function of degree 3, and $P'(x) = 3x^2 + 8x + 6 > 0$ for all real x . Hence $P(x)$ is strictly increasing for all x and therefore has exactly one real zero for any real number c .

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F. 2435. [1984: 76] *From Középiskolai Matematikai Lapok 67 (1983) 80.*

Let γ be a circle with centre O . Show that, of all the triangles ABC with incircle γ , it is the equilateral triangle for which the sum $OA^2 + OB^2 + OC^2$ is minimal.

Solution by M.S.K.

If r is the fixed inradius, then

$$OA^2 + OB^2 + OC^2 = r^2(\csc^2 \frac{A}{2} + \csc^2 \frac{B}{2} + \csc^2 \frac{C}{2}).$$

It is a known inequality that $\sum \csc^2(A/2) \geq 12$ with equality just when ABC is equilateral. For a proof, first note that $\csc x$ is convex for $x \in (0, \pi)$. Thus

$$\csc \frac{A}{2} + \csc \frac{B}{2} + \csc \frac{C}{2} \geq 3 \csc \frac{A+B+C}{6} = 6,$$

with equality just when $A = B = C$. Then, by the power mean inequality,

$$\frac{\csc^2 \frac{A}{2} + \csc^2 \frac{B}{2} + \csc^2 \frac{C}{2}}{3} \geq \left(\frac{\csc \frac{A}{2} + \csc \frac{B}{2} + \csc \frac{C}{2}}{3} \right)^2 \geq 4.$$

Whence $\min(OA^2 + OB^2 + OC^2) = 12r^2$, and this is attained just when ABC is equilateral.

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F. 2436. [1984: 76] *From Középiskolai Matematikai Lapok 67 (1983) 80.*

Prove that, for natural numbers $n > 1$,

$$\sqrt{1 + \sqrt{n + \sqrt{n^2 + \sqrt{n^3 + \dots + \sqrt{n^n}}}}} < n.$$

Solution by Gali Salvatore, Perkins, Québec.

Let $R(n)$ denote the expression on the left side of the proposed inequality.

For $n > 1$, we have

$$R(n) \equiv \sqrt{n} \cdot \sqrt{\frac{1}{n} + \sqrt{\frac{n}{n^2} + \sqrt{\frac{n^2}{n^4} + \sqrt{\frac{n^3}{n^8} + \dots + \sqrt{\frac{n^n}{n^{2^n}}}}}} < \sqrt{n} \cdot a_n,$$

where

$$a_n = \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots + \sqrt{1}}}} \quad (n+1 \text{ radicals}).$$

It is well known and easy to show (or see Problem 8 [1975: 19]) that the sequence $\{a_n\}$ is strictly increasing and converges to the golden ratio $g = (1+\sqrt{5})/2$. The inequality $R(n) < n$ therefore holds whenever $g\sqrt{n} \leq n$, or $n \geq g^2 \approx 2.618$, that is, for all $n \geq 3$. Furthermore, $R(2) = \sqrt{3} < 2$. Therefore $R(n) < n$ holds for all $n > 1$.

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F. 2437, [1984: 76] *From Középiskolai Matematikai Lapok 67 (1983) 80.*

Every point in space is coloured either red or blue. Prove that there is a unit square with four blue vertices, or else there is one with at least three red vertices.

Solution by John Morvay, Dallas, Texas.

If there are no red points, then all unit squares have four blue vertices. If there is a red point R_1 such that no red point is at unit distance from R_1 , then R_1 is the center of a sphere of unit radius all of whose points are blue, and there are then infinitely many unit squares with four blue vertices. Finally, suppose there is a red point R_1 and a red point R_2 at unit distance from R_1 , and let R_1R_2 be a lateral edge of a triangular prism with equilateral bases and square faces. The face opposite R_1R_2 either is a unit square with four blue vertices, or else it has a red vertex P_3 , and then R_1, R_2, P_3 are three red vertices of a unit square.

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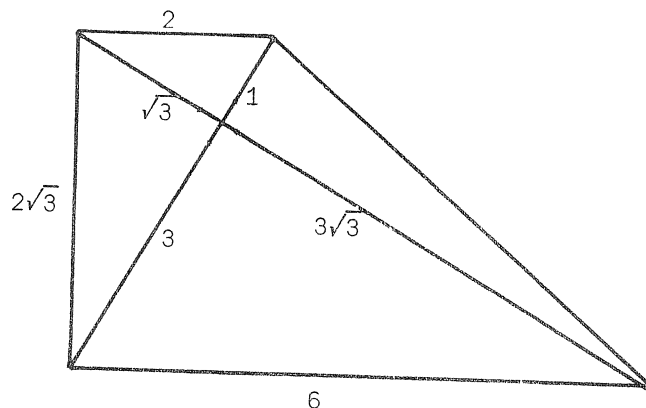
F. 2438, [1984: 76] *From Középiskolai Matematikai Lapok 67 (1983) 80.*

Drawing the diagonals of a convex quadrilateral, we find that, among the four triangles thus formed, three are similar to one another but not similar to the fourth. Is it true that then one of the acute angles of the fourth triangle is twice as large as an angle of the other triangles?

Solution by M.S.K.

No. Just consider the adjoining figure, where three of the triangles have acute angles 30° and 60° , while the fourth has acute angles approximately $79^\circ 06'$ and $10^\circ 54'$.

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.



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PROBLEMS - - PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk () after a number indicates a problem submitted without a solution.*

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly hand-written on signed, separate sheets, should preferably be mailed to the editor before September 1, 1985, although solutions received after that date will also be considered until the time when a solution is published.

1011, *Proposed by Charles W. Trigg, San Diego, California.*

In base six, find a nine-digit square of the form AAAAAXYZ, given that it is the square of a number whose central triad is XYZ.

1012, *Proposed by G.P. Henderson, Campbellcroft, Ontario.*

An amateur winemaker is siphoning wine from a carboy. To speed up the process, he tilts the carboy to raise the level of the wine. Naturally, he wants to maximize the height, H , of the surface of the liquid above the table on which the carboy rests. The carboy is actually a circular cylinder, but we will only assume that its base is the interior of a smooth closed convex curve, C , and that the generators are perpendicular to the base. P is a point on C , T is the line tangent to C at P , and the cylinder is rotated about T .

(a) Prove that H is a maximum when the centroid of the surface of the liquid is vertically above T .

(b) Let the volume of the wine be V and let the area inside C be A . Assume that $V \geq AW/2$, where W is the maximum width of C (i.e., the maximum distance between parallel tangents). Obtain an explicit formula for H_M , the maximum value of H . How should P be chosen to maximize H_M ?

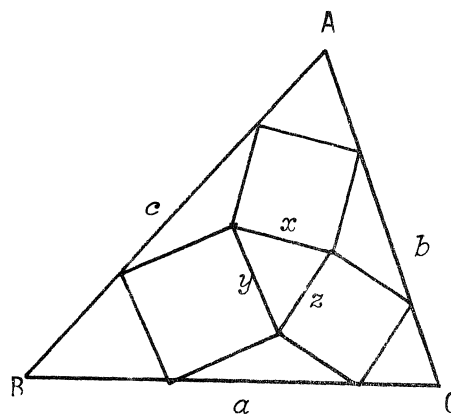
1013. *Proposed by Hidetosi Fukagawa, Yokosuka High School, Tokai City, Aichi, Japan.*

This problem is about "Malfatti" squares, named by analogy with Malfatti circles. The concept is illustrated in the adjoining figure.

(a) Given a triangle ABC, show how to construct its three Malfatti squares.

(b) *The Malfatti squares problem.* Given the sides a, b, c of a triangle, calculate the sides x, y, z of its Malfatti squares.

(c) *The reverse Malfatti squares problem.* Given the sides x, y, z of the Malfatti squares of a triangle, calculate the sides a, b, c of the triangle.

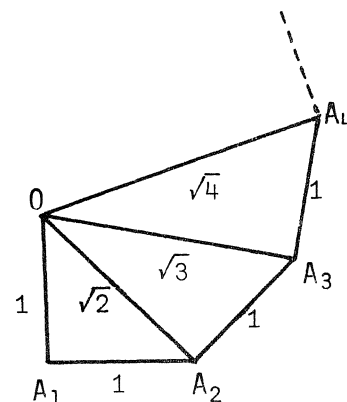


1014.* *Proposed by Shmuel Avital, Technion-Israel Institute of Technology, Haifa, Israel.*

The points A_1, A_2, A_3, \dots are chosen, by the familiar construction illustrated in the figure, in such a way that $OA_n = \sqrt{n}$, $n = 1, 2, 3, \dots$

(a) What is the nature of the smooth spiral that passes through A_1, A_2, A_3, \dots ?

(b) Find, in terms of n , an explicit formula for the measure of the rotation that ray OA_1 must undergo to bring it into coincidence with ray OA_n .



1015. *Proposed by Yang Lu, China University of Science and Technology, Hefei, Anhui, People's Republic of China.*

Let $A_1A_2A_3A_4$ be a convex quadrilateral, let a_{ij} denote the length of segment A_iA_j , ($i, j = 1, 2, 3, 4$), and let R_1, R_2, R_3, R_4 be the circumradii of triangles $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, and $A_1A_2A_3$, respectively. Prove that

$$(R_1R_2 + R_3R_4)a_{12}a_{34} + (R_1R_4 + R_2R_3)a_{14}a_{23} = (R_1R_3 + R_2R_4)a_{13}a_{24}. \quad (1)$$

(This is an extension of Ptolemy's Theorem, for if $A_1A_2A_3A_4$ is cyclic, then $R_1 = R_2 = R_3 = R_4$, and (1) is equivalent to $a_{12}a_{34} + a_{14}a_{23} = a_{13}a_{24}$.)

1016. *Proposed by Andrew P. Guinand, Trent University, Peterborough, Ontario.*

(a) Show that, for the triangle with angles $120^\circ, 30^\circ, 30^\circ$, the nine-point centre lies on the circumcircle.

(b) Characterize all the triangles for which the nine-point centre lies on the circumcircle.

1017. *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

If the figure on the left is a pandiagonal magic square, then so is the figure on the right.

| | | | |
|---|---|---|---|
| A | B | C | D |
| E | F | G | H |
| I | J | K | L |
| M | N | O | P |

| | | | |
|---|---|---|---|
| A | B | N | M |
| F | F | J | J |
| H | G | K | L |
| D | C | O | P |

Both figures are arrangements of the same 16 arbitrary numbers A, B, C, \dots, P , and both have A in the upper left corner cell. Enumerate all the ways the arbitrary A, B, C, \dots, P can be arranged to form pandiagonal magic squares in which A is fixed as shown.

1018. *Proposed by Kurt Schiffler, Schorndorf, Federal Republic of Germany.*

Let ABC be a triangle with incentre I . Prove that the Euler lines of triangles IBC , ICA , IAB , and ABC are all concurrent.

1019. *Proposed by Weixuan Li and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Determine the largest constant k such that the inequality

$$x \leq \alpha \sin x + (1-\alpha) \tan x$$

holds for all $\alpha \leq k$ and for all $x \in [0, \pi/2)$.

(The inequality obtained when α is replaced by $2/3$ is the Snell-Huygens inequality, which is fully discussed in Problem 115 [1976: 98-99, 111-113, 137-138].)

1020. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Solve, for $x \in [0, 2\pi)$, the equation

$$176 \cos x + 64 \sin x = 75 \cos 2x + 80 \sin 2x + 101.$$

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S O L U T I O N S

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

826. [1983: 79] *Proposed by Kent D. Boklan, student, Massachusetts Institute of Technology.*

It is a well-known consequence of the pigeonhole principle that, if six circles in the plane have a point in common, then one of the circles must entirely contain a radius of another.

Suppose n spherical balls have a point in common. What is the smallest value of n for which it can be said that one ball *must* entirely contain a radius of another?

I. *Comment by M.S. Klamkin and A. Meir, University of Alberta.*

Let P denote the common point and let C_r ($r = 1, 2, \dots, n$) be the respective centers of the balls. In order that one ball entirely contain a radius of another ball, at least one of the angles $\angle C_j P C_k$ ($j \neq k$) must not exceed 60° . In the planar case, since the sum of the six angles between consecutive rays PC_r is 360° , the desired result follows immediately by the pigeonhole principle (see Problem 2 [1984: 85-86]).

The spherical case considered here has a long history. It is a special case of the following more general space problem treated by L. Fejes Tóth in 1943 [1]: *Given n points on a unit sphere, determine the maximum of the least distance d_n between any two of the points.* He proved that

$$d_n \leq \sqrt{4 - \csc^2 \frac{n\pi}{6(n-2)}} \equiv D_n. \quad (1)$$

There is equality for the cases $n = 3, 4, 6, 12$. The last three cases correspond to the regular tetrahedron, octahedron, and icosahedron, respectively (in which the faces are equilateral triangles). From (1),

$$d_{12} = D_{12} \approx 1.0515, \quad D_{13} \approx 1.0139, \quad D_{14} \approx 0.9800.$$

Therefore the least n is either 13 or 14. According to Leech [3], it was conjectured by David Gregory in an unpublished notebook at Christ Church, Oxford, that a sphere can touch 13 nonoverlapping spheres congruent to it. Even if this conjecture were true, we could still have $n = 13$ or $n = 14$. However, Schütte and van der Waerden [2] and Leech [3] proved that there are no more than 12 such spheres. Consequently, the least n is 13. (The proofs in [2] and [3] are not easy.)

For an essentially equivalent problem giving the bounds $12 \leq n \leq 14$ by simple means, see Problem 6 in this issue (page 44).

II. *Comment by Edith Orr, Ottawa, Ontario.*

Another application of the pigeonhole principle: *Four Saints in Three Acts*, a 1934 surrealist opera with libretto by Gertrude Stein and music by Virgil Thomson.

One incorrect solution was received.

REFERENCES

1. L.L. Whyte, "Unique Arrangements of Points on a Sphere", *American Mathematical Monthly*, 59 (1952) 606-611.

2. K. Schütte and B.L. van der Waerden, "Das Problem der dreizehn Kugeln", *Math. Ann.*, 125 (1953) 325-334.

3. John Leech, "The Problem of the Thirteen Spheres", *The Mathematical Gazette*, (1956) 22-23.

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889, [1983: 277] Proposed by G.C. Giri, Midnapore College, West Bengal, India.

$A_1 A_2 \dots A_n$ is a regular n -gon ($n \geq 3$) inscribed in a circle of radius r ; M is the midpoint of the arc $A_1 A_n$; and, for $i = 1, 2, \dots, n$, P_i is the orthogonal projection of A_i upon a fixed diameter D of the circle. Prove the following:

$$(a) \sum_{i=1}^n A_i \vec{P}_i = \vec{0};$$

$$(b) \sum_{i=2}^n A_1 A_i = 2r \cot \frac{\pi}{2n} \text{ and } \prod_{i=2}^n A_1 A_i = nr^{n-1};$$

$$(c) \text{ if } n = 2m, \text{ then } \prod_{i=1}^m MA_i = \sqrt{2}r^m \text{ and } \prod_{i=2}^m A_1 A_i = \sqrt{m}r^{m-1};$$

$$(d) \text{ if } n = 2m+1, \text{ then } \prod_{i=1}^m MA_i = r^m.$$

Solution by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India.

We assume that the vertices are numbered consecutively in the counterclockwise sense and imbed the n -gon in the complex plane so that the centre is at the origin and the fixed diameter D is a subset of the real axis. If then $A_1 = r \exp i\alpha$, then

$$A_k = r \exp i(\alpha + \frac{2(k-1)\pi}{n}), \quad P_k = r \cos (\alpha + \frac{2(k-1)\pi}{n}), \quad k = 1, 2, \dots, n,$$

and

$$M = r \exp i(\alpha - \frac{\pi}{n}).$$

(a) Here we have

$$\begin{aligned} \sum_{k=1}^n A_k \vec{P}_k &= - \sum_{k=1}^n P_k \vec{A}_k = -ir \sum_{k=1}^n \sin (\alpha + \frac{2(k-1)\pi}{n}) \\ &= -ir \sin (\alpha + \frac{(n-1)\pi}{n}) \sin \pi \csc \frac{\pi}{n} = 0, \end{aligned}$$

where we have used formula (2) in [1, p. 91].

(b) First we obtain

$$\sum_{k=2}^n A_1 A_k = 2r \sum_{k=2}^n \sin \frac{(k-1)\pi}{n} = 2r \sin \frac{(n-1)\pi}{2n} \sin \frac{\pi}{2} \csc \frac{\pi}{2n}$$

$$= 2r \cos \frac{\pi}{2n} \csc \frac{\pi}{2n} = 2r \cot \frac{\pi}{2n}$$

from the same formula (2); and then

$$\begin{aligned} \prod_{k=2}^n A_1 A_k &= (2r)^{n-1} \prod_{k=2}^n \sin \frac{(k-1)\pi}{n} = (2r)^{n-1} \prod_{k=1}^n \sin \frac{k\pi}{n} \\ &= (2r)^{n-1} \cdot \frac{n}{2^{n-1}} = nr^{n-1} \end{aligned}$$

follows from formula (28) in [1, p. 119] and the fact that $\lim_{\theta \rightarrow 0} (\sin n\theta / \sin \theta) = n$.

(c) If $n = 2m$, we first obtain, from the second result in part (b),

$$\prod_{k=2}^m A_1 A_k = \sqrt{\frac{n}{\prod_{k=2}^n A_1 A_k / A_1 A_{m+1}}} = \sqrt{\frac{2mr^{2m-1}}{2r}} = \sqrt{mr}^{m-1}.$$

Next, since

$$MA_k = 2r \sin \frac{1}{2} \left\{ \alpha + \frac{2(k-1)\pi}{n} - \left(\alpha - \frac{\pi}{n} \right) \right\} = 2r \sin \frac{(2k-1)\pi}{2n}, \quad k = 1, 2, \dots, n,$$

we obtain, when $n = 2m$,

$$\prod_{k=1}^m MA_k = (2r)^m \sqrt{\frac{n}{\prod_{k=1}^n \sin \frac{(2k-1)\pi}{2n}}} = \frac{(2r)^m}{\sqrt{2^{n-1}}} = \sqrt{2r}^m$$

from formula (29) in [1, p. 119].

(d) From the same formula (29), we have, for any n ,

$$\prod_{k=1}^n MA_k = \frac{(2r)^n}{2^{n-1}};$$

hence, if $n = 2m+1$,

$$\prod_{k=1}^m MA_k = \sqrt{\frac{n}{\prod_{k=1}^n MA_k / MA_{m+1}}} = \sqrt{\frac{(2r)^n}{2^{n-1}} \cdot \frac{1}{2r}} = \sqrt{r^{n-1}} = r^m.$$

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California (part (a) only); WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

REFERENCE

1. F.W. Hobson, *A Treatise on Plane and Advanced Trigonometry*, Dover, New York, 1957.

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890, [1983: 277] *Proposed by Leroy F. Meyers, The Ohio State University.*

Construct triangle ABC, with straightedge and compass, given the lengths b and c of two sides, the midpoint M_α of the third side, and the foot H_α of the altitude to that third side.

Solution by the proposer.

If a nondegenerate triangle ABC determined by the given information is rotated 180° about the midpoint of the segment AM_α , the result is a triangle $M_\alpha QR$ in which $QR \parallel BC$ and A is the midpoint of side QR. Reflecting triangle $M_\alpha QR$ in the line AH_α produces a triangle PRQ in which Q and R have been merely interchanged, A remains the midpoint of RQ, and H_α is the midpoint of PM_α . Then

$$b = AC = M_\alpha R = PQ \quad \text{and} \quad c = AB = M_\alpha Q = PR.$$

Furthermore,

$$b + c > PM_\alpha = 2M_\alpha H_\alpha, \quad b + 2M_\alpha H_\alpha > c, \quad c + 2M_\alpha H_\alpha > b, \quad (1)$$

unless $b = c$ (and $M_\alpha = H_\alpha$), in which case $b = c > AH_\alpha$.

For the construction, if $b \neq c$ and $M_\alpha \neq H_\alpha$, and (1) holds, let P be the point on $M_\alpha H_\alpha$ produced so that $M_\alpha P = 2M_\alpha H_\alpha$, and (the notation $X(k)$ denoting the circle with center X and radius k) construct

$$Q \in M_\alpha(c) \cap P(b) \quad \text{and} \quad R \in M_\alpha(b) \cap P(c)$$

so that Q and R are on the same side of the line $M_\alpha H_\alpha$. Let L_1 be the line through M_α and H_α , and let L_2 be the line perpendicular to L_1 and passing through H_α . Let $A = L_2 \cap QR$, and let B and C be the intersections of L_1 with the lines through A which are parallel to $M_\alpha Q$ and $M_\alpha R$, respectively. There are two solution triangles in this case, symmetric with respect to the line $M_\alpha H_\alpha$.

If $b = c$ and $H_\alpha = M_\alpha$, let Q and R be any two distinct points on $H_\alpha(b)$ which are not diametrically opposite, let A be the foot of the perpendicular from H_α to (the midpoint of) QR, and let B and C be the feet of the perpendiculars from R and Q onto the diameter parallel to RQ. (This construction imitates that of the earlier case, with $P = M_\alpha$, but all four circles coincide.) There are infinitely many solutions, all of them isosceles (or equilateral).

If $b \neq c$ and $M_\alpha \neq H_\alpha$, but (1) does not hold, then the constructed circles will not intersect, or will be tangent, thus producing no triangle or a degenerate one.

If $b \neq c$ and $M_\alpha = H_\alpha$, the constructed circles (with $P = M_\alpha$) will be concentric, and so will not intersect, thus producing no triangle. If $b = c$ and $M_\alpha \neq H_\alpha$, the constructed circles will intersect (if at all) on L_2 and produce a degenerate triangle.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; CECILE M. COHEN, Horace Mann School, Bronx, N.Y.; JORDI DOU, Barcelona, Spain; JACK GARFUNKEL, Flushing, N.Y.; J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta; KESIRAJU SATY-ANARAYANA, Gagan Mahal Colony, Hyderabad, India; D.J. SMEENK, Zaltbommel, The Netherlands; DAN SOKOLOWSKY, Brooklyn, N.Y.; JORDAN B. TABOV, Sofia, Bulgaria; and DIMITRIS VATHIS, Chalcis, Greece.

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891. [1983: 312] *Proposed by Charles W. Trigg, San Diego, California.*

After the dog R A C E
A P A W
needed C A R E
using unguent from the E W E R.

Each letter in the square array uniquely represents a decimal digit. Identify the digits so that, when they replace the letters, each column and row will be a square integer.

Solution by Edwin M. Klein, University of Wisconsin-Whitewater.

Inspection of a table of squares reveals that there are only 4 four-digit squares with identical first and third digits, so

$$\{APAW, EWER\} \subset \{2025, 3136, 6561, 8281\}.$$

Since ER and RE are both endings of squares, we must have $EWER = 6561 = 81^2$. Hence

$$\begin{aligned} APAW &= APA5 = 2025 = 45^2, \\ RACE &= 12C6 = 1296 = 36^2, \\ CARE &= 9216 = 96^2. \end{aligned}$$

The unique reconstruction is therefore

1296
2025
9216
6561.

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay; the COPS of Ottawa; CLAYTON W. DODGE, University of Maine at Orono; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; ALLAN WM. JOHNSON JR., Washington, D.C.; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; JACK LESAGE, Eastview Secondary School, Barrie, Ontario; J.A. MCCALLUM, Medicine Hat, Alberta; TOM McDONOUGH and MIKE ROBERTS, students, Eastview Secondary School, Barrie, Ontario; GLEN E. MILLS, Pensacola Junior College, Florida; BOB PRIELIPP, University of Wisconsin-Oshkosh; RAM REKHA TIWARI, Radhaur, Bihar, India; W.R. UTZ, University of Missouri-Columbia; KENNETH M. WILKE, Topeka, Kansas; ANNELIESE ZIMMERMANN, Bonn, West Germany; and the proposer.

Editor's comment.

Compare with Crux 581 [1981: 254].

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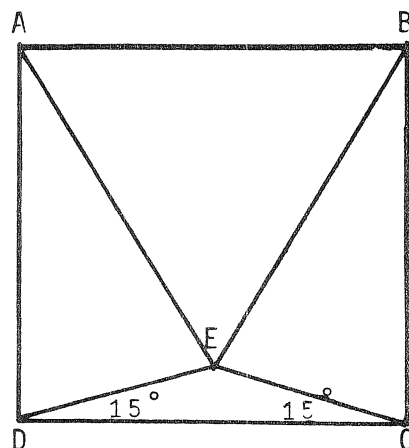
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892. [1983: 312] *Proposed by Stan Wagon, Smith College, Northampton, Massachusetts.*

ABCD is a square and ECD an isosceles triangle with base angles 15° , as shown in the figure. Prove that $\angle AFB = 60^\circ$ (and therefore triangle AFB is equilateral).

This problem is *very* well known, but all the published solutions use trigonometry and/or auxiliary lines. What is required here is a simple proof without trigonometry or any auxiliary lines (or circles).

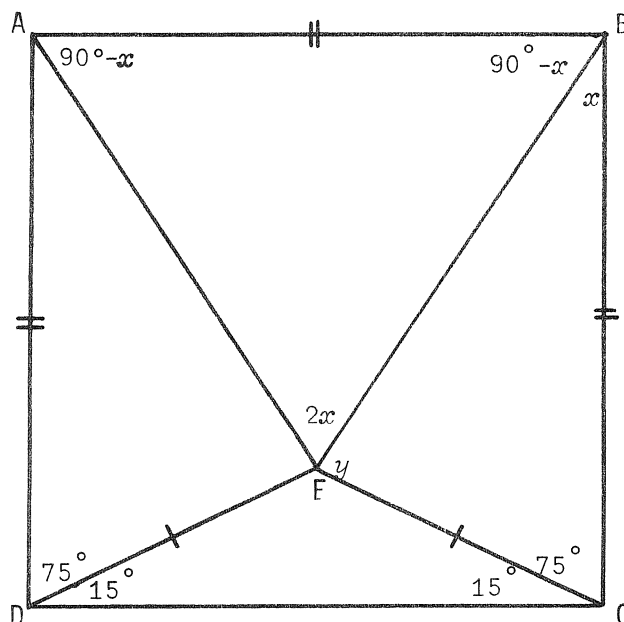


Solution by Kenneth S. Williams, Carleton University, Ottawa.

Triangles AED and BEC are congruent, so $AF = BF$. Let the degree measure of angle AEF be $2x$; then (see figure)

$$\begin{aligned} 2x \geq 60^\circ & \iff x \geq 30^\circ \\ & \iff y \leq 75^\circ \\ & \iff BC \leq BE \\ & \iff AB \leq BE \\ & \iff 2x \leq 90^\circ - x \\ & \iff 2x \leq 60^\circ. \end{aligned}$$

Therefore $\angle AFB = 2x = 60^\circ$.



Also solved by ELWYN ADAMS, Gainesville, Florida; HAYO AHLBURG, Benidorm, Alicante, Spain; LEON BANKOFF, Los Angeles, California; PAUL R. BEESACK, Carleton University, Ottawa; J.L. BRENNER, Palo Alto, California and HENRY E. FETTIS, Mountain View, California (jointly); the COPS of Ottawa; JORDI DOU, Barcelona, Spain; HENRY E. FETTIS, Mountain View, California (second solution); J.T. GROENMAN, Arnhem, The Netherlands; F.D. HAMMER, Palo Alto, California; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LEROY F. MEYERS, The Ohio State University; DONALD L. MURCH, St. John Fisher College, Rochester, N.Y.; DAN PEDOE, University of Minnesota; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; N. NARASIMHA RAO, Madras Institute of Technology, India; MALCOLM A. SMITH, Georgia Southern College, Statesboro; DAN SOKOLOWSKY, Brooklyn, N.Y.; GEORGE TSINTSIFAS, Thessaloniki, Greece; KENNETH M. WILKE, Topeka, Kansas; and the proposer. A comment was received from DIMITRIS VATHIS, Chalcis, Greece.

Editor's comment.

Vathis and Wilke noted that there *is* a published proof without trigonometry or auxiliary lines [1]. It is very nice, but we prefer our featured solution.

Most solvers adhered strictly to the rules of the game, but a few used statements of the type "It is clear that..." whose *proofs* would require auxiliary lines, which they virtuously did not draw.

REFERENCE

1. H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited*, New Mathematical Library No. 19, Mathematical Association of America, 1967, pp. 25, 158.

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893, [1983: 312] *Proposed by G.P. Henderson, Campbellcroft, Ontario.*

Let C be the centre of the ellipse

$$a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 + 2a_{13}x_1 + 2a_{23}x_2 + a_{33} = 0,$$

and let $P_i(x_{1i}, x_{2i})$, $i = 1, 2$, be two points on the ellipse. Find the area of the smaller of the regions bounded by CP_1 , CP_2 , and the ellipse.

Solution by the proposer.

Suppose the coordinate axes have been translated and rotated so that the equation of the ellipse is

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1$$

and let (u_i, v_i) be the new coordinates of P_i . Set $w = av/b$. In the uw -plane the image of the region CP_1P_2 is a sector of the circle $u^2 + w^2 = a^2$ bounded by the radii $C'P'_1$ and $C'P'_2$, where the coordinates of P'_i are $(u_i, av_i/b)$. The area of the sector is $\frac{1}{2}a^2\theta$, where θ is the smaller of the angles between $C'P'_1$ and $C'P'_2$. The scalar product of $C'P'_1$ and $C'P'_2$ yields

$$a^2\cos \theta = u_1u_2 + a^2v_1v_2/b^2.$$

Therefore

$$\theta = \text{Arccos}(u_1u_2/a^2 + v_1v_2/b^2).$$

The area of the sector is then $\frac{1}{2}a^2\text{Arccos}(u_1u_2/a^2 + v_1v_2/b^2)$ and the area of the region CP_1P_2 is

$$\frac{1}{2}ab \text{Arccos}(u_1u_2/a^2 + v_1v_2/b^2). \quad (1)$$

We now return to the original variables. Set

$$X_1^T = (x_{11}, x_{21}, 1), \quad X_2 = \begin{pmatrix} x_{12} \\ x_{22} \\ 1 \end{pmatrix},$$

$$A = (a_{ij}) = (a_{ji}),$$

$$\Delta = \det A, \quad \delta = a_{11}a_{22} - a_{12}^2 (> 0).$$

When we change to the uv axes, the equation has the form

$$\alpha'_{11}u^2 + \alpha'_{22}v^2 + \alpha'_{33} = 0, \quad (2)$$

where

$$\alpha'_{33} = \frac{\Delta}{\delta}, \quad \alpha'_{11} = -\frac{\alpha'_{33}}{a^2}, \quad \alpha'_{22} = -\frac{\alpha'_{33}}{b^2}.$$

The invariant δ can now be calculated from (2):

$$\delta = \alpha'_{11}\alpha'_{22} = \frac{\alpha'^2_{33}}{a^2b^2}.$$

Hence

$$ab = \frac{|\alpha'_{33}|}{\sqrt{\delta}} = |\Delta|\delta^{-3/2}.$$

We use (2) again to calculate the invariant $X_1^TAX_2$:

$$X_1^TAX_2 = \alpha'_{11}u_1u_2 + \alpha'_{22}v_1v_2 + \alpha'_{33} = \alpha'_{33}\left(1 - \frac{u_1u_2}{a^2} - \frac{v_1v_2}{b^2}\right).$$

Hence

$$\frac{u_1u_2}{a^2} + \frac{v_1v_2}{b^2} = 1 - \frac{X_1^TAX_2}{\alpha'_{33}} = 1 - \frac{\delta}{\Delta}X_1^TAX_2.$$

Making these changes in (1), we get the expression

$$\frac{1}{2}|\Delta|\delta^{-3/2}\text{Arccos}\left(1 - \frac{\delta}{\Delta}X_1^TAX_2\right)$$

for the required area.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and KESTIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India.

Editor's comment.

The other solvers showed in various ways how the required area could be calculated, but they did not in the end arrive at an explicit formula in terms of the given α_{ij} and p_i .

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894. [1983: 313] Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.

(a) Find necessary and sufficient conditions on the complex numbers a, b, ω so that the roots of

$$z^2 + 2az + b = 0 \quad \text{and} \quad z - \omega = 0$$

shall be collinear in the complex plane.

(b) Find necessary and sufficient conditions on the complex numbers a, b, c, d so that the roots of

$$z^2 + 2az + b = 0 \quad \text{and} \quad z^2 + 2cz + d = 0$$

shall all be collinear in the complex plane.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria,

(a) The roots of the two equations are

$$\omega, \quad z_1 = -a + \Delta_1, \quad z_2 = -a - \Delta_1, \quad (1)$$

where Δ_1 is (a fixed) one of the determinations of $\sqrt{a^2 - b}$. They are collinear if and only if

$$\begin{vmatrix} \omega & \bar{\omega} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0,$$

In terms of (1), this last equation is found to be equivalent to

$$(\omega + a)\bar{\Delta}_1 = (\overline{\omega + a})\Delta_1,$$

or to the requirement that $(\overline{\omega + a})\Delta_1$ be real.

(b) The roots of the second equation are

$$\omega_1 = -c + \Delta_2 \quad \text{and} \quad \omega_2 = -c - \Delta_2,$$

where Δ_2 is (a fixed) one of the determinations of $\sqrt{c^2 - d}$. It now follows from part

(a) that $z_1, z_2, \omega_1, \omega_2$ are all collinear if and only if

$$(\overline{\omega_1 + a})\Delta_1 \quad \text{and} \quad (\overline{\omega_2 + a})\Delta_1$$

are both real, that is, if and only if

$$(\overline{a - c + \Delta_2})\Delta_1 \quad \text{and} \quad (\overline{a - c - \Delta_2})\Delta_1$$

are both real.

Also solved by G.P. HENDERSON, Campbellcroft, Ontario; LEROY F. MEYERS, The Ohio State University; BASIL C. RENNIE, James Cook University of North Queensland, Australia; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and JORDAN B. TABOV, Sofia, Bulgaria.

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895. [1983: 313] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let ABC be a triangle with sides a, b, c in the usual order and circum-circle Γ . A line l through C meets the segment AB in D, Γ again in E, and the perpendicular bisector of AB in F. Assume that $c = 3b$.

(a) Construct the line l for which the length of DF is maximal.

(b) If DE has maximal length, prove that $DF = FE$.

(c) If DE has maximal length and also $CD = DF$, find a in terms of b and the measure of angle A.

I. *Solution to part (a) by the proposer.*

Let $\overline{AD} = \lambda c$, where $0 \leq \lambda \leq 1$; then $\overline{DB} = (1-\lambda)c$ and Stewart's Theorem gives

$$\overline{CD}^2 \cdot c = a^2 \lambda c + b^2 (1-\lambda) c - \lambda (1-\lambda) c^3,$$

from which

$$\overline{CD}^2 = a^2 \lambda + b^2 (1-\lambda) - c^2 \lambda (1-\lambda).$$

Moreover, $\overline{CD}^2 \cdot \overline{DE}^2 = \overline{AD}^2 \cdot \overline{DB}^2 = c^4 \lambda^2 (1-\lambda)^2$, so

$$\overline{DE}^2 = \frac{c^4 \lambda^2 (1-\lambda)^2}{a^2 \lambda + b^2 (1-\lambda) - c^2 \lambda (1-\lambda)}$$

and it suffices to maximize

$$f(\lambda) = \frac{\lambda^2 (1-\lambda)^2}{a^2 \lambda + b^2 (1-\lambda) - c^2 \lambda (1-\lambda)}.$$

It will be found that $f'(\lambda) = 0$ is equivalent to $\lambda(\lambda-1)g(\lambda) = 0$, where

$$g(\lambda) = a^2 \lambda (3\lambda-1) - b^2 (\lambda-1)(3\lambda-2) + c^2 \lambda (\lambda-1)(2\lambda-1). \quad (1)$$

Since $f'(\lambda) \geq 0$ and $f(0) = f(1) = 0$, the function attains its maximum value for some $\lambda \in (0,1)$ for which $g(\lambda) = 0$.

For $c = 3b$, (1) becomes

$$g(\lambda) = (3\lambda-1)h(\lambda),$$

where

$$h(\lambda) = a^2 \lambda + 2b^2 (\lambda-1)(3\lambda-1).$$

It is clear that $h(\lambda) > 0$ if $0 < \lambda \leq 1/3$; and consideration of the minimum value of $(\lambda-1)(3\lambda-1)$, which occurs when $\lambda = 2/3$, shows that when $1/3 < \lambda < 1$ we have

$$h(\lambda) \geq \frac{a^2 - 2b^2}{3} > \frac{2b^2}{3} > 0,$$

since $a > b = 2b < a$. Thus $h(\lambda)$ vanishes for no $\lambda \in (0,1)$ and hence the maximum value of $f(\lambda)$ occurs when $\lambda = 1/3$, the only value for which $g(\lambda)$ vanishes.

The point D and the line ℓ are now easily constructed.

II. *Comment on part (b) by Gali Salvatore, Perkins, Québec.*

It is known that DE has maximal length if and only if $DF = FE$ (whether or not $c = 3b$). See Problem 110 [1976: 84-88], where an analytic proof by H.G. Dworschak and a synthetic proof by Léo Sauvé are given. Sauvé also showed that the line ℓ is

not in general constructible by Euclidean means, although it may be in special cases (for example, when $c = 3b$).

III. *Solution to part (c) by Jordi Dou, Barcelona, Spain.*

Let C' be the foot of the altitude from C and C_1 the midpoint of AB . If $CD = DF$, then $C'D = DC_1 = \frac{1}{2}b = AC'$, and so $AC = CD$. Since also $AC = AD = b$, triangle ACD is equilateral and angle $A = 60^\circ$. Now

$$a^2 = b^2 + (3b)^2 - 2b(3b) \cos 60^\circ = 7b^2,$$

so $a = \sqrt{7}b$.

Also solved by JORDI DOU, Barcelona, Spain (also parts (a) and (b)); KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India (partial solution); D.J. SMEENK, Zaltbommel, The Netherlands; MALCOLM A. SMITH, Georgia Southern College, Statesboro; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer (also parts (b) and (c)).

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896. [1983: 313] *Proposed by Jack Garfunkel, Flushing, N.Y.*

Consider the inequalities

$$\sum \sin^2 \frac{A}{2} \geq 1 - \frac{1}{4} \prod \cos \frac{B-C}{2} \geq \frac{3}{4},$$

where the sum and product are cyclic over the angles A, B, C of a triangle. The inequality between the second and third members is obvious, and that between the first and third members is well known. Prove the sharper inequality between the first two members.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria and the proposer (independently).

The following identity is easily established (or see Hall and Knight's *Elementary Trigonometry*, p. 344, Ex. 68):

$$\sum \sin^2 \frac{A}{2} = 1 - 2 \prod \sin \frac{A}{2}.$$

Hence the inequality to be proved is equivalent to

$$\prod \cos \frac{B-C}{2} \geq 8 \prod \sin \frac{A}{2},$$

an inequality already established in Crux 585 [1981: 303].

Also solved by LEON BANKOFF, Los Angeles, California; W.J. BLUNDON, Memorial University of Newfoundland; J.T. GROENMAN, Arnhem, The Netherlands; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; BOB PRIELIPP, University of Wisconsin-Oshkosh; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and GEORGE TSINTSIFAS, Thessaloniki, Greece.

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897, [1983: 313] *Proposed by Vedula N. Murty, Pennsylvania State University, Capitol Campus.*

If $\lambda > \mu$ and $a \geq b \geq c > 0$, prove that

$$b^{2\lambda}c^{2\mu} + c^{2\lambda}a^{2\mu} + a^{2\lambda}b^{2\mu} \geq (bc)^{\lambda+\mu} + (ca)^{\lambda+\mu} + (ab)^{\lambda+\mu},$$

with equality just when $a = b = c$.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

If we divide throughout by $c^{2\lambda+2\mu}$ and set $\alpha = a/c$, $\beta = b/c$, the proposed inequality is equivalent to

$$f(\alpha) \equiv \beta^{2\lambda} + \alpha^{2\mu} + \alpha^{2\lambda}\beta^{2\mu} - \beta^{\lambda+\mu} - \alpha^{\lambda+\mu} - (\alpha\beta)^{\lambda+\mu} \geq 0, \quad \alpha \geq \beta \geq 1.$$

Differentiating, we obtain

$$f'(\alpha) = \alpha^{2\mu-1}g(\alpha),$$

where

$$g(\alpha) \equiv 2\mu + 2\lambda\alpha^{2\lambda-2\mu}\beta^{2\mu} - (\lambda+\mu)\alpha^{\lambda-\mu} - (\lambda+\mu)\alpha^{\lambda-\mu}\beta^{\lambda+\mu}.$$

Then

$$g'(\alpha) = (\lambda-\mu)\alpha^{\lambda-\mu-1}h(\alpha),$$

where

$$h(\alpha) \equiv 4\lambda\alpha^{\lambda-\mu}\beta^{2\mu} - (\lambda+\mu)(1 + \beta^{\lambda+\mu}).$$

Since $\lambda-\mu > 0$ and $\alpha \geq \beta \geq 1$, $h(\alpha)$ increases with α and so

$$\begin{aligned} h(\alpha) \geq h(\beta) &= 4\lambda\beta^{\lambda+\mu} - (\lambda+\mu)(1+\beta^{\lambda+\mu}) \geq 4\lambda\beta^{\lambda+\mu} - (\lambda+\mu) \cdot 2\beta^{\lambda+\mu} \\ &= 2(\lambda-\mu)\beta^{\lambda+\mu} > 0. \end{aligned}$$

Thus $g'(\alpha) > 0$, and hence

$$g(\alpha) \geq g(\beta) = 2\mu + (\lambda-\mu)\beta^{2\lambda} - (\lambda+\mu)\beta^{\lambda-\mu} \equiv k(\beta), \quad \beta \geq 1.$$

Now

$$k'(\beta) = (\lambda-\mu)\beta^{\lambda-\mu-1}\{2\lambda\beta^{\lambda+\mu} - (\lambda+\mu)\} \geq (\lambda-\mu)\beta^{\lambda-\mu-1}\{2\lambda - (\lambda+\mu)\} = (\lambda-\mu)2\beta^{\lambda-\mu-1} > 0.$$

Therefore $k(\beta) > k(1) = 0$, so $g(\alpha) \geq 0$, then $f'(\alpha) \geq 0$, and finally

$$f(\alpha) \geq f(\beta) = \beta^{2\lambda} - 2\beta^{\lambda+\mu} + \beta^{2\mu} = (\beta^{\lambda}-\beta^{\mu})^2 \geq 0,$$

as required.

Equality occurs just when $\alpha = \beta = 1$, that is, just when $a = b = c$.

Also solved by KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer.

Editor's comment.

The proposer noted that many interesting inequalities, some well-known and some not, are equivalent to special cases of our problem. For example, $\lambda = 1$ and $\mu = 0$ give

$$a^2 + b^2 + c^2 \geq bc + ca + ab;$$

$\lambda = 1/2$ and $\mu = -1/2$ give

$$\frac{b}{c} + \frac{c}{a} + \frac{a}{b} \geq 3;$$

$\lambda = 3/2$ and $\mu = 1/2$ give

$$b^2c(b-c) + c^2a(c-a) + a^2b(a-b) \geq 0,$$

an inequality given at the 1983 International Mathematical Olympiad [1983: 207; 1984: 73]; and for good measure we give (why not?)

$$b^{2\pi}c^{2e} + c^{2\pi}a^{2e} + a^{2\pi}b^{2e} \geq (bc)^{\pi+e} + (ca)^{\pi+e} + (ab)^{\pi+e},$$

which readers are invited to challenge their friends to prove directly (if they are not *CruX* subscribers).

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898. [1983: 313] *Proposed by S.C. Chan, Singapore.*

A fair coin is tossed n times. Let T_n be the number of times in the n tosses that a tail is followed by a head. Find (a) the expectation of T_n , (b) the variance of T_n .

Solution by G.P. Henderson, Campbellcroft, Ontario.

More generally, let the probability of a tail be p , where $0 \leq p \leq 1$, and set $x_r = 0$ or 1 according as the r th toss is head or tail. We then have the expectations

$$E(x_r) = E(x_r^2) = p$$

and if $r \neq s$, since x_r and x_s are independent,

$$E(x_r x_s) = p^2.$$

(a) Since

$$T_n = \sum_{r=1}^{n-1} x_r (1 - x_{r+1}),$$

we therefore have

$$E(T_n) = \sum_{r=1}^{n-1} \{E(x_r) - E(x_r x_{r+1})\} = (n-1)(p-p^2), \quad n = 1, 2, 3, \dots \quad (1)$$

In particular, if the coin is fair then $p = \frac{1}{2}$ and

$$E(T_n) = \frac{n-1}{4}.$$

(b) The required variance is

$$\text{Var}(T_n) = E(T_n^2) - \{E(T_n)\}^2. \quad (2)$$

Now

$$\begin{aligned} T_n^2 &= (\sum x_r - \sum x_r x_{r+1})(\sum x_s - \sum x_s x_{s+1}) \\ &= \sum \sum x_r x_s - 2 \sum \sum x_r x_{r+1} x_s + \sum \sum x_r x_{r+1} x_s x_{s+1}; \end{aligned}$$

so to evaluate $E(T_n^2)$ we need

$$\begin{aligned} E(\sum \sum x_r x_s) &= E(\sum_{r \neq s} x_r x_s) + E(\sum_r x_r^2) = (n-1)(n-2)p^2 + (n-1)p, \\ E(\sum \sum x_r x_{r+1} x_s) &= E(\sum_{r \neq s, s-1} x_r x_{r+1} x_s) + E(\sum_{r=1}^{n-1} x_r^2 x_{r+1}) + E(\sum_{r=1}^{n-2} x_r x_{r+1}^2) \\ &= (n-2)^2 p^3 + (2n-3)p^2, \end{aligned}$$

and

$$E(\sum \sum x_r x_{r+1} x_s x_{s+1}) = (n-2)(n-3)p^4 + 2(n-2)p^3 + (n-1)p^2.$$

Hence

$$E(T_n^2) = (n^2-5n+6)p^4 - 2(n^2-5n+6)p^3 + (n^2-6n+7)p^2 + (n-1)p. \quad (3)$$

Finally, substituting (3) and (1) into (2) gives

$$\text{Var}(T_n) = (n-1)p(1-p) - (3n-5)p^2(1-p)^2, \quad n = 2, 3, 4, \dots,$$

and of course $\text{Var}(T_1) = 0$. In particular, if $p = \frac{1}{2}$ then, for $n = 2, 3, 4, \dots$,

$$\text{Var}(T_n) = \frac{n+1}{16}.$$

Also solved by CURTIS COOPER, Central Missouri State University at Warrensburg; RICHARD I. HESS, Rancho Palos Verdes, California; EDWIN M. KLEIN, University of Wisconsin-Whitewater; LEROY F. MEYERS, The Ohio State University; and BASIL C. RENNIE, James Cook University of North Queensland, Australia.

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899, [1983: 314] Proposed by Loren C. Larson, St. Olaf College, Northfield, Minnesota.

Let $\{a_i\}$ and $\{b_i\}$, $i = 1, 2, \dots, n$, be two sequences of real numbers with the a_i all positive. Prove that

$$\sum_{i \neq j} a_i b_j = 0 \implies \sum_{i \neq j} b_i b_j \leq 0.$$

Solution by Geng-zhe Chang, University of Science and Technology, Hefei, Anhui, People's Republic of China.

We weaken the condition $a_i > 0$ to

$$a_i \geq 0 \quad \text{and} \quad \Sigma a_i > 0. \quad (1)$$

First we note that

$$\sum_{i \neq j} a_i b_j = 0 \implies (\Sigma a_i)(\Sigma b_j) = \Sigma a_j b_j \implies \Sigma b_j = \frac{\Sigma a_j b_j}{\Sigma a_i}.$$

Now

$$\sum_{i \neq j} b_i b_j + \Sigma b_j^2 = (\Sigma b_j)^2 = \frac{(\Sigma a_j b_j)^2}{(\Sigma a_i)^2} \leq \frac{(\Sigma a_i^2)(\Sigma b_j^2)}{(\Sigma a_i)^2} \leq \Sigma b_j^2, \quad (2)$$

where the first inequality in (2) is due to that of Cauchy-Schwarz and the second follows from (1). Hence

$$\sum_{i \neq j} b_i b_j \leq 0, \quad (3)$$

as required.

Equality holds throughout in (2), and hence in (3), if and only if either $\Sigma b_j^2 = 0$ or

$$a_i = k b_i \quad \text{and} \quad \Sigma a_i^2 = (\Sigma a_i)^2. \quad (4)$$

But if all $a_i > 0$, as in the proposal, then the second condition (4) is never satisfied, and equality holds in (3) just when $\Sigma b_j^2 = 0$, that is, just when each $b_j = 0$.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; G.P. HENDERSON, Campbellcroft, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; EDWIN M. KLEIN, University of Wisconsin-Whitewater; LEROY F. MEYERS, The Ohio State University; DONALD L. MUENCH, St. John Fisher College, Rochester, N.Y.; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; JORDAN B. TABOV, Sofia, Bulgaria; GEORGE TSINTSIFAS, Thessaloniki, Greece; KENNETH S. WILLIAMS, Carleton University, Ottawa; and the proposer.

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900. [1983: 314] *Proposed by W.R. Utz, University of Texas at Austin.*

Show that there are an infinite number of sets of three integers in arithmetic progression such that the sum of the square of the first, twice the square of the second, and three times the square of the third is a square.

Solution by Hayo Ahlburg, Benidorm, Alicante, Spain.

More generally, for a given positive integer t we find an infinite set of integer solutions to the Diophantine equation

$$(a+d)^2 + 2(a+2d)^2 + 3(a+3d)^2 + \dots + t(a+td)^2 = (b + \frac{t(t+1)}{2} \cdot d)^2. \quad (1)$$

Since $1^3+2^3+\dots+t^3 = t^2(t+1)^2/4$, it is seen from (1) that $a = 0$ if and only if $b = 0$, and one infinite set of integer solutions is given by

$$(a, b, d) = (0, 0, d), d \text{ an arbitrary integer.} \quad (2)$$

For $t = 3$, the infinite set $\{(d, 2d, 3d) | d \text{ any integer}\}$ therefore completely satisfies the very modest requirements of the proposal.

We will go further. We set $t = 3$ in (1) and completely solve the resulting equation

$$(a+d)^2 + 2(a+2d)^2 + 3(a+3d)^2 = (b+6d)^2. \quad (3)$$

In view of (2), we have left to find only the solutions for which $ab \neq 0$.

Accordingly, let (a, b, d) be an integer solution of (3) with $ab \neq 0$. Expanding the squares in (3) shows, first, that b is even, say $b = 2b'$; then that a is even, say $a = 2a'$; and finally that b' is even, say $b' = 2b''$. Equation (3) then turns out to be equivalent to

$$3a'^2 - 2b''^2 = (6b'' - 7a')d. \quad (4)$$

Since $3a'^2 - 2b''^2 \neq 0$ (otherwise $\sqrt{3/2}$ would be rational), it follows from (4) that there are uniquely determined relatively prime nonzero integers m and n , with $n > 0$, such that

$$\frac{b''}{a'} = \frac{m}{n} \neq \frac{7}{6}.$$

Now (4) is equivalent to $(3n^2 - 2m^2)a' = n(6m - 7n)d$, or, since $a = 2a'$, to

$$(3n^2 - 2m^2)a = 2n(6m - 7n)d,$$

or, since $b = 4b'' = 4(m/n)a' = 2(m/n)a$, to

$$(3n^2 - 2m^2)b = 4m(6m - 7n)d.$$

Thus

$$\frac{a}{2n(6m-7n)} = \frac{b}{4m(6m-7n)} = \frac{d}{3n^2-2m^2} = \frac{r}{s}, \text{ say,} \quad (5)$$

where s is the greatest common divisor of $2n(6m-7n)$, $4m(6m-7n)$, and $3n^2-2m^2$, and the nonzero integer r is then uniquely determined by (5). Therefore

$$(a, b, d) = (2n(6m-7n) \cdot \frac{r}{s}, 4m(6m-7n) \cdot \frac{r}{s}, (3n^2-2m^2) \cdot \frac{r}{s}), \quad (6)$$

where m, n, r, s are all uniquely determined nonzero integers.

At the risk of obganiating [1], we repeat that in (6) the integers m, n, r, s have the following properties: *each is nonzero; m and n are relatively prime, $n > 0$, and $m/n \neq 7/6$; and s is the greatest common divisor of $2n(6m-7n)$, $4m(6m-7n)$, and $3n^2-2m^2$.*

Conversely, let m, n, r, s be any four integers satisfying the italicized conditions. (Since s is uniquely determined by m and n , there are only three independent parameters: m , n , and r .) If we substitute from (6) into (3), we obtain

$$\begin{aligned} & \{(-2m^2+12mn-11n^2) \cdot \frac{r}{s}\}^2 + 2\{(-4m^2+12mn-8n^2) \cdot \frac{r}{s}\}^2 + 3\{(-6m^2+12mn-5n^2) \cdot \frac{r}{s}\}^2 \\ & = \{(12m^2-28mn+18n^2) \cdot \frac{r}{s}\}^2, \end{aligned}$$

and it is easily verified that this equation is true for all allowable values of m , n, r .

The complete solution of (3) therefore consists of the one-parameter family (2) and the three-parameter family (6); and the one-parameter family can be subsumed into the three-parameter family by allowing $r = 0$, and allowing $m/n = 7/6$ for all r .

We end with a few examples of one-parameter families of solutions.

| m | n | s | | | | | |
|-----|-----|-----|-------------|----------------|----------------|--------------|-----|
| 1 | 1 | 1 | $(-r)^2$ | $+ 2(0)^2$ | $+ 3(r)^2$ | $= (2r)^2$ | (7) |
| 7 | 6 | 10 | $(r)^2$ | $+ 2(2r)^2$ | $+ 3(3r)^2$ | $= (6r)^2$ | |
| 5 | 4 | 2 | $(7r)^2$ | $+ 2(6r)^2$ | $+ 3(5r)^2$ | $= (14r)^2$ | |
| 6 | 5 | 1 | $(13r)^2$ | $+ 2(16r)^2$ | $+ 3(19r)^2$ | $= (42r)^2$ | |
| 8 | 7 | 1 | $(5r)^2$ | $+ 2(24r)^2$ | $+ 3(43r)^2$ | $= (82r)^2$ | |
| 9 | 8 | 2 | $(-r)^2$ | $+ 2(14r)^2$ | $+ 3(29r)^2$ | $= (54r)^2$ | |
| -9 | 8 | 10 | $(-173r)^2$ | $+ 2(-170r)^2$ | $+ 3(-167r)^2$ | $= (414r)^2$ | |

There were 23 other solvers including the proposer (names omitted for lack of space).

Editor's comment.

The proposer noted that this problem generalizes one of Gregory Wulczyn [2]. If ever a problem cried out for a complete solution this one did. Yet fifteen of the other solvers, perhaps on the grounds that problems should be seen and not heard, gave only one or both of the trivial solutions (7).

REFERENCES

1. To obganiate: to irritate with reiteration. See *Mrs. Byrne's Dictionary of Unusual, Obscure, and Preposterous Words* (Washington Square Press, 1984), by Josefa Heifetz Byrne, daughter of the famed violinist Jascha Heifetz.
2. Gregory Wulczyn, (Proposer of) Problem 844, *Mathematics Magazine*, 46 (1973) 113-114.

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