Crux Mathematicorum

45: No 1 JANUARY / JANVIER 2019

Crux Mathematicorum is a problem-solving journal at the secondary and university undergraduate levels, published online by the Canadian Mathematical Society. Its aim is primarily educational; it is not a research journal. Online submission:

`https://publications.cms.math.ca/cruxbox/`

Crux Mathematicorum est une publication de résolution de problèmes de niveau secondaire et de premier cycle universitaire publiée par la Société mathématique du Canada. Principalement de nature éducative, le Crux n'est pas une revue scientifique. Soumission en ligne:

`https://publications.cms.math.ca/cruxbox/`

ISSN 1496-4309 (Online)

ISSN 1496-4309 (électronique)

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IN THIS ISSUE / DANS CE NUMÉRO

Crux Mathematicorum

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Crux Mathematicorum with Mathematical Mayhem

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EDITORIAL

Thanks to the financial support from the Intact Foundation, Volume 45 begins a new era for ***Crux*** as an open access online publication. With this new environment, we will be making changes to some of our existing sections and welcoming new initiatives.

The biggest change in the publication is the introduction of a new section called *MathemAttic*. This section is geared towards secondary school students: they will find it full of hidden mathematical treasures, while our more experienced problem solvers will see some of their old favourites. The problems in this section will replace the Contest Corner and be supplemented by articles and various other features under the name of *Problem Solving Vignettes*. Our longtime followers will find that *MathemAttic* will resemble the *Mathematical Mayhem* section of the journal that has been absent in recent years. *MathemAttic* is a shared project of contributing editors Shawn Godin and John McLoughlin, who will be supported by Kelly Paton as the main problem editor. We are truly excited to be bringing these materials into ***Crux*** and I hope that our readers will share our excitement by encouraging submissions from young mathematicians and teachers of junior and senior high school students.

Our second biggest change will be in the timelines. As an online publication, we will be tightening our deadlines to allow for problem solutions to appear closer to initial publication. With *MathemAttic* and the *Olympiad Corner*, we will be publishing solutions 5 issues after publishing problems. Here, I would like to thank Anamaria Savu for joining Alessandro Ventullo as *Olympiad Corner* editor and sharing the workload. With *Problems* sections, the solution to a problem will also be published 5 issues after its appearance. This will of course mean closer deadlines for solution submissions, so please watch out for those. With a couple of issues of 2019 a bit behind, we plan to quickly catch up and be on regular schedule in a couple of issues.

Finally, you will now see more colour and live links on our pages. We will also be looking for ways to introduce more dynamic content into ***Crux*** . For this Volume, the *Author Index* will appear as a standalone dynamic document online that will get updated with new solvers for each issue.

As I am always fascinated by the different places we receive correspondence from, a new short feature called *Snapshot* will showcase pictures of places where our contributors live, work, and do math. So if you would like to share your photos, please send them along.

If you have any ideas, suggestions or feedback (positive or constructive), please email me at `crux.eic@gmail.com`. I love hearing from you.

Kseniya Garaschuk

MathemAttic

No. 1

The problems featured in this section are intended for students at the secondary school level.

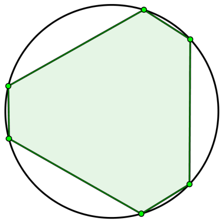
Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **April 15, 2019**.

MA1. How many two-digit numbers are there such that the difference of the number and the number with the digits reversed is a non-zero perfect square? Problem extension: What happens with three-digit numbers? four-digit numbers?

MA2. A sequence $t_1, t_2, \ldots$ beginning with any two positive numbers is defined such that for $n > 2$, $t_n = \frac{1+t_{n-1}}{t_{n-2}}$. Show that such a sequence must repeat itself with a period of 5.

MA3. A hexagon H is inscribed in a circle, and consists of three segments of length 1 and three segments of length 3. Find the area of H.



MA4. For what conditions on a and b is the line $x + y = a$ tangent to the circle $x^2 + y^2 = b$?

MA5. Point P lies in the first quadrant on the line $y = 2x$. Point Q is a point on the line $y = 3x$ such that PQ has length 5 and is perpendicular to the line $y = 2x$. Find the point P.

. . . .

Les problémes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

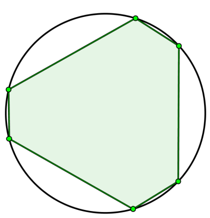
Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 avril 2019**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

MA1. Déterminer combien de nombres à deux chiffres sont tels que la différence entre le nombre et celui avec les chiffres renversés est un carré parfait. Généralisation du problème : qu'en est-il des nombres à trois chiffres ou à quatre chiffres ?

MA2. Une suite $t_1, t_2, \ldots$ débutant avec deux entiers positifs est telle que pour $n > 2$, on a $t_n = \frac{1+t_{n-1}}{t_{n-2}}$. Démontrer qu'une telle suite doit se répéter avec période 5.

MA3. Un hexagone H, dont trois côtés sont de longueur 1 et les trois autres de longueur 3, est inscrit dans un cercle. Déterminer la surface de cet hexagone.



MA4. Quelles conditions doit-on imposer à a et b de façon à ce que la ligne $x + y = a$ soit tangente au cercle $x^2 + y^2 = b$?

MA5. Un point P se situe dans le premier quadrant, sur la ligne $y = 2x$. Le point Q se situe sur la ligne $y = 3x$ et est tel que PQ est de longueur 5 et est perpendiculaire à la ligne $y = 2x$. Déterminer P.

CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2018: 44(1), p. 4–5; and 44(2), p. 47–48.

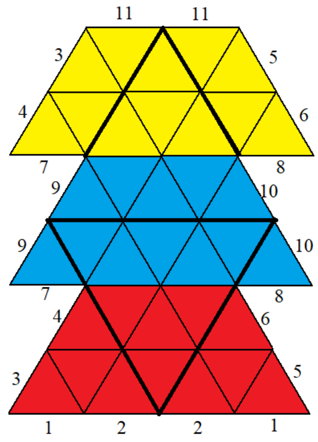
CC301. All natural numbers are coloured using 100 different colours. Prove that you can find several (no less than 2) different numbers, all of the same colour, that have a product with exactly 1000 different natural divisors.

Originally 2017 Savin Open Contest, Problem 7 (by E. Bakaev).

We received two solutions. We present the solution by Richard Hess.

Consider the set of numbers $p_1^9, p_2^9, \ldots, p_n^9$, where each p_i is a distinct prime and $n \geq 201$. If we colour each number in this set with any of 100 colours, then by the pigeonhole principle there will be at least three numbers with the same colour. The product of the three numbers has exactly 1000 natural divisors.

CC302. Nikolas used construction paper to make a regular tetrahedron (a pyramid consisting of equilateral triangles). Then he cut it in some ingenious way, unfolded it and this resulted in a Christmas tree-like shape consisting of three halves of a regular hexagon. How did Nikolas do this?



Originally 2017 Savin Open Contest, Problem 15 (by A. Domashenko).

We received one correct submission. We present the solution by the Missouri State University Problem Solving Group.

Folding the figure above along the heavily shaded lines and identifying edges labeled with the same number results in a regular tetrahedron.

CC303. Consider two convex polygons M and N with the following properties: polygon M has twice as many acute angles as polygon N has obtuse angles; polygon N has twice as many acute angles as polygon M has obtuse angles; each polygon has at least one acute angle; at least one of the polygons has a right angle.

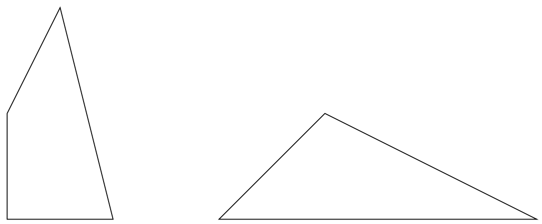
a) Give an example of such polygons.

b) How many right angles can each of these polygons have? Find the complete set of all the possibilities and prove that no others exist.

Originally 2017 Savin Open Contest, Problem 10 (by I. Akulich).

We received two solutions and present the slightly edited solution by Richard Hess.

If a convex polygon has at least one acute angle and the number of acute angles is even, then it has exactly two acute angles. Suppose it had four acute angles. Then each supplementary angle is greater than $90°$, and thus their sum is greater than $360°$, which is impossible for a convex polygon. Thus, M and N have each two acute angles and one obtuse angle. If either of M and N had two right angles, then the sum of the supplementary angles of the right angles and the acute angles would again be greater than $360°$, which is impossible. We are left with two possibilities: quadrilaterals with one right angle, one obtuse angle and two acute angles, and triangles with one obtuse angle and two acute angles. M and N can be two quadrilaterals or one quadrilateral and one triangle.



CC304. Consider a natural number n greater than 1 and not divisible by 10. Can the last digit of n and second last digit of n^4 (that is, the digit in the tens position) be of the same parity?

Originally 2017 Savin Open Contest, Problem 1 (by S. Dvoryaninov).

We received 5 submissions, all of which were correct and complete. We present the solution by David Manes.

The answer is no. We begin by determining the last digit of n^2. Assume the digits of n are abc so that $n = 100a + 10b + c$, a is any non-negative integer, $c \neq 0$ since n is not divisible by 10. Then

$$\begin{aligned} n^2 = (100a + 10b + c)^2 &= 100(100a^2 + b^2 + 20ab + 2ac) + (20bc + c^2) \\ &\equiv 20bc + c^2 \pmod{100}. \end{aligned}$$

Consequently, the last two digits of n^2 are determined by the least residue modulo 100 of $20bc + c^2$. Note that a, the hundredths digit of n, has no bearing on the last two digits of n^2. More precisely, the last two digits of n^2 are the digits of t such that

$$20bc + c^2 \equiv t \pmod{100}.$$

The 21 possibilities are

$$01, 04, 09, 16, 21, 24, 25, 29, 36, 41, 44, 49, 56, 61, 64, 69, 76, 81, 84, 89, 96.$$

To find the last two digits of n^4, it suffices to find the least residue modulo 100 of

$$t^2 \equiv (20bc + c^2)^2 \equiv 40bc^3 + c^4 \pmod{100}.$$

Furthermore, note that if $b = 0, 1, 2, 3,$ or 4, then

$$40(b + 5)c^3 + c^4 \equiv 40bc^3 + 200c^3 + c^4 \equiv 40bc^3 + c^4 \pmod{100}.$$

The results are summarized in the following table, where $c \neq 0$ is the units digit of n and b is the tens digit of n for values of $b = 0, 1, 2, 3$ and 4.

b/c	1	2	3	4	5	6	7	8	9
0	01	16	81	56	25	96	01	96	61
1	41	36	61	16	25	36	21	76	21
2	81	56	41	76	25	76	41	56	81
3	21	76	21	36	25	16	61	36	41
4	61	96	01	96	25	56	81	16	01

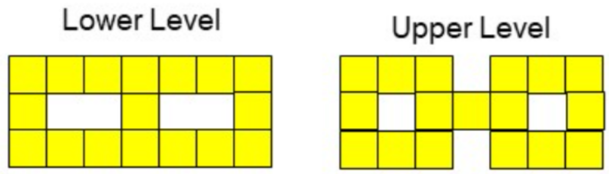
Therefore, the units digit of a positive integer n, not divisible by 10, is $c \neq 0$ which has the same parity as c^4, the units digit of n^4. The table then shows that in each case the tens digit of n^4 and c^4 have opposite parity. Accordingly, if n is not divisible by 10, then the units digit of n and the tens digit of n^4 have opposite parity.

CC305. Can you arrange n identical cubes in such a way that each cube has exactly three neighbours (cubes are considered to be neighbours if they have a common face)? Solve the problem for $n = 2016, 2017$ and 2018.

Originally 2017 Savin Open Contest, Problem 8 (by P. Kozhevnikov).

We received one submission. We present the solution by Richard Hess.

Putting 2016 or 2018 cubes together so each has exactly three neighbours can be done as shown below. Note that eight cubes in a two by two by two cube standing alone satisfies the requirement.

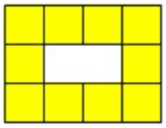


Case 1: 2016 *cubes.* Use 252 copies of the two by two by two cube.

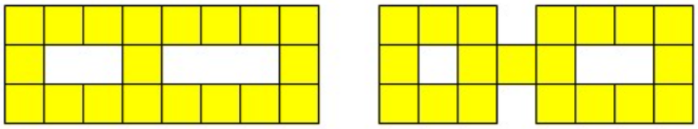
Case 2: 2018 *cubes.* Use 34 cubes in the lower and upper levels as shown along with 248 copies of the two by two by two cube.

Case 3: 2017 *cubes.* This is impossible. Consider an arrangement of any $2k+1$ cubes so each has exactly three neighbours. Then the number of shared faces is $F = 3(2k+1)/2$: three for each cube and divided by two because each shared face is counted twice. Expression for F is not an integer, producing a contradiction.

Additional thoughts. As shown in Case 2 above, we can get a solution for any $n \equiv 2 \pmod 8 \geq 34$ by adding copies of the two by two by two cube. We can get a solution for any $n \equiv 4 \pmod 8 \geq 20$ using two copies of the following frame, one above the other, and adding copies of the two by two by two cube:



Finally, we can get a solution for any $n \equiv 6 \pmod 8 \geq 38$ using the two layers shown below and adding copies of the two by two by two cube:

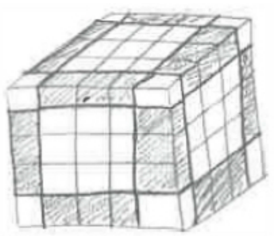


CC306. Consider a $5 \times 5 \times 5$ cube with the outside surface painted blue. Buzz cuts the cube into 5^3 unit cubes, then picks a cube at random. Given that the cube Buzz picked has at least one painted blue face, what is the probability that the cube has exactly two blue faces?

Originally problem 6 of Ciphering round of 2016 Georgia Tech High School Mathematics Competition.

We received 11 submissions of which 9 were correct and complete. We present the solution by Kathleen E. Lewis, accompanied by an image submitted along with the solution by the Quest University Math Literacy Class.

Since there are 125 small cubes altogether, including 27 interior cubes, there are 98 cubes with at least one blue face. Each of the six faces contains 9 cubes with only one blue face, making 54 such cubes altogether. There are 8 corner cubes with three blue faces and 36 edge cubes with two blue faces (3 on each of the 12 edges). So the probability we want is $36/98 = 18/49$.



CC307. Find (with proof) all integer solutions (x, y) to

$$x^2 - xy + 2017y = 0.$$

Originally Problem 3 from the 2017 Science Atlantic Math Competition.

We received 9 complete solutions and 4 incomplete submissions. We present the solution by Richard Hess, slightly edited.

The equation

$$x^2 - xy + 2017y = 0$$

can be converted to

$$x^2 = y(x - 2017).$$

For $x = 2017$ the equation is not satisfied, so there is no solution in this case. Let $z = (x-2017)$ and the equation can be rewritten as $yz = (z+2017)^2$. From this it is clear that z must divide 2017^2. Because 2017 is prime, this happens only when $z = \pm 1, \pm 2017$, or $\pm 2017^2$. The solutions for x are thus

$$x = 2017 \cdot 2018, 2 \cdot 2017, 2018, 2016, 0 \text{ and } -2017 \cdot 2016$$

with the following corresponding solutions for y:

$$y = 2018^2, 4 \cdot 2017, 2018^2, -2016^2, 0 \text{ and } -2016^2.$$

CC308. Define the $n \times n$ *Pascal matrix* as follows: $a_{1j} = a_{i1} = 1$, while $a_{ij} = a_{i-1,j} + a_{i,j-i}$ for $i, j > 1$. So, for instance, the 3×3 Pascal matrix is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}.$$

Show that every Pascal matrix is invertible.

Originally Problem 1 from the 2017 Science Atlantic Math Competition.

We received 9 correct solutions, out of which we present the one by Ángel Plaza, slightly edited.

Let P_n denote the $n \times n$ Pascal matrix. Then

$$P_n = [a_{ij}] = \left[\binom{i+j-1}{j-1}\right].$$

We will prove that $P_n = L_n \cdot U_n$, where L_n is a lower and U_n an upper triangular matrix with main diagonal equal to 1:

$$L_n = [l_{ij}] = \left[\binom{i-1}{j-1}\right] \quad \text{and} \quad U_n = [u_{ij}] = \left[\binom{j-1}{i-1}\right].$$

To prove the previous relations, it is enough to show that the i-th file of L_n multiplied by the j-th row of U_n is equal to the entry a_{ij} of P_n:

$$\begin{aligned}\sum_{k=1}^{n} l_{ik}u_{kj} &= \sum_{k=0}^{n-1}\binom{i-1}{k}\binom{j-1}{k}\\ &\overset{*}{=} \sum_{k=0}^{j-1}\binom{i-1}{k}\binom{j-1}{j-1-k}\\ &= \binom{i+j-2}{-1} = a_{ij}.\end{aligned}$$

Note that * in the above equation is Vandermonde's identity.

Thus $P_n = L_n \cdot U_n$ and therefore $\det(P_n) = \det(L_n)\cdot\det(U_n) = 1$.

CC309. Suppose $P(x)$ and $Q(x)$ are polynomials with real coefficients. Find necessary and sufficient conditions on N to guarantee that if the polynomial $P(Q(x))$ has degree N, there exists real x with $P(x) = Q(x)$.

Originally Problem 2 from the 2017 Science Atlantic Math Competition.

We received 4 solutions. We present the solution by Chun-Hao Huang.

We will show that the necessary and sufficient condition on N is that N is an odd positive integer which is not a perfect square. Note that there exists a real x with $P(x) = Q(x)$ if and only if $P(x) - Q(x) = 0$ has a real root. Denote by p and q the degrees of $P(x)$ and $Q(x)$ respectively; then, the degree N of $P(Q(x))$ is equal to pq.

Necessity. In the cases where N is even or an odd perfect square, one can find examples for $P(x)$ and $Q(x)$ for which $P(x) - Q(x)$ has no real root. Consider first the case when N is even. Choose

$$P(x) = x^N + x + 1 \quad\text{and}\quad Q(x) = x$$

(so $p = N$ and $q = 1$); then $P(x) - Q(x) = x^N + 1$, which has no real root. Next consider N an odd perfect square, and write $N = (k+1)^2$ with k even. Take

$$P(x) = x^{k+1} + x^k + 1 \quad\text{and}\quad Q(x) = x^{k+1}$$

(so $p = q = k+1$); then $P(x) - Q(x) = x^k + 1$, which again has no real root.

Sufficiency. Suppose N is an odd integer which is not a perfect square. From $N = pq$ it follows that p and q are both odd, and $p \neq q$. The degree of $P(x) - Q(x)$ is then $\max\{p, q\}$, which is odd; thus $P(x) - Q(x)$ is a polynomial with real coefficients and odd degree, and as such must have at least one real root.

This concludes the proof that $P(x) = Q(x)$ has a real solution if and only if N is an odd positive integer which is not a perfect square.

CC310. Suppose

$$\tan x + \cot x + \sec x + \csc x = 6.$$

Find the value of

$$\sin x + \cos x.$$

Originally problem 8 of Ciphering round of 2016 Georgia Tech High School Mathematics Competition.

We received 9 submissions, of which eight are correct, and one cannot be opened. We present the solution provided by Maria Mateo and Ángel Plaza.

Notice that for the correctness of the given equation, $\sin x \neq 0$ and $\cos x \neq 0$. The given equation may be written as

$$\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} + \frac{1}{\sin x} + \frac{1}{\cos x} = 6,$$

or

$$1 + \sin x + \cos x = 6 \sin x \cos x. \tag{1}$$

Since

$$(\sin x + \cos x)^2 = 1 + 2 \sin x \cos x,$$

then

$$\sin x \cos x = \frac{A^2 - 1}{2},$$

where $A = \sin x + \cos x$. Therefore (1) reads as

$$1 + A = 6\frac{A^2 - 1}{2},$$

which gives $A = -1$ or $A = 4/3$. Since $\sin x \neq 0$ and $\cos x \neq 0$, $A = -1$ is not a valid solution and so

$$A = \sin x + \cos x = 4/3.$$

PROBLEM SOLVING VIGNETTES

No.1

Shawn Godin

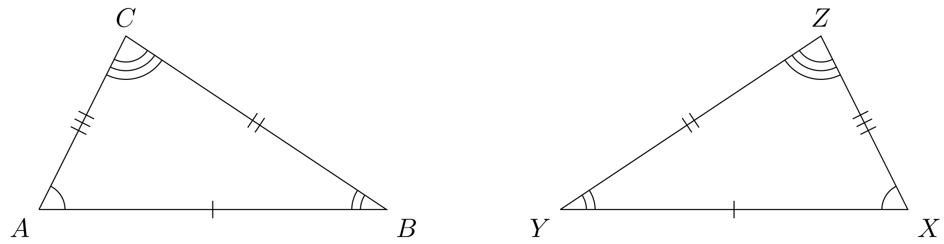
Congruent Triangles

Welcome *MathemAttic* readers! I am excited to see *Crux* reintroduce a section that is specifically aimed at a broader pre-university audience. This column will replace the column *Problem Solving 101* by expanding its scope. We will still, occasionally, focus on a particular problem and its solution. At other times, we will explore ideas from mathematics that will be of use to problem solvers.

Some of the ideas we will explore are things that have disappeared from the Canadian high school curriculum over the years. Other ideas will be things that students would run into later in high school or as an undergraduate, but that are accessible to younger readers without a lot of extra background. Over the years, in Canada anyway, Euclidean geometry has slowly all but disappeared from the curriculum. In this issue we will look at congruent triangles and their uses.

Two geometric figures in the plane are said to be *congruent* if they have the same size and shape; this means that they can be manipulated by use of translations, reflections, and rotations so that their corresponding parts coincide. For two triangles, we write $\triangle ABC \cong \triangle XYZ$ to indicate that triangles ABC and XYZ are congruent. If two triangles are congruent, their corresponding sides and angles are equal, so in our case we would have

$$\begin{aligned} AB &= XY & \qquad \angle A &= \angle X \\ BC &= YZ & \angle B &= \angle Y \\ CA &= ZX & \angle C &= \angle Z. \end{aligned}$$

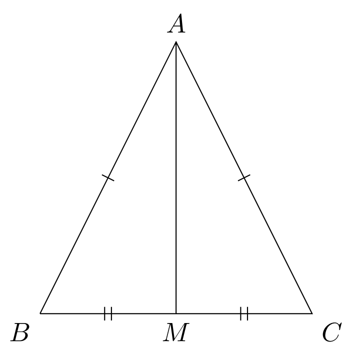


Notice that when we name the triangles in the congruence statement, the order of the vertices is important. The notation tells us how the vertices and sides match up.

We have several sufficient conditions for congruence:

- **side-side-side (SSS)** – when the three sides of one triangle equal the three sides of another (e.g. $AB = XY$, $BC = YZ$, and $CA = ZX$).
- **side-angle-side (SAS)** – when two sides and the *contained* angle of one triangle equal two sides and contained angle of another triangle (for example, $AB = XY$, $\angle B = \angle Y$, and $BC = YZ$). Note that angle-side-side is *not* a theorem — given ΔABC with $\angle A$ acute, there could be two triangles XYZ for which $\angle A = \angle X$, $AB = XY$, and $BC = YZ$.
- **angle-side-angle (ASA)** – when two angles and the *contained* side of one triangle equal two angles and the contained side of another triangle (e.g. $\angle A = \angle X$, $AB = XY$, and $\angle B = \angle Y$). Note if two angles and a non-contained side of one triangle are equal to those in another, then since the angles in a triangle add to $180°$, the third angles are equal as well, and we have ASA. Thus we can have ASA or AAS.
- **hypotenuse-side (HS)** – when the hypotenuse and a side of a right triangle are equal to the hypotenuse and side of another right triangle (e.g. $AB = XY$, $BC = YZ$, and $\angle C = 90° = \angle Z$).

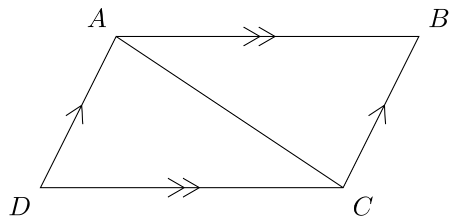
Let's see congruence in action. The word isosceles comes from Greek: *isos* meaning equal, and *skelos* meaning leg. Thus an isosceles triangle has (at least) two equal sides. Suppose $\triangle ABC$ is isosceles with $AB = AC$. Let M be the midpoint of BC and draw in the median AM as in the diagram below.



We now have two triangles $\triangle ABM$ and $\triangle ACM$. These triangles share the common side AM, have $AB = AC$ (since $\triangle ABC$ is isosceles) and $BM = CM$ since M is the midpoint of AB. Thus $\triangle ABM \cong \triangle ACM$ by SSS and the three angles must match up. Since $\angle B = \angle C$, we have shown that in any isosceles triangle, the angles opposite the equal sides are equal.

We can actually show more than that. From the congruence of the triangles we get that $\angle BAM = \angle CAM$, which means that AM bisects $\angle BAC$. Recall that AM was the median of $\triangle ABC$, so we have show that the median from the *apex*, the vertex between the two equal sides, also bisects the angle at the apex. Also, since $\angle BMA = \angle CMA$, by congruence, and $\angle BMA$ and $\angle CMA$ are supplementary, we can conclude that they are both right angles. Hence the altitude from the apex, the median from the apex and the angle bisector of the apex angle (the part not outside the triangle) all coincide.

Let's use congruent triangles to prove some other geometric properties. A *parallelogram* is a quadrilateral in which opposite sides are parallel. We will prove that the opposite sides are equal in length.



Let $ABCD$ be a parallelogram with $AB \parallel CD$ and $BC \parallel DA$. Draw in diagonal AC, creating two triangles. Since $AB \parallel CD$ then $\angle BAC = \angle DCA$ as they are alternate angles and similarly $\angle ACB = \angle CAD$. Finally, since the diagonal is a shared side of both triangles, $\triangle ABC \cong \triangle CDA$ by ASA, and hence $AB = CD$ and $BC = DA$.

You can also show that a quadrilateral where the opposite sides are equal in length is a parallelogram. This means that the conditions: "a quadrilateral has opposite sides equal in length" and "a quadrilateral has opposite sides that are parallel" are *equivalent conditions*. That means you cannot have one without the other as one implies the other.

The following are equivalent conditions on a quadrilateral $ABCD$:

1. $ABCD$ is a parallelogram (i.e. opposite sides are parallel),
2. opposite sides are equal in length,
3. opposite angles are equal to each other,
4. adjacent angles are supplementary,
5. one pair of opposite sides is equal in length and parallel,
6. the diagonals bisect each other,
7. the diagonals cut the quadrilateral into four triangles of equal area.

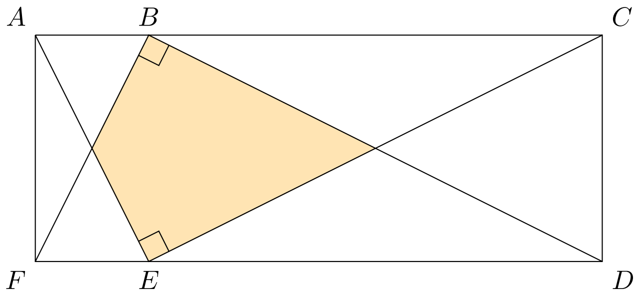
If you pick any two of these conditions you can show that they each imply the other. Also, you can number the conditions (1) to (7) in any order, and if you show that

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (7) \Rightarrow (1),$$

then every condition implies every other condition. For example, since $(1) \Rightarrow (2) \Rightarrow (3)$, we can conclude that $(1) \Rightarrow (3)$. Similarly, since $(3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (7) \Rightarrow (1)$, we can conclude that $(3) \Rightarrow (1)$, hence $(1) \Leftrightarrow (3)$. I will leave these proofs as an exercise, some will use congruent triangles while others will not.

We will end with one more example. The following problem is inspired by problem 7a from the *2015 Euclid Contest*:

> In the diagram, $ACDF$ is a rectangle. Also, $\triangle FBD$ and $\triangle AEC$ are congruent triangles which are right-angled at B and E, respectively. Show that the shaded area is one quarter the area of the rectangle.



As $\triangle FBD \cong \triangle AEC$ we have $FB = AE$. So in the two right angled triangles $\triangle AFE$ and $\triangle FAB$, their hypotenuses are equal and AF is a shared side so they are congruent by HS, thus $AB = FE$. Since $ACDF$ is a rectangle, $AC \parallel DF$ and so $\angle BAE = \angle FEA$ as they are alternate angles. Therefore, $\triangle ABE \cong \triangle EFA$ by SAS as we have already shown that $AB = EF$ and clearly $AE = EA$. It follows that $\angle EBA = \angle AFE = 90°$. Thus $ABEF$ and, similarly, $BCDE$ are both rectangles. Hence, by the last property in the list of equivalent properties of a parallelogram, one quarter of each rectangle is shaded, and thus one quarter of the whole figure is shaded.

Euclidean geometry offers many opportunities to play and yet is lightly treated in current curricula. We will explore other ideas from Euclidean geometry in future columns. Here are some practice problems:

1. Line segments AB and CD bisect each other at E. Prove that $AC = BD$.

2. Given line segment AB, draw two circles of equal radii centred at A and B such that the two circles intersect at two points X and Y. Prove that AB and XY are perpendicular and bisect each other.

3. Point P is in the interior of $\triangle ABC$ such that it is equidistant from the three sides of the triangle. Prove that P lies on the three angle bisectors of the triangle.

OLYMPIAD CORNER

No. 369

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **April 15, 2019**.

OC411. Show that for all integers $k > 1$ there is a positive integer m less than k^2 such that $2^m - m$ is divisible by k.

OC412. Find all the functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x, y

$$f(y - xy) = f(x)y + (x - 1)^2 f(y).$$

OC413. To each sequence consisting of n zeros and n ones is assigned a number which is the number of largest segments with the same digits in it (for example, the sequence 00111001 has 4 such segments 00, 111, 00, 1). For each n, add all the numbers assigned to each sequence. Prove that the resulting sum is equal to

$$(n + 1)\binom{2n}{n}.$$

OC414. Find all prime numbers p and all positive integers a and m such that $a \leq 5p^2$ and $(p - 1)! + a = p^m$.

OC415. Let n be a positive integer and let $f(x)$ be a polynomial of degree n with real coefficients and n distinct positive real roots. Is it possible for some natural integer $k \geq 2$ and for a real number a that the polynomial

$$x(x + 1)(x + 2)(x + 4)f(x) + a$$

is the k-th power of a polynomial with real coefficients?

. . . .

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 avril 2019**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

OC411. Démontrer que pour tout entier $k > 1$ il existe un entier positif m, plus petit que k^2, tel que $2^m - m$ est divisible par k.

OC412. Déterminer toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$ telles que

$$f(y - xy) = f(x)y + (x - 1)^2 f(y)$$

pour tous nombres réels x et y.

OC413. À chaque suite comprenant n zéros et n uns on associe un nombre égal au nombre de segments utilisant un seul chiffre (par exemple, la suite 00111001 possde 4 tels segments: 00, 111, 00, 1). Pour n donné, soit la somme des nombres associés à toutes ses suites. Démontrer que cette somme est égale à

$$(n + 1)\binom{2n}{n}.$$

OC414. Déterminer tous les nombres premiers p et tous les entiers a et m tels que $a \leq 5p^2$ et $(p - 1)! + a = p^m$.

OC415. Soit n un entier positif et soit $f(x)$ un polynôme de degré n à coefficients réels, ayant n racines réelles distinctes et positives. Est-ce possible que pour un certain entier $k \geq 2$ et pour un certain a réel le polynôme

$$x(x + 1)(x + 2)(x + 4)f(x) + a$$

soit la k-ième puissance dun polynôme à coefficients réels?

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2018: 44(1), p. 13–15.

OC351. Solve the system of equations

$$\begin{aligned} 6x - y + z^2 &= 3, \\ x^2 - y^2 - 2z &= -1, \\ 6x^2 - 3y^2 - y - 2z^2 &= 0. \end{aligned}$$

where $x, y, z \in \mathbb{R}$.

Originally Problem 1 of Day 1 of 2016 Vietnam National Olympiad.

We received 7 submissions of which 6 *were correct and complete. We present 2 solutions.*

Solution 1, by Dan Daniel.

First equation gives $y = z^2 + 6x - 3$; second equation gives $y^2 = x^2 - 2z + 1$. Substituting these values into the third equation, we get

$$6x^2 - 3(x^2 - 2z + 1) - (z^2 + 6x - 3) - 2z^2 = 0 \implies 3x^2 - 3z^2 - 6x + 6z = 0,$$

i.e.

$$(x - z)(3(x + z) - 6) = 0 \implies (x - z)(x + z - 2) = 0.$$

We have two cases.

(i) $x = 2 - z$. Then, the first equation of the system becomes

$$6(2 - z) - y + z^3 = 3 \implies y = (z - 3)^2$$

and the second equation of the system becomes

$$(2 - z)^2 - (z - 3)^4 - 2z + 1 = 0 \implies (z - 3)^4 - (z - 3)^2 + 4 = 0.$$

Setting $t = z - 3$, we have no solutions in this case.

(ii) $z = x$. Then, $y^2 = x^2 - 2x + 1$, i.e. $y = \pm(x - 1)$. If $y = x - 1$, then from the first equation of the system, we get

$$x^2 + 5x - 2 = 0 \implies (x, y, z) = \left(\frac{-5 \pm \sqrt{33}}{2}, \frac{-7 \pm \sqrt{33}}{2}, \frac{-5 \pm \sqrt{33}}{2}\right).$$

If $y = 1 - x$, then from the first equation of the system, we get

$$x^2 + 7x - 4 = 0 \implies (x, y, z) = \left(\frac{-7 \pm \sqrt{65}}{2}, \frac{9 \mp \sqrt{65}}{2}, \frac{-7 \pm \sqrt{65}}{2}\right).$$

Solution 2, by Oliver Geupel.

It is tedious but straightforward to check that the following four triplets (x, y, z) are solutions to the problem:

$$\left(\frac{-7+\sqrt{65}}{2}, \frac{9-\sqrt{65}}{2}, \frac{-7+\sqrt{65}}{2}\right), \left(\frac{-7-\sqrt{65}}{2}, \frac{9+\sqrt{65}}{2}, \frac{-7-\sqrt{65}}{2}\right),$$

$$\left(\frac{-5+\sqrt{33}}{2}, \frac{-7+\sqrt{33}}{2}, \frac{-5+\sqrt{33}}{2}\right), \left(\frac{-5-\sqrt{33}}{2}, \frac{-7-\sqrt{33}}{2}, \frac{-5-\sqrt{33}}{2}\right).$$

We show that there is no further solution.

Suppose (x, y, z) is any solution. Then,

$$0 = (3-6x+y-z^2)-3(x^2-y^2-2z+1)+(6x^2-3y^2-y-2z^2) = 3(x+z-2)(x-z).$$

Hence, $z = 2 - x$ or $z = x$. The case $z = 2 - x$ is impossible, because it would imply

$$x^2 + 2x + 1 - y = 6x - y + z^2 - 3 = 0 = x^2 - y^2 - 2z + 1 = x^2 + 2x - 3 - y^2;$$

hence $y - 1 = y^2 + 3$, that is,

$$(2y-1)^2 = -15,$$

a contradiction. Thus $z = x$.

We obtain

$$0 = x^2 - y^2 - 2z + 1 = (x + y - 1)(x - y - 1),$$

so that $y = 1 - x$ or $y = x - 1$. If $y = 1 - x$, then

$$0 = 6x - y + z^2 - 3 = x^2 + 7x - 4,$$

which yields the first and the second solution given above. If $y = x - 1$ then

$$0 = 6x - y + z^2 - 3 = x^2 + 5x - 2,$$

and we get the third and the fourth solution.

OC352. Let O_1 and O_2 intersect at P and Q. Their common external tangent touches O_1 and O_2 at A and B respectively. A circle Γ passing through A and B intersects O_1, O_2 at D, C. Prove that

$$\frac{CP}{CQ} = \frac{DP}{DQ}.$$

Originally Problem 2 of Day 1 of 2016 China Western Mathematical Olympiad.

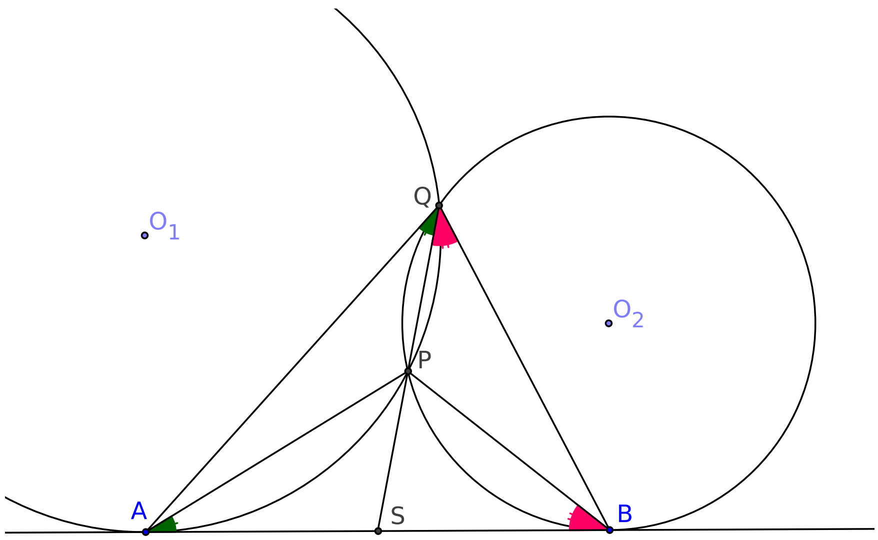
We received only one submission and we present the solution by Oliver Geupel.

The lines AB and PQ intersect at a point S which has equal powers with respect to the circles (O_1) and (O_2). It follows that $AS = BS$. We have

$$\angle BAP = \angle AQP = \angle AQS \quad \text{and} \quad \angle ABP = \angle BQP = \angle BQS$$

by the properties of inscribed angles. By the law of sines, we obtain

$$\frac{AP}{BP} = \frac{\sin \angle ABP}{\sin \angle BAP} = \frac{AS \cdot \dfrac{\sin \angle ASQ}{\sin \angle AQS}}{BS \cdot \dfrac{\sin \angle BSQ}{\sin \angle BQS}} = \frac{AQ}{BQ}. \tag{1}$$



Apply the inversion with centre at A and an arbitrary positive radius r; let X' denote the image of X. The circle (O_2) is transformed into a circle passing through C', P', B', Q'. By the properties of the inversion and by (1), we have

$$B'P' = \frac{r^2}{AB \cdot AP} \cdot BP = \frac{r^2}{AB \cdot AQ} \cdot BQ = B'Q'.$$

Hence, $C'B'$ bisects the angle $P'C'Q'$. The circle (O_1) with point A excluded is transformed into a line through P', D', Q'. The circle Γ without A is transformed into a line through C', D', B'. Thus, $C'D'$ is the internal bisector of angle C' in triangle $P'C'Q'$. By the properties of the inversion and by the angle bisector theorem, we obtain

$$\frac{CP}{CQ} = \frac{\dfrac{AC \cdot AP \cdot C'P'}{r^2}}{\dfrac{AC \cdot AQ \cdot C'Q'}{r^2}} = \frac{AP}{AQ} \cdot \frac{C'P'}{C'Q'} = \frac{AP}{AQ} \cdot \frac{D'P'}{D'Q'} = \frac{\dfrac{AD \cdot AP \cdot D'P'}{r^2}}{\dfrac{AD \cdot AQ \cdot D'Q'}{r^2}} = \frac{DP}{DQ}.$$

This completes the proof.

OC353. Prove that for any positive integer k,

$$(k^2)! \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$

is an integer.

Originally Problem 2 of Day 1 of 2016 USAMO. We received 2 solutions and present both of them.

Solution 1, by Ivko Dimitrić.

We proceed by induction. Let $N(k)$ denote the expression in the problem. Then $N(1) = 1$ and $N(2) = 4! \cdot \frac{0!}{2!} \cdot \frac{1!}{3!} = 2$ are integers. Assume $N(k)$ is an integer. Then,

$$\begin{aligned}
N(k+1) &= [(k+1)^2]! \cdot \prod_{j=0}^{k} \frac{j!}{(j+k+1)!} \\
&= [(k+1)^2]! \cdot \frac{k!}{(2k+1)!} \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k+1)(j+k)!} \\
&= [(k+1)^2]! \cdot \frac{k!}{(2k+1)!} \cdot \prod_{j=0}^{k-1} \frac{1}{j+k+1} \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!} \\
&= [(k+1)^2]! \cdot \frac{k!}{(2k+1)!} \cdot \frac{1}{(k+1)(k+2)\cdots(k+k)} \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!} \\
&= [(k+1)^2]! \cdot \frac{(k!)^2}{(2k+1)!\,(2k)!} N(k) \\
&= \frac{(2k+1+k^2)!\,(k!)^2}{(2k+1)!\,(2k)!} N(k) \\
&= \frac{(2k+2)(2k+3)\cdots(2k+k^2+1)}{(k+1)(k+2)\cdots(k+k)} (k!)\, N(k).
\end{aligned}$$

Since $k^2+1 > 2k$ for $k > 1$, then $2k+k^2+1 > 4k$ and the factors of the numerator of the last fraction include all doubled factors of the denominator,

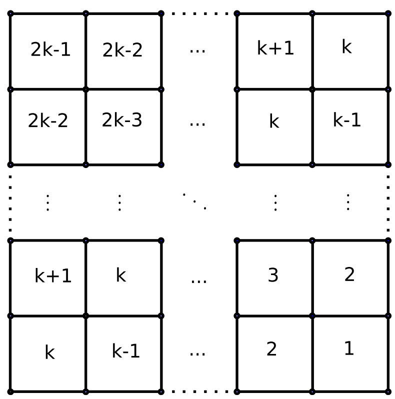
$$(2k+2) = 2(k+1),\ (2k+4) = 2(k+2),\ \cdots,\ 4k = 2(k+k),$$

so that the fraction is reduced to an integer quotient which is then multiplied by an integer $(k!)N(k)$ to produce integer value of $N(k+1)$, finishing the induction step and the proof.

Solution 2, by Oliver Geupel.

Let $n_1, n_2, \ldots, n_m$ be integers such that $n_1 \geq n_2 \geq \cdots \geq n_m > 0$, and let $N = n_1 + n_2 + \cdots + n_m$. A *Young tableau* of shape $(n_1, n_2, \ldots, n_m)$ is a collection of squares, arranged in m left-justified rows, with n_1 squares in the first row, n_2 squares in the second row, $\ldots$, n_m squares in the m-th row, in which the integers from 1 to N have to be inserted (one in each square) in such a way that all rows and columns are increasing. The *hook* corresponding to a square in a Young tableau is the union of the square and all squares to the right in the same row and the squares below in the same column. The *hook-length* is the number of squares in the hook.

Consider a Young tableau consisting of k rows of length k. The following figure shows in each square the corresponding hook-length:



The *hook length formula* expresses the number of Young tableaux of a given shape and a total of N squares as $N!$ divided by the product of all hook-lengths (see J.H. van Lint, R.M. Wilson, *A Course in Combinatorics*, 2nd ed., Cambridge University Press, 2001, Theorem 15.11, page 165).

By the hook length formula, the number of Young tableaux with k rows of length k is

$$\frac{(k^2)!}{(1\cdot 2\cdots k)\cdot(2\cdot 3\cdots(k+1))\cdots(k(k+1)\cdots(2k-1))} = (k^2)!\cdot\prod_{j=0}^{k-1}\frac{j!}{(j+k)!},$$

which is therefore an integer.

OC354. Solve the equation $1 + x^z + y^z = LCM(x^z, y^z)$ in the set of natural numbers.

Originally Problem 1 of 2016 Macedonia National Olympiad.

We received 4 solutions and will present 2 of them.

Solution 1, by Ivko Dimitrić.

Let $d = GCD(x, y)$ be the greatest common divisor of x and y. Then $x = dx_1,\ y = dy_1$, where $GCD(x_1, y_1) = 1$. Substituting these into the equation gives

$$1 + d^z x_1^z + d^z y_1^z = LCM(d^z x_1^z, d^z y_1^z) = d^z x_1^z y_1^z.$$

If $d > 1$ then all terms in this equation are divisible by d except 1, which is not possible. Thus, $GCD(x, y) = 1$ and $LCM(x^z, y^z) = x^z y^z$, so that the equation becomes

$$1 + x^z + y^z = x^z\, y^z,$$

which can be rewritten as $(x^z - 1)(y^z - 1) = 2$. In this product one of the factors is 1 and the other one is 2. If $x^z - 1 = 1$ and $y^z - 1 = 2$, we get $x = 2$, $y = 3$, $z = 1$, and if $x^z - 1 = 2$ and $y^z - 1 = 1$, we have $x = 3$, $y = 2$, $z = 1$, which are the only sets of solutions.

Solution 2, by the Missouri State University Problem Solving Group.

Suppose we have natural numbers x, y, z satisfying the equation. For simplicity, we consider the equation $1 + a + b = LCM(a, b)$, where $a = x^z$ and $b = y^z$. Since $1 + a + b$ is a multiple of a, we have a divides $1 + b$ and in particular $a \leq b + 1$. Similarly $b \leq a + 1$. Hence $a - 1 \leq b \leq a + 1$, so either (1) $b = a - 1$, (2) $b = a$, or (3) $b = a + 1$.

Note that in cases (1) and (3), a and b are relatively prime, so their least common multiple is their product.

The equation in case (1) reads $1 + a + a - 1 = a(a - 1)$, which gives $a = 3$, and then $b = 2$. The equation in case (2), $1 + a + a = a$ implies $a = -1$, which is not a natural number.

The equation in case (3) gives $1 + a + a + 1 = a(a + 1)$ whose only natural number solution is $a = 2$. Then $b = 3$ in this case.

Hence the only solutions in the natural numbers are $(x, y, z) = (3, 2, 1)$ and $(x, y, z) = (2, 3, 1)$.

Editor's Comments. Note that if we include zero in the set of natural numbers (as some authors in general do), it's easy to see that there are no solutions if $x = 0$ or $y = 0$ or $z = 0$ and the solutions are the ones already found.

OC355. Let $\mathbb{N}$ denote the set of natural numbers. Define a function $T : \mathbb{N} \to \mathbb{N}$ by $T(2k) = k$ and $T(2k + 1) = 2k + 2$. We write $T^2(n) = T(T(n))$ and in general $T^k(n) = T^{k-1}(T(n))$ for any $k > 1$.

a) Show that for each $n \in \mathbb{N}$, there exists k such that $T^k(n) = 1$.

b) For $k \in \mathbb{N}$, let c_k denote the number of elements in the set $\{n : T^k(n) = 1\}$. Prove that $c_{k+2} = c_{k+1} + c_k$, for $k \geq 1$.

Originally Problem 3 of 2016 India National Olympiad.

We only received 1 *submission and we present the solution by Oliver Geupel.*

For $n \in \mathbb{N}$, let $P(n)$ denote the assertion that there exists a natural number k with the property $T^k(n) = 1$. We prove $P(n)$ for all natural numbers n by mathematical induction. From $T^2(1) = 1$ we have $P(1)$. We prove $P(n)$ for $n > 1$ under the hypothesis that $P(m)$ holds for all natural numbers m less than the number n. In fact, if n is even, say $n = 2q$, then $P(n)$ follows from $T(n) = q$ and the hypothesis $P(q)$. If n is odd, say $n = 2q - 1$, then $P(n)$ follows from $T^2(n) = q$ and the hypothesis $P(q)$, because $q < n$. This completes the induction and the solution to part a).

Going on to part b), let $\#S$ denote the number of elements in the finite set S. By the recursion, we have for all natural numbers k and q:

$$T^{k+2}(2q) = 1 \Leftrightarrow T^{k+1}(q) = 1 \qquad \text{and} \qquad T^{k+2}(2q+1) = 1 \Leftrightarrow T^{k}(q) = 1.$$

Hence,

$$\begin{aligned} c_{k+2} &= \#\{n : T^{k+2}(n) = 1\} \\ &= \#\{q \in \mathbb{N} : T^{k+2}(2q) = 1\} + \#\{q \in \mathbb{N} : T^{k+2}(2q-1) = 1\} \\ &= \#\{q \in \mathbb{N} : T^{k+1}(q) = 1\} + \#\{q \in \mathbb{N} : T^{k}(q) = 1\} \qquad \text{(by recursion)} \\ &= c_{k+1} + c_k \end{aligned}$$

for all natural numbers k. This is the desired result b).

FOCUS ON...

No. 34

Michel Bataille

Some Trigonometric Relations

Introduction

Trigonometry is quite a vast topic! In this number, we will just consider a selection of problems involving the values of the circular functions at $\frac{m\pi}{n}$ for various natural numbers m, n. Solving these problems calls for a perfect knowledge of the classical trigonometric formulas, of course, but other subjects such as complex numbers or polynomials can often prove useful. We will see various techniques at work in the examples that follow.

Juggling with trig formulas

Let us start gently with two exercises that I used to set to my students:

$$\text{Evaluate} \quad \text{a)} \quad A = \sin^2\frac{\pi}{9} + \sin^2\frac{2\pi}{9} + \sin^2\frac{4\pi}{9}$$

$$\text{b)} \quad B = \sin\frac{\pi}{30} - \sin\frac{7\pi}{30} - \sin\frac{11\pi}{30} + \sin\frac{17\pi}{30}.$$

(In what follows, the classical formulas used in the calculations will easily be recognized by the reader and won't be explicated.)

a) We have

$$A = \frac{1}{2}\left(1 - \cos\frac{2\pi}{9}\right) + \frac{1}{2}\left(1 - \cos\frac{4\pi}{9}\right) + \frac{1}{2}\left(1 - \cos\frac{8\pi}{9}\right) = \frac{3}{2} - \frac{A'}{2}$$

where

$$A' = \cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{8\pi}{9} = 2\cos\frac{\pi}{3}\cos\frac{\pi}{9} - \cos\frac{\pi}{9} = 0$$

and so $A = \frac{3}{2}$.

b) A frequently used trick will ease the calculation of B; instead of B alone, we consider $2B\cos\frac{\pi}{10}$:

$$2B\cos\frac{\pi}{10} = \sin\frac{2\pi}{15} - \sin\frac{\pi}{15} - \sin\frac{\pi}{3} - \sin\frac{2\pi}{15} - \sin\frac{7\pi}{15} - \sin\frac{4\pi}{15} + \sin\frac{2\pi}{3} + \sin\frac{7\pi}{15}$$

and so

$$2B\cos\frac{\pi}{10} = -\sin\frac{4\pi}{15} - \sin\frac{\pi}{15} = -2\sin\frac{\pi}{6}\cos\frac{\pi}{10} = -\cos\frac{\pi}{10},$$

which gives $B = -\dfrac{1}{2}$.

Here is a more elaborated example:

Prove that $$\frac{\sin\frac{5\pi}{18}}{\sin\frac{7\pi}{18}}+\frac{\sin\frac{7\pi}{18}}{\sin\frac{\pi}{18}}-\frac{\sin\frac{\pi}{18}}{\sin\frac{5\pi}{18}}=6.$$

Let us rewrite the left-hand side as $\frac{N}{D}$ where

$$D=\sin\frac{\pi}{18}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18}\quad\text{and}\quad N=\sin\frac{\pi}{18}\sin^2\frac{5\pi}{18}+\sin\frac{5\pi}{18}\sin^2\frac{7\pi}{18}-\sin\frac{7\pi}{18}\sin^2\frac{\pi}{18}.$$

We use the trick seen above to compute D:

$$2D\cos\frac{\pi}{18}=\sin\frac{\pi}{9}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18}=\sin\frac{\pi}{9}\sin\frac{5\pi}{18}\cos\frac{\pi}{9}$$

(because $2\sin\frac{\pi}{18}\cos\frac{\pi}{18}=\sin\frac{\pi}{9}$ and $\frac{7\pi}{18}+\frac{\pi}{9}=\frac{\pi}{2}$). Continuing in the same vein, we obtain

$$4D\cos\frac{\pi}{18}=\sin\frac{2\pi}{9}\sin\frac{5\pi}{18}=\sin\frac{2\pi}{9}\cos\frac{2\pi}{9}$$

and finally $8D\cos\frac{\pi}{18}=\sin\frac{4\pi}{9}=\cos\frac{\pi}{18}$ so that $D=\frac{1}{8}$.

As for N, we transform products in sums and first get

$$N=\frac{1}{2}\sin\frac{5\pi}{18}\left(\cos\frac{2\pi}{9}-\cos\frac{\pi}{3}\right)+\frac{1}{2}\sin\frac{7\pi}{18}\left(\cos\frac{\pi}{9}-\cos\frac{2\pi}{3}\right)-\frac{1}{2}\sin\frac{\pi}{18}\left(\cos\frac{\pi}{3}-\cos\frac{4\pi}{9}\right)$$

and then

$$\begin{aligned}N&=\frac{1}{4}\left(\sin\frac{7\pi}{18}-\sin\frac{5\pi}{18}-\sin\frac{\pi}{18}\right)+\frac{1}{4}\left(1+\sin\frac{\pi}{18}\right)+\frac{1}{4}\left(1+\sin\frac{5\pi}{18}\right)+\frac{1}{4}\left(1-\sin\frac{7\pi}{18}\right)\\&=\frac{3}{4}.\end{aligned}$$

Thus $\frac{N}{D}=6$, as required.

With the help of additional tools

Complex numbers are closely related to trigonometry and this connection can often simplify proofs. To take a simple example, let us consider the following relation:

Show that

$$\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}+\cos\frac{6\pi}{5}+\cos\frac{8\pi}{5}=-1$$

and deduce the values of $\cos\frac{2\pi}{5}$ and $\cos\frac{4\pi}{5}$.

The reader is invited to prove the result with trig formulas as in the previous paragraph; however, it seems preferable to use the complex number $w=\exp(2\pi i/5)$ as follows:

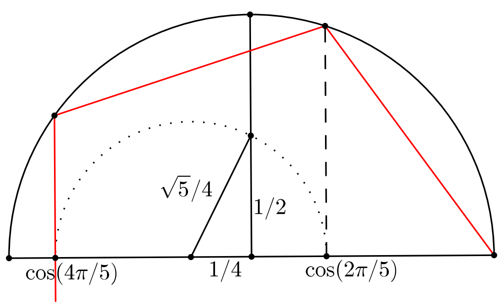
since $1+w+w^2+w^3+w^4 = \frac{1-w^5}{1-w} = 0$, we also have $\mathrm{Re}(1+w+w^2+w^3+w^4) = 0$, which gives the desired result (because $\mathrm{Re}(w^k) = \cos\frac{2k\pi}{5}$). The relation yields

$$-1 = 2\left(\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5}\right) = 2\left(\cos\frac{2\pi}{5} + 2\cos^2\frac{2\pi}{5} - 1\right)$$

and solving the quadratic equation $4x^2 + 2x - 1 = 0$, we obtain

$$\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}, \qquad \cos\frac{4\pi}{5} = \frac{-\sqrt{5}-1}{4}.$$

In passing, since $\frac{\sqrt{5}}{4}$ is the hypotenuse of a right-angled triangle whose other sides are $\frac{1}{4}$ and $\frac{1}{2}$, the results just obtained provide one of the simplest constructions of the regular pentagon.



In addition to complex numbers, polynomials often intervene in proofs. We illustrate this with two examples.

The first one is a problem set in the *American Math. Monthly* in 1999:

Prove that

$$\cos\frac{\pi}{7} = \frac{1}{6} + \frac{\sqrt{7}}{6}\left(\cos\left(\frac{1}{3}\arccos\frac{1}{2\sqrt{7}}\right) + \sqrt{3}\sin\left(\frac{1}{3}\arccos\frac{1}{2\sqrt{7}}\right)\right).$$

It seems wise to set $\theta = \arccos\frac{1}{2\sqrt{7}}$; with this notation, the proposed relation readily becomes

$$\cos\left(\frac{\pi-\theta}{3}\right) = \frac{6\cos\frac{\pi}{7} - 1}{2\sqrt{7}}.$$

Polynomials introduce themselves naturally if we recall the identity

$$\cos 3x = 4\cos^3 x - 3\cos x,$$

which shows that the number $u = \cos\left(\frac{\pi-\theta}{3}\right)$ satisfies

$$4u^3 - 3u = \cos(\pi - \theta) = -\frac{1}{2\sqrt{7}},$$

hence is a root of the polynomial $P(X) = 4X^3 - 3X + \frac{1}{2\sqrt{7}}$. Thus, we are led to seeking the roots of this polynomial. To this aim, a polynomial with $\cos\frac{\pi}{7}$ among its roots will be helpful. The role can be played by

$$Q(x) = x^3 - \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{8}$$

whose roots are $\cos\frac{\pi}{7}, \cos\frac{3\pi}{7}, \cos\frac{5\pi}{7}$. Indeed, if we set

$$\begin{aligned}
e_1 &= \cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7},\\
e_2 &= \cos\frac{\pi}{7}\cos\frac{3\pi}{7} + \cos\frac{3\pi}{7}\cos\frac{5\pi}{7} + \cos\frac{5\pi}{7}\cos\frac{\pi}{7}\\
e_3 &= \cos\frac{\pi}{7}\cos\frac{3\pi}{7}\cos\frac{5\pi}{7},
\end{aligned}$$

calculating $e_1 \sin\frac{\pi}{7}$ and $e_3 \sin\frac{\pi}{7}$ quickly provides $e_1 = \frac{1}{2}$ and $e_3 = -\frac{1}{8}$. In addition, transforming products into sums shows that $e_2 = -e_1$.

Now we have $Q(x) = \left(x - \frac{1}{6}\right)^3 - \frac{7}{12}\left(x - \frac{1}{6}\right) + \frac{7}{216}$ and so $Q\left(\frac{1}{6} + \frac{\sqrt{7}X}{3}\right) = \frac{7\sqrt{7}}{108} \cdot P(X)$. The roots of $P(X)$ are therefore

$$X_1 = \frac{6\cos\frac{\pi}{7} - 1}{2\sqrt{7}},\ X_2 = \frac{6\cos\frac{3\pi}{7} - 1}{2\sqrt{7}},\ X_3 = \frac{6\cos\frac{5\pi}{7} - 1}{2\sqrt{7}}.$$

It is easy to check that $X_2 < \frac{1}{2}, X_3 < 0$, while $u > \frac{1}{2}$ (since $0 < \frac{\pi-\theta}{3} < \frac{\pi}{3}$) and we can conclude that $u = X_1$, as desired.

Our second example, *Mathematics Magazine* problem 1562 posed in December 1998, mixes polynomials and complex calculations

$$\text{Prove that}\quad \tan\left(\frac{1}{4}\tan^{-1} 4\right) = 2\left(\cos\frac{6\pi}{17} + \cos\frac{10\pi}{17}\right).$$

With the help of the standard formulas, we readily obtain

$$\tan 4t = \frac{4\tan t - 4\tan^3 t}{1 - 6\tan^2 t + \tan^4 t}$$

and taking $t = \frac{1}{4}\tan^{-1} 4$ (so that $\tan 4t = 4$) we deduce that $x_0 = \tan\left(\frac{1}{4}\tan^{-1} 4\right)$ is a root of the polynomial $P(x) = x^4 + x^3 - 6x^2 - x + 1$.

Even better, noticing that $x_0 \in [0, 1]$ and that P is a decreasing function on $[0, 1]$, we see that x_0 is the only root of $P(x)$ in $[0, 1]$. Thus, if $a = 2\left(\cos\frac{6\pi}{17} + \cos\frac{10\pi}{17}\right)$, we just have to show that $P(a) = 0$ and $a \in [0, 1]$. The latter holds since

$$0 > \cos\frac{10\pi}{17} > \cos\frac{11\pi}{17} = -\cos\frac{6\pi}{17} > -\frac{1}{2}.$$

To prove the former, we introduce $u = \exp(2\pi i/17)$, so that the 17th roots of unity are the numbers u^k for $k = 0, 1, \ldots, 16$ and $a = u^3 + u^5 + u^{12} + u^{14}$. Conveniently, we also consider

$$b = u^2 + u^8 + u^9 + u^{15},$$
$$c = u^6 + u^7 + u^{10} + u^{11},$$
$$d = u + u^4 + u^{13} + u^{16}.$$

Using the equalities $u^{17} = 1$ and $-1 = \sum_{k=1}^{16} u^k = a + b + c + d$, and with a bit of algebra (and patience), we successively obtain:

$$\begin{aligned} a^2 &= 4 + 2b + c, \\ ab &= 2a + c + d = a - b - 1, \\ ac &= a + b + c + d = -1, \\ a^3 &= 4a + 2ab + ac = 6a - 2b - 3, \\ a^4 &= 6a^2 - 2ab - 3a = 26 - 5a + 14b + 6c. \end{aligned}$$

Lastly, a final calculation gives the expected result $a^4 + a^3 - 6a^2 - a + 1 = 0$.

It can be shown that the other roots of $P(x)$ are b, c, d. Relations analogous to the one of the statement can then be derived (left as an exercise to the reader).

The Quadratic Gauss Sums

Our last paragraph offers a few applications of the following beautiful theorem obtained by Gauss in 1805:

> **Theorem.** Let p be an odd prime number and $u = \exp(2\pi i/p)$. If p is of the form $4n+3$ then $\sum_{k=1}^{p-1} \left(\frac{k}{p}\right) u^k = i\sqrt{p}$, while if p is of the form $4n+1$, then $\sum_{k=1}^{p-1} \left(\frac{k}{p}\right) u^k = \sqrt{p}$.

[here $\left(\frac{k}{p}\right)$ is the Legendre symbol defined by $+1$ if k is a square modulo p and -1 otherwise.]

For a proof of this theorem, we refer the reader to **[1]** or **[2]**.

As a first application, we consider Problem 218 proposed in 1983 in the *College Mathematics Journal*. (It is also a part of problem **2463** [1999: 366 ; 2000: 379]):

> Prove that $\tan \dfrac{3\pi}{11} + 4 \sin \dfrac{2\pi}{11} = \sqrt{11}$.

We shall use two easy consequences of $e^{ix} = \cos x + i \sin x$:

$$4 \sin x = -2i(e^{ix} - e^{-ix}) \quad \text{and} \quad \tan x = i\left(\frac{2}{1 + e^{2ix}} - 1\right).$$

On the one hand, the theorem with $p = 11$ gives

$$u - u^2 + u^3 + u^4 + u^5 - u^6 - u^7 - u^8 + u^9 - u^{10} = i\sqrt{11}.$$

On the other hand, we have

$$\frac{2}{1+u^3}=\frac{1+(u^3)^{11}}{1+u^3}=\sum_{k=0}^{10}(-u^3)^k=1-u^3+u^6-u^9+u-u^4+u^7-u^{10}+u^2-u^5+u^8$$

and the formulas above lead to

$$\begin{aligned}&\tan\left(\frac{3\pi}{11}\right)+4\sin\left(\frac{2\pi}{11}\right)\\&=i(-u^3+u^6-u^9+u-u^4+u^7-u^{10}+u^2-u^5+u^8-2u+2u^{10})\\&=i(-i\sqrt{11})=\sqrt{11}.\end{aligned}$$

Our last example, extracted from problem **3305** [2008 : 45,47 ; 2009 : 52], may seem of the same category as the previous one. However, some extra work will be needed, as can be seen in the variant of solution proposed here.

Prove that $\tan\frac{5\pi}{13}+4\sin\frac{2\pi}{13}=\sqrt{13+2\sqrt{13}}$.

We mimic the solution above, first applying the theorem with $p=13$ to obtain $A-B=\sqrt{13}$ where

$$A=u+u^3+u^4+u^9+u^{10}+u^{12}\quad\text{and}\quad B=u^2+u^5+u^6+u^7+u^8+u^{11}.$$

In the same way as above, the calculations give

$$\tan\frac{5\pi}{13}+4\sin\frac{2\pi}{13}=i(C-D),$$

where

$$C=u^4+u^7+u^8+u^{10}+u^{11}+u^{12}\quad\text{and}\quad D=u+u^2+u^3+u^5+u^6+u^9.$$

Clearly, the connection is less direct than before! A look at the expected result prompts us to square and to show that

$$-(C-D)^2=13+2\sqrt{13},$$

that is,

$$4CD=14+2\sqrt{13}$$

(note that $C+D=-1$ so that $4CD=(C+D)^2-(C-D)^2=1-(C-D)^2$).

Using $u^{13}=1$, the calculation of CD is lengthy but easy and yields

$$CD=6+2(u^2+u^5+u^6+u^7+u^8+u^{11})+3(u+u^3+u^4+u^9+u^{10}+u^{12})=6+2B+3A.$$

Observing that $A+B=-1$ and $2A=(A+B)+(A-B)=-1+\sqrt{13}$, we conclude that

$$4CD=24+8(A+B)+4A=24-8-2+2\sqrt{13}=14+2\sqrt{13}.$$

As usual, we propose a few exercises for the reader's practice.

Exercises

1. Evaluate

a) $\sin \dfrac{\pi}{14} \sin \dfrac{3\pi}{14} \sin \dfrac{5\pi}{14}$

b) $\tan \dfrac{6\pi}{7} + 4 \sin \dfrac{5\pi}{7}$.

2. Prove that

a) $\cos \dfrac{7\pi}{15} = 4 \sin \dfrac{2\pi}{15} \cos \dfrac{2\pi}{5} \cos \dfrac{13\pi}{30}$

b) $1 + 6 \cos \dfrac{2\pi}{7} = 2\sqrt{7} \cos \left(\dfrac{1}{3} \cos^{-1} \dfrac{1}{2\sqrt{7}} \right)$

(Problem 974 of the *College Mathematics Journal*).

References

[1] M. Guinot, *Une époque de transition, Lagrange et Legendre*, Aléas, 1996, p. 260

[2] K. Ireland, M. Rosen, *A Classical Introduction to Modern Number Theory*, Springer, 1990, p. 70

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

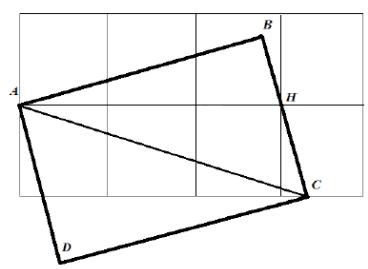
To facilitate their consideration, solutions should be received by **April 15, 2019**.

4401. *Proposed by Ruben Dario and Leonard Giugiuc.*

Let D and E be the centres of squares erected externally on the sides AB and AC, respectively, of an arbitrary triangle ABC, and define F and G to be the intersections of the line BC with lines perpendicular to ED at D and at E. Prove that the resulting segments BF and CG are congruent.

4402. *Proposed by Peter Y. Woo.*

Consider a rectangular carpet $ABCD$ lying on top of floor tiled with 8 square tiles with side length of 1 foot each (as shown in the diagram).



Suppose AH bisects $\angle BAC$. Express $\tan \angle BAH$ as the sum of a rational number and the square root of a rational number.

4403. *Proposed by Michel Bataille.*

Let m be an integer with $m > 1$. Evaluate in closed form

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n+1}{k+1} \frac{k}{m+k}.$$

4404. *Proposed by Nguyen Viet Hung.*

Let x, y and z be integers such that $x > 0, z > 0$ and $x + y > 0$. Find all the solutions to the equation

$$x^4 + y^4 + (x+y)^4 = 2(z^2 + 40).$$

4405. *Proposed by Kadir Altintas and Leonard Giugiuc.*

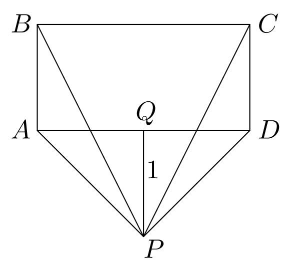
Let ABC be a triangle and let K be a point inside ABC. Suppose that BK intersects AC in F and CK intersects AB in E. Let M be the midpoint of BE, N be the midpoint of CF and suppose that MN intersects BK at P. Show that the midpoints of AF, EK and MP are collinear.

4406. *Proposed by Bill Sands.*

Four trees are situated at the corners of a rectangle. You are standing outside the rectangle, the nearest point of the rectangle being the midpoint of one of its sides, 1 metre away from you. To you in this position, the four trees appear to be equally spaced apart.

a) Find the side lengths of the rectangle, assuming that they are positive integers.

b) Suppose that the rectangle is a square. Find the length of its side.



4407. *Proposed by Mihaela Berindeanu.*

Circle C_1 lies outside circle C_2 and is tangent to it at E. Take arbitrary points B and D different from E on the common tangent line. Let the second tangent from B to C_1 touch it at M and to C_2 touch it at N, while the second tangents from D to those circles touch them at Q and P, respectively. If the orthocenters of the triangles MNQ and PNQ are H_1 and H_2, prove that $\overrightarrow{H_1H_2} = \overrightarrow{MP}$.

4408. *Proposed by Leonard Giugiuc, Dan Stefan Marinescu and Daniel Sitaru.*

Let $\alpha \in (0, 1] \cup [2, \infty)$ be a real number and let a, b and c be non-negative real numbers with $a + b + c = 1$. Prove that

$$a^\alpha + b^\alpha + c^\alpha + 1 \geq (a + b)^\alpha + (b + c)^\alpha + (c + a)^\alpha.$$

4409. *Proposed by Christian Chiser.*

Let A and B be two matrices in $M_2(\mathbb{R})$ such that $A^2 = O_2$ and B is invertible. Prove that the polynomial $P = \det(xB^2 - AB + BA)$ has all integer roots.

4410. *Proposed by Daniel Sitaru.*

Prove that

$$\int_0^{\frac{\pi}{4}} \sqrt{\sin 2x}dx < \sqrt{2} - \frac{\pi}{4}.$$

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposś dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 avril 2019***.*

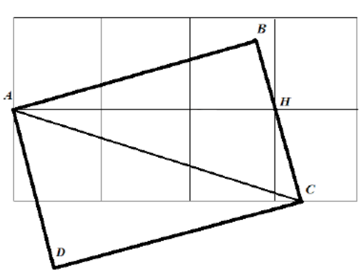
La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

4401. *Proposé par Ruben Dario et Leonard Giugiuc.*

Soient D et E les centres des carrés tracés à l'extérieur des côtés AB et AC d'un triangle arbitraire ABC et soient F et G les intersections de la ligne BC avec les perpendiculaires venant de ED, partant des points D et E, respectivement. Démontrer que les segments BF et CG sont congrus.

4402. *Proposé par Peter Y. Woo.*

Soit $ABCD$ un tapis rectangulaire, reposant sur un plancher formé de 8 tuiles carrées de côtés 1 pied de longueur chacune (comme indiqué au diagramme).



Supposons que AH bissecte $\angle BAC$. Exprimer $\tan \angle BAH$ comme somme d'un nombre rationnel et d'une racine carrée d'un nombre rationnel.

4403. *Proposé par Michel Bataille.*

Soit m un entier tel que $m > 1$. Évaluer en forme close l'expression

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n+1}{k+1}\frac{k}{m+k}.$$

4404. *Proposé par Nguyen Viet Hung.*

Soient x, y et z des entiers tels que $x > 0, z > 0$ et $x + y > 0$. Déterminer toutes les solutions à l'équation

$$x^4 + y^4 + (x + y)^4 = 2(z^2 + 40)$$

4405. *Proposé par Kadir Altintas et Leonard Giugiuc.*

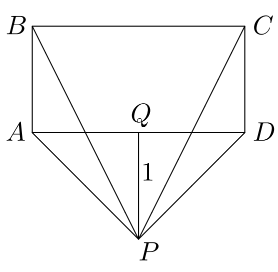
Soit ABC un triangle et K un point à son intérieur. Supposons que BK intersecte AC en F et que CK intersecte AB en E. Soient M le mi point de BE et N le mi point de CF; de plus, supposons que MN intersecte BK en P. Démontrer que les mi points de AF, EK et MP sont colinéaires.

4406. *Proposé par Bill Sands.*

Quatre arbres sont situés aux coins d'un rectangle. Vous êtes debout à l'extérieur du rectangle, le point du rectangle le plus près de vous étant le mi point d'un de ses côtés, à une distance de 1 mètre. De votre point de vue, les quatre arbres vous paraissent également espacés.

a) Déterminer les longueurs des côtés du rectangle, prenant pour acquis qu'elles sont entières et positives.

b) Supposant que le rectangle est effectivement un carré, déterminer la longueur de ses côtés.



4407. *Proposé par Mihaela Berindeanu.*

Le cercle $C1$ se situe à l'extérieur du cercle $C2$ et lui est tangent au point E. Soient B et D deux autres points sur la tangente en commun. La deuxième tangente de B vers $C1$ le rencontre en M et la deuxième tangente vers $C2$ le rencontre en N, tandis que les deuxièmes tangentes de D vers $C1$ et $C2$ les rencontrent en Q et P respectivement. Si les orthocentres des triangles MNQ et PNQ sont $H1$ et $H2$, démontrer que $\overrightarrow{H_1H_2} = \overrightarrow{MP}$.

4408. *Proposé par Leonard Giugiuc, Dan Stefan Marinescu et Daniel Sitaru.*

Soit $\alpha \in (0,1] \cup [2,\infty)$ un nombre réel et soient a, b et c des nombres réels non négatifs tels que $a+b+c=1$. Démontrer que

$$a^\alpha + b^\alpha + c^\alpha + 1 \geq (a+b)^\alpha + (b+c)^\alpha + (c+a)^\alpha.$$

4409. *Proposé par Christian Chiser.*

Soient A et B deux matrices dans $M_2(\mathbb{R})$ telles que $A^2 = O_2$ et que B est inversible. Démontrer que le polynôme $P = \text{dét}(xB^2 - AB + BA)$ n'a que des racines entières.

4410. *Proposé par Daniel Sitaru.*

Démontrer que

$$\int_0^{\frac{\pi}{4}} \sqrt{\sin 2x}\,dx < \sqrt{2} - \frac{\pi}{4}.$$

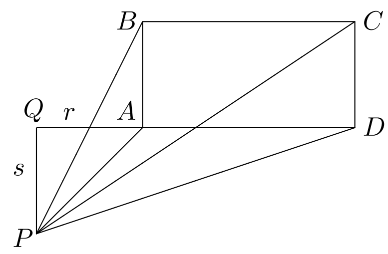
SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2018: 44(1), p. 28–32.

4301. *Proposed by Bill Sands.*

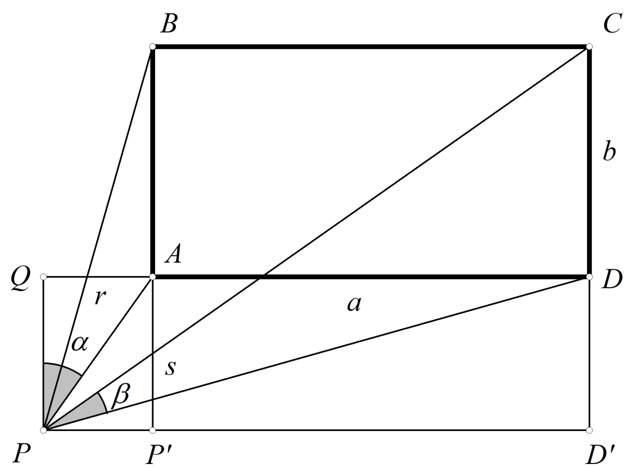
Four trees are situated at the corners of a rectangle $ABCD$. You are standing at a point P outside the rectangle, the nearest point of the rectangle to you being its corner A. To you in this position, the four trees, in the order B, A, C, D as in the diagram, appear to be equally spaced apart. Let Q be the foot of the perpendicular from P to line AD, and set $r = QA$, $s = PQ$.



a) Find the lengths of the sides of the rectangle in terms of r and s.

b) Find the range of $\angle APQ$.

We received 3 correct solutions and one incorrect submission. We present the solution by Cristobal Sanchez-Rubio.



(a) Calling $\beta = \angle BPA = \angle APC = \angle CPD$ and $\alpha = \angle QPA$, we have $2\beta = \angle BPC = \angle APD$. Also,

$$\tan\angle BPC = \tan\left(\angle BPP' - \angle CPD'\right) = \frac{\frac{b+s}{r} - \frac{b+s}{a+r}}{1 + \frac{(b+s)^2}{r(a+r)}} = \frac{a\left(b+s\right)}{r\left(a+r\right) + \left(b+s\right)^2}$$

and

$$\tan\angle APD = \tan\left(\angle APP' - \angle DPD'\right) = \frac{\frac{s}{r} - \frac{s}{a+r}}{1 + \frac{s^2}{r(a+r)}} = \frac{sa}{r\left(a+r\right) + s^2}.$$

Equating these expressions gives successively

$$\begin{aligned}\frac{b+s}{r\left(a+r\right)+\left(b+s\right)^2} &= \frac{s}{r\left(a+r\right)+s^2}\\ r(b+s)(a+r) + s^2(b+s) &= rs(a+r) + s(b+s)^2\\ rb(a+r) &= sb(b+s)\\ \frac{r}{s} &= \frac{b+s}{a+r}.\end{aligned}$$

Thus $\alpha = \angle CPD'$, implying that $\beta = 90^\circ - 2\alpha$, so that $\tan 2\alpha = \frac{1}{\tan\beta}$. From

$$\tan 2\alpha = \frac{2\frac{r}{s}}{1 - \frac{r^2}{s^2}} = \frac{2rs}{s^2 - r^2}$$

and

$$\begin{aligned}\tan\beta &= \tan\left(\angle BPA\right) = \tan(\angle BPP' - \angle APP')\\ &= \frac{\frac{b+s}{r} - \frac{s}{r}}{1 + \frac{bs+s^2}{r^2}} = \frac{br}{s^2 + r^2 + bs},\end{aligned}$$

we have

$$\frac{2rs}{s^2 - r^2} = \frac{s^2 + r^2 + bs}{br},$$

implying that

$$b = \frac{s^4 - r^4}{3r^2 s - s^3}.$$

Substituting this expression for b in

$$\frac{r}{s} = \frac{b+s}{a+r},$$

we now obtain

$$a = \frac{4\left(rs^2 - r^3\right)}{3r^2 - s^2}.$$

(b) From $3\beta < 90^\circ$, $2\alpha < 90^\circ$, and $\beta = 90^\circ - 2\alpha$, we have $30^\circ < \alpha < 45^\circ$.

4302. *Proposed by Martin Lukarevski.*

Let A be a $m \times n$ matrix with $m \geq n$ and X be any $n \times m$ matrix such that XA is invertible. Find the eigenvalues of the matrix $A(XA)^{-1}X$.

We received 3 solutions and will feature just one of them here. Solution by Missouri State University Problem Solving Group.

The eigenvalues are 0 (with multiplicity $m - n$) and 1 (with multiplicity n). We first note that both A and X have rank n: Since A is $m \times n$ and X is $n \times m$ and $m \geq n$, we have rank $A \leq n$ and rank $X \leq n$. But

$$n = \text{rank}(XA) \leq \min\{\text{rank } X, \text{rank}A)\}) \leq n.$$

Now, since X has rank n, there are $m-n$ vectors in a basis for the kernel (nullspace) of X. These vectors are also in the kernel of $A(XA)^{-1}X$, showing that 0 is an eigenvalue of $A(XA)^{-1}X$ with multiplicity at least $m - n$. Next, for any $\mathbf{v}$ in F^n (we are assuming these are matrices with entries in a field F),

$$A(XA)^{-1}X(A\mathbf{v}) = A(XA)^{-1}(XA)\mathbf{v} = A\mathbf{v},$$

so every vector in the range of A is an eigenvector with eigenvalue 1. But since $\text{rank}(A) = n$, there are n linearly independent such vectors, and hence 1 is an eigenvalue of $A(XA)^{-1}X$ of multiplicity n and 0 is an eigenvalue of multiplicity exactly $m - n$.

4303. *Proposed by Tung Hoang.*

Find the following limit

$$\lim_{n\to\infty} \left\{(6 + \sqrt{35})^n\right\},$$

where $\{x\} = x - [x]$ and $[x]$ is the greatest integer function.

We received 12 submissions, all correct. Almost all of these solutions made use of the Binomial Theorem and are very similar to one another. We present two solutions, both of which give stronger results and the second one is different from all the rest.

Solution 1, by Oliver Geupel.

We prove the more general result that for all $a, b \in \mathbb{N}$ with $(a - 1)^2 < b < a^2$, it is true that

$$\lim_{n\to\infty} \{(a + \sqrt{b})^n\} = 1.$$

The proposed problem is the special case when $a = 6$ and $b = 35$. By the Binomial Theorem, we have

$$(a + \sqrt{b})^n + (a - \sqrt{b})^n = \sum_{k=0}^{n} \binom{n}{k} \left(1 + (-1)^k\right) a^{n-k} b^{k/2}$$

which is an integer, say c_n. Since $0 < a - \sqrt{b} < 1$, we have

$$[(a+\sqrt{b})^n] = [c_n - (a-\sqrt{b})^n] = c_n - 1,$$

so

$$\begin{aligned}\{(a+\sqrt{b})^n\} &= (a+\sqrt{b})^n - [(a+\sqrt{b})^n]\\ &= c_n - (a-\sqrt{b})^n - (c_n - 1)\\ &= 1 - (a-\sqrt{b})^n\end{aligned}$$

from which $\lim\limits_{n\to\infty} \{(a+\sqrt{b})^n\} = 1$ follows.

Solution 2, by Missouri State University Problem Solving Group.

We prove in general that if $x \in \mathbb{R}, x > 1$ such that $x + \dfrac{1}{x} \in \mathbb{Z}$, then $\lim\limits_{n\to\infty} \{x^n\} = 1$. The proposed problem is the special case when $x = 6 + \sqrt{35}$.

We first prove that $x^n + \dfrac{1}{x^n} \in \mathbb{Z}$ for all integers $n \geq 0$. Since this is clear for $n = 0, 1$ it follows from the strong induction that for all $n \geq 2$,

$$x^n + \frac{1}{x^n} = \left(x + \frac{1}{x}\right)\left(x^{n-1} + \frac{1}{x^{n-1}}\right) - \left(x^{n-2} + \frac{1}{x^{n-2}}\right) \in \mathbb{Z}.$$

Let $x^n + \dfrac{1}{x^n} = m \in \mathbb{Z}$. Since $x > 1$, it is obvious that $m - 1 < x^n < m$, so $[x^n] = m - 1$. Hence,

$$\{x^n\} = x^n - [x^n] = x^n - m + 1 = 1 - \frac{1}{x^n}$$

from which $\lim\limits_{n\to\infty} \{x^n\} = 1$ follows.

4304. *Proposed by Michel Bataille.*

Evaluate

$$\cot\frac{\pi}{7} + \cot\frac{2\pi}{7} + \cot\frac{4\pi}{7} + \cot^3\frac{\pi}{7} + \cot^3\frac{2\pi}{7} + \cot^3\frac{4\pi}{7}.$$

We received 12 submissions, of which 11 were correct and one was incomplete. We present the solution by Kee-Wai Lau.

We show that the sum of the problem, denoted by S, equals $\dfrac{32}{\sqrt{7}}$. Let $\theta = \dfrac{\pi}{7}$. Since

$$\cot k\theta + \cot^3 k\theta = \frac{\cos k\theta}{\sin^3 k\theta}$$

for $k = 1, 2, 4$, we can write $S = \dfrac{A}{B^3}$, where

$$A = \cos\theta \sin^3 2\theta \sin^3 4\theta + \cos 2\theta \sin^3 4\theta \sin^3\theta + \cos 4\theta \sin^3\theta \sin^3 2\theta,$$
$$B = \sin\theta \sin 2\theta \sin 4\theta.$$

Using the formulas

$$4\sin^3 x = 3\sin x - \sin 3x,$$
$$2\sin x \sin y = \cos(x-y) - \cos(x+y),$$
$$2\cos x \cos y = \cos(x-y) + \cos(x+y),$$

we readily obtain $\cos x \sin^3 y \sin^3 z$ in terms of cosines only. In this way we get

$$64\cos\theta \sin^3 2\theta \sin^3 4\theta = 8 + 12\cos\theta + 9\cos 2\theta + 5\cos 3\theta$$
$$64\cos 2\theta \sin^3 4\theta \sin^3\theta = 8 + 5\cos\theta - 12\cos 2\theta - 9\cos 3\theta$$
$$64\cos 4\theta \sin^3\theta \sin^3 2\theta = 8 - 9\cos\theta - 5\cos 2\theta + 12\cos 3\theta$$

It follows that $A = \dfrac{3+C}{8}$, where $C = \cos\theta - \cos 2\theta + \cos 3\theta$. It is easy to check that

$$C^2 = \frac{3}{2} - \frac{5C}{2}.$$

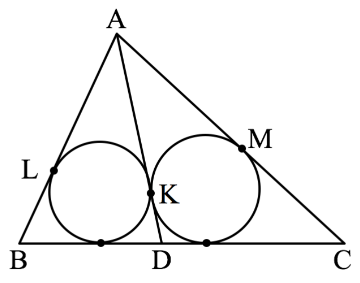
Since C is positive, $C = \dfrac{1}{2}$ and thus $A = \dfrac{7}{16}$. Hence to show that $S = \dfrac{32}{\sqrt{7}}$, it remains to show that $B = \dfrac{\sqrt{7}}{8}$. Changing product to sum, we obtain $4B = D$, where $D = \sin 3\theta + \sin 5\theta + \sin 8\theta$. It is again easy to check that

$$D^2 = \frac{3+C}{2} = \frac{7}{4}.$$

Since $D > 0$, we have $D = \dfrac{\sqrt{7}}{2}$ and thus $B = \dfrac{\sqrt{7}}{8}$ indeed. This completes the solution.

4305. *Proposed by Moshe Stupel and Avi Sigler.*

Find a nice description of the point D on side BC of a given triangle ABC so that the incircles of the resulting triangles ABD and ADC are tangent to one another at a point of their common tangent line AD.



We received 11 submissions of which all but one were correct and complete. We present the solution by Joel Schlosberg, together with a construction by the problem proposers Moshe Stupel and Avi Sigler.

We use the fact that in a triangle XYZ, the distance from vertex X to the point of tangency on side XY of the circle inscribed in XYZ is $(XY + XZ - YZ)/2$.

Under the assumption that D is a point on BC such that the incircles of triangles $\triangle ABD$ and $\triangle ACD$ are tangent to each other, and the segment AD passes through their point of tangency, we have

$$KD = \frac{DA + DB - AB}{2} = \frac{DA + DC - AC}{2}.$$

It follows that $DB - AB = DC - AC$, and

$$DB = \frac{(DC - AC + AB) + (BC - DC)}{2} = \frac{AB + BC - AC}{2}.$$

That is, D is the point of tangency on BC of the incircle of $\triangle ABC$.

Conversely, assume that D is the point of tangency on BC of the incircle of $\triangle ABC$, and let K_1 and K_2 be the points of tangency on AD of the incircles of $\triangle ABD$ and $\triangle ACD$, respectively. We have that

$$DK_1 = (DB + DA - AB)/2 \quad \text{and} \quad DK_2 = (DA + DC - AC)/2,$$

and their difference

$$\begin{aligned} DK_1 - DK_2 = \frac{DB - AB - DC + AC}{2} &= \frac{DB - AB + (BD - BC) + AC}{2} \\ &= DB - \frac{AB + BC - AC}{2} = 0. \end{aligned}$$

So, the points of tangency K_1 and K_2, are the same. Thus the incircles of $\triangle ABD$ and $\triangle ACD$ are tangent to each other.

In conclusion, the point D can be characterized as the point of tangency on BC of the circle inscribed in $\triangle ABC$.

Editor's comment. Moshe Stupel and Avi Sigler presented a construction of the point D. Their construction is because D can also be viewed as the foot of the perpendicular from the center of the circle inscribed in $\triangle ABC$ to the side BC. Specifically, construct the bisectors of angles $\angle ABC$ and $\angle ACB$ and take their intersection to find the center of the circle inscribed in $\triangle ABC$. From the incircle center we drop a perpendicular to the side BC and obtain the point D.

Most solvers, starting from the assumption that the incircles of triangles $\triangle ABD$ and $\triangle ACD$ are tangent to each other, showed that the point D is the point of tangency on BC of the incircle of $\triangle ABC$. Two authors, Joel Schlosberg and C.R. Pranesachar, added that the reverse statement holds, namely, if D is the point of tangency on BC of the incircle of $\triangle ABC$ then the incircles of triangles $\triangle ABD$ and $\triangle ACD$ are tangent to each other.

4306. *Proposed by Marius Drăgan.*

Prove that

$$\left[\sqrt{n}+\sqrt{n+1}+\sqrt{n+2}+\sqrt{n+3}\right]=\left[\sqrt{16n+20}\right]$$

for all $n \in \mathbb{N}$.

We received 9 solutions. We present the solution by Nghia Doan, slightly edited.

It is simple to verify the identity when $n = 0$, so assume that $n \geq 1$. By squaring both sides, one can easily check that $\sqrt{n}+\sqrt{n+3} \geq \sqrt{4n+5}$:

$$n+(n+3)+2\sqrt{n(n+3)} \geq 4n+5 \iff \sqrt{n(n+3)} \geq n+1 \iff n^2+3n \geq n^2+2n+1,$$

where the last inequality clearly holds for $n \geq 1$.

Moreover,

$$\sqrt{n+1}+\sqrt{n+2}-\sqrt{n}-\sqrt{n+3}=\frac{1}{\sqrt{n+1}+\sqrt{n}}-\frac{1}{\sqrt{n+3}+\sqrt{n+2}}>0,$$

whence $\sqrt{n+1}+\sqrt{n+2} > \sqrt{n}+\sqrt{n+3} \geq \sqrt{4n+5}$. Therefore

$$\sqrt{n}+\sqrt{n+1}+\sqrt{n+2}+\sqrt{n+3} > 2\sqrt{4n+5}=\sqrt{16n+20}.$$

On the other hand, Jensen's inequality applied to the square root function (which is a concave function) gives us that

$$\begin{aligned}\sqrt{n}+\sqrt{n+1}+\sqrt{n+2}+\sqrt{n+3} &\leq 4\sqrt{\frac{n+(n+1)+(n+2)+(n+3)}{4}}\\ &= \sqrt{16n+24}.\end{aligned}$$

Now suppose that there is an integer m such that

$$\sqrt{16n+20} < m \leq \sqrt{n}+\sqrt{n+1}+\sqrt{n+2}+\sqrt{n+3} \leq \sqrt{16n+24}.$$

Then, squaring, we get $16n+20 < m^2 \leq 16n+24$, which means m^2 must be one of $16n+21, \ldots, 16n+24$. However, the quadratic residues mod 16 are 0, 1, 4 and 9, which eliminates all the listed possibilities. So there is no integer m such that

$$\sqrt{16n+20} < m \leq \sqrt{16n+24},$$

which means that

$$[\sqrt{16n+20}]=[\sqrt{16n+24}].$$

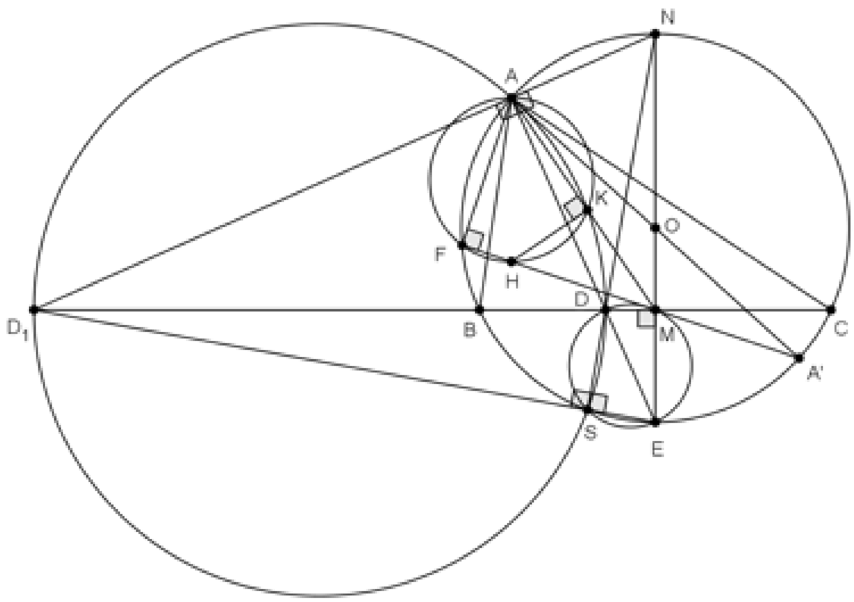
Therefore,

$$[\sqrt{n}++\sqrt{n+1}+\sqrt{n+2}+\sqrt{n+3}]=[\sqrt{16n+20}].$$

4307. *Proposed by Adnan Ibric and Salem Malikic.*

In a non-isosceles triangle ABC, let H and M denote the orthocenter and the midpoint of side BC, respectively. The internal angle bisector of $\angle BAC$ intersects BC and the circumcircle of triangle ABC at points D and E, $E \neq A$. If K is the foot of the perpendicular from H to AM and S is the intersection (other than E) of the circumscribed circles of triangles ABC and DEM, prove that quadrilateral $ASDK$ is cyclic.

We received five submissions, all of which were correct, and will feature the one from Stefan Lozanovski.



Let N be the midpoint of the arc BC that contains A. Then, because E is the midpoint of the arc BC opposite A, M lies on the diameter EN and $MD \perp EN$. Also, since

$$\angle DSE = 180^\circ - \angle DME = 90^\circ = \angle NSE,$$

point D must lie on the line SN and $ES \perp ND$. Thirdly, because $\angle EAN = 90^\circ$, we have $NA \perp DE$. Therefore the lines CD (which coincides with MD), ES, and NA concur at the orthocenter of triangle DEN, which we denote by D_1. Moreover, because of the right angles at S and at A, the quadrangle $SDAD_1$ is cyclic. It remains to show that K lies on this circle.

Since D and D_1 are the intersections of the line BC with the internal and external bisectors of $\angle BAC$, we know that D and D_1 are harmonic conjugates with respect to B and C; consequently, since M is the center of the circle with diameter BC, we get

$$MD \cdot MD_1 = MB^2.$$

Let A' be the antipode of A on the circumcircle (ABC). An old theorem of triangle geometry says that M is the midpoint of HA':

$$MH = MA'.$$

Let F be the second intersection of MH with (ABC). Because AA' is a diameter, $\angle AFA' = 90°$; this with the given assumption that $\angle AKH = 90°$ implies that $AFHK$ is cyclic and, therefore

$$MH \cdot MF = MK \cdot MA.$$

Note, finally, that the power of M with respect to (ABC) equals

$$MB \cdot MC = MF \cdot MA'.$$

In conclusion, we have

$$MD \cdot MD_1 = MB^2 = MB \cdot MC = MF \cdot MA' = MF \cdot MH = MK \cdot MA,$$

which implies that KAD_1D is cyclic. Since both S and K lie on the circle determined by A, D, D_1, we conclude that $ASDK$ is cyclic.

Comment by the proposers. The point K of our problem is known to have several nice properties. These were investigated in a recently published article, "A Special Point on the Median" by Anant Mudgal and Gunmay Handa [*Mathematical Reflections*, issue 2, 2017]. Among other things, the line BC is the perpendicular bisector of the segment KS, and AS is a symmedian of triangle ABC. While these results would quickly establish our result, we independently came upon our problem.

4308. *Proposed by Leonard Giugiuc and Sladjan Stankovik.*

Let a, b and c be positive real numbers. Prove that

$$27abc(a^2b + b^2c + c^2a) \leq (a + b + c)^2(ab + bc + ca)^2.$$

Eight correct solutions were received. One additional solution used Maple *and two others were wrong. We present the solution by AN-anduud Problem Solving Group.*

Let

$$A = a^2b + b^2c + c^2a, \quad B = ab^2 + bc^2 + ca^2, \quad C = 3abc.$$

The inequality can be written as

$$9CA \leq (A + B + C)^2.$$

Using the inequality

$$(u + v + w)^2 \geq 3(uv + vw + wu),$$

we find that

$$\begin{aligned} B &= \sqrt{(ca^2+ab^2+bc^2)(ab^2+bc^2+ca^2)} \\ &\geq \sqrt{3abc(a^2b+b^2c+c^2a)} = \sqrt{CA}, \end{aligned}$$

so that, by the arithmetic-geometric means inequality,

$$(A+B+C)^2 \geq (A+C+\sqrt{CA})^2 \geq (2\sqrt{CA}+\sqrt{CA})^2 = 9CA,$$

as desired.

4309. *Proposed by Daniel Sitaru.*

Let a, b and c be real numbers such that $a+b+c=3$. Prove that

$$2(a^4+b^4+c^4) \geq ab(ab+1)+bc(bc+1)+ca(ca+1).$$

Fifteen correct solutions were received. While it was not clear that everyone took on board the possibility that some of the variables could be negative, the inequalities they invoked turned out to be justifiable. We present three solutions.

Solution 1, by Šefket Arslanagić.

Using the inequality

$$3(x^2+y^2+z^2) \geq (x+y+z)^2,$$

we have that

$$\begin{aligned} a^4+b^4+c^4 &\geq \frac{1}{3}(a^2+b^2+c^2)(a^2+b^2+c^2) \\ &\geq \frac{1}{9}(a+b+c)^2(a^2+b^2+c^2) = a^2+b^2+c^2 \\ &\geq ab+bc+ca. \end{aligned}$$

Also $a^4+b^4+c^4 \geq a^2b^2+b^2c^2+c^2a^2$, so that the desired inequality holds.

Solution 2, by the AN-anduud Problem Solving Group.

Using the inequality

$$x^2+y^2+z^2 \geq xy+yz+zx,$$

we have that

$$a^4+b^4+c^4 \geq (ab)^2+(bc)^2+(ca)^2.$$

Since

$$x^4-4x+3 = (x-1)^2[(x+1)^2+2] \geq 0,$$

we find that

$$\begin{aligned} a^4+b^4+c^4 &\geq 4(a+b+c)-9 = 3 = \frac{1}{3}(a+b+c)^2 \\ &= \frac{1}{6}[(a-b)^2+(b-c)^2+(c-a)^2+6(ab+bc+ca)] \\ &\geq ab+bc+ca. \end{aligned}$$

Adding these two inequalities for $a^4 + b^4 + c^4$ yields the desired result. Equality holds iff $a = b = c = 1$.

Solution 3, by Paolo Perfetti and Angel Plaza, independently.

Recall the Muirhead Inequalities for three variables. For $a, b, c > 0$ and $p \geq q \geq r$, let

$$[p, q, r] = \frac{1}{6}(a^p b^q c^r + a^p b^r c^q + a^q b^p c^r + a^q b^r c^p + a^r b^p c^q + a^r b^q c^p).$$

Then,

$$p \geq u, \quad p + q \geq u + v \quad \text{and} \quad p + q + r = u + v + w$$

together imply that $[p, q, r] \geq [u, v, w]$.

Make the given inequality homogeneous by replacing each 1 by $\frac{1}{9}(a+b+c)^2$. Thus we have to prove that

$$\begin{aligned} 18(a^4 + b^4 + c^4) \geq 11(a^2b^2 + b^2c^2 + a^2c^2) + (a^3b + ab^3 + b^3c + bc^3 + a^3c + ac^3) \\ + 5(a^2bc + ab^2c + abc^2), \end{aligned}$$

or, equivalently,

$$9[4, 0, 0] \geq \frac{11}{2}[2, 2, 0] + [3, 1, 1] + \frac{5}{2}[2, 1, 1].$$

This is true since

$$\begin{aligned} [4, 0, 0] &\geq [2, 2, 0], \\ [4, 0, 0] &\geq [3, 1, 1], \\ [4, 0, 0] &\geq [2, 1, 1]. \end{aligned}$$

4310. *Proposed by Steven Chow.*

Let $\triangle A_1B_1C_1$ be the incentral triangle with respect to $\triangle ABC$, i.e., A_1 is the point of intersection of $\overline{BC}$ and $\overleftrightarrow{AI}$ where I is the incentre of $\triangle ABC$, with B_1 and C_1 similarly defined. Let r be the inradius of $\triangle ABC$.

a) Prove that $AA_1 \cdot BB_1 \cdot CC_1 \geq \frac{3\sqrt{3}}{2}(BC + CA + AB)\, r^2$.

b) $\star$ Prove or disprove that $B_1C_1 \cdot C_1A_1 \cdot A_1B_1 \geq 3r^3\sqrt{3}$.

Remark. Curiously, this problem was discovered by the proposer when he misread problem 4203. See problem 4203 to appreciate the connection.

We received 6 submissions of which 4 were correct and complete. We present one solution for part a) and two solutions for part b).

Solution 1 to part a), by Michel Bataille.

Let

$$a = BC, \; b = CA, \; c = AB \text{ and } s = (a + b + c)/2.$$

Since AA_1, BB_1, and CC_1 are the angle bisectors of triangle $\triangle ABC$, we know that their lengths are

$$AA_1 = \frac{2\sqrt{bcs(s-a)}}{b+c}, \quad BB_1 = \frac{2\sqrt{cas(s-b)}}{c+a}, \quad CC_1 = \frac{2\sqrt{abs(s-c)}}{a+b}.$$

Therefore,

$$AA_1 \cdot BB_1 \cdot CC_1 = \frac{8s \cdot abc\sqrt{s(s-a)(s-b)(s-c)}}{(b+c)(c+a)(a+b)}.$$

Due to the link between the area of a triangle, the length of its three sides (Heron's formula) and the radius of its inscribed circle, r, we have

$$Area(\triangle ABC) = \sqrt{s(s-a)(s-b)(s-c)} = rs,$$

and

$$AA_1 \cdot BB_1 \cdot CC_1 = \frac{8s^2r \cdot abc}{(b+c)(c+a)(a+b)}.$$

Moreover, $abc = 4srR$, where R is the radius of circumscribed circle of $\triangle ABC$, leading to

$$AA_1 \cdot BB_1 \cdot CC_1 = \frac{32s^3r^2R}{(b+c)(c+a)(a+b)}.$$

From the concavity of the sine function on $[0, \pi]$, we have

$$2s = 2R(\sin A + \sin B + \sin C) \leq 2R \cdot 3 \cdot \sin\left(\frac{A+B+C}{3}\right) = 3R\sqrt{3} \tag{1}$$

and, consequently

$$AA_1 \cdot BB_1 \cdot CC_1 \geq \frac{64s^4r^2}{3\sqrt{3}(b+c)(c+a)(a+b)}.$$

Based on AM-GM inequality

$$(b+c)(c+a)(a+b) \leq \left(\frac{b+c+c+a+a+b}{3}\right)^3 = \frac{64s^3}{27} \tag{2}$$

we obtain as required

$$AA_1 \cdot BB_1 \cdot CC_1 \geq \frac{64s^4r^2}{3\sqrt{3}} \cdot \frac{27}{64s^3} = \frac{3\sqrt{3}}{2}(BC + CA + AB)r^2.$$

Solution 2 to part b), by Michel Bataille.

We prove the inequality. The angle bisector theorem can be used to calculate the following lengths:

$$BA_1 = ac/(b+c), \; CA_1 = ab/(b+c), \; AB_1 = bc/(a+c),$$
$$CB_1 = ab/(a+c), \; AC_1 = bc/(a+b), \; BC_1 = ac/(a+b).$$

Consequently,

$$\begin{aligned} B_1C_1^2 &= AC_1^2 + AB_1^2 - 2AC_1 \cdot AB_1 \cos A \\ &= \frac{b^2c^2}{(a+b)^2(a+c)^2}\left[(a+b)^2 + (a+c)^2 - 2(a+b)(a+c)\cos A\right] \\ &= \frac{b^2c^2}{(a+b)^2(a+c)^2}\left[(1-\cos A)(2a^2+2ac+2ab+2bc) + (b-c)^2\right] \\ &\geq \frac{4b^2c^2}{(a+b)(a+c)}\sin^2(A/2). \end{aligned}$$

Above we used $1 - \cos A = 2\sin^2(A/2)$. Similar inequalities are valid for C_1A_1 and A_1B_1, implying that

$$B_1C_1 \cdot C_1A_1 \cdot A_1B_1 \geq \frac{8a^2b^2c^2 \sin(A/2)\sin(B/2)\sin(C/2)}{(a+b)(a+c)(b+c)}.$$

Since

$$r = 4R\sin(A/2)\sin(B/2)\sin(C/2),$$

$abc = 4srR$, and inequalities (1) and (2) at part a) we obtain, as claimed

$$B_1C_1 \cdot C_1A_1 \cdot A_1B_1 \geq \frac{27}{64s^3} \cdot 16s^2r^2R^2 \cdot \frac{2r}{R} = 27r^3 \cdot \frac{R}{2s} \geq 27r^3 \cdot \frac{1}{3\sqrt{3}} = 3r^3\sqrt{3}.$$

Solution 3 to part b), by Leonard Giugiuc.

Notations used in part a) are maintained in what follows. We prove the inequality using two known results.

Lemma 1. If $\triangle XYZ$ is an arbitrary triangle then

$$(XY \cdot YZ \cdot ZX)^2 \geq \frac{64}{3\sqrt{3}}(Area(\triangle XYZ))^3.$$

Lemma 2. In $\triangle ABC$

$$Area(\triangle A_1B_1C_1) = \frac{2abc}{(a+b)(b+c)(c+a)} Area(\triangle ABC).$$

After combining the two lemmas we obtain

$$(B_1C_1 \cdot C_1A_1 \cdot A_1B_1)^2 \geq \frac{64}{3\sqrt{3}}\left(\frac{2abc}{(a+b)(b+c)(c+a)} Area(\triangle ABC)\right)^3.$$

Due to relations $abc = 4srR$ and $Area(\triangle ABC) = rs$ the last inequality becomes

$$(B_1C_1 \cdot C_1A_1 \cdot A_1B_1)^2 \geq \frac{64}{3\sqrt{3}}\left(\frac{8s^2r^2R}{(a+b)(b+c)(c+a)}\right)^3.$$

Moreover, using inequalities (1) and (2) mentioned in part a) we have, as required

$$(B_1C_1 \cdot C_1A_1 \cdot A_1B_1)^2 \geq \frac{64}{3\sqrt{3}} \left(\frac{16s^3r^2/(3\sqrt{3})}{64s^3/27} \right)^3 = 27r^6.$$

Editor's comment. Šefket Arslanagić pointed out that inequalities (1) and (2) were crucial to establishing the inequalities in parts a) and b). If part a) or part b) displays equality then (1) and (2) must become equalities as well. Then the triangle must be an equilateral triangle. In addition he mentioned that inequality (1) was established before and provided us the reference: Bottema O., Djordjević R.Ž., Janić R.R., Mitrinović D.S., and Vasić P.M. (1969) *Geometric inequalities*, Wolters-Noordhoff Publishing, Groningen, The Netherlands.

SNAPSHOT

The home of Miguel Amengual Covas, Cala Figuera is a village on the island of Mallorca in Spain.