Crux Mathematicorum

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EDITORIAL

A year ago, in my last editorial of 2017, I compared Volume 45 of ***Crux*** to Schrödinger's Cat: just like the cat may or may not be alive, it wasn't clear whether Volume 45 was going to happen or not. The discussions about closing ***Crux*** due to its costs were brought up at the CMS Board of Directors meeting every year since I've been in this role and probably before me, and maybe even before I was alive since ***Crux*** is actually older than I am and has never turned a profit. But this time it was serious: if the CMS could not obtain funding to cover operational costs to offer ***Crux*** as a free open access online journal, then it would not offer it at all.

Unfortunately for all of those who were hoping to never hear from me regarding ***Crux*** again (this includes several members of my immediate family), ***Crux*** will live another day. In fact, it will live at least another 3 years due to the generous support of the sponsor who will be announced in the New Year.

To say I am thrilled would be a gross understatement. Not only does this mean that ***Crux*** will live on, it also means that it will be more accessible as no subscriptions will be necessary. Online format will also allow for inclusion of more graphical, potentially dynamic content and, at the very minimum, more colours. We are already planning some reshuffling of the existing sections and introduction of new ones. However, this will be work in progress from January 2019 on. See, it would have been too easy and un-suspensful if a sponsor were found in, say, March or July or even October of 2018. That would have allowed time for planning and sorting out the details and so on. Instead, it was November 27th when I first heard the news. So with minimal time to plan the transition, changes will be implemented on an ongoing basis once ***Crux*** goes online in 2019.

In the meantime, I want to thank all of our devoted readers for your concerned emails and your support. This publication consists of your efforts as much as ours and I look forward to our future together.

Kseniya Garaschuk

THE CONTEST CORNER

No. 69

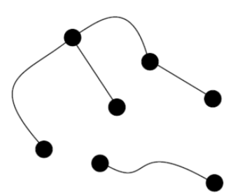
John McLoughlin

The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **April 1, 2019**.

The editor thanks Valérie Lapointe, Carignan, QC, for translations of the problems.

CC341. The graph below shows 7 vertices (the dots) and 5 edges (the lines) connecting them. An edge here is defined to be a line that connects 2 vertices together. In other words, an edge cannot loop back and connect to the same vertex. Edges are allowed to cross each other, but the crossing of 2 edges does not create a new vertex. What is the least number of edges that could be added to the graph, in addition to the 5 already present, so that each of the 7 vertices has the same number of edges?

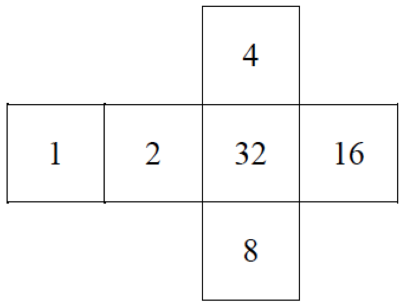


CC342. You are given the 5 points $A = (1,1), B = (2,-1), C = (-2,-1), D = (0,0)$ and $P_0 = (0,2)$. P_1 equals the rotation of P_0 around A by 180°, P_2 equals the rotation of P_1 around B by 180°, P_3 equals the rotation of P_2 around C by 180°, P_4 equals the rotation of P_3 around D by 180°, P_5 equals the rotation of P_4 around A by 180°, and so on repeating this pattern. If $P_{2016} = (a,b)$, then what is the value of $a + b$?

CC343. In the following long division problem, most of the digits (26 in fact) are hidden by the symbol X. What is the sum of all of the 26 hidden digits?

```
          X X 8 X X
     ______________
X X | X X X X X X X
      X X X
      _____________
            X X
            X X
            _______
              X X X
              X X X
              _____
                  1
```

CC344. Take the pattern below and form a cube. Then take three of these exact same cubes and stack them one on top of another on a table so that exactly 13 numbers are visible. What is the greatest possible sum of these 13 visible numbers?



CC345. Your teacher asks you to write down five integers such that the median is one more than the mean, and the unique mode is one greater than the median. You then notice that the median is 10. What is the smallest possible integer that you could include in your list?

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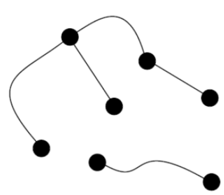
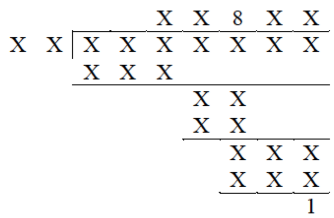
CC341. Le graphe ci-dessous possède 7 sommets (les points) et 5 arêtes (les lignes) qui les relient. Une arête est définie comme un segment reliant 2 sommets. En d'autres mots, une arête ne peut relier un sommet à lui-même. Les arêtes peuvent se croiser, mais ce croisement ne crée pas un nouveau sommet. Quel est le plus petit nombre d'arêtes pouvant être ajoutées au graphe en tenant compte des 5 arêtes déjà présentes de façon à ce que les 7 sommets possèdent le même nombre d'arêtes.

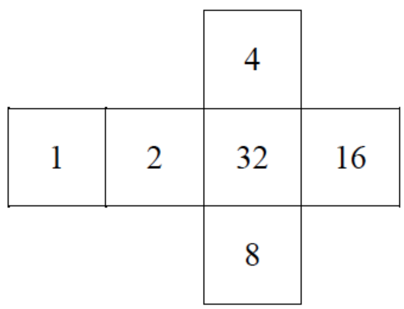


CC342. Soit les 5 points $A = (1, 1), B = (2, -1), C = (-2, -1), D = (0, 0)$ et $P_0 = (0, 2)$. P_1 est la rotation de P_0 autour de A de 180°, P_2 est la rotation de P_1 autour de B de 180°, P_3 est la rotation de P_2 autour de C de 180°, P_4 est la rotation de P_3 autour de D de 180°, P_5 est la rotation de P_4 autour de A de 180° et ainsi de suite. Si $P_{2016} = (a, b)$, quelle est la valeur de $a + b$?

CC343. Dans la division suivante, presque tous les chiffres (26 exactement) sont cachés par le symbole X. Quelle est la somme de ces 26 chiffres cachés?



CC344. On forme un cube à partir du motif ci-dessous. On prend ensuite trois de ces mêmes cubes et on les empile les uns sur les autres sur une table de façon à ce qu'exactement 13 nombres soient visibles. Quelle est la plus grande somme possible de ces 13 nombres visibles ?



CC345. Votre professeur vous demande d'écrire cinq entiers tels que la médiane est supérieure de un à la moyenne et l'unique mode est supérieur de un à la médiane. Il vous dit maintenant que la médiane est 10. Quel est le plus petit entier que vous pouvez inclure dans cette liste ?

CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2017: 43(9), p. 375–377.

CC291. The point P is on the parabola $x^2 = 4y$. The tangent at P meets the line $y = -1$ at the point A. For the point $F(0, 1)$, prove that $\angle AFP = 90^\circ$ for all positions of P, except $(0, 0)$.

Originally Problem C3 from the 1998 Descartes Contest.

We received 11 submissions, all correct. We present the solution by Dan Daniel.

Suppose the point P has coordinates $\left(a, \frac{a^2}{4}\right)$. For a parabola of the form $x^2 = 2py$, here $p = 2$, point F is the focus and the line $y = -1$ is the directrix. Computing the tangent at P, we have $xa = 2\left(y + \frac{a^2}{4}\right)$ and $y = -1$, which implies $x = \frac{a}{2} - \frac{2}{a}$, so point A has coordinates $(\frac{a}{2} - \frac{2}{a}, -1)$. Then the slope of AF is

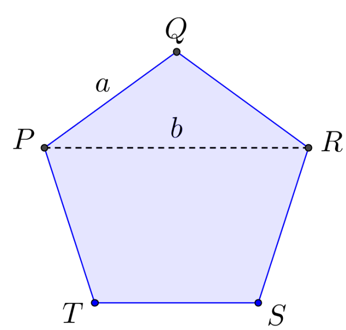
$$s_1 = \frac{1 - (-1)}{0 - (\frac{a}{2} - \frac{2}{a})} = \frac{4a}{4 - a^2}$$

and the slope of PF is

$$s_2 = \frac{\frac{a^2}{4} - 1}{a - 0} = \frac{a^2 - 4}{4a}.$$

Since $s_1 s_2 = -1$, AF is perpendicular to PF and so $\angle AFP = 90^\circ$.

CC292. Let a be the length of a side and b be the length of a diagonal in the regular pentagon $PQRST$ as shown.



Prove that

$$\frac{b}{a} - \frac{a}{b} = 1.$$

Originally Problem D1 from the 1998 Descartes Contest.

We received 12 solutions. We present the solution by Titu Zvonaru.

$PQRT$ is a cyclic quadrilateral, so by Ptolemy's theorem

$$PR \cdot QT = PQ \cdot TR + PT \cdot QR;$$

in other words, $b^2 = ab + a^2$. Dividing both sides by ab and rearranging gives us $\frac{b}{a} - \frac{a}{b} = 1$, as desired.

CC293. The transformation $T : (x, y) \mapsto \left(-\frac{1}{2}(3x - y), -\frac{1}{2}(x + y)\right)$ is applied repeatedly to the point $P_0(3, 1)$, which produces a sequence of points $P_1, P_2, \ldots$. Show that the area of the convex quadrilateral defined by any four consecutive points is constant.

Originally Problem D2 from the 1998 Descartes Contest.

We received five submissions to this problem, all of which were correct and complete. We present the composite solution by Dan Daniel and Šefket Arslanagić (done independently), modified by the editor.

Given $P_n(x_n, y_n)$, it follows that

$$P_{n+2}(2x_n - y_n, x_n). \tag{1}$$

The slope of P_nP_{n+2} is

$$\frac{y_n - x_n}{x_n - 2x_n + y_n} = 1.$$

Inductively, it follows that $P_0, P_2, P_4, \ldots$ are collinear along the line $y = x - 2$. Similarly, $P_1, P_3, P_5, \ldots$ are collinear points along the line $y = x + 2$. We note that these lines are parallel and that any convex quadrilateral defined by four consecutive points is a trapezoid with bases P_nP_{n+2} and $P_{n+1}P_{n+3}$. Without loss of generality, assume that P_nP_{n+2} lies along $y = x - 2$ and $P_{n+1}P_{n+3}$ lies along $y = x + 2$.

Given $P_n(x_n, x_n - 2)$, it follows from (1) that $P_{n+2}(x_n + 2, x_n)$. Therefore,

$$|P_nP_{n+2}| = \sqrt{2^2 + 2^2} = 2\sqrt{2}.$$

The same methodology can be used to show that $|P_{n+1}P_{n+3}| = 2\sqrt{2}$. Therefore, the convex quadrilateral formed by the points $P_n, P_{n+1}, P_{n+2}, P_{n+3}$ has base lengths of $2\sqrt{2}$.

As $y = x - 2$ and $y = x + 2$ are parallel, we solve for the distance between the two lines with the points $(1, -1)$ and $(-1, 1)$, found on each line respectively:

$$\sqrt{(1 + 1)^2 + (-1 - 1)^2} = 2\sqrt{2}.$$

Therefore, the height of the convex quadrilateral formed by the points $P_n, P_{n+1}, P_{n+2}, P_{n+3}$ is $2\sqrt{2}$.

The area of the convex quadrilateral formed by the points $P_n, P_{n+1}, P_{n+2}, P_{n+3}$ is therefore

$$\frac{\left(2\sqrt{2}+2\sqrt{2}\right)2\sqrt{2}}{2} = 8.$$

CC294.

a) Prove that $\sin 2A = \dfrac{2\tan A}{1+\tan 2A}$, where $0 < A < \pi/2$.

b) If $\sin 2A = 4/5$, find $\tan A$.

Originally Problem B2 from the 1998 Descartes Contest.

We received 15 correct solutions. We present the solution by Ivko Dimitrić.

The statement, as printed, is false, due to a typo. Namely, if we let $A \to \pi/4$ then the left-hand side approaches 1, whereas the right-hand side approaches 0. The term $\tan 2A$ in the denominator should be corrected to $\tan^2 A$ for the statement to hold. Thus, we prove that

$$\sin 2A = \frac{2\tan A}{1+\tan^2 A},$$

where $0 < A < \pi/2$. Beginning with the right-hand side we have

$$\frac{2\tan A}{1+\tan^2 A} = \frac{2\tan A}{\sec^2 A} = 2\cdot\frac{\sin A}{\cos A}\cdot\cos^2 A = 2\sin A\cos A = \sin 2A,$$

proving (a). Now let $x = \tan A$. Using the relation in part (a), we solve

$$\frac{4}{5} = \frac{2x}{1+x^2} \Longleftrightarrow 4+4x^2 = 10x \Longleftrightarrow 2(2x-1)(x-2) = 0,$$

with two solutions $x = 2$ and $x = 1/2$. Thus $\tan A \in \{2, 1/2\}$.

CC295.

In how many ways is it possible to choose four distinct integers from $1, 2, 3, 4, 5, 6$ and 7, so that their sum is even?

Originally Problem A5 from the 1998 Descartes Contest.

We received 10 submissions, all correct and complete. We present the solution by Kathleen E. Lewis.

Since there are four odd numbers and three even numbers to choose from, one must either pick all four odd numbers or two odds and two evens. There is only one way to choose four odds, and $\binom{4}{2}\binom{3}{2} = 18$ ways to choose two odds and two evens. So altogether there are 19 ways to get four numbers whose sum is even.

THE OLYMPIAD CORNER

No. 367

Alessandro Ventullo

The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **April 1, 2019**.

The editor thanks Valérie Lapointe, Carignan, QC, for translations of the problems.

OC401. Determine all polynomials $P(x) \in \mathbb{R}[x]$ satisfying the following two conditions:

(a) $P(2017) = 2016$;

(b) $(P(x)+1)^2 = P(x^2+1)$ for all real numbers x.

OC402. Find all natural numbers n that satisfy the following property: for each integer $k \geq n$ there is a multiple of n whose digits sum up to k.

OC403. Let S be the point of tangency of the incircle of a triangle ABC with the side AC. Let Q be a point such that the midpoints of the segments AQ and QC lie on the incircle. Prove that QS is the angle bisector of $\angle AQC$.

OC404. Let $(A, +, \cdot)$ be a ring simultaneously satisfying the conditions:

(i) A is not a division ring;

(ii) $x^2 = x$ for every invertible element $x \in A$.

Prove that:

(a) $a + x$ is not invertible for any $a, x \in A$, where a invertible and $x \neq 0$ is not invertible;

(b) $x^2 = x$ for all $x \in A$.

OC405. Each cell of a 100×100 table is painted either black or white and all the cells adjacent to the border of the table are black. It is known that in every 2×2 square there are cells of both colours. Prove that in the table there is 2×2 square that is coloured in the chessboard manner.

. . . .

OC401. Déterminer tous les polynomes $P(x) \in \mathbb{R}[x]$ qui satisfont aux deux conditions suivantes :

(a) $P(2017) = 2016$;

(b) $(P(x)+1)^2 = P(x^2+1)$ pour tous les nombres réels x.

OC402. Trouver tous les nombres naturels n qui satisfont à la propriété suivante: pour chaque entier $k \geq n$, il y a un multiple de n dont la somme des chiffres est k.

OC403. Soit S le point de rencontre entre le cercle inscrit du triangle ABC et le côté AC. Soit Q un point tel que les points milieux des segments AQ et QC soient sur le cercle inscrit. Montrer que QS est la bissectrice de $\angle AQC$.

OC404. Soit $(A, +, \cdot)$ un anneau qui satisfait simultanément aux conditions suivantes :

(i) A n'est pas un anneau pour la division;

(ii) $x^2 = x$ pour tout élément inversible $x \in A$.

Montrer que :

(a) $a + x$ n'est pas inversible pour tout $a, x \in A$, où a est inversible et $x \neq 0$ n'est pas inversible;

(b) $x^2 = x$ pour tout $x \in A$.

OC405. Chaque cellule d'une grille 100×100 est colorée en noir ou en blanc et toutes les cellules adjacentes à la bordure de la grille sont noires. On sait que dans tous les carrés 2×2 il y a autant de cellules de chaque couleur. Montrer que dans la grille, il y a un carré 2×2 coloré à la façon d'un échequier.

OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2017: 43(7), p. 285–287.

OC341. There are 30 teams in the NBA and every team plays 82 games in the year. Owners of the NBA teams want to divide all teams into Western and Eastern Conferences (not necessarily equally), such that the number of games between teams from different conferences is half the number of all games. Can they do it?

Originally 2016 AllRussian Olympiad Grade 11, Day 1, Problem 1.

We received only one solution. We present the solution by Mohammed Aassila.

We prove, by contradiction, that they cannot do it. We use graph theory. We construct a graph such that its vertices are the teams and we draw an edge between two teams if they have played with each other (it may be more than one edge between two teams). Now let A be the subgraph with Western teams and name B the subgraph with Eastern teams. We know that there are 615 edges between A and B. We know that in each graph, there are exactly an even number of vertices with odd degree, so we get that there are an even number of vertices in A with odd degree, and since the degree of each vertex is 82, each of these vertices send an odd number of edges to B, so there must be an even number of edges between A and B, contradiction.

OC342. Consider the second-degree polynomial $P(x) = 4x^2 + 12x - 3015$. Define the sequence of polynomials $P_1(x) = \frac{P(x)}{2016}$ and $P_{n+1}(x) = \frac{P(P_n(x))}{2016}$ for every integer $n \geq 1$.

a) Show that there exists a real number r such that $P_n(r) < 0$ for every positive integer n.

b) For how many integers m does $P_n(m) < 0$ hold for infinitely many positive integers n?

Originally 2016 Brazil National Olympiad Day 2, Problem 5.

We received 3 solutions. We present the solution by Mohammed Aassila.

Let $Q(x) = \dfrac{P(x)}{2016}$. Then, $P_1(x) = Q(x)$ and $P_{n+1}(x) = Q(P_n(x))$.

Define $P_0(x) = x$. We have

$$Q(x) = \frac{(2x+3)^2 - 3024}{2016} = \frac{\left(x+\frac{3}{2}\right)^2}{504} - \frac{3}{2}, \qquad \frac{Q(x)+\frac{3}{2}}{504} = \left(\frac{x+\frac{3}{2}}{504}\right)^2.$$

Then, for every positive integer n

$$\frac{P_n(x)+\frac{3}{2}}{504}=\left(\frac{P_{n-1}(x)+\frac{3}{2}}{504}\right)^2=\ldots=\left(\frac{P_0(x)+\frac{3}{2}}{504}\right)^{2^n}.$$

Hence,

$$P_n(x)=504\cdot\left(\frac{x+\frac{3}{2}}{504}\right)^{2^n}-\frac{3}{2}.$$

(a) $P_n\left(-\frac{3}{2}\right)=-\frac{3}{2}<0$. Take $r=-\frac{3}{2}$.

(b) $P_n(m)<0$ if and only if $\left(\frac{m+\frac{3}{2}}{504}\right)^{2^n}<\frac{1}{336}$. This holds for infinitely many positive integers n if and only if $\left|\frac{m+\frac{3}{2}}{504}\right|<1$, i.e. if and only if $-505.5<m<502.5$. Therefore there are 1008 integers with this property.

OC343. Determine all pairs of positive integers (a,n) with $a\geq n\geq 2$ for which $(a+1)^n+a-1$ is a power of 2.

Originally 2016 Italian Mathematical Olympiad, Problem 4.

We received 3 solutions. We present the solution by Oliver Geupel.

The pair $(a,n)=(4,3)$ is a solution, and we prove that there are no other ones.

Let (a,n) be any solution of the problem and let

$$(a+1)^n+a-1=2^m. \tag{1}$$

Since a divides $(a+1)^n+a-1$, it follows from (1) that $a=2^k$ for some positive integer k. We have $2k\leq nk<m$, whence $2^m\equiv 0 \pmod{2^{2k}}$. By the binomial theorem,

$$(a+1)^n+a-1=\sum_{j=2}^{n}\binom{n}{j}2^{jk}+(n+1)2^k\equiv(n+1)2^k \pmod{2^{2k}},$$

Hence,

$$(n+1)2^k\equiv 0 \pmod{2^{2k}},$$

which implies that $a=2^k$ is a divisor of $n+1$. Since $n\leq a$, we obtain $n=2^k-1$. Thus, $n\geq 3$.

We have $3k \leq nk < m$, whence $2^m \equiv 0 \pmod{2^{3k}}$. By the binomial theorem,

$$\begin{aligned}(a+1)^n + a - 1 &= \binom{n}{2}2^{2k} + (n+1)2^k + \sum_{j=3}^{n}\binom{n}{j}2^{jk} \\ &\equiv \binom{2^k-1}{2}2^{2k} + 2^{2k} \\ &\equiv (2^{2k-1} - 2^k - 2^{k-1} + 2)2^{2k} \pmod{2^{3k}}.\end{aligned}$$

Hence,

$$(2^{2k-1} - 2^k - 2^{k-1} + 2)2^{2k} \equiv 0 \pmod{2^{3k}}.$$

Thus, $2 - 2^{k-1} \equiv 0 \pmod{2^k}$. It follows that $k = 2$. Consequently, $a = 4$ and $n = 3$.

OC344. Find all $a \in \mathbb{R}$ such that there exists a function $f : \mathbb{R} \to \mathbb{R}$ satisfying

- $f(1) = 2016$;
- $f(x + y + f(y)) = f(x) + ay \quad \forall x, y \in \mathbb{R}$.

Originally 2016 Vietnam National Olympiad Day 2, Problem 1.

We received one solution. We present the solution by Mohammed Aassila.

Let $P(x, y)$ be the assertion : $f(x + y + f(y)) = f(x) + ay$. We have two cases.

Case 1. If $a \neq 0$, we immediately get that $f(x)$ is bijective. Then $P(x, 0)$ and injectivity imply that $f(0) = 0$. $P(0, y)$ implies $f(y + f(y)) = ay$ and so $x \to x + f(x)$ is surjective. $P(x, y)$ may then be written as

$$f(x + (y + f(y))) = f(x) + f(y + f(y)).$$

and so, since $x \to x + f(x)$ is surjective, $f(x)$ is additive. So

$$f(f(1)) = f(2016) = 2016f(1) = 2016^2.$$

and $P(0, 1)$ implies $a = 2016 + 2016^2$ and then $f(x) = 2016x$ is a solution.

Case 2. If $a = 0$, then $f(x) = 2016$ for all x fits.

In conclusion the answer is $a \in \{0, 2016 \cdot 2017\}$.

OC345. Let $\triangle ABC$ be an acute triangle, and let I_B, I_C, and O denote its B-excenter, C-excenter, and circumcenter, respectively. Points E and Y are selected on $\overline{AC}$ such that $\angle ABY = \angle CBY$ and $\overline{BE} \perp \overline{AC}$. Similarly, points F and Z are selected on $\overline{AB}$ such that $\angle ACZ = \angle BCZ$ and $\overline{CF} \perp \overline{AB}$. Lines $\overleftrightarrow{I_B F}$ and $\overleftrightarrow{I_C E}$ meet at P. Prove that $\overline{PO}$ and $\overline{YZ}$ are perpendicular.

Originally 2016 USAMO Day 1, Problem 3.

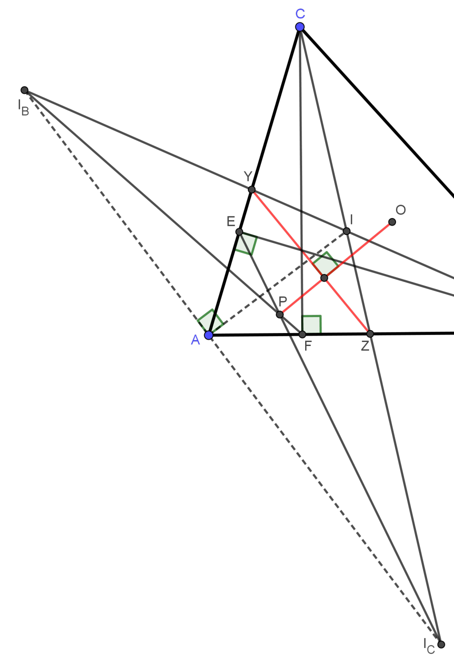
We received 2 solutions and will present both of them.

Solution 1, by Andrea Fanchini.

We use barycentric coordinates and the usual Conway's notations with reference to the triangle ABC. Points I_B, I_C, O, Y, Z, E and F have coordinates

$$I_B(a:-b:c),\quad I_C(a:b:-c),\quad O(a^2S_A:b^2S_B:c^2S_C)$$

$$Y(a:0:c),\quad Z(a:b:0),\quad E(S_C:0:S_A),\quad F(S_B:S_A:0)$$



Let us find coordinates of point P. Lines I_BF and I_CE have equations

$$I_BF: cS_Ax-cS_By-(aS_A+bS_B)z=0,\qquad I_CE: bS_Ax-(aS_A+cS_C)y-bS_Cz=0.$$

Then the point P is

$$\begin{aligned}P&=I_BF\cap I_CE\\&=(a(aS_A+bS_B+cS_C):b(aS_A+bS_B-cS_C):c(aS_A-bS_B+cS_C)).\end{aligned}$$

We will next show that PO and YZ are perpendicular. Line PO has equation

$$PO: bc(cS_C-bS_B)x+ac(aS_A+cS_C)y-ab(aS_A+bS_B)z=0$$

and it has infinite point

$$PO_\infty(-a(abc+bS_B+cS_C):b(abc-aS_A+cS_C):c(abc-aS_A+bS_B)).$$

now line YZ has equation

$$YZ: bcx-acy-abz=0$$

and it has infinite perpendicular point

$$YZ_{\infty\perp}(-a(abc+bS_B+cS_C):b(abc-aS_A+cS_C):c(abc-aS_A+bS_B)),$$

as needed.

Solution 2, by Oliver Geupel.

Let us apply trilinear coordinates relative to $\triangle ABC$. For convenience, let us use shortcuts $\alpha=\cos A$, $\beta=\cos B$ and $\gamma=\cos C$. We have

$$E=\gamma:0:\alpha,\quad F=\beta:\alpha:0,\quad I_B=1:-1:1,\quad I_C=1:1:-1,$$
$$O=\alpha:\beta:\gamma,\quad Y=1:0:1,\quad Z=1:1:0.$$

Since point $P=p:q:r$ lies on the lines I_BF and I_CE, we have

$$\begin{vmatrix} 1 & -1 & 1 \\ \beta & \alpha & 0 \\ p & q & r \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ \gamma & 0 & \alpha \\ p & q & r \end{vmatrix} = 0,$$

Solving this system of two linear equations with repertoire methods, yields

$$P=p:q:r=(\alpha+\beta+\gamma):(\alpha+\beta-\gamma):(\alpha-\beta+\gamma).$$

Equations for points $x:y:z$ on the lines YZ and OP are

$$-x+y+z=\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ x & y & z \end{vmatrix}=0$$

and

$$(\beta r-\gamma q)x+(\gamma p-\alpha r)y+(\alpha q-\beta p)z=\begin{vmatrix} \alpha & \beta & \gamma \\ p & q & r \\ x & y & z \end{vmatrix}=0.$$

It is well-known that two lines $\ell x+my+nz=0$ and $\ell' x+m'y+n'z=0$ are perpendicular if

$$T=\ell\ell'+mm'+nn'-(mn'+m'n)\alpha-(n\ell'+n'\ell)\beta-(\ell m'+\ell' m)\gamma$$

vanishes. With the equations of lines YZ and OP as above, T becomes

$$\begin{aligned}
&-\beta(\alpha-\beta+\gamma)+\gamma(\alpha+\beta-\gamma)+\gamma(\alpha+\beta+\gamma)-\alpha(\alpha-\beta+\gamma)\\
&+\alpha(\alpha+\beta-\gamma)-\beta(\alpha+\beta+\gamma)\\
&-\alpha^2(\alpha+\beta-\gamma)+\alpha\beta(\alpha+\beta+\gamma)-\alpha\gamma(\alpha+\beta+\gamma)+\alpha^2(\alpha-\beta+\gamma)\\
&-\beta^2(\alpha-\beta+\gamma)+\beta\gamma(\alpha+\beta-\gamma)+\alpha\beta(\alpha+\beta-\gamma)-\beta^2(\alpha+\beta+\gamma)\\
&+\gamma^2(\alpha+\beta+\gamma)-\alpha\gamma(\alpha-\beta+\gamma)-\beta\gamma(\alpha-\beta+\gamma)-\gamma^2(\alpha+\beta-\gamma)\\
&=0.
\end{aligned}$$

Consequently, $PO\perp YZ$.

FOCUS ON...

No. 33

Michel Bataille

Solutions to Exercises from Focus On... No. 27 – 31

From Focus On... No. 27

1. *a) Establish the formula* $\sin 2A + \sin 2B + \sin 2C = \dfrac{abc}{2R^3}$ *and deduce an expression of* $a^2 \sin 2A + b^2 \sin 2B + c^2 \sin 2C$.

b) Prove that

$$a^3 \cos A + b^3 \cos B + c^3 \cos C = \frac{abc}{2R^2} \cdot (a^2 + b^2 + c^2 - 6R^2)$$

and that

$$a \cos^3 A + b \cos^3 B + c \cos^3 C = \frac{abc}{8R^4} \cdot (10R^2 - (a^2 + b^2 + c^2)).$$

c) From the latter, deduce that if ΔABC *is not obtuse then* $a \cos^3 A + b \cos^3 B + c \cos^3 C \leq \frac{abc}{4R^2}$.

a) From the law of sines, we have

$$\begin{aligned}\sin 2A + \sin 2B + \sin 2C &= 2 \sin A \cos A + 2 \sin B \cos B + 2 \sin C \cos C \\ &= \frac{1}{R}(a \cos A + b \cos B + c \cos C).\end{aligned}$$

The result then follows from $a \cos A + b \cos B + c \cos C = \dfrac{abc}{2R^2}$ (see Focus On... No. 27, p. 293-294). With the help of this relation, we deduce

$$\begin{aligned}&a^2 \sin 2A + b^2 \sin 2B + c^2 \sin 2C \\ &= (a^2 + b^2 + c^2)(\sin 2A + \sin 2B + \sin 2C) - \sum_{\text{cyclic}} (b^2 + c^2) \sin 2A \\ &= (a^2 + b^2 + c^2)\frac{abc}{2R^3} - \sum_{\text{cyclic}} (b^2 \sin 2A + a^2 \sin 2B) \\ &= (a^2 + b^2 + c^2)\frac{abc}{2R^3} - 12F \quad (\textit{ibid.} \text{ formula (4) p. 294}) \\ &= (a^2 + b^2 + c^2)\frac{abc}{2R^3} - 12 \cdot \frac{abc}{4R}\end{aligned}$$

and so $a^2 \sin 2A + b^2 \sin 2B + c^2 \sin 2C = \dfrac{abc}{R}\left(\dfrac{a^2 + b^2 + c^2}{2R^2} - 3\right)$.

b) From the latter, we obtain

$$\begin{aligned}a^3\cos A + b^3\cos B + c^3\cos C &= \sum_{\text{cyclic}} a^2 \cdot 2R\sin A\cdot\cos A\\ &= R(a^2\sin 2A + b^2\sin 2B + c^2\sin 2C)\\ &= abc\left(\frac{a^2+b^2+c^2}{2R^2} - 3\right)\\ &= \frac{abc}{2R^2}\cdot(a^2+b^2+c^2-6R^2),\end{aligned}$$

as required. Then, using this relation, we get

$$\begin{aligned}&a\cos^3 A + b\cos^3 B + c\cos^3 C =\\ &= \sum_{\text{cyclic}} a\cos A(1-\sin^2 A)\\ &= (a\cos A + b\cos B + c\cos C) - \sum_{\text{cyclic}}\left(a\cos A\cdot\frac{a^2}{4R^2}\right)\\ &= \frac{abc}{2R^2} - \frac{1}{4R^2}\cdot\frac{abc}{2R^2}\cdot(a^2+b^2+c^2-6R^2)\end{aligned}$$

and the result follows.

c) The desired inequality is equivalent to $a^2+b^2+c^2 \geq 8R^2$, that is, to

$$\sin^2 A + \sin^2 B + \sin^2 C \geq 2 \quad\text{or}\quad \cos 2A + \cos 2B + \cos 2C \leq -1.$$

But the latter holds since the left-hand side is equal to

$$-1 - 4\cos A\cos B\cos C$$

and $\cos A, \cos B, \cos C$ are positive numbers.

2. *Prove the inequality*

$$\sum_{\textit{cyclic}} a\cos\frac{B-C}{2} \geq s\left(1+\frac{2r}{R}\right)$$

(use (5) *or give a look at problem **696**) and deduce that*

$$\frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c} \geq \frac{1}{2r}+\frac{1}{R}.$$

We observe that

$$\cos B + \cos C = 2\sin\frac{A}{2}\cos\frac{B-C}{2} \leq 2\sin\frac{A}{2}$$

and then, using (5), we deduce

$$\sum_{\text{cyclic}} a\cos\frac{B-C}{2}$$
$$= \sum_{\text{cyclic}} (b+c)\sin\frac{A}{2}$$
$$\geq \frac{1}{2}\sum_{\text{cyclic}} (b+c)(\cos B+\cos C)$$
$$= \frac{1}{2}\left(2(a\cos A+b\cos B+c\cos C)+a+b+c\right) \quad (\text{since } b\cos C+c\cos B=a,\ etc.)$$
$$= s\left(1+\frac{2r}{R}\right) \text{ (note that } \frac{abc}{2R^2}=\frac{4rsR}{2R^2}).$$

But, since $\cos\frac{B-C}{2}=\frac{h_a}{w_a}$ (*ibid.* p. 297), we have $a\cos\frac{B-C}{2}=\frac{2F}{w_a}=\frac{2rs}{w_a}$, hence

$$2rs\left(\frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c}\right)\geq s+\frac{2rs}{R}$$

and the desired inequality follows.

From Focus On... No. 28

1. *Prove the formulas*

$$\left(\frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c}\right)\left(\frac{a+b+c}{r_a+r_b+r_c}\right)=4$$

and

$$\frac{1}{rr_br_c}+\frac{1}{rr_cr_a}+\frac{1}{rr_ar_b}=\frac{8}{h_ah_bh_c}+\frac{1}{r_ar_br_c}.$$

From formulas to be found in Focus On... No 28 p. 391, we deduce

$$\frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c}=\frac{a(s-a)+b(s-b)+c(s-c)}{F}=\frac{2s^2-(a^2+b^2+c^2)}{rs}=\frac{2r(r+4R)}{rs}.$$

This yields the first result since $\dfrac{a+b+c}{r_a+r_b+r_c}=\dfrac{2s}{r+4R}$.

As for the second one, we first observe that $\dfrac{8}{h_ah_bh_c}=\dfrac{8abc}{8F^3}=\dfrac{4R}{F^2}$ and we calculate

$$\frac{1}{rr_br_c}+\frac{1}{rr_cr_a}+\frac{1}{rr_ar_b}-\frac{1}{r_ar_br_c}=\frac{r_a+r_b+r_c-r}{rr_ar_br_c}=\frac{4R}{\frac{F}{s}\cdot\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}}=\frac{4R}{F^2}.$$

The identity follows.

2. *Prove that in a scalene triangle, the following inequality holds*

$$\frac{w_a^4(m_a^2-h_a^2)}{h_a^2(w_a-h_a)\sqrt{w_a\cdot h_a}}+\frac{w_b^4(m_b^2-h_b^2)}{h_b^2(w_b-h_b)\sqrt{w_b\cdot h_b}}+\frac{w_c^4(m_c^2-h_c^2)}{h_c^2(w_c-h_c)\sqrt{w_c\cdot h_c}}\geq 24R^2.$$

With the help of formula (10) (*ibid.* p. 392), the desired inequality rewrites as

$$\frac{w_a+h_a}{\sqrt{w_a\cdot h_a}}+\frac{w_b+h_b}{\sqrt{w_b\cdot h_b}}+\frac{w_c+h_c}{\sqrt{w_c\cdot h_c}}\geq 6,$$

which readily follows from the familiar inequality $x+y\geq 2\sqrt{xy}$ for positive x,y.

From Focus On... No. 30

1. *Let f be a function from $\mathbb{R}$ to $\mathbb{R}$. Prove that $f(x+y+f(y))=f(x)+2f(y)$ for all real x and y if and only if $f(x+y)=f(x)+f(y)$ and $f(f(x))=f(x)$ for all real x and y.*

First, suppose that $f(x+y)=f(x)+f(y)$ and $f(f(x))=f(x)$ for all x,y. Then, we also have

$$f(x+y+f(y))=f(x+y)+f(f(y))=f(x)+f(y)+f(y)=f(x)+2f(y)$$

for all x,y.

Conversely, suppose that the relation

$$E(x,y):f(x+y+f(y))=f(x)+2f(y)$$

holds for all x,y. Substituting $-f(y)$ for x gives $f(-f(y))=-f(y)$ for all y and then $E(f(x),-f(x))$ leads to $-f(x)=f(f(x))-2f(x)$. Thus, $f(f(x))=f(x)$ for all x.

From this relation and $E(x,-x)$, we deduce that f is odd; also, a comparison of the relations $E(x-f(y),y)$ and $E(x-f(y),f(y))$ leads to $f(x+y)=f(x+f(y))$, from which, using $E(x-y,y)$, we obtain that f satisfies the relation

$$f(x+y)=f(x-y)+2f(y)$$

for all x,y. Now, f being odd, we have $f(0)=0$, hence $f(2y)=2f(y)$ for all y and (with $x=\frac{u-v}{2}$ and $y=\frac{u+v}{2}$), $f(u)+f(v)=f(u+v)$ for all u,v, which completes the proof.

2. *Find every function $f:\mathbb{R}\to\mathbb{R}$ that is continuous at 0 and satisfies*

$$f(x+2f(y))=f(x)+y+f(y)$$

for all real numbers x and y.

It is readily checked that the functions $x\mapsto x$ and $x\mapsto -\frac{x}{2}$ are solutions. We show that there are no other solutions. Let f be any function satisfying the conditions and let x, y be arbitrary real numbers. Substituting $x+2f(y)$ for y in the equation leads to $f(x+2y+2f(x)+2f(y)) = x+y+2f(x)+3f(y)$. But we also have

$$f(x+2y+2f(x)+2f(y)) = f(x+2y+2f(x))+y+f(y) = f(x+2y)+x+f(x)+y+f(y)$$

and by comparison we obtain $f(x+2y) = f(x)+2f(y)$. With $x = y = 0$, this gives $f(0) = 0$ and with $x = 0$ only, $f(2y) = 2f(y)$ so that $f(x+2y) = f(x)+f(2y)$. Now, f is continuous at 0 and satisfies Cauchy's equation, hence f is of the form $x \mapsto cx$ for some real number c. Since c must satisfy $c(x+2cy) = cx+y+cy$ for all x, y, we have $c = 1$ or $c = -\frac{1}{2}$, as desired.

From Focus On... No. 31

1. *Assume that f is nonnegative and has a finite third derivative f''' in the open interval $(0,1)$. If $f(x) = 0$ for at least two values of x in $(0,1)$, prove that $f'''(c) = 0$ for some c in $(0,1)$.*

By hypothesis, we have $f(x_1) = f(x_2) = 0$ for some numbers $x_1, x_2 \in (0,1)$ such that $x_1 < x_2$. Rolle's theorem then provides $c_0 \in (x_1, x_2)$ such that $f'(c_0) = 0$. Now, being nonnegative, the function f attains its minimum at x_1 and at x_2, so that $f'(x_1) = f'(x_2) = 0$. Rolle's theorem again gives $f''(c_1) = 0$ and $f''(c_2) = 0$ where $c_1 \in (x_1, c_0)$ and $c_2 \in (c_0, x_2)$. Applying Rolle's theorem one more time finally gives $f'''(c) = 0$ for some $c \in (c_1, c_2)$.

2. *Let $f : [0,1] \to \mathbb{R}$ be continuous and such that*

$\int_0^1 f(t)\,dt = 0$. *Prove that there exists $c \in (0,1)$ such that* $\int_0^c tf(t)\,dt = 0$.

The function F defined on $[0,1]$ by $F(x) = \int_0^x f(t)\,dt$ is continuously differentiable (since its derivative is f). Integrating by parts gives

$$\int_0^x tf(t)\,dt = \int_0^x tF'(t)\,dt = xF(x) - \int_0^x F(t)\,dt.$$

Thus, if we set $G(x) = \int_0^x F(t)\,dt$, all boils down to proving that

$$\frac{G(c)-G(0)}{c-0} = G'(c)$$

for some $c \in (0,1)$. Since $G'(0) = G'(1)$, this directly results from the lemma established in Focus On... No 31, p. 205.

Avoiding the Pointwise Trap in Functional Equations

Shuborno Das

Abstract

A *functional equation* is an equation where the unknown is a function instead of a variable. In solving functional equations, solvers often unknowingly commit a mistake called the *pointwise trap*. This mistake is quite subtle and even after recognition takes significant ingenuity to resolve it. The name "pointwise trap" was originally given by WOOTers [1] and since then is commonly used in the student community.

1 Introduction

1.1 Understanding the trap

There is no better way to understand this than through an example.

Example 1 *Find all functions* $f : \mathbb{R} \to \mathbb{R}$ *satisfying the condition* $f(x)^2 = x^2$ $\forall x \in \mathbb{R}$.

Post factorisation, we get $(f(x) + x)(f(x) - x) = 0$ for all x. So this gives us two solutions: $f(x) = x$ for all x and $f(x) = -x$ for all x and we are done. □

Hold on! Do we have only two solutions? No. Consider $f(x) = |x|$; this also satisfies the equation. In fact, there are uncountably many solutions; for any subset S of the reals, there's a solution f_S such that $f_S(x) = -x$ for $x \in S$ and x otherwise; this is the general solution to this problem. Refer Figure 1 as an example of such an $f_S(x)$. This error is often missed and has been given a special name in Olympiad folklore as the *pointwise trap.*

1.2 Why is this trap called *pointwise*?

This error is called the pointwise trap because the solution, while valid pointwise, does not give a complete description of the entire function. The example above showed us that $f(x) = x$ or $f(x) = -x$, for all x, does not mean $f(x) = x$ for all x, or $f(x) = -x$ for all x, are the only solutions.

A functional equation often has infinitely many solutions which are patched together from pointwise solutions; failing to take this properly into account is called the pointwise trap.

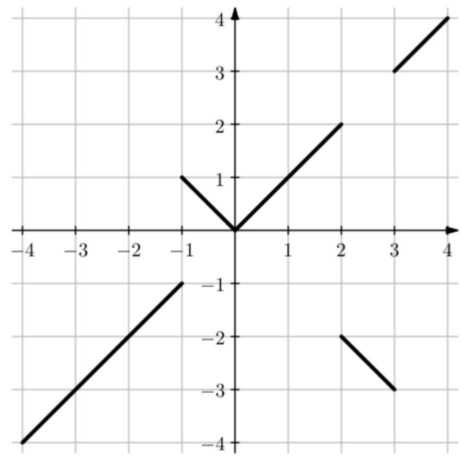


Figure 1: Example illustrating the pointwise trap

2 Recognizing the pointwise trap

Consider a functional equation of the form $g(f(x)) = 0$ (like Example 1.1) with more than one solution. Such a functional equation is *always* subject to the pointwise trap, because the functional equation can be worked separately for each value of x. Additional conditions are required to reduce the solutions to a smaller set.

Consider a functional equation in only one variable like $f(2x) = 2f(x)$. Such a functional equation will often have an orbit where the solutions are linked (in this case $\dots, 4a, 2a, a, \frac{a}{2}, \frac{a}{4}, \dots$). As real numbers are uncountable, no such orbit can cover the entire set. Again, such a problem will always need other conditions if it is to have a smaller set of solutions.

Consider a functional equation in two real variables like $f(xy) = xf(y)$. Such a functional equation can be simplified in many ways by setting one variable to a constant or to a function of the other, each substitution yielding a one-variable functional equation. By letting these interact, we can often work our way from the pointwise trap to the correct small set of solutions.

2.1 Dealing with the pointwise trap

Some contest problems give additional conditions such as continuity, monotonicity, injectivity or surjectivity which can be used to select the smaller set of solutions from the infinite pointwise solutions (see example 2).

However, some other problems do not give these explicit conditions and we need to derive them from other given conditions.

- If $g(f(x)) = x$, then the left inverse of $f(x)$ exists – which also implies $f(x)$ is injective.
- If $f(g(x)) = x$, then the right inverse of $f(x)$ exists – which implies $f(x)$ is surjective.

In a multivariable functional equation which is subject to the pointwise trap, contradiction is usually helpful (see example 3 and example 4) to select the smaller set of solutions from the infinite pointwise solutions.

3 Example Problems

Example 2 *Find all continuous functions* $f : \mathbb{R} \to \mathbb{R}$ *satisfying the condition* $f(x)^2 = x^2 \ \forall x \in \mathbb{R}$.

Post factorisation, we get $(f(x) - x)(f(x) + x) = 0$ or $f(x) = \pm x$. We have $f(0) = 0$. We also have $f(1) = 1$ or -1 (we can take any other positive number also). First, let us assume $f(1) = 1$, and we claim that $f(x) = x$ for all $x \geq 0$.

Let's try to show this using contradiction. Suppose there is a real $a > 0$ and not equal to 1 such that $f(a) = -a$. Now $f(1) = 1 > 0$ and $f(a) < 0$. Using continuity, $f(x)$ must be 0 for some x between 1 and a, say b. But $b > 0$ and $0 = f(b) = \pm b$, which is not possible. So $f(x) = x$ for all $x \geq 0$. Similarly, we can consider $f(1) = -1$, $f(-1) = -1$ and $f(-1) = 1$, which together gives us four solutions:

$$\begin{aligned} f(x) &= x \qquad \forall x, \\ f(x) &= -x \qquad \forall x, \\ f(x) &= |x| \qquad \forall x, \\ f(x) &= -|x| \qquad \forall x. \end{aligned}$$

□

Example 3 (Japan MO Final 2004) *Find all functions* $f : \mathbb{R} \to \mathbb{R}$ *such that*

$$f(xf(x) + f(y)) = f(x)^2 + y \qquad \forall x, y \in \mathbb{R}.$$

Let $P(x, y)$ be the assertion of the problem statement. Notice an isolated term of y in the RHS, which indicates that f may be bijective.

Claim: f is bijective.

Proof: If $f(a) = f(b)$, $P(x, a)$ and $P(x, b)$ give $f(x)^2 + a = f(x)^2 + b$ which gives $a = b$. So f is injective. $P(0, y)$ gives $f(f(y)) = f(0)^2 + y$. Since RHS can span the entire range, f is surjective. Therefore f is bijective.

Claim: $f(0) = 0$.

Proof: Let $f(0) = c$ and let $f(a) = 0$ (why?). Now, $P(a, a)$ gives

$$c = f(0) = f(af(a) + f(a)) = f(a)^2 + a = a.$$

Note that $P(0, 0)$ gives $f(c) = f(f(0)) = c^2$. So, we have $c^2 = f(c) = f(a) = 0$. So $f(0) = c = 0$.

Now that we know $f(0)$, we can utilise it by plugging it in the original equation. $P(x, 0)$ gives $f(xf(x)) = f(x)^2$ and $P(0, y)$ gives $f(f(y)) = y$. Replacing x with $f(x)$ in the former equation gives

$$x^2 = f(f(x))^2 = f(f(x)f(f(x))) = f(f(x)x) = f(x)^2 \Rightarrow f(x) = \pm x.$$

This is a pointwise trap which we will avoid using contradiction. For the sake of contradiction suppose $f(a) = a$ and $f(b) = -b$ for $a, b \neq 0$ (why can we assume this?). Now $P(a, b)$ gives $a^2 + b = f(a^2 - b) = a^2 - b$ or $b - a^2$. In the first case, $2b = 0 \Rightarrow b = 0$ and in the second case, $2a^2 = 0 \Rightarrow a = 0$, both of which are impossible.

So solutions are $f(x) = x$ for all x and $f(x) = -x$ for all x, which indeed are solutions to the problem. □

Example 4 (Iran 1999) *Find all functions* $f : \mathbb{R} \to \mathbb{R}$ *such that*

$$f(x^2 + y) = f(f(x) - y) + 4f(x)y \qquad \forall x, y \in \mathbb{R}.$$

Let $P(x, y)$ be the assertion of the problem statement. We proceed by a slightly different way. There is a term with only f in the LHS and one on the RHS: $f(x^2 + y)$ and $f(f(x) - y)$ respectively. If these be equal, the other term in RHS will be 0 which might prove to be helpful.

$P\left(x, \frac{x^2 - f(x)}{2}\right)$ gives

$$0 = 4f(x) \cdot \left(\frac{x^2 - f(x)}{2}\right).$$

So, $f(x)(x^2 - f(x)) = 0 \Rightarrow f(x) = 0$ or x^2.

This is a pointwise trap which we will avoid using contradiction. Let $a, b \in \mathbb{R}_{\neq 0}$ (why?) and $a^2 \neq 2b$ such that $f(a) = 0, f(b) = b^2$. Let's prove why such a choice is possible. Let $A = \{a | f(a) = 0\}$ and $B = \{b | f(b) = b^2\}$. For the sake of contradiction, assume that there does not exist any a and b such that $a^2 \neq 2b$ and $f(a) = 0$ and $f(b) = b^2$. We have also assumed that A and B are not null sets.

If $a \in A$, then all reals except $\frac{a^2}{2}$ must also be in A. Now since B is not empty, $\frac{a^2}{2} = b \in B$. Consider any real c other than a and $\frac{a^2}{2}$. We have shown that c also lies in A. But note that we assumed that there is no pair a, b satisfying $a^2 \neq 2b$ and $a \in A$, $b \in B$. Consider the pair c and b. Clearly $c^2 \neq a^2 = 2b$ and $c \in A$, $b \in B$. Contradiction.

So we can assume that there is a pair a, b satisfying $f(a) = 0, f(b) = b^2$, $a, b \neq 0$ and $a^2 \neq 2b$.

$$P(a, -b) \to b^2 = f(b) = f(f(a) + b) = f(a^2 - b) + 4f(a)b = f(a^2 - b).$$

So $b^2 = 0$ or $(a^2 - b)^2 = a^4 + b^2 - 2a^2b$. We know that $b \neq 0$, so $b^2 = a^4 + b^2 - 2a^2b$ or $a^4 = 2a^2b \Rightarrow a^2 = 2b$ which is impossible.

So solutions are $f(x) = 0$ for all x and $f(x) = x^2$ for all x which indeed are solutions to the problem. □

4 Practice Problems

1. (Kyrgyzstan 2012) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(f(x)^2 + f(y)) = xf(x) + y \qquad \forall x, y \in \mathbb{R}.$$
2. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that
$$f(n^2) = f(n + m)f(n - m) + m^2 \qquad \forall m, n \in \mathbb{Z}.$$
3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x)f(x - y) + f(y)f(x + y) = x^2 + f(y)^2 \qquad \forall x, y \in \mathbb{R}.$$
4. Find all $f : \mathbb{R} \to \mathbb{R}$ such that:
$$xf(y) + yf(x) = 2f(x)f(y) \qquad \forall x, y \in \mathbb{R}.$$
5. (USA(J)MO 2016) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$(f(x) + xy) \cdot f(x - 3y) + (f(y) + xy) \cdot f(3x - y) = (f(x + y))^2 \qquad \forall x, y \in \mathbb{R}.$$

References

[1] https://artofproblemsolving.com/school/course/woot

PROBLEMS

Readers are invited to submit solutions, comments and generalizations to any problem in this section. Moreover, readers are encouraged to submit problem proposals. Please see submission guidelines inside the back cover or online. To facilitate their consideration, solutions should be received by **April 1, 2019**.

The editor thanks Rolland Gaudet, retired professor of Université de Saint-Boniface in Winnipeg, for translations of the problems.

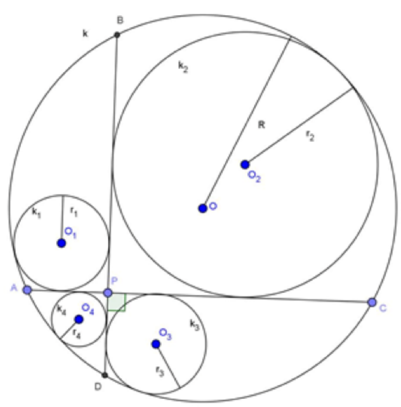
4381. *Proposed by Mihaela Berindeanu.*

Let ABC be an acute triangle with circumcircle Γ_1 and circumcenter O. Suppose the open ray AO intersects Γ_1 at point D and E is the middle point of BC. The perpendicular bisector of BE intersects BD in P and the perpendicular bisector of EC intersects CD in Q. Finally suppose that circle Γ_2 with center P and radius PE intersects the circle Γ_3 with center Q and radius QE in X. Prove that AX is a symmedian in $\triangle ABC$.

4382. *Proposed by Borislav Mirchev and Leonard Giugiuc.*

Let $ABCD$ be an orthogonal cyclic quadrilateral with $AC \perp BD$. Let O and R be the circumcenter and the circumradius of $ABCD$ respectively and let P be the intersection of AC and BD. Denote by r_1, r_2, r_3 and r_4 the inradii of the minor circular sectors PAB, PBC, PCD and PDA respectively. Prove that

$$r_1 + r_2 + r_3 + r_4 + 8R = (R^2 - OP^2)\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}\right).$$



4383. *Proposed by Michel Bataille.*

Evaluate the integral

$$\int_0^1 (\ln x) \cdot \sqrt{\frac{x}{1-x}}\, dx.$$

4384. *Proposed by Michel Bataille.*

Let n be an integer with $n \geq 2$. Find all real numbers x such that

$$\sum_{0\leq i<j\leq n-1} \left\lfloor x+\frac{i}{n}\right\rfloor \cdot \left\lfloor x+\frac{j}{n}\right\rfloor = 0.$$

4385. *Proposed by Miguel Ochoa Sanches and Leonard Giugiuc.*

Let ABC be a triangle with circumcircle ω and $AB < AC$. The tangent at A to ω intersects the line BC at P. The internal bisector of $\angle APB$ intersects the sides AB and AC at E and F, respectively. Show that

$$\frac{PE}{PF} = \sqrt{\frac{EB}{FC}}.$$

4386. *Proposed by Thanos Kalogerakis.*

Let $ABCD$ be a cyclic quadrilateral with $AD > BC$, where $X = AB \cap CD$ and $Y = BC \cap AD$. The bisectors of angles X and Y intersect BC and CD at P and S, respectively. Finally, let Q and T be points on the sides AD and AB such that $PQ \perp AD$ and $ST \perp AB$. Prove that $ABCD$ is bicentric if and only if $PQ = ST$.

4387. *Proposed by Nguyen Viet Hung.*

Let

$$a_n = \sum_{k=1}^{n} \sqrt[k]{1+\frac{k^2}{(k+1)!}}, \; n = 1, 2, 3, \ldots$$

Determine $\lfloor a_n \rfloor$ and evaluate $\lim\limits_{n\to\infty} \dfrac{a_n}{n}$.

4388. *Proposed by Marian Cucoanes and Leonard Giugiuc.*

For positive real numbers a, b and c, prove

$$8abc(a^2 + 2ac + bc)(b^2 + 2ab + ac)(c^2 + 2bc + ab) \leq [(a + b)(b + c)(c + a)]^3.$$

4389. *Proposed by Daniel Sitaru.*

Consider the real numbers a, b, c and d. Prove that

$$a(c + d) - b(c - d) \leq \sqrt{2(a^2 + b^2)(c^2 + d^2)}.$$

4390. *Proposed by Marius Drăgan and Neculai Stanciu.*

Let x, y and z be positive real numbers with $x + y + z = m$. Find the minimum value of the expression

$$\frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+z^2}.$$

. . . .

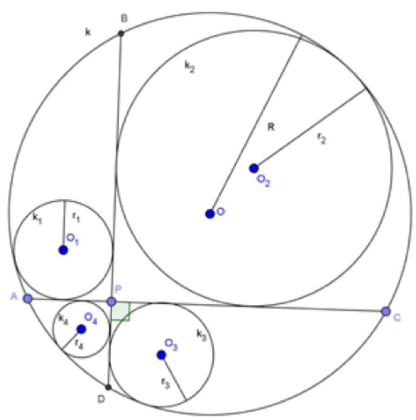
4381. *Proposé par Mihaela Berindeanu.*

Soit ABC un triangle acutangle avec cercle circonscrit Γ_1, de centre O. Supposer que le rayon ouvert AO intersecte Γ_1 au point D. De plus, supposer que E est le mi point de BC, que la bissectrice perpendiculaire de BE intersecte BD en P, puis que la bissectrice perpendiculaire de EC intersecte CD en Q. Enfin, supposer que le cercle Γ_2 de centre P et rayon PE intersecte le cercle Γ_3 de centre Q et rayon QE en X. Démontrer que AX est sym-médian dans le triangle ABC.

4382. *Proposé par Borislav Mirchev et Leonard Giugiuc.*

Soit $ABCD$ un quadrilatère orthogonal et cyclique tel que $AC \perp BD$. Soient O et R le centre et le rayon du cercle circonscrit, respectivement, et soit P l'intersection de AC et BD. Dénoter par r_1, r_2, r_3 et r_4 les rayons des cercles inscrits de PAB, PBC, PCD et PDA, respectivement. Démontrer que

$$r_1 + r_2 + r_3 + r_4 + 8R = (R^2 - OP^2)\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}\right).$$



4383. *Proposé par Michel Bataille.*

Évaluer

$$\int_0^1 (\ln x) \cdot \sqrt{\frac{x}{1-x}}\, dx.$$

4384. *Proposé par Michel Bataille.*

Soit n un entier tel que $n \geq 2$. Déterminer tous les nombres réels x tels que

$$\sum_{0 \leq i < j \leq n-1} \left\lfloor x + \frac{i}{n} \right\rfloor \cdot \left\lfloor x + \frac{j}{n} \right\rfloor = 0.$$

4385. *Proposé par Miguel Ochoa Sanches et Leonard Giugiuc.*

Soit ABC un triangle avec cercle circonscrit ω, tel que $AB < AC$. La tangente à ω au point A intersecte la ligne BC en P. La bissectrice interne de $\angle APB$ intersecte les côtés AB et AC en E et F, respectivement. Démontrer que

$$\frac{PE}{PF} = \sqrt{\frac{EB}{FC}}.$$

4386. *Proposé par Thanos Kalogerakis.*

Soit $ABCD$ un quadrilatère cyclique tel que $AD > BC$, où $X = AB \cap CD$ et $Y = BC \cap AD$. Les bissectrices des angles X et Y intersectent BC et CD en P et S, respectivement. Enfin, soient Q et T les points sur les côtés AD et AB tels que $PQ \perp AD$ et $ST \perp AB$. Démontrer que $ABCD$ est bicentrique si et seulement si $PQ = ST$.

4387. *Proposé par Nguyen Viet Hung.*

Soit

$$a_n = \sum_{k=1}^{n} \sqrt[k]{1 + \frac{k^2}{(k+1)!}}, \ n = 1, 2, 3, \ldots$$

Déterminer $\lfloor a_n \rfloor$ et évaluer $\displaystyle\lim_{n\to\infty} \frac{a_n}{n}$.

4388. *Proposé par Marian Cucoanes et Leonard Giugiuc.*

Démontrer la suivante, pour a, b et c des nombres réels positifs :

$$8abc(a^2 + 2ac + bc)(b^2 + 2ab + ac)(c^2 + 2bc + ab) \le [(a+b)(b+c)(c+a)]^3.$$

4389. *Proposé par Daniel Sitaru.*

Soient a, b, c et d des nombres réels. Démontrer que

$$a(c+d) - b(c-d) \le \sqrt{2(a^2+b^2)(c^2+d^2)}.$$

4390. *Proposé par Marius Drăgan et Neculai Stanciu.*

Soient x, y et z des nombres réels positifs tels que $x + y + z = m$. Déterminer la valeur minimale de l'expression

$$\frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+z^2}.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2017: 43(9), p. 398–402.

An asterisk (⋆) after a number indicates that a problem was proposed without a solution.

4281⋆. *Proposed by Šefket Arslanagić.*

Prove or disprove the following inequalities:

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\sqrt{3(a^2+b^2+c^2)},\quad (a,b,c>0), \tag{1}$$

$$\frac{a_1^2}{a_2}+\frac{a_2^2}{a_3}+\cdots+\frac{a_n^2}{a_1}\geq\sqrt{n(a_1^2+a_2^2+\cdots+a_n^2)},\quad (a_i>0, n\geq 3). \tag{2}$$

We received three correct solutions to (1) and we present two of them.

One reader submitted a Maple analysis for small cases of (2), and another an incorrect solution. The inequality (2) remains open, except for the cases $n = 3$ and $n = 4$.

Solution 1, by Kevin Soto Palaccios.

We begin with two inequalities. The generalized Hölder inequality states that

$$\left(\sum x_i^p\right)^{1/p}\left(\sum y_i^q\right)^{1/q}\left(\sum z_i^r\right)^{1/r}\geq\sum x_iy_iz_i,$$

where the summation is from 1 to n, the variables are nonnegative, and p, q, r are positive reals with $(1/p) + (1/q) + (1/r) = 1$.

The second inequality is

$$n\left(\sum_{i=1}^{n}x_ix_{i+1}\right)\leq\left(\sum_{i=1}^{n}x_i\right)^2$$

for nonnegative x_i, where $x_{n+1} = x_1$. This holds for $n = 3$ and $n = 4$, but fails for $n \geq 6$ (as can be seen by letting $(x_1, x_2, \ldots, x_n) = (0, 1, 2, \ldots, n-1)$).

The example $(x_1, x_2, x_3, x_4, x_5) = (0, 1, 3, 5, 7)$ shows that the inequality fails for $n = 5$; the value of the left side is 265 and of the right is 256. (Interestingly enough, equality occurs in the $n = 5$ case for five entries in arithmetic progression.)

For $n = 3$, the difference between the two sides is

$$\frac{1}{2}((x_1-x_2)^2+(x_2-x_3)^2+(x_3-x_1)^2).$$

For $n = 4$, the difference is $(x_1 - x_2 + x_3 - x_4)^2$.

Applying the Hölder inequality with $n = 3$, $p = q = r = 1/3$, $x_i = y_i = \left(\dfrac{a_i^2}{a_{i+1}}\right)^{1/3}$ and $z_i = (a_i a_{i+1})^{2/3}$, we obtain that

$$\left(\sum_{i=1}^{n} \frac{a_i^2}{a_{i+1}}\right)\left(\sum_{i=1}^{n} \frac{a_i^2}{a_{i+1}}\right)\left(\sum_{i=1}^{n} a_i^2 a_{i+1}^2\right) \geq (a_1^2 + a_2^2 + \cdots + a_n^2)^3.$$

(Here $a_{n+1} = a_1$.)

When $n = 3$ and $n = 4$, we use this inequality followed by the second one with $x_i = a_I^2$ and sum from 1 to n to obtain

$$\left(\sum \frac{a_i^2}{a_{i+1}}\right)^2 \geq \frac{\left(\sum a_i^2\right)^3}{\sum a_i^2 a_{i+1}^2} \geq \frac{\left(\sum a_i^2\right)^3}{\left(\sum a_i^2\right)^2 / n} = n\left(\sum a_i^2\right).$$

This establishes the inequality for these two cases.

Solution 2, by Oliver Geupel.

We establish the result for $n = 3$. Note that, by the Cauchy-Schwarz inequality

$$\left(\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}\right)(a^2 b + b^2 c + c^2 a) \geq (a^2 + b^2 + c^2)^2.$$

Also, we have

$$(a^2+b^2+c^2)(a+b+c) - 3(a^2 b + b^2 c + c^2 a) = a(a-b)^2 + b(b-c)^2 + c(c-a)^2 \geq 0,$$

whence

$$a^2 b + b^2 c + c^2 a \leq \frac{1}{3}(a^2 + b^2 + c^2)(a + b + c).$$

Finally, by the root-mean-square, arithmetic mean inequality,

$$\sqrt{3(a^2 + b^2 + c^2} \geq (a + b + c).$$

Putting all of this together, we find that

$$\begin{aligned} \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} &\geq \frac{(a^2 + b^2 + c^2)^2}{a^2 b + b^2 c + c^2 a} \\ &\geq \frac{3(a^2 + b^2 + c^2)}{a + b + c} \\ &\geq \sqrt{3(a^2 + b^2 + c^2)}, \end{aligned}$$

as desired. Equality holds if and only if $a = b = c$.

4282. *Proposed by Michel Bataille.*

Find $\lim_{n\to\infty} u_n$ where the sequence $(u_n)_{n\geq 0}$ is defined by $u_0 = 1$ and the recursion

$$u_{n+1} = \frac{1}{2}\left(u_n + \sqrt{u_n^2 + \frac{u_n}{4^n}}\right)$$

for every nonnegative integer n.

There were two correct and three incorrect solutions submitted. We present the solution of the AN-Anduud Problem Solving Group and the proposer, obtained independently.

For each nonnegative integer n, let $u_n = \frac{1}{4v_n^2}$, with $v_n > 0$. Then $v_0 = 1$ and v_n satisfies the recursion

$$v_{n+1}^2 = \frac{\sqrt{1+v_n^2} - 1}{2}.$$

Let $v_n = \sinh x_n$ for some $x_n \geq 0$. Then

$$(\sinh x_{n+1})^2 = \frac{\cosh x_n - 1}{2} = \left(\sinh \frac{x_n}{2}\right)^2,$$

whence $x_{n+1} = x_n/2$ for each nonnegative integer n. Therefore

$$x_n = \frac{x_0}{2^n} = \frac{\sinh^{-1} 1}{2^n} = \frac{\ln(1+\sqrt{2})}{2^n}.$$

Therefore,

$$\begin{aligned} u_n &= \frac{1}{4^n[\sinh(2^{-n}\ln(1+\sqrt{2}))]^2} \\ &= 4^{-n}\left(\frac{2^{-n}\ln(1+\sqrt{2})}{\sinh(2^{-n}\ln(1+\sqrt{2}))}\right)^2\left(\frac{2^n}{\ln(1+\sqrt{2})}\right)^2. \end{aligned}$$

Letting n tend to infinity yields the desired limit $(\ln(1+\sqrt{2}))^{-2}$.

The limit is approximately equal to 1.2873.

4283. *Proposed by Margarita Maksakova.*

We are given a convex polygon, whose vertices are coloured with three colours so that adjacent vertices get different colours. If all three colours are used in the colouring, prove that you can divide this polygon into triangles using non-intersecting diagonals in such a way that all the resulting triangles have vertices of all three different colours.

We received 5 solutions. We present the solution by Missouri State University Problem Solving Group.

We use induction on the number of vertices.

If there are three vertices, the result follows immediately.

Consider a convex n-gon with $n > 3$ and assume that any convex polygon with fewer vertices satisfies the property. Choose three vertices that are coloured different colours, say colour 1, colour 2 and colour 3. At least one of the edges between these vertices is an interior edge (since $n > 3$ and the polygon is convex). Without loss of generality, we may assume that the vertices are coloured with colours 1 and 2. We may also assume that the vertices are labeled consecutively (i.e. v_i is adjacent to v_{i+1} and v_n is adjacent to v_1) and that the vertex coloured 1 is v_1, the vertex coloured 2 is v_j, and the vertex coloured 3 is v_k with $1 < j < k \leq n$.

By induction, the polygon with vertices $\{v_1, v_j, v_{j+1}, \dots, v_n\}$ can be divided into triangles with each triangle having vertices of all three colours.

If one of the vertices $v_2, \dots, v_{j-1}$ is coloured 3, then the polygon with these vertices can also be divided into triangles with each triangle having vertices of all three colours, giving the desired triangulation for the original n-gon.

If none of the vertices $v_2, \dots, v_{j-1}$ is coloured 3, then j must be even, $v_2, v_4, \dots, v_{j-2}$ must be coloured 2, and $v_3, v_5, \dots, v_{j-1}$ must be coloured 1. Since the polygon with vertices $\{v_1, v_j, v_{j+1}, \dots, v_n\}$ can be divided into triangles with each triangle having vertices of all three colours, there must be a triangle with vertices v_1, v_j, v_ℓ ($j < \ell \leq n$) with v_ℓ colored 3. Removing the diagonal from v_1 to v_j and adding those from v_ℓ to v_1, from v_ℓ to v_2,…, and from v_ℓ to v_j gives the desired triangulation.

4284. *Proposed by Daniel Sitaru.*

Prove that if a, b and c are real numbers greater than 3 and

$$\log_a 2 + \log_b 2 + \log_c 2 = \log_a \frac{1}{b} + \log_b \frac{1}{c} + \log_c \frac{1}{a}, \tag{1}$$

then

$$\log_{a-1}(a^2 + b^2) + \log_{b-1}(b^2 + c^2) + \log_{c-1}(c^2 + a^2) > 3.$$

We received 10 submissions from nine people; three of them proved that (1) is not satisfied for any triple a, b, c of real numbers greater than 3, making the implication vacuously true. Michel Bataille, Leonard Giugiuc, and Digby Smith all showed in the same way that the left side of the condition is positive and the right side is negative. We present their solution here.

We have $a, b, c > 3$, thus

$$\log_a 2,\ \log_b 2,\ \log_c 2 > 0,$$

and

$$\log_a \frac{1}{b},\ \log_b \frac{1}{c},\ \log_c \frac{1}{a} < 0.$$

4285. *Proposed by Shafiqur Rahman and Leonard Giugiuc.*

Find the following limit:

$$\lim_{n\to\infty}\left(\sqrt[n]{n!}-\frac{n}{e}-\frac{\ln n}{2e}\right).$$

We received 10 submissions, of which 7 were correct and complete. We present the solution by Michel Bataille, slightly modified by the editor.

We will show that the limit is $\dfrac{\ln(2\pi)}{2e}$.

First, recall the following asymptotic expansion as $n\to\infty$ (a consequence of Stirling's formula):

$$\ln(n!)=n\cdot(\ln(n)-1)+\frac{\ln(2\pi n)}{2}+o(1).$$

Hence

$$\sqrt[n]{n!}=\exp\left(\frac{\ln(n!)}{n}\right)=\frac{n}{e}\exp\left(\frac{\ln(2\pi n)}{2n}+o(1/n)\right). \tag{1}$$

Note that

$$\lim_{n\to\infty}\left(\frac{\ln(2\pi n)}{2n}+o(1/n)\right)=0.$$

From the Taylor expansion, we have

$$e^x=1+x+\frac{x^2}{2}+o(x^2)\quad\text{as}\quad x\to 0.$$

Now use this facts in (1) with $x=\frac{\ln(2\pi n)}{2n}+o(1/n)$. Since

$$\left(\frac{\ln(2\pi n)}{2n}+o(1/n)\right)^2=\frac{(\ln(2\pi n))^2}{4n^2}+\frac{\ln(2\pi n)}{2n}\cdot o(1/n)+o(1/n^2)=o(1/n),$$

from (1) we get

$$\begin{aligned}\sqrt[n]{n!}&=\frac{n}{e}\left(1+\frac{\ln(2\pi n)}{2n}+o(1/n)\right)\\&=\frac{n}{e}+\frac{\ln(2\pi n)}{2e}+o(1)\\&=\frac{n}{e}+\frac{\ln(2\pi)}{2e}+\frac{\ln(n)}{2e}+o(1);\end{aligned}$$

we can rearrange this to

$$\sqrt[n]{n!}-\frac{n}{e}-\frac{\ln n}{2e}=\frac{\ln(2\pi)}{2e}+o(1),$$

and the announced result follows.

4286. *Proposed by Marian Cucoaneş and Marius Drăgan.*

Let x, y and z be positive real numbers such that $xyz = 1$. Prove that

$$(x^7 + y^7 + z^7)^2 \geq 3(x^9 + y^9 + z^9).$$

We received 7 submissions all of which are correct if we consider proofs based on computer assisted computations or Maple outputs as valid. We present the proof by AN-anduud Problem Solving Group of Ulaanbaatar, Mongolia.

Let

$$f(t) = t^{14} + \frac{2}{t^7} - 3t^9 + 27\log t,\ \ t > 0.$$

Then

$$f'(t) = 14t^{13} - \frac{14}{t^8} - 27t^8 + \frac{27}{t} = \frac{1}{t^8}\left(14t^{21} - 14 - 27t^{16} + 27t^7\right).$$

Let $g(t) = 14t^{21} - 14 - 27t^{16} + 27t^7,\ t > 0$. Then

$$g'(t) = 294t^{20} - 27 \cdot 16 \cdot t^{15} + 189t^6 = 3t^6(98t^{14} - 144t^9 + 63).$$

By the AM-GM inequality we have

$$\begin{aligned} 98t^{14} + 63 &= 9\left(\frac{98t^{14}}{9}\right) + 5 \cdot \left(\frac{63}{5}\right) \\ &\geq 14 \cdot \sqrt[14]{\left(\frac{98t^{14}}{9}\right)^9 \cdot \left(\frac{63}{5}\right)^5} \\ &= 98 \cdot \sqrt[14]{\frac{2^9 \cdot 7^9}{3^8 \cdot 5^5}} \cdot t^9 > 144t^9, \end{aligned}$$

so $g'(t) > 0$. Hence, $g(t)$ is increasing on $(0, \infty)$. But $g(0) = -14 < 0$ and $g(1) = 0$, so $f'(t) < 0$ on $(0, 1)$, and $f'(t) > 0$ on $(1, \infty)$. Hence $f(t)$ is decreasing on $(0, 1)$, and increasing on $(1, +\infty)$, from which we get

$$\min_{t>0} f(t) = f(1) = 0.$$

It follows that

$$\begin{aligned} &f(x) + f(y) + f(z) \geq 0 \\ \Leftrightarrow\ \ & x^{14} + y^{14} + z^{14} + \frac{2}{x^7} + \frac{2}{y^7} + \frac{2}{z^7} + 27\log(xyz) \geq 3(x^9 + y^9 + z^9) \\ \Leftrightarrow\ \ & x^{14} + y^{14} + z^{14} + \frac{2x^7y^7 + 2y^7z^7 + 2z^7x^7}{(xyz)^7} + 27\log 1 \geq 3(x^9 + y^9 + z^9) \\ \Leftrightarrow\ \ & (x^7 + y^7 + z^7)^2 \geq 3(x^9 + y^9 + z^9), \end{aligned}$$

completing the proof.

4287. *Proposed by Van Khea and Leonard Giugiuc.*

Let ABC be a triangle inscribed in a circle O and let K be a point inside ABC. Suppose that AK, BK and CK cut the circle O in points D, E and F, respectively. Prove that

$$\frac{BD\cdot CE}{BC\cdot DE}+\frac{CE\cdot AF}{CA\cdot EF}+\frac{AF\cdot BD}{AB\cdot FD}=1.$$

We received 5 submissions, of which 4 were correct and complete, and 1 used Maple. Of the four correct solutions, two used an inversion method, and two used barycentric coordinates. We present one of each.

Solution 1, by Modak Madhav.

We need the following result: Let P, Q, R, and S be four distinct points, in no particular order, on the circle with centre O' and diameter d'. Let $r' > d'$ and invert in point P with power of inversion r'^2. Let points Q', R', S' be the images of Q, R, S respectively. Then

$$\text{(i)}\quad \frac{PQ\cdot RS}{PR\cdot QS}=\frac{R'S'}{Q'S'}.$$

This follows from

$$\text{(a)}\quad \frac{QR\cdot r^2}{PQ\cdot PR}=Q'R',\quad \text{(b)}\quad \frac{RS\cdot r^2}{PR\cdot PS}=R'S',\quad \text{(c)}\quad \frac{QS\cdot r^2}{PQ\cdot PS}=Q'S'.$$

For proving (a), note that the circle O' inverts into a line which is perpendicular to line PO' and on which points Q', R', S' lie and $PR\cdot PR'=r^2$.

Also, $\triangle PQR\sim\triangle PR'Q'$, so that $PQ/PR'=QR/Q'R'$ or $Q'R'=QR\cdot PR'/PQ$ or $Q'R'=QR\cdot r^2/PQ\cdot PR$. So (a) holds. Similarly, (b), (c) can be proved.

Let d be the diameter of circle O in the given figure. Since $\triangle EKC$ and $\triangle EFK$ have EK as their common base, we have

$$\frac{CK}{KF}=\frac{CE\sin(\angle CEB)}{EF\sin(\angle FEB)}.$$

Similarly, since $\triangle AKC$ and $\triangle AFK$ have AK as their common base, we have

$$\frac{CK}{KF}=\frac{CA\sin(\angle CAD)}{AF\sin(\angle FAD)}.$$

Using these two values of CK/KF, we get

$$\frac{CE\cdot AF}{CA\cdot EF}=\frac{\sin(\angle CAD)\sin(\angle FEB)}{\sin(\angle FAD)\sin(\angle CEB)}=\frac{DC\cdot FB}{DF\cdot CB};\qquad(1)$$

the last equality following from $\sin(\angle FEB)=BF/d$, etc. Similarly,

$$\frac{AF\cdot BD}{AB\cdot FD}=\frac{DC\cdot EA}{DE\cdot CA}.\qquad(2)$$

Now let $r > d$ and invert in point D with power of inversion r^2. Let points B', F', A', E', C' be the images of B, F, A, E, C respectively. Then circle O inverts into a line which is perpendicular to line DO and on which points B', F', A', E', C' lie, in that order. Hence, we can write by (i) for points D, B, E, and C

$$\frac{BD \cdot CE}{BC \cdot DE} = \frac{DB \cdot EC}{DE \cdot BC} = \frac{E'C'}{B'C'}, \tag{3}$$

by (1) and (i) for points D, C, F, and B

$$\frac{CE \cdot AF}{CA \cdot EF} = \frac{DC \cdot FB}{DF \cdot CB} = \frac{F'B'}{C'B'}, \tag{4}$$

by (i) for points D, B, F, and A

$$\frac{AF \cdot BD}{AB \cdot FD} = \frac{DB \cdot FA}{DF \cdot BA} = \frac{F'A'}{B'A'}, \tag{5}$$

by (2) and (i) for points D, C, E, and A

$$\frac{AF \cdot BD}{AB \cdot FD} = \frac{DC \cdot EA}{DE \cdot CA} = \frac{E'A'}{C'A'}. \tag{6}$$

By (5), (6), and the order of B', F', A', E', C' on the line, we get

$$\frac{AF \cdot BD}{AB \cdot FD} = \frac{F'A'}{B'A'} = \frac{E'A'}{C'A'} = \frac{F'A' + A'E'}{B'A' + A'C'} = \frac{F'E'}{B'C'}. \tag{7}$$

Adding (3), (4), (7), and the order of B', F', A', E', C' on the line, we finally get

$$\begin{aligned} \frac{BD \cdot CE}{BC \cdot DE} + \frac{CE \cdot AF}{CA \cdot EF} + \frac{AF \cdot BD}{AB \cdot FD} &= \frac{E'C'}{B'C'} + \frac{F'B'}{C'B'} + \frac{F'E'}{B'C'} \\ &= \frac{B'F' + F'E' + E'C'}{B'C'} = 1. \end{aligned}$$

Solution 2, by Oliver Geupel.

Let $BC = a$, $CA = b$, $AB = c$, $\vec{K} = \alpha\vec{A} + \beta\vec{B} + \gamma\vec{C}$ with positive real numbers α, β, and γ that sum up to 1. Let lines AD and BC intersect in point L. By the properties of barycentric coordinates and by the power of point L, we have

$$BL = \frac{\gamma a}{\beta + \gamma}, \qquad CL = \frac{\beta a}{\beta + \gamma} \qquad AL \cdot DL = BL \cdot CL.$$

By Stewart's theorem, for triangle ABC and the cevian AL it holds

$$AC^2 \cdot BL + AB^2 \cdot CL = BC(AL^2 + BL \cdot CL),$$

which yields

$$AL = \frac{\sqrt{\beta\gamma(b^2 + c^2 - a^2) + \gamma^2 b^2 + \beta^2 c^2}}{\beta + \gamma}.$$

Then,

$$DL = \frac{BL \cdot CL}{AL} = \frac{\beta\gamma a^2}{(\beta+\gamma)\sqrt{\beta\gamma(b^2+c^2-a^2)+\gamma^2 b^2+\beta^2 c^2}},$$

$$AD = AL + DL = \frac{\gamma b^2 + \beta c^2}{\sqrt{\beta\gamma(b^2+c^2-a^2)+\gamma^2 b^2+\beta^2 c^2}}.$$

By Stewart's theorem, for triangle ABD and the cevian BL it holds

$$AB^2 \cdot DL + BD^2 \cdot AL = AD(BL^2 + AL \cdot DL),$$

which gives

$$BD = \frac{\gamma ab}{\sqrt{\beta\gamma(b^2+c^2-a^2)+\gamma^2 b^2+\beta^2 c^2}}.$$

Similarly,

$$CE = \frac{\alpha bc}{\sqrt{\gamma\alpha(c^2+a^2-b^2)+\alpha^2 c^2+\gamma^2 a^2}}.$$

By the properties of barycentric coordinates, we have

$$AK = \frac{(\beta+\gamma)AL}{\alpha+\beta+\gamma} = \frac{\sqrt{\beta\gamma(b^2+c^2-a^2)+\gamma^2 b^2+\beta^2 c^2}}{\alpha+\beta+\gamma}.$$

Similarly,

$$BK = \frac{\sqrt{\gamma\alpha(c^2+a^2-b^2)+\alpha^2 c^2+\gamma^2 a^2}}{\alpha+\beta+\gamma}.$$

Also,

$$DK = AD - AK = \frac{\beta\gamma a^2 + \gamma\alpha b^2 + \alpha\beta c^2}{(\alpha+\beta+\gamma)\sqrt{\beta\gamma(b^2+c^2-a^2)+\gamma^2 b^2+\beta^2 c^2}}.$$

By the power of point K, the triangles ABK and EDK are similar. Thus,

$$\begin{aligned} DE &= \frac{AB \cdot DK}{BK} \\ &= \frac{(\beta\gamma a^2 + \gamma\alpha b^2 + \alpha\beta c^2)c}{\sqrt{\beta\gamma(b^2+c^2-a^2)+\gamma^2 b^2+\beta^2 c^2} \cdot \sqrt{\gamma\alpha(c^2+a^2-b^2)+\alpha^2 c^2+\gamma^2 a^2}} \end{aligned}$$

Therefore,

$$\frac{BD \cdot CE}{BC \cdot DE} = \frac{\gamma\alpha b^2}{\beta\gamma a^2 + \gamma\alpha b^2 + \alpha\beta c^2}.$$

With similar terms for the remaining summands, the result follows.

4288. *Proposed by Hung Nguyen Viet.*

A sequence $\{a_n\}$ is defined as follows:

$$a_1 = 1, \quad a_2 = 2, \quad a_{n+2} = \frac{a_{n+1}^2 + 1}{a_n}$$

for every $n \geq 1$. Let $b_n = a_n a_{n+1}$, $n = 1, 2, \ldots$ Prove that for every positive integer n, the number $5b_n^2 - 6b_n + 1$ is a perfect square.

We received 19 correct solutions. We present the solution by Nghia Doan.

For $n \geq 2$, $a_{n+2}a_n = a_{n+1}^2 + 1$ and $a_{n+1}a_{n-1} = a_n^2 + 1$. Hence we have

$$a_{n+2}a_n + a_n^2 = a_{n+1}^2 + a_{n+1}a_{n-1}$$

and

$$\frac{a_{n+2} + a_n}{a_{n+1}} = \frac{a_{n+1} + a_{n-1}}{a_n} = \cdots = \frac{a_3 + a_1}{a_2} = 3.$$

Therefore, $a_{n+2} = 3a_{n+1} - a_n$.

Using the above formula, by induction, with $\{F_n\}$ the Fibonacci sequence defined by $F_1 = F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$, we have $a_n = F_{2n-1}$. Hence, using the Cassini identity, we get

$$b_n = a_n a_{n+1} = F_{2n+1}F_{2n-1} = F_{2n}^2 + 1,$$

which gives $b_n - 1 = F_{2n}^2$. On the other hand,

$$a_{n+2} = 3a_{n+1} - a_n = \frac{a_{n+1}^2 + 1}{a_n} \Rightarrow 3a_{n+1}a_n - a_n^2 = a_{n+1}^2 + 1$$
$$5a_{n+1}a_n - 1 = a_{n+1}^2 + a_n^2 + 2a_{n+1}a_n = (a_{n+1} + a_n)^2$$

which implies that

$$5b_n - 1 = (F_{2n+1} + F_{2n-1})^2.$$

Therefore

$$5b_n^2 - 6b_n + 1 = (b_n - 1)(5b_n - 1) = F_{2n}^2 (F_{2n+1} + F_{2n-1})^2$$

is a perfect square for every $n \geq 1$.

4289. *Proposed by George Apostolopoulos.*

Prove that in any triangle ABC, we have

$$\sqrt[3]{\frac{r_a}{h_a}} + \sqrt[3]{\frac{r_b}{h_b}} + \sqrt[3]{\frac{r_c}{h_c}} \leq \frac{3R}{2r},$$

where r_a, r_b, r_c are lengths of the exradii, h_a, h_b, h_c are the lengths of the altitudes and R and r are circumradius and inradius, respectively, of the triangle ABC.

We received 10 correct submissions. We present the solution by Dan Daniel.

Let $t = \frac{R}{r}$. It is well known that

$$r_a + r_b + r_c = r + 4R$$

and

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r}.$$

Hence, by the generalized Hölder's Inequality, we have

$$\begin{aligned}\sum_{cyc} \sqrt[3]{\frac{r_a}{h_a}} &= \sum_{cyc} \left(1 \cdot \sqrt[3]{r_a} \cdot \frac{1}{\sqrt[3]{h_a}}\right) \\ &= (1+1+1)^{1/3}(r_a + r_b + r_c)^{1/3} \left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right)^{1/3} \\ &= \left(3(r+4R)\left(\frac{1}{r}\right)\right)^{1/3} \\ &= (3(1+4t))^{1/3}.\end{aligned}$$

Hence, it suffices to prove that

$$\begin{aligned}&(3(1+4t))^{1/3} \le \frac{3t}{2} \\ \iff\ & 8(1+4t) \le 9t^3 \\ \iff\ & 9t^3 - 32t - 8 \ge 0 \\ \iff\ & (t-2)(9t^2 + 18t + 4) \ge 0,\end{aligned}$$

which is true since $t - 2 \ge 0$ by the well known Euler's Inequality $R \ge 2r$.

4290. *Proposed by Dao Thanh Oai and Leonard Giugiuc.*

Let $A_1A_2A_3A_4A_5$ be a cyclic convex pentagon and let $B_1B_2B_3B_4B_5$ be a regular pentagon, both inscribed in the same circle. Prove that

$$\sum_{1 \le i < j \le 5} A_iA_j \le \sum_{1 \le i < j \le 5} B_iB_j.$$

The two submissions that we received were correct; we feature the solution by Digby Smith. We shall see that a stronger result holds: The perimeter of a convex pentagon inscribed in a circle is less than or equal to the perimeter of a regular pentagon inscribed in the same circle; furthermore, the same holds for the perimeters of the pentagrams that share those same vertices. The first claim is a known theorem; although the Editor cannot provide a handy reference, the proof of Problem 1 in L. Kurlyandchik's "Approaching the Extremum" [2016: 69-74] can easily be modified by replacing the word area *everywhere by* perimeter*.*

Assume that the common circumcircle of the given pentagons has radius $\frac{1}{2}$ and centre C. For $j = 1, 2, 3, 4, 5$, let $\alpha_j = \angle A_jCA_{j+1}$ (where subscripts are reduced modulo 5). Note that

$$\sum_{j=1}^{5} \alpha_j = 2\pi,$$

while $\angle B_j C B_{j+1} = \frac{2\pi}{5}$ for all j; furthermore, any chord XY has length $\sin \frac{\angle XCY}{2}$ with $0 < \frac{\angle XCY}{2} < \pi$. Because the function $\sin x$ is strictly concave on the domain $(0, \pi)$, Jensen's inequality applies so that

$$\begin{aligned}\sum_{j=1}^{5} A_j A_{j+1} &= \sum_{j=1}^{5} \sin \frac{\alpha_j}{2} \\ &\leq 5 \sin \left(\frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5}{10} \right) \\ &= 5 \sin \frac{2\pi}{10} \\ &= \sum_{j=1}^{5} B_j B_{j+1}.\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{j=1}^{5} A_j A_{j+2} &= \sum_{j=1}^{5} \sin \frac{\alpha_j + \alpha_{j+1}}{2} \\ &\leq 5 \sin \left(\frac{\alpha_1 + \alpha_2 + \alpha_2 + \alpha_3 + \alpha_3 + \alpha_4 + \alpha_4 + \alpha_5 + \alpha_5 + \alpha_1}{10} \right) \\ &= 5 \sin \frac{4\pi}{10} \\ &= \sum_{j=1}^{5} B_j B_{j+2}.\end{aligned}$$

Putting these two inequalities together yields the desired result; specifically,

$$\begin{aligned}\sum_{1 \leq j < k \leq 5} A_j A_k &= \sum_{j=1}^{5} A_j A_{j+1} + \sum_{j=1}^{5} A_j A_{j+2} \\ &\leq \sum_{j=1}^{5} B_j B_{j+1} + \sum_{j=1}^{5} B_j B_{j+2} \\ &= \sum_{1 \leq j < k \leq 5} B_j B_k.\end{aligned}$$

Equality holds if and only if $\alpha_j = \frac{2\pi}{5}$ for all j, which holds if and only if $A_1 A_2 A_3 A_4 A_5$ is a regular pentagon (because an equilateral pentagon inscribed in a circle is necessarily regular).