Crux Mathematicorum

VOLUME 43, NO. 9 November/Novembre 2017

IN THIS ISSUE / DANS CE NUMÉRO

Crux Mathematicorum

Founding Editors / Rédacteurs-fondateurs: Léopold Sauvé & Frederick G.B. Maskell

Former Editors / Anciens Rédacteurs: G.W. Sands, R.E. Woodrow, Bruce L.R. Shawyer, Shawn Godin

Crux Mathematicorum with Mathematical Mayhem

Former Editors / Anciens Rédacteurs: Bruce L.R. Shawyer, James E. Totten, Václav Linek, Shawn Godin

THE CONTEST CORNER

No. 59

John McLoughlin

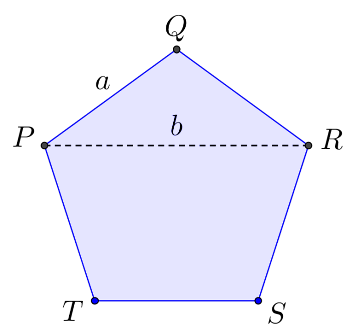
The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **May 1, 2018**.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

CC291. The point P is on the parabola $x^2 = 4y$. The tangent at P meets the line $y = -1$ at the point A. For the point $F(0,1)$, prove that $\angle AFP = 90^\circ$ for all positions of P, except $(0,0)$.

CC292. Let a be the length of a side and b be the length of a diagonal in the regular pentagon $PQRST$ as shown.



Prove that
$$\frac{b}{a} - \frac{a}{b} = 1.$$

CC293. The transformation $T : (x, y) \mapsto \left(-\frac{1}{2}(3x - y), -\frac{1}{2}(x + y)\right)$ is applied repeatedly to the point $P_0(3, 1)$, which produces a sequence of points $P_1, P_2, \ldots$. Show that the area of the convex quadrilateral defined by any four consecutive points is constant.

CC294.

a) Prove that $\sin 2A = \dfrac{2\tan A}{1 + \tan 2A}$, where $0 < A < \pi/2$.

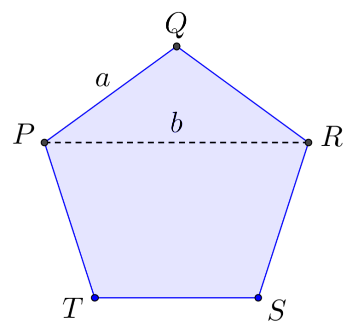
b) If $\sin 2A = 4/5$, find $\tan A$.

CC295. In how many ways is it possible to choose four distinct integers from $1, 2, 3, 4, 5, 6$ and 7, so that their sum is even?

. . . .

CC291. On considère un point P sur la parabole d'équation $x^2 = 4y$. La tangente au point P coupe la droite d'équation $y = -1$ au point A. Soit le point $F(0, 1)$. Démontrer que $\angle AFP = 90°$ pour toutes les positions de P, à l'exception de $(0, 0)$.

CC292. Dans la figure suivante, $PQRST$ est un pentagone régulier, a est la longueur de ses côtés et b est la longueur de ses diagonales.



Démontrer que

$$\frac{b}{a} - \frac{a}{b} = 1.$$

CC293. On fait subir la transformation $T : (x, y) \mapsto \left(-\frac{1}{2}(3x - y), -\frac{1}{2}(x + y)\right)$ de façon répétée au point $P_0(3, 1)$ et aux images successives, ce qui produit une suite de points $P_1, P_2, \ldots$ Démontrer que l'aire du quadrilatère convexe défini par n'importe quels quatre points consécutifs est constante.

CC294.

a) Démontrer que $\sin 2A = \dfrac{2 \tan A}{1 + \tan 2A}$, où $0 < A < \pi/2$.

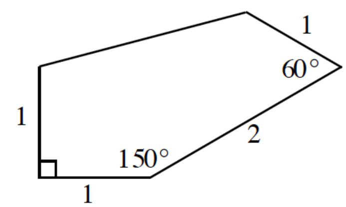
b) Sachant que $\sin 2A = \dfrac{4}{5}$, déterminer $\tan A$.

CC295. De combien de façons est-il possible de choisir quatre entiers distincts, parmi les entiers $1, 2, 3, 4, 5, 6$ et 7, de manière que leur somme soit paire?

CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2016: 42(9), p. 373–375.

CC241. Over many centuries, tilings have fascinated mathematicians and the society in general. Particularly interesting are tilings of the plane that use a single type of tile. You can tile the plane with some regular polygons (such as equilateral triangles, squares, regular hexagons). On the other hand, you cannot tile the plane using regular pentagons. Now, we know that some non-regular pentagons can be used to tile the plane, although not all of them are yet known. It was therefore with great enthusiasm that in August 2015, the world welcomed the discovery of a new pentagonal tiling, illustrated below.

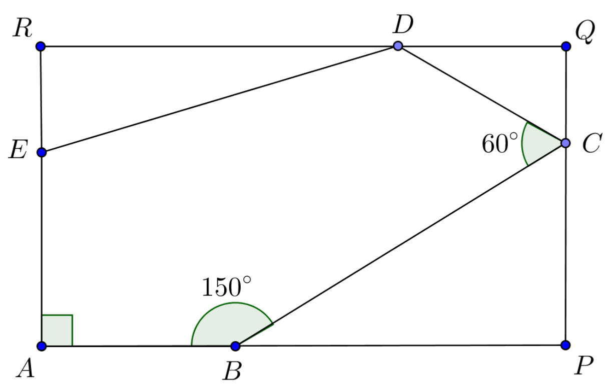


Use the lengths given and angle sizes to calculate the exact area of this pentagon.

Originally Problem 4 from the Scottish Mathematical Council Mathematical Challenge 2016–2017.

We received nine correct solutions and one incorrect submission. We present the solution by Kathleen E. Lewis, slightly expanded by the editor.

We inscribe the pentagon in a rectangle $APQR$ as shown in the diagram, with A set at the origin. Then the area of the pentagon is the area of the rectangle minus the area of the three corner triangles.



By definition of the pentagon, $\angle CBP = 30^\circ$. Since $BC = 2$, we get $CP = 1$, $BP = \sqrt{3}$, and thus $AP = 1 + \sqrt{3}$. Furthermore

$$\angle DCQ = 180^\circ - \angle BCD - \angle PCB = 60^\circ,$$

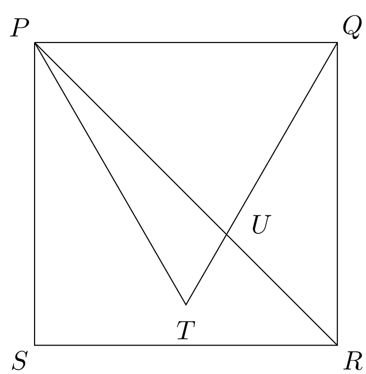
implying $CQ = \frac{1}{2}$ (thus $PQ = \frac{3}{2}$) and $DQ = \frac{\sqrt{3}}{2}$. Since $APQR$ is a rectangle we obtain

$$DR = 1 + \frac{\sqrt{3}}{2} \quad \text{and} \quad ER = \frac{1}{2}.$$

The area of the pentagon is therefore

$$\begin{aligned}
[ABCDE] &= AP \cdot PQ - \frac{1}{2}(BP \cdot CP + CQ \cdot DQ + DR \cdot ER) \\
&= \frac{3(1+\sqrt{3})}{2} - \frac{1}{2}\left(\sqrt{3} + \frac{\sqrt{3}}{4} + \frac{2+\sqrt{3}}{4}\right) \\
&= \frac{5+3\sqrt{3}}{4}.
\end{aligned}$$

CC242. The diagram below shows square $PQRS$ with sides of length 1 unit. Triangle PQT is equilateral. Show that the area of triangle UQR is $(\sqrt{3}-1)/4$ square units.



Originally Problem 5 from the Scottish Mathematical Council Mathematical Challenge 2016–2017.

We received twelve correct solutions. We present the solution of John Heuver.

Note that $\angle RQT = 30^\circ$ and $\angle PUQ = 75^\circ$ where $\sin\ 75^\circ = \frac{\sqrt{6}+\sqrt{2}}{4}$. Using the Law of Sines in $\triangle PUQ$ we find that $UQ = \sqrt{3} - 1$. Letingt $[UQR]$ denote area $\triangle UQR$, we get

$$[UQR] = \frac{1}{2} \cdot UQ \cdot QR \cdot \sin\ 30^\circ = \frac{1}{2}(\sqrt{3}-1) \cdot 1 \cdot \frac{1}{2} = \frac{1}{4}\ (\sqrt{3}-1).$$

CC243. Eight islands each have one or more air services. An air service consists of flights to and from another island, and no two services link the same pair of islands. There are 17 air services in all between the islands. Show that it must be possible to use these air services to fly between any pair of islands.

Originally Problem 2 from the Scottish Mathematical Council Mathematical Challenge 2016–2017.

We received six correct solutions. We present the solution of Joel Schlosberg.

Select an arbitrary island I. Let A be the set of a islands it is possible to reach from I via the network of air services (including both those that require multiple flights and I itself). Since there is at least one air service from I to another island, $a \geq 2$.

Suppose that $a < 8$. Then there is at least one island $J \notin A$. At least one air service connects J with another island K. If $K \in A$, there is a way to fly from I to K and then to J, contradicting the assumption that $J \notin A$. Therefore $J, K \notin A$, so $a \leq 6$. Since $2 \leq a \leq 6$, $|a - 4| \leq 2$.

If $J \in A$, $K \notin A$, then any air service connecting J and K would make it possible to fly from I to J to K, contradicting $K \notin A$. Therefore, no air service can connect an island in A to an island not in A. Since each pair of islands has at most one air service between them, at most $\binom{a}{2}$ air services connect the a islands in A, and at most $\binom{8-a}{2}$ air services connect the $8 - a$ islands not in A. Then the total number of air services is at most

$$\binom{a}{2} + \binom{8-a}{2} = a^2 - 8a + 28 = 12 + (a-4)^2 \leq 12 + 2^2 = 16,$$

contradicting the assumption that there are 17 air services. Therefore, $a = 8$. That is, all 8 islands are connected to I by some sequence of flights. Then for any pair of islands J, K, it is possible to fly from J to I to K.

CC244. How many distinct solutions consisting of positive integers does this system of equations have?

$$\begin{aligned} x_1 + x_2 + x_3 &= 5, \\ y_1 + y_2 + y_3 &= 5, \\ z_1 + z_2 + z_3 &= 5, \\ x_1 + y_1 + z_1 &= 5, \\ x_2 + y_2 + z_2 &= 5, \\ x_3 + y_3 + z_3 &= 5. \end{aligned}$$

Originally Problem 3 from the Scottish Mathematical Council Mathematical Challenge 2016–2017.

We received 7 submissions of which 3 were correct and complete. We present the solution by Ivko Dimitrić.

We think of a solution of this system as a set of nine numbers (assigned to the variables) placed in nine cells of a 3×3 grid (a table, or a matrix) so that each row sum and each column sum equals 5. Clearly, since all integers are positive, no number greater than or equal to 4 can be considered since the other two numbers in the same row are at least one, producing a row sum of at least 6. Therefore, the numbers that can be used are 1, 2, and 3. There are only two ways (up to permutation) to represent 5 as the sum of three positive integers and they are $5 = 3 + 1 + 1 = 2 + 2 + 1$. The number of 3s that can be placed in the grid can be only 0, 1, or 3. Indeed, the number of 3s used cannot be exactly two, for otherwise the row without 3 is filled with two 2s and one 1, whereas one of the 2s would be in the same column with one of the 3s, making that column sum at least 6.

x_1	x_2	x_3
y_1	y_2	y_3
z_1	z_2	z_3

If the number of 3s used is exactly three, the other six cells must be filled with 1s, and, as long as 3s are placed so that no two are in the same row or the same column, the other cells are filled with 1s to produce a solution to the system. Thus, in this case the number of solutions equals the number of different ways to arrange three 3s in a 3×3 grid so that in any given row or column there is only one 3. That can be done in $3! = 6$ ways since we can place the first 3 in the first row in any of the three cells of the first row, and when that has been done there are two choices to place the next number 3 in the second row and after that the cell for the last 3 in the third row is uniquely determined by the previous choices, which is the cell not in the columns containing the first two 3s chosen.

If the number of 3s is zero (no 3 used in the grid) then the grid is filled with 2s and 1s so that of the three 1s used no two are in the same row or column. The number of ways to arrange three 1s that way is the same as the number of ways to arrange three 3s previously discussed, so it equals $3! = 6$ ways.

There remains the case of having exactly one 3 in the grid. In the column and the row containing that 3 the other four entries are 1s, the remaining four cells in the grid to be filled with 2s. That sole number 3 can be placed in any of the nine cells and when that has been done, use four 1s for the four cells in the same row and the same column where the 3 is and four 2s for the remaining four cells of the grid. So, the cells where 1s and 2s go are completely determined by the choice of the cell for the only 3. Thus there are nine ways in this case.

Altogether, the number of different ways to fill the grid with a combination of 1s, 2s, and 3s so that each row and column sum is 5 equals $6 + 6 + 9 = 21$, so the number of solutions to the system is also 21.

CC245. A pyramid stands on horizontal ground. Its base is an equilateral triangle with sides of length a, the other three edges of the pyramid are of length b and its volume is V. Show that

$$V = \frac{1}{12}a^2\sqrt{3b^2 - a^2}.$$

The pyramid is then placed so that a non-equilateral face lies on the ground. Find the height of the pyramid in this position.

Originally Problem 1 from the Scottish Mathematical Council Mathematical Challenge 2016–2017.

We received seven submissions, out of which six were correct and complete. We present the solution by Joel Schlosberg.

The volume of a pyramid with base B is

$$V = \frac{1}{3}A_B h_B, \tag{1}$$

where A_B is the area of the base and h_B the height with respect to B. First we use (1) with the base B_1 that is an equilateral triangle. Then $A_{B_1} = \sqrt{3}a^2/4$. The apex point of the pyramid, the center of the equilateral triangle (which by symmetry is also the foot of the altitude through the apex point), and an arbitrary vertex of the equilateral triangle form a right triangle with hypotenuse length b and legs of lengths h_{B_1} and $a/\sqrt{3}$ (the distance between the center and any vertex of an equilateral triangle of length a). By the Pythagorean theorem,

$$h_{B_1} = \sqrt{b^2 - \frac{a^2}{3}}.$$

By (1) we obtain

$$V = \frac{1}{3}A_{B_1}h_{B_1} = \frac{1}{3}\cdot\frac{\sqrt{3}a^2}{4}\cdot\sqrt{b^2 - \frac{a^2}{3}} = \frac{1}{12}a^2\sqrt{3b^2 - a^2}. \tag{2}$$

Now let B_2 be one of the bases with sidelengths a, b, and b. The height of this triangle over the side with sidelength a is $\sqrt{b^2 - (a/2)^2}$ and its area thus

$$A_{B_2} = \frac{a}{2}\cdot\sqrt{b^2 - \frac{a^2}{4}} = \frac{1}{4}a\sqrt{4b^2 - a^2}. \tag{3}$$

Using (1), (2), and (3), we obtain

$$\frac{1}{12}a^2\sqrt{3b^2 - a^2} = V = \frac{1}{3}A_{B_2}h_{B_2} = \frac{1}{12}a\sqrt{4b^2 - a^2}\cdot h_{B_2},$$

so the height of the pyramid from a non-equilateral face is

$$a\sqrt{\frac{3b^2 - a^2}{4b^2 - a^2}}.$$

THE OLYMPIAD CORNER

No. 357

Carmen Bruni

The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **May 1, 2018**.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

OC351. Solve the system of equations

$$\begin{aligned} &6x - y + z^2 = 3, \\ &x^2 - y^2 - 2z = -1, \\ &6x^2 - 3y^2 - y - 2z^2 = 0. \end{aligned}$$

where $x, y, z \in \mathbb{R}$.

OC352. Let O_1 and O_2 intersect at P and Q. Their common external tangent touches O_1 and O_2 at A and B respectively. A circle Γ passing through A and B intersects O_1, O_2 at D, C. Prove that $\dfrac{CP}{CQ} = \dfrac{DP}{DQ}$.

OC353. Prove that for any positive integer k,

$$(k^2)! \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$

is an integer.

OC354. Solve the equation $1 + x^z + y^z = LCM(x^z, y^z)$ in the set of natural numbers.

OC355. Let $\mathbb{N}$ denote the set of natural numbers. Define a function $T : \mathbb{N} \to \mathbb{N}$ by $T(2k) = k$ and $T(2k+1) = 2k+2$. We write $T^2(n) = T(T(n))$ and in general $T^k(n) = T^{k-1}(T(n))$ for any $k > 1$.

a) Show that for each $n \in \mathbb{N}$, there exists k such that $T^k(n) = 1$.

b) For $k \in \mathbb{N}$, let c_k denote the number of elements in the set $\{n : T^k(n) = 1\}$. Prove that $c_{k+2} = c_{k+1} + c_k$, for $k \geq 1$.

. . . .

OC351. Résoudre le système d'équations

$$\begin{aligned} &6x - y + z^2 = 3 \\ &x^2 - y^2 - 2z = -1 \\ &6x^2 - 3y^2 - y - 2z^2 = 0, \end{aligned}$$

x, y et z étant des réels.

OC352. Soit O_1 et O_2 deux cercles qui se coupent en P et Q. Leur tangente commune extérieure touche O_1 et O_2 aux points respectifs A et B. Un cercle Γ passe aux points A et B et coupe O_1 et O_2 aux points respectifs D et C. Démontrer que $\dfrac{CP}{CQ} = \dfrac{DP}{DQ}$.

OC353. Démontrer que pour tout entier strictement positif k,

$$(k^2)! \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$

est un entier.

OC354. Résoudre l'équation $1 + x^z + y^z = PPCM(x^z, y^z)$ dans l'ensemble des entiers strictement positifs.

OC355. On définit une fonction $T : \mathbb{N} \to \mathbb{N}$ par $T(2k) = k$ et $T(2k+1) = 2k+2$. On définit aussi $T^2(n) = T(T(n))$ et $T^k(n) = T^{k-1}(T(n))$ lorsque $k > 1$.

a) Démontrer que pour tout n ($n \in \mathbb{N}$), il existe une valeur de k pour laquelle $T^k(n) = 1$.

b) Pour tout k ($k \in \mathbb{N}$), soit c_k le nombre d'éléments de l'ensemble $\{n : T^k(n) = 1\}$. Démontrer que $c_{k+2} = c_{k+1} + c_k$, pour tout k ($k \geq 1$).

OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2016 : 42(7), p. 297–298.

OC291. Let the integer $n \geq 2$, and $x_1, x_2, \cdots, x_n$ be positive real numbers such that $\sum_{i=1}^n x_i = 1$. Prove that

$$\left(\sum_{i=1}^{n} \frac{1}{1-x_i}\right)\left(\sum_{1\leq i<j\leq n} x_i x_j\right) \leq \frac{n}{2}.$$

Originally problem 3 from day 1 of the 2015 China Western National Olympiad.

We received 9 correct submissions. We present the solution by Michel Bataille.

Let

$$X = \sum_{1\leq i<j\leq n} x_i x_j.$$

From

$$\begin{aligned} X &= x_1(x_2+\cdots+x_n) + \sum_{2\leq i<j\leq n} x_i x_j \\ &= x_1(1-x_1) + \sum_{2\leq i<j\leq n} x_i x_j \end{aligned}$$

we deduce

$$\frac{X}{1-x_1} = x_1 + \frac{\sum_{2\leq i<j\leq n} x_i x_j}{x_2+\cdots+x_n} = x_1 + \frac{\sum_{2\leq i<j\leq n} x_i x_j}{1-x_1}. \tag{1}$$

But we have

$$\begin{aligned} \sum_{2\leq i<j\leq n} x_i x_j &= \frac{1}{2}\left((x_2+\cdots+x_n)^2 - (x_2^2+\cdots+x_n^2)\right) \\ &\leq \frac{1}{2}\left((x_2+\cdots+x_n)^2 - \frac{1}{n-1}(x_2+\cdots+x_n)^2\right), \end{aligned}$$

where the inequality follows from

$$(n-1)(x_2^2+\cdots+x_n^2) \geq (x_2+\cdots+x_n)^2$$

(by the Cauchy-Schwarz inequality).

Thus we obtain

$$\sum_{2\leq i<j\leq n} x_i x_j \leq \frac{n-2}{2(n-1)} \cdot (x_2+\cdots+x_n)^2 = \frac{(n-2)(1-x_1)^2}{2(n-1)}$$

and from (1)

$$\frac{X}{1-x_1} \le x_1 + \frac{(n-2)(1-x_1)}{2(n-1)}.$$

In the same way, with obvious changes, we get

$$\begin{aligned}
\frac{X}{1-x_2} &\le x_2 + \frac{(n-2)(1-x_2)}{2(n-1)}, \\
\frac{X}{1-x_3} &\le x_3 + \frac{(n-2)(1-x_3)}{2(n-1)}, \quad \dots \\
\frac{X}{1-x_n} &\le x_n + \frac{(n-2)(1-x_n)}{2(n-1)}
\end{aligned}$$

and so

$$\begin{aligned}
\left(\sum_{i=1}^{n} \frac{1}{1-x_i}\right)\left(\sum_{1\le i<j\le n} x_i x_j\right) &= \sum_{i=1}^{n} \frac{X}{1-x_i} \\
&\le (x_1 + x_2 + \cdots + x_n) + \frac{n-2}{2(n-1)} \cdot \sum_{i=1}^{n}(1-x_i) \\
&= 1 + \frac{n-2}{2(n-1)}(n-1) = \frac{n}{2}
\end{aligned}$$

as required.

OC292. On the graph of a polynomial with integer coefficients, two points are chosen with integer coordinates. Prove that if the distance between them is an integer, then the segment that connects them is parallel to the horizontal axis.

Originally problem 1 from day 1 of the 2015 Spain Mathematical Olympiad.

We received 5 correct submissions. We present the solution by Steven Chow.

Let the x-axis be the horizontal axis. Let $f(x)$ be the polynomial function. Let a and b be the x-coordinates of the 2 points.

Since a, b, and the coefficients of $f(x)$ are integers, $a - b \mid f(a) - f(b)$.

Since the distance between the 2 points is an integer, from the Pythagorean Theorem, the following is a square number :

$$(a-b)^2 + (f(a) - f(b))^2 = (a-b)^2\left(1 + \left(\frac{f(a)-f(b)}{a-b}\right)^2\right).$$

Therefore $1 + \left(\frac{f(a)-f(b)}{a-b}\right)^2$ is a square number and $\frac{f(a)-f(b)}{a-b} = 0$ implies that $f(a) = f(b)$.

Thus, the segment that connects the 2 points is parallel to the horizontal axis.

OC293. You are given N such that $N \geq 3$. We call a set of N points on a plane acceptable if their abscissae are unique, and each of the points is coloured either red or blue. Let's say that a polynomial $P(x)$ divides a set of acceptable points either if there are no red dots above the graph of $P(x)$, and below, there are no blue dots, or if there are no blue dots above the graph of $P(x)$ and there are no red dots below. Keep in mind, dots of both colors can be present on the graph of $P(x)$ itself. For what least value of k is an arbitrary acceptable set of N points divisible by a polynomial of degree k ?

Originally problem 4 of the 2015 All Russian Olympiad Grade 11.

We present the solution by Oliver Geupel. There were no other submissions.

The answer is $k = N - 2$.

Given any acceptable N-set, the interpolation polynomial of $N-1$ out of the N points is a polynomial of degree not greater than $N-2$ which divides that set. Hence $k \leq N - 2$.

It is now sufficient to specify an acceptable N-set $\mathcal{S}_N$ such that every polynomial that divides $\mathcal{S}_N$ is of degree at least $N-2$. The blue points in our set $\mathcal{S}_N$ are $A_{-1}(-1, 1)$, $A_0(0, -1)$, and $A_{2k}(2k, 0)$ where $k = 1, 2, \ldots, \lfloor N/2 \rfloor - 1$. The red points in $\mathcal{S}_N$ are $A_{2k-1}(2k-1, 0)$ where $k = 1, 2, \ldots, \lceil N/2 \rceil - 1$. Those are in total $\lfloor N/2 \rfloor + \lceil N/2 \rceil = N$ points as required. Let $P(x)$ be a polynomial that divides $\mathcal{S}_N$. If $N = 3$, then $P(x)$ cannot be a constant, which shows that $k \geq 1 = N - 2$. It remains to consider $N \geq 4$.

We show that $\deg P(x) > 1$. Assume $\deg P(x) \leq 1$. Since the vertices of triangle $A_{-1}A_0A_2$ are of the same colour (blue), the graph of $P(x)$ cannot contain any interior point of the triangle. Thus, $P(x)$ cannot separate the interior red point A_1 from the blue vertices, which is a contradiction. Consequently, $\deg P(x) \geq 2$.

We have two cases : Either there are no red dots above the graph of $P(x)$ and no blue dots below, or there are no blue dots above the graph of $P(x)$ and no red dots below.

Let us first suppose that there are no red dots above the graph of $P(x)$ and no blue dots below. We then have $P(0) \leq P(1)$, $P(1) \geq P(2)$, $P(2) \leq P(3)$, continued with alternating order relations until either $P(N-3) \leq P(N-2)$ or $P(N-3) \geq P(N-2)$. By the Mean Value Theorem, there are real numbers $x_1 \in (0, 1)$, $x_2 \in (1, 2)$, …, $x_{N-2} \in (N-3, N-2)$ such that $P'(x_1)$, $P'(x_2)$, …, $P'(x_{N-2})$ have alternating signs. Hence, $\deg P(x) \geq N - 2$.

It remains to consider the case where there are no blue dots above the graph of $P(x)$ and no red dots below that graph. We then readily see that $P(-1) \geq P(1)$, $P(1) \leq P(2)$, $P(2) \geq P(3)$, continued with alternating order relations until either $P(N-3) \leq P(N-2)$ or $P(N-3) \geq P(N-2)$. By the Mean Value Theorem, there are real numbers $x_1 \in (-1, 1)$, $x_2 \in (1, 2)$, …, $x_{N-2} \in (N-3, N-2)$ such that $P'(x_1), P(x_2), \ldots, P'(x_{N-2})$ have alternating signs. Hence, $\deg P(x) \geq N - 2$.

OC294. In given triangle $\triangle ABC$, difference between sizes of each pair of sides is at least $d > 0$. Let G and I be the centroid and incenter of $\triangle ABC$ and r be its inradius. Show that

$$|AIG| + |BIG| + |CIG| \geq \frac{2}{3}dr,$$

where $|XYZ|$ is the area of triangle $\triangle XYZ$.

Originally problem 5 from Round 3 Category A of the 2015 Czech and Slovak National Olympiad.

We received 2 correct submissions. We present the solution by Mohammed Aassila.

We use barycentric coordinates in triangle ABC. We know that $G = (1 : 1 : 1)$ and $I = (a : b : c)$ where $a = BC, b = CA$ and $c = AB$. We have :

$$[AIG] = \frac{[ABC]}{3(a+b+c)} \cdot \begin{vmatrix} 1 & 0 & 0 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = \frac{[ABC] \cdot |b-c|}{3(a+b+c)}.$$

Similar equations for $[BIG]$ and $[CIG]$ can be obtained in the same manner. Hence :

$$\begin{aligned} [AIG] + [BIG] + [CIG] &= \frac{[ABC] \cdot (|b-c| + |c-a| + |a-b|)}{3(a+b+c)} \\ &= \frac{r \cdot (|b-c| + |c-a| + |a-b|)}{6}. \end{aligned}$$

Assume without loss of generality that $a \leq b \leq c$. Since $b - a \geq d$ and $c - b \geq d$ we have that $c - a \geq 2d$. Thus

$$\frac{r \cdot (|b-c| + |c-a| + |a-b|)}{6} \geq \frac{2}{3} \cdot dr.$$

OC295. Let $\mathbb{N} = \{1, 2, 3, ...\}$ be the set of positive integers. Let $f : \mathbb{N} \to \mathbb{N}$ be a function that gives a positive integer value, to every positive integer. Suppose that f satisfies the following conditions :

$$f(1) = 1, \quad f(a + b + ab) = a + b + f(ab).$$

Find the value of $f(2015)$.

Originally problem 3 from day 1 of the 2015 Mexico National Olympiad.

We received 9 correct submissions. We present the solution by Missouri State University Problem Solving Group.

We will show that $f(n) = n$ for every $n \in \mathbb{N}$. Let $S = \{n \in \mathbb{N} : f(n) = n\}$. First we note that for any $a, b \in \mathbb{N}$, we have that

$$ab \in S \iff a + b + ab \in S$$

Since $2n+1 = 1+n+1\cdot n$, then for any $n \in \mathbb{N}$, we have

$$n \in S \iff 2n+1 \in S$$

By repeating this argument, we can then extend this to

$$n \in S \iff 2n+1 \in S \iff 4n+3 \in S \iff 8n+7 \in S$$

and so on. In particular, $1 \in S$ if and only if $8(1)+7 = 15 \in S$ and $2 \in S$ if and only if $8(2)+7 = 23 \in S$. But, by the first displayed equation above, $15 \in S$ if and only if $5+3+5(3) = 23 \in S$ and so combining gives $1 \in S$ if and only if $2 \in S$. In a similar manner, combining the third displayed equation above with the fact that $4n+3 = 3+n+3n$ gives that

$$n \in S \iff 3n \in S.$$

So when $n = 1$, we see that $1 \in S \iff 3 \in S$. In particular, since $f(1) = 1$, we see that $1 \in S$ and hence so are both 2 and 3. Now, we prove that $n \in S$ for all $n \in \mathbb{N}$ by strong induction. The base cases have already been proven so we suppose that $n \in S$ for all $1 \le n \le k$ for some $k \in \mathbb{N}$ with $k \ge 3$. We will show that $k+1 \in S$.

If $k+1 \equiv 0 \mod 3$, then $k+1 = 3n$ for some $n \le k$ and by assumption, $n \in S$ and thus, from above, we have that $3n \in S$.

If $k+1 \equiv 1 \mod 3$, then $2k+3 = 2(k+1)+1 \equiv 0 \mod 3$. Then $2k+3 = 3n$ for some integer n. Since $k \ge 3$, we see that $n \le k$ and so as before, $2k+3 \in S$. Finally, since $2(k+1)+1 \in S$, the second displayed equation gives us that $k+1 \in S$.

If $k+1 \equiv 2 \mod 3$, then $k+2 = 2n$ for some $n \in \mathbb{N}$ with $n \ge 2$ since $k \ge 3$. Now, since $2(n-1) \le 3(n-1) = k-1 \le k$, we know that $2(n-1) \in S$ and so $2+(n-1)+2(n-1) = 3n-1 = k+1 \in S$.

Thus, by strong induction, we see that $S = \mathbb{N}$. In particular, $f(2015) = 2015$.

FOCUS ON...

No. 28

Michel Bataille

Some relations in the triangle (II)

Introduction

In this number, we continue our selection of relations in the triangle, focusing on formulas involving lengths related to the classical cevians. As in part I, the notations are standard and borrowed from **[2]**.

About the altitudes h_a, h_b, h_c

Besides the easy

$$\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r}$$

(which readily follows from $\frac{1}{h_a}=\frac{a}{2F}=\frac{a}{2rs}$ and similar relations), we consider a less known, easy-to-remember formula :

$$(h_a+h_b+h_c)\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)=(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \tag{1}$$

(again an obvious consequence of $2F=ah_a=bh_b=ch_c$).

With the help of the identities

$$(x+y+z)(xy+yz+zx)-3xyz=\sum_{\text{cyclic}}x^2(y+z)=(x+y)(y+z)(z+x)-2xyz, \tag{2}$$

we may equivalently write (1) as

$$\frac{(h_a+h_b)(h_b+h_c)(h_c+h_a)}{h_ah_bh_c}=\frac{(a+b)(b+c)(c+a)}{abc} \tag{3}$$

and give a solution to problem **3453** [2009 : 325,328 ; 2010 : 342] that asked for the inequality

$$8\left(\sum_{\text{cyclic}}h_a^2(h_b+h_c)\right)+16h_ah_bh_c\leq 3\sqrt{3}\left(\sum_{\text{cyclic}}a^2(b+c)\right)+6\sqrt{3}abc.$$

Indeed, a consequence of (2) is that this inequality is equivalent to $Q\leq\frac{3\sqrt{3}}{8}$ where

$$Q=\frac{(h_a+h_b)(h_b+h_c)(h_c+h_a)}{(a+b)(b+c)(c+a)}.$$

But from (3) we have

$$Q=\frac{h_ah_bh_c}{abc}=\frac{8F^3}{(abc)^2}=\frac{abc}{8R^3}=\sin A\sin B\sin C,$$

hence, using AM-GM and the concavity of the Sine function on $(0,\pi)$,

$$Q \leq \left(\frac{\sin A + \sin B + \sin C}{3}\right)^3 \leq \left(\sin\left(\frac{A+B+C}{3}\right)\right)^3 = \frac{3\sqrt{3}}{8}.$$

About the distances IA, IB, IC

Prompted by the intervention of IA in part I, we now present a couple of interesting relations connecting the distances IA, IB, IC to other elements of the triangle.

First we consider the product $IA\cdot IB\cdot IC$ and show that

$$sIA\cdot IB\cdot IC = r\cdot abc. \tag{4}$$

The proof is easy :

$$IA\cdot IB\cdot IC = \frac{r}{\sin\frac{A}{2}}\cdot\frac{r}{\sin\frac{B}{2}}\cdot\frac{r}{\sin\frac{C}{2}} = \frac{r^3}{\frac{r}{4R}} = 4Rr^2 = \frac{r}{s}\cdot abc.$$

At this point, it is worth mentioning a beautiful formula that also involves the excenters I_a, I_b, I_c :

$$IA\cdot IB\cdot IC\cdot I_aA\cdot I_bB\cdot I_cC = (abc)^2. \tag{5}$$

To see this, we first remark that

$$bc\cos^2\frac{A}{2} = \frac{bc(1+\cos A)}{2} = \frac{bc}{2}\left(1+\frac{b^2+c^2-a^2}{2bc}\right) = s(s-a), \tag{6}$$

from which we deduce that

$$IA\cdot I_aA = \frac{s-a}{\cos\frac{A}{2}}\cdot\frac{s}{\cos\frac{A}{2}} = bc.$$

Similarly, $IB\cdot I_bB = ca,\ IC\cdot I_cC = ab$ and (5) follows.

Relation (5) reminds us of the known relation $w_aw_bw_cW_aW_bW_c = (abc)^2$ where W_a, W_b, W_c denote the lengths of the angle bisectors extended until they are chords of the circumcircle (see problem **168** [1976 : 136 ; 1977 : 233]). It is interesting to notice that we even have $w_aW_a = bc = IA\cdot I_aA$ and similar relations.

We conclude this paragraph with the formula

$$aIA^2 + bIB^2 + cIC^2 = abc, \tag{7}$$

from which we will derive a general inequality.

Using (6), we obtain

$$\begin{aligned} aIA^2 + bIB^2 + cIC^2 &= a\left(\frac{s-a}{\cos\frac{A}{2}}\right)^2 + b\left(\frac{s-b}{\cos\frac{B}{2}}\right)^2 + c\left(\frac{s-c}{\cos\frac{C}{2}}\right)^2 \\ &= a(s-a)^2\frac{bc}{s(s-a)} + b(s-b)^2\frac{ca}{s(s-b)} + c(s-c)^2\frac{ab}{s(s-c)} \\ &= \frac{abc}{s}(s-a+s-b+s-c) \end{aligned}$$

and (7) follows.

A nice application is the inequality

$$aIA \cdot PA + bIB \cdot PB + cIC \cdot PC \geq abc$$

that holds for any point P in the plane of the triangle ABC. To prove it, we use the dot product and the Cauchy-Schwarz inequality as follows :

$$\begin{aligned}
&aPA \cdot IA + bPB \cdot IB + cPC \cdot IC \\
&\qquad = \|a\overrightarrow{IA}\|\|\overrightarrow{IA} - \overrightarrow{IP}\| + \|b\overrightarrow{IB}\|\|\overrightarrow{IB} - \overrightarrow{IP}\| + \|c\overrightarrow{IC}\|\|\overrightarrow{IC} - \overrightarrow{IP}\| \\
&\qquad \geq a\overrightarrow{IA} \cdot (\overrightarrow{IA} - \overrightarrow{IP}) + b\overrightarrow{IB} \cdot (\overrightarrow{IB} - \overrightarrow{IP}) + c\overrightarrow{IC} \cdot (\overrightarrow{IC} - \overrightarrow{IP}) \\
&\qquad = aIA^2 + bIB^2 + cIC^2 - \overrightarrow{IP} \cdot (a\overrightarrow{IA} + b\overrightarrow{IB} + c\overrightarrow{IC}) \\
&\qquad = abc
\end{aligned}$$

(since $a\overrightarrow{IA} + b\overrightarrow{IB} + c\overrightarrow{IC} = \overrightarrow{0}$).

About the exradii r_a, r_b, r_c

Faced with the proof of a relation between the exradii r_a, r_b, r_c, the first move is often to use the equalities

$$F = r_a(s-a) = r_b(s-b) = r_c(s-c). \tag{8}$$

For examples, the striking formulas

$$\frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}, \qquad r_a r_b + r_b r_c + r_c r_a = s^2 = \frac{r_a r_b r_c}{r}$$

and

$$\sqrt{\frac{r r_b r_c}{r_a}} + \sqrt{\frac{r r_c r_a}{r_b}} + \sqrt{\frac{r r_a r_b}{r_c}} = s$$

are straightforwardly deduced from (8) and $F = rs = \sqrt{s(s-a)(s-b)(s-c)}$.

With the additional known formulas

$$ab + bc + ca = s^2 + r^2 + 4rR \quad \text{and} \quad a^2 + b^2 + c^2 = 2s^2 - 2r^2 - 8rR,$$

we easily obtain

$$r_a + r_b + r_c = r + 4R \quad \text{and} \quad r^2 + r_a^2 + r_b^2 + r_c^2 + a^2 + b^2 + c^2 = 16R^2$$

that were at work in problem **3570** [2010 : 397,399 ; 2011 : 402].

Since $2F = ah_a = bh_b = ch_c$, one can expect some connections with h_a, h_b, h_c. A good example is

$$\frac{h_b + h_c}{r_a} + \frac{h_b + h_c}{r_a} + \frac{h_b + h_c}{r_a} = 6 \tag{9}$$

which is mentioned but not proved in **[1]**. Here is a quick proof. Since

$$h_b + h_c = 2F\left(\frac{1}{b}+\frac{1}{c}\right) = \frac{2F(ab+ac)}{abc},$$

the left-hand side of (9) rewrites as

$$\frac{2}{abc}\left((ab+ac)(s-a)+(bc+ba)(s-b)+(ca+cb)(s-c)\right) = \frac{2}{abc}\cdot(ab(c)+bc(a)+ca(b))$$

and (9) follows.

The reader will find other formulas of the same kind in exercise **1**.

A mixed formula

A long time ago, I came across the following impressive formula in an old copy of the 1886 *Journal of mathématiques élémentaires Vuibert*,

$$\frac{w_a^2}{h_a}\cdot\sqrt{\frac{m_a^2-h_a^2}{w_a^2-h_a^2}} = 2R. \tag{10}$$

(Of course, a similar result holds if the subscript a is replaced by b or c.). This formula was given with a typo (r instead of R) and without proof!

A possible proof is as follows. With the help of the known formulas

$$w_a^2 = \frac{bc(a+b+c)(b+c-a)}{(b+c)^2}, \qquad h_a = \frac{2F}{a} = \frac{bc}{2R}, \qquad 4m_a^2 = 2b^2+2c^2-a^2$$

and

$$16F^2 = 2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4) = (a+b+c)(b+c-a)(c+a-b)(a+b-c),$$

we first obtain

$$\begin{aligned} w_a^4(4m_a^2-4h_a^2) &= \frac{b^2c^2}{(b+c)^4}\,(a+b+c)^2(b+c-a)^2\left(2b^2+2c^2-a^2-\frac{16F^2}{a^2}\right) \\ &= \frac{b^2c^2(b-c)^2(a+b+c)^2(b+c-a)^2}{a^2(b+c)^2} \end{aligned}$$

and, second,

$$\begin{aligned} &16R^2h_a^2(w_a^2-h_a^2) \\ &= 4b^2c^2\left(\frac{bc(a+b+c)(b+c-a)}{(b+c)^2}-\frac{16F^2}{4a^2}\right) \\ &= \frac{b^2c^2(a+b+c)(b+c-a)}{a^2(b+c)^2}\,(4a^2bc-(b+c)^2(c+a-b)(a+b-c)). \end{aligned}$$

Then (10) follows from $4a^2bc-(b+c)^2(c+a-b)(a+b-c) = (b-c)^2(a+b+c)(b+c-a)$ (as it is readily checked). (Variants of proofs can be found in **[3]**.)

A new youth was recently granted to this relation through several problems composed by Panagiote Ligouras. A typical example is problem **1847** posed in *Mathematics Magazine* in June 2010. Here is the slightly arranged statement :

Prove that in a scalene triangle the following inequality holds

$$\frac{w_a^4(m_a^2-h_a^2)}{h_a^3r_a(w_a^2-h_a^2)}+\frac{w_b^4(m_b^2-h_b^2)}{h_b^3r_b(w_b^2-h_b^2)}+\frac{w_c^4(m_c^2-h_c^2)}{h_c^3r_c(w_c^2-h_c^2)}>\frac{16}{3}.$$

Formula (10) allows a quick proof by immediately transforming the required inequality into

$$\frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c}>\frac{4r(a+b+c)}{3R^2}. \tag{11}$$

From exercise **1** below, we deduce

$$\frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c}=\frac{4(4R+r)}{a+b+c}$$

so that (11) is equivalent to $3R^2(4R+r)>r(a+b+c)^2$. The proof is easily completed by recalling that $R>2r$ and $a+b+c<3\sqrt{3}R$.

Exercises

1. Prove the formulas

$$\left(\frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c}\right)\left(\frac{a+b+c}{r_a+r_b+r_c}\right)=4$$

and

$$\frac{1}{rr_br_c}+\frac{1}{rr_cr_a}+\frac{1}{rr_ar_b}=\frac{8}{h_ah_bh_c}+\frac{1}{r_ar_br_c}.$$

2. (from *College Math. Journal* Problem 937) Prove that in a scalene triangle, the following inequality holds

$$\frac{w_a^4(m_a^2-h_a^2)}{h_a^2(w_a-h_a)\sqrt{w_a\cdot h_a}}+\frac{w_b^4(m_b^2-h_b^2)}{h_b^2(w_b-h_b)\sqrt{w_b\cdot h_b}}+\frac{w_c^4(m_c^2-h_c^2)}{h_c^2(w_c-h_c)\sqrt{w_c\cdot h_c}}\geq 24R^2.$$

References

[1] T. Lalesco, *La géométrie du triangle*, J. Gabay, 2003, p. 101-120

[2] O. Bottema *et al.*, *Geometric Inequalities*, Wolters-Noordhoff, 1968, p. 9-10

[3] Solution to J136, *Mathematical Reflections*, **5**, (2009), p. 10

Quadratic Allemands

Ted Barbeau

1 Sequences with complimentary properties

The sequence of natural numbers, $x_n = n$, has two complementary familiar properties :

$$x_{n+1} + x_{n-1} = 2x_n, \qquad \text{and} \qquad x_{n-1}x_{n+1} = x_n^2 - 1.$$

These two facts can be wrapped up in the fact that, for each n, x_{n-1} and x_{n+1} are the solutions of the quadratic equation

$$h(x, x_n) = 0,$$

where

$$h(x, y) = x^2 - 2xy + (y^2 - 1).$$

In particular, note that this bivariate polynomial is quadratic and *symmetric*. One consequence of this is that, for each n, $h(x_{n-1}, x_n) = h(x_n, x_{n+1})$ so that all the points (x_n, x_{n+1}) lie on the curve $h(x, y) = 0$ in the plane.

The sequence of natural numbers is not the only sequence that exhibits these complementary properties involving the sum and product of the terms adjacent to a given term. As an exercise, discover analogous equations for each of the sequences

$$\{0, 1, 3, 8, 21, 55, 144, \ldots\}$$

and

$$\{1, 2, 5, 13, 34, 89, 233, \ldots\}$$

formed by taking alternate terms of the Fibonacci sequence. For each of them, determine a symmetric quadratic polynomial $h(x, y)$ such that x_{n-1} and x_{n+1} are the solutions of the polynomial equation $h(x, x_n) = 0$. Identify the curves that contain the two sets of points

$$(0, 1), (1, 3), (3, 8), (8, 21), (21, 55), (55, 144), \ldots$$

and

$$(1, 2), (2, 5), (5, 13), (13, 34), (34, 89), (89, 233), \ldots.$$

2 Quadratic allemands

We are led to generalize the situation as follows. Start with the general symmetric quadratic polynomial

$$h(x, y) = \alpha(x^2 + y^2) + \beta xy + \gamma(x + y) + \delta.$$

Since our interest will be in the solutions of the equation $h(x,y) = 0$, we will assume that $\alpha = 1$. We define a *quadratic allemand* as a bilateral sequence

$$\{x_n : n = 0, \pm 1, \pm 2, \pm 3, \ldots\}$$

for which a *seed* x_0 is given. x_{-1} and x_1 are defined to be the two solutions of the quadratic equation $h(x, x_0) = 0$.

We can define the remaining terms recursively going in both directions from x_0. Suppose that we have determined $x_0, x_1, \ldots, x_m$ for $m > 0$, so that

$$0 = h(x_m, x_{m-1}) = h(x_{m-1}, x_m).$$

Then x_{m-1} is one solution of $h(x, x_m) = 0$ and we define x_{m+1} to be the second solution of this equation. A similar definition can be used for negative indices.

Write

$$h(x,y) = x^2 + y^2 + \beta xy + \gamma(x+y) + \delta = x^2 + (\beta y + \gamma)x + (y^2 + \gamma y + \delta).$$

Any allemand corresponding to this function must satisfy both of the recursions

$$x_{n+1} + x_{n-1} = -(\beta x_n + \gamma) \tag{1}$$

$$x_{n+1}x_{n-1} = x_n^2 + \gamma x_n + \delta \tag{2}$$

However, it turns out that any sequence that satisfies either of the recursions (1) and (2) are allemands. If (1) is satisfied, then it is straightforward to show, for each n, that

$$x_{n+1}x_{n-1} - x_n^2 - \gamma x_n = x_n x_{n-2} - x_{n-1}^2 - \gamma x_{n-1}$$

so that $x_{n+1}x_{n-1} - x_n^2 - \gamma x_{n-1}$ is an invariant for any recursion satisfying (1) alone. If we let δ be the value of this invariant, then we have (2) holding as well, so that any sequence (1) turns out to be an allemand.

With a little more trouble, it can be shown that, if (2) holds, then

$$x_{n+1} + x_{n-1} + \gamma = \frac{x_n}{x_{n-1}}(x_n + x_{n-2} + \gamma)$$

so that $x_n^{-1}(x_{n+1} + x_{n-1} + \gamma)$ is an invariant $-\beta$ and

$$x_{n+1} + \beta x_n + x_{n-1} + \gamma = 0.$$

Thus, any sequence defined by (2) alone is in fact an allemand.

We note the relations :

$$h(y, -(\beta y + x + \gamma)) = h(x,y)$$

and

$$h\left(y, \frac{y^2 + \gamma y + \delta}{x}\right) = \frac{y^2 + \gamma y + \delta}{x^2} h(x,y)$$

with the result that $h(x,y)$ is an invariant for two consecutive terms of sequence (1) and $h(x,y)/xy$ is an invariant for two consecutive terms of (2).

3 Special cases

There are a number of special cases worth investigating. In each case, determine the recursions (1) and (2), when there is an allemand consisting completely of real terms, when the sequence is periodic, and the curve that contains all points (x_n, x_{n+1}).

(1) $\beta = -2$.

(2) $\beta = 0$.

(3) $\beta = 2$. Consider the cases that $\gamma^2 - \delta$ is positive, negative and zero.

(4) $(\beta, \gamma, \delta) = (2, -3, 2)$.

(5) $\gamma = \delta = 0$.

(6) $\gamma = 0, \delta = -1$.

4 Application

One application of this theory is determining when a sequence that satisfies a recursion relation (2) has all integer entries. Consider for example the sequence defined by

$$x_0 = 1, \qquad x_1 = -1, \qquad \text{and} \qquad x_{n+1} = (x_n^2 - x_n + 1)/x_{n-1}$$

for $n \geq 1$. The first few terms of this sequence are

$$1, -1, 3, -7, 19, -49, 129, -337, 883, -2311.$$

Show that this is the positive part of a quadratic allemand with seed 1 and thus prove that each of its entries is an integer.

5 Cubic and quartic allemands

We can also talk about cubic and quartic allemands, where the function $h(x, y)$ is symmetric and respectively cubic and quartic while being quadratic in each of its variables. In the cubic case, we have

$$\begin{aligned} h(x, y) &= x^2y + xy^2 + \alpha(x^2 + y^2) + \beta xy + \gamma(x + y) + \delta \\ &= (y + \alpha)x^2 + (y^2 + \beta y + \gamma)x + (\alpha y^2 + \gamma y + \delta). \end{aligned}$$

and, in the quartic case,

$$\begin{aligned} h(x, y) &= x^2y^2 + \alpha xy(x + y) + \beta(x^2 + y^2) + \gamma xy + \delta(x + y) + \epsilon \\ &= (y^2 + \alpha y + \beta)x^2 + (\alpha y^2 + \gamma y + \delta)x + (\beta y^2 + \delta y + \epsilon). \end{aligned}$$

As for quadratic allemands, the allemands produced by these functions have a related pair of recursion relations corresponding to (1) and (2). But this is a story for another day.

6 Appendix

We provide some brief notes on the questions of the foregoing sections. The sequence $\{0, 1, 3, 8, 21, \ldots\}$ corresponds to the quadratic $h(x, y) = x^2 - 3xy + y^2 - 1$ and the sequence $\{1, 2, 5, 13, 34, \ldots\}$ corresponds to $h(x, y) = x^2 - 3xy + y^2 + 1$.

Here are the computations in full relating (1) and (2) in Section 2.

$$\begin{aligned}
x_{n+1}x_{n-1} - x_n^2 - \gamma x_n &= -x_{n-1}(\beta x_n + \gamma + x_{n-1}) - x_n^2 - \gamma x_n \\
&= -x_n(\beta x_{n-1} + x_n + \gamma) - \gamma x_{n-1} - x_{n-1}^2 \\
&= x_n x_{n-2} - x_{n-1}^2 - \gamma x_{n-1};
\end{aligned}$$

$$\begin{aligned}
x_{n+1} + x_{n-1} + \gamma &= \frac{x_n^2 + \gamma x_n + \delta}{x_{n-1}} + x_{n-1} + \gamma \\
&= \frac{1}{x_{n-1}}(x_n^2 + \gamma x_n + \delta + x_{n-1}^2 + \gamma x_{n-1}) \\
&= \frac{x_n}{x_{n-1}}\left(x_n + \gamma + \frac{x_{n-1}^2 + \gamma x_{n-1} + \delta}{x_n}\right) \\
&= \frac{x_n}{x_{n-1}}(x_n + x_{n-2} + \gamma).
\end{aligned}$$

For some special cases in Section 3, we have the following :

(1) The allemand is an arithmetic progression with common difference d given by $d^2 = -\delta$.

(2) The allemand has period 4 and the points (x_n, x_{n+1}) lie on a circle with centre $(-\frac{1}{2}, -\frac{1}{2})$ and radius $\sqrt{\frac{1}{2}\gamma^2 - \delta}$.

(5) This is a geometric progression with common ratio r satisfying $r^2 + \beta r + 1$.

In Section 4, the sequence satisfies the recursion

$$x_{n+1} + x_{n-1} = -3x_n + 1$$

and the allemand is produced by

$$h(x, y) = x^2 + y^2 + 3xy - (x + y) + 1.$$

PROBLEMS

Readers are invited to submit solutions, comments and generalizations to any problem in this section. Moreover, readers are encouraged to submit problem proposals. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **May 1, 2018**.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

An asterisk ($\star$) *after a number indicates that a problem was proposed without a solution.*

4281$\star$. *Proposed by Šefket Arslanagić.*

Prove or disprove the following inequalities :

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \geq \sqrt{3(a^2+b^2+c^2)}, \quad (a,b,c>0), \tag{1}$$

$$\frac{a_1^2}{a_2}+\frac{a_2^2}{a_3}+\cdots+\frac{a_n^2}{a_1} \geq \sqrt{n(a_1^2+a_2^2+\cdots+a_n^2)}, \quad (a_i>0, n\geq 3). \tag{2}$$

4282. *Proposed by Michel Bataille.*

Find $\lim\limits_{n\to\infty} u_n$ where the sequence $(u_n)_{n\geq 0}$ is defined by $u_0=1$ and the recursion

$$u_{n+1}=\frac{1}{2}\left(u_n+\sqrt{u_n^2+\frac{u_n}{4^n}}\right)$$

for every nonnegative integer n.

4283. *Proposed by Margarita Maksakova.*

We are given a convex polygon, whose vertices are coloured with three colours so that adjacent vertices get different colours. If all three colours are used in the colouring, prove that you can divide this polygon into triangles using non-intersecting diagonals in such a way that all the resulting triangles have vertices of all three different colours.

4284. *Proposed by Daniel Sitaru.*

Prove that if a, b and c are real numbers greater than 3 and

$$\log_a 2+\log_b 2+\log_c 2=\log_a \frac{1}{b}+\log_b \frac{1}{c}+\log_c \frac{1}{a},$$

then

$$\log_{a-1}(a^2+b^2)+\log_{b-1}(b^2+c^2)+\log_{c-1}(c^2+a^2)>3.$$

4285. *Proposed by Shafiqur Rahman and Leonard Giugiuc.*

Find the following limit :

$$\lim_{n\to\infty}\left(\sqrt[n]{n!}-\frac{n}{e}-\frac{\ln n}{2e}\right).$$

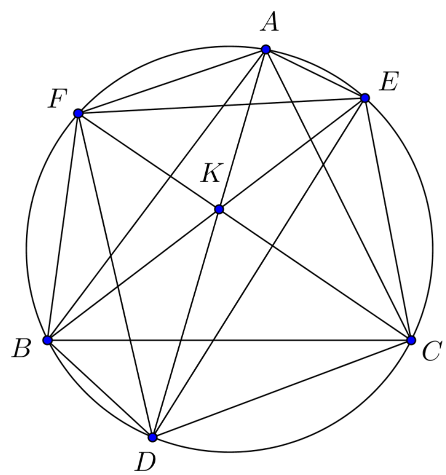
4286. *Proposed by Marian Cucoaneş and Marius Drăgan.*

Let x, y and z be positive real numbers such that $xyz = 1$. Prove that

$$(x^7+y^7+z^7)^2 \geq 3(x^9+y^9+z^9).$$

4287. *Proposed by Van Khea and Leonard Giugiuc.*

Let ABC be a triangle inscribed in a circle O and let K be a point inside ABC. Suppose that AK, BK and CK cut the circle O in points D, E and F, respectively.



Prove that

$$\frac{BD\cdot CE}{BC\cdot DE}+\frac{CE\cdot AF}{CA\cdot EF}+\frac{AF\cdot BD}{AB\cdot FD}=1.$$

4288. *Proposed by Hung Nguyen Viet.*

A sequence $\{a_n\}$ is defined as follows :

$$a_1=1,\quad a_2=2,\quad a_{n+2}=\frac{a_{n+1}^2+1}{a_n}$$

for every $n \geq 1$. Let $b_n = a_n a_{n+1}$, $n = 1, 2, \ldots$ Prove that for every positive integer n, the number $5b_n^2 - 6b_n + 1$ is a perfect square.

4289. *Proposed by George Apostolopoulos.*

Prove that in any triangle ABC, we have

$$\sqrt[3]{\frac{r_a}{h_a}}+\sqrt[3]{\frac{r_b}{h_b}}+\sqrt[3]{\frac{r_c}{h_c}}\leq\frac{3R}{2r},$$

where r_a, r_b, r_c are lengths of the exradii, h_a, h_b, h_c are the lengths of the altitudes and R and r are circumradius and inradius, respectively, of the triangle ABC.

4290. *Proposed by Dao Thanh Oai and Leonard Giugiuc.*

Let $A_1A_2A_3A_4A_5$ be a cyclic convex pentagon and let $B_1B_2B_3B_4B_5$ be a regular pentagon, both inscribed in the same circle. Prove that

$$\sum_{1\leq i<j\leq 5}A_iA_j\leq\sum_{1\leq i<j\leq 5}B_iB_j.$$

. . . .

4281★. *Proposé par Šefket Arslanagić.*

Démontrer ou infirmer les inégalités suivantes :

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\sqrt{3(a^2+b^2+c^2)},\quad (a,b,c>0), \tag{1}$$

$$\frac{a_1^2}{a_2}+\frac{a_2^2}{a_3}+\cdots+\frac{a_n^2}{a_1}\geq\sqrt{n(a_1^2+a_2^2+\cdots+a_n^2)},\quad (a_i>0, n\geq 3). \tag{2}$$

4282. *Proposé par Michel Bataille.*

Déterminer $\lim\limits_{n\to\infty}u_n$, la suite $(u_n)_{n\geq 0}$ étant définie par $u_0=1$ et

$$u_{n+1}=\frac{1}{2}\left(u_n+\sqrt{u_n^2+\frac{u_n}{4^n}}\right),$$

pour tous entiers non négatifs n.

4283. *Proposé par Margarita Maksakova.*

On considère un polygone dont chaque sommet est colorié d'une de trois couleurs, de manière que les sommets adjacents soient de couleurs différentes. Si les trois couleurs ont été utilisées, démontrer qu'il est possible de diviser le polygone en triangles à l'aide de diagonales qui ne se coupent pas, de manière que chaque triangle ait des sommets de trois couleurs différentes.

4284. *Proposé par Daniel Sitaru.*

Soit a, b et c des réels supérieurs à 3 tels que

$$\log_a 2 + \log_b 2 + \log_c 2 = \log_a \frac{1}{b} + \log_b \frac{1}{c} + \log_c \frac{1}{a}.$$

Démontrer que

$$\log_{a-1}(a^2 + b^2) + \log_{b-1}(b^2 + c^2) + \log_{c-1}(c^2 + a^2) > 3.$$

4285. *Proposé par Shafiqur Rahman et Leonard Giugiuc.*

Déterminer la limite suivante :

$$\lim_{n\to\infty} \left(\sqrt[n]{n!} - \frac{n}{e} - \frac{\ln n}{2e} \right).$$

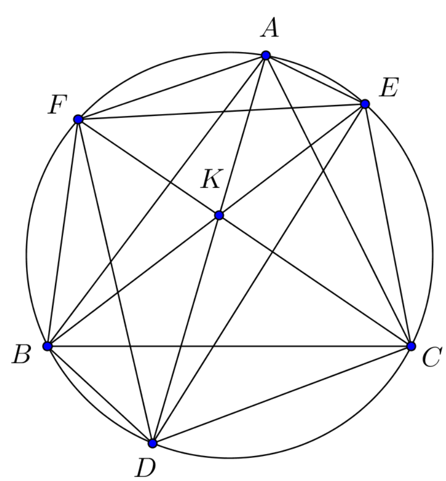
4286. *Proposé par Marian Cucoaneş et Marius Drăgan.*

Soit x, y et z des réels positifs tels que $xyz = 1$. Démontrer que

$$(x^7 + y^7 + z^7)^2 \geq 3(x^9 + y^9 + z^9).$$

4287. *Proposé par Van Khea et Leonard Giugiuc.*

Soit ABC un triangle inscrit dans un cercle et K un point à l'intérieur de ABC. Les demi-droites AK, BK, CK coupent le cercle aux points respectifs D, E et F.



Démontrer que

$$\frac{BD \cdot CE}{BC \cdot DE} + \frac{CE \cdot AF}{CA \cdot EF} + \frac{AF \cdot BD}{AB \cdot FD} = 1.$$

4288. *Proposé par Hung Nguyen Viet.*

Une suite $\{a_n\}$ est définie comme suit :

$$a_1 = 1, \quad a_2 = 2, \quad a_{n+2} = \frac{a_{n+1}^2 + 1}{a_n}$$

pour tous n ($n \geq 1$). Soit $b_n = a_n a_{n+1}$, pour $n = 1, 2, \ldots$ Démontrer que pour tout entier strictement positif n, le nombre $5b_n^2 - 6b_n + 1$ est un carré parfait.

4289. *Proposé par George Apostolopoulos.*

Démontrer que pour tout triangle ABC, on a

$$\sqrt[3]{\frac{r_a}{h_a}} + \sqrt[3]{\frac{r_b}{h_b}} + \sqrt[3]{\frac{r_c}{h_c}} \leq \frac{3R}{2r},$$

h_a, h_b et h_c étant les hauteurs du triangle, R étant le rayon du cercle circonscrit au triangle, r étant le rayon du cercle inscrit dans le triangle et r_a, r_b et r_c étant les rayons des cercles exinscrits.

4290. *Proposé par Dao Thanh Oai et Leonard Giugiuc.*

Soit $A_1A_2A_3A_4A_5$ un pentagone convexe inscriptible et $B_1B_2B_3B_4B_5$ un pentagone régulier, tous deux inscrits dans un même cercle. Démontrer que

$$\sum_{1\leq i<j\leq 5} A_iA_j \leq \sum_{1\leq i<j\leq 5} B_iB_j.$$

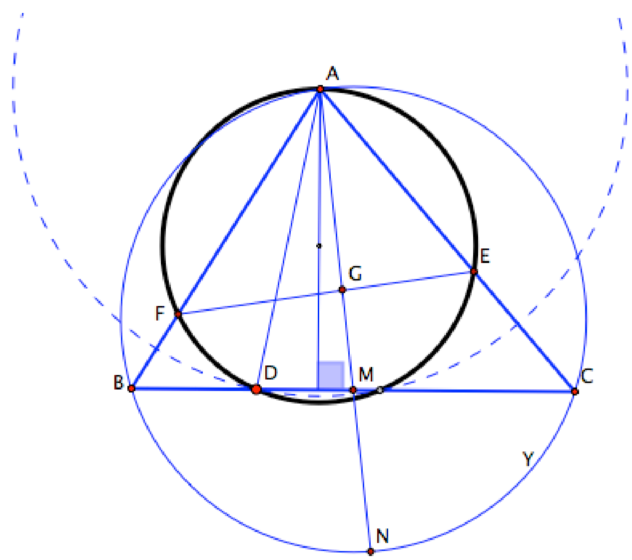
SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2016: 42(9), p. 400–404.

4181. *Proposed by Marius Stănean.*

Let $D \in BC$ be the foot of the A-symmedian of triangle ABC with centroid G (where the A-symmedian is the reflection of the median at A in the bisector of angle A). The circle passing through A, D and tangent to the line parallel to BC passing through A intersects sides AB and AC at F and E, respectively. If $3AD^2 = AB^2 + AC^2$, prove that G lies on EF.



We received 6 solutions, all of which were correct; we feature the similar solutions by Peter Woo and the proposer, combined and expanded upon by the editor.

It turns out that properties of the symmedian are irrelevant to the problem — we shall prove a stronger result, namely:

Given a triangle ABC with centroid G, let D be a point of the line BC, and define E and F to be the points where the circle through A and D that is tangent to the line parallel to BC passing through A intersects sides AC and AB. Then G lies on EF if and only if the length of AD satisfies $3AD^2 = AB^2 + AC^2$.

Denote by $\mathcal{J}$ inversion in the circle with center A and radius AD. Because the given circle AEF is defined to be symmetric about the line through A that is perpendicular to BC, while $D \in BC$ is fixed by the inversion, $\mathcal{J}$ must interchange the line BC with the circle AEF. Specifically, C and E are interchanged by the inversion as are B and F. Consequently, $\mathcal{J}$ interchanges the line EF with the circumcircle of ΔABC, call it γ. Denoting by M the midpoint of BC and by N the point where the extension AM of the median intersects γ, we have, in particular, the image N' of N under $\mathcal{J}$ always lies on EF. Consequently, we are required to prove that N' coincides with the centroid G if and only if $3AD^2 = AB^2 + AC^2$. To that end we introduce the standard notation a, b, c for the lengths of sides BC, AC, AB, and $m = AM$ for the length of the median. As a consequence of Stewart's theorem we have

$$4m^2 = 2b^2 + 2c^2 - a^2.$$

Moreover, the chords of γ that intersect at M yield $AM \cdot MN = BM \cdot MC = \frac{a^2}{4}$, or

$$MN = \frac{a^2}{4m}.$$

Thus,

$$AN = m + MN = \frac{4m^2 + a^2}{4m} = \frac{b^2 + c^2}{2m}.$$

Because the circle of inversion has radius AD, the inverse N' of N satisfies $AN' = \frac{AD^2}{AN} = \frac{2mAD^2}{b^2+c^2}$. But the centroid of the given triangle is the point between A and N that satisfies $AG = \frac{2}{3}m$. Thus $N' = G$ if and only if $\frac{2}{3}m = \frac{2mAD^2}{b^2+c^2}$, which immediately reduces to the required equation.

Editor's comments. The other four solvers began with coordinates for the foot of the symmedian, and thereby failed to discover the more general result. Their computations suggest that there *might* exist a one-parameter family of triangles ABC with a given side BC for which D is the foot of the symmedian and $G \in EF$; this conjecture is supported (but, of course, not proved) by computer graphics. By analyzing isosceles triangles it is easily seen that there exist triangles for which $G \in EF$ even though AD is not the symmedian (which for isosceles triangles coincides with the perpendicular bisector of BC). Indeed, we have seen that so long as $\sqrt{\frac{b^2+c^2}{3}}$ exceeds the altitude, there will be two positions of D on BC for which $G \in EF$.

4182. *Proposed by Michel Bataille.*

Let F_m be the mth Fibonacci number (defined by $F_0 = 0, F_1 = 1, F_{m+2} = F_{m+1} + F_m$ for all integers $m \geq 0$) and let n be a positive integer. For $k = 1, 2, \ldots, n$, let

$$U_k = \frac{k}{F_{n+1-k}F_{n+3-k}} + (-1)^{k+1}\frac{2F_k}{F_{k+2}}.$$

Prove that $|U_1 + U_2 + \cdots + U_n - n|$ is the quotient of two Fibonacci numbers.

We received 6 solutions, all of which were correct. We present the solution by the proposer.

For positive integer n, let $S_n = U_1 + U_2 + \cdots + U_n$. Since we will use induction on n and U_k is dependent on n as well as k, we express S_n in a form that makes its dependence on n more amenable to managing the induction step. Note that

$$S_1 = 1/F_1F_3 + 2F_1/F_3 = 3/2 = 1 + F_1/F_3,$$

and, for $n \geq 2$,

$$\begin{aligned}
S_n &= \sum_{k=1}^{n} \frac{k}{F_{n+1-k}F_{n+3-k}} + \sum_{k=1}^{n} (-1)^{k+1} \frac{2F_k}{F_{k+2}} \\
&= \sum_{k=1}^{n} \frac{n+1-k}{F_kF_{k+2}} + \sum_{k=1}^{n} (-1)^{k+1} \frac{2F_k}{F_{k+2}} \\
&= \sum_{k=1}^{n} \frac{1}{F_kF_{k+2}} + \sum_{k=1}^{n-1} \frac{n-k}{F_kF_{k+2}} + \sum_{k=1}^{n} (-1)^{k+1} \frac{2F_k}{F_{k+2}} \\
&= S_{n-1} + \sum_{k=1}^{n} \frac{1}{F_kF_{k+2}} + (-1)^{n+1} \frac{2F_n}{F_{n+2}}.
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{F_kF_{k+2}} = \sum_{k=1}^{n} \frac{F_{k+2} - F_k}{F_kF_{k+1}F_{k+2}} &= \sum_{k=1}^{n} \left(\frac{1}{F_kF_{k+1}} - \frac{1}{F_{k+1}F_{k+2}} \right) \\
&= 1 - \frac{1}{F_{n+1}F_{n+2}},
\end{aligned}$$

$$S_n = S_{n-1} + 1 - \frac{1}{F_{n+1}F_{n+2}} + (-1)^{n+1} \frac{2F_n}{F_{n+2}}.$$

We prove that

$$S_n - n = (-1)^{n+1} \frac{F_n}{F_{n+2}}$$

for $n \geq 1$. This holds for $n = 1$. Suppose it holds for $n = m - 1$. Then

$$\begin{aligned}
S_m - m &= S_{m-1} - (m-1) - \frac{1}{F_{m+1}F_{m+2}} + (-1)^{m+1} \frac{2F_m}{F_{m+2}} \\
&= (-1)^m \frac{F_{m-1}}{F_{m+1}} - \frac{1}{F_{m+1}F_{m+2}} + (-1)^{m+1} \frac{2F_m}{F_{m+2}} \\
&= \frac{(-1)^{m+1}[F_mF_{m+1} + (F_mF_{m+1} - F_{m-1}F_{m+2})] - 1}{F_{m+1}F_{m+2}} \\
&= \frac{(-1)^{m+1}[F_mF_{m+1} + (-1)^{m+1}] - 1}{F_{m+1}F_{m+2}} = (-1)^{m+1} \frac{F_m}{F_{m+2}},
\end{aligned}$$

as desired.

Therefore, $|S_n - n|$ is equal to F_n/F_{n+2}, the quotient of two Fibonacci numbers.

Editor's comments. Joel Schlosberg gave a direct argument, by obtaining a double sum:

$$\sum_{j=1}^{n} \frac{n+1-j}{F_j F_{j+2}} = \sum_{k=1}^{n} \sum_{j=1}^{k} \frac{1}{F_j F_{j+2}} = \sum_{k=1}^{n} \left(\frac{1}{F_1 F_2} - \frac{1}{F_{k+1} F_{k+2}} \right)$$
$$= n - \sum_{k=1}^{n} \frac{1}{F_{k+1} F_{k+2}}.$$

Then $U_1 + \cdots + U_n - n$ can be evaluated similarly to the foregoing solution.

4183. *Proposed by Lorian Saceanu and Leonard Giugiuc.*

Let ABC be a non obtuse triangle with orthocenter H and circumradius R. Prove that
$$(3\sqrt{3} - 4) \cdot AH \cdot BH \cdot CH \geq abc - 4R^3$$
and determine when the equality holds.

We received 3 correct solutions and one incorrect solution. We present the solution by Leonard Giugiuc, submitted independently.

This proof is based on Blundon's theorem, which states that in any triangle,
$$s \leq 2R + (3\sqrt{3} - 4)r,$$
where the notations are as customary. Equality holds if and only if the triangle is equilateral. For a reference, see
`https://www.emis.de/journals/JIPAM/images/220_08_JIPAM/220_08.pdf`.

Since triangle ABC is non obtuse, we have
$$AH = 2R\cos A, \quad BH = 2R\cos B, \quad CH = 2R\cos C.$$

Also, by the law of sines,
$$a = 2R\sin A, \quad b = 2R\sin B, \quad c = 2R\sin C.$$

The required inequality is thus equivalent to
$$2(\sqrt{3} - 4\cos A\cos B\cos C \geq 2\sin A\sin B\sin C - 1.$$

Case 1. Triangle ABC is right-angled. Without loss of generality, $A = \frac{\pi}{2}$. The required inequality is equivalent to
$$1 \geq 2\sin A\sin B\sin C,$$
implying that $1 \geq \sin 2B$, which is true.

Case 2. Triangle ABC is acute-angled. Let DEF be the orthic triangle of ABC; that is, D, E, and F are the endpoints of the altitudes of ABC. Standard facts for orthic triangles give

$$R_{DEF} = \frac{R}{2}, \quad s_{DEF} = 2R\sin A\sin B\sin C, \quad \text{and} \quad r_{DEF} = 2R\cos A\cos B\cos C.$$

By Blundon's Theorem,

$$s_{DEF} \leq 2R_{DEF} + (3\sqrt{3}-4)r_{DEF},$$

or equivalently

$$2(3\sqrt{3}-4)\cos A\cos B\cos C \geq 2\sin A\sin B\sin C - 1.$$

Equality holds if and only if DEF is equilateral, i.e., if and only if ABC is equilateral.

4184. *Proposed by Mihaela Berindeanu.*

Evaluate the following integral

$$\int_{32}^{63} \frac{\ln 2016x}{x^2+2016}\mathrm{d}x.$$

We received 12 submissions of which 8 were correct and complete. We present 2 solutions.

Solution 1, by Michel Bataille.

Let I be the given integral. We show that

$$I = \frac{3\ln(2016)}{2\sqrt{2016}}\left(2\arctan\left(\sqrt{\frac{63}{32}}\right) - \frac{\pi}{2}\right).$$

We shall use the following lemma: if $\alpha > 0$, then

$$\int_{\alpha}^{1/\alpha} \frac{\ln x}{x^2+1}\,dx = 0.$$

Proof. The change of variables $x = \frac{1}{u}$, $dx = \frac{-du}{u^2}$ gives

$$\int_{\alpha}^{1/\alpha} \frac{\ln x}{x^2+1}\,dx = \int_{1/\alpha}^{\alpha} \frac{-\ln u}{1+\frac{1}{u^2}} \cdot \frac{-du}{u^2} = -\int_{\alpha}^{1/\alpha} \frac{\ln u}{u^2+1}\,du$$

and the result follows. $\square$

Now, $I = I_1 + I_2$ with

$$I_1 = \ln(2016)\int_{32}^{63} \frac{dx}{x^2+2016} = \frac{\ln(2016)}{\sqrt{2016}}\left(\arctan\left(\sqrt{\frac{63}{32}}\right) - \arctan\left(\sqrt{\frac{32}{63}}\right)\right)$$

and

$$I_2 = \int_{32}^{63} \frac{\ln x}{x^2+2016}\,dx = \frac{\ln(2016)}{2\sqrt{2016}} \int_{\sqrt{\frac{32}{63}}}^{\sqrt{\frac{63}{32}}} \frac{dy}{1+y^2} + \frac{1}{\sqrt{2016}} \int_{\sqrt{\frac{32}{63}}}^{\sqrt{\frac{63}{32}}} \frac{\ln y}{1+y^2}\,dy$$

with the change of variables $x = y\sqrt{2016}$.

Using the lemma,

$$\int_{\sqrt{\frac{32}{63}}}^{\sqrt{\frac{63}{32}}} \frac{\ln y}{1+y^2}\,dy = 0,$$

hence we have

$$I_2 = \frac{\ln(2016)}{2\sqrt{2016}} \left(\arctan\left(\sqrt{\frac{63}{32}}\right) - \arctan\left(\sqrt{\frac{32}{63}}\right)\right).$$

In conclusion,

$$\begin{aligned} I &= \frac{3\ln(2016)}{2\sqrt{2016}} \left(\arctan\left(\sqrt{\frac{63}{32}}\right) - \arctan\left(\sqrt{\frac{32}{63}}\right)\right) \\ &= \frac{3\ln(2016)}{2\sqrt{2016}} \left(2\arctan\left(\sqrt{\frac{63}{32}}\right) - \frac{\pi}{2}\right), \end{aligned}$$

where the latter is true because $\arctan a + \arctan(1/a) = \frac{\pi}{2}$ when $a > 0$.

Solution 2, by Leonard Giugiuc.

Let a and b be real numbers with $b > a > 0$. We will evaluate

$$I = \int_a^b \frac{\ln abx}{x^2+ab}\,dx.$$

Making the substitution $\dfrac{ab}{x} \to x$, we get

$$I = \int_a^b \frac{\frac{ab}{x^2}\ln\frac{(ab)^2}{x}}{\frac{ab}{x^2}(x^2+ab)}\,dx = \int_a^b \frac{\ln\frac{(ab)^2}{x}}{x^2+ab}\,dx.$$

So,

$$\begin{aligned} 2I &= \int_a^b \frac{\ln abx}{x^2+ab}\,dx + \int_a^b \frac{\ln\frac{(ab)^2}{x}}{x^2+ab}\,dx \\ &= 3\ln ab \int_a^b \frac{1}{x^2+ab}\,dx \\ &= \frac{3\ln ab}{\sqrt{ab}} \left(\arctan\sqrt{\frac{b}{a}} - \arctan\sqrt{\frac{a}{b}}\right) \\ &= \frac{3\ln ab}{\sqrt{ab}} \arctan\frac{b-a}{2\sqrt{ab}}. \end{aligned}$$

Setting $= 32$ and $b = 63$, we obtain

$$\int_{32}^{63} \frac{\ln 2016x}{x^2 + 2016}\, dx = \frac{3\ln 2016}{2\sqrt{2016}} \arctan \frac{31}{2\sqrt{2016}}.$$

4185. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Prove that for any positive real numbers a, b, c and k, we have

$$\left[\frac{a^{k-1}(a^2+bc)}{(b+c)^{k+1}}\right]^{\frac{1}{k}} + \left[\frac{b^{k-1}(b^2+ca)}{(c+a)^{k+1}}\right]^{\frac{1}{k}} + \left[\frac{c^{k-1}(c^2+ab)}{(a+b)^{k+1}}\right]^{\frac{1}{k}} \geq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}.$$

We received 3 correct solutions. We present the solution by Sĕfket Arslanagić.

Without loss of generality we will take $a \geq b \geq c$; it follows then that

$$\frac{a}{b+c} \geq \frac{b}{c+a} \geq \frac{c}{a+b}$$

and

$$\frac{a^2+bc}{a(b+c)} \geq \frac{b^2+ca}{b(c+a)} \geq \frac{c^2+ab}{c(a+b)},$$

and we have by Chebyshev's inequality,

$$\sum_{\text{cyc}} \frac{a}{b+c}\left(\frac{a^2+bc}{a(b+c)}\right)^{1/k} \geq \frac{1}{3}\left(\sum_{\text{cyc}} \frac{a}{b+c}\right)\left(\sum_{\text{cyc}}\left(\frac{a^2+bc}{a(b+c)}\right)^{1/k}\right). \qquad (1)$$

We now make use of the inequality

$$(a^2+bc)(b^2+ca)(c^2+ab) \geq abc(a+b)(b+c)(c+a),$$

which holds since, upon expansion, it becomes the sum of the Muirhead inequalities $(3,3,0) \succcurlyeq (3,2,1)$ and $(4,1,1) \succcurlyeq (3,2,1)$. Thus, by the AM-GM inequality,

$$\sum_{\text{cyc}}\left(\frac{a^2+bc}{a(b+c)}\right)^{1/k} \geq 3 \cdot \sqrt[3]{\left[\frac{(a^2+bc)(b^2+ca)(c^2+ab)}{a(b+c)(b(c+a)(c(a+b)}\right]^{1/k}} \geq 3. \qquad (2)$$

The claimed inequality now follows from (1) and (2). Equality holds if and only if $a = b = c$.

4186. *Proposed by Florin Stanescu.*

Let $f, g : [0,1] \to [0,\infty)$, $f(0) = g(0) = 0$ be two continuous functions such that f is convex and g is concave. If $h : [0,1] \to \mathbb{R}$ is an increasing function, show that

$$\int_0^1 g(x)h(x)dx \cdot \int_0^1 f(x)dx \leq \int_0^1 g(x)dx \cdot \int_0^1 h(x)f(x)dx.$$

We received 2 correct solutions and feature the one by Leonard Giugiuc.

If $\int_0^1 f(x)dx = 0$, then, since f is continuous and $f(x) \geq 0, \forall\, x \in [0,1]$, $f \equiv 0$, and hence the inequality is proved. The same conclusion holds if $\int_0^1 g(x)dx = 0$. So we assume that

$$I = \int_0^1 f(x)dx > 0 \text{ and } J = \int_0^1 g(x)dx > 0.$$

The required inequality is equivalent to

$$\int_0^1 h(x)\left(\frac{f(x)}{I} - \frac{g(x)}{J}\right) dx \geq 0.$$

Define the function $\phi : [0,1] \to \mathbb{R}$ as $\phi(x) = \frac{f(x)}{I} - \frac{g(x)}{J}$, $\forall\, x \in [0,1]$. Clearly, ϕ is continuous and convex. Moreover,

$$\int_0^1 \phi(x) = \frac{1}{I}\int_0^1 f(x) - \frac{1}{J}\int_0^1 g(x) = 1 - 1 = 0,$$

and $\phi(0) = 0$. Now, the required inequality is equivalent to $\int_0^1 h(x)\phi(x)dx \geq 0$.

If ϕ is increasing on $[0,1]$, then since $\phi(0) = 0$ and $\int_0^1 \phi(x) = 0$, we deduce $\phi \equiv 0$ and we are done. A similar conclusion holds if ϕ is decreasing. Otherwise, since ϕ is convex, we know $\exists\, t \in (0,1)$ such that ϕ is decreasing on $[0,t]$ and ϕ is increasing on $[t,1]$. Thus it follows that $\phi(1) \geq 0$ and that $\exists\, c \in (0,1)$ such that $\phi(x) \leq 0$ $\forall\, x \in [0,c]$ and $\phi(x) \geq 0$ $\forall\, x \in [c,1]$. Then $\forall\, x \in [0,c]$, we have $h(x) - h(c) \leq 0$ and $\phi(x) \leq 0$, so that $h(x)\phi(x) \geq h(c)\phi(x)$. Similarly, $\forall\, x \in [c,1]$, we have $h(x) - h(c) \geq 0$ and $\phi(x) \geq 0$, so that $h(x)\phi(x) \geq h(c)\phi(x)$.

In conclusion, $h(x)\phi(x) \geq h(c)\phi(x)$, $\forall\, x \in [0,1]$, and hence

$$\int_0^1 h(x)\phi(x) \geq \int_0^1 h(c)\phi(x) = 0.$$

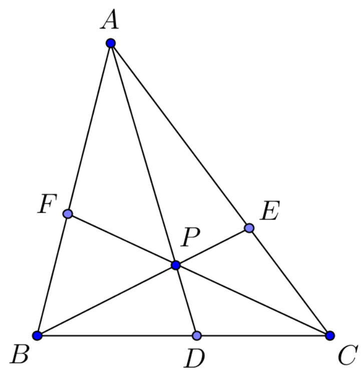
The proof is complete.

4187. *Proposed by Avi Sigler and Moshe Stupel.*

A point P inside triangle ABC divides the three cevians AD, BE, CF through P into segments whose harmonic means are

$$K_A = \frac{2AP \cdot PD}{AP + PD}, \quad K_B = \frac{2BP \cdot PE}{BP + PE}, \quad K_C = \frac{2CP \cdot PF}{CP + PF}.$$

Prove that these three harmonic means, each associated with a cevian, are proportional to the sines of the angles $\angle CPE, \angle APF, \angle EPA$ formed between the other two cevians.



We received nine solutions, all of which were correct, and feature the solution whose similar versions came independently from Steven Chow and Oliver Geupel.

Let square brackets represent areas. From the relation

$$\frac{PD}{AD} = \frac{[PBC]}{[ABC]} = \frac{BP \cdot CP \sin \angle CPE}{2[ABC]},$$

we obtain

$$\sin \angle CPE = \frac{2[ABC] \cdot PD}{AD \cdot BP \cdot CP}.$$

Hence,

$$\frac{K_A}{\sin \angle CPE} = \frac{AP \cdot BP \cdot CP}{[ABC]}.$$

Similarly,

$$\frac{K_B}{\sin \angle APF} = \frac{K_C}{\sin \angle EPA} = \frac{AP \cdot BP \cdot CP}{[ABC]}.$$

As a consequence, the three harmonic means of the segments on one cevian are proportional to the sines of the angles formed between the other two cevians.

4188. *Proposed by Daniel Sitaru.*

Let $0 < x < y < z < \frac{\pi}{2}$. Prove that

$$(x + y) \sin z + (x - z) \sin y < (y + z) \sin x.$$

There were eight correct solutions. Four of them gave essentially the solution below.

Because $\frac{\sin t}{t}$ is a decreasing function on $(0, \pi/2)$, we find that

$$x \sin y < y \sin x, \quad y \sin z < z \sin y, \quad \text{and} \quad x \sin z < z \sin x.$$

Therefore

$$\begin{aligned}(y + z) \sin x &- (x - z) \sin y - (x + y) \sin z \\ &= (y \sin x - x \sin y) + (z \sin y - y \sin z) + (z \sin x - x \sin z) > 0.\end{aligned}$$

4189. *Proposed by Mihaela Berindeanu.*

Prove that the equation

$$3y^2 = -2x^2 - 2z^2 + 5xy + 5yz - 4xz + 1$$

has infinitely many solutions in integers.

We received 15 solutions, most of which solved the given equation. We present the solution by Arkady Alt.

Rearrange and factor the given equation:

$$\begin{aligned}
&2x^2 + 4xz + 2z^2 + 3y^2 - 5xy - 5yz = 1 \Leftrightarrow \\
&2(x+z)^2 - 5y(x+z) + 3y^2 = 1 \Leftrightarrow \\
&(2(x+z) - 3y)(x+z-y) = 1.
\end{aligned}$$

Let $w = x + z$; if x, z are integers then so is w. To find the integer solutions to the equation $(2w - 3y)(w - y) = 1$ we consider two cases:

$$\begin{gathered}
2w - 3y = w - y = 1 \text{ OR} \\
2w - 3y = w - y = -1.
\end{gathered}$$

The first system of equations yields $w = 2$ and $y = 1$, and the second $w = -2$ and $y = -1$. From $x + y = w$, we get that the solutions to the original equation are $\{(t, 1, 2-t) : t \in \mathbb{Z}\} \cup \{(t, -1, -2-t) : t \in \mathbb{Z}\}$, showing that there are infinitely many solutions.

4190. *Proposed by Leonard Giugiuc.*

Let a, b, c, d and e be real numbers such that $a + b + c + d + e = 20$ and $a^2 + b^2 + c^2 + d^2 + e^2 = 100$. Prove that

$$625 \leq abcd + abce + abde + acde + bcde \leq 945.$$

Only the proposer supplied a solution.

Suppose, wolog, that $a \geq b \geq c \geq d \geq e$. We first verify that a, b, c, d, e must be all nonnegative. Since

$$\begin{aligned}
4(a^2 + b^2 + c^2 + d^2) - (a+b+c+d)^2 \\
&= 3(a^2 + b^2 + c^2 + d^2) - 2(ab + ac + ad + bc + bd + be) \\
&= (a-b)^2 + (a-c)^2 + (a-d)^2 \\
&\quad + (b-c)^2 + (b-d)^2 + (c-d)^2 \geq 0,
\end{aligned}$$

then

$$(a+b+c+d)^2 \leq 4(a^2 + b^2 + c^2 + d^2) \leq 400.$$

Thus $a+b+c+d \leq 20$ and $e \geq 0$. Equality occurs if and only if $(a,b,c,d,e) = (5,5,5,5,0)$.

The foregoing inequality can be rewritten $(20-e)^2 \leq 4(100-e^2)$, whence $0 \leq e \leq 8$. An analogous argument certifies that a,b,c,d also lie in the closed interval $[0,8]$.

Consider the quintic polynomial

$$p(x) = x^5 - 20x^4 + 150x^3 - 5qx^2 + 5rx - s,$$

whose roots are a,b,c,d,e, so that $5q = \sum abc$, $5r = \sum abcd$ and $s = abcde$. (We note that $r \neq 0$, for otherwise $d = e = 0$ and $300 = 3(a^2+b^2+c^2) \geq (a+b+c)^2 = 400$, a contradiction.)

By Rolle's theorem,

$$p'(x) = 5(x^4 - 16x^3 + 90x^2 - 2qx + r)$$

has four real roots in the open interval $(0,8)$, as does the function

$$f(x) = \frac{p'(x)}{5x} = x^3 - 16x^2 + 90x - 2q + \frac{r}{x}.$$

We have that $f'(x) = x^{-2}g(x)$, where $g(x) = 3x^4 - 32x^3 + 90x^2 - r$. By Rolle's theorem, $f'(x)$ and so $g(x)$ have three positive roots in $(0,8)$. Since

$$g'(x) = 12x(x-3)(x-5),$$

the function $g(x)$ is monotone on and has a root in each of the intervals $(0,3)$, $(3,5)$ and $(5,8)$. We have $0 > g(0) = -r$, $0 \leq g(3) = 189 - r$ and $0 \geq 125 - r$. Therefore $125 \leq r \leq 189$ and

$$625 \leq 5r = abcd + abce + abde + acde + bcde \leq 945.$$

Editor's comment. For each quintuple (a,b,c,d,e) satisfying the two conditions, there is an associate $(8-e, 8-d, 8-c, 8-b, 8-a)$ that also satisfies them. Two associate quintuples are $(5,5,5,5,0)$ and $(8,3,3,3,3)$. In the former case, $p(x) = x(x-5)^4$, $p'(x) = 5(x-1)(x-5)^3$, $g(x) = 3x^4 - 32x^3 + 90x^2 - 125 = (x-5)^2(x+1)(3x-5)$ and $\sum abcd = 625$. In the latter case, $p(x) = (x-8)(x-3)^4$, $p'(x) = 5(x-7)(x-3)^3$, $f(x) = x^3 - 16x^2 + 90x - 216 + (189/x)$, $g(x) = 3x^4 - 32x^3 + 90x^2 - 189 = (x-3)^2(3x^2 - 14x - 21)$, and $\sum abcd = 945$.

There is one other quintuple of integers $(7,5,4,3,1)$ satisfying the conditions which is its own associate. In this case $\sum abcd = 809$.

AUTHORS' INDEX

Solvers and proposers appearing in this issue

(Bold font indicates featured solution.)

Proposers

Solvers - individuals

Solvers - collaborations