Crux Mathematicorum

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Editorial Board

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Crux Mathematicorum

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Crux Mathematicorum with Mathematical Mayhem

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EDITORIAL

As I am writing this in issue 1 of Volume 43, I am thinking of Volume 44. And not just because I like planning ahead, but because ***Crux*** will be undergoing some serious changes. For many years now, ***Crux*** has been running a deficit and the budget-makers are running out of red ink. As a result, the Society is taking year 2017 to decide on how ***Crux*** can proceed in a sustainable way while also reaching more readers. Or whether it can.

Not surprisingly, the production of ***Crux*** requires administrative support. Furthermore, the non-trivial printing and distribution costs limit our reach and available budget. The Canadian Mathematical Society is currently fundraising to help support all of its educational activities. CMS is a non-profit, charitable organization, and ***Crux*** in particular requires its readership to show support for the publication in order for it to continue. If you think that ***Crux*** is something you would personally like to donate money to, please consider doing so. If you know of a potential donor whose interests align with the purpose of this publication, please let them know about us and let me know about them.

The future of ***Crux*** is most likely to be electronic only and that provides an exciting opportunity for the journal to grow and become available to more people in Canada and around the globe. While the future format and directions are still to be decided, I would like to have your input. Do you have ideas for what we can do? What would you want to see in ***Crux*** going forward?

My email is `crux-editors@cms.math.ca`. Feel free to drop me a line about any of the above or any not of the above.

We have almost an entire year to shape Volume 44. For now, back to Volume 43...

I don't like forecasting things. It is an ungrateful job. Just think about all the wrong weather forecasts that we receive: no matter how many times meteorologists get it right and no matter how much you understand that a behaviour of a live dynamical system cannot be predicted with full certainty, it is still maddening to see snow on what was supposed to be a sunny day. Yet, I can honestly say that Volume 43 of ***Crux*** will be an adventure that will not disappoint.

Ever since I started as ***Crux*** EIC, I have been asked to foresee things: when will we be out of backlog, when will we get through inequality submissions, what will our deficit be this year, how many problem proposals and articles will we receive. Backlog has been on top of this list for a while and now I can say with confidence: May 2017. We will be out of backlog in May 2017. My daughter will be 1 and backlog will be 0. From now on, only one of these numbers is allowed to increase.

This volume has many things to offer other than just being published on time. Issue 4 is a Ross Honsberger Commemorative issue, which is shaping up to be rich in content and in memories. We have many interesting articles scheduled to appear, several written by student authors. We will launch the inequality submission system to help streamline the problem proposals process. Our Open Access sections have been quite popular and have attracted new readers and contributors, so I am happy to see new names appear on the pages of the magazine. And as usual, this volume will feature 50 Contest problems, 50 Olympiad problems and 100 original proposed problems. We will work hard so you have plenty of things to do on a free Saturday night.

Kseniya Garaschuk

THE CONTEST CORNER

No. 51

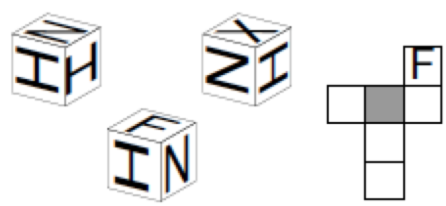
John McLoughlin

The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **September 1, 2017**.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

CC251. The six faces of a cube are labeled F, H, I, N, X and Z. Three views of the labelled cube are shown. Note that the H and the N on the die are indistinguishable from the rotated I and Z, respectively. The cube is then unfolded to form the lattice shown, with F shown upright. What letter should be drawn upright on the shaded square?



CC252. There are ten coins, each blank on one side and numbered on the other side with numbers 1 through 10. All ten coins are tossed and the sum of the numbers landing face up is calculated. What is the probability that this sum is at least 45?

CC253. Let $A(n)$ represent the number of ways n pennies can be arranged in any number of rows, where each row starts at the same position as the row below it and has fewer pennies than the row below it. For example, $A(6) = 4$, as shown below:

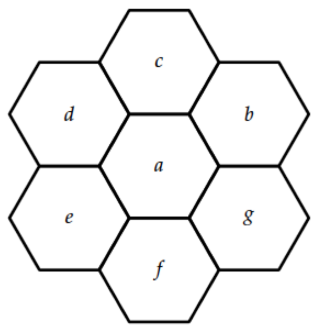
1. Show that $A(9) = 8$.
2. Find the smallest number k which is not equal to $A(n)$ for any n.

CC254. Hayden has a lock with a combination consisting of two 8s separated by eight digits, two 7s separated by seven digits, all the way down to two 1s separated by one digit. Unfortunately, Hayden spilled coffee on the paper that the combination was written on, and all that can be read of the combination is

$$* * * 584 * * * * * * * * * *$$

Determine all the possible combinations of the lock.

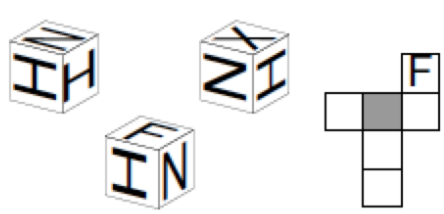
CC255. Antonino is instructed to colour the honeycomb pattern shown, which is made up of labelled hexagonal cells:



If two cells share a common side, they are to be coloured with different colours. Antonino has four colours available. Determine the number of ways he can colour the honeycomb, where two colourings are different if there is at least one cell that is a different colour in the two colourings.

. . . .

CC251. Les six faces d'un cube sont étiquetées F, H, I, N, X et Z. Trois vues du cube sont données ci-dessous. On remarque que le H et le I sont identiques si on fait subir à l'un ou l'autre une rotation de 90° et il en est de même avec le N et le Z. Le cube est ensuite déplié et son développement est donné à droite, le F étant en position verticale. Quelle lettre devrait paraître en position verticale dans l'espace ombré?



CC252. Dix jetons ont une face vide et portent un numéro de 1 à 10 sur l'autre face, les numéros étant tous différents. On lance les jetons et on compte ensuite la somme des numéros qui paraissent sur les faces supérieures. Quelle est la probabilité pour que cette somme soit supérieure ou égale à 45?

CC253. Soit $A(n)$ le nombre de façons de placer n jetons identiques en rangées, où le début de chaque rangée est aligné avec le début de la rangée en dessous et chaque rangée contient moins de jetons que la rangée en dessous. Par exemple, $A(6) = 4$, comme on le voit dans la figure suivante:

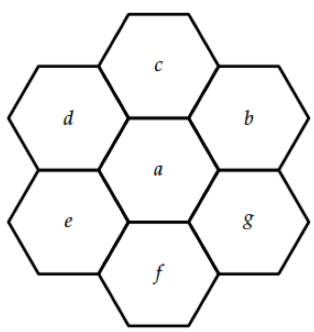
1. Montrer que $A(9) = 8$.
2. Déterminer le plus petit entier strictement positif k qui n'est pas égal à $A(n)$, quelle que soit la valeur de n.

CC254. Hector a un cadenas à combinaison dont le code comprend deux 8 séparés par huit chiffres, deux 7 séparés par sept chiffres et ainsi de suite jusqu'à deux 1 séparés par un chiffre. Malheureusement, Hector a renversé du café sur une feuille de papier sur laquelle le code est écrit et voici ce qu'il peut lire:

$$***584**********$$

Déterminer tous les codes possibles du cadenas.

CC255. Antonino doit colorier la figure suivante composée d'hexagones réguliers:



Lorsque deux hexagones ont un côté commun, ils doivent être coloriés de couleurs différentes. Antonino a quatre couleurs disponibles. Déterminer le nombre de façons qu'il y a de colorier la figure. On considère que deux coloriages sont différents s'il existe au moins un hexagone dont les couleurs sont différentes dans les deux coloriages.

CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2016: 42(1), p. 4–5.

CC201. An expedition to the planet Bizarro finds the following equation scrawled in the dust.

$$3x^2 - 25x + 66 = 0 \quad \Longrightarrow \quad x = 4 \text{ or } x = 9.$$

What base is used for the number system on Bizarro?

Originally Question 3 of the 2007 Maritime Mathematics Competition.

We received nine submissions. We present the solution by Digby Smith.

Let b be the base of the number system on Bizarro.

Since the roots of the equation in base b arithmetic are $x = 4$ and $x = 9$ it follows that $b > 9$. Since $b > 9$ the equation in base 10 arithmetic is

$$3(x-4)(x-9) = 0,$$
$$3x^2 - 39x + 108 = 0.$$

It then follows that

$$25_b = (2b+5)_{10} = 39_{10} \quad \text{and} \quad 66_b = (6b+6)_{10} = 108_{10}.$$

That is $2b+5 = 39$ and $6b+6 = 108$. Solving for b in both equations, $b = 17$. The base which is used for arithmetic on Bizarro is 17.

CC202. The positive integers from 1 to n inclusive are written on a blackboard. After one number is erased, the average (arithmetic mean) of the remaining $n-1$ numbers is $46\frac{20}{23}$. Determine n and the number that was erased.

Originally Question 5 of the 2007 Maritime Mathematics Competition.

We received twelve solutions, of which eleven were correct and complete. We present a slightly edited version of the similar yet independently-submitted solutions by Somasundar Muralidharan and Steven Chow.

We will prove that $n = 93$ and the number erased is 59.

When we erase one number from $1, 2, \ldots, n$, the minimum sum of the remaining numbers is $\frac{(n-1)n}{2}$, and the maximum sum is

$$\frac{n(n+1)}{2} - 1 = \frac{(n+2)(n-1)}{2}.$$

Hence the average of the remaining numbers lies between $\frac{n}{2}$ and $\frac{n+2}{2}$:

$$\frac{n}{2} \leq 46\frac{20}{23} \leq \frac{n+2}{2} \quad \Longrightarrow \quad 92 \leq n \leq 93.$$

Also, if the number erased is k, we have

$$\frac{\frac{n(n+1)}{2} - k}{n-1} = 46\frac{20}{23} \quad \Longrightarrow \quad \frac{n(n+1)}{2} - k = (n-1)\frac{46 \cdot 23 + 20}{23}.$$

Since 23 is not a factor of $46 \cdot 23 + 20$, it follows that 23 must divide $n - 1$. Since $92 \leq n \leq 93$, and $n \equiv 1 \bmod 23$, it follows that $n = 93$. Now, if k is the number erased, we have

$$\frac{\frac{93 \cdot 94}{2} - k}{92} = 46\frac{20}{23}.$$

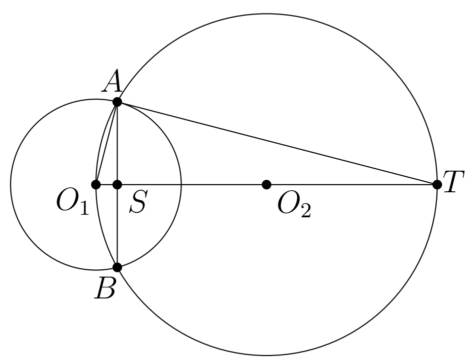
Solving for k, we get $k = 59$.

CC203. Two circles, one of radius 1, the other of radius 2, intersect so that the larger circle passes through the centre of the smaller circle. Find the distance between the two points at which the circles intersect.

Originally Question 4 of the 2007 Maritime Mathematics Competition.

We received 15 correct submissions. We provide the solution of Šefket Arslanagić.

Let O_1 be the centre of the circle of radius 1, and O_2 the centre of the circle of radius 2. Let A and B be the points of intersection of the circles. Let T be the point diametrically opposite O_1 on the larger circle, and let S be the intersection of AB and O_1T.



The triangles O_1AT and O_1SA are both right-angled and are similar. This gives

$$\frac{|AO_1|}{|AS|} = \frac{|O_1T|}{|AT|}.$$

We have $|O_1T| = 4$ as it is a diameter of the larger circle and $|O_1A| = 1$ as it is a diameter of the smaller circle. By the Pythagorean Theorem,

$$|AT| = \sqrt{4^2 - 1^2} = \sqrt{15}.$$

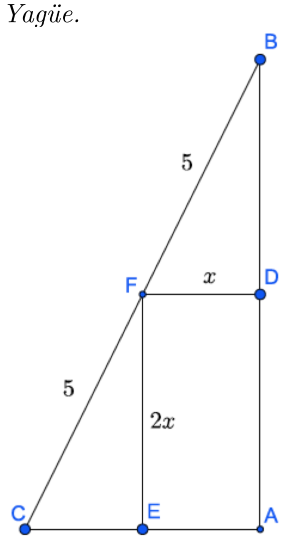
So we have

$$|AB| = 2|AS| = 2\frac{|AT||AO_1|}{|O_1T|} = \frac{\sqrt{15}}{2}.$$

CC204. A 10 metre ladder rests against a vertical wall. The midpoint of the ladder is twice as far from the ground as it is from the wall. At what height on the wall does the ladder reach?

Originally Question 3 of the 2003 Maritime Mathematics Competition.

We received 14 solutions, all correct and complete. We present the solution submitted by Fernando Ballesta Yagüe.



The triangles CEF and FDB are congruent, since all their angles are equal and they have the same hypotenuse. Therefore $CE = FD = x$ and $FE = BD = 2x$. By Pythagoras' Theorem,

$$10^2 = (2x)^2 + (4x)^2 \quad \longrightarrow \quad x = \sqrt{5},$$

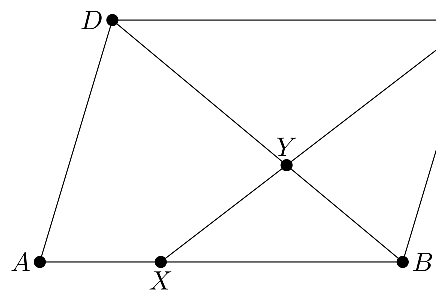
so we have $AB = 4x = 4\sqrt{5}$. The ladder touches the wall at a height of $4\sqrt{5}$m.

CC205. In the parallelogram $ABCD$, point X lies on AB such that XB is twice the length of AX. Let Y be the point of intersection of XC and BD. What fraction is the area of the triangle DCY of the area of the parallelogram $ABCD$?

Originally Question 3 of the 2008 Maritime Mathematics Competition.

We received nine correct submissions. We present the solution by Ricard Peiró.

We will use $[ab\ldots c]$ to represent the area of polygon $ab\ldots c$. Let $p = [XBY]$.



Triangles XBY and CDY are similar in the ratio

$$\frac{3}{2} = \frac{CD}{XB} = \frac{CY}{XY}.$$

So

$$\frac{[CDY]}{[XBY]} = \left(\frac{3}{2}\right)^2,$$

which gives

$$[CDX] = \frac{9}{4}[XBY] = \frac{9}{4}p.$$

The triangles XBY, CBY have the same height on the base CX, so the areas of the two triangles are proportional to the bases:

$$\frac{[CBY]}{[XBY]} = \frac{CY}{XY} = \frac{3}{2}.$$

So $[CBY] = \frac{3}{2}p$.

Now $[CDB] = [CDY] + [CBY] = \frac{9}{4}p + \frac{3}{2}p = \frac{15}{4}p$.

The area of the parallelogram is twice this, so $[ABCD] = 2 \cdot [CDB] = \frac{15}{2}p$.

Finally, the ratio between the areas of the triangle CDY and the parallelogram $ABCD$ is

$$\frac{[CDY]}{[ABCD]} = \frac{\frac{9}{4}p}{\frac{15}{2}p} = \frac{3}{10}.$$

THE OLYMPIAD CORNER

No. 349

Carmen Bruni

The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **September 1, 2017**.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

OC311. Let $\triangle ABC$ be an acute-scalene triangle, and let N be the center of the circle which passes through the feet of the altitudes. Let D be the intersection of the tangents to the circumcircle of $\triangle ABC$ at B and C. Prove that A, D and N are collinear if and only if $\measuredangle BAC = 45^\circ$.

OC312. Let a, b, c be nonnegative real numbers.Prove that

$$\frac{(a-bc)^2+(b-ca)^2+(c-ab)^2}{(a-b)^2+(b-c)^2+(c-a)^2} \geq \frac{1}{2}.$$

OC313. Let $x_1, x_2, \cdots, x_n \in (0,1)$, $n \geq 2$. Prove that

$$\frac{\sqrt{1-x_1}}{x_1}+\frac{\sqrt{1-x_2}}{x_2}+\cdots+\frac{\sqrt{1-x_n}}{x_n} < \frac{\sqrt{n-1}}{x_1x_2\cdots x_n}.$$

OC314. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all reals x, y, z, we have

$$(f(x)+1)(f(y)+f(z)) = f(xy+z)+f(xz-y)$$

OC315. Suppose that a is an integer and that $n!+a$ divides $(2n)!$ for infinitely many positive integers n. Prove that $a = 0$.

. . . .

OC311. Soit ABC un triangle scalène acutangle et N le centre du cercle qui passe aux pieds des trois hauteurs du triangle. Soit D le point d'intersection des tangentes au cercle circonscrit au triangle ABC aux sommets B et C. Démontrer que les points A, D et N sont alignés si et seulement si $\measuredangle BAC = 45^\circ$.

OC312. Soit a, b et c des réels positifs ou nuls. Démontrer que

$$\frac{(a-bc)^2+(b-ca)^2+(c-ab)^2}{(a-b)^2+(b-c)^2+(c-a)^2} \geq \frac{1}{2}.$$

OC313. Soit $x_1, x_2, \cdots, x_n \in (0,1)$, $n \geq 2$. Démontrer que

$$\frac{\sqrt{1-x_1}}{x_1}+\frac{\sqrt{1-x_2}}{x_2}+\cdots+\frac{\sqrt{1-x_n}}{x_n} < \frac{\sqrt{n-1}}{x_1x_2\cdots x_n}.$$

OC314. Déterminer toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$ telles que

$$(f(x)+1)(f(y)+f(z)) = f(xy+z)+f(xz-y)$$

pour tous réels x, y et z.

OC315. Soit a un entier tel que $n!+a$ soit un diviseur de $(2n)!$ pour un nombre infini d'entiers strictement positifs n. Démontrer que $a = 0$.

OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2015: 41(9), p. 377–378.

OC251. Let a, b, c, d be real numbers such that $b-d \geq 5$ and all zeros x_1, x_2, x_3, and x_4 of the polynomial $P(x) = x^4+ax^3+bx^2+cx+d$ are real. Find the smallest value the product $(x_1^2+1)(x_2^2+1)(x_3^2+1)(x_4^2+1)$ can take.

Originally problem 1 from day 1 of the 2014 USAMO.

We present the solution by Missouri State University Problem Solving Group. There were no other submissions.

We have that $P(x) = (x-x_1)(x-x_2)(x-x_3)(x-x_4)$ and so

$$\begin{aligned}
&(x_1^2+1)(x_2^2+1)(x_3^2+1)(x_4^2+1)\\
&= (i-x_1)(i-x_2)(i-x_3)(i-x_4)\cdot(-i-x_1)(-i-x_2)(-i-x_3)(-i-x_4)\\
&= P(i)P(-i)\\
&= (1-ai-b+ci+d)(1+ai-b-ci+d)\\
&= ((1-b+d)-(a-c)i)((1-b+d)+(a-c)i)\\
&= (1-b+d)^2+(a-c)^2\\
&= (b-d-1)^2+(a-c)^2\\
&\geq 4^2+(a-c)^2
\end{aligned}$$

This is smallest when $a = c$ and in that case, the minimum value is 16. Note that $(x-1)^4$ shows that this minimum can be obtained.

OC252. In an obtuse triangle ABC ($AB > AC$), O is the circumcentre and D, E, F are the midpoints of BC, CA, AB respectively. Median AD intersects OF and OE at M and N respectively. BM meets CN at point P. Prove that $OP \perp AP$.

Originally problem 3 from day 1 of the 2014 South East Mathematical Olympiad.

We received 2 correct submissions. We present the solution by Andrea Fanchini.

We use barycentric coordinates and the usual Conway's notations with reference to triangle ABC.

Coordinates of points D, E, F, O. These points are well known

$$D(0:1:1), \qquad E(1:0:1), \qquad F(1:1:0), \qquad O(a^2S_A : b^2S_B : c^2S_C).$$

Equations of lines AD, OE, OF. Now the equations of these lines are

$$AD : y-z=0, \quad OE : b^2x+(a^2-c^2)y-b^2z=0, \quad OF : c^2x-c^2y+(a^2-b^2)z=0.$$

Coordinates of points M and N. We have

$$M = AD \cap OF = (2S_A : c^2 : c^2), \qquad N = AD \cap OE = (2S_A : b^2 : b^2).$$

Equations of lines BM, CN. Now the equations of these lines are

$$BM : c^2x - 2S_Az = 0, \qquad CN : b^2x - 2S_Ay = 0.$$

Coordinates of point P. We have

$$P = BM \cap CN = (2S_A : b^2 : c^2)$$

Equations of lines AP, OP. Now the equations of these lines are

$$AP : c^2y - b^2z = 0, \qquad OP : b^2c^2x - c^2S_Ay - b^2S_Az = 0.$$

Perpendicularity of AP and OP. The infinite perpendicular point of line AP is

$$AP_{\infty\perp}\left(S_A(b^2-c^2) : -b^2(S_A+c^2) : c^2(S_A+b^2)\right).$$

then the infinite point of OP is

$$OP_{\infty}\left(S_A(b^2-c^2) : -b^2(S_A+c^2) : c^2(S_A+b^2)\right).$$

so $AP_{\infty\perp} = OP_{\infty}$, therefore AP and OP are perpendicular.

OC253. Prove that there exist infinitely many positive integers n such that $3^n + 2$ and $5^n + 2$ are all composite numbers.

Originally problem 8 of the 2014 China Northern Mathematical Olympiad.

We received 9 correct submissions. We present the solution by Ali Adnan.

It is easily seen using Fermat's Little Theorem that for all $k \geq 0$,

$$3^{16k+6} + 2 \equiv 3^6 + 2 = 731 \equiv 0 \pmod{17}$$
$$5^{16k+6} + 2 \equiv (-1)^{16k+6} + 2 \equiv 0 \pmod{3}$$

which shows that there are infinitely many $n \in \mathbb{N}$ such that both $3^n + 2$ and $5^n + 2$ are composite ($n = 16k + 6$, $k \geq 0$).

OC254. Find all non-negative integers k, n which satisfy $2^{2k+1} + 9 \cdot 2^k + 5 = n^2$.

Originally problem 5 of the 2014 Balkan Mathematical Olympiad Team Selection Test.

We received 9 correct submissions. We present the solution by Titu Zvonaru.

The only solution is when $k = 0$ and $n = 4$ which one can easily check is a solution. By inspection, we can plug in the values for $k = 0$, $k = 1$, and $k = 2$ and see that the only solution arises when $k = 0$ which we have already considered. Now, suppose that $k > 2$. Since the left hand side is odd, it follows that n is odd. Let $n = 2p + 1$ where p is a positive integer. The given equation is equivalent to

$$2^{2k+1} + 9 \cdot 2^k + 5 = 4p^2 + 4p + 1$$
$$2^{2k-1} + 9 \cdot 2^{k-2} + 5 = p(p + 1)$$

As $k > 2$, the left hand side above is odd however the right hand side above is even and thus we do not obtain any solutions.

OC255. Let n be a positive integer and $x_1, x_2, \ldots, x_n$ be positive reals. Show that there are numbers $a_1, a_2, \ldots, a_n \in \{-1, 1\}$ such that the following holds:

$$a_1x_1^2 + a_2x_2^2 + \cdots + a_nx_n^2 \geq (a_1x_1 + a_2x_2 + \cdots + a_nx_n)^2$$

Originally problem 6 of the 2014 France Team Selection Test.

We received 1 incorrect submission.

Editor's Note: There is a requirement that $|a_i| = 1$ for each i.

FOCUS ON...

No. 25

Michel Bataille

The Long Division of Polynomials

Introduction

Let R be a commutative ring and $R[x]$ the ring of polynomials with coefficients in R. With any given elements $a(x)$ and $b(x) \neq 0$ of $R[x]$, the leading coefficient of $b(x)$ being a unit, the division algorithm associates a unique pair $(q(x), r(x))$ of polynomials of $R[x]$ such that

$$a(x) = b(x)q(x) + r(x)$$

and $\deg(r(x)) < \deg(b(x))$. This well-known result is rich in consequences. For example, in the familiar case when R is a field F, it leads to the development of an arithmetic in $F[x]$ analogous to the arithmetic in $\mathbb{Z}$. In this number, we will just examine direct applications through a selection of problems.

Exploiting the result of a long division

As a warm-up exercise, consider the following:

> Let α be the real root of $x^3 + x + 1$. Write $\frac{1}{2\alpha+1}$ with no α in the denominator, that is, as $k(\alpha)$ for some polynomial $k(x)$.

Everyone is able to answer the question if α is replaced by, say, $\sqrt{2}$. The way we solve the problem in that case leads to the idea of dividing $x^3 + x + 1$ by $2x + 1$. A long division yields

$$8(x^3 + x + 1) = (2x + 1)(4x^2 - 2x + 5) + 3$$

and the substitution $x = \alpha$ then shows that

$$k(x) = -\frac{4}{3}x^2 + \frac{2}{3}x - \frac{5}{3}$$

is a suitable polynomial. The method generalizes to $\frac{1}{p(\alpha)}$ where $p(x)$ is a polynomial coprime with $x^3 + x + 1$ with the help of a Bezout relation found after successive long divisions.

We turn to an intriguing problem posed by Nick Lord in *The Mathematical Gazette* in 1999, here slightly reworded:

> Let a and b be positive integers such that $\sqrt{a} = \sqrt{\sqrt[3]{b} - 1} + \sqrt{\sqrt[3]{b^2} - \sqrt[3]{b}}$. If b is not a sixth power, find a and b.

We propose a variant of solution in which long divisions of polynomials play a central role.

It is well-known that $r = \sqrt[6]{b}$ is either an integer or an irrational, so we may suppose that r is not in $\mathbb{Q}$. Squaring the given relation between a and b, we readily obtain that

$$a = r^2 - 1 + r^2(r^2 - 1) + 2r(r^2 - 1), \qquad (1)$$

so that r is a real root of the polynomial $s(x) = x^4 + 2x^3 - 2x - (a+1)$. Long divisions in $\mathbb{Z}[x]$ produce polynomials $q_0(x), q_1(x), t(x), u(x)$ with integral coefficients satisfying

$$x^6 - b = s(x)q_0(x) - t(x), \qquad 36s(x) = t(x)q_1(x) + u(x),$$

where

$$u(x) = (a-3)^2x^2 - 2(a^2 - 6a - 3 + 3b)x + (4a + 4a^2 - ab - 9b).$$

For later use, note that r is a root of $u(x)$.

Now, assume that $a \neq 3$, so that $\deg(u(x)) = 2$. Then $u(x)$ divides $x^6 - b$ since otherwise the remainder in the long division of $x^6 - b$ by $u(x)$ would be of degree 1 with r as a root, which implies that r is rational. Since $x^6 - b$ has no other real roots than r and $-r$, the roots of $u(x)$ are r and $-r$. Thus,

$$a^2 - 6a - 3 + 3b = 0 \quad \text{and} \quad u(x) = (a-3)^2x^2 + \frac{(a-3)(a^2 + 18a + 9)}{3}.$$

Expressing that $u(r) = 0$, we see that r^2 would be a rational number and so would r (by (1)), a contradiction. As a result, $a = 3$ and $u(x) = 6(4-b)x + 12(4-b)$ so that b must be equal to 4.

Conversely, $a = 3, b = 4$ is a solution, as it is readily checked (using (1)).

Polynomials often intervene in linear algebra. Here is an example inspired by a problem set in *Mathematics Magazine* in 2008, where long divisions prove useful.

> Let A be a 2×2 matrix over $\mathbb{Z}_p$, where p is prime. Suppose that the powers A^k, $k = 1, 2, \ldots, p^2 - 1$ are distinct and that $A^{p^2-1} \neq O_2$. Show that if $i, j \in \{1, 2, \ldots, p^2 - 1\}$, then $A^i + A^j$ is O_2 or a power of A.

Let $p(x)$ be the characteristic polynomial of A: the degree of $p(x)$ is 2 and (from the Hamilton-Cayley theorem) $p(A) = O_2$. Let $G = \{\alpha x + \beta : \alpha, \beta \in \mathbb{Z}_p\}$. Clearly, the cardinality of G is p^2. For each $k \in \{1, 2, \ldots, p^2 - 1\}$, consider the long division of x^k by $p(x)$:

$$x^k = p(x)q_k(x) + r_k(x),$$

where the remainder $r_k(x)$ is in G. No $r_k(x)$ can be the zero polynomial since otherwise

$$A^k = p(A)q_k(A) = O_2$$

and we would have $A^{p^2-1} = O_2$. In addition, $r_i(x) = r_j(x)$ for some $i \neq j$ implies

$$A^i = r_i(A) = r_j(A) = A^j,$$

contradicting the hypotheses. Thus, the polynomial $r_k(x),\ k = 1, 2, \ldots, p^2 - 1$ and the zero polynomial form a subset of G with cardinality p^2. It follows that

$$\{r_k(x) :\ k = 1, 2, \ldots, p^2 - 1\} \cup \{0\} = G.$$

Observing that G is closed under addition, we obtain that

$$A^i + A^j = r_i(A) + r_j(A),$$

if not O_2, is equal to $r_\ell(A) = A^\ell$ for some $\ell \in \{1, 2, \ldots, p^2 - 1\}$.

Division by a polynomial with degree 1

Consider the case when $b(x) = x - \alpha$, a monic polynomial of degree 1 ($\alpha \in R$). Then the remainder $r(x)$ is a constant $\rho \in R$ and substituting α for x in the relation $a(x) = (x-\alpha)q(x)+\rho$, we obtain $\rho = a(\alpha)$. It follows that $a(x)$ is divisible by $x-\alpha$ if and only if α is a root of $a(x)$. Here is a very simple example of application:

> Let $p(x) \in R[x]$. Prove that $p(x)-x$ divides the polynomial $p(p(x))-x$.

This nice exercise can be solved in various ways and the reader is invited to find at least two proofs different from the following one: Consider the polynomial $p(y)-x$, an element of $(R[x])[y]$. From the result above, the remainder in the division of $p(y) - x$ by $y - x$ is $p(x) - x$ and we have the relation

$$p(y) - x = (y - x)q_x(y) + p(x) - x$$

for some polynomial $q_x(y)$ in $(R[x])[y]$. Substituting $p(x)$ for y, we obtain

$$p(p(x)) - x = (p(x) - x)(q_x(p(x)) + 1)$$

and the desired result follows.

Our next example, again in the domain of linear algebra, is problem 1053 of *The College Mathematics Journal* (a particular case is **3860** [2013 : 276 ; 2014 : 269]).

> Let $n \geq 3$ be an odd integer, let A be an $n \times n$ integer matrix, and let $\alpha, \beta \in \mathbb{Z}$. Prove that the determinant of $\alpha A + \beta A^T$ is a multiple of $\alpha + \beta$.

We propose a variant of solution. Consider the polynomial $p(x) = \det(xA+\beta A^T)$. Since β and the entries of A are integers, we have $p(x) \in \mathbb{Z}[x]$. In addition, $p(x)$ satisfies

$$p(-\beta) = \det(\beta(A^T - A)) = \beta^n \det(A^T - A).$$

But the matrix $A^T - A$ is skew-symmetric since

$$(A^T - A)^T = A^{TT} - A^T = A - A^T = -(A^T - A)$$

and n is odd, hence $\det(A^T - A) = 0$. It follows that $p(-\beta) = 0$ and therefore $p(x)$ is divisible by $(x+\beta)$ in $\mathbb{Z}[x]$, that is, $p(x) = (x+\beta)q(x)$ for some polynomial $q(x)$ of $\mathbb{Z}[x]$. Thus,

$$\det(\alpha A + \beta A^T) = p(\alpha) = (\alpha + \beta)q(\alpha)$$

and since $q(\alpha) \in \mathbb{Z}$, we can conclude that $\det(\alpha A + \beta A^T)$ is a multiple of $\alpha + \beta$.

We shall now illustrate the natural and readily proved generalization: if R is an integral domain, distinct elements $\alpha_1, \ldots, \alpha_n$ of R are roots of $a(x)$ if and only if $a(x)$ is divisible by the product $(x-\alpha_1)\cdots(x-\alpha_n)$. We start with a shortened and slightly modified version of a problem proposed by M. Klamkin in 2001 (problem 711 of *The College Mathematics Journal*).

> Let n be a positive integer. Prove that there exist infinitely many polynomials $p(x) \in \mathbb{Z}[x]$ such that
>
> $$p(n) = n^2,\ p(2n) = 2n^2,\ p(3n) = n^2.$$

If $p(x)$ satisfies the conditions, polynomial $p(x) - n^2$ is divisible by $(x-n)(x-3n)$, hence

$$p(x) = (x-n)(x-3n)s(x) + n^2$$

for some $s(x) \in \mathbb{Z}[x]$. The condition $p(2n) = 2n^2$ then gives $s(2n) = -1$ so that, similarly,

$$s(x) = (x-2n)t(x) - 1$$

for some $t(x) \in \mathbb{Z}[x]$. Finally, we obtain that

$$p(x) = (x-n)(x-2n)(x-3n)t(x) + (-x^2 + 4nx - 2n^2).$$

Conversely, any polynomial $p(x)$ given by (1) (on page 14) with $t(x) \in \mathbb{Z}[x]$ meets the required conditions, as it is easily checked.

An appeal to the above generalization sometimes occurs rather unexpectedly. A typical example is a problem of the 2006 Croatian Mathematical Olympiad:

> Let k and n be positive integers. Prove that
>
> $$(n^4 - 1)(n^3 - n^2 + n - 1)^k + (n+1)n^{4k-1}$$
>
> is divisible by $n^5 + 1$.

A proof involving only divisibility for integers is possible: see [2010 : 443] for a neat proof by induction on k. Here is a solution resorting to polynomials.

Let $A(n) = (n^4 - 1)(n^3 - n^2 + n - 1)^k + (n+1)n^{4k-1}$. Then, remarking that $n^4 - 1 = (n+1)(n^3 - n^2 + n - 1)$, we obtain

$$A(n) = (n+1)\left((n^3 - n^2 + n - 1)^{k+1} + n^{4k-1}\right) = (n+1)P(n),$$

where the polynomial $P(x)$ is defined by $P(x) = (x^3 - x^2 + x - 1)^{k+1} + x^{4k-1}$.

Let $\omega = e^{2\pi i/5}$. Then, for $j = 1, 2, 3, 4$, we have

$$1 + \omega^j + \omega^{2j} + \omega^{3j} + \omega^{4j} = \frac{(\omega^5)^j - 1}{\omega^j - 1} = 0.$$

Hence

$$\begin{aligned} P(-\omega^j) &= (-(\omega^j)^3 - (\omega^j)^2 - (\omega^j) - 1)^{k+1} - (\omega^j)^{4k-1} \\ &= (\omega^{4j})^{k+1} - (\omega^j)^{4k-1} \\ &= \omega^{(4k-1)j}((\omega^5)^j - 1) \\ &= 0. \end{aligned}$$

Thus,

$$P(x) = (x+\omega)(x+\omega^2)(x+\omega^3)(x+\omega^4)Q(x)$$

for some polynomial $Q(x) \in \mathbb{C}[x]$.

Now, we observe that

$$\begin{aligned} &(x+\omega)(x+\omega^2)(x+\omega^3)(x+\omega^4) \\ &= ((-x)-\omega)((-x)-\omega^2)((-x)-\omega^3)((-x)-\omega^4) \\ &= x^4 - x^3 + x^2 - x + 1. \end{aligned}$$

Since $P(x)$ and $x^4 - x^3 + x^2 - x + 1$ have integer coefficients, the same is true of $Q(x)$ and $Q(n)$ is an integer. Finally,

$$A(n) = (n+1)(n^4 - n^3 + n^2 - n + 1)Q(n) = (n^5 + 1)Q(n)$$

and the result follows.

For more examples in the same vein, we refer the reader to **3704** [2012 : 24 ; 2013 : 42] and to problem 1 of the 1994 Iranian National Mathematical Olympiad [1998 : 6 ; 1999 : 141].

As usual, we conclude with a couple of exercises.

Exercises

1. Find all complex numbers λ such that the product of two roots of

$$x^4 - 2x^2 + \lambda x + 3$$

is 1.

2. Find real numbers a_k, b_k $(k = 1, 2, \ldots, 2017)$ such that

$$\frac{3x^5 - 3x^4 - 2x^2 + 2x + 4}{(x^2 + x + 1)^{2017}} = \sum_{k=1}^{2017} \frac{a_k x + b_k}{(x^2 + x + 1)^k}.$$

Products that are Powers

Ted Barbeau

This investigation was originally designed for school students to gain fluency with the factoring of positive integers as a product of prime powers through the challenge of optimizing sets of numbers with square products. However, like many such things, it morphed into something larger and generated some interesting more general questions.

1 Square products

A diverting pastime is to begin with a positive integer n and multiply it by any number of distinct larger integers until the product is a square. It is easy to find examples. Simply multiply n by $4n$, for instance, to get the square $(2n)^2$. So to make it a bit more challenging, ask that the largest integer that we introduce into the product is *as small as possible.*

For example, let $n = 5$. We can do better than $\{5, 20\}$, by selecting the set of integers in the product to be $\{5, 10, 18\}$ or better yet $\{5, 12, 15\}$. But as you can readily see, the set $\{5, 8, 10\}$ is one whose largest number is minimum, since we need at least two multiples of 5.

For each positive integer n, define the function $f_2(n)$ to be the smallest number k exceeding n for which there is a set of at least two distinct integers, including n, in the closed interval $[n, k]$ whose product is a square. Thus, $f_2(5) = 10$. Just to get you into the proper spirit, check that

$$f_2(2) = 6, \quad f_2(3) = 8 \quad \text{and} \quad f_2(8) = 15.$$

What is $f_2(12)$?

If $n = m^2$ is itself a square, then we could choose the set $\{m^2, (m + 1)^2\}$, thus showing that $f_2(m^2) \leq (m + 1)^2$. Is it possible for $f_2(m^2)$ to be strictly less than $(m+1)^2$? Or, to ask a stronger question, are there only finitely many values of m for which $f_2(m^2) = (m + 1)^2$? Or for which $f_2(m^2) < (m + 1)^2$?

It is easy to see that $f_2(n) \leq 4n$ for each positive integer n. However, empirical investigation suggests that $f_2(n) \leq 2n$ except for $1 \leq n \leq 4$. This turns out to be so, and there are at least two approaches that you can take to establish this. First, you can show that, for sufficiently large n, the open interval $(n, 2n)$ contains a number of the form $2k^2$, and check small values of n individually. Alternatively, you can try to put inside each open interval $(n, 2n)$ a number equal to an odd power of 2 multipled by a small square. Note that, with this result in hand, that $f_2(n) = 2n$ whenever n is a prime exceeding 3.

This leads to other questions. What rational values are assumed by $f_2(n)/n$ infinitely often? Is $f_2(2m) \leq 3m$ for sufficiently large values of m?

2 Higher power products

We can play the same game with higher powers, and define, for each positive integer $r \geq 2$, the function $f_r(n)$ to be the smallest value of k exceeding n for which there are at least two distinct integers, including n, in the closed interval $[n, k]$ whose product is an rth power. You may wish to verify that

$$f_3(6) = f_4(6) = f_5(6) = 18, \quad \text{while} \quad f_6(6) = 24.$$

We can ask questions analogous to those posed for the case $n = 2$. For $n \geq 2$, show that $f_3(n) \leq 3n$. ($f_3(1) = 4$.) More generally, is it true that $f_r(n) \leq rn$ for n sufficiently large, with equality occurring when n is a prime exceeding r?

Prove that $f_r(m^r)$ is always strictly less than $(m+1)^r$ when $r \geq 3$.

What values of $f_r(n)/n$ are assumed for infinitely many values of n?

Finally, here is a conjecture: For any integers r and s exceeding 1, there are at most finitely many integer solutions n to the inequality $f_r(n) \leq n + s$.

3 Comments

In finding $f_2(12)$, we must be sure that our set contains another multiple of an odd power of 3. We can try to see if we can put in either 15 or 21 (we may decide whether to include 18 depending on straightening out the power of 2). We can find the set $\{12, 14, 18, 21\}$; but $\{12, 15, 20\}$ is better. For a set with a smaller maximum, we see that we have to exclude any number divisible by a prime greater than 3, and $\{12, 18\}$ does not work. Therefore $f_2(12) = 20$.

To show that $f(n) \leq 2n$ for $n \geq 5$, we can check the cases $5 \leq n \leq 9$ by hand. Now let

$$x_1 = 18 = 2 \times 3^2, x_2 = 32 = 2^5, x_3 = 50 = 2 \times 5^2, x_m = 4 \times x_{m-3}$$

for $m \geq 4$. Each x_m is the product of an odd power of 2 and a square and $x_m < x_{m+1} < 2x_m$ for each $m \geq 1$. Then for each $n \geq 10$, we can show that for some m, $n < x_m < 2n$ for some value of $m \geq 1$.

To show that $f_3(n) \leq 3n$ for $n \geq 2$, we can check the small cases by hand and then show that for n sufficiently large, there is always a value of k for which $n < 36k^3 < 3n$ so that

$$n \times 2n \times 3n \times 36k^3 = (6kn)^3$$

is a cube. The only fly in the ointment is that $36k^3$ might equal $2n$, so that the case $n = 18k^3$ needs special attention. However, $f_3(18) = 25 < 54$, a suitable set being $\{18, 20, 24, 25\}$, and we can derive a set that works for $18k^3$.

There is another approach to this result. It appears to be the case that, for $n \geq 9$, we can find a set of distinct integers in the closed interval $[n, 3n]$ whose cardinality is a multiple of 3, which contains n, $2n$ and $3n$, and whose elements multiply to give a cubic product. We can check this for an initial tranche of integers n and then try to prove it in general by an induction argument that involves multiplying each element in a set for n by a number u to get a set for un. The attempt to construct an argument is beset by various annoyances.

In order to show that $f_r(n) \leq rn$ for sufficiently large n, we can try to find numbers v and k such that $n < vk^r < rn$ such that $r!v$ is a rth power, vk^r is not a multiple of n, and

$$n \times 2n \times 3n \times \cdots \times rn \times (vk^r)$$

is an rth power.

The evaluation of $f_2(m^2)$ turns out to be more interesting than it first appears. Investigation of small values of m reveals that the open interval $(m^2, (m+1)^2)$ contains a set of distinct integers with a square product more often than not. It seems plausible that $f_2(m^2)$ could equal, or not equal, $(m+1)^2$ each infinitely often. Where the inequality is strict, determination of the set of integers seems to be a highly idiosyncratic process and it is hard to see how one can devise a systematic process that will cover infinitely many cases. Sometimes we find that we only need three numbers for a square product. For example, between 7^2 and 8^2, we have the triple $(50, 56, 63)$; between 12^2 and 13^2, we have the triple $(147, 150, 162)$. Does this happen infinitely often? However, it is not possible to find a pair of numbers strictly between two consecutive squares whose product is square. Can you see why?

Since

$$m^3 \times m^2(m+1) \times m(m+1)^2 = (m^2(m+1))^3$$

is cube, $f_3(m^3) \leq m(m+1)^2$. Incidentally, note that $f_3(m^2) \leq (m+1)^2$ since

$$m^2 \times m(m+1) \times (m+1)^2 = (m(m+1))^3.$$

More generally, since

$$m^r \times m^s(m+1)^{r-s} \times m^{r-s}(m+1)^s$$

is an rth power for $2s \geq r$, we see that $f_r(m^r) < (m+1)^r$ for all m.

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PROBLEMS

Readers are invited to submit solutions, comments and generalizations to any problem in this section. Moreover, readers are encouraged to submit problem proposals. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **September 1, 2017**.

The editor thanks Rolland Gaudet, retired professor of Université de Saint-Boniface in Winnipeg, for translations of the problems.

4201. *Proposed by Florin Stanescu.*

Let M be a point in the interior of a regular polygon $A_1A_2\ldots A_n$ inscribed in the unit circle centered at O, and let A_kB_k be the chord from the vertex A_k through M. Prove that

$$\frac{A_1B_1^2 + A_2B_2^2 + \cdots + A_nB_n^2}{n} \geq \frac{4}{1+OM^2}.$$

4202. *Proposed by Roy Barbara.*

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$f(a^2 + b^2) = f(a)^2 + f(b)^2$$

for all $a, b \in \mathbb{N}$.

4203. *Proposed by Michel Bataille.*

The incircle of a triangle ABC has centre I, radius r and intersects the line segments AI, BI, CI at A', B', C', respectively. Prove that

(a) $AA' \cdot BB' \cdot CC' \leq \frac{\sqrt{3}}{18}(AB + BC + CA)r^2$;

(b) $A'B' \cdot B'C' \cdot C'A' \leq 3\sqrt{3}\,r^3$.

4204. *Proposed by Leonard Giugiuc and Diana Trailescu.*

Let ABC be a triangle with $AB \neq AC$ and let I be the incenter of ABC. Suppose the lines AI, BI and CI intersect the sides BC, CA and AB in D, E and F, respectively. If $DE = DF$ and $\angle ABC = 2\angle ACB$, find $\angle ACB$.

4205. *Proposed by Daniel Sitaru.*

Prove that for $0 < a < c < b, a, b, c \in \mathbb{R}$, we have

$$\frac{1}{c\sqrt{ab}} \int_a^b x \arctan x dx > \frac{(c-a)\arctan\sqrt{ac}}{\sqrt{bc}} + \frac{(b-c)\arctan\sqrt{bc}}{\sqrt{ac}}.$$

4206. *Proposed by Gheorghe Alexe and George-Florin Serban.*

Find positive integers p and q that are relatively prime to each other such that $p + p^2 = q + q^2 + 3q^3$.

4207. *Proposed by Mihaela Berindeanu.*

Let x, y and z be real numbers such that $x + y + z = 3$. Show that

$$\frac{1}{1 + 2^{4-3x}} + \frac{1}{1 + 2^{4-3y}} + \frac{1}{1 + 2^{4-3z}} \geq 1.$$

4208. *Proposed by Leonard Giugiuc, Daniel Sitaru and Marian Dinca.*

Let x, y and z be positive real numbers such that $x \leq y \leq z$. Prove that for any real number $k > 2$, we have:

$$xy^k + yz^k + zx^k \geq x^2y^{k-1} + y^2z^{k-1} + z^2x^{k-1}.$$

4209. *Proposed by Nguyen Viet Hung.*

Let m and n be distinct positive integers. Evaluate

$$\lim_{x \to 0} \frac{(1 + nx)^m - (1 + mx)^n}{\sqrt[m]{1 + mx} - \sqrt[n]{1 + nx}}.$$

4210. *Proposed by Van Khea and Leonard Giugiuc.*

Let ABC be a triangle in which the circumcenter lies on the incircle. Furthermore, let $BC = a, CA = b$ and $AB = c$. For which triangles does the expression $\dfrac{a + b + c}{\sqrt[3]{abc}}$ attain its minimum?

. . . .

4201. *Proposé par Florin Stanescu.*

Soit M un point à l'intérieur d'un polygone régulier $A_1A_2 \ldots A_n$ inscrit dans le cercle unitaire centré à O et soit A_kB_k la corde passant par A_k et M. Démontrer que

$$\frac{A_1B_1^2 + A_2B_2^2 + \cdots + A_nB_n^2}{n} \geq \frac{4}{1 + OM^2}.$$

4202. *Proposé par Roy Barbara.*

Soit $\mathbb{N} = \{0, 1, 2, \ldots\}$. Déterminer toutes les fonctions $f : \mathbb{N} \to \mathbb{N}$ satisfaisant

$$f(a^2 + b^2) = f(a)^2 + f(b)^2$$

pour tout $a, b \in \mathbb{N}$.

4203. *Proposé par Michel Bataille.*

Le cercle inscrit du triangle ABC possède I comme centre et r comme rayon; il intersecte les segments AI, BI et CI en A', B' et C' respectivement. Démontrer que

(a) $AA' \cdot BB' \cdot CC' \leq \frac{\sqrt{3}}{18}(AB + BC + CA)r^2$;

(b) $A'B' \cdot B'C' \cdot C'A' \leq 3\sqrt{3}\, r^3$.

4204. *Proposé par Leonard Giugiuc et Diana Trailescu.*

Soit ABC un triangle tel que $AB \neq AC$ et soit I le centre de son cercle inscrit. Supposer que les lignes AI, BI et CI intersectent les côtés BC, CA et AB en D, E et F respectivement. Si $DE = DF$ et $\angle ABC = 2\angle ACB$, déterminer $\angle ACB$.

4205. *Proposé par Daniel Sitaru.*

Pour $0 < a < c < b$ où $a, b, c \in \mathbb{R}$, démontrer que

$$\frac{1}{c\sqrt{ab}} \int_a^b x \arctan x dx > \frac{(c-a)\arctan\sqrt{ac}}{\sqrt{bc}} + \frac{(b-c)\arctan\sqrt{bc}}{\sqrt{ac}}.$$

4206. *Proposé par Gheorghe Alexe et George-Florin Serban.*

Déterminer des entiers positifs p et q, relativement premiers, tels que

$$p + p^2 = q + q^2 + 3q^3.$$

4207. *Proposé par Mihaela Berindeanu.*

Soient x, y et z des nombres réels tels que $x + y + z = 3$. Démontrer que

$$\frac{1}{1 + 2^{4-3x}} + \frac{1}{1 + 2^{4-3y}} + \frac{1}{1 + 2^{4-3z}} \geq 1.$$

4208. *Proposé par Leonard Giugiuc, Daniel Sitaru et Marian Dinca.*

Soient x, y et z des nombres réels positifs tels que $x \leq y \leq z$. Démontrer que pour tout nombre réel $k > 2$, l'inégalité suivante tient:

$$xy^k + yz^k + zx^k \geq x^2y^{k-1} + y^2z^{k-1} + z^2x^{k-1}.$$

4209. *Proposé par Nguyen Viet Hung.*

Soient m et n des entiers positifs distincts. Évaluer

$$\lim_{x\to 0} \frac{(1+nx)^m - (1+mx)^n}{\sqrt[m]{1+mx} - \sqrt[n]{1+nx}}.$$

4210. *Proposé par Van Khea and Leonard Giugiuc.*

Soit ABC un triangle dont le centre du cercle circonscrit se trouve sur le cercle inscrit. Soient $BC = a, CA = b$ et $AB = c$. Pour quels triangles est-ce que l'expression $\dfrac{a+b+c}{\sqrt[3]{abc}}$ atteint son minimum?

Massachusetts Institute of Technology
Entrance Examination, 1869-70
Geometry

1. Prove that the sum of the three angles of a plane triangle equals two right angles.
2. Prove that the diagonal of a parallelogram divides it into two equal triangles.
3. Prove that the area of a trapezoid is equal to the half sum of its parallel bases multiplied by its altitude.
4. Prove that the side of a regular hexagon inscribed in a circle is equal to its radius.
5. The radius of a circle equals 10. Find its area.
6. The perpendicular dropped from the vertex of the right angle upon the hypothenuse divides it into two segments of 9 and 16 feet respectively. Find the lengths of the perpendicular, and the two legs of the triangle.
7. Define similar polygons. To what are their areas proportional?

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SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2016: 42(1), p. 28–30.

4101. *Proposed by Max Alekseyev.*

Let n be an integer such that $3^n \equiv 7 \pmod{n}$. Show that 127 cannot divide n.

There were two correct solutions submitted and two incorrect submissions. We present the solution by Max Alekseyev.

Suppose there exists an n satisfying the given congruence such that $127|n$. Then we have

$$3^n \equiv 7 \pmod{127}.$$

3 is a primitive root modulo 127, so the above congruence holds if and only if

$$n \equiv 115 \pmod{126}$$

and in particular n is odd.

Since n is odd, 3^{n+1} is a square, so

$$3^{n+1} \equiv 21 \pmod{n}$$

is a square modulo n.

Using the Jacboi symbol, we have that $\left(\frac{21}{n}\right) = 1$ and by quadratic reciprosity:

$$\left(\frac{n}{21}\right) = (-1)^{\frac{21-1}{2}\frac{n-1}{2}} \left(\frac{21}{n}\right) = 1.$$

However,

$$n \equiv 115 \pmod{126} \Rightarrow n \equiv 10 \pmod{21},$$

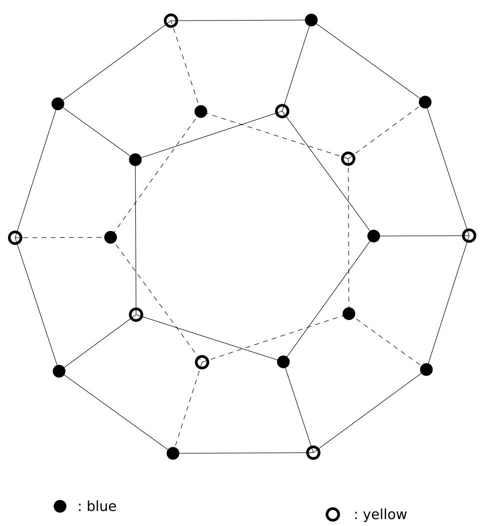
and 10 is not a square modulo 21, so $\left(\frac{n}{21}\right) = -1$ which gives us a contradiction. Thus, 127 cannot divide n.

4102. *Proposed by Kimberly D. Apple and Eugen J. Ionascu.*

Suppose the faces of a regular icosahedron are coloured with blue or yellow in such a way that every blue face shares an edge with at most one other blue face. What is the maximum possible number of blue faces?

We received four correct solutions, of which we present the solution by Michel Bataille.

We show that the maximal number of blue faces is 12. For convenience, we shall use the dodecahedral graph $\mathcal{D}$ whose vertices correspond to the faces of the icosahedron, two vertices being linked by an edge when the corresponding faces are adjacent. A coloring in blue or yellow of the faces of the icosahedron corresponds to a coloring in blue or yellow of the vertices of $\mathcal{D}$. Each of the 20 vertices of $\mathcal{D}$ has degree 3 (see figure) and the coloring will respect the constraint of the problem if and only if at most one blue vertex is adjacent to each blue vertex. A suitable coloring with 12 blue vertices is presented on the figure.



Now, we show that a suitable coloring cannot have more than 12 blue vertices. Each of the twelve faces of the dodecahedron has five vertices of which three at most are blue vertices (otherwise at most one vertex would be yellow so that a blue vertex would necessarily be adjacent to two blue vertices). If we count the blue vertices by adding the blue vertices obtained face after face, this provides a totality of at most 3×12 blue vertices. However, each vertex of $\mathcal{D}$ is a vertex of exactly three faces, hence each of the blue vertices counted just before is counted three times. Thus, we actually have at most 12 blue vertices.

4103. *Proposed by Dan Stefan Marinescu, Leonard Giugiuc and Daniel Sitaru.*

Let x, y and z be positive numbers such that $x + y + z = 1$. Show that

$$\sum_{\text{cyc}}[(1-x)\sqrt{3yz(1-y)(1-z)}] \geq 4\sqrt{xyz}.$$

There were six correct solutions. Three of the solutions used algebraic inequalities while the remaining three employed trigonometry. We present three different solutions here.

Solution 1, by Kee-Wai Lau.

The inequality is equivalent to

$$\sum\left[(1-x)\sqrt{\frac{x+yz}{x}}\right]\geq\frac{4}{\sqrt{3}},$$

or

$$\sum\sqrt{1+\frac{yz}{x}}\geq\frac{4}{\sqrt{3}}+\sum\sqrt{x(x+yz)},$$

where each sum is cyclic with three terms. Thus it suffices to show that

$$\sum\sqrt{1+\frac{yz}{x}}\geq 2\sqrt{3} \tag{1}$$

and

$$\sum\sqrt{x(x+yz)}\leq\frac{2}{\sqrt{3}}. \tag{2}$$

For (1), we have that

$$\begin{aligned}
&\left(\sum\sqrt{1+\frac{yz}{x}}\right)^2\\
&=\sum\left(1+\frac{yz}{x}\right)+2\sum\sqrt{\left(1+\frac{yz}{x}\right)\left(1+\frac{zx}{y}\right)}\\
&=3+\sum\frac{yz}{x}+2\left(\sum\sqrt{(1+z)^2+\frac{z(x-y)^2}{xy}}\right)\\
&=3+\left(\sum x+\frac{1}{2xyz}\sum x^2(y-z)^2\right)+2\left(\sum\sqrt{(1+z)^2+\frac{z(x-y)^2}{xy}}\right)\\
&\geq 4+2\sum(1+z)\\
&=12.
\end{aligned}$$

For (2), the Cauchy-Schwarz inequality implies that

$$\begin{aligned}
\left(\sum\sqrt{x(x+yz)}\right)^2&\leq\left(\sum x\right)\left(\sum(x+yz)\right)\\
&=1+\sum yz\\
&=1+\frac{1}{6}\left[2(x+y+z)^2-(x-y)^2-(y-z)^2-(z-x)^2\right]\\
&\leq 1+\frac{1}{6}\cdot 2=\frac{4}{3}.
\end{aligned}$$

The result follows, with equality occurring iff $x=y=z=1/3$.

Solution 2, by AN-anduud Problem Solving Group.

There exists a triangle ABC with sides of lengths $a = y + z$, $b = z + x$, $c = x + y$. Let R be its circumradius and s its semi-perimeter; $s = x + y + z = 1$. The inequality is equivalent to

$$\sqrt{3}\left[a\sqrt{\frac{bc}{s(s-a)}} + b\sqrt{\frac{ca}{s(s-b)}} + c\sqrt{\frac{ab}{s(s-c}}\right] \geq 2(a+b+c).$$

Noting that $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$, $a = 2R\sin A = 4R\cos\frac{A}{2}\sin\frac{A}{2}$, *etc.*, we have to establish that

$$4\sqrt{3}R\left(\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}\right) \geq 8R\left(\cos\frac{A}{2}\sin\frac{A}{2} + \cos\frac{B}{2}\sin\frac{B}{2} + \cos\frac{C}{2}\sin\frac{C}{2}\right).$$

Recall that

$$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \leq \frac{3\sqrt{3}}{2}$$

(by Jensen's Theorem, for example) and that

$$3(u_1v_1 + u_2v_2 + u_3v_3) \leq (u_1 + u_2 + u_3)(v_1 + v_2 + v_3)$$

when $u_1 \geq u_2 \geq u_3 > 0$ and $0 < v_1 \leq v_2 \leq v_3$ (Chebyshev's sum inequality). Since the cosine and sine functions are monotonely opposite on $(0, \pi/2)$.

$$\begin{aligned}
&8R(\cos\frac{A}{2}\sin\frac{A}{2} + \cos\frac{B}{2}\sin\frac{B}{2} + \cos\frac{C}{2}\sin\frac{C}{2}) \\
&\leq \frac{8R}{3}\left(\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\right)\left(\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}\right) \\
&\leq 4\sqrt{3}R\left(\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}\right),
\end{aligned}$$

as desired.

Solution 3, by Digby Smith.

We first establish that, for $x, y, z > 0$,

$$\sum \frac{x(y+z)}{(x+y)(x+z)} \leq \frac{3}{2},$$

with equality iff $x = y = z$. (The sum is cyclic with three terms.) This follows from

$$\begin{aligned}
&3(x+y)(y+z)(z+x) - 2[x(y+z)^2 + y(z+x)^2 + z(x+y)^2] \\
&= 3[x(y^2+z^2) + y(z^2+x^2) + z(x^2+y^2) + 2xyz] \\
&\quad - 2[x(y^2+z^2) + y(z^2+x^2) + z(x^2+y^2) + 6xyz] \\
&= x(y^2+z^2) + y(z^2+x^2) + z(x^2+y^2) - 6xyz \\
&\geq 2xyz + 2xyz + 2xyz - 6xyz = 0.
\end{aligned}$$

Because $x+y+z=1$, the inequality is equivalent to

$$\left(\sum(y+z)\sqrt{\frac{(x+y)(x+z)}{x}}\right)^2 \geq \frac{16}{3}.$$

Recall the Hölder inequality with u_i and v_i nonnegative for $i=1,2,3$:

$$\left(\sum_{i=1}^{3} u_i^3\right)^{1/3}\left(\sum_{i=1}^{3} v_i^{3/2}\right)^{2/3} \geq \sum_{i=1}^{3} u_i v_i.$$

Applying this to the triples

$$(u_1,u_2,u_3)=\left(\left(\frac{x(y+z)}{(x+y)(x+z)}\right)^{1/3},\left(\frac{y(z+x)}{(y+z)(y+x)}\right)^{1/3},\left(\frac{z(x+y)}{(z+x)(z+y)}\right)^{1/3}\right)$$

and (v_1,v_2,v_3)

$$=\left(\left((y+z)\sqrt{\frac{(x+y)(x+z)}{x}}\right)^{\frac{2}{3}},\left((z+x)\sqrt{\frac{(y+z)(y+x)}{y}}\right)^{\frac{2}{3}},\left((x+y)\sqrt{\frac{(z+x)(z+y)}{z}}\right)^{\frac{2}{3}}\right),$$

and using the preliminary result, we find that

$$\begin{aligned}
&\frac{3}{2}\left(\sum(y+z)\sqrt{\frac{(x+y)(x+z)}{x}}\right)^2\\
&\geq\left(\sum\frac{x(y+z)}{(x+y)(x+z)}\right)\left(\sum(y+z)\sqrt{\frac{(x+y)(x+z)}{x}}\right)^2\\
&\geq\left(\sum(y+z)\right)^3=8.
\end{aligned}$$

The desired inequality follows directly.

4104. *Proposed by Daniel Sitaru.*

Prove that for $0<a\leq b\leq c\leq d<2$, we have

$$5(ab^4+bc^4+cd^4+16d)<5(b^5+c^5+d^5+16a)+128.$$

There were five correct solutions. We present two different ones here.

Solution 1, by Šefket Arslanagić; and Salem Malikić (independently).

By the arithmetic-geometric means inequality, we have

$$a^5+4b^5\geq 5ab^4,\quad b^5+4c^5\geq 5bc^4,\quad c^5+4d^5\geq 5cd^4$$

and

$$d^5+128=d^5+4\cdot 2^5\geq 5\cdot 2^4 d=80d.$$

Adding these along with the positive $4a^5$ yields that

$$5(a^5 + b^5 + c^5 + d^5) + 128 > 5(ab^4 + bc^4 + cd^4 + 16d).$$

Solution 2, by the proposer.

The function $f(x) = 16 - x^4$ is nonnegative, decreasing and concave on $[0, 2]$. Therefore

$$(b - a)f(b) + (c - b)f(b) + (d - c)f(d) < \int_0^2 f(x)dx.$$

Hence

$$16(d - a) - (b^5 + c^5 + d^5) + (ab^4 + bc^4 + cd^4) < 128/5.$$

Multiplying by 5 and rearranging the terms gives the result.

4105. *Proposed by Mihaela Berindeanu.*

Let ABC be a triangle with centroid G. Let A', B' and C' be the feet of altitudes on the sides of the triangle from the vertices A, B and C, respectively. Let G' be the centroid of $A'B'C'$. If $GG' \parallel BC$, find all possible values of angle A.

We received five submissions, of which four were correct. We present Michel Bataille's solution supplemented by ideas from the similar solution of Steven Chow.

We shall see that A can take all values between $\frac{\pi}{3}$ and $\frac{\pi}{2}$. In barycentric coordinates relative to (A, B, C), $G = (1 : 1 : 1)$ and the point at infinity on BC is $(0 : 1 : -1)$, hence the equation of the parallel ℓ to BC through G is $2x = y + z$. On the other hand, we have

$$\begin{aligned} aA' &= (b \cos C)B + (c \cos B)C, \\ bB' &= (a \cos C)A + (c \cos A)C, \\ cC' &= (a \cos B)A + (b \cos A)B, \end{aligned}$$

hence

$$\begin{aligned} &(3abc)G' \\ &= abc(A' + B' + C') \\ &= a^2(b \cos B + c \cos C)A + b^2(a \cos A + c \cos C)B + c^2(a \cos A + b \cos B)C. \end{aligned}$$

When $G \neq G'$, the condition $GG' \parallel BC$ is equivalent to $G' \in \ell$; that is,

$$\begin{aligned} 2a^2(b \cos B + c \cos C) &= b^2(a \cos A + c \cos C) + c^2(a \cos A + b \cos B) \\ &= a(b^2 + c^2) \cos A + bc(b \cos C + c \cos B). \end{aligned}$$

Because $a = b \cos C + c \cos B$, this equation reduces to

$$(b^2 + c^2) \cos A + bc = 2ab \cos B + 2ac \cos C,$$

or equivalently (using the Law of Cosines),

$$(b^2+c^2)\frac{b^2+c^2-a^2}{2bc}+bc=2ab\frac{a^2+c^2-b^2}{2ac}+2ac\frac{a^2+b^2-c^2}{2ab},$$

which reduces to

$$a^2(b^2+c^2)=(b^2+c^2)^2-2b^2c^2,$$

so that

$$a^2=b^2+c^2-\frac{2b^2c^2}{b^2+c^2}.$$

But $b^2+c^2-a^2=2bc\cos A$, so we conclude finally that $GG'\parallel BC$ (when $G\neq G'$) is equivalent to

$$\cos A=\frac{bc}{b^2+c^2}.$$

Because $b^2+c^2\geq 2bc$, $\cos A$ must lie between 0 and $\frac{1}{2}$. From the geometry the sides b and c cannot be zero; nor can $\cos A=1/2$ since that would force $b=c$ (in which case the triangle would necessarily be equilateral and the line GG' would be undefined). We thus conclude that $\frac{\pi}{3}<A<\frac{\pi}{2}$.

Conversely, suppose that $\frac{\pi}{3}<A<\frac{\pi}{2}$. Since $x\mapsto\frac{x}{1+x^2}$ is a bijection from $(0,1)$ to $(0,\frac{1}{2})$, for any value of $\cos A$ between 0 and $\frac{1}{2}$ there exists an $\alpha\in(0,1)$ for which $\cos A=\frac{\alpha}{1+\alpha^2}$. We can then construct a triangle ABC with a given angle A between $\frac{\pi}{3}$ and $\frac{\pi}{2}$ such that $\frac{b}{c}=\alpha$. Then

$$\cos A=\frac{\frac{b}{c}}{1+\frac{b^2}{c^2}}=\frac{bc}{b^2+c^2},$$

so that $GG'\parallel BC$ holds.

4106. *Proposed by D.M. Bătineţu-Giurgiu and Neculai Stanciu.*

Let ABC be a triangle with $BC=a$, $AC=b$, $AB=c$ and circumradius R. Show that

$$\frac{b+c}{a^5}+\frac{c+a}{b^5}+\frac{a+b}{c^5}\geq\frac{2}{3R^4}.$$

We received 14 submissions all of which are correct. We present two solutions, the second of which gives a stronger result.

Solution 1, by Marija Milošević.

Let L denote the left side of the given inequality. Then by the AM-GM inequality we have

$$L\geq 2\left(\frac{\sqrt{bc}}{a^5}+\frac{\sqrt{ca}}{b^5}+\frac{\sqrt{ab}}{c^5}\right)=\frac{2((bc)^{\frac{11}{2}}+(ca)^{\frac{11}{2}}+(ab)^{\frac{11}{2}})}{(abc)^5}$$
$$\geq\frac{2\cdot 3\sqrt[3]{(abc)^{11}}}{(abc)^5}=\frac{6}{(abc)^{\frac{4}{3}}}.$$

Since it is known (cf. item 5.27 on p.55 of *Geometric Inequalities* by O. Bottema, et al, Wolters Noordhoff, Groningen, 1969.) that $abc \le (\sqrt{3}R)^3$, we then have

$$L \ge \frac{6}{(\sqrt{3}R)^4} = \frac{2}{3R^4}.$$

Solution 2, by Titu Zvonaru.

Without loss of generality, assume that $a \le b \le c$. Then

$$\frac{1}{a} \ge \frac{1}{b} \ge \frac{1}{c} \quad \text{and} \quad b+c \ge c+a \ge a+b.$$

Let s and r denote the semiperimeter and the inradius of ΔABC, respectively.

Using Chebyshev's Inequality, the trivial fact that $x^2+y^2+z^2 \ge xy+yz+zx$ (applied to $\frac{1}{a}$, $\frac{1}{b}$, and $\frac{1}{c}$), the AM-GM inequality, and the well known identity $abc = 4Rrs$ we obtain:

$$\begin{aligned}
\frac{b+c}{a^5} + \frac{c+a}{b^5} + \frac{a+b}{c^5} &\ge \frac{1}{3}(b+c+c+a+a+b)\left(\frac{1}{a^2}\cdot\frac{1}{a^3} + \frac{1}{b^2}\cdot\frac{1}{b^3} + \frac{1}{c^2}\cdot\frac{1}{c^3}\right)\\
&\ge \frac{4s}{9}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)\\
&\ge \frac{4s}{9}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)\left(\frac{3}{abc}\right)\\
&= \frac{8s^2}{3(abc)^2} = \frac{8s^2}{3(4Rrs)^2} = \frac{1}{6R^2r^2}. \qquad (1)
\end{aligned}$$

Note that (1) is stronger than the given inequality since by Euler's Inequality, we have $2r \le R$ so

$$\frac{1}{6R^2r^2} \ge \frac{1}{6R^2(\frac{R}{2}^2)} = \frac{2}{3R^4}.$$

Note also that equality holds if and only if $a = b = c$.

4107. *Proposed by Lorian Saceanu.*

Let ABC be an acute triangle with inradius r, circumradius R and semiperimeter s. Prove that

$$\sqrt{\left(\frac{9}{4}+\frac{r}{2R}\right)^2 + \frac{r}{R}} \le \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \le \frac{s}{\frac{R}{2}+r}.$$

We received no submissions apart from the proposer's original solution which we, after modification and expansions, present below.

Let $I, r, R,$ and s denote the incentre, inradius, circumradius, and the semiperimeter of ΔABC, respectively. We also let $L, M,$ and N denote the points where AI, BI, and CI intersect the circumcircle of ΔABC, respectively. Further, let q denote the radius of the incircle of ΔLMN.

The following identities are known:

$$\sin\left(\frac{A}{2}\right)+\sin\left(\frac{B}{2}\right)+\sin\left(\frac{C}{2}\right)=1+\frac{q}{R}, \tag{1}$$

$$\cos\left(\frac{A}{2}\right)+\cos\left(\frac{B}{2}\right)+\cos\left(\frac{C}{2}\right)=1+\frac{s}{2q}. \tag{2}$$

We first establish the following inequality:

$$4q \geq R+2r. \tag{3}$$

To do this, we first start with Popoviciu's Inequality which states that:

if $f(x): J \to \mathbb{R}$ is a concave function, then for all $a,b,c \in J$,

$$2\sum_{\text{cyc}} f\left(\frac{a+b}{2}\right) \geq \sum_{\text{cyc}} f(a) + 3f\left(\frac{a+b+c}{3}\right). \tag{4}$$

Now, let $f(x) = \cos x$ where $x \in \left(0, \frac{\pi}{2}\right)$. Then since $f''(x) = -\cos x < 0$, f is concave on $\left(0, \frac{\pi}{2}\right)$. We have

$$f(a)=\cos A, \quad f(b)=\cos B, \quad f(c)=\cos C, \quad f\left(\frac{a+b+c}{3}\right)=\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}.$$

Furthermore,

$$f\left(\frac{a+b}{2}\right)=\cos\left(\frac{A+B}{2}\right)=\sin\left(\frac{C}{2}\right),\ f\left(\frac{b+c}{2}\right)=\sin\left(\frac{A}{2}\right),\ f\left(\frac{c+a}{2}\right)=\sin\left(\frac{B}{2}\right).$$

Then (4) implies

$$2\sum_{\text{cyc}}\sin\left(\frac{A}{2}\right) \geq \left(\sum_{\text{cyc}}\cos A\right)+\frac{3}{2} \tag{5}$$

Using (5) and the familiar fact that

$$\sum_{\text{cyc}}\cos A = 1+4\prod_{\text{cyc}}\left(\sin\left(\frac{A}{2}\right)\right)=1+\frac{r}{R}.$$

We then have

$$\sum_{\text{cyc}}\sin\left(\frac{A}{2}\right) \geq \frac{1}{2}+\frac{r}{2R}+\frac{3}{4}=\frac{5}{4}+\frac{r}{2R} \tag{6}$$

From (1) and (6) we have $1+\frac{q}{R} \geq \frac{5}{4}+\frac{r}{2R}$ or $q \geq \frac{R}{4}+\frac{r}{2}$, so $4q \geq R+2r$, which proves (3).

Using (2) and (4) we then have

$$\sum_{\text{cyc}}\cos\left(\frac{A}{2}\right)=\frac{s}{2q} \leq \frac{s}{\frac{R}{2}+r}$$

so the right half of the given inequality is true.

To prove the left half of the given inequality, we need the following results:

$$a^2+b^2+c^2 = 2(s^2-4Rr-r^2) \tag{7}$$

$$\sum_{\text{cyc}} \cos^2\left(\frac{A}{2}\right) = 2+\frac{r}{2R}. \tag{8}$$

Identity (7) is known and is available in the literature. To prove (8), we use (7) and the well known facts that $\cos^2\left(\frac{A}{2}\right)=\frac{s(s-a)}{bc}$ and $abc=4Rrs$ to obtain

$$\begin{aligned}\sum_{\text{cyc}} \cos^2\left(\frac{A}{2}\right) &= \sum_{\text{cyc}} \frac{s(s-a)}{bc} = \frac{1}{abc}\sum_{\text{cyc}} a(s-a)\\ &= \frac{s}{abc}\left[s(a+b+c)-(a^2+b^2+c^2)\right] = \frac{s}{abc}\left[2s^2-2(s^2-4Rr-r^2)\right]\\ &= \frac{s}{abc}(8Rr+2r^2) = 2+\frac{r}{2R}.\end{aligned}$$

Now we let a', b', c', r', R', and s' denote the quantities associated with ΔLMN which correspond to a, b, c etc. associated with ΔABC.

Then

$$a' = 2R\cos\left(\frac{A}{2}\right),\ b' = 2R\cos\left(\frac{B}{2}\right),\ c' = 2R\cos\left(\frac{C}{2}\right),\ r'=q,\ R'=R,$$

and $s' = R\sum_{\text{cyc}}\cos\left(\frac{A}{2}\right)$, so from (7) we obtain

$$4R^2\sum_{\text{cyc}}\cos^2\left(\frac{A}{2}\right) = 2\left[R^2\left(\sum_{\text{cyc}}\cos\left(\frac{A}{2}\right)\right)^2 - 4Rq - q^2\right].$$

Hence from (8) we obtain, in sequence,

$$\begin{aligned}&4R^2\left(2+\frac{r}{2R}\right) = 2\left[R^2\left(\sum_{\text{cyc}}\cos\left(\frac{A}{2}\right)\right)^2 - 4Rq - q^2\right],\\ &R^2\left(\sum_{\text{cyc}}\cos\left(\frac{A}{2}\right)\right)^2 = 4R^2+Rr+4Rq+q^2,\\ &\left(\sum_{\text{cyc}}\cos\left(\frac{A}{2}\right)\right)^2 = 4+\frac{r}{R}+\frac{4q}{R}+\frac{q^2}{R^2} = \left(2+\frac{q}{R}\right)^2+\frac{r}{R}.\end{aligned} \tag{9}$$

Finally, since $4q \geq R+2r$ by (3) we have

$$\left(2+\frac{q}{R}\right)^2+\frac{r}{R} \geq \left(2+\frac{R+2r}{4R}\right)^2+\frac{r}{R} = \left(\frac{9}{4}+\frac{r}{2R}\right)^2+\frac{r}{R}. \tag{10}$$

From (9) and (10), it follows that

$$\sqrt{\left(\frac{9}{4}+\frac{r}{2R}\right)^2+\frac{r}{R}} \le \sum_{\text{cyc}} \cos\left(\frac{A}{2}\right),$$

which completes the proof.

4108. *Proposed by Alessandro Ventullo.*

a) Write 2010 as a sum of consecutive squares.

b) Is it possible to write 2014 as the sum of several consecutive squares?

We received 15 submissions, out of which 14 were correct. We present the solution by Joel Schlosberg.

For part a), we have $2010 = 18^2 + 19^2 + 20^2 + 21^2 + 22^2$.

For part b), suppose that 2014 is the sum of n consecutive squares, that is

$$2014 = s^2 + (s+1)^2 + \cdots + (s+n-1)^2 = ns^2 + (n-1)ns + \frac{1}{6}(n-1)n(2n-1).$$

for some $s \in \mathbb{N}$ [Ed.: As some readers pointed out, we could even allow squares of negative numbers, that is $s \in \mathbb{Z}$]. Then n divides both

$$6 \cdot 2014 = 2^2 \cdot 3 \cdot 19 \cdot 53 \quad \text{and} \quad 2014 - \frac{1}{6}(n-1)n(2n-1).$$

Since any sum of 18 or more consecutive squares is at least $1^2+2^2+\cdots+18^2 > 2014$ we conclude $n \le 17$, which leaves us with $n \in \{1, 2, 3, 4, 6, 12\}$. As 2014 is not a perfect square, $n \ne 1$. If $n \in \{2, 3, 6, 12\}$, then

$$n \nmid 2014 - \frac{1}{6}(n-1)n(2n-1).$$

For the only remaining possibility, $n = 4$, we obtain

$$2014 = 4s^2 + 12s + 14 \quad \text{or} \quad (2s+3)^2 = 2009,$$

which is impossible as 2009 is not a perfect square. Thus 2014 cannot be written as the sum of consecutive squares.

4109. *Proposed by Mehtaab Sawhney.*

Let k and n be positive integers. Compute the following sum in closed form:

$$\sum_{r=1}^{k}\sum_{\ell=r}^{k}(-1)^{k-r}\binom{k}{r}\binom{nr}{k+\ell}\binom{k-r}{k-\ell}n^{k-\ell}.$$

We received no submissions to this problem besides the original solution by the proposer which we present below.

Let S denote the given double summation. Following common practice, we let $[x^6]f(x)$ denote the coefficient of the term x^6 in the polynomial $f(x)$. Then

$$\begin{aligned} S &= \sum_{r=0}^{k}\binom{k}{r}\sum_{l=r}^{k}(-1)^{k-r}\binom{nr}{k+l}\binom{k-r}{k-l}n^{k-l} \\ &= \sum_{r=0}^{k}\binom{k}{r}\sum_{l=r}^{k}(-1)^{k-r}\left([x^{k+l}](1+x)^{nr}\right)\left([x^{k-l}](1+nx)^{k-r}\right) \\ &= \sum_{r=0}^{k}\binom{k}{r}\left([x^{2k}](-1)^{k-r}\left((1+x)^{nr}(1+nx)^{k-r}\right)\right) \\ &= [x^{2k}]\left((1+x)^n-(1+nx)\right)^k \\ &= [x^{2k}]\left(\binom{n}{2}x^2+0(x^3)\right)^k = \binom{n}{2}^k. \end{aligned}$$

4110. *Proposed by Michel Bataille.*

Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$xf(x+y)=(x+y)f\left(\frac{x}{y}f(y)\right)$$

for all real numbers x, y with $y\neq 0$.

We received four solutions. We present the solution by Kee-Wai Lau.

We show that either $f(x)=0$ or $f(x)=x$ for all real numbers x. Denote the given functional equation by (*). By putting $x=0$ and $y=1$ into (*) we see that $f(0)=0$. Consider two cases:

Case 1: $f(s)=0$ for some real $s\neq 0$. Substitute $y=s$ into (*) to obtain $xf(x+s)=0$ for all real numbers x. Hence $f(x+s)=0$ for all non-zero real numbers x; combined with $f(s)=0$ this gives us $f(x)=0$ for all $x\in\mathbb{R}$.

Case 2: $f(x)\neq 0$ for all $x\neq 0$. In this case we will show that $f(x)=x$ for all x. Suppose on the contrary that there exists some non-zero t such that $f(t)\neq t$. Consider $x=\frac{t^2}{f(t)-t}$ and $y=t$; then $x+y=\frac{tf(t)}{f(t)-t}$, and note that $x+y\neq 0$. Substituting these values for x and y into (*) we get

$$\frac{t^2}{f(t)-t}\cdot f\left(\frac{tf(t)}{f(t)-t}\right)=\frac{tf(t)}{f(t)-t}\cdot f\left(\frac{tf(t)}{f(t)-t}\right).$$

Since f is only zero at zero we can cancel common factors to obtain $t=f(t)$, contradicting our choice of t.

Therefore the only functions which satisfy (*) are the zero function and the identity function.

AUTHORS' INDEX

Solvers and proposers appearing in this issue

(Bold font indicates featured solution.)

Proposers

Solvers - individuals

Solvers - collaborations