Crux Mathematicorum

VOLUME 42, NO. 5 May / Mai 2016

IN THIS ISSUE / DANS CE NUMÉRO

Crux Mathematicorum

Founding Editors / Rédacteurs-fondateurs: Léopold Sauvé & Frederick G.B. Maskell

Former Editors / Anciens Rédacteurs: G.W. Sands, R.E. Woodrow, Bruce L.R. Shawyer, Shawn Godin

Crux Mathematicorum with Mathematical Mayhem

Former Editors / Anciens Rédacteurs: Bruce L.R. Shawyer, James E. Totten, Václav Linek, Shawn Godin

EDITORIAL

Oral exams – yea or nay? They are a standard practice in many European countries. In North America, however, this format seems to be reserved for PhD defences. One explanation might be big class sizes of undergraduate universities, which make it infeasible to administer individual oral tests; another – potential lack of consistency and accountability on the part of the examiner as they can choose to save students with helpful comments or to sink them with killer questions. This latter impartiality was a point of contention in USSR for years with circulating claims of discrimination against various groups. The biggest arguments surrounded admissions of Moskovites versus non-Moscovites as well as Jewish versus non-Jewish students. An article on this topic that drew my attention was by Vershik and Shen (*The Mathematical Intelligencer* (1994), 16: 4) as it includes examples of killer questions used by examiners entrance exams to Mechanical-Mathematical Faculty of the Moscow State University in the 1980s. Here are some of them. Do you think you would have been able to answer these?

1. The faces of a triangular pyramid have the same area. Show that they are congruent.

2. Draw a straight line that halves the area and the perimeter of a triangle.

3. Show that

$$\frac{1}{\sin^2 x} \leq \frac{1}{x^2} + 1 - \frac{4}{\pi^2}$$

for $0 < x \leq \frac{\pi}{2}$.

4. Let K be a point on the base AB of a trapezoid $ABCD$. Find a point M on the base CD that maximizes the area of the quadrangle formed by the intersection of triangles AMB and CDK.

5. Compare $\log_3 4 \cdot \log_3 6 \cdot \ldots \cdot \log_3 80$ and $2 \log_3 3 \cdot \log_3 5 \cdot \ldots \cdot \log_3 79$.

6. Given k segments on the plane, give an upper bound on the number of triangles all of whose sides belong to the given set of segments.

I think these make for excellent challenge problems. If I find a particularly elegant solution to any of the above, I will consider using them in my courses.

Kseniya Garaschuk

P. S. My editors recognized some of these problems as appearing in Canadian sources before. For example, Problem 1 was mentioned in the solution to ***Crux*** problem **478**, which was published as an open problem and later settled by Tomasz Ciesla (***Crux*** 38(2), p. 68–70). Also, Problem 2 is discussed here `http://math.fau.edu/Yiu/GJARCMG2016.pdf` and here `http://mathcentral.uregina.ca/mp/previous2011/apr12sol.php`. Finally, all the killer problems from Vershik and Shen's article have been analyzed by Ilan Vardi in his 1999 essay, which you can find here `http://www.lix.polytechnique.fr/Labo/Ilan.Vardi/mekh-mat.ps`

THE CONTEST CORNER

No. 45

John McLoughlin

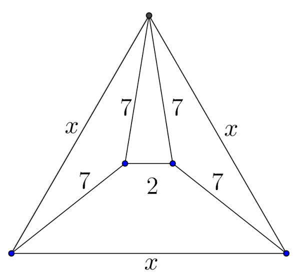
The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **February 1, 2017**.

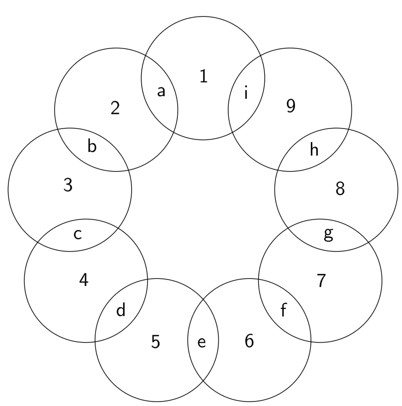
The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

CC221. What is the smallest positive integer n such that if S is any set containing n or more integers, then there must be three integers in S whose sum is divisible by 3?

CC222. What is the value of x in the plane figure shown?

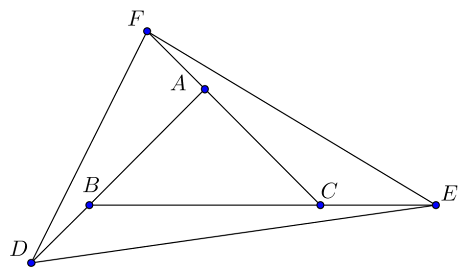


CC223. The letters a, b, c, d, e, f, g, h and i in the figure below represent the numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9 in a certain order. In each of the nine circles, we sum the three numbers so that nine sums are obtained. Suppose that all nine sums are equal. What is the value of $a + d + g$?



CC224. What is the smallest positive integer n such that 31 divides $5^n + n$?

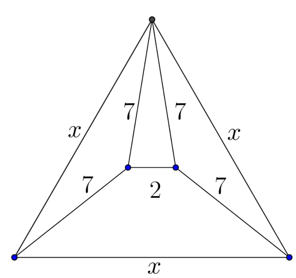
CC225. The three sides of triangle ABC are extended as shown so that $BD = \frac{1}{2}AB, CE = \frac{1}{2}BC$ and $AF = \frac{1}{2}CA$. What is the ratio of the area of triangle DEF to that of triangle ABC?



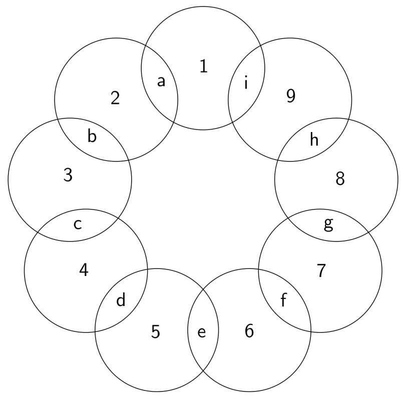
. . . .

CC221. Quel est le plus petit entier strictement positif n pour lequel étant donné n'importe quel ensemble S contenant au moins n entiers, S doit contenir trois entiers dont la somme est divisible par 3?

CC222. Quelle est la valeur de x dans la figure plane suivante?

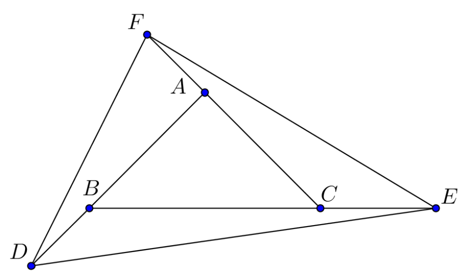


CC223. Les lettres a, b, c, d, e, f, g, h et i dans la figure suivante représentent les nombres 1, 2, 3, 4, 5, 6, 7, 8 et 9 dans un ordre quelconque. Dans chacun des neuf cercles on additionne les trois nombres contenus dans le cercle. On obtient ainsi neuf sommes. Sachant que les neuf sommes sont égales, quelle est la valeur de $a + d + g$?



CC224. Quel est le plus petit entier positif n pour lequel $5^n + n$ est divisible par 31?

CC225. Les côtés du triangle ABC sont prolongés comme dans la figure suivante de manière que $BD = \frac{1}{2}AB, CE = \frac{1}{2}BC$ et $AF = \frac{1}{2}CA$. Quel est le rapport de l'aire du triangle DEF à celle du triangle ABC?



CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2015: 41(5), p. 192–193.

CC171. The zeroes of the polynomial $f(x) = x^2 - ax + 2a$ are integers. What is the sum of all the possible values of the number a?

Originally Problem 21 from the 2015 level 4 phase 2 of the Primavera contest in Spain.

We received nine submissions of which eight were correct and complete. We present two solutions.

Solution 1, by Henry Ricardo.

If r_1 and r_2 are the roots of f, we may write $x^2 - ax + 2a = (x - r_1)(x - r_2) = x^2 - (r_1 + r_2)x + r_1 r_2$, so that (1) $r_1 + r_2 = a$ and (2) $r_1 r_2 = 2a$. Substituting (2) in (1), we get $r_1 + r_2 = (r_1 r_2)/2$, or $r_1(2 - r_2) + 2r_2 = 0$. Subtracting 4 from each side of the last equation gives us $r_1(2 - r_2) + 2r_2 - 4 = -4$, or $(2 - r_1)(2 - r_2) = 4$. It is easy to verify that the only (unordered) pairs (r_1, r_2) satisfying this equation are $(0, 0), (1, -2), (3, 6)$, and $(4, 4)$, yielding $a = 0, -1, 9$, and 8, respectively, so that the sum of the values of a is 16.

Solution 2, by Titu Zvonaru.

The discriminant of the quadratic equation $x^2 - ax + 2a = 0$ is $a^2 - 8a$. Since the roots are integers, let k be an integer such that

$$a^2 - 8a = k^2 \iff (a - 4)^2 = k^2 + 16 \iff (a - 4 - k)(a - 4 + k) = 16.$$

As $(a - 4 - k) + (a - 4 + k) = 2(a - 4)$, the numbers $a - 4 - k$ and $a - 4 + k$ have the same parity. We have the possibilities:

(i) $a - 4 - k = -8$, $a - 4 + k = -2 \implies a = -1$. The equation $x^2 + x - 2 = 0$ has the roots 1 and -2;

(ii) $a - 4 - k = -2, a - 4 + k = -8 \implies a = -1$;

(iii) $a - 4 - k = -4$, $a - 4 + k = -4 \implies a = 0$. The equation $x^2 = 0$ has the root 0;

(iv) $a - 4 - k = 2$, $a - 4 + k = 8 \implies a = 9$. The equation $x^2 - 9x + 18 = 0$ has the roots 3 and 6;

(v) $a - 4 - k = 8, a - 4 + k = 2 \implies a = 9$;

(vi) $a - 4 - k = 4$, $a - 4 + k = 4 \implies a = 8$. The equation $x^2 - 8x + 16 = 0$ has the root 0.

Hence, the sum of all the possible values of the number a is $-1+0+9+8=16$.

CC172. What is the area of regular hexagon $ABCDEF$ with $A(0,0)$ and $C(7,1)$?

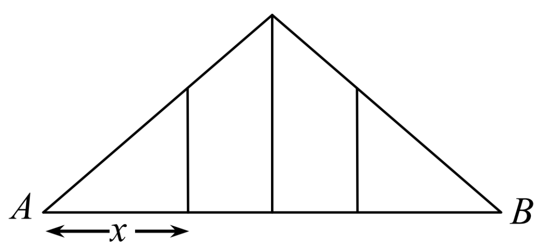
Originally Problem 11 of the 2014 level 4 phase 2 of the Primavera contest in Spain.

There were ten correct solutions for this problem and one incorrect solution. We present the solution by Lucia Ma Li and Angel Plaza.

Let d be the distance between A and C. Then $d=\sqrt{7^2+1^2}=\sqrt{50}$. Notice that d is the side length of the equilateral triangle ACE, which by Heron's Formula has area $\frac{d^2}{4}\sqrt{3}=\frac{25\sqrt{3}}{2}$.

Let O be the centre of the hexagon. Then OD bisects CE and so triangles ODE and CDE have equal areas. Repeating this cyclically, we see that the area of ACE is half the area of $ABCDEF$, so the area of $ABCDEF$ is $25\sqrt{3}$.

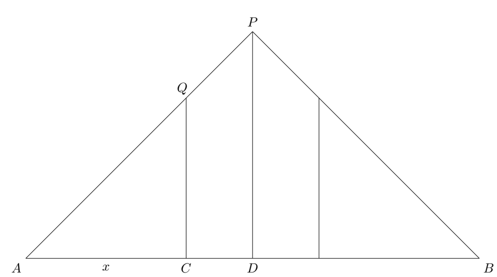
CC173. In the following figure, an isosceles triangle with $AB=12$ is divided into 4 polygons of equal area using segments perpendicular to AB. Find x.



Originally Problem 15 of the 2014 level 4 phase 2 of the Primavera contest in Spain.

We received eleven correct solutions. We present the solution of Kathleen Lewis.

First, let us label our triangle as follows:



Triangle AQC is similar to triangle APD with sides in the ratio $x:6$ and therefore areas in the ratio $x^2:36$. But AQC has the same area as the trapezoid $PQCD$, so it has half the area of triangle APD, which says that $\frac{x^2}{36}=\frac{1}{2}$. Therefore $x=3\sqrt{2}$.

CC174. Evaluate $\sqrt{1111-22}$ and $\sqrt{111111-222}$. Conjecture the result for $\sqrt{111111111111111111111111-222222222222}$ and prove it.

Originally Problem 11 of the Swiss "Mathématiques sans frontières" Épreuve de découverte 2015-16.

There were nine correct solutions for this problem. We present the solution by Titu Zvonaru.

$\sqrt{1111-22}=33$, $\sqrt{111111-222}=333$.

Let $A=\underbrace{11\ldots1}_{n}=(10^n-1)/9$.

We have $\underbrace{11\ldots1}_{2n}-\underbrace{22\ldots2}_{n}=A\cdot10^n+A-2A=A(10^n-1)=9A^2$.

The result is that $\sqrt{\underbrace{11\ldots1}_{2n}-\underbrace{22\ldots2}_{n}}=\underbrace{33\ldots3}_{n}$.

CC175. Twenty-two mathematics contests were held with five prizes given out for each one. The organizers notice that for each pair of contests, there is exactly one participant who has won a prize in both contests. Show that one of the participants has won a prize in each of the contests.

Originally Problem 4 of the Swiss preliminary contest for the Swiss Math Olympiad 2016.

We received no solutions to this problem.

THE OLYMPIAD CORNER

No. 343

Carmen Bruni

The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **February 1, 2017**.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

OC281. Find all polynomials $P(x)$ with real coefficients such that

$$P(P(x)) = (x^2 + x + 1) \cdot P(x)$$

where $x \in \mathbb{R}$.

OC282. Let x, y, z be three nonzero real numbers satisfying $x + y + z = xyz$. Prove that

$$\sum\left(\frac{x^2 - 1}{x}\right)^2 \geq 4.$$

OC283. In isosceles $\triangle ABC$, $AB = AC$, I is its incenter, D is a point inside $\triangle ABC$ such that I, B, C, D are concyclic. The line through C parallel to BD meets AD at E. Prove that $CD^2 = BD \cdot CE$.

OC284. A positive integer n is given. If there exist sets $F_1, F_2, \ldots, F_m$ satisfying the following conditions, prove that $m \leq n$.

1. For all $1 \leq i \leq m$, $F_i \subseteq \{1, 2, \cdots, n\}$
2. For all $1 \leq i < j \leq m$, $\min(|F_i - F_j|, |F_j - F_i|) = 1$

OC285. Show that from a set of 11 square integers one can select six numbers $a^2, b^2, c^2, d^2, e^2, f^2$ such that $a^2 + b^2 + c^2 \equiv d^2 + e^2 + f^2 \pmod{12}$.

. . . .

OC281. Déterminer tous les polynômes $P(x)$ à coefficients réels tels que

$$P(P(x)) = (x^2 + x + 1) \cdot P(x)$$

pour tous réels x.

OC282. Soit x, y, z trois réels non nuls tels que $x + y + z = xyz$. Démontrer que

$$\sum (\frac{x^2 - 1}{x})^2 \geq 4.$$

OC283. Soit un triangle isocèle ABC où $AB = AC$ et soit I le centre du cercle inscrit dans le triangle. Soit D un point à l'intérieur du triangle tel que I, B, C et D soient cocycliques. La droite qui passe au point C et qui est parallèle à BD coupe AD en E. Démontrer que $CD^2 = BD \cdot CE$.

OC284. Soit n un entier strictement positif. Sachant qu'il existe des ensembles $F_1, F_2, \cdots, F_m$ qui satisfont aux deux conditions suivantes, démontrer que $m \leq n$.

1. Pour tous i $(1 \leq i \leq m)$, on a $F_i \subseteq \{1, 2, \cdots, n\}$
2. Pour tous i et j $(1 \leq i < j \leq m)$, on a $\min(|F_i - F_j|, |F_j - F_i|) = 1$

OC285.

Étant donné un ensemble des carrés de 11 entiers, démontrer qu'il est possible de choisir six de ces carrés, a^2, b^2, c^2, d^2, e^2 et f^2, tels que

$$a^2 + b^2 + c^2 \equiv d^2 + e^2 + f^2 \pmod{12}.$$

OLYMPIAD SOLUTIONS

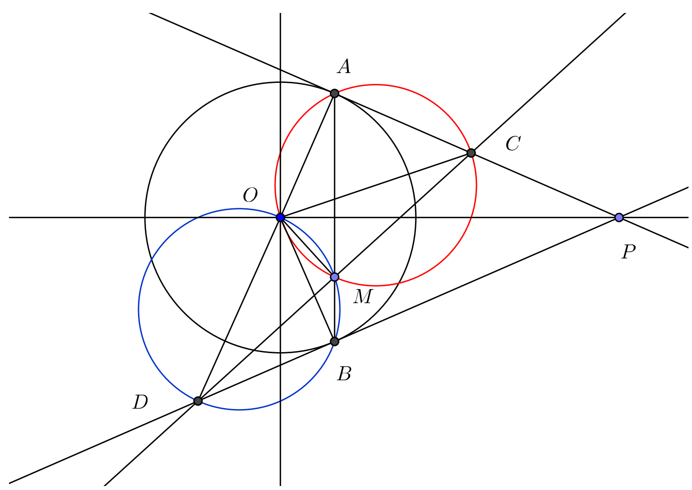
Statements of the problems in this section originally appear in 2015: 41(3), p. 101–102.

OC221. From the point P outside a circle ω with center O draw the tangents PA and PB where A and B belong to ω. In a random point M in the chord AB we draw the perpendicular to OM, which intersects PA and PB in C and D. Prove that M is the midpoint of CD.

Originally problem 3 of the 2014 Balkan Mathematical Olympiad Team Selection Test.

We received 10 correct submissions, consisting of a variety of solutions from many new readers which is fantastic!

We present the solution by Somasundaram Muralidharan, who actually gave 4 different solutions to this problem. The editor chose the shortest of the solutions to present.



Since $\angle OAC = \angle OMC = 90°$, O, A, C, M are concylic. Similarly, since $\angle OMD = \angle OBD = 90°$, the points O, M, B, D are concylic. Now,

$$\begin{aligned}\angle MOC &= \angle MAC\\ &= \angle MBP \qquad (PA, PB \text{ are tangents from an external point})\\ &= \angle MOD\end{aligned}$$

Since OM is perpendicular to CD, it follows that M is the midpoint of CD.

OC222. Let a, b be natural numbers with $ab > 2$. Suppose that the sum of their greatest common divisor and least common multiple is divisible by $a + b$. Prove that the quotient is at most $\frac{a+b}{4}$. When is this quotient exactly equal to $\frac{a+b}{4}$?

Originally problem 3 of the 2014 India National Olympiad.

We present the solution by Šefket Arslanagić. There were no other submissions.

First, if $a = b$ then $\mathrm{lcm}(a, b) = \gcd(a, b) = a$ and thus the given condition is

$$\frac{\mathrm{lcm}(a,b) + \gcd(a,b)}{a+b} = \frac{a+a}{a+a} = 1 \le \frac{a+b}{4} = \frac{a}{2}$$

which holds whenever $a \ge 2$ which must hold since $ab > 2$. Equality holds here when $a = b = 2$. Now, suppose without loss of generality that $a < b$. Now, suppose that $\gcd(a, b) = d$ and write $a = a_1 d$ and $b = b_1 d$ where a_1 and b_1 are coprime. Then $\mathrm{lcm}(a, b) = a_1 b_1 d$ and thus,

$$\frac{\mathrm{lcm}(a,b) + \gcd(a,b)}{a+b} = \frac{a_1 b_1 d + d}{a_1 d + b_1 d} = \frac{a_1 b_1 + 1}{a_1 + b_1}$$

If $b_1 = a_1 + 1$, then the above becomes

$$\frac{a_1 b_1 + 1}{a_1 + b_1} = \frac{a_1^2 + a_1 + 1}{2a_1 + 1}$$

which is a natural number. Hence, this value times 2 must also be a natural number. However,

$$\frac{2a_1^2 + 2a_1 + 2}{2a_1 + 1} = a_1 + \frac{a_1 + 2}{2a_1 + 1}$$

and thus, $a_1 + 2 \ge 2a_1 + 1$. This implies that $a_1 = 1$ and so $a = d$ and $b = 2d$ and $d > 1$ since $ab > 2$. Hence, in this case,

$$\frac{\mathrm{lcm}(a,b) + \gcd(a,b)}{a+b} = \frac{d + 2d}{d + 2d} = 1 \le \frac{a+b}{4} = \frac{3d}{4}$$

holding since $d > 1$. Now, suppose that $b_1 \ge a_1 + 2$. Then $2 \le b_1 - a_1$ and so $4 \le b_1^2 - 2a_1 b_1 + a_1^2$. Rearranging shows that $a_1 b_1 + 1 \le \frac{(a_1+b_1)^2}{4}$ and hence

$$\frac{a_1 b_1 + 1}{a_1 + b_1} \le \frac{a_1 + b_1}{4}.$$

Hence, the given inequality holds. Equality holds in the cases

$$(a, b) \in \{(2, 2), (a_1 d, (a_1 + 2)d), ((a_1 + 2)d, a_1 d)\}$$

where d is a natural number and a_1 is an arbitrary odd number (If it were even, then a_1 and $a_1 + 2$ are not coprime and so we could factor out another 2).

OC223. Let $\mathbb{Z}$ be the set of integers. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$xf(2f(y)-x)+y^2f(2x-f(y))=\frac{f(x)^2}{x}+f(yf(y))$$

for all $x, y \in \mathbb{Z}$ with $x \neq 0$.

Originally problem 3 from day 1 of the 2014 USAJMO.

We received 2 correct submissions. We present the solution by Oliver Geupel.

It is straightforward to check that the two functions

$$f : x \mapsto 0 \qquad \text{and} \qquad f : x \mapsto x^2$$

are solutions. We prove that there are no other ones.

Suppose that f is any solution. For integers x and y, let $P(x, y)$ denote the assertion that x and y satisfy the proposed functional equation.

For every $x \neq 0$, the number x divides $f(x)^2$ by $P(x, y)$. Assume towards a contradiction that $f(0) \neq 0$. Then from $P(2f(0), 0)$, we see that

$$2f(0)f(2f(0)-2f(0))+0^2f(4f(0)-f(0))=\frac{f(2f(0))^2}{2f(0)}+f(0f(0))$$

which simplifies to

$$4f(0)-2=\left(\frac{f(2f(0))}{f(0)}\right)^2.$$

This is a contradiction since the left hand side is divisible by exactly one copy of 2 whereas the right hand side must be divisible by 4. Hence $f(0) = 0$. From the assertions $P(x, 0)$ and $P(-x, 0)$ for $x \neq 0$, we obtain $f(-x) = \frac{f(x)^2}{x^2}$ and $f(x) = \frac{f(-x)^2}{x^2}$. Therefore, $f(x)^4 = x^6 f(x)$, so that $f(x)$ is either 0 or x^2. Also, $f(x) = f(-x)$. Let us assume that a and b are non-zero integers such that $f(a) = 0$ and $f(b) = b^2$. All that remains to be done is to show that this is impossible.

By $P(x, a)$ for $x \neq 0$, we have $xf(-x) + a^2 f(2x) = \frac{f(x)^2}{x}$. Thus, $f(2x) = 0$. Hence b is odd. For every integer $x \neq 0$, we obtain $b^2 f(4x - b^2) = f(b^3)$ applying $P(2x, b)$; whence $f(4x-b^2) = \frac{f(b^3)}{b^2}$. Since this holds for all nonzero x, we deduce that

$$f(4x-b^2)=0.$$

If $b \equiv -1 \pmod 4$ where $b \neq -1$, then putting $x = \frac{b^2+b}{4}$ leads to

$$0=f(4x-b^2)=f(b)=b^2\neq 0, \tag{1}$$

a contradiction. If $b \equiv 1 \pmod 4$, $b \neq 1$, then choosing $x = \frac{b^2-b}{4}$ gives (1), which is impossible. As a consequence, $f(-1) = f(1) = 1$ and $f(x) = 0$ for $x \neq \pm 1$. By $P(2,1)$, we have $0 = 2f(0) + f(3) = f(1) = 1$, which is the desired contradiction.

OC224. Let $n > 1$ be an integer. An $n \times n$-square is divided into n^2 unit squares. Of these unit squares, n are coloured green and n are coloured blue, and all remaining ones are coloured white. Are there more such colourings for which there is exactly one green square in each row and exactly one blue square in each column; or colourings for which there is exactly one green square and exactly one blue square in each row?

Originally problem 5 of the 2014 South Africa National Olympiad.

We received 2 correct submissions. We present the solution by Kathleen Lewis.

There are more colourings with one green and one blue in each row. To see this, think of first placing one green square in each row; for both methods there are n^n ways to do that. If we want to place a blue square in each row, there would be $(n-1)^n$ to accomplish this, since each row has one square already coloured green. But if we wish to put a blue square in each column, the number of possibilities depends on the arrangement already made of the green squares. Suppose that there are a_i blank squares in column i. Then the number of possible arrangements of the blue squares is $\prod_{i=1}^n a_i$. The total number of available squares is $n^2 - n = n(n-1)$, so $\sum_{i=1}^n a_i = n(n-1)$. But for variables with a fixed sum, the product is greatest when all the factors are equal. So, the maximum value of $\prod_{i=1}^n a_i$ occurs when $a_1 = a_2 = \cdots = a_n = n-1$ and $\prod_{i=1}^n a_i = (n-1)^n$. In other cases, the product would be smaller, even as small as zero if the green squares were all placed in the same column. So the number of ways of placing a blue square in each column is always less than or equal to the number of ways to place the blue squares with one in each row.

OC225. Find the maximum value of real number k such that

$$\frac{a}{1+9bc+k(b-c)^2} + \frac{b}{1+9ca+k(c-a)^2} + \frac{c}{1+9ab+k(a-b)^2} \geq \frac{1}{2}$$

holds for all non-negative real numbers $a,\ b,\ c$ satisfying $a+b+c=1$.

Originally problem 5 of the 2014 Japan Mathematical Olympiad.

We received 3 correct submissions. We present the solution by Arkady Alt.

Let k be such that the original inequality holds for any non-negative real numbers a, b, c satisfying $a+b+c=1$. Then, in particular, if $a=0$ and $b=c=1/2$, we get

$$\frac{1/2}{1+k(1/2)^2} + \frac{1/2}{1+k(1/2)^2} \geq \frac{1}{2} \iff \frac{4}{k+4} \geq \frac{1}{2} \iff k \leq 4.$$

Let $k \leq 4$. By Cauchy's Inequality

$$\begin{aligned}\sum_{cyc} \frac{a}{1+9bc+k(b-c)^2} &= \sum_{cyc} \frac{a^2}{a\,(1+9bc+k(b-c)^2)}\\ &\geq \frac{(a+b+c)^2}{\sum\limits_{cyc} a\,(1+9bc+k(b-c)^2)}\\ &= \frac{1}{\sum\limits_{cyc} a\,(1+9bc+k(b-c)^2)}\\ &= \frac{1}{1+9abc\,(3-k)+k(ab+bc+ca)}\\ &= \frac{1}{1+9q\,(3-k)+kp},\end{aligned}$$

where $p := ab+bc+ca$ and $q := abc$. We have

$$\begin{aligned}&p = ab+bc+ca \leq \frac{(a+b+c)^2}{3} = 1/3,\\ &9q = 9abc \leq (ab+bc+ca)\,(a+b+c) = p,\\ &9q \geq 4p-1.\end{aligned}$$

(Schur's Inequality $\sum\limits_{cyc} a\,(a-b)\,(a-c) \geq 0$ in p, q notation with normalization by $a+b+c=1$).

If $k \leq 3$, then

$$9q\,(3-k)+kp \leq p\,(3-k)+kp = 3p \leq 3 \cdot \frac{1}{3} = 1.$$

If $3 < k \leq 4$, then

$$9q\,(3-k)+kp \leq (4p-1)\,(3-k)+kp = k+3p\,(4-k)-3 \leq k+3\cdot\frac{1}{3}\,(4-k)-3 = 1.$$

Thus,

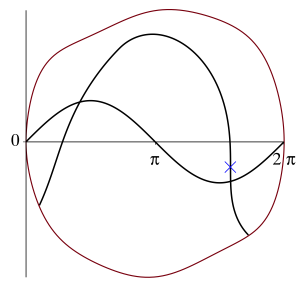
$$\sum_{cyc} \frac{a}{1+9bc+k(b-c)^2} \geq \frac{1}{1+9q\,(3-k)+kp} \geq \frac{1}{1+1} = \frac{1}{2}$$

for any $k \leq 4$ and, therefore, the maximum value of k is 4.

Wobbling Bicycle: Solution

Proposed by Luis Goddyn, Simon Fraser University, Burnaby, BC. The problem originally appeared in ***Crux*** *42(1), p. 10.*

Problem. A wobbling bicycle passes through a mud patch. One of its wheels traces a part of the curve $y = \sin x$. The other wheel makes a curve with a vertical inflection point. How long is the bicycle?



In order to eliminate effects due to bicycle geometry, tilting and wheel size, assume the bicycle has vanishingly thin tires with its front axle always positioned directly below a vertical headset. Assume also that both wheels were in the mud patch when the inflection point is traversed. Determine the distance ℓ between its axles.

Solution. One readily sees that the rear wheel traced the sine curve. Suppose the bicycle traveled from left to right on the coordinate plane, and that the front wheel is at (X, Y) when the rear wheel is at $(x, \sin x)$, for $x \in [0, 2\pi]$. Then $(X, Y) = (x+\ell_x,\ \sin x+\ell_y)$, where (ℓ_x, ℓ_y) is a translation vector of length ℓ. Since the rear wheel of a bicycle is aligned with its frame, the line through $(x, \sin x)$ and (X, Y) is tangent to the curve $y = \sin x$. This gives two equations,

$$\ell^2 = \ell_x^2 + \ell_y^2, \qquad \frac{\ell_y}{\ell_x} = \frac{d}{dx}\sin x = \cos x.$$

Noting that $\ell_x > 0$, these imply $\ell_x = \ell/\sqrt{1+\cos^2 x}$ and $X = x + \ell/(\sqrt{1+\cos^2 x}\,)$.

At the vertical inflection, the derivative function

$$\frac{dX}{dx} = 1 + \frac{\ell \cos x\ \sin x}{(1+\cos^2 x)^{3/2}}$$

attains a local minimum value of 0. Since $1 + \cos^2 x$ is positive, the function

$$(1+\cos^2 x)^3 - (\ell \cos x\ \sin x)^2$$

also attains a local minimum value of 0 at the vertical inflection. We change the variable to $C = \cos^2 x$ (so $\sin^2 x = 1 - C$). At the local minimum, C is both a zero and a critical point of the resulting function $f(C)$.

$$f(C) = (1+C)^3 - \ell^2 C(1-C) = 0 \tag{2}$$
$$f'(C) = 3(1+C)^2 - \ell^2(1-2C) = 0. \tag{3}$$

To solve for ℓ^2, we subtract $1+C$ times equation (3) from 3 times equation (2) to get

$$\ell^2(1-4C+C^2) = 0.$$

Since $\ell > 0$, the second factor equals zero. We "complete the square" in two ways,

$$(1+C)^2 = (1-4C+C^2) + 6C \ = \ 6C,$$
$$(1-2C)^2 = (1-4C+C^2) + 3C^2 = 3C^2.$$

Using these with (3), we find

$$\ell^2 = \frac{3(1+C)^2}{1-2C} = \frac{3 \cdot 6C}{\pm\sqrt{3C^2}} = \pm 6\sqrt{3} = \pm\sqrt{108}.$$

Thus the bicycle's "length" is $\ell = \sqrt[4]{108} \approx 3.223709$. □

Note: With a bit more work, one finds that

$$C = 2 - \sqrt{3} = \frac{1}{4}\left(\sqrt{6} - \sqrt{2}\right)^2 \approx 0.268,$$

and that at the vertical inflection we have the following vallues.

$$x = \arccos\left(-\sqrt{2-\sqrt{3}}\right) = \arccos\frac{\sqrt{2}-\sqrt{6}}{2} = \pi - \arcsin\sqrt{\sqrt{3}-1} \approx 2.115,$$
$$y = \sqrt{\sqrt{3}-1} \approx 0.856,$$
$$l_x = \sqrt{3\sqrt{3}+3} \approx 2.863,$$
$$l_y = -\sqrt{3\sqrt{3}-3} \approx -1.482,$$
$$X = \arccos\frac{\sqrt{2}-\sqrt{6}}{2} + \sqrt{3\sqrt{3}+3} \approx 4.978,$$
$$Y = \sqrt{\sqrt{3}-1} - \sqrt{3\sqrt{3}-3} = -\left(\sqrt{3}-1\right)^{3/2} \approx -0.626.$$

It is notable that, at the vertical inflection, the vertical positions of the tires satisfy

$$Y + y^3 = 0.$$

FOCUS ON...

No. 22

Michel Bataille

Constructions on the Sides

Introduction

On the sides of an arbitrary triangle, equilateral triangles are erected externally. Then the centres of these equilateral triangles are the vertices of a new equilateral triangle. This result, often called Napoleon's theorem, if well-known, likely comes as a surprise to students seeing it for the first time! We will not give a proof here (see **[1]** or **[2]** if necessary), but present a selection of examples in the same vein. We limit ourselves to configurations involving triangles and/or quadrilaterals and favour proofs using transformations or complex numbers.

Napoleon's configuration, slightly modified

We begin with an exercise adapted from problem **2815** [2003 : 111; 2004 : 113]:

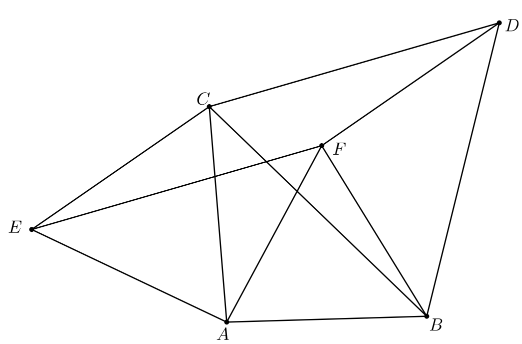
> Let ABC be a triangle with $\angle ACB \neq 60^\circ$ and D, E, F such that BCD, CAE, ABF are equilateral triangles with D, E external to ΔABC and F on the same side of AB as C. Prove that $DCEF$ is a parallelogram.

Without loss of generality, we suppose that ABC is anti-clockwise oriented. Let $\mathbf{r}_M$ denote the anti-clockwise rotation with centre M and angle 60°. With this notation, we may write $A = \mathbf{r}_C(E) = \mathbf{r}_B(F)$ and $C = \mathbf{r}_B(D)$, from which we deduce

$$D = \mathbf{r}_B^{-1}(C) = \mathbf{r}_B^{-1} \circ \mathbf{r}_C(C)$$

and

$$F = \mathbf{r}_B^{-1}(A) = \mathbf{r}_B^{-1} \circ \mathbf{r}_C(E).$$



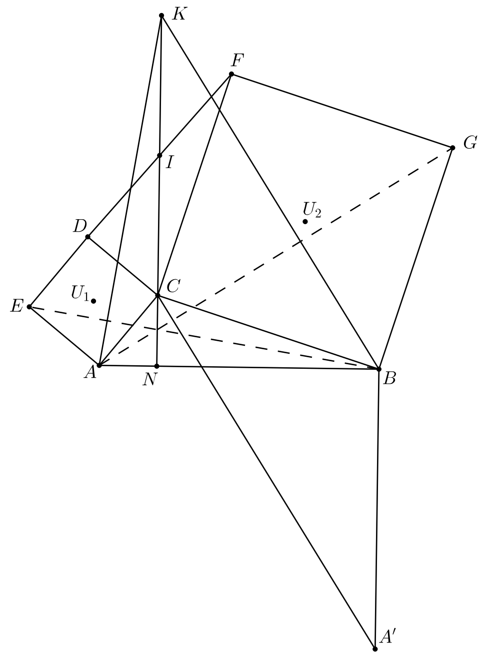
Since $\mathbf{r}_B^{-1} \circ \mathbf{r}_C$ is a translation, we must have $\overrightarrow{DF} = \overrightarrow{CE}$ and the result follows.

Another long time known problem

The following problem, which generalizes problem **2658** [2001 : 337 ; 2002 : 347], seems to date back to the eighteenth century:

> Squares $ACDE$ and $CBGF$ are drawn externally to an arbitrary triangle ABC. Suppose that AG and BE intersect at M. Show that M lies on the altitude CN.

The leading idea is to bring out a triangle whose altitudes are the lines CN, AG, BE (triangle AKB on the figure).



Let U_1 and U_2 denote the centres of the squares $ACDE$ and $CBGF$, respectively and let $\mathbf{r_1}, \mathbf{r_2}$ be the rotations with respective centres U_1, U_2 such that $\mathbf{r_1}(C) = D$ and $\mathbf{r_2}(F) = C$. Then $\mathbf{r_1} \circ \mathbf{r_2}(F) = D$ and $\mathbf{r_1} \circ \mathbf{r_2}$ is a rotation with angle 180° so that $\mathbf{r_1} \circ \mathbf{r_2} = \mathbf{h}$, the half-turn around the midpoint I of DF. Let $K = \mathbf{h}(C)$. In addition to $\mathbf{r_1}(A) = C$, we have $\mathbf{r_1}(B) = \mathbf{h} \circ \mathbf{r_2}^{-1}(B) = K$, hence $CK = AB$ and $CK \perp AB$. It follows that the lines KN and CN coincide and KN is an altitude of ΔAKB.

Now, we introduce A' such that $\overrightarrow{BA'} = \overrightarrow{KC}$ and the rotation $\mathbf{r}$ with centre B such that $\mathbf{r}(G) = C$. From $\mathbf{r}(A) = A'$, we deduce $AG \perp CA'$. However, $CKBA'$ being a parallelogram, BK is parallel to CA' and so AG is perpendicular to BK as well. Thus, AG is an altitude of ΔAKB. Similarly, BE is the third altitude and therefore intersects AG on the line $KN = CN$.

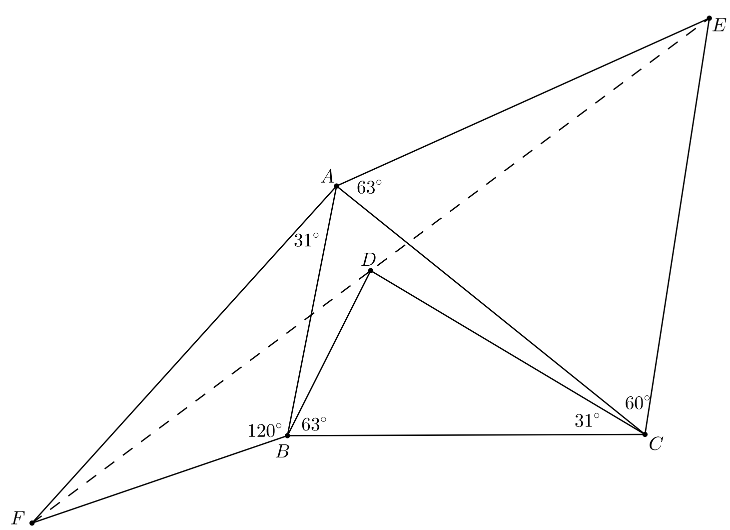
An example with triangles only

In the following example, unlike in Napoleon's theorem, quite irregular triangles are constructed on the sides of an arbitrary triangle. However, the angles of the new triangles are so carefully chosen that some nice results hold. This interesting problem, proposed in *The Mathematical Gazette* in November 2014, is slightly reformulated.

> In the plane of a triangle ABC, ΔBCD is described anti-clockwise with $\angle CBD = 63°, \angle BCD = 31°$, ΔCAE is described clockwise with $\angle ACE = 60°, \angle CAE = 63°$ and ΔABF is described clockwise with $\angle BAF = 31°, \angle ABF = 120°$. Prove that D, E, F are collinear and that the ratio $DE : EF$ is independent of the triangle ABC.

We work with complex numbers, the small letter k denoting the affix of the point K. The notation $\angle(\cdot,\cdot)$ is used for directed angles. The solution rests upon the following interesting, general result: Let M, N, P, Q be points of the circle with centre O and radius 1, with $M \neq N$. If $\angle(MN, MP) = \alpha$ and $\angle(NM, NQ) = \beta$, the affix of R, the point of intersection of the lines MP and NQ, is given by

$$r = \frac{ne^{i\alpha}\sin\beta - me^{i\beta}\sin\alpha}{\sin(\beta - \alpha)}. \tag{1}$$



The proof is straightforward: Since $\angle(\overrightarrow{ON}, \overrightarrow{OP}) = 2\alpha$, the affix of P is $ne^{2i\alpha}$. Thus, the equation of MP is $z + mne^{2i\alpha}\overline{z} = m + ne^{2i\alpha}$ and similarly, the equation of NQ is $z + mne^{2i\beta}\overline{z} = n + me^{2i\beta}$. The affix r of R is the solution of the system formed by the two equations: we are first led to

$$r(e^{2i\beta} - e^{2i\alpha}) = me^{2i\beta}(1 - e^{2i\alpha}) + ne^{2i\alpha}(e^{2i\beta} - 1)$$

and then to (1) after multiplication by $e^{-i(\alpha+\beta)}$.

Back to the problem, without loss of generality we suppose that $|a| = |b| = |c| = 1$. We set $\sigma = 63°, \tau = 31°$ and $\omega = e^{2\pi i/3}$. Applying the result above three times yields

$$d = \frac{ce^{i\sigma}\sin\tau + be^{-i\tau}\sin\sigma}{\sin(\sigma+\tau)}, \quad e = \frac{c(\sqrt{3}/2)e^{i\sigma} - a\omega\sin\sigma}{\sin(\sigma+60°)}, \quad f = \frac{a\omega\sin\tau + b(\sqrt{3}/2)e^{-i\tau}}{\sin(60°-\tau)}.$$

We calculate

$$e-d = \frac{\sin\sigma}{\sin(\sigma+60°)\sin(\sigma+\tau)} \cdot (ce^{i\sigma}\sin(60°-\tau) - a\omega\sin(\sigma+\tau) - be^{-i\tau}\sin(\sigma+60°))$$

(using $\frac{\sqrt{3}}{2}\sin(\sigma+\tau) - \sin\tau\sin(\sigma+60°) = \sin\sigma\sin(60°-\tau)$) and

$$f-e = \frac{\sqrt{3}/2}{\sin(60°-\tau)\sin(\sigma+60°)} \cdot (a\omega\sin(\sigma+\tau) + be^{-i\tau}\sin(\sigma+60°) - ce^{i\sigma}\sin(60°-\tau))$$

from which we deduce that $\frac{e-d}{e-f} = \lambda$ where $\lambda = \frac{\sin\sigma\sin(60°-\tau)}{(\sqrt{3}/2)\sin(\sigma+\tau)}$.
The result follows since λ is a positive real number independent of ΔABC.

A general result

Another recent problem of *The Mathematical Gazette* proposed to construct on the sides of a parallelogram $ABCD$ equilateral triangles ABP, BCQ, CDR, DAS with P, Q, R, S external to $ABCD$ and to show that $PQRS$ is a parallelogram. The problem can be generalized as follows:

> Let $ABCD$ be an arbitrary quadrilateral and let α be a nonzero complex number with modulus ρ and argument θ. Denote by $\mathbf{S}_M$ the spiral similarity with centre M, factor ρ and angle θ and let $\mathbf{S}_A(B) = P, \mathbf{S}_B(C) = Q, \mathbf{S}_C(D) = R, \mathbf{S}_D(A) = S$. Show that $PQRS$ is a parallelogram if and only if $\alpha = \frac{1}{2}$ or $ABCD$ is a parallelogram.

(This problem was set in a French high school final exam in the 1980s.)

Note that triangles ABP, BCQ, CDR, DAS are directly similar, not necessarily equilateral.

Once again, complex numbers work wonders! We denote by m the complex affix of M. From the way P is constructed, we have $p - a = \alpha(b-a)$, that is $p = (1-\alpha)a + \alpha b$. Similarly, $q = (1-\alpha)b + \alpha c, r = (1-\alpha)c + \alpha d, s = (1-\alpha)d + \alpha a$. Now, $PQRS$ is a parallelogram if and only if $q + s = p + r$. A short calculation shows that the latter is equivalent to

$$(1-2\alpha)\left((a+c) - (b+d)\right) = 0.$$

The conclusion immediately follows. In the case $\alpha = \frac{1}{2}$, $PQRS$ is the Varignon parallelogram of the quadrilateral $ABCD$.

Exercises

Our first exercise is based on problem 858 of *The College Mathematics Journal*.

1. Let $ABCD$ be a convex quadrilateral that is not a parallelogram. On the sides AB, BC, CD, DA, construct isosceles triangles KAB, MBC, LCD, NDA external to $ABCD$ such that the angles at K, L, M, N are right angles. Show that if O is the midpoint of BD, then one of the triangles MON or LOK is a 90° rotation of the other around O.

2. Let $ABCD$ be a square and O, P be such that DOC and BCP are equilateral triangles with O inside $ABCD$ and P external to $ABCD$. Show that A, O, P are collinear. [A possible solution follows from the value of $\angle AOB$ found in problem **3458** [2009 : 326 ; 2010 : 347]; preferably solve the problem with the help of a well-chosen rotation.]

References

[**1**] H.S.M. Coxeter, S.L. Greitzer, *Geometry Revisited*, MAA, 1967, p. 61.

[**2**] M. Bataille, *Géométrie plane, avec des nombres*, ATOM Vol. XV, p. 36.

The Engel-Titu Inequality

Eeshan Banerjee

In this article, we present a proof of the 'Engel-Titu Inequality' (It's a long name, let's call it **E-T**!), and some applications of it.

The Engel-Titu Inequality was discovered independently by Arthur Engel in 1998 and Titu Andreescu in 2001. In their book *Problems from the Book*, Titu Andreescu and Gabriel Dopinescu wrote the following note about the inequality:

> "[The Engel-Titu Inequality] is clearly a direct application of the Cauchy-Schwarz inequality. Some will say that it is actually the Cauchy-Schwarz inequality and they are not wrong. Anyway, this particular lemma has become very popular among the American students who attended the training of the USA IMO team. This happened after a lecture delivered by the first author at the Mathematical Olympiad Summer Program (MOSP) held at Georgetown University in June, 2001."

Theorem 1 (E-T) *For positive real numbers $a_k, b_k, k \in [1, 2, \ldots, n]$, the following inequality holds.*

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n} \ .$$

Equality occurs if and only if all the $\dfrac{a_i}{b_i}$ are equal.

Proof. Let us induct on k.

For $k = 1$, the statement becomes $\frac{a^2}{b} \geq \frac{(a)^2}{b}$, which is obviously true.

For $k = 2$, the statement becomes $\frac{a^2}{c} + \frac{b^2}{d} \geq \frac{(a+b)^2}{c+d}$. Cross-multiplication yields

$$\begin{aligned}\left(\frac{a^2}{c} + \frac{b^2}{d}\right)(c+d) &\geq (a+b)^2 \implies \\ a^2 + b^2 + \frac{a^2 d}{c} + \frac{b^2 c}{d} &\geq a^2 + 2ab + b^2 \implies \\ \frac{a^2 d}{c} + \frac{b^2 c}{d} &\geq 2ab. \qquad (1)\end{aligned}$$

which is obviously true by AM-GM, as $\dfrac{a^2 d}{c} + \dfrac{b^2 c}{d} \geq 2\sqrt{\dfrac{a^2 d}{c} \cdot \dfrac{b^2 c}{d}} = 2ab$.
Now, let us assume that for some positive integer k, the statement is true. That is, the following inequality is true.

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_k^2}{b_k} \geq \frac{(a_1 + a_2 + \cdots + a_k)^2}{b_1 + b_2 + \cdots + b_k} \ . \qquad (2)$$

Now by (1) and (2) we have

$$\frac{a_1^2}{b_1}+\frac{a_2^2}{b_2}+\cdots+\frac{a_{k+1}^2}{b_{k+1}} \geq \frac{(a_1+a_2+\cdots+a_k)^2}{b_1+b_2+\cdots+b_k}+\frac{a_{k+1}^2}{b_{k+1}} \geq \frac{(a_1+a_2+\cdots+a_{k+1})^2}{b_1+b_2+\cdots+b_{k+1}}\ .$$

The E-T Inequality follows by induction; the case of equality is easily verified. □

There's also a very straightforward proof of the E-T inequality using the Cauchy-Schwarz inequality. We leave that proof to the readers.

To illustrate the use of the E-T Inequality, let's work on some problems using this result.

Example 1 (IMO 1995) *Let a, b, c be positive real numbers with product 1. Then prove that*

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)} \geq \frac{3}{2}\ .$$

Solution. Let us substitute $a=\dfrac{1}{x}\ , b=\dfrac{1}{y}\ , c=\dfrac{1}{z}$. As $abc=1, xyz=1$. We get

$$\sum_{\text{cyc}}\frac{1}{a^3(b+c)}=\sum_{\text{cyc}}\frac{1}{\dfrac{1}{x^3}\cdot\left(\dfrac{1}{y}+\dfrac{1}{z}\right)}=\sum_{\text{cyc}}\frac{1}{\left(\dfrac{y+z}{x^3yz}\right)}=\sum_{\text{cyc}}\frac{x^2}{y+z}\ .$$

By E-T and AM-GM inequality, we get

$$\sum_{\text{cyc}}\frac{1}{a^3(b+c)}=\sum_{\text{cyc}}\frac{x^2}{y+z} \geq \frac{(x+y+z)^2}{2(x+y+z)}=\frac{x+y+z}{2} \geq \frac{3\sqrt[3]{xyz}}{2}=\frac{3}{2}\ .$$

Hence we get

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)} \geq \frac{3}{2}\ .$$

This completes the proof. □

The next example is believed to be new.

Example 2 *Again letting a, b, c be positive real numbers with product 1, prove that* $\sum_{cyc}\dfrac{a^3}{b+c} \geq \frac{3}{2}$.

Solution. We may write $\sum_{\text{cyc}}\dfrac{a^3}{b+c}=\sum_{\text{cyc}}\dfrac{a^4}{ab+ac}$. Now by E-T, we get

$$\sum_{\text{cyc}}\frac{a^4}{ab+ac} \geq \frac{\left(a^2+b^2+c^2\right)^2}{2(ab+bc+ca)} \implies$$

$$\sum_{\text{cyc}}\frac{a^3}{b+c} \geq \frac{\left(a^2+b^2+c^2\right)^2}{2(ab+bc+ca)} \geq \frac{\left(a^2+b^2+c^2\right)^2}{2\left(a^2+b^2+c^2\right)}=\frac{\left(a^2+b^2+c^2\right)}{2}.$$

Therefore, by Power Mean Inequality and AM-GM Inequality with $abc = 1$, we have

$$\sum_{\text{cyc}} \frac{a^3}{b+c} = \frac{(a^2+b^2+c^2)}{\geq} \left(\frac{a+b+c}{3}\right)^2 \cdot \frac{3}{2} \geq \left(\frac{3\sqrt[3]{abc}}{3}\right)^2 \cdot \frac{3}{2} = \left(\frac{3}{3}\right)^2 \cdot \frac{3}{2} = \frac{3}{2},$$

which completes the proof. $\square$

Example 3 (Turkey 1997) *Let $n \geq 2$. Find the minimal value of*

$$\frac{x_1^5}{x_2+x_3+\cdots+x_n} + \frac{x_2^5}{x_1+x_3+\cdots+x_n} + \cdots + \frac{x_n^5}{x_1+x_2+\cdots+x_{n-1}},$$

where $x_1, x_2, \cdots x_n$ are positive real numbers satisfying $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$.

Solution. Let $s = x_1 + x_2 + \cdots + x_n$. By E-T inequality we have

$$\sum_{i=1}^{n} \frac{x_i^5}{s-x_i} = \sum_{i=1}^{n} \frac{x_i^6}{sx_i - x_i^2} \geq \frac{(x_1^3 + x_2^3 + \cdots + x_n)^3}{\sum\limits_{i=1}^{n}(sx_i - x_i^2)} = \frac{\left(\sum\limits_{i=1}^{n} x_i^3\right)^2}{s^2-1}.$$

Now, by the Power-Mean Inequality we have

$$\left(\sum_{i=1}^{n} \frac{x_i^3}{n}\right)^{\frac{1}{3}} \geq \left(\sum_{i=1}^{n} \frac{x_i^2}{n}\right)^{\frac{1}{2}} \geq \sum_{i=1}^{n} \frac{x_i}{n}$$

and hence

$$\left(\sum_{i=1}^{n} \frac{x_i^3}{n}\right)^{\frac{2}{3}} \geq \frac{1}{n} \geq \frac{s^2}{n^2}.$$

So we have

$$\frac{1}{n} \geq \frac{s^2}{n^2} \implies s^2 \leq n \implies s^2 - 1 \leq n-1 \implies \frac{1}{s^2-1} \leq \frac{1}{n-1}$$

and

$$\left(\sum_{i=1}^{n} \frac{x_i^3}{n}\right)^2 \geq \frac{1}{n^3} \implies \left(\sum_{i=1}^{n} x_i^3\right)^2 \geq \frac{1}{n}.$$

So

$$\frac{\left(\sum\limits_{i=1}^{n} x_i^3\right)^2}{s^2-1} \geq \frac{1}{n(n-1)}.$$

Thus, the minimal value of the given expression is $\frac{1}{n(n-1)}$. $\square$

Problems for the Reader

Problem 1 *For positive real numbers a, b, c, prove that*

$$\frac{a}{2a+b} + \frac{b}{2b+c} + \frac{c}{2c+a} \le 1.$$

Problem 2 (IMOSL 1993) *For arbitrary positive real numbers a, b, c, d prove the inequality*

$$\frac{a}{b+2c+3d} + \frac{b}{c+2d+3a} + \frac{c}{d+2a+3b} + \frac{d}{a+2b+3c} \ge \frac{2}{3}.$$

Problem 3 (Japan 1997) *Prove that for any positive real numbers a, b, c, the following inequality holds*

$$\frac{(b+c-1)^2}{a^2+(b+c)^2} + \frac{(c+a-1)^2}{b^2+(c+a)^2} + \frac{(a+b-1)^2}{c^2+(a+b)^2} \ge \frac{3}{5}$$

Problem 4 (Gabriel Dopinescu) *Prove that if $a, b, c, d > 0$ satisfy $abc + bcd + cda + dab = a + b + c + d$, then*

$$\sqrt{\frac{a^2+1}{2}} + \sqrt{\frac{b^2+1}{2}} + \sqrt{\frac{c^2+1}{2}} + \sqrt{\frac{d^2+1}{2}} \le a + b + c + d.$$

Problem 5 (MOSP 2001) *Prove that for any positive real numbers a_1, a_2, a_3, a_4, a_5,*

$$\frac{a_1}{a_2+a_3} + \frac{a_2}{a_3+a_4} + \frac{a_3}{a_4+a_5} + \frac{a_4}{a_5+a_1} + \frac{a_5}{a_1+a_2} \ge \frac{5}{2}.$$

References

[1] AoPS Community - www.artofproblemsolving.com/community

[2] Samin Riasat, *Basics of Olympiad Inequalities*, Unpublished.

[3] Arthur Engel, *Problem Solving Strategies*, Problem Book in Mathematics, Springer, 1995.

[4] Titu Andreescu, Gabriel Dopinescu, *Problems from the Book*, XYZ Press, 2008

PROBLEMS

Readers are invited to submit solutions, comments and generalizations to any problem in this section. Moreover, readers are encouraged to submit problem proposals. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **February 1, 2017**.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

4141. *Proposed by Leonard Giugiuc, Daniel Sitaru and Oai thanh Dao; modified by the editor.*

a) Let $A_0A_1\dots A_{n-1}$ be a convex n-gon for which there exists an interior point T such that $\angle A_{i-1}TA_i = \frac{2\pi}{n}, i = 1, 2, \dots n$ (with $A_n \equiv A_0$). Construct regular n-gons Π_i externally on the sides $A_{i-1}A_i$. Prove that

$$[A_0A_1\dots A_{n-1}] \le \frac{1}{n}\sum_{i=1}^{n}[\Pi_i]$$

(where square brackets denote area).

b) $\star$ Does the inequality continue to hold if the given convex polygon is arbitrary?

4142. *Proposed by Daniel Sitaru.*

Prove that if $a, b, c \in (0, \infty)$ then:

$$\left(1+\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^{\frac{(a+b+c)^2}{a^2+b^2+c^2}} \le \left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right).$$

4143. *Proposed by Roy Barbara.*

For any real number $x \ge 1$, let $y = x^{1/2} + x^{-1/2}$.

a) Express x in terms of y by a radical formula and check that no rational fraction $F(t)$ can exist such that $x = F(y)$. (A rational fraction is an expression of the form $f(t)/g(t)$, where $f(t)$ and $g(t)$ are polynomials with rational coefficients.)

b) Find a closed form formula $x = F(y)$ containing no radicals.

c) $\star$ Is there a *complex* fraction such that $x = F(y)$? (A complex fraction is a function of the form $f(z)/g(z)$, where $f(t)$ and $g(t)$ are polynomials with complex coefficients.)

4144. *Proposed by George Apostolopoulos.*

Let a, b and c be positive real numbers such that $a+b+c=1$. Find the maximum value of the expression

$$\left(a-\frac{1}{2}\right)^3+\left(b-\frac{1}{2}\right)^3+\left(c-\frac{1}{2}\right)^3.$$

4145. *Proposed by Leonard Giugiuc.*

Prove that the system

$$\begin{cases} A^3+A^2B+AB^2+ABA=\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \\ \\ B^3+B^2A+BA^2+BAB=\begin{bmatrix} -1 & 0 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{cases}$$

has no solutions in the set of 3×3 matrices over complex numbers.

4146. *Proposed by Mehmet Berke Işler.*

Let a, b, c be non-negative real numbers such that $a^2+b^2+c^2=2(ab+bc+ca)$ and $\sqrt{a}+\sqrt{b}+\sqrt{c}=2$. Prove that at least one of the numbers a, b, c is equal to 1.

4147. *Proposed by Mehtaab Sawhney.*

Let $\{a_i\}$ be a sequence of real numbers. Suppose that $|a_i-a_j| \geq 2^{i-j}$ if $i>j$, then find the minimal value of

$$\sum_{1\leq i<j\leq n}(a_j-a_i)^2.$$

4148. *Proposed by Lorian Saceanu.*

For positive real numbers x, y and z, show that

$$\begin{aligned}&\sqrt{xy(x+y)}+\sqrt{yz(y+z)}+\sqrt{xz(x+z)}\\ &\qquad\geq \sqrt{(x+y)(y+z)(z+x)}+(x+y+z)\sqrt{\frac{2xyz}{3(xy+yz+xz)}}.\end{aligned}$$

4149. *Proposed by Daniel Sitaru.*

Prove that if $[a,b] \subset \left[0, \frac{\pi}{4}\right]$ then:

$$3(a \tan b + b \tan a) \geq ab(6 + a \tan a + b \tan b).$$

4150. *Proposed by Leonard Giugiuc.*

Let $(x_n)_{n\geq 1}$ be a sequence of positive real numbers such that

$$\lim_{n\to\infty} \left(x_n^2 + 2x_n + \frac{32}{x_n^3} \right) = 12.$$

Show that $\lim_{n\to\infty} x_n$ exists and find its value.

. . . .

4141. *Proposé par Leonard Giugiuc, Daniel Sitaru et Oai thanh Dao; modifié par l'éditeur.*

a) Soit $A_0A_1 \dots A_{n-1}$ un n-gone convexe avec un point intérieur T tel que $\angle A_{i-1}TA_i = \frac{2\pi}{n}, i = 1, 2, \dots, n$ (avec $A_n \equiv A_0$). On construit des n-gones réguliers externes Π_i sur les côtés $A_{i-1}A_i$. Démontrer que

$$[A_0A_1 \dots A_{n-1}] \leq \frac{1}{n} \sum_{i=1}^{n} [\Pi_i]$$

(les crochets représentent l'aire).

b) $\star$ L'inégalité est-elle toujours vérifiée si le polygone convexe donné est plutôt un polygone arbitraire?

4142. *Proposé par Daniel Sitaru.*

Sachant que $a, b, c \in (0, \infty)$, démontrer que

$$\left(1 + \frac{a^2+b^2+c^2}{ab+bc+ca}\right)^{\frac{(a+b+c)^2}{a^2+b^2+c^2}} \leq \left(1 + \frac{a}{b}\right)\left(1 + \frac{b}{c}\right)\left(1 + \frac{c}{a}\right).$$

4143. *Proposé par Roy Barbara.*

Étant donné un réel x ($x \geq 1$), soit $y = x^{1/2} + x^{-1/2}$.

a) Exprimer x en fonction de y au moyen d'une expression contenant des radicaux et montrer qu'il n'existe aucune fraction rationnelle $F(t)$ telle que $x = F(y)$. (Une fraction rationnelle est une expression de la forme $f(t)/g(t)$, $f(t)$ et $g(t)$ étant des polynômes avec coefficients rationnels.)

b) Déterminer une formule explicite $x = F(y)$ qui ne contient aucun radical.

c) $\star$ Est-il possible d'exprimer $x = F(y)$ au moyen d'une fraction *complexe*? (Une fraction complexe est une expression de la forme $f(z)/g(z)$, $f(t)$ et $g(t)$ étant des polynômes avec coefficients complexes.)

4144. *Proposé par George Apostopoulos.*

Soit a, b et c des réels strictement positifs tels que $a + b + c = 1$. Déterminer la valeur maximale de l'expression

$$\left(a - \frac{1}{2}\right)^3 + \left(b - \frac{1}{2}\right)^3 + \left(c - \frac{1}{2}\right)^3.$$

4145. *Proposé par Leonard Giugiuc.*

Démontrer que le système

$$\begin{cases} A^3 + A^2B + AB^2 + ABA = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \\ \\ B^3 + B^2A + BA^2 + BAB = \begin{bmatrix} -1 & 0 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{cases}$$

n'admet aucune solution, A et B étant des matrices 3×3 de nombres complexes.

4146. *Proposé par Mehmet Berke Işler.*

Soit a, b et c des réels non négatifs tels que $a^2 + b^2 + c^2 = 2(ab + bc + ca)$ et $\sqrt{a} + \sqrt{b} + \sqrt{c} = 2$. Démontrer qu'au moins un des nombres a, b et c est égal à 1.

4147. *Proposé par Mehtaab Sawhney.*

Soit $\{a_i\}$ une suite de réels. Sachant que $|a_i - a_j| \geq 2^{i-j}$ lorsque $i > j$, déterminer la valeur minimale de

$$\sum_{1 \leq i < j \leq n} (a_j - a_i)^2.$$

4148. *Proposé par Lorian Saceanu.*

Soit x, y et z des réels strictement positifs. Démontrer que

$$\sqrt{xy(x+y)} + \sqrt{yz(y+z)} + \sqrt{xz(x+z)}$$
$$\geq \sqrt{(x+y)(y+z)(z+x)} + (x+y+z)\sqrt{\frac{2xyz}{3(xy+yz+xz)}}.$$

4149. *Proposé par Daniel Sitaru.*

Soit $[a, b] \subset \left[0, \frac{\pi}{4}\right]$. Démontrer que

$$3(a \tan b + b \tan a) \geq ab(6 + a \tan a + b \tan b).$$

4150. *Proposé par Leonard Giugiuc.*

Soit $(x_n)_{n\geq 1}$ une suite de réels strictement positifs tels que

$$\lim_{n\to\infty} \left(x_n^2 + 2x_n + \frac{32}{x_n^3} \right) = 12.$$

Démontrer que $\lim_{n\to\infty} x_n$ existe et déterminer sa valeur.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2015: 41(5), p. 212–215.

4041. *Proposed by Arkady Alt.*

Let a, b and c be the side lengths of a triangle ABC. Let AA', BB' and CC' be the heights of the triangle and let $a_p = B'C', b_p = C'A'$ and $c_p = A'B'$ be the sides of the orthic triangle. Prove that:

a) $a^2(b_p + c_p) + b^2(c_p + a_p) + c^2(a_p + b_p) = 3abc$;

b) $a_p + b_p + c_p \leq s$, where s is the semiperimeter of ABC.

We received 15 correct solutions and present the solution by Michel Bataille.

We show (a) and (b) in the case when ΔABC has no obtuse angle and provide a counter-example in the opposite case.

First, suppose that angles A, B, C are acute. Since $\Delta AB'B$ is right-angled with $\angle AB'B = 90^\circ$, we have $AB' = c \cdot \cos A$. Similarly, $AC' = b \cdot \cos A$, and it follows that

$$\begin{aligned} B'C'^2 &= c^2 \cos^2 A + b^2 \cos^2 A - 2bc \cos^3 A \\ &= (c^2 + b^2 - 2bc \cos A) \cos^2 A = a^2 \cos^2 A \end{aligned}$$

and so $a_p = B'C' = a \cos A$. In a similar way, we obtain $b_p = A'C' = b \cos B$ and $c_p = A'B' = c \cos C$.

Now we calculate $X = a^2(b_p + c_p) + b^2(c_p + a_p) + c^2(a_p + b_p)$ as follows:

$$\begin{aligned} X &= a^2 b \cos B + a^2 c \cos C + b^2 c \cos C + b^2 a \cos A + bc^2 \cos B + c^2 a \cos A \\ &= ab(a \cos B + b \cos A) + bc(b \cos C + c \cos B) + ca(c \cos A + a \cos C) \\ &= abc + bca + cab = 3abc, \end{aligned}$$

as desired. Denoting by r and R the inradius and the circumradius of ΔABC and using the Law of Sines, we get

$$\begin{aligned} a_p + b_p + c_p &= a \cos A + b \cos B + c \cos C \\ &= R(\sin 2A + \sin 2B + \sin 2C) \\ &= 4R \sin A \sin B \sin C \\ &= 4R \cdot \frac{abc}{8R^3} = \frac{4rRs}{2R^2} = s \cdot \frac{2r}{R} \end{aligned}$$

and the result $a_p + b_p + c_p \leq s$ follows from Euler's inequality $2r \leq R$.

If ΔABC is right-angled, say $\angle BAC = 90°$, results (a) and (b) continue to hold if we take, as is natural, $a_p = 0$, $b_p = c_p = h$, where $h = AA'$. Indeed, we have $3abc = 3a \cdot ah = 3a^2h$ and

$$\begin{aligned} a^2(b_p + c_p) + b^2(c_p + a_p) + c^2(a_p + b_p) &= a^2 \cdot 2h + b^2 \cdot h + c^2 \cdot h \\ &= h(b^2 + c^2 + 2a^2) = 3a^2h. \end{aligned}$$

Also, the inequality $a_p + b_p + c_p \leq s$ rewrites as $4h \leq a+b+c$ or $4bc \leq a^2 + a(b+c)$. Since $b + c \geq 2\sqrt{bc}$ and $a^2 = b^2 + c^2 \geq 2bc$, we have

$$a^2 + a(b+c) \geq 2bc + 2\sqrt{2bc} \cdot 2\sqrt{bc} = (2 + 2\sqrt{2})bc \geq 4bc.$$

None of these results is correct, however, if an angle of ΔABC is obtuse, as the following example shows. Consider a triangle ABC with $\angle BAC = 120°$ and $AB = AC$. Then $b = c$, $a = c\sqrt{3}$, and $a_p = b_p = c_p = \frac{a}{2} = \frac{c\sqrt{3}}{2}$. One easily finds that $3abc = 3c^3\sqrt{3}$, while

$$a^2(b_p + c_p) + b^2(c_p + a_p) + c^2(a_p + b_p) = 5c^3\sqrt{3}.$$

Also,

$$a_p + b_p + c_p = \frac{3c\sqrt{3}}{2} > \frac{(2+\sqrt{3})c}{2} = s.$$

4042. *Proposed by Leonard Giugiuc and Diana Trailescu.*

Let a, b and c be real numbers in $[0, \pi/2]$ such that $a + b + c = \pi$. Prove the inequality

$$2\sqrt{2}\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2} \geq \sqrt{\cos a \cos b \cos c}.$$

We received 14 correct solutions. We present the solution by Scott Brown. Similar solutions came from Arslanagić Šefket, Michel Bataille, Andrea Fanchini, and John Heuvel.

In [1] and [2] respectively, we find the identities

$$\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2} = \frac{r}{4R} \tag{1}$$

and

$$\cos a \cos b \cos c = \frac{s^2 - 4R^2 - 4Rr - r^2}{4R^2}, \tag{2}$$

where R, r, and s are the circumradius, inradius, and semiperimeter of the triangle. We square both sides of the original inequality to obtain the equivalent statement

$$8\sin^2\frac{a}{2}\sin^2\frac{b}{2}\sin^2\frac{c}{2} \leq \cos a \cos b \cos c,$$

into which we substitute the identities (1) and (2). The resulting inequality is equivalent to one due to Gerretsen [3]:

$$s^2 \leq 4R^2 + 4Rr + 3r^2.$$

References

[1] Anders Bager. "A family of goniometric inequalities." Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz., No. 338 352 (1971), p. 10.

[2] Anders Bager. "Another family of goniometric inequalities." Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz., No. 412 460 (1973), p. 209.

[3] D. S. Mitrinovic et al. *Recent Advances in Geometric Inequalities.* Kluwer, Dordrecht, 1989.

Editor's Comment. Digby Smith pointed out that the inequality is equivalent to ***Crux*** Problem **974**, proposed by Jack Garfunkel in Volume 10, (8), October 1984, and solved by Murray Klamkin in Volume 11 (10), December 1985. The solution to **974** is based on ***Crux*** Problem **836**, proposed by Vedula N. Murty in Volume 9 (4), April 1983, and solved, again by Klamkin, in Volume 10 (7), August 1984.

4043. *Proposed by Michel Bataille.*

Suppose that the lines m and n intersect at A and are not perpendicular. Let B be a point on n, with $B \neq A$. If F is a point of m, distinct from A, show that there exists a unique conic $\mathcal{C}_F$ with focus F and focal axis BF, intersecting n orthogonally at A. Given $\epsilon > 0$, how many of the conics $\mathcal{C}_F$ have eccentricity ϵ?

We recieved two correct solutions and present the solution submitted by the proposer.

Since $m \neq n$, the perpendicular to m through F and the perpendicular t to n at A intersect, say at K. Note that K is distinct from both F and A (since $F \neq A$). Define p to be the perpendicular to BF through K. Then $A \notin p$ (otherwise we would have $t \perp BF$, implying $n \parallel BF$, a contradiction). We also have $F \notin p$ (otherwise $KF \perp BF$, implying $BF = m$ and $B \in m$, contradicting $B \neq A$).

We first show uniqueness: Suppose that $\mathcal{C}_F$ exists. Note that t is the tangent to $\mathcal{C}_F$ at A. Since $\angle KFA = 90^\circ$ and $K \in t$, K must be on the directrix of $\mathcal{C}_F$ associated with F (see [2], Theorem 1 p. 14). Thus, $\mathcal{C}_F$ must be the unique conic with focus F, directrix p and eccentricity $\frac{AF}{d(A,p)}$. Conversely, because the line p misses the distinct points A and F, we can consider the unique conic with focus F, directrix p and eccentricity $\frac{AF}{d(A,p)}$. This conic passes through A (by the definition of eccentricity) and is tangent to AK at A (since $K \in p$ and $\angle KFA = 90^\circ$); it therefore intersects n orthogonally at A. Also, its focal axis is BF (since $BF \perp p$). Thus, this conic satisfies the required conditions for $\mathcal{C}_F$.

Note that the eccentricity of $\mathcal{C}_F$ is also equal to $\frac{FB}{FA}$ (see [2], Theorem 4, p. 18) and that if F, F' are two distinct points on m ($F, F' \neq A$), then the conics $\mathcal{C}_F$ and

$\mathcal{C}_{F'}$ are distinct (their focal axes are distinct). From these remarks, we see that there are as many conics $\mathcal{C}_F$ with eccentricity ϵ as points of $M \in m$ that belong to the locus $\mathcal{E}$ of points for which $\frac{MB}{MA} = \epsilon$. If $\epsilon = 1$, $\mathcal{E}$ is the perpendicular bisector of AB; it intersects m (since m and n are not perpendicular), so that exactly one conic $\mathcal{C}_F$ is a parabola. If $\epsilon \neq 1$, then $\mathcal{E}$ is a circle—the circle of Apollonius—which can intersect m in at most two points. The collection of all these circles (as ϵ varies over the positive real numbers except 1) forms a nonintersecting pencil of circles with limiting points A and B, one through each point of the plane not on the perpendicular bisector of AB (see [1], Section 6.6). It follows that there are at most two conics $\mathcal{C}_F$ corresponding to a given value of ϵ.

To be more specific, $\mathcal{E}$ has diameter JJ' where J, J' are the points of n defined by $(1+\epsilon)\overrightarrow{AJ} = \overrightarrow{AB}$ and $(1-\epsilon)\overrightarrow{AJ'} = \overrightarrow{AB}$. The centre U of $\mathcal{E}$ is such that $(1-\epsilon^2)\overrightarrow{AU} = \overrightarrow{AB}$; its radius is $\rho = JU = \epsilon AU = \frac{\epsilon AB}{|1-\epsilon^2|}$. If H, H' are the orthogonal projections of B, U onto m, respectively, then $\frac{UH'}{BH} = \frac{AU}{AB} = \frac{1}{|1-\epsilon^2|}$; hence $UH' = \frac{BH}{|1-\epsilon^2|}$ (where BH is the distance from B to m which, of course, is always less than AB). We conclude that no, one, or two conics $\mathcal{C}_F$ have eccentricity ϵ according as UH' is greater than, equal to, or less than ρ, which is equivalent to ϵ less than, equal to, or greater than $\frac{BH}{AB}$. So, for example, $\mathcal{C}_F$ could never be a circle (for which $\epsilon = 0$).

References

[1] H.S.M. Coxeter, *Introduction to Geometry*, Wiley, 1961.

[2] C. V. Durell, *A Concise Geometrical Conics*, MacMillan, 1952.

4044. *Proposed by Dragoljub Milošević.*

Let x, y, z be positive real numbers such that $x + y + z = 1$. Prove that

$$\frac{x+1}{x^3+1} + \frac{y+1}{y^3+1} + \frac{z+1}{z^3+1} \leq \frac{27}{7}.$$

We received 24 submissions, of which 22 were correct and complete. There were two main approaches: Jensen's inequality, or comparing each term to a linear function. We present two solutions, one for each approach.

Solution 1, by Fernando Ballesta Yagüe.

As $x + y + z = 1$ and x, y, z are positive, we have $x, y, z \in (0,1)$. For x in the interval $(0,1)$, consider the rational function

$$f(x) = \frac{x+1}{x^3+1} = \frac{1}{x^2 - x + 1}.$$

Let's take the second derivative to check its convexity:

$$f'(x) = \frac{-2x+1}{(x^2-x+1)^2},$$
$$f''(x) = \frac{-2(x^2-x+1)^2 - (-2x+1)\cdot 2\cdot (x^2-x+1)\cdot(2x-1)}{(x^2-x+1)^4} = \frac{6x(x-1)}{(x^2-x+1)^3}.$$

Since $x \in (0,1)$, we have $x-1<0$, but $6x>0$ and $x^2-x+1>0$ (note that $-x+1>0$ for $x\in(0,1)$). So in the interval $x\in(0,1)$ we have $f''(x)<0$ and hence f is concave. By Jensen's Inequality for concave functions,

$$\frac{1}{3}(f(x)+f(y)+f(z)) \le f\left(\frac{1}{3}(x+y+z)\right) = f\left(\frac{1}{3}\right) = \frac{9}{7};$$

in other words,

$$\frac{1}{x^2-x+1} + \frac{1}{y^2-y+1} + \frac{1}{z^2-z+1} \le \frac{27}{7},$$

which is equivalent to the inequality we wanted to prove. Note that equality holds when $x=y=z=\frac{1}{3}$.

Solution 2, by Paul Bracken.

As in the previous solution, define $f(x) = \frac{1}{x^2-x+1}$ and show that f is concave for $x\in(0,1)$. Therefore, if $t(x)$ is a tangent line to $f(x)$ at some point $x_0\in(0,1)$ then the inequality $f(x)\le t(x)$ holds for $x\in(0,1)$. Let us calculate the tangent line to $f(x)$ at $x_0=\frac{1}{3}$:

$$t(x) = f'\left(\frac{1}{3}\right)\cdot\left(x-\frac{1}{3}\right) + f\left(\frac{1}{3}\right) = \frac{27}{49}x + \frac{54}{49}.$$

The inequality $f(x)\le t(x)$ then gives us $\frac{x+1}{x^3+1} \le \frac{27}{49}x + \frac{54}{49}$ for $x\in(0,1)$. We obtain similar inequalities by replacing x by y and z respectively, then add the three inequalities to get

$$\frac{x+1}{x^3+1} + \frac{y+1}{y^3+1} + \frac{z+1}{z^3+1} \le \frac{27}{49}(x+y+z) + 3\cdot\frac{54}{49} = \frac{27}{7},$$

where for the last equality we used $x+y+z=1$.

4045. *Proposed by Galav Kapoor.*

Suppose that we have a natural number n such that $n\ge 10$. Show that by changing at most one digit of n, we can compose a number of the form $x^2+y^2+10z^2$, where x, y, z are integers.

We received two correct solutions. We present the solution by Roy Barbara.

Recall that Legendre's three-square theorem states that a natural number is a sum of three squares if and only if it is not of the form $4^m(8k+7)$. In particular, any natural number of the form $4k+2$ is a sum of three squares.

Now let $2k+1$ be any odd natural number. Then we can write $4k+2 = a^2+b^2+c^2$. Using $a^2+b^2+c^2 \equiv 2 \pmod 4$, it is clear that exactly one of a, b, c is even, say c. Setting $x = \frac{1}{2}(a+b)$, $y = \frac{1}{2}(a-b)$, $c = 2z$ yields

$$4k+2 = 2x^2+2y^2+4z^2,$$

whence

$$2k+1 = x^2+y^2+2z^2.$$

Finally let $n \geq 10$. By changing the last digit of n to a 5 (if necessary), we obtain a number of the form $10k+5$ for which we have

$$10k+5 = 5x^2+5y^2+10z^2 = (2x+y)^2+(2y-x)^2+10z^2.$$

4046. *Proposed by Michel Bataille.*

Let a, b, c be nonnegative real numbers such that $\sqrt{a}+\sqrt{b}+\sqrt{c} \geq 1$. Prove that

$$a^2+b^2+c^2+7(ab+bc+ca) \geq \sqrt{8(a+b)(b+c)(c+a)}.$$

Two correct solutions were received. A purported counterexample that was submitted had an error. We present both solutions.

Solution 1, by Madhav R. Modak.

$$\begin{aligned}
&\sqrt{8(a+b)(b+c)(c+a)}\\
&\leq (\sqrt{a}+\sqrt{b}+\sqrt{c})\sqrt{8(a+b)(b+c)(c+a)}\\
&= \sqrt{(4ab+4ca)[2(a+b)(c+a)]} + \sqrt{(4bc+4ab)[2(a+b)(b+c)]}\\
&\qquad\qquad + \sqrt{(4ca+4bc)[2(b+c)(c+a)]}\\
&\leq \frac{1}{2}[(4ab+4ca)+2(a+b)(c+a)] + \frac{1}{2}[(4bc+4ab)+2(a+b)(b+c)]\\
&\qquad\qquad + \frac{1}{2}[(4ca+4bc)+2(b+c)(c+a)]\\
&= 4(ab+bc+ca)+(a^2+ab+ca+bc)+(b^2+ab+bc+ca)+(c^2+ca+bc+ab)\\
&= a^2+b^2+c^2+7(ab+bc+ca),
\end{aligned}$$

which yields the desired result.

Solution 2, by the proposer.

Since

$$(a^2+3ab+3ca+bc)^2 = 8a(a+b)(b+c)(c+a)+(a-b)^2(a-c)^2,$$

it follows that

$$a^2 + 3ab + 3ca + bc \geq \sqrt{a}(\sqrt{8(a+b)(b+c)(c+a)}).$$

Similarly

$$b^2 + 3ab + 3bc + ca \geq \sqrt{b}(\sqrt{8(a+b)(b+c)(c+a)})$$

and

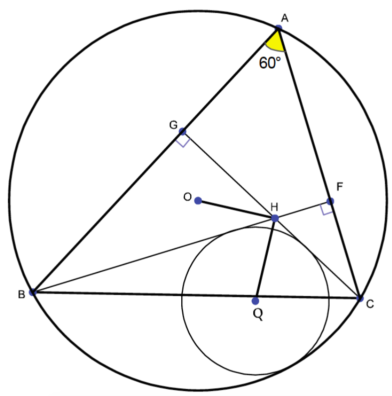
$$c^2 + 3ca + 3bc + ab \geq \sqrt{c}(\sqrt{8(a+b)(b+c)(c+a)}).$$

Adding these three inequalities yields the result.

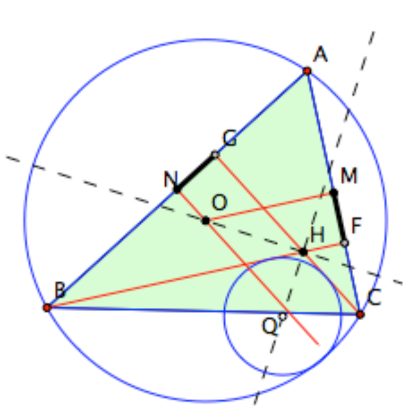
Editor's comment. Equality holds if and only if $a = b = c = 1/9$.

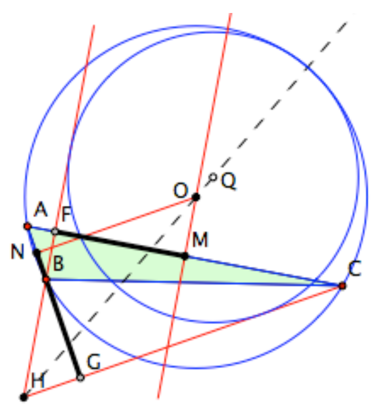
4047$\star$. *Proposed by Abdilkadir Altintaş.*

Let ABC be a triangle with circumcircle O, orthocenter H and $\angle BAC = 60^\circ$. Suppose the circle with centre Q is tangent to BH, CH and the circumcircle of ABC. Show that $OH \perp HQ$.



All 14 submissions we received were correct. We feature two solutions.





Solution 1 is a composite of solutions by Václav Konečný and Edmund Swylan.

The statement of the problem is faulty: Because the plane is partitioned into as many as eight regions by the circumcircle of triangle ABC and the lines HB and HC, there could be eight tritangent circles and, consequently, eight choices for Q, of which some lie on the Euler line OH (in which case the lines OH and HQ would be coincident, not perpendicular).

[*Editor's comment*: Since the centres of all circles tangent to HB and HC would lie on a bisector of $\angle BHC$, the requirement that the circle be tangent also to the circumcircle was perhaps included to limit the choice of tritangent circle to the incircle of the curvilinear triangle HBC (formed by the line segments HB and HC and the circular arc BC). Then the problem has been correctly stated for an acute ΔABC, but it is still not correct when there is an obtuse angle at B or C.]

The exact location of Q is not relevant to the correct theorem:

> For any circle tangent to the lines HB and HC, its centre Q must belong to one of the two bisectors of $\angle BHC$, and so must O.

The claim for Q is a familiar theorem, while the claim for O depends on $\angle BAC = 60^\circ$ and must be proved.

As in the figure, denote the midpoints of AC and AB by M and N, respectively, and the feet of the altitudes to these lines by F and G. Then the segments FM and GN are congruent:

$$FM = |FA - MA| = \left| AB \cdot \cos 60^\circ - \frac{AC}{2} \right| = \frac{1}{2}|AB - AC|$$

and

$$GN = |GA - NA| = \left| AC \cdot \cos 60^\circ - \frac{AB}{2} \right| = \frac{1}{2}|AC - AB|.$$

Then the lines BF, CG, MO, NO form the sides of a rhombus for which the line OH is a diagonal. Thus OH bisects one of the angles formed by the lines HF and HG, as claimed.

Solution 2 is a composite of similar solutions by Šefket Arslanagić, Ricardo Barroso Campos, Prithwijit De (done independently), and Adnan Ibrić with Salem Malikić.

As in the figure that accompanies the statement of the problem, we assume that the given triangle is acute, and that F and G are the feet of the altitudes from B and C, respectively. Observe that $\angle BHC = \angle FHG = 120^\circ$ (since $\angle A$ is 60° and is opposite $\angle FHG$ in the circle whose diameter is AH). Furthermore, $\angle BOC = 120^\circ$ also (because O is the centre of the circumcircle so that the angle there is twice the angle $BAC = 60^\circ$ which is inscribed in that circle). Because O and H are on the same side of BC, it follows that B, O, H, C are concyclic. Finally, note that because ΔBOC is isosceles, $\angle OCB = 30^\circ$. Since HQ is the bisector of $\angle BHC$,

$$\angle BHQ = \frac{1}{2}\angle BHC = 60^\circ.$$

Combine this with

$$\angle OHB = \angle OCB = 30^\circ,$$

and conclude that

$$\angle OHQ = \angle OHB + \angle BHQ = 90^\circ.$$

Editor's Comments. Essentially the same problem has appeared before in ***Crux*** [1988: 165; 1990: 103] as Problem **M1046**, which was taken from the 1987 U.S.S.R journal *Kvant*:

> If $\angle A = 60^\circ$ then one of the bisectors of the angle between the altitudes from B and C passes through O.

This and related properties were discussed under the heading "Property 3" in the article "Recurring ***Crux*** Configurations 3: Triangles Whose Angles Satisfy $2B = C + A$" [2011: 350].

4048. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Let $n \geq 2$ be an integer and let $a_k \geq 1$ be real numbers, $1 \leq k \leq n$. Prove the inequality

$$a_1a_2\cdots a_n - \frac{1}{a_1a_2\cdots a_n} \geq \left(a_1 - \frac{1}{a_1}\right) + \left(a_2 - \frac{1}{a_2}\right) + \cdots + \left(a_n - \frac{1}{a_n}\right)$$

and study equality cases.

Thirteen solutions were received, all of which established the inequality. Two of them did not get all the possible conditions for equality, while three others neglected to consider when equality occurred. The solutions were all similar to the one presented below.

Let

$$f(x) = x - \frac{1}{x}$$

and observe that, for $x, y \geq 1$,

$$f(xy) - f(x) - f(y) = (xy)^{-1}(xy-1)(x-1)(y-1) \geq 0$$

with equality if and only if at least one of x and y is equal to 1.

We establish the result by induction.

The foregoing shows that it is true for $n = 2$. Suppose that the inequality holds for $n = m \geq 2$ with equality iff all but at most one of $a_1, a_2, \ldots, a_m$ is equal to 1. Then, by the foregoing property of f and the result for $n = m$,

$$f(a_1a_2\cdots a_ma_{m+1}) \geq f(a_1a_2\cdots a_m) + f(a_{m+1}) \geq \sum_{k=1}^{m} f(a_k) + f(a_{m+1}) = \sum_{k=1}^{m+1} f(a_k).$$

Equality holds if and only if either

- $a_1a_2\cdots a_m = 1$, in which case $a_1 = a_2 = \cdots = a_m = 1$, or
- $a_{m+1} = 1$ and $f(a_1a_2\cdots a_m) = \sum_{k=1}^{m} f(a_k)$.

In either case, all but at most one of $a_1, a_2, \ldots, a_{m+1}$ is equal to 1.

Editor's comments. One can also peel off the last two terms in the product so that the induction step becomes

$$f(a_1a_2\ldots a_m a_{m+1}) \geq \sum_{k=1}^{m-1} f(a_k) + f(a_m a_{m+1}) \geq \sum_{k=1}^{m+1} f(a_k).$$

Edmund Swyland observed that if, for any i and j, you replaced the pair (a_i, a_j) by $(a_ia_j, 1)$, the left side $f(a_1a_2\cdots a_n)$ of the inequality remained unchanged, but the right side increased. Thus we can reduce the problem to establishing that it holds when all but two of the a_i are equal to 1, and this now involves dealing with the case $n = 2$.

Kee-Wai Lau pointed out that an easy induction argument yields

$$f(a_1a_2\cdots a_n) - \sum_{k=1}^{n} f(a_k) = \sum_{k=2}^{n} \frac{(a_1a_2\cdots a_{k-1} - 1)(a_k - 1)(a_1a_2\cdots a_k - 1)}{a_1a_2\cdots a_k}.$$

4049. *Proposed by Mihaela Berindeanu.*

Evaluate

$$\int \frac{\sin x - x\cos x}{(x + \sin x)(x + 2\sin x)} dx$$

for all $x \in (0, \pi/2)$.

We received 16 submissions all of which were correct. We present a composite of the nearly identical solutions given by Adnan Ali, Michel Bataille, Prithwijit De, Joseph Ling and Albert Stadler, all done independently

Let I denote the given integral. Since it is readily checked that

$$(1 + \cos x)(x + 2\sin x) - (1 + 2\cos x)(x + \sin x) = \sin x - x\cos x,$$

we have

$$\begin{aligned} I &= \int \left(\frac{1 + \cos x}{x + \sin x} - \frac{1 + 2\cos x}{x + 2\sin x} \right) dx \\ &= \ln(x + \sin x) - \ln(x + 2\sin x) + C \\ &= \ln\left(\frac{x + \sin x}{x + 2\sin x} \right) + C, \end{aligned}$$

where C is an arbitrary constant.

4050. *Proposed by Mehtaab Sawhney.*

Prove that
$$\sum_{k=0}^{2n} \binom{4n}{k, k, 2n-k, 2n-k} = \binom{4n}{2n}^2$$
for all nonnegative integers n.

We received twelve correct solutions which were split between an arithmetic proof and a proof by double counting, so we present a solution of each type.

Solution 1, by C.R. Pranesachar.

We have
$$\begin{aligned}\sum_{k=0}^{2n} \binom{4n}{k, k, 2n-k, 2n-k} &= \sum_{k=0}^{2n} \binom{4n}{2n}\binom{2n}{k}^2 \\ &= \binom{4n}{2n}\sum_{k=0}^{2n}\binom{2n}{k}\binom{2n}{2n-k} \\ &= \binom{4n}{2n}^2,\end{aligned}$$
where the last equality is due to Vandermonde's identity.

Solution 2, by Joseph DiMuro.

Let's say we have a classroom with $4n$ students. The teacher wants to choose $2n$ of them to work on one project and $2n$ of them to work on a second project (independently of each other – students may be assigned to both or neither of the projects). In how many ways can the teacher assign students to the projects?

On one hand there are $\binom{4n}{2n}$ ways to choose the students for each of the two projects, thus $\binom{4n}{2n}^2$ possibilities altogether.

On the other hand note that if k students are assigned to both projects then $2n-k$ will be assigned to just the first project, $2n-k$ to just the second project, and k to neither project. So the teacher can proceed as follows: first decide on the number k of students that will be assigned to both projects, then partition the class into four groups – of size k (both projects), $2n-k$ (first project), $2n-k$ (second project), and k (neither project). There are
$$\sum_{k=0}^{2n} \binom{4n}{k, k, 2n-k, 2n-k}$$
ways to do this. Thus the two sides in the problem are equal.

AUTHORS' INDEX

Solvers and proposers appearing in this issue

(Bold font indicates featured solution.)

Proposers

Solvers - individuals

Solvers - collaborations