Crux Mathematicorum

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IN THIS ISSUE / DANS CE NUMÉRO

YEAR-END FINALE

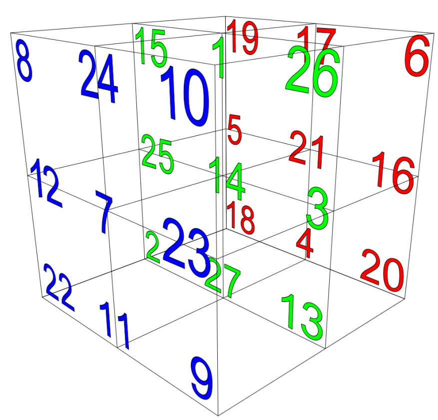
42 is a number that fascinated many. Lewis Carroll, for example, was a big fan of this number: *Alice's Adventures in Wonderland* has 42 illustrations; there is Rule 42 in *Alice's Adventures in Wonderland* and in *The Hunting of the Snark*; the combined age of the White and Red Queens is 74088 days, which is $42 \times 42 \times 42$. The list goes on. And it wasn't just Lewis Carroll, because according to Douglas Adams and the *The Hitchhiker's Guide to the Galaxy*, 42 is **The Answer to the Ultimate Question of Life, The Universe, and Everything**.

But it is not just the number that makes this Volume 42 special. This Volume serves as a major milestone in the recent ***Crux*** history: this is the last Volume with backlog as we plan to be fully caught up and start publishing our journal on time starting in April 2017.

This milestone comes as a result of the hard work on behalf of my amazing Editorial Board. I cannot say enough good things about my editors who are the driving force behind each issue of ***Crux*** . I am truly privileged to work with this group of people, whose expertise and commitment continues to impress me. I thank you all for supporting ***Crux*** and me personally during this past year through my productive and reproductive activities. On this note, I really appreciate the support from the CMS office who happily agreed to provide me with an assistant during my maternity leave in order to keep ***Crux*** progress on track.

To ***Crux*** readers, you are the reason this journal exists and I am extremely happy to see more and more submissions and letters coming in. Thank you for supporting ***Crux*** and actively participating in its production. I am always happy to hear from you, so drop me a line.

So here is to Volume 43 and, to commemorate Volume 42, check out this $3 \times 3 \times 3$ magic cube with magic constant 42:



Kseniya Garaschuk

THE CONTEST CORNER

No. 50

John McLoughlin

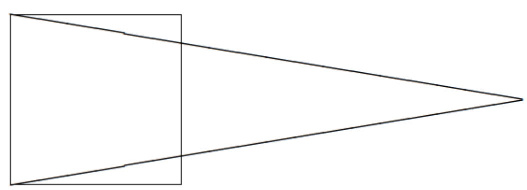
Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1 mai 2017**.

La rédaction souhaite remercier André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.

CC246. Les entiers $1, 2, \ldots, 9$ sont placés au hasard de manière à remplir un tableau 3×3. Quelle est la probabilité pour que les sommes des nombres de chaque rangée et de chaque colonne soient toutes impaires?

CC247. Un triangle isocèle et un carré partagent une même base. L'aire du triangle est le double de l'aire du carré. Le côté du carré opposé à la base commune coupe le triangle de manière à former un petit triangle et un trapèze. Quel est le rapport de l'aire du petit triangle à l'aire du trapèze?



CC248. Neuf points sont placés dans $\mathbb{R}^8$ de manière qu'il y ait une distance de 1 entre chaque paire de points. (Il s'agit donc d'un simplexe régulier ayant des arêtes de longueur 1.) Déterminer le rayon de la plus petite hypersphère qui contient tous les 9 points.

CC249. Soit S un ensemble d'entiers. L'ensemble de toutes les sommes possibles de deux éléments distincts de S est $\{7, 8, 10, 11, 13, 14, 16, 19, 20, 22\}$. Chacune de ces sommes ne peut être produite que d'une seule façon. Soit X la moyenne de S et Y la médiane de S. Déterminer $X + Y$.

CC250. Un tournoi d'échecs est organisé pour deux élèves de 9^e année et n élèves de 10^e année. Chacun de ces élèves joue une partie contre chaque autre élève. Une victoire vaut un point, un match nul vaut un demi-point et une défaite vaut zéro point. Les deux élèves de 9e année remportent un total de 8 points. Tous les élèves de 10e année remportent le même nombre de points. Chaque élève de 9e année obtient moins de points que n'importe quel élève de 10e année. Combien y a-t-il d'élèves de 10e année?

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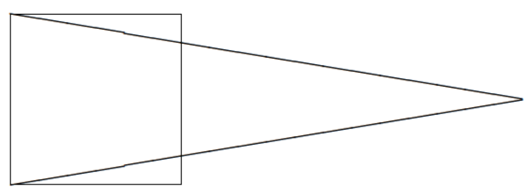
CC246. Place the numbers $1, 2, \ldots, 9$ at random so that they fill a 3×3 grid. What is the probability that each of the row sums and each of the column sums is odd?

CC247. An isosceles triangle and a square share the same base. The area of the triangle is twice the area of the square. The square splits the larger triangle into a smaller triangle and a trapezoid. What is the ratio of the area of that smaller triangle to the area of the trapezoid?



CC248. Nine points are arranged in $\mathbb{R}^8$ so that each pair of points is distance 1 apart. (That is, this is a regular simplex of edge length 1.) Find the radius of the smallest hypersphere that contains all 9 points.

CC249. Let S be a set of integers. The set of all possible sums of two different elements of S is $\{7, 8, 10, 11, 13, 14, 16, 19, 20, 22\}$. Each of these sums happens in only one way. If X is the mean of the set S and Y is the median of the set S, find $X + Y$.

CC250. Two 9th graders and n 10th graders play a chess tournament. Every student plays every other student once. A student scores one point for winning a match, one half of a point for drawing a match, and zero points for losing a match. The total number of points scored by the two 9th graders was 8. Each 10th grader scored the same number of points as each other. The two 9th grade students each had scores lower than any 10th grader. How many 10th grade students were there?

CONTEST CORNER SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2015: 41(10), p. 418–419.

CC196. You are given nine square tiles, with sides of lengths 1, 4, 7, 8, 9, 10, 14, 15 and 18 units, respectively. They can be used to tile a rectangle without gaps or overlaps. Find the lengths of the sides of the rectangle, and show how to arrange the tiles.

Originally question 1 from the 2007 University of New South Wales School Mathematics Competition, Junior Division.

We received four correct solutions. The solution presented below uses the reasoning given by the Missouri State University Problem Solving Group.

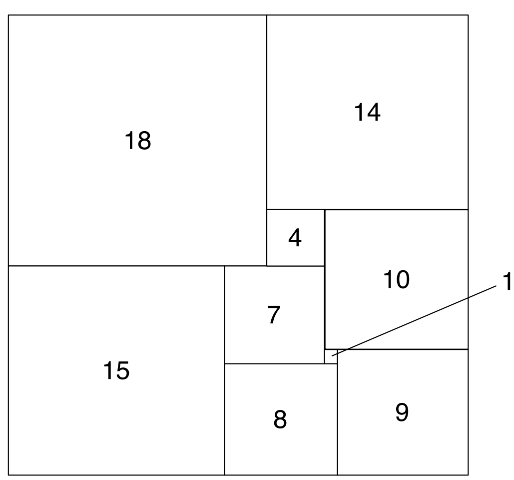
The area of the rectangle is

$$1^2 + 4^2 + 7^2 + 8^2 + 9^2 + 10^2 + 14^2 + 15^2 + 18^2 = 1056.$$

Let its dimensions be x and y, with $x \geq y$. Then x and y must be positive integers such that $x \cdot y = 1056$, and x must be greater than or equal to the side of the largest square: $x \leq 18$. The only possible dimensions are

$$22 \times 48, \quad 24 \times 44 \quad \text{and} \quad 32 \times 33.$$

In the first two cases, placing the square of side length 18 in the rectangle leaves strips of width 4 and 6, respectively, that cannot be tiled. The 32×33 rectangle can be tiled as shown below.



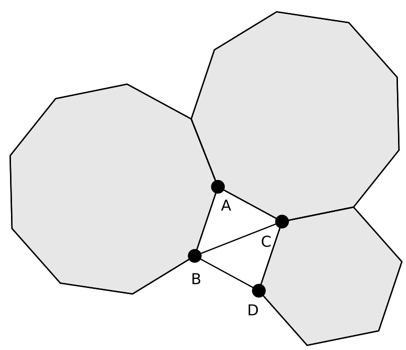
CC197. Let a and b be two randomly chosen positive integers (not necessarily distinct) such that $a, b \leq 100$. What is the probability that the units digit of $3^a + 7^b$ is 6?

Originally question 5 of the Junior-Senior 5-Person Team Test portion of 2011 John O'Bryan Mathematical Competition.

We received four correct solutions and present the solution by Kathleen Lewis.

A number of the form 3^a can have a units digit of 3, 9, 7, or 1. These occur in a cycle of length four, so each occurs equally often in the interval $1 \leq a \leq 100$. In the same way, the powers of 7 have final digits of 7, 9, 3, or 1, occurring equally often in this interval. The sum $3^a + 7^b$ has a units digit of 6 if both terms end in a 3, or if one term ends in 9 and the other in 7 (in either order). Since a and b are chosen independently, each pair of final digits occurs with probability $1/4 \cdot 1/4 = 1/16$. Since there are three pairs giving a sum ending in 6, the probability of this is $3/16$.

CC198. The diagram shows five polygons placed together edge to edge: two triangles, a regular hexagon and two regular nonagons.



Prove that each of the triangles is isosceles.

Originally question 2 of the 2015 Mathematical Olympiad for Girls organised by the United Kingdom Mathematics Trust.

We received three solutions. We present the solution by Titu Zvonaru.

Let ABC and BCD be the given triangles, with AB, AC sides of the nonagons and DC the side of the hexagon. It easy to see that

$$\angle BAC = 360^\circ - 140^\circ - 140^\circ = 80^\circ.$$

Since the triangle ABC is isosceles, it follows that $\angle ABC = \angle ACB = 50^\circ$. We deduce that

$$\angle BCD = 360^\circ - 140^\circ - 120^\circ - 50^\circ = 50^\circ.$$

Then

$$\angle BAC + \angle ACB + \angle BCD = 180^\circ,$$

which yields $AB||DC$. Now, $AB = CD$, hence the quadrilateral $ABDC$ is a parallelogram. Because $AC = CD$, $ABDC$ is a rhombus; it results that $DC = BD$.

CC199. For any real number u, let $\{u\} = u - \lfloor u \rfloor$ denote the fractional part of u (here, $\lfloor u \rfloor$ denotes the greatest integer less than or equal to u). For example, $\{\pi\} = \pi - 3$ and $\{-2.4\} = -2.4 - (-3) = 0.6$. Find all real numbers x such that $\{(x+1)^3\} = \{x^3\}$.

Originally question 7 from the 2005 Eleventh Annual Iowa Intercollegiate Mathematics Competition.

We received one correct solution. We present the solution of the Missouri State University Problem Solving Group.

If $\{(x+1)^3\} = \{x^3\}$, then $(x+1)^3$ and x^3 must differ by an integer. Therefore, $(x+1)^3 - x^3$ is an integer and it follows that $3x^2 + 3x = n$ is also an integer. Solving for x gives

$$x = \frac{-3 \pm \sqrt{9+12n}}{6}$$

with $n \geq 0$ in order that x be real.

CC200. Find all positive integers m and n such that $m! + 76 = n^2$, where $m! = m \times (m-1) \times \cdots \times 2 \times 1$.

Originally question 5b of the 2015 Mathematical Olympiad for Girls organised by the United Kingdom Mathematics Trust.

We received eight correct solutions. We present the solution of Billy Jin.

The only solutions are $(m, n) = (4, 10)$ and $(5, 14)$.

Let $f(m) = m! + 76$. Since $f(1) = 77$, $f(2) = 78$, $f(3) = 82$, $f(4) = 100 = 10^2$, $f(5) = 196 = 14^2$, and $f(6) = 796$, we see that for $m \leq 6$, $f(m)$ is a square only when $m = 4$ or $m = 5$.

For any $m \geq 7$ we have $f(m) \equiv 6 \pmod 7$ since $m! \equiv 0 \pmod 7$. However, direct calculations show that for $n \equiv 0, 1, 2, 3, 4, 5, 6 \pmod 7$, we have

$$n^2 \equiv 0, 1, 4, 2, 2, 4, 1 \pmod 7,$$

respectively. Hence, $n^2 \not\equiv 6 \pmod 7$ for any integer n. Thus, $f(m)$ is not a square for any $m \geq 7$, and our proof is complete.

THE OLYMPIAD CORNER

No. 348

Carmen Bruni

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1 mai 2017**.

La rédaction souhaite remercier André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.

OC306. Déterminer tous les entiers strictement positifs n pour lesquels

$$\frac{10^n}{n^3+n^2+n+1}$$

est un entier.

OC307. Vingt villages sont situés sur une rive d'une rivière droite et quinze villages sont situés sur l'autre rive. On aimerait construire des ponts qui relieraient chacun un village sur une rive à un village sur l'autre rive. Les ponts ne doivent pas se croiser et il doit être possible de se rendre de n'importe quel village à n'importe quel autre village en n'utilisant que ces ponts (donc sans utiliser des chemins qui existent entre les villages sur une même rive). Combien existe-t-il de façons de placer les ponts?

OC308. Soit n un entier strictement positif et soit $d_1, d_2, \ldots, d_k$ ses diviseurs positifs. On considère le nombre

$$f(n) = (-1)^{d_1}d_1 + (-1)^{d_2}d_2 + \cdots + (-1)^{d_k}d_k.$$

On suppose que $f(n)$ est une puissance de 2. Démontrer que si m est un entier supérieur à 1, alors m^2 n'est pas un diviseur de n.

OC309. Soit A, B, D, E, F et C six points situés dans cet ordre sur un cercle de manière que $AB = AC$. Soit $P = AD \cap BE$, $R = AF \cap CE$, $Q = BF \cap CD$, $S = AD \cap BF$ et $T = AF \cap CD$. Soit K un point sur ST tel que $\angle QKS = \angle ECA$. Démontrer que

$$\frac{SK}{KT} = \frac{PQ}{QR}.$$

OC310. Soit k un entier strictement positif, $n = \left(2^k\right)!$ et $\sigma(n)$ la somme des diviseurs positifs de n. Démontrer que $\sigma(n)$ admet au moins un diviseur premier supérieur à 2^k.

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OC306. Find all positive integers n such that

$$\frac{10^n}{n^3 + n^2 + n + 1}$$

is an integer.

OC307. Several small villages are situated on the banks of a straight river. On one side, there are 20 villages in a row, and on the other there are 15 villages in a row. We would like to build bridges, each of which connects a village on the one side with a village on the other side. The bridges must not cross, and it should be possible to get from any village to any other village using only those bridges (and not any roads that might exist between villages on the same side of the river). How many different ways are there to build the bridges?

OC308. Let n be a positive integer and let $d_1, d_2, \ldots, d_k$ be its positive divisors. Consider the number

$$f(n) = (-1)^{d_1} d_1 + (-1)^{d_2} d_2 + \cdots + (-1)^{d_k} d_k.$$

Assume $f(n)$ is a power of 2. Show that if m is an integer greater than 1, then m^2 does not divide n.

OC309. Let A, B, D, E, F, C be six points lie on a circle (in order) satisfy $AB = AC$. Let $P = AD \cap BE$, $R = AF \cap CE$, $Q = BF \cap CD$, $S = AD \cap BF$ and $T = AF \cap CD$. Let K be a point lying on ST satisfying $\angle QKS = \angle ECA$. Prove that

$$\frac{SK}{KT} = \frac{PQ}{QR}.$$

OC310. For a positive integer k, let $n = \left(2^k\right)!$ and let $\sigma(n)$ denote the sum of all positive divisors of n. Prove that $\sigma(n)$ has at least one prime divisor larger than 2^k.

OLYMPIAD SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2015: 41(8), p. 332–333.

OC246. Find all positive integers x, y, z satisfying the equation $x^3 = 3^y 7^z + 8$.

Originally problem 2 from the second round of day 1 of the 2014 Turkey Mathematical Olympiad.

We received 3 correct submissions and 1 incorrect submission. We present the solution by Oliver Geupel.

A solution is $(x, y, z) = (11, 3, 2)$ and we show that it is unique.

Assume (x, y, z) is a solution. Then $3^y 7^z = x^3 - 8 = (x-2)(x^2+2x+4)$. Since $x^2+2x+4-x(x-2) = 4(x+1)$, every common divisor of $x-2$ and x^2+2x+4 is also a divisor of $4(x+1)$. Hence, neither 7 nor 9 are common divisors of $x-2$ and x^2+2x+4. As a consequence, we obtain four cases, which we will consider in succession:

1. $\{x-2,\ x^2+2x+4\} = \{1,\ 3^y 7^z\}$,
2. $\{x-2,\ x^2+2x+4\} = \{3,\ 3^{y-1} 7^z\}$,
3. $\{x-2,\ x^2+2x+4\} = \{3^y,\ 7^z\}$,
4. $\{x-2,\ x^2+2x+4\} = \{3^{y-1},\ 3\cdot 7^z\}$.

Cases 1 and 2. We obtain $x = 3$ or $x = 5$, which is impossible.

Case 3. If $x = 3^y + 2$ then $x \equiv -1 \pmod 3$; whence

$$7^z = x^2 + 2x + 4 \equiv 0 \pmod 3,$$

a contradiction.

On the other hand, if $x = 7^z + 2$, then $x \equiv 0 \pmod 3$; hence

$$3^y = x^2 + 2x + 4 \equiv 1 \pmod 3,$$

a contradiction.

Case 4. If $x = 3 \cdot 7^z + 2$ then

$$3^{y-1} = x^2 + 2x + 4 = 9 \cdot 7^{2z} + 18 \cdot 7^z + 12 \equiv 3 \pmod 9;$$

whence $y = 2$ and $x^2 + 2x + 4 = 3$, which is impossible.

Therefore it must hold $x = 3^{y-1} + 2$. Then,

$$(-1)^{z+1} \equiv 3 \cdot 7^z = x^2 + 2x + 4 = 9^{y-1} + 2 \cdot 3^y + 12 \equiv 1 + 2 \cdot (-1)^y \equiv -1 \pmod 4.$$

Thus z is even, say $z = 2u$. We obtain $7^{2u} = 3^{2y-3} + 2 \cdot 3^{y-1} + 4$, that is, $y \geq 2$ and $(7^u - 2)(7^u + 2) = 3^{y-1}(3^{y-2} + 2)$. Since $7^u - 2 \equiv -1 \pmod 3$, 3 does not divide $7^u - 2$, so that $7^u + 2 = 3^{y-1}k$ where k is a divisor of $3^{y-2} + 2$. We have either $k \geq 5$ or $k = 1$. If $k \geq 5$ then

$$75 \cdot 3^{y-2} \leq 5k \cdot 3^{y-1} = 5(7^u + 2) = 5(7^u - 2 + 4) = 5\left(\frac{3^{y-2} + 2}{k} + 4\right) \leq 3^{y-2} + 22,$$

a contradiction. Hence $k = 1$ and

$$3^{y-1} = 7^u + 2 = (7^u - 2) + 4 = (3^{y-2} + 2) + 4 = 3^{y-2} + 6,$$

so that $y = 3$, $u = 1$, $z = 2$, and $x = 11$.

OC247. Let $a_1 \leq a_2 \leq \cdots$ be a non-decreasing sequence of positive integers. A positive integer n is called good if there is an index i such that $n = \frac{i}{a_i}$. Prove that if 2013 is good, then so is 20.

Originally problem 3 from day 1 of the 2014 German Team Selection Test.

We received 3 correct submissions. We present the solution by Missouri State University Problem Solving Group.

We prove a more general result, that if a is good and $b < a$, then b is good. Assume towards a contradiction that $b < a$ is not good. Then we see that $a_{nb} \neq n$ for all positive integers n, for otherwise,

$$b = \frac{nb}{n} = \frac{nb}{a_{n,b}}$$

contradicting that b is not good. It follows that $a_b \neq 1$ and so $a_b \geq 2$. Now, let's assume that $a_{nb} \geq n + 1$ for some positive integer n. Then

$$n + 1 \leq a_{nb} \leq a_{(n+1)b} \neq n + 1,$$

and hence $a_{(n+1)b} \geq n + 2$. Therefore, by induction, $a_{nb} \geq n + 1$ for all positive integers n. Let j be any positive integer. If $j < b$, then $a_j \geq 1 > \frac{j}{b}$. If on the other hand, $j \geq b$, then there exists a positive integer n such that $nb \leq j < (n + 1)b$. Thus,

$$a_j \geq a_{nb} \geq n + 1 > \frac{j}{b} > \frac{j}{a}$$

whence $a \neq \frac{j}{a_j}$. Also, since for every positive integer j,

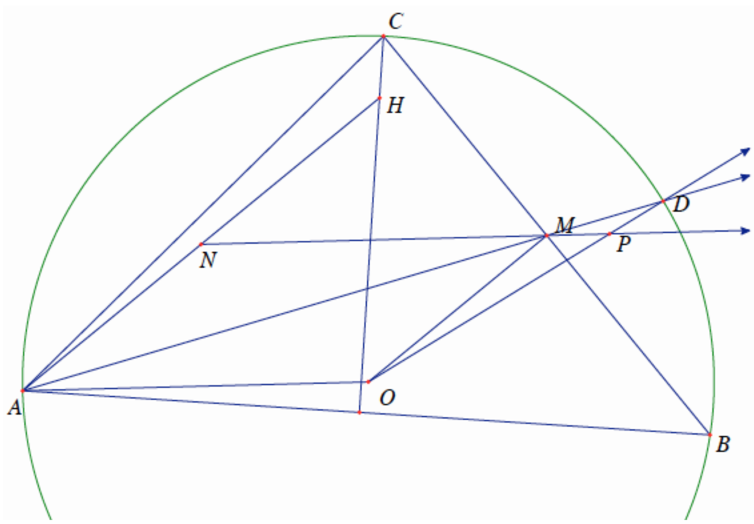
$$a_j > \frac{j}{b} > \frac{j}{a},$$

we have that $a \neq \frac{j}{a_j}$ for every positive integer n showing that a is not good and completing the proof.

OC248. Let B and C be two fixed points on a circle centered at O that are not diametrically opposed. Let A be a variable point on the circle distinct from B and C and not belonging to the perpendicular bisector of BC. Let H be the orthocenter of $\triangle ABC$, and M and N be the midpoints of the segments BC and AH, respectively. The line AM intersects the circle again at D, and finally, NM and OD intersect at P. Determine the locus of points P as A moves around the circle.

Originally problem 3 from day 1 of the 2014 Spain Mathematical Olympiad.

We received 3 correct submissions. We present the solution by John Heuver.



Let R be the circumradius of triangle ABC with fixed points O and M. We will show that the locus is an ellipse with focal points O and M and $PO + PM = R$. Note that AH and OM are perpendicular to BC hence $AH \parallel OM$ while

$$AH = 2R\cos\angle A \quad \text{and} \quad OM = R\cos\angle A.$$

It follows that

$$AN = \frac{1}{2}OH = OM,$$

which lets us conclude that $AOMN$ is a parallelogram.

Triangle AOD is isosceles hence

$$\angle OAD = \angle ODA \quad \text{and} \quad \angle AOD = \angle AMN = \angle PMD.$$

This lets us conclude that $\triangle PMD$ is isosceles and thus $PM = PD$. Consequently

$$OP + PD = OP + PM = R,$$

which implies that the locus is an ellipse.

OC249. Determine all the functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the following:

$$f(xf(x) + f(x)f(y) + y - 1) = f(xf(x) + xy) + y - 1.$$

Originally problem 2 from day 1 of the 2014 Spain National Olympiad.

We received 4 correct submissions. We present the solution by Billy Jin.

Observe that if $f(x) = x$, then both sides of the above equation are equal to $x^2 + xy + y - 1$. We claim that $f(x) = x$ is the only function satisfying the given equation.

First, plug in $x = 0$ to get

$$f(f(0)f(y) + y - 1) = f(0) + y - 1. \tag{1}$$

Note that the right hand side of the above equation ranges over all of $\mathbb{R}$ as y ranges over $\mathbb{R}$. Hence, f is surjective.

Next, plug $y = 1$ in the original equation to get

$$f(xf(x) + f(x)f(1)) = f(xf(x) + x). \tag{2}$$

Since f is surjective, there exists some $x_0 \in \mathbb{R}$ with $f(x_0) = 0$. Substitute $x = x_0$ in (2) to get $f(0) = f(x_0)$. Hence, $f(0) = 0$.

Finally, substitute $x = 0$ in the original equation and use the fact that $f(0) = 0$ to get

$$f(y - 1) = y - 1.$$

Thus, $f(x) = x$ for all $x \in \mathbb{R}$, completing the proof.

OC250. Alice and Bob play a number game. Starting with a positive integer n they take turns changing the number with Alice going first. Each player may change the current number k to either $k - 1$ or $\lceil k/2 \rceil$. The person who changes 1 to 0 wins. Determine all n such that Alice has a winning strategy.

Originally problem 5 from the second round of the 2014 Singapore Senior Math Olympiad.

We received 2 correct submissions. We present the solution by Oliver Guepel.

For every odd number $m \geq 3$ there are unique integers $a \geq 1$ and $b \geq 0$ such that $m = 2^a(2b + 1) + 1$. If a is odd then let us say m has type 1 when $b = 0$, whereas m has type 2 when $b > 0$. If a is even then let us say m has type 3 when $b = 0$, whereas m has type 4 if $b > 0$.

For any specific positive integer n let $P(n)$ denote the assertion that Alice has a winning strategy if and only if either $n = 1$ or n is an even number greater than 2 or n is an odd number of type 1 or type 4; otherwise Bob has a winning strategy.

We prove $P(n)$ for every n by mathematical induction.

Alice wins when $n = 1$ or $n = 3$ and loses when $n = 2$; hence $P(1)$, $P(2)$, and $P(3)$ are valid.

Let n be at least 4 and suppose $P(m)$ holds whenever $1 \leq m < n$. We have to prove $P(n)$.

First consider the case where n is even. If $n - 1$ is type 2 or type 3, then Alice changes the number to $n-1$, for which Bob has no winning strategy by induction; hence Alice will win. If $n - 1$ is type 1, then there is a nonnegative integer k such that $n - 1 = 2^{2k+1} + 1$; whence $n/2 = 2^{2k} + 1$. This is either 2 or a type 3 number, which does not give Bob a winning strategy by induction. Therefore, Alice has a winning strategy when changing n to $n/2$. Finally, if $n - 1$ is type 4, then there is a nonnegative k and $b \geq 1$ such that $n - 1 = 2^{2k+2}(2b + 1) + 1$; whence $n/2 = 2^{2k+1}(2b+1)+1$, which is type 2. By induction, Alice has a winning strategy when changing n to $n/2$. Consequently, it holds $P(n)$.

It remains to consider the case when $n \geq 5$ is odd. We consider the four subcases when n is type 1 or type 2 or type 3 or type 4 in succession.

Subcase 1: n has type 1. Then there is an integer $k \geq 1$ such that $n = 2^{2k+1} + 1$. By induction, Alice has a winning strategy when changing to $\lceil n/2 \rceil = 2^{2k} + 1$, a type 3 number.

Subcase 2: n has type 2. Then there are integers $k \geq 0$ and $b \geq 1$ such that $n = 2^{2k+1}(2b+1)+1$. But $n-1 = 2^{2k+1}(2b+1)$ is an even number greater than 2. So Bob has a winning strategy when Alice changes to $n-1$ by induction. Moreover $\lceil n/2 \rceil = 2^{2k}(2b + 1) + 1$. This is an even number greater than 2 when $k = 0$ and it is a type 4 odd number when $k > 0$. As a consequence, Alice changing to $\lceil n/2 \rceil$ will give Bob a winning strategy.

Subcase 3: n has type 3. Then there is a nonnegative integer k such that $n = 2^{2k+2} + 1$. By induction, Alice will lose when changing to $n - 1 = 2^{2k+2}$, an even number greater than 2. Alice will also lose when changing to the type 1 odd number $\lceil n/2 \rceil = 2^{2k+1} + 1$ by induction.

Subcase 4: n has type 4. Then n has the form $n = 2^{2k+2}(2b + 1) + 1$ with $k \geq 0$ and $b \geq 1$. By induction, Alice has a winning strategy when changing to the type 2 number $\lceil n/2 \rceil = 2^{2k+1}(2b + 1) + 1$.

We have proved that $P(n)$ holds when $n \geq 5$ is odd.

This completes the induction.

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Editor's Note: Congratulations to Oliver Geupel for submitting 5 correct solutions and special note to Missouri State University Problem Solving Group for solving 4 of 5.

Selected Problems from the Early Years of the Moscow Mathematical Olympiad: Solutions

Zhi Kin Loke

Here we present solutions to the selected Moscow Mathematical Olympiad problems published in ***Crux*** 42(9).

1. Solve the system of equations $x^2 + y^2 - 2z^2 = 2a^2,\ x + y + 2z = 4(a^2 + 1)$ and $z^2 - xy = a^2$ where a is a real constant. (1935)

Solution. We have

$$0 = (x^2 + y^2 - 2z^2) - 2(z^2 - xy) = (x + y)^2 - (2z)^2 = (x + y + 2z)(x + y - 2z).$$

Since $x + y + 2z = 4(a^2 + 1)$, $x + y - 2z = 0$, so that $z = a^2 + 1$. Hence, $x + y = 2(a^2 + 1)$. Now

$$(x - y)^2 = x^2 + y^2 - 2xy = (x^2 + y^2 - 2z^2) + (z^2 - 2xy) = 4a^2,$$

so that $|x - y| = 2|a|$. Hence $(x, y, z) = (a^2 + a + 1, a^2 - a + 1, a^2 + 1)$ or $(a^2 - a + 1, a^2 + a + 1, a^2 + 1)$.

2. Which is larger, 300! or 100^{300}? (1940)

Solution. Clearly, $k! > 2^{k-1}$ for $k \geq 3$. Because

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} < \frac{n^k}{k!},$$

it follows that

$$\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \frac{\binom{n}{k}}{n^k} < \sum_{k=0}^{n} \frac{1}{k!} \leq 1 + 1 + \sum_{k=2}^{n} \frac{1}{2^{k-1}} < 3.$$

Writing $1 + \frac{1}{n}$ as $\frac{\frac{n+1}{3}}{\frac{n}{3}}$, the above inequality becomes $\left(\frac{n}{3}\right)^n > \frac{1}{3}\left(\frac{n+1}{3}\right)^n$.

We now prove by induction that $n! > \left(\frac{n}{3}\right)^n$ for $n \geq 1$. For $n = 1$, we have that $1! = 1 > \frac{1}{3} = \left(\frac{1}{3}\right)^1$. Suppose the result holds for some $n \geq 1$. Then

$$(n + 1)! = (n + 1)n! > \frac{n+1}{3}\left(\frac{n+1}{3}\right)^n = \left(\frac{n+1}{3}\right)^{n+1}.$$

In particular, when $n = 300$, we have $300! > 100^{300}$.

3. a) Find all possible values of a such that for all x and some integers b and c, $(x-a)(x-10)+1=(x-b)(x-c)$.

b) Find all possible triples (a,b,c) of distinct non-zero integers such that the expression $x(x-a)(x-b)(x-c)+1$ is the product of two non-constant polynomials with integer coefficients. (1941)

Solution.

a) We have $x^2-(a+10)x+10a=x^2-(b+c)x+bc$. Since this is true for all x, we must have $a+10=b+c$ and $10a+1=bc$. Hence a must also be an integer. Let $b=a+t$ for some integer t. Then $c=10-t$. We have

$$10a+1=(a+t)(10-t)=10a+(10-a)t-t^2.$$

Hence $1=(10-a)t-t^2$. Since the right side is divisible by t, so is the left side, which is 1. Hence $t=\pm1$. If $t=1$, we have $10-a=2$ so that $a=12$. If $t=-1$, we have $a-10=2$ so that $a=8$.

b) Let $x(x-a)(x-b)(x-c)+1=P(x)Q(x)$, where $P(x)$ and $Q(x)$ are polynomials with integer coefficients. Setting $x=0,\ a,\ b$ and c in turns, we have

$$\begin{aligned} & P(0)Q(0)=P(a)Q(a)=P(b)Q(b)=P(c)Q(c)=1 \\ \Longrightarrow\quad & P(0)=Q(0)=P(a)=Q(a)=P(b)=Q(b)=P(c)=Q(c)=\pm1. \end{aligned}$$

Since the degree of $P(x)Q(x)$ is 4, $P(x)$ must be identical to $Q(x)$, so that both are quadratic. Let $x(x-a)(x-b)(x-c)+1=(px^2+qx+r)^2$. We must have $p^2=r^2=1$. We may assume that $p=1$. Then

$$(x-a)(x-b)(x-c)=\frac{(x^2+qx\pm1)^2-1}{x}=(x+q)(x^2+qx\pm2).$$

By symmetry, we may assume that $q=-a$ as well as $b<c$. Since we know that $(x-b)(x-c)=x^2-ax\pm2$, we have $b+c=a$ and $bc=\pm2$. Hence $(b,c)=(1,2),\ (-1,-2),\ (-1,2)$ or $(-2,1)$. Overall, we have four solutions, namely, $(a,b,c)=(3,1,2),\ (-3,-1,-2),\ (1,-1,2)$ and $(-1,-2,1)$.

4. How many planes are equidistant from four given points not all in a plane? (1938)

Solution. The given points are not all in a plane, so they cannot all be on the same side of any plane equidistant from them. Thus, there are two cases to consider.

Case 1. Three of them are on one side while the fourth is on the other side. The three points on the same side determine a plane Π_1, and let Π_2 be the plane through the fourth point parallel to Π_1. Then Π is the plane halfway between Π_1 and Π_2. There are four such partitions of the four points.

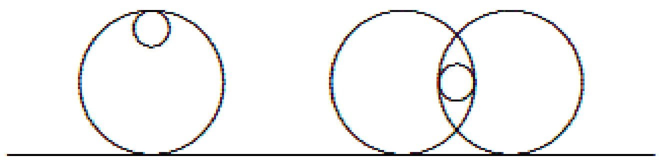
Case 2. Two of the points are on each side of Π. They determine two skew lines, and we can construct two parallel planes each containing one of these lines. Then Π is the plane halfway between these two planes. There are three such partitions of the four points.

Overall, the total number of solutions is seven.

5. Given a line and a circle, construct a unit circle tangent to both. How many solutions are there? (1940)

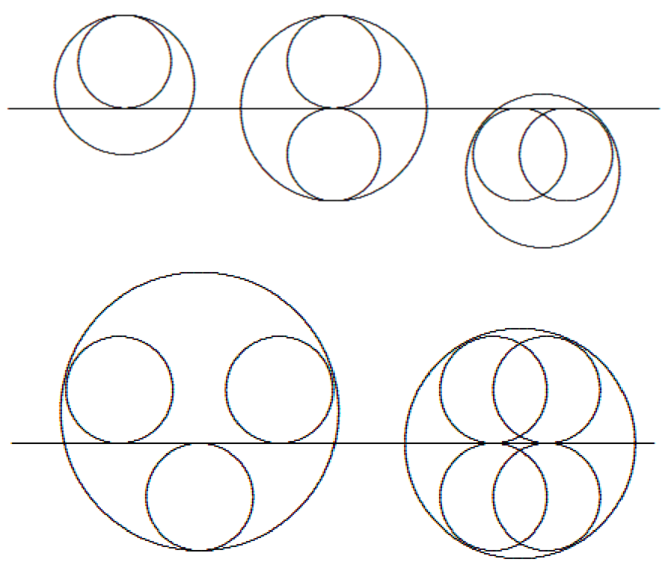
Solution. We consider three cases.

Case 1. The line and the circle are disjoint.
If the circle is too far away from the line, there are no solutions. If its nearest point to the line is exactly 2 away, there is a unique solution. If it is closer to the line than that, there are 2 solutions outside the circle, one on either side of it. In addition, we may also have 1 or 2 solutions which contain the circle, as shown in the diagram below.



Case 2. The line and the circle are tangent.
There are always 4 solutions, two tangent to both the line and the circle at their point of tangency, and one on either side outside the circle and on the same side of the line as the circle.

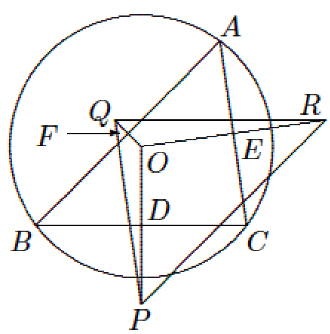
Case 3. The line and the circle intersect.
There are 4 solutions outside the circle, two on each side of the line. In addition, we may also have 1 to 4 solutions inside the circle, as shown in the diagram below.



In summary, the number of solutions may be any number from 0 to 8.

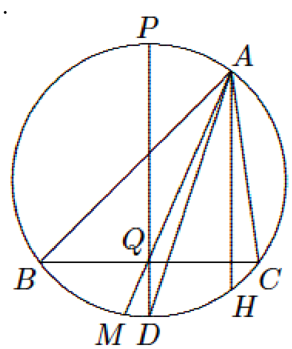
6. O is the circumcentre of triangle ABC. P, Q and R are the points symmetric to O about BC, CA and AB respectively. Construct ABC given only the points P, Q and R. (1940)

Solution. Let E and F be the respective midpoints of CA and AB. They are also the respective midpoints of OQ and OR. Hence QR is parallel to EF, which is parallel to BC. Since OP is perpendicular to BC, it lies along an altitude of triangle PQR. By symmetry, O is the orthocentre of PQR. Hence it can be constructed, along with E, F and the midpoint D of BC. Triangle ABC will be formed from the lines through D, E and F and perpendicular to OP, OQ and OR respectively.



7. Construct a triangle given the points of intersection of its circumcircle with the extensions of the altitude, angle bisector and median from the same vertex. (1935)

Solution. Let the triangle be ABC. Let the points of intersection of its circumcircle with the extensions of the altitude, angle bisector and median from A be H, D and M respectively. If two of these points coincide, then they all coincide and ABC is an arbitrary isosceles triangle. If these three points are distinct and collinear, there are no solutions. Otherwise, we construct the circle through them. This will be the circumcircle of ABC.

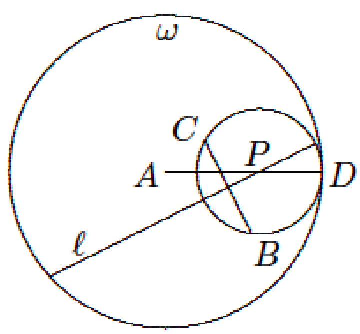


Construct the diameter PD and the line through H parallel to it, intersecting the circumcircle again at A (setting $A = H$ should that line be a tangent). If A and M are on the same side of the diameter PD, then no triangle exists. Otherwise, let AM intersect PD at Q, and construct the line through Q perpendicular to PD.

This line (if it does not contain A) will intersect the circumcircle at the desired vertices B and C (because the diameter that is perpendicular to the chord BC will intersect it at its midpoint Q, which makes AM the extended median as required and, furthermore, by construction AH is perpendicular to BC). Finally, if A and D are on opposite sides of BC, then D is the midpoint of the arc BC, so that AD is the extension of the internal angle bisector and ABC is the desired triangle. On the other hand, should A and D lie on the same side of BC then no triangle would exist (because then AD would be an external angle bisector).

8. Given two points A and B not on a line ℓ, construct a point P on ℓ such that $AP + BP = 1$. (1937)

Solution. Construct the circle ω with center A and radius 1. If B is outside ω, there is no solution. Suppose B lies on ω. If ℓ does not intersect AB, there is no solution. Otherwise, P is the point of intersection. Suppose B is inside ω. Reflect B across ℓ to C. If C is outside ω, there is no solution. If C lies on ω, then ℓ must intersect AC and P is the point of intersection. Suppose C is also inside ω. Construct either circle passing through B and C and tangent to ω. Then P is the center of this circle.



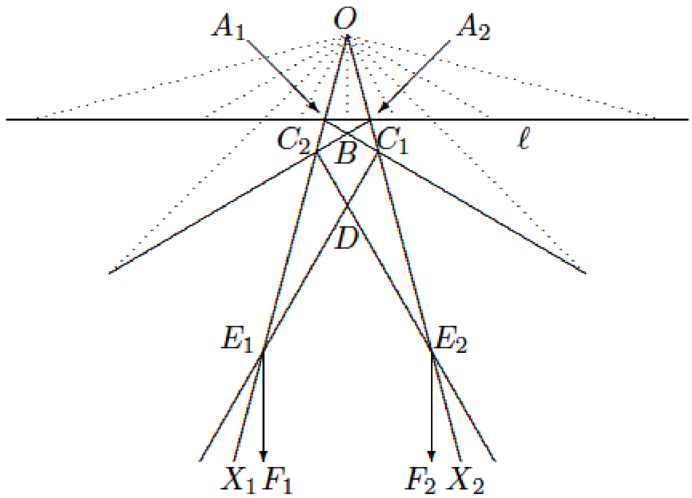
9. Given three non-collinear points, construct three circles, each passing through two of them, such that every two circles intersect, and the tangents to the circles at each point of intersection are perpendicular to each other. (1937)

Solution. Let A, B and C be the given points. Construct the lines AH and BK perpendicular to AB. Construct the line CU symmetric to AH about the perpendicular bisector of CA. Construct the line CV symmetric to BK about the perpendicular bisector of BC. Construct the bisector CO of $\angle UCV$. Construct the lines CX and CY, making $45°$ angles with CO. Let Q be the point of intersection of CX with the perpendicular bisector of CA, and let P be the point of intersection of CY with the perpendicular bisector of BC.

Finally, let R be the point of intersection of the line through A perpendicular to QA and the line through B perpendicular to PB. We claim that the circles with centers P, Q and R, passing respectively through C, A and B, are the desired circles.

10. When an infinite circular cone is cut along a line through its vertex, its surface opens up into a circular sector. A straight line ℓ is drawn perpendicular to the bisector of the central angle which has measure θ. When the cone is reconstructed, determine the number of points of self-intersection of ℓ in terms of θ. (1940)

Solution. Let the central angle be $\angle X_1OX_2 = \theta$. If $\theta \geq 180°$, the line ℓ will not intersect itself on the surface of the cone. The diagram below illustrates the case $\theta = 30°$. The angles between dotted lines are all $15°$.



The line ℓ intersects OX_1 at A_1, and its reflection about OA_1 intersects OX_2 at C_1. Similarly, ℓ intersects OX_2 at A_2, and its reflection about OA_2 intersects OX_1 at C_2. Moreover, A_1C_1 intersects A_2C_2 at B. The points E_1, E_2 and D are defined analogously. Finally, C_1E_1 is reflected about OX_1 to E_1F_1 and C_2E_2 is reflected about OX_2 to E_2F_2. These two rays are parallel to each other. There are altogether five points of self-intersection of ℓ on the surface of the cone, namely, $A_1 = A_2$, B, $C_1 = C_2$, D and $E_1 = E_2$. In general, the number of points of intersection is the largest integer n such that $n\theta < 180°$.

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Constructing Paradoxical Sequences

P. Samovol, M. Appelbaum, A. Zhukov

Mathematical olympiads sometimes include seemingly paradoxical problems. Consider the following example:

Problem 1. For the last year, Shane tracked his monthly income and his monthly expenses. Is it possible that for all sets of five consecutive months his expenses exceeded his income, but his yearly income exceeded his overall expenses?

At first sight, this seems impossible. However, after some consideration, we can construct an example that satisfies the conditions of the problem. Suppose that the monthly balance (the difference between the income and the expenses) takes on only two values $x > 0$ and $y < 0$. Let us construct a sequence of length 12 using x and y, which contains only one y for any 5 consecutive spots. For example,

$$x, x, x, x, y, x, x, x, x, y, x, x.$$

Next, we need to make sure that the sum of any 5 consecutive numbers is negative, that is $y + 4x < 0$, and that the sum of all 12 numbers is positive, so $2y + 10x > 0$. Overall, we get

$$-5x < y < -4x.$$

So we can take, say, $x = 2$ and $y = -9$.

This problem used the sequence of length 12, but how long can a sequence be with such conditions imposed on it? Let us take a look at a more general problem.

Problem 2. (Inspired by one of the problems from the XIX International Mathematical Olympiad). Consider a sequence of real numbers in which the sum of any seven consecutive elements is negative, but the sum of any eleven consecutive elements is positive.

a) What is the maximum possible length of such a sequence?

b) Give an example of such a sequence of length 16.

You can approach part b) of Problem 2 using trial and error method, but with some creativity. However, you can also approach it quite systematically. For example, let us find a sequence $x_1, x_2, \ldots, x_{16}$ that satisfies the conditions

$$\begin{aligned}
&x_1 + x_2 + \cdots + x_7 = -b,\\
&\cdots\\
&x_{10} + x_{11} + \cdots + x_{16} = -b,\\
&x_1 + x_2 + \cdots + x_{11} = a,\\
&\cdots\\
&x_6 + x_7 + \cdots + x_{16} = a,
\end{aligned}$$

where a and b are some positive constants.

Exercise 1. Solve this system of equations.

The reader who struggles with this system will appreciate how difficult it is to solve it. Below we will show another method that will help us easily find an example for Problem 2b. But first we will take a closer look at Problem 2a.

Consider a more general problem and define some terms. Given a sequence of m numbers, we say that a *q-sum* is a sum of some q consecutive numbers in this sequence. A sequence where any n-sum has one sign and any k-sum has the opposite sign is called an *$\{n, k\}$-sequence*.

Problem 3. Prove that the length m of an $\{n, k\}$-sequence does not exceed $n + k - d - 1$, where d is the greatest common divisor of n and k.

First of all, note that that neither of n nor k can be divisible by the other (prove this). We will start off by proving that there cannot be an $\{n, k\}$-sequence of length m if $m > n + k - d - 1$.

Suppose that we found a sequence of length $n + k - d$ that satisfies the necessary conditions. Take the smaller of the two numbers n and k, suppose it is k. Remove the first k numbers from our sequence. In the remaining sequence of $n-d$ numbers, all k-sums still have the same sign as before and all $(n-k)$-sums have the opposite signs (prove this statement by contradiction). Since $n - d = (n - k) + k - d$, we transformed the $\{n, k\}$-sequence into a $\{n-k, k\}$-sequence. Repeating the process, we get a chain of sequences of decreasing lengths $\{n, k\} \to \{n_1, k_1\} \to \{n_2, k_2\} \to \cdots \to \{n_l, k_l\} \to$, where one of the numbers n_l or k_l in the end equals to d and the other one is divisible by d. But this is a contradiction.

Obviously, if there is no sequence of length $n + k - d$ satisfying the conditions, there is no longer sequence satisfying these conditions.

We can give an example of such sequence of length $n + k - d - 1$. We look for this example among the sequences whose elements can take only one of two values x and y, which we will pick later. As our "guiding star", we take the chain $\{n, k\} \to \{n_1, k_1\} \to \{n_2, k_2\} \to \cdots \to \{n_l, k_l\} \to$ from above; we will call it *the defining sequence*. This chain realizes the familiar Euclidean algorithm computing the greatest common divisor d of two numbers n and k. Indeed, $d = |n_l - k_l|$. Here, we will use this chain for constructing our sequence and we will do so by moving from the end of the chain to its beginning. First, using the pair (n_l, k_l), let us construct a sequence

$$x, x, \ldots, x, y, x, x, \ldots, x, \tag{1}$$

the number of x's on the left of y is the same as on the right and equals

$$\begin{cases} n_l - 1, & \text{if } k_l > n_l, \\ k_l - 1, & \text{if } n_l > k_l. \end{cases}$$

In general, we say a sequence is *$[p, q]$-sequence* if all of its subsequences of length p contains the same number of y's and all of its subsequences of length q contains the same number of y's (the number of y's need not be the same in both cases). Clearly, the sequence (1) is $[n_l, k_l]$-sequence: any subsequence of length n_l or k_l contains exactly one y. This definition characterizes the uniformity of distributions of y's in the sequence.

Starting with the base sequence (1), we will add to it according to the following rule. Suppose we already have a $[n_i, k_i]$-sequence of x's and y's, which corresponds to the pair of numbers (n_i, k_i) from the defining sequence. We can get to this sequence from the previous pair (n_{i-1}, k_{i-1}) in two ways: either $n_i = n_{i-1} - k_{i-1}$ or $k_i = k_{i-1} - n_{i-1}$. Let us consider the first case (the second one is analogous). To increase the length of the sequence, fix the first k_{i-1} symbols and append them to the sequence from the left. We will prove that this results in a $[n_i, k_i]$-sequence.

Any subsequence of the resulting sequence consisting of $k_{i-1} = k_i$ consecutive symbols contains the same number of y's. Any subsequence of the resulting sequence consisting of $n_{i-1} = n_i + k_{i-1}$ consecutive symbols can be thought of as composed of two parts: right subsequence of length n_i and left subsequence of length k_{i-1}. Each of these subsequences contains a fixed number of y's, so the entire sequence contains a fixed number of y's.

We now come back to solve Problem 2b). Since $(7, 11) \to (7, 4) \to (3, 4)$, the first step of the algorithm gives

$$x, x, y, x, x.$$

The second step gives

$$x, x, y, x, x, x, y, x, x,$$

and the third step produces

$$x, x, y, x, x, x, y, x, x, y, x, x, x, y, x, x.$$

Since the sum of any seven consecutive elements must be negative and the sum of any eleven consecutive elements must be positive, the following inequalities must be satisfied:

$$\begin{cases} 2y + 5x < 0, \\ 3y + 8x > 0, \end{cases}$$

which gives $-\frac{5}{2}x > y > -\frac{8}{3}x$. We can take, for example, $x = 5$ and $y = -13$.

To prove the validity of the above algorithm, it only remains to prove that the general system of inequalities for x and y always has solutions. We will leave this proof as an exercise for the reader.

Exercise 2. Suppose the aforementioned algorithm gives a sequence of x's and y's. Let a denote the number of y's in any subsequence of length n and let b denote the number of y's in any subsequence of length k. Prove that the following two systems of inequalities always have solutions:

$$\begin{cases} ay + (n-a)x > 0, \\ by + (k-b)x < 0, \end{cases} \quad \text{and} \quad \begin{cases} ay + (n-a)x < 0, \\ by + (k-b)x > 0. \end{cases}$$

Exercise 3. It is known that for some sequence of length 23, the sum of any ten consecutive elements is negative and the sum of any k consecutive elements is positive. Find k, if the length of this sequence is the largest possible. Give an example of such a sequence.

Exercise 4. It is known that for some sequence the sum of any n consecutive elements is negative and the sum of any k consecutive elements is positive. It is also known that the maximum possible length of this sequence is 30. Find the maximum possible value of the difference $|k - n|$. Give an example of such a sequence.

Problem 4. Consider an integrable function $f(x)$ on the interval $[0, 23]$. Suppose that the definite integral of $f(x)$ on any interval of length n is positive, while the definite integral of $f(x)$ on any interval of length m is negative, where $c > m > n > 0$. Suppose the fraction $\frac{m}{n}$ reduces to $\frac{q}{p}$ in lowest terms; that is, p and q are mutually prime positive integers. Prove that $c < m + n - \frac{m}{q}$.

Let $d = \frac{m}{q} = \frac{n}{p}$, so that $n = pd$ and $m = qd$. Suppose that $c \geq m + n - \frac{m}{q} = (p + q - 1)d$. For any positive integer k such that $kd \leq c$, let

$$S_k = \int_{(k-1)d}^{kd} f(x)dx$$

and consider the following sequence of length $p + q - 1$: $S_1, S_2, \ldots, S_{p+q-1}$. It is not hard to check that this sequence satisfies the properties that the sum of any p consecutive elements is positive and the sum of any q consecutive elements is negative. By using the result from Problem 3, we see that such a sequence has maximum possible length of $p + q - 2$. This contradiction finishes the proof.

Exercise 5. Consider an integrable function $f(x)$ on the interval $[0, 23]$. Suppose that the definite integral of $f(x)$ on any interval of length 10 is positive, while the definite integral of $f(x)$ on any interval of length 16 is negative. Is this possible? What if $x \in [0, 24]$?

Exercise 6. Does there exist a continuous function $y = f(x)$ such that any definite integral of $f(x)$ on any interval of length 3 is negative, while the definite integral of $f(x)$ on any interval of length 5 is positive?

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This article appeared in Russian in Kvant, 2005(1), p. 37–39. It has been translated and adapted with permission.

PROBLEMS

Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème présenté dans cette section. De plus, nous les encourageons à soumettre des propositions de problèmes. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1 mai 2017**.

La rédaction souhaite remercier André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.

4191. *Proposé par Mehmet Berke Işler.*

Soit a, b, c des réels strictement positifs tels que $a + b + c \geq 3$. Démontrer que

$$abc + 2 \geq \frac{9}{a^3 + b^3 + c^3}.$$

4192. *Proposé par Florin Stanescu.*

On considère un polynôme $P = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ ($P \in \mathbb{C}[x]$) dont les zéros ont tous un module égal à 1. Sachant que $\sum_{k=0}^{n} a_k \neq 0$, montrer que

$$Re\left(\frac{a_1 + 2a_2 + \cdots + na_n}{a_0 + a_1 + \cdots + a_n}\right) = \frac{n}{2},$$

$Re(z)$ étant la partie réelle de z ($z \in \mathbb{C}$).

4193. *Proposé par Dan Stefan Marinescu, Leonard Giugiuc et Daniel Sitaru.*

Soit $f : [0, \infty) \mapsto \mathbb{R}$ une fonction dérivable dont la dérivée f' est convexe et telle que $f(0) = 0$. Démontrer que

$$f(x) + f(y) + f(z) + f(x + y + z) \geq f(x + y) + f(x + z) + f(y + z)$$

pour tous x, y, z non négatifs.

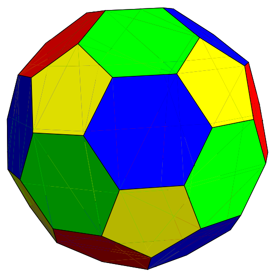
4194. *Proposé par Mihaela Berindeanu.*

Soit c_1 et c_2 deux cercles tangents extérieurement en A. La tangente à c_2 en un point B coupe c_1 en D et E. La droite AB coupe le cercle c_1 une deuxième fois en C et la bissectrice de l'angle DCE coupe c_1 en M. Soit Q le point d'intersection de AM et de BE et P le point d'intersection de BM et de CQ. Démontrer que

$$\frac{BP}{BC} = \frac{BA}{BM}.$$

4195. *Proposé par Eugen Ionascu.*

On considère un icosaèdre tronqué (qui comprend 12 faces pentagonales régulières et 20 faces hexagonales régulières, comme dans la figure ci-dessous). Sur chaque face, on écrit un entier strictement positif de manière que les nombres écrits sur les hexagones aient une somme de 39 et les nombres écrits sur les pentagones aient une somme de 25. Démontrer que deux faces ne peuvent partager un même sommet et porter le même entier.



4196. *Proposé par Leonard Giugiuc et Daniel Sitaru.*

Démontrer que

$$1 \leq \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} \leq 2$$

pour tous réels strictement positifs a, b et c.

4197. *Proposé par Michel Bataille.*

Soit x, y, z des réels strictement positifs tels que $xy+yz+zx+2xyz = 1$. Démontrer que

(a) $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} \geq 6$,

(b) $x + y + z \geq \dfrac{3}{2}$.

4198. *Proposé par Leonard Giugiuc.*

Soit un triangle ABC pour lequel $\sin A \sin B \sin C = \frac{2+\sqrt{3}}{8}$. Déterminer la valeur maximale possible de $\cos A \cos B \cos C$.

4199. *Proposé par Michel Bataille.*

Soit Γ_1 et Γ_2 deux cercles de centres respectifs O_1 et O_2 qui se coupent en A et B et soit ℓ la bissectrice interne de l'angle O_1AO_2. Soit M_1 et M_2 deux points sur Γ_2 ($M_1, M_2 \neq A, B$). La droite BM_k et l'image de AM_k par une réflexion par rapport à ℓ coupent Γ_1 une seconde fois aux points respectifs N_k et P_k ($k = 1, 2$). Démontrer que $N_1P_2 = N_2P_1$.

4200. *Proposé par Van Khea et Leonard Giugiuc.*

Soit AD, BE et CF les céviennes d'un triangle ABC qui se coupent en Q. Soit M et N des points sur les côtés respectifs AB et AC et soit P le point d'intersection de MN et de AD. Démontrer que

$$\frac{BM}{MA}\cdot\frac{AF}{FB}+\frac{CN}{NA}\cdot\frac{AE}{EC}=\frac{AQ}{QD}\cdot\frac{DP}{PA}.$$

3500. *Proposé par Paul Bracken.*

Soit $\beta=-f(1)+\frac{1}{4}f(\frac{1}{2})-\frac{1}{4}f(-\frac{1}{2})$, la fonction f étant définie comme suit:

$$f(a)=\sum_{k=1}^{\infty}\frac{\log(k)}{k(k+a)},\quad a\in(-1,\infty).$$

Démontrer que

$$\prod_{k=1}^{\infty}\frac{(2k-1)^{\frac{1}{2k}}}{(2k)^{\frac{1}{2k-1}}}=2^{-\frac{3}{2}\log(2)+1-\gamma}\cdot e^{\beta},$$

γ étant la constante d'Euler.

Note de la rédaction. Le problème 3500 ci-dessus a paru la première fois dans ***Crux*** 35(8), p. 519, en décembre 2009. Sa solution devait paraître dans ***Crux*** 36(8) en décembre 2010. Or, cette solution semble avoir été perdue lors de la transition d'un rédacteur en chef à un autre à ce moment-là. Nous remercions Paul Bracken d'avoir porté cette situation à notre attention et nous invitons les lectrices et les lecteurs de soumettre leurs solutions directement à `crux-editors@cms.math.ca`, tout en plaçant le numéro du problème sur la ligne indiquant l'objet.

. . . .

4191. *Proposed by Mehmet Berke Işler.*

Let a, b, c be positive real numbers such that $a+b+c\geq 3$. Show that

$$abc+2\geq\frac{9}{a^3+b^3+c^3}.$$

4192. *Proposed by Florin Stanescu.*

Consider a polynomial $P=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0\in\mathbb{C}[x]$ such that all roots of P are equal in modulus to 1. If $\sum_{k=0}^{n}a_k\neq 0$, show that

$$Re\left(\frac{a_1+2a_2+\cdots+na_n}{a_0+a_1+\cdots+a_n}\right)=\frac{n}{2},$$

where $Re(z)$ represents the real part of $z\in\mathbb{C}$.

4193. *Proposed by Dan Stefan Marinescu, Leonard Giugiuc and Daniel Sitaru.*

Let $f : [0, \infty) \mapsto \mathbb{R}$ be a differentiable function such that its derivative f' is convex and $f(0) = 0$. Prove that for any nonnegative numbers x, y, z, we have

$$f(x) + f(y) + f(z) + f(x + y + z) \geq f(x + y) + f(x + z) + f(y + z).$$

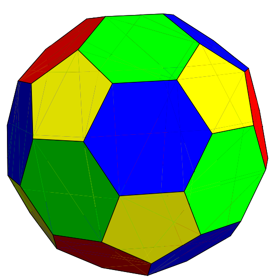
4194. *Proposed by Mihaela Berindeanu.*

Given circles c_1 and c_2 that are externally tangent at A, let the tangent to c_2 at B intersect c_1 at D and E. Furthermore, let c_1 intersect AB again at C and the bisector of $\angle DCE$ at M, and define Q and P to be the points where AM intersects BE and BM intersects CQ. Show that

$$\frac{BP}{BC} = \frac{BA}{BM}.$$

4195. *Proposed by Eugen Ionascu.*

On the faces of a regular truncated icosahedron (12 faces are regular pentagons, and 20 faces are regular hexagons, see figure below), a positive integer is written such that the sum of the numbers on the hexagons is 39 and the sum of the numbers on the pentagons is 25. Show that there are two faces that share a vertex and have the same integer written on them.



4196. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Show that for all positive real numbers a, b and c, we have

$$1 \leq \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} \leq 2.$$

4197. *Proposed by Michel Bataille.*

Let x, y, z be positive real numbers such that $xy + yz + zx + 2xyz = 1$. Prove that

(a) $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} \geq 6,$

(b) $x + y + z \geq \dfrac{3}{2}.$

4198. *Proposed by Leonard Giugiuc.*

In a triangle ABC, we have that $\sin A \sin B \sin C = \frac{2+\sqrt{3}}{8}$. Find the maximum possible value of $\cos A \cos B \cos C$.

4199. *Proposed by Michel Bataille.*

Let two circles Γ_1, Γ_2, with respective centres O_1, O_2, intersect at A and B and let ℓ be the internal bissector of $\angle O_1AO_2$. Let $M_1, M_2 \neq A, B$ be points on Γ_2. For $k = 1, 2$, the line BM_k and the reflection of AM_k in ℓ intersect Γ_1 again at N_k and P_k , respectively. Prove that $N_1P_2 = N_2P_1$.

4200. *Proposed by Van Khea and Leonard Giugiuc.*

Let the cevians AD, BE, and CF of triangle ABC intersect at Q. Let points M and N lie on sides AB and AC, respectively, and let P be the point where MN intersects AD. Prove that

$$\frac{BM}{MA} \cdot \frac{AF}{FB} + \frac{CN}{NA} \cdot \frac{AE}{EC} = \frac{AQ}{QD} \cdot \frac{DP}{PA}.$$

3500. *Proposed by Paul Bracken.*

Let $\beta = -f(1) + \frac{1}{4}f(\frac{1}{2}) - \frac{1}{4}f(-\frac{1}{2})$, where the function f is defined as follows:

$$f(a) = \sum_{k=1}^{\infty} \frac{\log(k)}{k(k+a)}, \quad a \in (-1, \infty).$$

Show that

$$\prod_{k=1}^{\infty} \frac{(2k-1)^{\frac{1}{2k}}}{(2k)^{\frac{1}{2k-1}}} = 2^{-\frac{3}{2}\log(2)+1-\gamma} \cdot e^{\beta},$$

where γ is Euler's constant.

Editor's comment. Problem 3500 above was originally published in ***Crux*** 35(8), p. 519, in December 2009. Its solution was supposed to appear in ***Crux*** 36(8) in December 2010. However, there is no mentioning of the problem in that issue. We assume that the solution to the problem got lost in the transition between the editors taking place at the same time. We thank the proposer to bringing it our attention and invite readers to submit solutions to this problem directly to `crux-editors@cms.math.ca` with the problem number in the subject line.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2015: 41(10), p. 441–445.

4091. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Find the greatest positive number k such that

$$a+b+c+3k-3 \geq k\left(\sqrt[3]{\frac{b}{a}}+\sqrt[3]{\frac{c}{b}}+\sqrt[3]{\frac{a}{c}}\right)$$

for any positive numbers a, b and c with $abc=1$.

We received three submissions, two of which were correct. We present the solution of the proposers, modified by the editor.

Using $abc=1$ we can rewrite the inequality as

$$a+b+c+3(k-1)-k(\sqrt[3]{a^2b}+\sqrt[3]{b^2c}+\sqrt[3]{c^2a}) \geq 0.$$

By substituting $a=x^3$, $b=y^3$, and $c=z^3$ (and thus $xyz=1$) and defining the homogeneous cyclic polynomial $f:[0,\infty)^3 \to \mathbb{R}$ by

$$f(x,y,z)=x^3+y^3+z^3+3(k-1)xyz-k(x^2y+y^2z+z^2x),$$

we thus need to find the greatest positive number k such that

$$f(x,y,z) \geq 0$$

for all positive $x,y,z \geq 0$ with $xyz=1$. We will proceed by first finding the greatest k such that $f(x,y,z) \geq 0$ for all $x,y,z \geq 0$ and then showing that this k is optimal for the case of $xyz=1$ as well. To accomplish the first part we will use a lemma proven by Pham Kim Hung in his book *Secrets in Inequalities (Volume 2)*. As the lemma might be of interest to the reader but is not a common reference we will give a proof here.

Lemma. *Let $P(x,y,z)$ be a cyclic homogeneous polynomial of degree* 3. *Then $P(x,y,z) \geq 0$ for all $x,y,z \geq 0$ if and only if $P(1,1,1) \geq 0$ and $P(u,1,0) \geq 0$ for all $u \geq 0$.*

Proof of the lemma: Clearly the given conditions are necessary. So assume that $P(1,1,1)$ and $P(u,1,0)$ are nonnegative for all $u \geq 0$. We can write

$$P(x,y,z)=m\sum_{cyc}x^3+n\sum_{cyc}x^2y+p\sum_{cyc}xy^2+qxyz$$

for some $m, n, p, q \in \mathbb{R}$. Then $P(1,1,1)$ gives us

$$3m + 3n + 3p + q \geq 0,$$

while setting $u = 0$ and $u = 1$ respectively yields $m \geq 0$ and $2m + n + p \geq 0$. The derivative of $P(x, y, z)$ in the direction $(1, 1, 1)$ is

$$\begin{aligned} 3m \sum_{cyc} x^2 + n\Big(\sum_{cyc} x^2 + 2\sum_{cyc} xy\Big) + p\Big(\sum_{cyc} x^2 + 2\sum_{cyc} xy\Big) + q\sum_{cyc} xy \\ = (3m + n + p)\sum_{cyc} x^2 + (2n + 2p + q)\sum_{cyc} xy \end{aligned}$$

Note that

$$3m + n + p = m + (2m + n + p) \geq 0$$

and

$$(3m + n + p) + (2n + 2p + q) = 3m + 3n + 3p + q \geq 0,$$

thus

$$(3m + n + p)\sum_{cyc} x^2 + (2n + 2p + q)\sum_{cyc} xy \geq (3m + n + p)\Big(\sum_{cyc} x^2 - \sum_{cyc} xy\Big) \geq 0$$

by the rearrangement inequality. Since the derivative in the direction $(1, 1, 1)$ is nonnegative for all $x, y, z \geq 0$ we only need to show that $P(x, y, z) \geq 0$ on the boundary, that is where at least one variable is equal to 0. This now follows from the cyclicity of $P(x, y, z)$ and $P(a, b, 0) = b^3 P(\frac{a}{b}, 1, 0)$ for $b \neq 0$ which finishes the proof of the lemma.

We can now apply the lemma to show that $f(x, y, z) \geq 0$ for all $x, y, z \geq 0$ if and only if $f(1, 1, 1) \geq 0$ (which clearly holds), and $f(u, 1, 0) \geq 0$. Defining

$$g(u) = f(u, 1, 0) = u^3 - ku^2 + 1,$$

we obtain $g'(u) = u(3u - 2k)$. Thus the minimum of $g(u)$ occurs at $u = \frac{2k}{3}$ and $g(u)$ is positive for all $u \geq 0$ if and only if $g(\frac{2k}{3}) \geq 0$, which occurs for $k \leq \frac{3}{\sqrt[3]{4}}$.

It remains to be shown that this is best possible for the case of $xyz = 1$ as well. Suppose $k > \frac{3}{\sqrt[3]{4}}$ and set $s = \frac{2k}{3}$. Consider $h(w) = f(s, 1, w)$. Since $h(w)$ is continuous and $h(0) = g(s) < 0$ there exists a positive t such that $h(t) < 0$. Set

$$x = \frac{s}{\sqrt[3]{st}}, y = \frac{1}{\sqrt[3]{st}} \quad \text{and} \quad z = \frac{t}{\sqrt[3]{st}}.$$

Then $x, y, z \geq 0$ with $xyz = 1$ and

$$f(x, y, z) = stf(s, 1, t) = sth(t) < 0.$$

It follows that the answer to the question is $k = \frac{3}{\sqrt[3]{4}}$.

4092. *Proposed by Mihaela Berindeanu.*

Show that

$$\left[\frac{a^2+16a+80}{16\,(a+4)}+\frac{2}{\sqrt{2\,(b^2+16)}}\right]\left[\frac{b^2+16b+80}{16\,(b+4)}+\frac{2}{\sqrt{2\,(a^2+16)}}\right]\geq\frac{9}{4}$$

for all $a,b>0$. When does equality hold?

We received ten correct submissions. We present two solutions.

Solution 1, by Arkady Alt.

Since

$$\frac{a^2+16a+80}{16\,(a+4)}=1+\frac{a^2+4^2}{16\,(a+4)},$$

we have

$$\begin{aligned}&\left(\frac{a^2+16a+80}{16\,(a+4)}+\frac{2}{\sqrt{2\,(b^2+16)}}\right)\left(\frac{b^2+16b+80}{16(b+4)}+\frac{2}{\sqrt{2\,(a^2+16)}}\right)\\&=\left(1+\frac{a^2+4^2}{16\,(a+4)}+\frac{2}{\sqrt{2\,(b^2+4^2)}}\right)\left(1+\frac{b^2+4^2}{16\,(b+4)}+\frac{2}{\sqrt{2\,(a^2+4^2)}}\right)\end{aligned}$$

and, combining Cauchy-Schwarz Inequality and inequality $\sqrt{2\,(u^2+v^2)}\geq u+v$, we obtain

$$\begin{aligned}&\left(1+\frac{a^2+4^2}{16\,(a+4)}+\frac{2}{\sqrt{2\,(b^2+4^2)}}\right)\left(1+\frac{2}{\sqrt{2\,(a^2+4^2)}}+\frac{b^2+4^2}{16\,(b+4)}\right)\\&\geq\left(1\cdot 1+\sqrt{\frac{a^2+4^2}{16\,(a+4)}}\cdot\sqrt{\frac{2}{\sqrt{2\,(a^2+4^2)}}}+\sqrt{\frac{2}{\sqrt{2\,(b^2+4^2)}}}\cdot\sqrt{\frac{b^2+4^2}{16\,(b+4)}}\right)^2\\&=\left(1+\frac{1}{4}\sqrt{\frac{\sqrt{2\,(a^2+4^2)}}{a+4}}+\frac{1}{4}\sqrt{\frac{\sqrt{2\,(b^2+4^2)}}{b+4}}\right)^2\\&\geq\left(1+\frac{1}{4}+\frac{1}{4}\right)^2=\frac{9}{4}.\end{aligned}$$

Since in inequality $\sqrt{2\,(u^2+v^2)}\geq u+v$ equality occurs if and only if $u=v$, it is easy to see that the equality holds if and only if $a=b=4$.

Solution 2, by AN-anduud Problem Solving Group.

Using AM-GM inequality, we get

$$\sqrt{2(b^2+16)}=\sqrt{8\cdot\frac{b^2+16}{4}}\leq\frac{1}{2}\left(8+\frac{b^2+16}{4}\right)=\frac{b^2+48}{8}. \qquad (1)$$

Applying AM-GM inequality and using (1), we have

$$\begin{aligned}
\frac{a^2+16a+80}{16(a+4)}+\frac{2}{\sqrt{2(b^2+16)}} &\geq \frac{a^2+16a+80}{16(a+4)}+\frac{16}{b^2+48}\\
&= \frac{(a^2+48)+(a+4)^2+24(a+4)}{32(a+4)}+\frac{16}{b^2+48}\\
&= \frac{a^2+48}{32(a+4)}+\frac{a+4}{32}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{16}{b^2+48}\\
&\geq 6\sqrt[6]{\frac{a^2+48}{32(a+4)}\cdot\frac{a+4}{32}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{16}{b^2+48}}\\
&= \frac{3}{2}\cdot\sqrt[6]{\frac{a^2+48}{b^2+48}}. \qquad (2)
\end{aligned}$$

Similarly,

$$\frac{b^2+16b+80}{16(b+4)}+\frac{2}{\sqrt{2(a^2+16)}} \geq \frac{3}{2}\cdot\sqrt[6]{\frac{b^2+48}{a^2+48}}. \qquad (3)$$

Multiplying (2) and (3), we obtain the desired inequality. Equality holds only when $a=b=4$.

4093. *Proposed by Dragolijub Milošević.*

Let ABC be an arbitrary triangle. Let r and R be the inradius and the circumradius of ABC, respectively. Let m_a be the length of the median from vertex A to side BC and let w_a be the length of the internal bisector of $\angle A$ to side BC. Define m_b, m_c, w_b and w_c similarly. Prove that

$$\frac{a^2}{m_a w_a}+\frac{b^2}{m_b w_b}+\frac{c^2}{m_c w_c} \leq 4\left(\frac{R}{r}-1\right).$$

We received five correct solutions and present the solution by Andrea Fanchini.

We have that

$$4\left(\frac{R}{r}-1\right)=\frac{abcs-4K^2}{K^2}$$

where s is the semiperimeter and K is the area of the triangle. We know (see Wei-Dong Jiang and Mihály Bencze, *JMI Volume 5 Number 3*, 2011, p. 365) that $m_a w_a \geq s(s-a)$, so we have to prove

$$\frac{a^2}{s(s-a)}+\frac{b^2}{s(s-b)}+\frac{c^2}{s(s-c)} \leq \frac{abcs-4K^2}{K^2},$$

that is

$$a^2(s-b)(s-c)+b^2(s-a)(s-c)+c^2(s-a)(s-b) \leq s\left(abc-4(s-a)(s-b)(s-c)\right).$$

In fact, the inequality can be replaced by equality since both sides equal:

$$\begin{aligned}
&s\left(abc-4(s-a)(s-b)(s-c)\right)\\
&=\frac{1}{4}\left(a^4+b^4+c^4+2a^2bc+2ab^2c+2abc^2-2a^2b^2-2b^2c^2-2c^2a^2\right).
\end{aligned}$$

4094. *Proposed by Michel Bataille.*

Let $x_1, x_2, \ldots, x_n$ be real numbers such that $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$. Prove that

$$n-1+\cosh\left(\sum_{k=1}^{n}(-1)^{k-1}x_k\right) \leq \sum_{k=1}^{n}\cosh x_k \leq n-1+\cosh\left(\sum_{k=1}^{n}x_k\right).$$

We received four correct and complete solutions. We present the solution by the proposer.

Let f be the function defined on $\mathbb{R}$ by $f(x) = (\cosh x) - 1$. As a lemma, we show that

$$f(a)+f(b) \leq f(a+b) \text{ if } ab \geq 0 \text{ and } f(a)+f(b) \geq f(a+b) \text{ if } ab \leq 0.$$

The result clearly holds if $b = 0$. Fix $b > 0$ and consider the function $\phi : x \mapsto f(x+b) - f(x)$. This function ϕ is differentiable and its derivative, defined by $\phi'(x) = f'(x+b) - f'(x)$, is nonnegative (since $f' = \sinh$ is nondecreasing). Thus $\phi(a) \geq \phi(0) = f(b)$ if $a \geq 0$ and $\phi(a) \leq f(b)$ if $a \leq 0$. The result follows when $b > 0$. In a similar way, we see that it also holds when $b < 0$.

This lemma and an easy induction lead to

$$f(x_1) + f(x_2) + \cdots + f(x_n) \leq f(x_1 + x_2 + \cdots + x_n)$$

if $x_1, x_2, \ldots, x_n \geq 0$. The right-hand inequality immediately follows.

We also prove the left-hand inequality by induction. The case $n = 1$ is obvious. Consider the case $n = 2$: Since $x_1(-x_2) \leq 0$, we have

$$f(x_1) + f(-x_2) \geq f(x_1 - x_2) \quad \text{or} \quad \cosh(x_1) + \cosh(x_2) - \cosh(x_1 - x_2) \geq 1,$$

the desired inequality. Now, assume that for some $n \geq 2$

$$\left(\sum_{k=1}^{n}\cosh x_k\right) \geq n-1+\cosh\left(\sum_{k=1}^{n}(-1)^{k-1}x_k\right) \tag{1}$$

whenever $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$.

Let $x_1, x_2, \ldots, x_n, x_{n+1}$ be such that $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1}$. Then we observe that if n is even, then

$$x_{n+1}(x_1 - x_2 + \cdots + x_{n-1} - x_n) \leq 0$$

and if n is odd, then

$$(-x_{n+1})(x_1 - x_2 + \cdots - x_{n-1} + x_n) \leq 0.$$

Applying the lemma and using $\cosh(-x) = \cosh(x)$, we obtain

$$\cosh x_{n+1} - 1 + \cosh\left(\sum_{k=1}^{n}(-1)^{k-1}x_k\right) - 1 \geq \cosh\left(\sum_{k=1}^{n+1}(-1)^{k-1}x_k\right) - 1$$

in either case. With the help of (1), we are led to

$$\begin{aligned}\left(\sum_{k=1}^{n+1}\cosh x_k\right) &\geq \cosh x_{n+1} + n - 1 + \cosh\left(\sum_{k=1}^{n}(-1)^{k-1}x_k\right)\\ &\geq n + \cosh\left(\sum_{k=1}^{n+1}(-1)^{k-1}x_k\right).\end{aligned}$$

This completes the induction step and the proof.

4095. *Proposed by George Apostolopoulos.*

Let a, b and c be positive real numbers with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 3$. Prove that

$$ab(a+b) + bc(b+c) + ac(a+c) \geq \frac{2}{3}(a^2 + b^2 + c^2) + 4abc.$$

There were 14 correct solutions. We present six different ones here.

Solution 1, by Arkady Alt.

With $1 = ax = by = cz$ and $x + y + z = 3$, the inequality is equivalent to

$$\begin{aligned}&(x+y+z)[z^2(x+y) + x^2(y+z) + y^2(z+x)]\\ &\geq 2(x^2y^2 + y^2z^2 + z^2x^2) + 4(x+y+z)(xyz)\\ &= 2(xy+yz+zx)^2.\end{aligned}$$

The difference between the two sides is

$$\begin{aligned}&(x+y+z)[(x+y+z)(xy+yz+zx) - 3xyz] - 2(xy+yz+zx)^2\\ &= (x+y+z)^2(xy+yz+zx) - 2(xy+yz+zx)^2 - 3xyz(x+y+z)\\ &\geq 3(xy+yz+zx)(xy+yz+zx) - 2(xy+yz+zx)^2 - 3xyz(x+y+z)\\ &= (xy+yz+zx)^2 - 3xyz(x+y+z) \geq 0\end{aligned}$$

(from two applications of the inequality $(u+v+w)^2 \geq 3(uv+vw+wu)$). Equality occurs iff $1 = x = y = z = a = b = c$.

Solution 2, by Andrew Siefker and Digby Smith (done independently).

When a, b, c, x, y, z are all positive, we have

$$\begin{aligned}&(a^2yz + b^2zx + c^2xy)(x+y+z) - (a+b+c)^2xyz\\ &= z[(a^2y^2 + b^2x^2 - 2abxy] + y[a^2z^2 + c^2x^2 - 2acxz] + x[b^2z^2 + c^2y^2 - 2bcyz] \geq 0,\end{aligned}$$

so that

$$\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} \geq \frac{(a+b+c)^2}{x+y+z}.$$

Equality occurs iff $a : b : c = x : y : z$. (This also results from Cauchy's Inequality.)

Let $bx = cy = az = 1$. Then (since $ab + bc + ca = 3abc$),

$$a^2b + b^2c + c^2a \geq \frac{abc(a+b+c)^2}{ab+bc+ca} = \frac{(a+b+c)^2}{3} = \frac{a^2+b^2+c^2}{3} + 2abc.$$

Let $cx = ay = bz = 1$. Then,

$$a^2c + b^2a + c^2b = \frac{(a+b+c)^2}{3} = \frac{a^2+b^2+c^2}{3} + 2abc.$$

Adding these two inequalities yields the result. Equality occurs iff $a = b = c = 1$.

Solution 3, by Prithwijit De.

Since $ab + bc + ca = 3abc$, it follows that

$$2(a^2 + b^2 + c^2) = (b-c)^2 + (c-a)^2 + (a-b)^2 + 6abc.$$

Also,

$$ab(a+b) + bc(b+c) + ac(a+c) = a(b-c)^2 + b(c-a)^2 + c(a-b)^2 + 6abc$$

and

$$a\left(\frac{1}{b} + \frac{1}{c}\right) = a\left(3 - \frac{1}{a}\right) = 3a - 1,$$

$$b\left(\frac{1}{c} + \frac{1}{a}\right) = 3b - 1 \qquad \text{and} \qquad c\left(\frac{1}{a} + \frac{1}{b}\right) = 3c - 1.$$

Therefore,

$$\begin{aligned}
&3[ab(a+b) + bc(c+a) + ca(a+c)] - 2(a^2+b^2+c^2) - 12abc \\
&= 3[a(b-c)^2 + b(c-a)^2 + c(a-b)^2] + 18abc \\
&\quad - [(b-c)^2 + (c-a)^2 + (a-b)^2 + 6abc] - 12abc \\
&= (3a-1)(b-c)^2 + (3b-1)(c-a)^2 + (3c-1)(a-b)^2 \\
&= a\left(\frac{1}{b} + \frac{1}{c}\right)(b-c)^2 + b\left(\frac{1}{c} + \frac{1}{a}\right)(c-a)^2 + c\left(\frac{1}{a} + \frac{1}{b}\right)(a-b)^2 \\
&\geq 0,
\end{aligned}$$

with equality iff $a = b = c = 1$. The desired result follows.

Solution 4, by Titu Zvonaru.

Replacing the 3 in the denominator by $(ab + bc + ca)/abc$ and multiplying by $ab + bc + ca$, we obtain the equivalent homogeneous inequality

$$\begin{aligned}
&a^2b^2(a+b) + b^2c^2(b+c) + c^2a^2(c+a) + abc[(a+b)^2 + (b+c)^2 + (c+a)^2] \\
&\geq 2(a^3bc + b^3ca + c^3ab) + 4(ab^2c^2 + bc^2a^2 + ca^2b^2)
\end{aligned}$$

which in turn is equivalent to

$$a^3b^2 + a^2b^3 + b^3c^2 + b^2c^3 + c^3a^2 + c^2a^3 \geq 2(ab^2c^2 + bc^2a^2 + ca^2b^2).$$

By the arithmetic-geometric means inequality, we have that

$$2a^3b^2 + b^2c^3 \geq 3ca^2b^2; \quad 2b^3c^2 + c^2a^3 \geq 3ab^2c^2; \quad 2c^3a^2 + a^2b^3 \geq 3bc^2a^2;$$

$$a^3b^2 + 2b^2c^3 \geq 3ab^2c^2; \quad b^3c^2 + 2c^2a^3 \geq 3bc^2a^2; \quad c^3a^2 + 2a^2b^3 \geq 3ca^2b^2.$$

Adding these inequalities yields the desired result, with equality iff $a = b = c = 1$.

Solution 5, by AN-Anduud Problem Solving Group; and Dionne Bailey, Elsie Campbell and Charles R. Diminnie (independently).

Since $a^2 + b^2 + c^2 \geq ab + bc + ca = 3abc$ and $x + (1/x) \geq 2$ for $x > 0$, we have that

$$\begin{aligned}
&3[ab(a+b) + bc(b+c) + ca(c+a)] \\
&= \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)[ab(a+b) + bc(b+c) + ca(c+a)] \\
&= 2(a^2 + b^2 + c^2) + 2(ab + bc + ca) + \frac{bc}{a}(b+c) + \frac{ca}{b}(c+a) + \frac{ab}{c}(a+b) \\
&= 2(a^2 + b^2 + c^2) + 6abc + a^2\left(\frac{b}{c} + \frac{c}{b}\right) + b^2\left(\frac{c}{a} + \frac{a}{c}\right) + c^2\left(\frac{a}{b} + \frac{b}{a}\right) \\
&\geq 6abc + 6abc + 2(a^2 + b^2 + c^2),
\end{aligned}$$

yielding the desired result. Equality holds iff $a = b = c = 1$.

Solution 6, by the proposer.

Applying the arithmetic-harmonic means inequality to the three pairs $(1/a, 1/b)$, $(1/b, 1/c)$ and $(1/c, 1/a)$, we find that

$$3 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{c+a}.$$

This is equivalent to

$$\begin{aligned}
&3(a+b)(b+c)(c+a) \geq 2[(b+c)(c+a) + (c+a)(a+b) + (a+b)(b+c)] \iff \\
&6abc + 3[ab(a+b) + bc(b+c) + ca(c+a)] \geq 2(a^2 + b^2 + c^2) + 6(ab + bc + ca) \\
&\qquad\qquad = 2(a^2 + b^2 + c^2) + 18abc \iff \\
&ab(a+b) + bc(b+c) + ca(c+a) \geq \frac{2}{3}(a^2 + b^2 + c^2) + 4abc.
\end{aligned}$$

Equality holds iff $a = b = c = 1$.

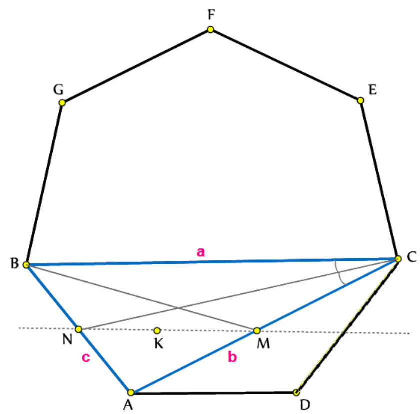
Editor's Comments. Four solvers reformulated the inequality as in Solution 4 and applied Muirhead's Inequality $[3, 2, 0] \geq [2, 2, 1]$, where

$$[p, q, r] = a^pb^qc^r + a^pb^rc^q + a^qb^pc^r + a^qb^rc^p + a^rb^pc^q + a^rb^qc^p.$$

Students Ahmad Talafha and Kevin Wunderlich gave a variant of Solution 2.

4096. *Proposed by Abdilkadir Altintaş.*

Let ABC be a heptagonal triangle with $BC = a$, $AC = b$ and $AB = c$. Suppose CN is the internal angle bisector of $\angle BCA$, BM is the median of triangle ABC and K is the symmedian point of ABC. Show that N, K and M are collinear.



We received six correct submissions. We present a combined solution based on those received from Michel Bataille and Titu Zvonaru.

In barycentric coordinates with respect to the vertices A, B, C of the triangle we have $N = (a : b : 0)$, $K = (a^2 : b^2 : c^2)$, $M = (1 : 0 : 1)$. The points N, K and M are collinear if and only if

$$\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 0 \\ 0 & c^2 & 1 \end{vmatrix} = 0,$$

which is equivalent to

$$c^2 + ab = a^2. \tag{1}$$

Consider the cyclic quadrilateral $ABEC$. Since $ADCEFGB$ is a regular heptagon, $|CE| = |AB| = c$ and $|EB| = |EA| = |BC| = a$. Applying Ptolemy's theorem to quadrilateral $ABEC$ we have

$$|AB| \cdot |CE| + |AC| \cdot |BE| = |BC| \cdot |EA|,$$

which gives us (1). Therefore the points N, K and M are collinear.

4097. *Proposed by Leonard Giugiuc.*

Let $a_i, 1 \leq i \leq 6$ be real numbers such that

$$\sum_{i=1}^{6} a_i = \frac{15}{2} \quad \text{and} \quad \sum_{i=1}^{6} a_i^2 = \frac{45}{4}.$$

Prove that $\prod_{i=1}^{6} a_i \leq \dfrac{5}{2}$.

We received five submissions, four of which were incorrect for various reasons. We present the proposer's solution, modified by the editor.

Using Jensen's inequality with the function $g(x) = x^2$ we have

$$5\left(\sum_{i=1}^{5} a_i^2\right) \geq \left(\sum_{i=1}^{5} a_i\right)^2$$

and thus using the assumptions from the question,

$$5\left(\frac{45}{4} - a_6^2\right) \geq \left(\frac{15}{2} - a_6\right)^2,$$

which lets us conclude $0 \leq a_6 \leq \frac{5}{2}$. By symmetry we obtain $0 \leq a_i \leq \frac{5}{2}$ for $i = 1, \ldots, 6$. If any of the six variables is zero, then $\prod_{i=1}^{6} a_i = 0 \leq \frac{5}{2}$, and we are done. Thus we assume that all a_i are positive and at most $\frac{5}{2}$.

Note that

$$\sum_{1 \leq i < j \leq 6} a_i a_j = \frac{1}{2}\left[\left(\sum_{i=1}^{6} a_i\right)^2 - \sum_{i=1}^{6} a_i^2\right] = \frac{1}{2}\left[\left(\frac{15}{2}\right)^2 - \frac{45}{4}\right] = \frac{45}{2}.$$

Define the polynomial $P(x) = \prod_{i=1}^{6}(x - a_i)$ which can be written as

$$P(x) = x^6 - \frac{15}{2}x^5 + \frac{45}{2}x^4 - mx^3 + nx^2 - qx + p$$

for $p = \prod_{i=1}^{6} a_i$ and suitable $m, n, q \in \mathbb{R}$. Set $f(x) = \frac{P(x)}{x}$ and define the sequence of polynomials

$$P(x) = P_0(x), P_1(x), P_2(x), P_3(x) = Q(x),$$

where $P_i(x) = x^{i+1} f^{(i)}(x)$ is the numerator of the i-th derivative of $f(x)$. Note that we can calculate

$$P_{i+1}(x) = x^{i+2} f^{(i+1)}(x) = x^{i+2} \frac{d}{dx} \frac{P_i(x)}{x^{i+1}} = xP_i'(x) - (i+1)P_i(x).$$

Now consider the positive roots of $P_{i+1}(x)$. If $P_i(x)$ has a root at α with multiplicity k, then $P_i'(x)$ has a root at α with multiplicity $k-1$, and therefore so does $P_{i+1}(x)$. On the other hand suppose we have two distinct positive roots of $P_i(x)$. Then they are also roots of $f^{(i)}(x)$. By Rolle's theorem (note that $f^{(i)}(x)$ is continuous for $x > 0$), there exists a root of $f^{(i+1)}(x)$ (and thus of $P_{i+1}(x)$) between these two roots. Using those two facts we can conclude that if $P_i(x)$ has n positive roots then $P_{i+1}(x)$ has at least $n-1$ positive roots. As $P(x)$ has six positive roots, $Q(x)$ must have at least three positive roots.

We can calculate

$$Q(x) = 60x^6 - 180x^5 + 135x^4 - 6p$$

and from there

$$Q'(x) = 180x^3(x-1)(2x-3).$$

If $Q(x)$ had two roots in the interval $(0,1)$, then $Q'(x)$ would have a root in the interval $(0,1)$ (by Rolle's theorem and our remarks about roots with multiplicity), which is not the case. Similarly $Q(x)$ cannot have more than one root greater than $\frac{3}{2}$. Since $Q(x)$ has at least three positive roots, though, it has a root in the interval $[1,\frac{3}{2}]$. Looking at $Q'(x)$ we can see that $Q(x)$ is decreasing on this interval and we obtain $Q(1)\ge 0$ and therefore $p\le \frac{5}{2}$, which finishes the proof.

The bound can be obtained by setting one variable to $\frac{5}{2}$ and the others to 1.

4098. *Proposed by Ardak Mirzakhmedov.*

Let α,β and γ be acute angles such that $\alpha+\beta=\gamma$. Show that

$$\cos\alpha+\cos\beta+\cos\gamma-1\ge 2\sqrt{\cos\alpha\cdot\cos\beta\cdot\cos\gamma}.$$

We received six correct submissions. We present the solution by Arkady Alt.

Note first that for all $a,b,c,d\in\mathbb{R}$, we have $(ac-bd)^2-(ad-bc)^2=(a^2-b^2)(c^2-d^2)$, so $(a^2-b^2)(c^2-d^2)\le(ac-bd)^2$. In particular, if $a>b>0$ and $c>d>0$, then

$$ac-bd\ge\sqrt{a^2-b^2}\cdot\sqrt{c^2-d^2}. \qquad (1)$$

Next,

$$\begin{aligned}\cos\alpha+\cos\beta+\cos\gamma-1&=2\cos\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}-2\sin^2\frac{\gamma}{2}\\&=2\left(\cos\frac{\gamma}{2}\cdot\cos\frac{\alpha-\beta}{2}-\sin\frac{\gamma}{2}\cdot\sin\frac{\alpha+\beta}{2}\right). \qquad (2)\end{aligned}$$

Since $\frac{\alpha+\beta}{2}=\frac{\gamma}{2}\in(0,\frac{\pi}{4})$ we have

$$\cos\frac{\alpha-\beta}{2}>\cos\frac{\alpha+\beta}{2}=\cos\frac{\gamma}{2}>\sin\frac{\gamma}{2}.$$

Hence, if we let

$$a=\cos\frac{\gamma}{2}, b=d=\sin\frac{\gamma}{2}\quad\text{and}\quad c=\cos\frac{\alpha-\beta}{2},$$

then $a>b>0$ and $c>d>0$ so applying (1) we obtain

$$\begin{aligned}\cos\frac{\gamma}{2}\cdot\cos\frac{\alpha-\beta}{2}-\sin\frac{\gamma}{2}\cdot\sin\frac{\alpha+\beta}{2}&\ge\sqrt{\cos^2\frac{\gamma}{2}-\sin^2\frac{\gamma}{2}}\cdot\sqrt{\cos^2\frac{\alpha-\beta}{2}-\sin^2\frac{\alpha+\beta}{2}}\\&=\sqrt{\cos\gamma}\sqrt{\frac{1}{2}\left(1+\cos(\alpha-\beta)-(1-\cos(\alpha+\beta)\right)}\\&=\sqrt{\cos\gamma}\sqrt{\cos\alpha\cdot\cos\beta}\\&=\sqrt{\cos\alpha\cdot\cos\beta\cdot\cos\gamma}. \qquad (3)\end{aligned}$$

Substituting (3) into (2), we then have

$$\cos\alpha+\cos\beta+\cos\gamma-1\ge 2\sqrt{\cos\alpha\cdot\cos\beta\cdot\cos\gamma},$$

thus completing the proof.

4099. *Proposed by Lorian Saceanu.*

Let ABC be an acute angle triangle. Suppose the internal bisectors of angles A, B and C intersect the sides of ABC in points A', B' and C' and they intersect the circumcircle of ABC in points L, M and N respectively. Let I be the point of intersection of all internal bisectors. Show that:

a) $\dfrac{AI}{IL} = \dfrac{IA'}{A'L}$,

b) $\sqrt{\dfrac{AI}{IL}} + \sqrt{\dfrac{BI}{IM}} + \sqrt{\dfrac{CI}{IN}} \geq 3$.

We received seven submissions, all correct, and will feature parts from two of them.

Solution to part a), by Prithwijit De De and B.J. Venkatachala (together).

In triangle IBL, $\angle IBL = \angle BIL = \frac{A+B}{2}$. Thus, $IL = BL = 2R\sin\frac{A}{2}$, where R is the circumradius of ΔABC. Moreover, since $AI = \frac{r}{\sin(A/2)}$, where r is the inradius of ΔABC,

$$\frac{AI}{IL} = \frac{r}{BL\sin\frac{A}{2}}.$$

Observe that

$$\frac{IA'}{A'L} = \frac{[BIA']}{[BLA']} = \frac{BI\sin\frac{B}{2}}{BL\sin\frac{A}{2}} = \frac{r}{BL\sin\frac{A}{2}} = \frac{AI}{IL}.$$

This settles part a).

Solution to part b), by Salem Malikić, modified by the editor.

We first transform the form of $\frac{AI}{IL}$ obtained above into something more helpful. In particular, we use area formulas

$$r = \frac{[ABC]}{s} = \frac{bc\sin^2 A}{2s},$$

the sine law $2R = \dfrac{a}{\sin A}$, and the half-angle formulas together with the cosine law

$$\frac{\sin^2 A}{\sin^2\frac{A}{2}} = 4\cos^2\frac{A}{2} = \frac{2s(b+c-a)}{bc}$$

in turn to get

$$\frac{AI}{IL} = \frac{r}{BL\sin\frac{A}{2}} = \frac{r}{2R\sin^2\frac{A}{2}} = \frac{bc\sin^2 A}{2sa\sin^2\frac{A}{2}} = \frac{b+c-a}{a}.$$

This, with analogous expressions for $\frac{BI}{IM}$ and $\frac{CI}{IN}$, reduces the inequality of part b) to

$$\sqrt{\frac{b+c-a}{a}} + \sqrt{\frac{c+a-b}{b}} + \sqrt{\frac{a+b-c}{c}} \geq 3, \tag{1}$$

which is an inequality due to Sorin Rădulescu that holds for all acute triangles. [*Editor's comment.* Malikić found a proof on the web page
`www.artofproblemsolving.com/community/c6h1102804p5008234`,
which he reproduced while adding some helpful details as follows.] After squaring both sides of (1) we need to prove that

$$\sum_{cyclic} \frac{b+c-a}{a} + 4\sum_{cyclic}\sqrt{\frac{(a+b-c)(a+c-b)}{4bc}} \ge 9,$$

which is

$$\sum_{cyclic} \frac{b+c-a}{a} + 4\left(\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}\right) \ge 9.$$

Since cosine is a concave function on $[0, \frac{\pi}{2}]$, Popoviciu's inequality tells us that

$$\cos A + \cos B + \cos C + 3\cos\left(\frac{A+B+C}{3}\right) \le 2\left(\cos\frac{A+B}{2} + \cos\frac{B+C}{2} + \cos\frac{C+A}{2}\right),$$

or equivalently,

$$2\left(\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}\right) \ge \sum_{cyclic}\frac{b^2+c^2-a^2}{2bc} + \frac{3}{2}. \tag{2}$$

With the help of inequality (2) it remains to prove that

$$\sum_{cyclic}\frac{b+c-a}{a} + \sum_{cyclic}\frac{b^2+c^2-a^2}{bc} \ge 6.$$

Setting $a = y+z, b = x+z$ and $c = x+y$ reduces the last inequality to the equivalent

$$\sum_{cyclic}(x^3 - x^2y - x^2z + xyz) \ge 0,$$

which directly follows from Schur's inequality, thus completing the proof. Moreover, from the last step we see that for equality one must have $x = y = z$, which implies that ΔABC must be equilateral. It is easily seen that equality is indeed achieved for an equilateral triangle.

Editor's Comments. Barroso Campos observed that the inequality of part b) might fail when ΔABC has an obtuse angle. One can easily construct counterexamples using the left-hand-side of (1); for example, with sides $a = 9, b = 10, c = 18$ we get a sum slightly smaller than 2.993.

4100. *Proposed by Daniel Sitaru and Leonard Giugiuc.*

Let ABC be an arbitrary triangle with area S, $\angle A < 90^\circ$ and sides $BC = a$, $AC = b$ and $AB = c$. Show that

$$\frac{c\cos B}{ac+2S} + \frac{b\cos C}{ab+2S} < \frac{a}{2S}.$$

We received nine submissions, of which eight were correct. We present the solution by Salem Malikić, slightly modified by the editor.

Using the area formula $S = \frac{1}{2}ac\sin B$ we get

$$\frac{c\cos B}{ac+2S} = \frac{c\cos B}{ac+ac\sin B} = \frac{\cos B}{a(1+\sin B)};$$

treating the second term similarly, the left hand side of the inequality can be re-written as

$$\frac{1}{a}\cdot\left(\frac{\cos B}{1+\sin B}+\frac{\cos C}{1+\sin C}\right).$$

On the right hand side of the given inequality we have

$$\begin{aligned}
\frac{a}{2S} &= \frac{a}{ac\sin B} && \text{(area formula)}\\
&= \frac{a}{a\cdot\frac{a\sin C}{\sin A}\cdot\sin B} && \text{(sine law)}\\
&= \frac{\sin(B+C)}{a\sin C\sin B} && (A+B+C=\pi,\ \text{so } \sin(A)=\sin(B+C))\\
&= \frac{1}{a}\cdot\left(\frac{\cos C}{\sin C}+\frac{\cos B}{\sin B}\right) && \text{(sum of angles formula for sine)}
\end{aligned}$$

Hence, since $a > 0$, the given inequality is equivalent to

$$\frac{\cos B}{1+\sin B}+\frac{\cos C}{1+\sin C} < \frac{\cos B}{\sin B}+\frac{\cos C}{\sin C},$$

and, by moving all the terms to one side, this is in turn equivalent to

$$\frac{\cos B}{\sin B(1+\sin B)}+\frac{\cos C}{\sin C(1+\sin C)} > 0. \qquad (\dagger)$$

If neither of $\angle B$ and $\angle C$ is obtuse then all the trigonometric functions are non-negative and the last inequality clearly holds (note that both terms cannot be simultaneously zero, as that would imply $\angle B = \angle C = 90^\circ$).

Assume now that one of the angles B and C, say without loss of generality C, is obtuse. As $\cos(C) = -\cos(\pi - C)$ and $\sin(C) = \sin(\pi - C)$, the inequality ($\dagger$) holds if and only if

$$\frac{\cos B}{\sin B(1+\sin B)} > \frac{\cos(\pi-C)}{\sin(\pi-C)(1+\sin(\pi-C))}.$$

However, $A+B+C=\pi$ and C obtuse imply $0 < B < \pi - C < \frac{\pi}{2}$; since on the interval $(0, \pi/2)$ cosine is decreasing and sine is increasing, this last inequality follows, concluding the proof of ($\dagger$) and thus also of the desired inequality.

AUTHORS' INDEX

Solvers and proposers appearing in this issue

(Bold font indicates featured solution.)

Proposers

Solvers - individuals

Solvers - collaborations

Math Quotes

Mathematics is often erroneously referred to as the science of common sense. Actually, it may transcend common sense and go beyond either imagination or intuition. It has become a very strange and perhaps frightening subject from the ordinary point of view, but anyone who penetrates into it will find a veritable fairyland, a fairyland which is strange, but makes sense, if not common sense.

E. Kasner and J. Newman in "Mathematics and the Imagination", New York: Simon and Schuster, 1940.

INDEX TO VOLUME 42, 2016

Articles *Robert Dawson, Kseniya Garaschuk*

From the archives *Kseniya Garaschuk*

Problems

Solutions

Authors' Index

Miscellaneous

PROPOSERS' INDEX

List of proposers to problems appearing in Volume 42