Crux Mathematicorum

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Editorial Board

IN THIS ISSUE / DANS CE NUMÉRO

EDITORIAL

In academia, we write a lot. As students, we are normally told how long our writing pieces should be. But academic adulthood often comes with more nebulous guidelines in terms of length of various writing pieces. So my question is, does the size matter? In spirit, it should not (or should it?); in reality, we have some, often untold, expectations about the "reasonable" minimum and maximum.

An abstract for a talk is expected to be no more than 200 words, an abstract for a paper normally spans no more than a third of a page, a conventional Ph.D. thesis spreads 100 or more pages. But what about articles? Do norms change depending on the area or the journal? A thesis is supposed to be somewhat comprehensive and include the necessary background for the reader not to be bothered looking up all the sources. An article, on the other hand, assumes a more specialized audience and hence does not require an in-depth background or literature review. But who really decides on the necessary length as a measure of an article's worth? And is there such a thing as not long enough? Maybe not. John H. Conway and Alexander Soifer win the prize for the shortest math article ever published: their "Can $n^2 + 1$ unit equilateral triangles cover an equilateral triangle of side $> n$, say $n + \epsilon$?" published in the January 2005 edition of *The American Mathematical Monthly* consists of just two words. (For the full story behind this article and its publication, read the piece by Alexander Soifer in *Mathematics Competitions*, 23 (1), available here: `http://www.openculture.com/2015/04/shortest-known-paper-in-a-serious-math-journal.html`).

At ***Crux*** we have our own standards: an average article spans 5-6 pages while each issue consists of approximately 45–50 pages. But problems and solutions come in all shapes and sizes: short and long, pure and applied, "from the book" and brute force. Flip through the following pages of ***Crux*** to see it all.

Kseniya Garaschuk

THE CONTEST CORNER

No. 41

John McLoughlin

The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by the editor by **December 1, 2016**, *although late solutions will also be considered until a solution is published.*

CC201. An expedition to the planet Bizarro finds the following equation scrawled in the dust.

$$3x^2 - 25x + 66 = 0 \quad \Longrightarrow \quad x = 4 \text{ or } x = 9.$$

What base is used for the number system on Bizarro?

CC202. The positive integers from 1 to n inclusive are written on a blackboard. After one number is erased, the average (arithmetic mean) of the remaining $n - 1$ numbers is $46\frac{20}{23}$. Determine n and the number that was erased.

CC203. Two circles, one of radius 1, the other of radius 2, intersect so that the larger circle passes through the centre of the smaller circle. Find the distance between the two points at which the circles intersect.

CC204. A 10 metre ladder rests against a vertical wall. The midpoint of the ladder is twice as far from the ground as it is from the wall. At what height on the wall does the ladder reach?

CC205. In the parallelogram $ABCD$, point X lies on AB such that XB is twice the length of AX. Let Y be the point of intersection of XC and BD. What fraction is the area of the triangle DCY of the area of the parallelogram $ABCD$?

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CC201. Une expédition à la planète Bizarro découvre l'énoncé suivant inscrit dans le sable.

$$3x^2 - 25x + 66 = 0 \quad \Longrightarrow \quad x = 4 \text{ ou } x = 9.$$

Quelle est la base du système de numération de la planète Bizarro?

CC202. On écrit au tableau les entiers positifs de 1 à n. Un des nombres est éffacé. La moyenne des $n-1$ nombres qui restent est $46\frac{20}{23}$. Déterminer la valeur de n ainsi que le nombre éffacé.

CC203. Trouver la distance entre les deux points d'intersection de deux cercles, de rayon 1 et 2 respectivement, qui se coupent de sorte le plus grand passe par le centre du plus petit.

CC204. Une échelle longue de dix mètres est placée contre un mur vertical. Si le milieu de l'échelle est deux fois plus distant du sol que du mur, à quelle hauteur l'échelle s'appuie-t-elle contre le mur?

CC205. Dans le parallélogramme $ABCD$, soit X le point du segment AB tel que XB est deux fois plus long que AX. Soit Y le point d'intersection de XC et BD. Trouver le rapport de l'aire du triangle DCY à celle du parallélogramme $ABCD$.

Math Quotes

Unfortunately what is little recognized is that the most worthwhile scientific books are those in which the author clearly indicates what he does not know; for an author most hurts his readers by concealing difficulties.

Evariste Galois, in N. Rose (ed.) "Mathematical Maxims and Minims", Raleigh NC: Rome Press Inc., 1988.

CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2015: 41(1), p. 4–5.

CC151. Consider a non-zero integer n such that $n(n+2013)$ is a perfect square.

a) Show that n cannot be prime.

b) Find a value of n such that $n(n+2013)$ is a perfect square.

Originally question 1 from 2013 Pan African Mathematics Olympiad.

We received eleven correct solutions. We present two of the solutions.

Solution 1, by Andrea Fanchini.

a) We denote $m = n(n+2013)$. If m is a perfect square and n is prime then n must divide $n+2013$. By the divisibility properties n must then also be a factor of $2013 = 3 \cdot 11 \cdot 61$. Thus there are three possibilities for m:

$$m = 3(3+2013), \qquad m = 11(11+2013), \qquad m = 61(61+2013).$$

None of these numbers are square, so n cannot be prime.

b) We know that the sum of odd numbers gives a perfect square. If we set

$$n = 1+3+\cdots+2011 = 1006^2,$$

then

$$n+2013 = 1+3+\cdots+2011+2013 = 1007^2$$

and $m = 1006^2 \cdot 1007^2$ is a perfect square.

Solution 2, by Albert Stadler.

Put $t = \gcd(n, 2013)$. Note that $\gcd(n, n+2013) = \gcd(n, 2013) = t$. Then

$$n(n+2013) = t^2 \cdot \frac{n}{t} \cdot \frac{n+2013}{t}$$

and $\gcd(\frac{n}{t}, \frac{n+2013}{t}) = 1$. So $n(n+2013)$ is a perfect square if and only if both $\frac{n}{t}$ and $\frac{n+2013}{t}$ are perfect squares. Set $\frac{n}{t} = a^2$ and $\frac{n+2013}{t} = b^2$. Then

$$\frac{2013}{t} = b^2 - a^2 = (b-a)(b+a).$$

Since $2013 = 3 \cdot 11 \cdot 61$, we have $t \in \{1, 3, 11, 33, 61, 183, 671, 2013\}$.

$$\begin{aligned}
t = 2013 &: (b-a)(b+a) = 1 \text{ has no solutions in positive integers } a, b.\\
t = 671 &: (b-a)(b+a) = 3 \text{ implies } (a,b) = (1,2).\\
t = 183 &: (b-a)(b+a) = 11 \text{ implies } (a,b) = (5,6).\\
t = 61 &: (b-a)(b+a) = 33 \text{ implies } (a,b) = (16,17) \text{ or } (a,b) = (4,7).\\
t = 33 &: (b-a)(b+a) = 61 \text{ implies } (a,b) = (30,31).\\
t = 11 &: (b-a)(b+a) = 183 \text{ implies } (a,b) = (91,92) \text{ or } (a,b) = (29,32).\\
t = 3 &: (b-a)(b+a) = 671 \text{ implies } (a,b) = (335,336) \text{ or } (a,b) = (25,36).\\
t = 1 &: (b-a)(b+a) = 2013 \text{ implies } (a,b) = (1006,1007) \text{ or}\\
&\quad (a,b) = (334,337) \text{ or } (a,b) = (86,97) \text{ or } (a,b) = (14,47).
\end{aligned}$$

With $n = a^2t$ we obtain that $n(n+2013)$ is a perfect square if and only if n is one of 196, 671, 976, 1875, 4575, 7396, 9251, 15616, 29700, 91091, 111556, 336675, or 1012036, none of which is prime.

CC152. A square of an $n \times n$ chessboard with $n \geq 5$ is coloured in black and white in such a way that three adjacent squares in either a line, a column or a diagonal are not all the same colour. Show that for any 3×3 square inside the chessboard, two of the squares in the corners are coloured white and the two others are coloured black.

Originally question 5 from 2013 Pan African Mathematics Olympiad.

We received only one incorrect submission.

CC153. A sequence $a_0, a_1, \ldots, a_n, \ldots$ of positive integers is constructed as follows:

- if the last digit of a_n is less than or equal to 5, then this digit is deleted and a_{n+1} is the number consisting of the remaining digits; if a_{n+1} contains no digits, the process stops;
- otherwise, $a_{n+1} = 9a_n$.

Can one choose a_0 so that we can obtain an infinite sequence?

Originally question 5 from 2010 Pan African Mathematics Olympiad.

We received two correct solutions and one incomplete submission. We present the solution by Titu Zvonaru.

It is not possible to obtain an infinite sequence. If the last digit of a_n is less than or equal to 5, then it is obvious that $a_{n+1} < a_n$. If the last digit of a_n is greater than 5, then the last digit of a_{n+1} is less than 5. It results that

$$a_{n+2} = [a_{n+1}/10] = [9a_n/10] < a_n.$$

So if we had an infinite sequence (a_n) of positive integers we would find an infinite strictly decreasing subsequence, a contradiction.

CC154. The numbers $\frac{1}{1}, \frac{1}{2}, \ldots, \frac{1}{2012}$ are written on the blackboard. Alice chooses any two numbers from the blackboard, say x and y, erases them and instead writes the number $x + y + xy$. She continues to do so until there is only one number left on the board. What are the possible values of the final number?

Originally question 4 from 2012 Pan African Mathematics Olympiad.

There was one correct solution for this problem and two incomplete submissions. We present the solution by Konstantine Zelator.

Note that $x = (x + 1) - 1$ and $xy + x + y = (x + 1)(y + 1) - 1$.

We use the following lemma:

> If $X = (a_1 + 1) \cdots (a_k + 1) - 1$ and $Y = (a_{k+1} + 1) \cdots (a_{k+m} + 1) - 1$, then $XY + X + Y = (a_1 + 1) \cdots (a_{k+m} + 1) - 1$.

By writing $XY + X + Y = (X + 1)(Y + 1) - 1$, the lemma follows immediately.

From this lemma, it follows that if the board starts with numbers $n_1, n_2, \ldots, n_t$ and the given operation is applied to the numbers in any order until a single number remains, that number will be $(n_1 + 1)(n_2 + 1) \cdots (n_t + 1) - 1$.

For the set of numbers $1, \frac{1}{2}, \ldots, \frac{1}{2012}$ the final answer will thus be

$$\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{2013}{2012} - 1 = 2012.$$

CC155. Find all real solutions x to the equation $[x^2 - 2x] + 2[x] = [x]^2$. Here $[a]$ denotes the largest integer less than or equal to a.

Originally question 3 from 2012 Pan African Mathematics Olympiad.

There were three correct solutions for this problem and one incorrect submission. We present the solution by the Missouri State University Problem Solving Group.

The equation is true for any integer x, so we need only fnid the non-integer solutions. Suppose x is a non-integer solution and $\lfloor x \rfloor = n$. Then $x = n + \epsilon$ for some ϵ with $0 < \epsilon < 1$. We will make use of the fact that for any integer k, $\lfloor a + k \rfloor = \lfloor a \rfloor + k$. We have:

$$\begin{aligned}
\lfloor (x-1)^2 - 1 \rfloor + 2\lfloor x \rfloor &= \lfloor x \rfloor^2 \\
\lfloor (x-1)^2 \rfloor &= \lfloor x \rfloor^2 - 2\lfloor x \rfloor + 1 \\
\lfloor (n - 1 + \epsilon)^2 \rfloor &= (n-1)^2 \\
\lfloor (n-1)^2 + 2\epsilon(n-1) + \epsilon^2 \rfloor &= (n-1)^2 \\
\lfloor 2\epsilon(n-1) + \epsilon^2 \rfloor &= 0
\end{aligned}$$

Equivalently,

$$0 \leq 2\epsilon(n-1) + \epsilon^2 < 1. \tag{1}$$

Since we assumed x is not an integer, $\epsilon \neq 0$. Taking both sides of inequality (1) we get:

$$1 - \frac{\epsilon}{2} \leq n \leq \frac{1 + 2\epsilon - \epsilon^2}{2\epsilon}.$$

Since $0 < \epsilon < 1$, we have $0 < 1 - \epsilon/2 < 1$, implying $n > 0$.

Taking the right side of inequality (1) and completing the square gives:

$$\begin{aligned} (\epsilon + (n-1))^2 &< (n-1)^2 + 1 \\ \epsilon + n - 1 &< \sqrt{(n-1)^2 + 1} \\ x = n + \epsilon &< \sqrt{(n-1)^2 + 1} + 1 \end{aligned}$$

For any positive integer n, this gives the following interval for the solution x:

$$(n, \sqrt{(n-1)^2 + 1} + 1).$$

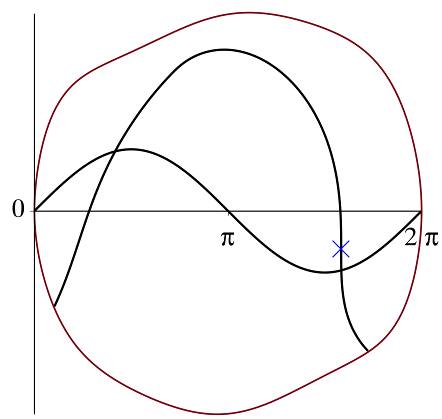
Thus, the set of all solutions is given by

$$\bigcup_{n=1}^{\infty} \left(n, \sqrt{(n-1)^2 + 1} + 1\right) \bigcup \mathbb{Z}.$$

WOBBLING BICYCLE

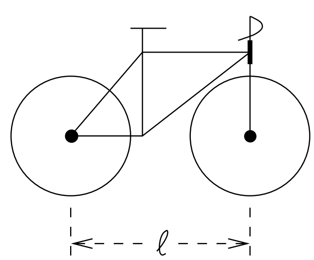
Proposed by Luis Goddyn, Simon Fraser University, Burnaby, BC.

A wobbling bicycle passes through a mud patch. One of its wheels traces a part of the curve $y = \sin x$. The other wheel makes a curve with a vertical inflection point.



How long is the bicycle?

In order to eliminate effects due to bicycle geometry, tilting and wheel size, assume the bicycle has vanishingly thin tires with its front axle always positioned directly below a vertical headset. Assume also that both wheels were in the mud patch when the inflection point is traversed. Determine the distance ℓ between its axles.



Editor's Comment. This problem is related to a famous puzzle of determining in which way a bicycle was going from its tracks. You can find it in the book *Which Way Did the Bicycle Go?: And Other Intriguing Mathematical Mysteries* by Joseph D. E. Konhauser, Dan Velleman and Stan Wagon.

The solution to this puzzle will appear in ***Crux*** 42(5).

THE OLYMPIAD CORNER

No. 339

Carmen Bruni

The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by the editor by **December 1, 2016**, *although late solutions will also be considered until a solution is published.*

The editor thanks The editor thanks Rolland Gaudet, retired professor of the University College of Saint Boniface, for translations of the problems.

OC261. Show that there are no 2-tuples (x, y) of positive integers satisfying the equation $(x+1)(x+2)\cdots(x+2014) = (y+1)(y+2)\cdots(y+4028)$.

OC262. In obtuse triangle ABC, with the obtuse angle at A, let D, E, F be the feet of the altitudes through A, B, C respectively. DE is parallel to CF, and DF is parallel to the angle bisector of $\angle BAC$. Find the angles of the triangle.

OC263. An integer $n \geq 3$ is called *special* if it does not divide

$$(n-1)!\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right).$$

Find all special numbers n such that $10 \leq n \leq 100$.

OC264. A positive integer is called *beautiful* if it can be represented in the form $\frac{x^2+y^2}{x+y}$ for two distinct positive integers x, y. A positive integer that is not beautiful is *ugly*.

1. Prove that 2014 is a product of a beautiful number and an ugly number.
2. Prove that the product of two ugly numbers is also ugly.

OC265. Five airway companies operate in a country consisting of 36 cities. Between any pair of cities exactly one company operates two way flights. If some air company operates between cities A, B and B, C we say that the ordered triple A, B, C is properly-connected. Determine the largest possible value of k such that no matter how these flights are arranged there are at least k properly-connected triples.

. . . .

OC261. Démontrer qu'il n'existe aucun couple d'entiers positifs (x, y) satisfaisant à l'équation $(x+1)(x+2)\cdots(x+2014) = (y+1)(y+2)\cdots(y+4028)$.

OC262. Soit un triangle obtus ABC, où l'angle obtus se situe à A, et soient D, E, F les pieds des altitudes provenant de A, B, C respectivement. DE est parallèle à CF et DF est parallèle à la bissectrice de $\angle BAC$. Déterminer les angles du triangle.

OC263. Un entier $n \geq 3$ est dit *spécial* s'il ne divise pas

$$(n-1)!\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right).$$

Déterminer tous les nombres spéciaux n tels que $10 \leq n \leq 100$.

OC264. Un entier est dit *adorable* s'il peut être représenté sous la forme $\frac{x^2+y^2}{x+y}$ pour deux entiers positifs distincts x, y. Un entier positif qui n'est pas adorable est dit *moche*.

1. Démontrer que 2014 est le produit d'un nombre adorable et un nombre moche.

2. Démontrer que le produit de deux nombres moches est moche.

OC265. Cinq compagnies aériennes opèrent dans un pays comprenant 36 villes. Entre toute paire de villes, exactement une compagnie aérienne opère un vol aller-retour. Si une compagnie aérienne opère entre les villes A, B puis B, C, on dit que le triplet A, B, C est proprement connecté. Déterminer la plus grande valeur possible de k telle que, quelle que soit l'organisation des vols, il y aura toujours au moins k triplets proprement connectés.

OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2014: 40(9), p. 374–375.

OC201. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) \in \mathbb{Q}$ and

$$f(x + f(y)^2) = f(x + y)^2.$$

Originally problem 2 from the third round algebra of the 2013 Iran National Mathematical Olympiad.

We received one correct solution. We present the solution by Oliver Geupel.

The two constant functions $f(x) = 0$ and $f(x) = 1$ have the required property, and we show that there are no other solutions.

Let us refer to the functional equation as $F(x, y)$. Let f be a solution and let $f(0) = q = a/b$ where a, b are integers and $b \geq 1$. From $F(x+y-q^2, 0)$ we obtain $f(x + y) = f(x + y - q^2 + f(0)^2) = f(x + y - q^2)^2$. Specialising $x = 0$, we also have $f(y) = f(y - q^2)^2$. Hence, using $F(x, y - q^2)$, we see that

$$f(x + f(y)) = f(x + f(y - q^2)^2) = f(x + y - q^2)^2 = f(x + y). \tag{1}$$

Setting $y = 0$ in (1), we obtain $f(x + q) = f(x)$ and therefore

$$f(x + nq) = f(x) \tag{2}$$

for every integer n. A further consequence of (1) is

$$f(f(x)) = f(x).$$

By $F(x, 0)$, we have $f(x + q^2) = f(x)^2$. Mathematical induction yields

$$f(x + nq^2) = f(x)^{2^n} \qquad (n \in \mathbb{N}). \tag{3}$$

From $F(x - q^2, 0)$, we see that $f(x) = f(x - q^2 + f(0)^2) = f(x - q^2)^2 \geq 0$. By (2) and (3), $f(x) = f(x + aq) = f(x + bq^2) = f(x)^{2^b}$, so that $f(x) \in \{0, 1\}$; whence also $q \in \{0, 1\}$. We consider the cases $q = 1$ and $q = 0$ in succession.

Case $q = 1$. We show that f is the constant function $f(x) = 1$. The proof is by contradiction. Suppose for some real number t it holds $f(t) = 0$. Then,

$$1 = f(0) = f(f(t)) = f(t) = 0,$$

a contradiction which shows that f is the constant function $f(x) = 1$.

Case $q = 0$. We prove that f is the constant function $f(x) = 0$. The proof is again by contradiction. Assume $f(t) = 1$ for some real number t. By (1),

$$f\left(\frac{1}{2} + f\left(\frac{1}{2}\right)\right) = f\left(\frac{1}{2} + \frac{1}{2}\right) = f(1) = f(f(t)) = f(t) = 1.$$

If $f\left(\frac{1}{2}\right) = 0$, then $f\left(\frac{1}{2} + f\left(\frac{1}{2}\right)\right) = 0$, which is impossible. Thus $f\left(\frac{1}{2}\right) = 1$. Hence, by (1), $f\left(\frac{3}{2}\right) = f\left(\frac{1}{2} + 1\right) = f\left(\frac{1}{2} + f\left(\frac{1}{2}\right)\right) = 1$. Moreover, $0 = f\left(\frac{1}{2} - \frac{1}{2}\right) = f\left(\frac{1}{2} + f\left(-\frac{1}{2}\right)\right)$. Note that $f\left(-\frac{1}{2}\right)$ is either 0 or 1. So $f\left(\frac{1}{2} + f\left(-\frac{1}{2}\right)\right)$ is equal to either $f\left(\frac{1}{2}\right) = 1$ or $f\left(\frac{3}{2}\right) = 1$. This is impossible and the proof is complete.

OC202. Let a, b be real numbers such that the equation $x^3 - ax^2 + bx - a = 0$ has three positive real roots. Find the minimum of $\frac{2a^3 - 3ab + 3a}{b+1}$.

Originally problem 1 from day 1 of the 2013 South East Mathematical Olympiad.

We received six correct solutions. We present the solution by Michel Bataille.

First, let S be the set of all pairs of real numbers (a, b) such that the equation $x^3 - ax^2 + bx - a = 0$ has three positive real roots and let $R(a, b) = \frac{2a^3 - 3ab + 3a}{b+1}$. We show that $\min\limits_{(a,b)\in S} R(a, b) = 9\sqrt{3}$.

Now, if $a = 3\sqrt{3}$, $b = 9$, the equation becomes $(x - \sqrt{3})^3 = 0$ whose roots are clearly positive real numbers and it is readily checked that $R(3\sqrt{3}, 9) = 9\sqrt{3}$. Thus, there just remains to prove that $R(a, b) \geq 9\sqrt{3}$ whenever $(a, b) \in S$.

Let $(a, b) \in S$ and let x_1, x_2, x_3 be positive roots of $x^3 - ax^2 + bx - a = 0$. Then,

$$x_1 + x_2 + x_3 = a, \quad x_1x_2 + x_2x_3 + x_3x_1 = b, \quad x_1x_2x_3 = a.$$

Note that the above shows that $a, b > 0$ since the roots are positive real numbers. Observing that

$$(x_1 + x_2 + x_3)^3 = x_1^3 + x_2^3 + x_3^3 + 3(x_1 + x_2 + x_3)(x_1x_2 + x_2x_3 + x_3x_1) - 3x_1x_2x_3$$

we obtain $2a^3 = 2(x_1^3 + x_2^3 + x_3^3) + 6ab - 6a$, hence

$$2a^3 - 3ab + 3a = 2(x_1^3 + x_2^3 + x_3^3) + 3ab - 3a \geq 2\cdot 3x_1x_2x_3 + 3ab - 3a = 3ab + 3a = 3a(b+1)$$

(using AM-GM for the inequality). It follows that

$$R(a, b) \geq \frac{3a(b+1)}{b+1} = 3a. \tag{1}$$

Now, by AM-GM again, $\left(\frac{x_1+x_2+x_3}{3}\right)^3 \geq x_1x_2x_3$, hence $\frac{a^3}{27} \geq a$ and so $a \geq 3\sqrt{3}$. With (1), we deduce that $R(a, b) \geq 9\sqrt{3}$, as desired.

OC203. Find all positive integers m and n satisfying $2^n + n = m!$.

Originally problem 1 from day 2 of the 2013 Turkey Mathematical Olympiad.

We received one correct solution. We present the solution by Oliver Geupel.

A solution is

$$(m, n) = (3, 2)$$

and we show that it is unique.

Suppose that (m,n) is any solution. Then there exists a nonnegative integer a and a positive odd integer b such that $n = 2^a b$. The exact power of 2 that divides $m! = 2^n + n = 2^a(2^{n-a} + b)$ is 2^a. Thus $m \le 2a+1$. If $a \le 4$ then $m \le 9$. A straightforward inspection shows that when $m \le 9$, the only solution is $(m,n) = (3,2)$. We now consider the case $a \ge 5$.

We prove that for every $a \ge 5$ it holds

$$2^{2^a} > (2a+1)!.$$

The proof is by mathematical induction on a. The base case $a = 5$ is satisfied since

$$2^{32} > 2^{27} = 2^8 \cdot 2^7 \cdot 2^5 \cdot 2^3 \cdot 2^4 > 2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 = 11!$$

Suppose that for some $a \ge 6$ we have $2^{2^{a-1}} > (2a-1)!$. Then

$$2^{2^a} > (2a-1)!^2 > (2a-1)! \cdot 2(2a-2) \cdot 3(2a-1) > (2a-1)! \cdot 2a \cdot (2a+1) = (2a+1)!,$$

which completes the induction.

We conclude $2^n + n > 2^{2^a} > (2a+1)! \ge m!$, a contradiction.

OC204. Let ABC be a triangle. Find all points P on segment BC satisfying the following property: If X and Y are the intersections of line PA with the common external tangent lines of the circumcircles of triangles PAB and PAC, then

$$\left(\frac{PA}{XY}\right)^2 + \frac{PB \cdot PC}{AB \cdot AC} = 1.$$

Originally problem 6 from day 2 of the 2013 USA Mathematical Olympiad.

We received one correct solution. We present the solution by Titu Zvonaru.

There are only two such points, namely the intersection of the internal bisector of $\angle BAC$ with BC or its reflection with respect to the midpoint of BC.

Let $a = BC$, $b = CA$, $c = AB$ and let $\Gamma_1(O_1, R_1)$ and $\Gamma_2(O_2, R_2)$ be the circumcircles of triangles PAB and PAC respectively. Let M be the midpoint of AP (and XY) and let T_1 and T_2 be the points of tangency of the common external tangent through X with the circles Γ_1 and Γ_2 respectively.

By the power of a point with respect to a circle, we have $XT_1^2 = XA \cdot XP = XT_2^2$. Hence X is the midpoint of T_1T_2. Since the point M lies on O_1O_2 and $\angle XMO_2 = \angle O_2T_2X = \pi/2$, we obtain the following equivalences:

$$(XM)^2 + (MO_2)^2 = (O_2T_2)^2 + (T_2X)^2,$$

$$(XM)^2 = -(R_2\cos(C))^2 + R_2^2 + \frac{(T_1T_2)^2}{4},$$

$$4(XM)^2 = 4R_2^2\sin^2(C) + (O_1O_2^2 - (R_1 - R_2)^2),$$

$$(XY)^2 = 4R_2^2\sin^2(C) + (R_1\cos(B) + R_2\cos(C))^2 - R_1^2 - R_2^2 + 2R_1R_2,$$

$$(XY)^2 = 4R_2^2\sin^2(C) - R_1^2\sin^2(B) - R_2^2\sin^2(C) + 2R_1R_2\cos(B)\cos(C) + 2R_1R_2.$$

Applying the law of sines, it follows that

$$(XY)^2 = (AP)^2 - \frac{(AP)^2}{4} - \frac{(AP)^2}{4} + \frac{(AP)^2\cos(B)\cos(C)}{2\sin(B)\sin(C)} + \frac{AP^2}{2\sin(B)\sin(C)},$$
$$(XY)^2 = \frac{(AP)^2(1+\cos(B)\cos(C)+\sin(B)\sin(C))}{2\sin(B)\sin(C)},$$
$$\frac{(AP)^2}{(XY)^2} = \frac{2\sin(B)\sin(C)}{1+\cos(B)\cos(C)+\sin(B)\sin(C)}.$$

Letting $x = BP/PC$, we get $BP = ax/(x+1)$ and $PC = a/(x+1)$. This yields

$$\begin{aligned}&\left(\frac{PA}{XY}\right)^2 + \frac{PB\cdot PC}{AB\cdot AC} = 1\\
&\quad\Leftrightarrow \frac{2\sin(B)\sin(C)}{1+\cos(B)\cos(C)+\sin(B)\sin(C)} + \frac{a^2x}{bc(x+1)^2} = 1\\
&\quad\Leftrightarrow \frac{x\sin^2(A)}{(x+1)^2\sin(B)\sin(C)} = \frac{1+\cos(B)\cos(C)-\sin(B)\sin(C)}{1+\cos(B)\cos(C)+\sin(B)\sin(C)}\\
&\quad\Leftrightarrow \frac{x(1-\cos^2(A))}{(x+1)^2\sin(B)\sin(C)} = \frac{1+\cos(B+C)}{1+\cos(B)\cos(C)+\sin(B)\sin(C)}\\
&\quad\Leftrightarrow x^2\sin(B)\sin(C) + x(-1+\cos(B-C)\cos(B+C)) + \sin(B)\sin(C) = 0.\end{aligned}$$

Since

$$\begin{aligned}-1+\cos(B-C)\cos(B+C) &= -1 + \frac{\cos(2B)+\cos(2C)}{2}\\
&= -1 + \frac{1-2\sin^2(B)+1-2\sin^2(C)}{2},\end{aligned}$$

we obtain the equation

$$x^2\sin(B)\sin(C) - x(\sin^2(B)+\sin^2(C)_+\sin(B)\sin(C) = 0,$$

which is a quadratic equation with roots $x = \sin(C)/\sin(B)$ and $x = \sin(B)/\sin(C)$. Thus, the points P are the intersection of the internal bisector of $\angle BAC$ with BC or its reflection with respect to the midpoint of BC.

OC205. For each positive integer n determine the maximum number of points in space creating the set A which has the following properties:

1. the coordinates of every point from the set A are integers from the range $[0, n]$;
2. for each pair of different points $(x_1, x_2, x_3), (y_1, y_2, y_3)$ belonging to the set A at least one of the following inequalities $x_1 < y_1, x_2 < y_2, x_3 < y_3$ is satisfied and at least one of the following inequalities $x_1 > y_1, x_2 > y_2, x_3 > y_3$ is satisfied.

Originally problem 6 from day 2 of the 2013 Polish Mathematical Olympiad.

We received one correct solution. We present the solution by Oliver Geupel.

The answer is
$$a_n = \left\lfloor \frac{3(n+1)^2+1}{4} \right\rfloor.$$
We show that an a_n-element set with the desired properties is $A = \{(x,y,z) : x+y+z = \lfloor 3n/2 \rfloor\}$. In fact, if n is an even number, $n = 2m$, then members of A are points $(x, y, 3m-x-y)$ where $0 \le x \le m$ and $m - x \le y \le 2m$, as well as points $(x, y, 3m-x-y)$ where $m+1 \le x \le 2m$ and $0 \le y \le 3m-x$, the total number of elements being
$$\sum_{x=0}^{m}(m+1+x) + \sum_{x=m+1}^{2m}(3m+1-x) = 3m^2+3m+1 = a_n.$$
If n is odd, $n = 2m+1$, then members of A are points $(x, y, 3m+1-x-y)$ where $0 \le x \le m$ and $m-x \le y \le 2m+1$, and points $(x, y, 3m+1-x-y)$ where $m+1 \le x \le 2m+1$ and $0 \le y \le 3m+1-x$, with the total number of elements
$$\sum_{x=0}^{m}(m+2+x) + \sum_{x=m+1}^{2m+1}(3m+2-x) = 3(m+1)^2 = a_n.$$

It remains to show that every set A with the required properties has not more than a_n elements. Let us define subsets $B_0, \ldots, B_n$ of the lattice cube $[0,n]^3$. The members of B_k are the points $(x, n-k, z)$ where $0 \le x \le k-1$ and $0 \le z \le n$, as well as the points (k, y, z) where $n-k \le y \le n$ and $0 \le z \le n$. So B_k consists of $2k+1$ classes of $n+1$ elements each, where the members of a single class vary only in the third coordinate. Let $P = (x,y,z) \in [0,n]^3$. It follows that, if $x+y < n$ then $P \in B_{n-y}$, whereas if $x+y \ge n$ then $P \in B_x$. Hence the sets $B_0, \ldots, B_n$ constitute a disjoint partition of the lattice cube.

Let A be a set with the required properties. Then $A \cap B_k$ has not more than $2k+1$ elements because it cannot contain any two members from the same class by the given property 2. Also by property 2., the elements in $A \cap B_k$ have distinct z-coordinates. Thus $A \cap B_k$ has not more than $n+1$ elements. We obtain $|A| = \sum_{k=0}^{n} |A \cap B_k| \le \sum_{k=0}^{n} \min(2k+1, n+1)$. If n is an even number, $n = 2m$, then
$$\sum_{k=0}^{n} \min(2k+1, n+1) = \sum_{k=0}^{m}(2k+1) + \sum_{k=m+1}^{2m}(2m+1) = 3m^2+3m+1 = a_n.$$
If n is odd, say $n = 2m+1$, then
$$\sum_{k=0}^{n} \min(2k+1, n+1) = \sum_{k=0}^{m}(2k+1) + \sum_{k=m+1}^{2m+1}(2m+2) = 3(m+1)^2 = a_n.$$
Consequently, A has at least a_n elements, which completes the proof.

BOOK REVIEWS

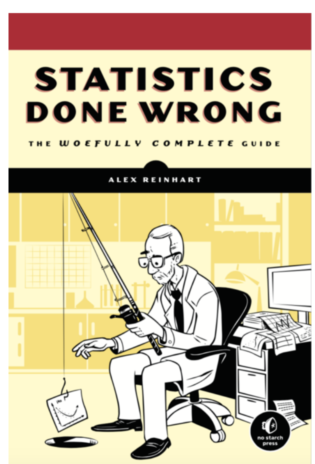
Robert Bilinski

Statistics Done Wrong: The Woefully Complete Guide by Alex Reinhart.
ISBN 9789-1-59327-620-1, 158 pages
Published by No starch press, 2015

Reviewed by Robert Bilinski, Collège Montmorency.

In his graduate studies in physics, M. Reinhart discovered that unintentional statistical errors are more common than all-out fraud. Ultimately, this physicist got hooked on statistics and reoriented his career path: now his main research interest is finding models that predict where crimes occur. This book, the author's first, is the result of his newfound passion for uncovering statistical fallacies.

Statistics Done Wrong is split into 12 chapters with titles such as "Statistical Power and Underpowered Statistics", "Pseudoreplication: Choose Your Data Wisely", "The p Value and the Base Rate Fallacy", "Double Dipping in the Data". If you are unfamiliar with these terms, you should probably read this book. It will give you an idea of how to apply critical thinking to published research and recognize many possible sources of errors such as poorly planned experiments, bad data collection and errors in compilation. At the beginning, the book assumes that all the correct steps have been taken. The author then tries to answer "What is statistics as a field?", "What are the various statistics and how can they be used correctly?", "What do they mean and what do they not mean?". Later, more fundamental questions are broached: "How do you measure what you want to study?", "What answers does your data give you and does it answer the questions you asked?".



And if the titles of the chapters are familiar to you, it probably means you have done some statistical analysis, but I still recommend this book. The content will offer some interesting tidbits. The review of the possible errors is quite extensive and the variety of the examples of misuse is not only wide ranging, but also stemming from many fields. The last few chapters cover the unethical side of the research industry as well as structural flaws that encourage errors in publishing. The very last chapter offers a few guidelines on how to be more conscious and try to avoid the pitfalls in statistical research.

Books that talk about what not to do are rare whereas I feel that they should occupy a bigger place in a scientist's reading diet. It keeps the mind sharp; it is all too easy to be stuck in a rut without knowing it. Reading a book like this recharges the good "doubting of oneself and one's approach" that makes science advance in objectivity towards truth. Naturally, this book will be of much more apparent use to an applied mathematician dealing with authentic data and modelling. However, the current overspecialization and the decrease in mathematical and statistical content in other scientific fields make it all the more important that all mathematicians and statisticians get more knowledgeable about resources like this book. One becomes wise when reading the very subtle shortcomings of others; as Socrates was quoted by Plato to have said : "The beginning of wisdom is to know that one knows nothing." Actually, the crux of the book comes down to "even if a scientist is well meaning, he can make subtle methodological errors that make his results unusable". As an example, let us read (p.37) the analysis of the error on "menstrual synchronization" apparently established in a 1971 Nature article on (M. K. McClintock (1971), "Menstrual Synchrony and Suppression", *Nature* 229, p. 244-245):

> Unfortunately, the statistical test they used assumed that if there was no synchronization, the deviation would randomly increase or decrease from one period to another. But imagine two women in the study who start with aligned cycles. One has an average gap of 28 days between periods and the other a gap of roughly 30 days. Their cycles will diverge consistently over the course of the study, starting 2 days apart, then four days, and so on, with only a bit of random variation because periods are not perfectly timed. Similarly, two women can start the study not aligned but gradually align.

This example shows us that statistical problems contain mathematics, that their solution requires problem-solving skills much like ***Crux*** problems, that measuring things is not particularly easy, and, moreover, as a society we have erred in lowering math standards.

Statistics Done Wrong is not all about errors; it is also about avoiding them. Each chapter ends with a brief list of do's and don'ts that should limit the mistakes exposed in the chapter. Naturally, these lists are not exhaustive and failsafe, but they offer stepping stones to a better statistical practise.

The above example is one of many in the book. The reader will not learn statistics while reading it though. There are no formulas or graphs or even data for that matter. *Statistics Done Wrong* is a general interest book that should be read by researchers in all fields. The style and writing is fluid and enjoyable, but, as is true for all books of this level, one needs to be available mentally to fully benefit from it. It is not light reading material, but oh so necessary. If this book review entices you to read the book under scrutiny and if you find it interesting, I also recommend *Common Errors in Statistics and How to Avoid Them* published at Wiley. This second book is technical, with formulas, graphs and data. The next evolution would be statistical case study books, supposing you already have knowledge of advanced statistical techniques. If this kind of book is well written, you can have a statistical apprentice's journey which will further your skills. *Statistics Done Wrong* has been done right. Good reading!

FOCUS ON...

No. 20

Michel Bataille

Inequalities *via* Complex Numbers

Introduction

Consider the following famous inequality: If A, B, C, D are four points in the plane, then $AB.CD + BC.AD \geq AC.BD$ (Ptolemy's inequality). A very short proof uses complex numbers: introducing the affixes a, b, c, d of A, B, C, D, the equality $(b-a)(d-c)+(c-b)(d-a) = (c-a)(d-b)$ is readily checked. The familiar properties of the modulus of a complex number (in particular the triangle inequality) then give

$$|c-a|.|d-b| = |(b-a)(d-c)+(c-b)(d-a)| \leq |b-a|.|d-c| + |c-b|.|d-a|$$

and Ptolemy's inequality follows at once! This gem of a proof, now well-known, seems to date back to 1914 ([**1**]). In this number, we present some results in the same vein, related to more or less recent problems.

Hayashi's Inequality

Hayashi's inequality, although less known, appears in problems from time to time. To name a couple of recent examples, it is the main argument of the solutions to the *American Mathematical Monthly* problem 11536 proposed in November 2010 and to problem **OC41** [2011 : 424 ; 2012 : 361]. The inequality can be stated as follows:

If P is a point in the plane of a triangle ABC, then

$$\frac{PA \cdot PB}{CA \cdot CB} + \frac{PB \cdot PC}{AB \cdot AC} + \frac{PC \cdot PA}{BC \cdot BA} \geq 1.$$

Various identities for complex numbers can be taken as the starting point of the proof. My favourite one follows from a decomposition in partial fractions which leads to

$$\frac{1}{(p-a)(p-b)(p-c)} = \frac{1}{(b-a)(c-a)} \cdot \frac{1}{p-a} + \frac{1}{(c-b)(a-b)} \cdot \frac{1}{p-b} + \frac{1}{(a-c)(b-c)} \cdot \frac{1}{p-c}.$$

The proof then proceeds by multiplying by $(p-a)(p-b)(p-c)$ and taking moduli as in the proof of Ptolemy's inequality above.

Let us connect Hayashi's inequality to a close one which involves, besides triangle ABC, two points M, N:

$$\frac{CM \cdot CN}{CA \cdot CB} + \frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BC \cdot BA} \geq 1. \tag{1}$$

The method is similar, the inequality being deduced as above from the identity

$$\frac{(m-a)(n-a)}{(b-a)(c-a)}+\frac{(m-b)(n-b)}{(c-b)(a-b)}+\frac{(m-c)(n-c)}{(a-c)(b-c)}=1. \qquad (2)$$

However, the proof of this identity is a good opportunity to apply some results obtained in a prior Focus On (No 7). Indeed, introducing $D(z)=(z-a)(z-b)(z-c)$, the left-hand side L of (2) is

$$\frac{(m-a)(n-a)}{D'(a)}+\frac{(m-b)(n-b)}{D'(b)}+\frac{(m-c)(n-c)}{D'(c)}$$

that is,

$$mn\left(\frac{1}{D'(a)}+\frac{1}{D'(b)}+\frac{1}{D'(c)}\right)-(m+n)\left(\frac{a}{D'(a)}+\frac{b}{D'(b)}+\frac{c}{D'(c)}\right)$$
$$+\left(\frac{a^2}{D'(a)}+\frac{b^2}{D'(b)}+\frac{c^2}{D'(c)}\right)$$

and finally $L=mn\times 0-(m+n)\times 0+1=1$.

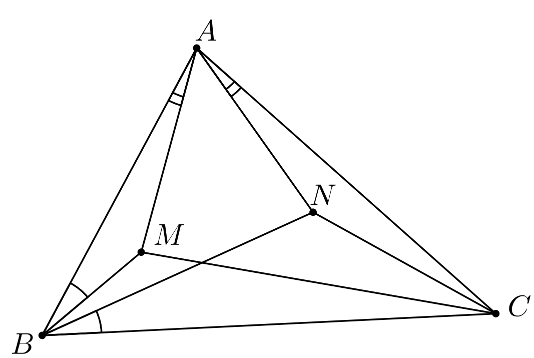
Incidentally, another interesting application of identity (2) is a variant of solution to problem **2595** ([2000 : 498 ; 2001 : 557]), which offers a case of equality in (1).

> Given that M and N are points inside the triangle ABC such that $\angle MAB=\angle NAC$ and $\angle MBA=\angle NBC$, prove that
>
> $$\frac{AM\cdot AN}{AB\cdot AC}+\frac{BM\cdot BN}{BC\cdot BA}+\frac{CM\cdot CN}{CA\cdot CB}=1.$$

Keeping the notations a,b,c,m,n for the affixes of A,B,C,M,N, the additional hypothesis on M and N leads to

$$\arg\left(\frac{(m-a)(n-a)}{(b-a)(c-a)}\right)=\arg\left(\frac{(m-b)(n-b)}{(c-b)(a-b)}\right)=0$$

(since $\angle(\overrightarrow{BM},\overrightarrow{BA})=\angle(\overrightarrow{BC},\overrightarrow{BN})$ and $\angle(\overrightarrow{AM},\overrightarrow{AB})=\angle(\overrightarrow{AC},\overrightarrow{AN})$).



As a result, both $\dfrac{(m-a)(n-a)}{(b-a)(c-a)}$ and $\dfrac{(m-b)(n-b)}{(c-b)(a-b)}$ are positive real numbers.

From (2), $\dfrac{(m-c)(n-c)}{(a-c)(b-c)}$ is a real number and because M and N are interior to the triangle, we must have $\angle(\overrightarrow{CA}, \overrightarrow{CN}) = \angle\overrightarrow{CM}, \overrightarrow{CB})$, so that $\dfrac{(m-c)(n-c)}{(a-c)(b-c)}$ is a positive real number as well. Taking moduli in (2) then yields the desired equality. Note in passing that M, N are isogonal conjugates with respect to ABC.

More examples

Prompted by expressions evoking the modulus of a complex number, one can sometimes introduce complex numbers advantageously. Here are two examples.

We start with problem **3092**, part (a) [2005 : 544,546 ; 2006 : 526]:

> Let a, b, and c be positive real numbers such that $a+b+c = abc$. Find the minimum value of $\sqrt{1+a^2} + \sqrt{1+b^2} + \sqrt{1+c^2}$.

The statement then referred to the previous problem **2814** of which one of the featured solutions (by Guersenzvaig) used complex numbers. We can mimic the method as follows.

Since $\sqrt{1+a^2} + \sqrt{1+b^2} + \sqrt{1+c^2} = |1+ia| + |1+ib| + |1+ic| \geq |3+i(a+b+c)|$ (by the triangle inequality), we have

$$\sqrt{1+a^2} + \sqrt{1+b^2} + \sqrt{1+c^2} \geq \sqrt{9+(a+b+c)^2}.$$

Now, since $\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = 1$ (from $a+b+c = abc$), the harmonic-arithmetic mean inequality gives

$$ab + bc + ca \geq \frac{3}{\frac{1}{3}\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right)} = 9.$$

Thus, $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca) \geq 3(ab+bc+ca) \geq 27$ and we finally obtain

$$\sqrt{1+a^2} + \sqrt{1+b^2} + \sqrt{1+c^2} \geq 6.$$

Observing that for $a = b = c = \sqrt{3}$, we have $a+b+c = abc$ and the equality case in the above inequality, we conclude that the required minimum value is 6.

Another example is provided by problem **3686** that I proposed in 2011 [2011 : 456, 458 ; 2012 : 391].

> Let a, b, and c be real numbers such that $abc = 1$. Show that
>
> $$\left(a - \frac{1}{a} + b - \frac{1}{b} + c - \frac{1}{c}\right)^2 \leq 2\left(a + \frac{1}{a}\right)\left(b + \frac{1}{b}\right)\left(c + \frac{1}{c}\right).$$

The problem attracted various methods and three solutions were featured. My proposed solution rested upon complex numbers: Since $abc = 1$, we have

$$\begin{aligned} 2\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right) &= 2(a^2+1)(b^2+1)(c^2+1) \\ &= 2|(a+i)(b+i)(c+i)|^2 \\ &= 2|(1-a-b-c)+i(ab+bc+ca-1)|^2 \end{aligned}$$

so that

$$2\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right) = 2\left[(a+b+c-1)^2+(1-(ab+bc+ca))^2\right].$$

But, $2(X^2+Y^2) \geq (X+Y)^2$ for all real numbers X, Y, hence

$$\begin{aligned} 2\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right) &\geq (a+b+c-ab-bc-ca)^2 \\ &= \left(a-\frac{1}{a}+b-\frac{1}{b}+c-\frac{1}{c}\right)^2. \end{aligned}$$

As usual, we conclude this number with a couple of exercises.

Exercises

1. Prove the identity

$$vw(v-w)+wu(w-u)+uv(u-v)+(v-w)(w-u)(u-v) = 0$$

where u, v, w are complex numbers and deduce another proof of Hayashi's inequality.

2. Using complex numbers, prove the identity

$$(b^2+c^2)(c^2+a^2)(a^2+b^2) = (a^2b+b^2c+c^2a-abc)^2+(ab^2+bc^2+ca^2-abc)^2$$

for real numbers a, b, c. Deduce that if a, b, c are the side lengths of a triangle, then

$$2(b^2+c^2)(c^2+a^2)(a^2+b^2) > (a^3+b^3+c^3)^2.$$

References

[1] T. Hayashi, Two Theorems on Complex Numbers, *Tôhoku Math. Journal*, p. 75–77, (1913/1914).

Generating inequalities using Schweitzer's theorem

Daniel Sitaru and Claudia Nănuţi

In 1914 P. Schweitzer published a theorem (see [1]) that later featured in the 1978 Russian Olympiad. Romanian mathematician Daniel Culea has since proposed several applications of Schweitzer's theorem [2]. In this article, we will present other applications of this theorem.

Theorem. (Kantorovic) If $p_k \in (0,\infty); k \in 1\dots n; x_k \in \mathbb{R}; 0 < m \le x_k \le M$ then

$$\Big(\sum_{k=1}^{n} p_k x_k\Big)\Big(\sum_{k=1}^{n} \frac{p_k}{x_k}\Big) \le \frac{(m+M)^2}{4mM}\Big(\sum_{k=1}^{n} p_k\Big)^2 - \frac{(m-M)^2}{4mM}\cdot \min_A \Big(\sum_{i\in A} p_i - \sum_{j\in B p_j} p_j\Big)^2,$$

where $A\cup B = \{1,2,\cdots,n\}; A\cap B = \emptyset$

Proof. From $(x_k - m)(x_k - M) \le 0$ we obtain successively:

$$\begin{aligned}
& x_k^2 - (m+M)x_k + mM \le 0,\\
& x_k + \frac{mM}{x_k} \le m+M,\\
& \frac{mM}{x_k} \le m+M-x_k,\\
& \frac{1}{x_k} \le \frac{m+M-x_k}{mM},\\
& \frac{p_k}{x_k} \le \frac{(m+M)p_k - p_k x_k}{mM}, \quad k \in 1\dots n,\\
& \sum_{k=1}^{n} \frac{p_k}{x_k} \le \frac{1}{mM}\sum_{k=1}^{n}\Big((m+M)p_k - p_k x_k\Big),
\end{aligned}$$

$$\Big(\sum_{k=1}^{n} p_k x_k\Big)\Big(\sum_{k=1}^{n} \frac{p_k}{x_k}\Big) \le \Big(\sum_{k=1}^{n} p_k x_k\Big)\frac{1}{mM}\Big((m+M)\sum_{k=1}^{n} p_k - \sum_{k=1}^{n} p_k x_k\Big). \tag{1}$$

We compute the maximum value of the right-hand side from (1). Let $x_i = m, i \in A, x_j = M, j \in B, A\cap B = \emptyset$, $A\cap B = \{1,2,...,n\}$ and $\alpha = \sum_{i\in A} p_i; \beta = \sum_{j\in B} p_j$. Then

$$\begin{aligned}
& \frac{1}{mM}\Big(m\sum_{i\in B} p_i\Big)\Big[(m+M)\Big(\sum_{i\in A} p_i + \sum_{j\in B} p_j\Big) - m\sum_{i\in A} p_i - M\sum_{j\in B} p_j\Big]\\
&= \Big(m\sum_{i\in A} p_i + M\sum_{j\in B} p_j\Big)\Big(\frac{\sum_{i\in A} p_i}{m} + \frac{\sum_{j\in B} p_j}{M}\Big)\\
&= (m\alpha + M\beta)\Big(\frac{\alpha}{m} + \frac{\beta}{M}\Big) = \frac{(2m\beta + 2M\alpha)(2m\alpha + 2M\beta)}{4mM} =
\end{aligned}$$

$$= \frac{(m\alpha+m\beta+M\alpha+M\beta-m\alpha+m\beta+M\alpha-M\beta)(m\alpha+m\beta+M\alpha+M\beta+m\alpha-m\beta-M\alpha+M\beta)}{4mM}$$
$$= \frac{[(m+M)(\alpha+\beta)-(m-M)(\alpha-\beta)][(m+m)(\alpha+\beta)+(m-M)(\alpha-\beta)]}{4mM}$$
$$= \frac{(m+M)^2(\alpha+\beta)^2-(m-M)^2(\alpha-\beta)^2}{4mM}$$
$$= \frac{(m+M)^2(\alpha+\beta)^2}{4mM} - \frac{(m-M)^2(\alpha-\beta)^2}{4mM}.$$

The maximum value of the right-hand side from (1) is obtained when $(\alpha-\beta)^2$ is minimum, namely when

$$\min_A\Big(\sum_{i\in A} p_i - \sum_{j\in B} p_j\Big), \quad A\cup B=\{1,2,...,n\}, \quad A\cap B=\emptyset.$$

□

Theorem. (Schweitzer) If $x_k\in\mathbb{R}; k\in 1\dots n$ and $0<m\le x_k\le M$ then

$$\Big(\sum_{k=1}^n x_k\Big)\Big(\sum_{k=1}^n \frac{1}{x_k}\Big) \le \frac{(m+M)^2n^2}{4mM} - \frac{(m-M)^2[1+(-1)^{n+1}]}{8mM}.$$

Proof. In the Kantorovic theorem, let the weights be $p_k=1, k\in 1\dots n, x=|A|, n-x=|B|$. It follows that:

$$\Big(\sum_{k=1}^n x_k\Big)\Big(\sum_{k=1}^n \frac{1}{x_k}\Big) \le \frac{(m+M)^2n^2}{4mM} - \frac{(m-M)^2}{4mM}\cdot\min[4(x^2-nx)+n^2]$$

and

$$\min(4x^2-4nx) = -\frac{16n^2}{16} = -n^2.$$

The minimum value is reached for $x=\frac{n}{2}$ if n is even when $\min(4x^2-4nx+n^2)=0$. If n is odd, the minimum is reached when $x=\frac{n-1}{2}<\frac{n}{2}$.

$$\begin{aligned}\min[4(x^2-nx)+n^2] &= 4\Big[\Big(\frac{n-1}{2}\Big)^2 - n\frac{n-1}{2}\Big]+n^2\\ &= 4\Big(\frac{n^2-2n+1-2n^2+2n}{4}\Big)+n^2\\ &= -n^2+1+n^2=1.\end{aligned}$$

For n even:

$$\Big(\sum_{k=1}^n x_k\Big)\Big(\sum_{k=1}^n \frac{1}{x_k}\Big) \le \frac{(m+M)^2n^2}{4mM}.$$

For n odd:

$$\Big(\sum_{k=1}^n x_k\Big)\Big(\sum_{k=1}^n \frac{1}{x_k}\Big) \le \frac{(m+M)^2n^2}{4mM} - \frac{(m-M)^2}{4mM}.$$

For $n = 2, 3, 4$ the inequality becomes, respectively,

$$(x_1 + x_2)\Big(\frac{1}{x_1} + \frac{1}{x_2}\Big) \le \frac{(m+M)^2}{mM},$$

$$(x_1 + x_2 + x_3)\Big(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\Big) \le \frac{(m+M)^2 \cdot 9 - (m-M)^2}{4mM} = 5 + 2\Big(\frac{m}{M} + \frac{M}{m}\Big),$$

$$(x_1 + x_2 + x_3 + x_4)\Big(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}\Big) \le \frac{(m+M)^2 \cdot 16}{4mM} = 8 + 4\Big(\frac{m}{M} + \frac{M}{m}\Big).$$

Let $0 < a \le b, m = a, M = b$ and $x_1, x_2, x_3, x_4 \in [a, b]$. The inequality is

$$(x_1 + x_2)\Big(\frac{1}{x_1} + \frac{1}{x_2}\Big) \le \frac{(a+b)^2}{ab}, \tag{2}$$

$$(x_1 + x_2 + x_3)\Big(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\Big) \le 5 + 2\Big(\frac{a}{b} + \frac{b}{a}\Big), \tag{3}$$

$$(x_1 + x_2 + x_3 + x_4)\Big(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}\Big) \le 8 + 4\Big(\frac{a}{b} + \frac{b}{a}\Big). \tag{4}$$

The following inequality is well known:

$$0 < a \le \sqrt{\frac{2a^2b^2}{a^2+b^2}} \le \frac{2ab}{a+b} \le \sqrt{ab} \le \frac{a+b}{2} \le \sqrt{\frac{a^2+b^2}{2}} \le b \tag{5}$$

□

Problem. Prove that if $x, y, z, t \in [a, b], 0 < a \le b$ then:

$$\frac{x+y+z+t}{\sqrt{xy} + \sqrt{yz} + \sqrt{zt} + \sqrt{tx}} \le \frac{a+b}{2\sqrt{ab}}.$$

Proof. From (2) for $m = a, M = b, x_1 = x, x_2 = y$ it follows that

$$(x+y)\Big(\frac{1}{x} + \frac{1}{y}\Big) \le \frac{(a+b)^2}{ab},$$
$$\frac{(x+y)^2}{xy} \le \frac{(a+b)^2}{ab}$$
$$ab(x+y)^2 \le xy(a+b)^2,$$
$$(x+y)\sqrt{ab} \le \sqrt{xy}(a+b).$$

Analogously, $(y+z)\sqrt{ab} \le \sqrt{yz}(a+b)$, $(z+t)\sqrt{ab} \le \sqrt{zt}(a+b)$ and $(t+x)\sqrt{ab} \le \sqrt{tx}(a+b)$ and by adding

$$2(x+y+z+t)\sqrt{ab} \le (a+b)(\sqrt{xy} + \sqrt{yz} + \sqrt{zt} + \sqrt{tx}),$$

and we obtain the result. □

Problem. In triangle ABC, let $M, N, P \in [BC]$. Prove that

$$\sqrt[3]{AM \cdot AN \cdot AP}\Big(\frac{1}{AM} + \frac{1}{AN} + \frac{1}{AP}\Big) \leq \frac{5}{3} + \frac{2}{3}\Big(\frac{AB}{AC} + \frac{AC}{AB}\Big).$$

Proof. WLOG we assume that $AB < AC$. In (3) we take $m = AB, M = AC$ and then $AM, AN, AP \in [m, M]$. Let be $x_1 = AM, x_2 = AN, x_3 = AP$. Then

$$(AM + AN + AP)\Big(\frac{1}{AM} + \frac{1}{AN} + \frac{1}{AP}\Big) \leq 5 + 2\Big(\frac{AB}{AC} + \frac{AC}{AB}\Big)$$

From AM-GM inequality, we obtain:

$$AM + AN + AP \geq 3\sqrt[3]{AM \cdot AN \cdot AP}.$$

It follows that

$$3\sqrt[3]{AM \cdot AN \cdot AP}\Big(\frac{1}{AM} + \frac{1}{AN} + \frac{1}{AP}\Big) \leq 5 + 2\Big(\frac{AB}{AC} + \frac{AC}{AB}\Big),$$
$$\sqrt[3]{AM \cdot AN \cdot AP}\Big(\frac{1}{AM} + \frac{1}{AN} + \frac{1}{AP}\Big) \leq \frac{5}{3} + \frac{2}{3}\Big(\frac{AB}{AC} + \frac{AC}{AB}\Big).$$

□

Problem. Prove that if $0 < a \leq b$ then

$$(a + \sqrt{ab} + \frac{a+b}{2} + b)\Big(\frac{1}{a} + \frac{1}{\sqrt{ab}} + \frac{2}{a+b} + \frac{1}{b}\Big) \leq 8 + 4\Big(\frac{a}{b} + \frac{b}{a}\Big), \tag{6}$$

$$\Big(\frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2} + \sqrt{\frac{a^2+b^2}{2}}\Big)\Big(\frac{a+b}{2ab} + \frac{1}{\sqrt{ab}} + \frac{2}{a+b} + \sqrt{\frac{2}{a^2+b^2}}\Big) \leq 8 + 4\Big(\frac{a}{b} + \frac{b}{a}\Big). \tag{7}$$

Proof. For (6), in (4) take $m = a, M = b, x_1 = a, x_2 = \sqrt{ab}, x_3 = \frac{a+b}{2}, x_4 = b$.

For (7), in (4) take $m = a, M = b, x_1 = \frac{2ab}{a+b}, x_2 = \sqrt{ab}, x_3 = \frac{a+b}{2}, x_4 = \sqrt{\frac{a^2+b^2}{2}}$.

References.

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3. Daniel Sitaru, *Math Phenomenon*, Paralela 45 Publishing House, Piteşti, 2016.

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6. Romanian Mathematical Gazette, A and B series.

PROBLEMS

Readers are invited to submit solutions, comments and generalizations to any problem in this section. Moreover, readers are encouraged to submit problem proposals. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by the editor by **December 1, 2016**, *although late solutions will also be considered until a solution is published.*

The editor thanks Rolland Gaudet, retired professor of the University College of Saint Boniface for translations of the problems.

4101. *Proposed by Max Alekseyev.*

Let n be an integer such that $3^n \equiv 7 \pmod{n}$. Show that 127 cannot divide n.

4102. *Proposed by Kimberly D. Apple and Eugen J. Ionascu.*

Suppose the faces of a regular icosahedron are coloured with blue or yellow in such a way that every blue face shares an edge with at most one other blue face. What is the maximum possible number of blue faces?

4103. *Proposed by Dan Stefan Marinescu, Leonard Giugiuc and Daniel Sitaru.*

Let x, y and z be positive numbers such that $x + y + z = 1$. Show that

$$\sum_{\text{cyc}}[(1-x)\sqrt{3yz(1-y)(1-z)}] \geq 4\sqrt{xyz}.$$

4104. *Proposed by Daniel Sitaru.*

Prove that for $0 < a \leq b \leq c \leq d < 2$, we have

$$5(ab^4 + bc^4 + cd^4 + 16d) < 5(b^5 + c^5 + d^5 + 16a) + 128.$$

4105. *Proposed by Mihaela Berindeanu.*

Let ABC be a triangle with centroid G. Let A', B' and C' be the feet of altitudes on the sides of the triangle from the vertices A, B and C, respectively. Let G' be the centroid of $A'B'C'$. If $GG' \parallel BC$, find all possible values of angle A.

4106. *Proposed by D.M. Bătineţu-Giurgiu and Neculai Stanciu.*

Let ABC be a triangle with $BC = a$, $AC = b$, $AB = c$ and circumradius R. Show that

$$\frac{b+c}{a^5} + \frac{c+a}{b^5} + \frac{a+b}{c^5} \geq \frac{2}{3R^4}.$$

4107. *Proposed by Lorian Saceanu.*

Let ABC be an acute triangle with inradius r, circumradius R and semiperimeter s. Prove that

$$\sqrt{\left(\frac{9}{4}+\frac{r}{2R}\right)^2+\frac{r}{R}} \leq \cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2} \leq \frac{s}{\frac{R}{2}+r}.$$

4108. *Proposed by Alessandro Ventullo.*

a) Write 2010 as a sum of consecutive squares.

b) Is it possible to write 2014 as the sum of several consecutive squares?

4109. *Proposed by Mehtaab Sawhney.*

Let k and n be positive integers. Compute the following sum in closed form:

$$\sum_{r=1}^{k}\sum_{\ell=r}^{k}(-1)^{k-r}\binom{k}{r}\binom{nr}{k+\ell}\binom{k-r}{k-\ell}n^{k-\ell}.$$

4110. *Proposed by Michel Bataille.*

Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$xf(x+y)=(x+y)f\left(\frac{x}{y}f(y)\right)$$

for all real numbers x, y with $y \neq 0$.

. . . .

4101. *Proposé par Max Alekseyev.*

Soit n un entier tel que $3^n \equiv 7 \pmod{n}$. Démontrer que 127 ne peut pas diviser n.

4102. *Proposé par Kimberly D. Apple and Eugen J. Ionascu.*

Supposons que les faces d'un icosaèdre sont colorées bleu ou jaune de façon à ce que toute face bleue partage une arête avec au plus une autre face bleue. Quel est le nombre maximum possible de faces bleues?

4103. *Proposé par Dan Stefan Marinescu, Leonard Giugiuc and Daniel Sitaru.*

Soient x, y et z des nombres positifs tels que $x+y+z=1$. Démontrer que

$$\sum_{\text{cyc}}[(1-x)\sqrt{3yz(1-y)(1-z)}] \geq 4\sqrt{xyz}.$$

4104. *Proposé par Daniel Sitaru.*

Démontrer que si $0 < a \leq b \leq c \leq d < 2$, alors la suivante tient

$$5(ab^4 + bc^4 + cd^4 + 16d) < 5(b^5 + c^5 + d^5 + 16a) + 128.$$

4105. *Proposé par Mihaela Berindeanu.*

Soit ABC un triangle avec centroïde G. Soient A', B' et C' les pieds des altitudes émanant de A, B et C respectivement. Soit G' le centroïde de $A'B'C'$. Si $GG' \parallel BC$, déterminer toute valeur possible pour l'angle A.

4106. *Proposé par D.M. Bătineţu-Giurgiu and Neculai Stanciu.*

Soit ABC un triangle où $BC = a$, $AC = b$, $AB = c$ et où R dénote le rayon du cercle circonscrit. Démontrer que

$$\frac{b+c}{a^5} + \frac{c+a}{b^5} + \frac{a+b}{c^5} \geq \frac{2}{3R^4}.$$

4107. *Propoé par Lorian Saceanu.*

Soit ABC un triangle aigu, où r est le rayon du cercle inscrit, R est le rayon du cercle circonscrit et s est le demi périmètre. Démontrer que

$$\sqrt{\left(\frac{9}{4} + \frac{r}{2R}\right)^2 + \frac{r}{R}} \leq \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \leq \frac{s}{\frac{R}{2} + r}.$$

4108. *Proposé par Alessandro Ventullo.*

a) Représenter 2010 comme somme de carrés consécutifs.

b) Est-ce possible de représenter 2014 comme somme de plusieurs carrés consécutifs.

4109. *Proposé par Mehtaab Sawhney.*

Soient k et n des entiers positifs. Déterminer

$$\sum_{r=1}^{k}\sum_{\ell=r}^{k}(-1)^{k-r}\binom{k}{r}\binom{nr}{k+\ell}\binom{k-r}{k-\ell}n^{k-\ell}$$

en forme close.

4110. *Proposé par Michel Bataille.*

Déterminer toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$ telles que

$$xf(x+y) = (x+y)f\left(\frac{x}{y}f(y)\right)$$

pour tous nombres réels x, y tels que $y \neq 0$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2015: 41(1), p. 27–30.

4001. *Proposed by Cristinel Mortici and Leonard Giugiuc.*

Let $a, b, c, d \in \mathbb{R}$ with $d > 2$ such that

$$(2d+1) \cdot \frac{a}{6} + \frac{b}{2} + \frac{c}{d+1} = 0.$$

Prove that there exists $t \in (0, d)$ such that $at^2 + bt + c = 0$.

We received four correct solutions and one solution that was almost complete. The first solution is due to Digby Smith and the second consists of ingredients of others.

Solution 1, by Digby Smith.

The result actually holds when $d > 1$. Let

$$2v = d+1 \quad \text{and} \quad 3w = 2d+1.$$

Then $0 < v < w < d$ with $6(w-v) = d-1$. The given condition can be rewritten as

$$0 = avw + bv + c = v(aw+b) + c.$$

Let $f(t) = at^2 + bt + c$. Then

$$f(w) = aw^2 + bw + c = w(aw+b) + c = -\frac{wc}{v} + c = -\frac{c(w-v)}{v} = -\frac{c(d-1)}{6v}.$$

If $c \neq 0$, then it follows that $f(0) = c$ and $f(w)$ have opposite signs, so that $f(t)$ has a real root in the interval $(0, w) \subseteq (0, d)$.

If $c = 0$, then $f(w) = w(aw+b) = 0$ since $v(aw+b) = 0$.

Solution 2.

Use the notation of Solution 1. Again $0 < v < w < d$. Furthermore, when $a = 0$, $f(v) = 0$. Otherwise, we may assume that $a > 0$, in which case $f(v) < avw + bv + c = 0$.

When $c = 0$, then $f(w) = 0$. When $c > 0$, then $f(0) > 0$ and $f(t)$ has a root in $(0, v)$. Finally, when $c < 0$, then $avw + bv = -c > 0$ and

$$f(w) = aw^2 + bw - avw - bv = (w-v)(aw+b) > 0$$

and $f(t)$ has a root in (v, w).

4002. *Proposed by Henry Aniobi.*

Let f be a convex function on an interval I. Let $x_1 \leq x_2 \leq \ldots \leq x_n$ and $y_1 \leq y_2 \leq \ldots \leq y_n$ be numbers such that $x_i + y_j$ is always in I for all $1 \leq i, j \leq n$. Let $z_1, z_2, \ldots, z_n$ be an arbitrary permutation of $y_1, y_2, \ldots, y_n$. Show that

$$\begin{aligned} f(x_1+y_1)+\ldots+f(x_n+y_n) &\geq f(x_1+z_1)+\ldots+f(x_n+z_n) \\ &\geq f(x_1+y_n)+f(x_2+y_{n-1})+\ldots+f(x_n+y_1); \end{aligned}$$

We received five submissions of which four were correct and complete. We present the solution by Joseph DiMuro.

We can prove the above statement by proving the following simpler statement:

Claim. Let $x_1 < x_2$ and $y_1 < y_2$ be numbers such that $x_i + y_j$ is always in I. Then

$$f(x_1+y_1)+f(x_2+y_2) \geq f(x_1+y_2)+f(x_2+y_1).$$

The reason why this suffices: if we choose a permutation $z_1, z_2, \ldots, z_n$ such that $z_i > z_j$ for some $i < j$, then we will have

$$f(x_i+z_i)+f(x_j+z_j) \leq f(x_i+z_j)+f(x_j+z_i).$$

We would then be able to interchange z_i and z_j without decreasing the overall sum. Thus, a permutation $z_1, z_2, \ldots, z_n$ that gives us the largest overall sum is one where $z_i \leq z_j$ whenever $i < j$; that is, $z_i = y_i$ for all i. Similarly, a permutation $z_1, z_2, \ldots, z_n$ that gives us the smallest overall sum is one where $z_i \geq z_j$ whenever $i < j$; that is, $z_i = y_{n-i+1}$ for all i.

Proof of claim. By the definition of convexity, for all $a, b \in I$ and for all $t \in [0,1]$, we have

$$f(ta+(1-t)b) \leq tf(a)+(1-t)f(b).$$

Let $a = x_1 + y_1$ and $b = x_2 + y_2$. Then $a < x_1 + y_2 < b$, so for some $t \in [0,1]$, we have $x_1 + y_2 = ta + (1-t)b$. From that, we have:

$$\begin{aligned} x_2+y_1 &= (x_1+y_1+x_2+y_2)-(x_1+y_2) \\ &= (a+b)-(ta+(1-t)b) \\ &= (1-t)a+tb. \end{aligned}$$

Therefore,

$$\begin{aligned} f(x_1+y_2)+f(x_2+y_1) &= f(ta+(1-t)b)+f((1-t)a+tb) \\ &\leq (tf(a)+(1-t)f(b))+((1-t)f(a)+tf(b)) \\ &= f(a)+f(b) \\ &= f(x_1+y_1)+f(x_2+y_2), \end{aligned}$$

completing the proof of the claim.

4003. *Proposed by Martin Lukarevski.*

Show that for any triangle ABC, the following inequality holds

$$\sin A \sin B \sin C \left(\frac{1}{\sin A + \sin B} + \frac{1}{\sin B + \sin C} + \frac{1}{\sin C + \sin A} \right)$$
$$\leq \frac{3}{4}(\cos A + \cos B + \cos C).$$

We received 13 correct solutions. We present the solution by John G. Heuver, modified slightly by the editor.

Let r, R and s denote the inradius, the circumradius and the semiperimeter of $\triangle ABC$, respectively. The following identities and inequalities are well known:

$$\sum \sin^2 A = \frac{s^2 - 4Rr - r^2}{2R^2}, \qquad (1)$$
$$\sum \sin A \sin B = \frac{s^2 + 4Rr + r^2}{4R^2}, \qquad (2)$$
$$\sum \cos A = \frac{R + r}{R}, \qquad (3)$$
$$R \geq 2r \quad \text{Euler's inequality} \qquad (4)$$
$$s^2 \leq 4R^2 + 4Rr + 3r^2 \quad \text{Gerretsen's inequality}, \qquad (5)$$

(where all the summations are taken over all angles of $\triangle ABC$).

Let L denote the left-hand side of the given inequality. By the AM-GM Inequality, we have $\sin A + \sin B \geq 2\sqrt{\sin A \sin B}$. Hence, by Cauchy-Schwarz Inequality we have

$$L \leq \frac{1}{2} \sum (\sin A)\sqrt{\sin B \sin C} \leq \frac{1}{2}\sqrt{\sum \sin^2 A}\sqrt{\sum \sin A \sin B}. \qquad (6)$$

By (1) and (5), we have

$$\sum \sin^2 A \leq \frac{4R^2 + 2r^2}{2R^2}. \qquad (7)$$

By (2) and (5), we have

$$\sum \sin A \sin B \leq \frac{4R^2 + 8Rr + 4r^2}{4R^2}. \qquad (8)$$

Using (6), (7) and (8) followed by (3) and (4), we then have

$$L \leq \frac{1}{2}\sqrt{\frac{4R^2 + 2r^2}{2R^2}} \cdot \sqrt{\frac{4R^2 + 8Rr + 4r^2}{4R^2}} = \frac{1}{2}\sqrt{2 + \left(\frac{r}{R}\right)^2} \cdot \frac{R + r}{R}$$
$$\leq \frac{1}{2}\sqrt{2 + \left(\frac{1}{2}\right)^2} \cdot \sum \cos A$$
$$= \frac{3}{4} \sum \cos A,$$

which completes the proof.

Editor's comment. Digby Smith remembered that the following problem proposed by Jack Garfunkel and George Tsintsifas appeared in the August–September 1982 issue (Vol. 8, no. 7, p. 210) of ***Crux*** and a solution given by Vedula N. Murty appeared in the November 1983 issue (Vol. 9, no. 9, p. 282):

$$\frac{4}{9}\sum \sin B \sin C \leq \prod \cos\frac{B-C}{2} \leq \frac{2}{3}\sum \cos A.$$

Smith gave a proof by first showing that $2L \leq \sum \sin B \sin C$, which together with the above inequality yields the result.

4004. *Proposed by George Apostolopoulos.*

Let x, y, z be positive real numbers such that $x+y+z=2$. Prove that

$$\frac{x^5}{yz(x^2+y^2)}+\frac{y^5}{zx(y^2+z^2)}+\frac{z^5}{xy(z^2+x^2)} \geq 1.$$

We received 16 correct submissions. We present 3 solutions.

Solution 1, by Arkady Alt.

Since by Cauchy's Inequality

$$\sum_{cyc}\frac{x^5}{yz\left(x^2+y^2\right)}=\sum_{cyc}\frac{x^6}{xyz\left(x^2+y^2\right)} \geq \frac{\left(x^3+y^3+z^3\right)^2}{\sum\limits_{cyc} xyz\left(x^2+y^2\right)},$$

it suffices to prove the inequality

$$\frac{\left(x^3+y^3+z^3\right)^2}{\sum\limits_{cyc} xyz\left(x^2+y^2\right)} \geq 1.$$

We have the following equivalences:

$$\begin{aligned}\frac{\left(x^3+y^3+z^3\right)^2}{\sum\limits_{cyc} xyz\left(x^2+y^2\right)} \geq 1 \quad &\Longleftrightarrow \quad \left(x^3+y^3+z^3\right)^2 \geq 2xyz\left(x^2+y^2+z^2\right) \\ &\Longleftrightarrow \quad \left(x^3+y^3+z^3\right)^2 \geq xyz\,(x+y+z)\left(x^2+y^2+z^2\right),\end{aligned}$$

where the latter inequality holds because by AM-GM Inequality

$$x^3+y^3+z^3 \geq 3xyz$$

and by Chebyshev's Inequality

$$x^3+y^3+z^3 \geq \frac{(x+y+z)\left(x^2+y^2+z^2\right)}{3}.$$

Solution 2, by Michel Bataille.

Let $a = \frac{x}{2}$, $b = \frac{y}{2}$ and $c = \frac{z}{2}$. With these notations, we are required to prove

$$\frac{a^6}{a^2+b^2} + \frac{b^6}{b^2+c^2} + \frac{c^6}{c^2+a^2} \geq \frac{abc}{2} \tag{1}$$

under the conditions $a, b, c > 0$ and $a + b + c = 1$.

The Cauchy-Schwarz inequality gives

$$\left(\frac{a^6}{a^2+b^2} + \frac{b^6}{b^2+c^2} + \frac{c^6}{c^2+a^2}\right)((a^2+b^2)+(b^2+c^2)+(c^2+a^2)) \geq (a^3+b^3+c^3)^2.$$

Hence, (1) will follow if we prove

$$\frac{(a^3+b^3+c^3)^2}{a^2+b^2+c^2} \geq abc.$$

Since $abc \leq \frac{a^3+b^3+c^3}{3}$, it is sufficient to show that

$$3(a^3+b^3+c^3) \geq a^2+b^2+c^2.$$

Now, the latter follows from

$$\begin{aligned} 3(a^3+b^3+c^3) &\geq 2(a^3+b^3+c^3) + 3abc \\ &= a^3+b^3+c^3+(a^3+b^3+c^3+3abc) \\ &\geq a^3+b^3+c^3+ab^2+a^2b+bc^2+b^2c+ca^2+c^2a \quad \text{(Schur's ineq.)} \\ &= (a^2+b^2+c^2)(a+b+c) = a^2+b^2+c^2 \quad (\text{since } a+b+c=1) \end{aligned}$$

so we are done.

Solution 3, by Oliver Geupel.

By hypothesis $x + y + z = 2$ and by the Cauchy-Schwarz inequality we have

$$\begin{aligned} &\left(\frac{x^5}{yz(x^2+y^2)} + \frac{y^5}{zx(y^2+z^2)} + \frac{z^5}{xy(z^2+x^2)}\right)(x+y+z)xyz(x^2+y^2+z^2) \\ &= \left(\sum_{\text{cyc}} \frac{x^5}{yz(x^2+y^2)}\right)\left(\sum_{\text{cyc}} xyz(x^2+y^2)\right) \geq (x^3+y^3+z^3)^2. \end{aligned}$$

By the power mean inequality, it holds

$$\left(\frac{x^3+y^3+z^3}{3}\right)^{1/3} \geq \left(\frac{x^2+y^2+z^2}{3}\right)^{1/2} \geq \frac{x+y+z}{3} \geq (xyz)^{1/3}.$$

Putting together we obtain

$$\begin{aligned}
&\frac{x^5}{yz(x^2+y^2)}+\frac{y^5}{zx(y^2+z^2)}+\frac{z^5}{xy(z^2+x^2)}\\
&\qquad\ge \frac{(x^3+y^3+z^3)^2}{(x+y+z)xyz(x^2+y^2+z^2)}\\
&\qquad= \frac{(x^3+y^3+z^3)^{1/3}}{x+y+z}\cdot\frac{x^3+y^3+z^3}{xyz}\cdot\frac{(x^3+y^3+z^3)^{2/3}}{x^2+y^2+z^2}\\
&\qquad\ge 3^{-2/3}\cdot 3\cdot 3^{-1/3}=1.
\end{aligned}$$

Hence the result. By the equality condition of the power mean inequality, the equality holds if and only if $x=y=z=2/3$.

4005. *Proposed by Michel Bataille.*

Let a,b,c be the sides of a triangle with area F. Suppose that some positive real numbers x,y,z satisfy the equations

$$x+y+z=4 \quad\text{and}$$

$$2xb^2c^2+2yc^2a^2+2za^2b^2-\left(\frac{4-yz}{x}\,a^4+\frac{4-zx}{y}\,b^4+\frac{4-xy}{z}\,c^4\right)=16F^2.$$

Show that the triangle is acute and find x,y,z.

We present the proposer's solution — no others were submitted.

The second equation gives

$$\begin{aligned}
&(xyz)(16F^2)\\
&=xyz(2xb^2c^2+2yc^2a^2+2za^2b^2)-yz(4-yz)a^4-zx(4-zx)b^4-xy(4-xy)c^4\\
&=(a^2yz+b^2zx+c^2xy)^2-(4a^4yz+4b^4zx+4c^4xy)\\
&=\left(\frac{x}{2}(b^2z+c^2y)+\frac{y}{2}(c^2x+a^2z)+\frac{z}{2}(b^2x+a^2y)\right)^2-(4a^4yz+4b^4zx+4c^4xy).
\end{aligned}$$

Since $t\mapsto t^2$ is a convex function and $x+y+z=4$, Jensen's inequality yields

$$\begin{aligned}
&\frac{x}{4}(b^2z+c^2y)^2+\frac{y}{4}(c^2x+a^2z)^2+\frac{z}{4}(b^2x+a^2y)^2\\
&\qquad\ge\left(\frac{x}{4}(b^2z+c^2y)+\frac{y}{4}(c^2x+a^2z)+\frac{z}{4}(b^2x+a^2y)\right)^2 \qquad (1)
\end{aligned}$$

and it follows that

$$\begin{aligned}
&(xyz)(16F^2)\\
&\le x(b^2z+c^2y)^2+y(c^2x+a^2z)^2+z(b^2x+a^2y)^2-(4a^4yz+4b^4zx+4c^4xy)\\
&=a^4yz(y+z-4)+b^4zx(z+x-4)+c^4xy(x+y-4)+xyz(2b^2c^2+2c^2a^2+2a^2b^2)\\
&=xyz(2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4)\\
&=(xyz)(16F^2).
\end{aligned}$$

Thus, equality must hold in (1) and because $t \mapsto t^2$ is a strictly convex function, this calls for

$$b^2z + c^2y = c^2x + a^2z = b^2x + a^2y.$$

Setting these three expressions equal to λ and solving for x, y, z yields

$$\begin{aligned} x &= \lambda\frac{b^2+c^2-a^2}{2b^2c^2} = \frac{\lambda\cos A}{bc}, \\ y &= \lambda\frac{c^2+a^2-b^2}{2c^2a^2} = \frac{\lambda\cos B}{ca}, \\ z &= \lambda\frac{a^2+b^2-c^2}{2a^2b^2} = \frac{\lambda\cos C}{ab}. \end{aligned}$$

(As usual, A, B, C denote the angles of the triangle opposite sides a, b, c, respectively.) Since at most one of A, B, C is not acute and x, y, z are positive, we conclude that $\cos A, \cos B, \cos C$, and λ are positive. Thus, the triangle is acute.

In addition, we have

$$4 = \frac{\lambda\cos A}{bc} + \frac{\lambda\cos B}{ca} + \frac{\lambda\cos C}{ab}.$$

Since $a\cos A + b\cos B + c\cos C = \dfrac{2F}{R}$ and $4RF = abc$ (where R is the circumradius of the triangle), we readily find $\lambda = \dfrac{a^2b^2c^2}{2F^2}$ and obtain

$$x = \frac{a^2(b^2+c^2-a^2)}{4F^2}, \quad y = \frac{b^2(c^2+a^2-b^2)}{4F^2}, \quad z = \frac{c^2(a^2+b^2-c^2)}{4F^2}.$$

Note that conversely, if given an acute triangle, then these numbers x, y, z are positive and satisfy the two equations: $x + y + z = 4$ is readily checked; also we have $b^2z + c^2y = c^2x + a^2z = b^2x + a^2y = \lambda$, hence the calculations made at the beginning (with equality in (1)) show that the second equation holds as well.

4006. *Proposed by Dragolijub Milošević.*

Let x, y, z be positive real numbers such that $xyz = 1$. Prove that

$$\frac{2}{xy + yz + zx} - \frac{1}{x + y + z} \le \frac{1}{3}.$$

We received 15 correct solutions from 14 submitters. Ten of these solutions were along the lines of the solution presented below, with variations in how they justified the ancillary inequalities and how straightforwardly they handled the algebra. In addition, there was a MAPLE-based solution, which seemed heavy-handed for this problem. There were four other solutions that were defective in some way. We present the solution by Henry Ricardo.

Let $p = x+y+z$, $q = xy+yz+zx$ and $r = xyz = 1$. Observe that $q^2 \geq 3rp = 3p$, since, by the arithmetic-geometric means inequality,

$$\begin{aligned} q^2 &= \frac{1}{2}(x^2y^2 + y^2z^2) + \frac{1}{2}(y^2z^2 + z^2x^2) + \frac{1}{2}(z^2x^2 + x^2y^2) + 2xyz(x+y+z) \\ &\geq xy^2z + yz^2x + zx^2y + 2xyz(x+y+z) \\ &= 3xyz(x+y+z) = 3rp = 3p. \end{aligned}$$

The difference between the two sides of the inequality is one-third of

$$1 - \frac{6}{q} + \frac{3}{p} \geq 1 - \frac{6}{q} + \frac{9}{q^2} = \left(1 - \frac{3}{q}\right)^2 \geq 0,$$

and the result follows with equality if and only if $x = y = z = 1$.

Editor's comment. Oliver Geupel notes that this problem is equivalent to a problem proposed by Vasile Cîrtoaje and Mircea Lascu for the Junior TST 2003 Romania. It is also Problem 72 in Chapter 20 of *Inequalities, Theorems, Techniques and Selected Problems* by Zdravko Cvetkovski (Springer, 2012).

4007. *Proposed by Mihaela Berindeanu.*

Show that for any numbers $a, b, c > 0$ such that $a^2 + b^2 + c^2 = 12$, we have

$$(a^3 + 4a + 8)(b^3 + 4b + 8)(c^3 + 4c + 8) \leq 24^3.$$

We received nine submission of which eight were correct and complete. We present two solutions.

Solution 1, by Ángel Plaza.

By taking logarithms, the proposed inequality may be written as

$$\frac{\ln\left(a^3 + 4a + 8\right) + \ln\left(b^3 + 4b + 8\right) + \ln\left(c^3 + 4c + 8\right)}{3} \leq \ln 24.$$

Changing variables $a^2 = x$, $b^2 = y$, $c^2 = z$ the problem becomes:

For any $x, y, z > 0$ such that $x + y + z = 12$, prove that

$$\frac{\ln\left(x^{3/2} + 4x^{1/2} + 8\right) + \ln\left(y^{3/2} + 4y^{1/2} + 8\right) + \ln\left(z^{3/2} + 4z^{1/2} + 8\right)}{3} \leq \ln 24.$$

Let us consider function $f(x) = \ln\left(x^{3/2} + 4x^{1/2} + 8\right)$ for $x > 0$. Then

$$f''(x) = \frac{-8x^{3/2} - 3x^{5/2} + 12x - 16\sqrt{x} - 16}{2x^{3/2}\left(x^{3/2} + 4\sqrt{x} + 8\right)^2}$$

and since $f''(x) < 0$ for $x > 0$, the function f is concave. By Jensen's inequality

$$\frac{f(x) + f(y) + f(z)}{3} \leq f\left(\frac{x+y+z}{3}\right) = f(12/3) = f(4) = \ln 24.$$

Solution 2, by the proposer.

Observe that $(a-2)^4 \geq 0$ implies that $a^4 - 8a^3 + 24a^2 - 32a + 16 \geq 0$, that is

$$a^4 + 24a^2 + 80 \geq 8a^3 + 32a + 64,$$

which gives

$$(a^2+4)(a^2+20) \geq 8(a^3+4a+8)$$

and hence

$$a^3 + 4a + 8 \leq \frac{(a^2+4)(a^2+20)}{8}.$$

So,

$$\begin{aligned}(a^3+4a+8)&(b^3+4b+8)(c^3+4c+8)\\ &\leq \frac{(a^2+4)(a^2+20)}{8} \cdot \frac{(b^2+4)(b^2+20)}{8} \cdot \frac{(c^2+4)(c^2+20)}{8},\end{aligned}$$

but we know that $\sqrt[3]{xyz} \leq \dfrac{x+y+z}{3}$, therefore

$$(a^2+4)(b^2+4)(c^2+4) \leq \left(\frac{a^2+b^2+c^2+12}{3}\right)^3 = 8^3$$

and

$$(a^2+20)(b^2+20)(c^2+20) \leq \left(\frac{a^2+b^2+c^2+60}{3}\right)^3 = 24^3.$$

Finally,

$$\left(a^3+4a+8\right)\left(b^3+4b+8\right)\left(c^3+4c+8\right) \leq \frac{8^3 \cdot 24^3}{8^3} = 24^3.$$

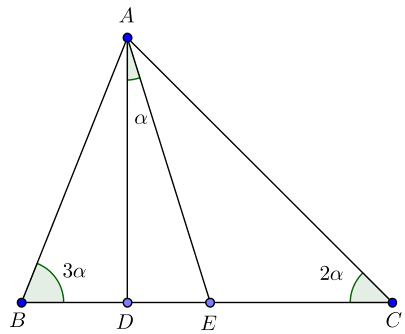
Editor's Comments. Ángel Plaza sent two solutions: the second solution consists in taking the logarithms of the given inequality, setting $a^2 = x, b^2 = y, c^2 = z$, considering the concave function (on $x \in (0,12)$) $f(x) = \ln(x^{3/2} + 4x^{1/2} + 8)$ and using Jensen's Inequality. A very similar approach was also used by Šefket Arslanagić.

4008. *Proposed by Mehmet Şahin.*

Let ABC be a triangle with $\angle ACB = 2\alpha$, $\angle ABC = 3\alpha$, AD is an altitude and AE is a median such that $\angle DAE = \alpha$. If $|BC| = a, |CA| = b, |AB| = c$, prove that

$$\frac{a}{b} = 1 + \sqrt{2\left(\frac{c}{b}\right)^2 - 1}.$$

We received 15 correct solutions and one incorrect submission. We present the solution given by Titu Zvonaru, modified slightly by the editor.



We have $AD = c\sin 3\alpha, BD = c\cos 3\alpha$, so $DE = \dfrac{a}{2} - \cos 3\alpha$. By the law of sines, we have $\dfrac{a}{\sin(180^\circ - 5\alpha)} = \dfrac{c}{\sin 2\alpha}$, so $a = \dfrac{c\sin 5\alpha}{\sin 2\alpha}$. Then

$$\begin{aligned}\frac{\sin\alpha}{\cos\alpha} = \tan\alpha = \frac{DE}{AD} = \frac{\frac{a}{2} - \cos 3\alpha}{c\sin 3\alpha} &= \frac{\sin 5\alpha - 2\sin 2\alpha\cos 3\alpha}{2\sin 2\alpha\sin 3\alpha} \\ &= \frac{\sin 5\alpha - (\sin 5\alpha + \sin(-\alpha))}{2\sin 2\alpha \sin 3\alpha} \\ &= \frac{\sin\alpha}{2\sin 2\alpha\sin 3\alpha},\end{aligned}$$

so that

$$\cos\alpha = 2\sin 2\alpha\sin 3\alpha = \cos\alpha - \cos 5\alpha,$$

which implies that

$$\cos 5\alpha = 0, \quad \text{so} \quad 5\alpha = 90^\circ, \quad \text{or} \quad \alpha = 18^\circ.$$

Hence, $\angle BAC = 180^\circ - 5\alpha = 90^\circ$, $\angle ABC = 3\alpha = 54^\circ$ and $\angle ACB = 2\alpha = 36^\circ$. Since $\cos 36^\circ = \dfrac{1+\sqrt{5}}{4}$, we have

$$b = a\cos 2\alpha = \left(\frac{1+\sqrt{5}}{4}\right)a,$$

so

$$c = \sqrt{a^2 - b^2} = \sqrt{a^2 - \frac{3+\sqrt{5}}{8}a^2} = a\sqrt{\frac{5-\sqrt{5}}{8}}.$$

Now, $\dfrac{a}{b} = \dfrac{4}{1+\sqrt{5}} = \sqrt{5} - 1$ and

$$2\left(\frac{c}{b}\right)^2 - 1 = 2\left(\frac{5-\sqrt{5}}{8}\right)\left(\frac{4}{1+\sqrt{5}}\right)^2 - 1 = \frac{14-6\sqrt{5}}{6+2\sqrt{5}} = \frac{7-3\sqrt{5}}{3+\sqrt{5}}.$$

Therefore, we have the following equivalences:

$$\begin{aligned}
&\frac{a}{b} = 1 + \sqrt{2\left(\frac{c}{b}\right)^2 - 1} \\
\iff \quad & \sqrt{5} - 1 = 1 + \sqrt{\frac{7 - 3\sqrt{5}}{3 + \sqrt{5}}} \\
\iff \quad & (\sqrt{5} - 2)^2 = \frac{7 - 3\sqrt{5}}{3 + \sqrt{5}} \\
\iff \quad & (9 - 4\sqrt{5})(3 + \sqrt{5}) = 7 - 3\sqrt{5} \\
\iff \quad & 7 - 3\sqrt{5} = 7 - 3\sqrt{5},
\end{aligned}$$

which is true and our proof is complete.

4009. *Proposed by George Apostolopoulos.*

Let m_a, m_b, m_c be the lengths of the medians of a triangle ABC. Prove that

$$\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} \leq \frac{R}{2r^2},$$

where r and R are inradius and circumradius of ABC, respectively.

We received eleven solutions, of which ten were correct. We present two solutions.

Solution 1, by Arkady Alt.

Let F, s and h_a, h_b, h_c be the area, semiperimeter, and altitudes of the triangle. Since $m_x \geq h_x, x \in \{a, b, c\}$ and

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{a}{2F} + \frac{b}{2F} + \frac{c}{2F} = \frac{s}{2F} = \frac{1}{r}$$

then

$$\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} \leq \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r} \leq \frac{R}{2r^2}$$

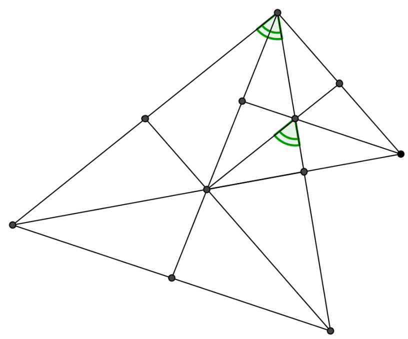
because

$$\frac{1}{r} \leq \frac{R}{2r^2} \iff 2r \leq R,$$

by Euler's Inequality.

Solution 2, by Edmund Swylan.

We take it as known that the triangle with side lengths $2m_a$, $2m_b$, $2m_c$ has medians of lengths $\frac{3}{2}a$, $\frac{3}{2}b$, $\frac{3}{2}c$. (See the drawing below.)



Let the area of $\triangle ABC$ be F. The area of the big triangle is then $3F$. Let the altitudes of the big triangle be H_a, H_b, H_c.

We have that $\dfrac{6F}{2m_x} = H_x$ and $H_x \le \dfrac{3}{2}x$, for each $x \in \{a, b, c\}$. Therefore,

$$3F(\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c}) \le \frac{3}{2}(a+b+c);$$

equality occurs if and only if the big triangle, and consequently $\triangle ABC$ too, is equilateral. Finally,

$$\frac{3}{2}(a+b+c) = 3F\frac{1}{r} \le 3F\frac{1}{r}\frac{R}{2r} = 3F\frac{R}{2r^2};$$

equality occurs if and only if $\triangle ABC$ is equilateral.

4010. *Proposed by Ovidiu Furdui.*

Let $f : [0, \frac{\pi}{2}] \to \mathbb{R}$ be a continuous function. Calculate

$$\lim_{n\to\infty} n \int_0^{\frac{\pi}{2}} \left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)^{2n} f(x)\mathrm{d}x.$$

There were eight submitted solutions for this problem, all of which were correct. We present two solutions.

Solution 1, by the group of M. Bello, M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal, expanded slightly by the editor.

The value of the required limit is $\dfrac{1}{4}\left(f(0) + f\left(\dfrac{\pi}{2}\right)\right)$. Indeed, if we denote by L the limit, then from the identity

$$\frac{\cos(x) - \sin(x)}{\cos(x) + \sin(x)} = \tan\left(\frac{\pi}{4} - x\right),$$

we have

$$\begin{aligned}
L &= \lim_{n\to\infty} n \int_0^{\pi/2} \left(\tan\left(\frac{\pi}{4} - x\right)\right)^{2n} f(x)\; dx \\
&= \lim_{n\to\infty} n \int_{-\pi/4}^{\pi/4} (\tan(s))^{2n} f\left(\frac{\pi}{4} - s\right)\; ds \\
&= \lim_{n\to\infty} n \int_0^{\pi/4} (\tan(s))^{2n} \left(f\left(\frac{\pi}{4} - s\right) + f\left(\frac{\pi}{4} + s\right)\right)\; ds \\
&= \lim_{n\to\infty} \int_0^1 \frac{r^{1+1/n}}{1+r^{2/n}} \left(f\left(\frac{\pi}{4} - \arctan(r^{1/n})\right) + f\left(\frac{\pi}{4} + \arctan(r^{1/n})\right)\right)\; dr,
\end{aligned}$$

where we have used symmetry, and in the last step we have used the change of variable $r = (\tan(s))^n$.

Since f is a continuous function, $\exists M$ such that $|f(x)| \leq M$, for $x \in [0, \frac{\pi}{2}]$, and

$$\left|\frac{r^{1+1/n}}{1+r^{2/n}} \left(f\left(\frac{\pi}{4} - \arctan(r^{1/n})\right) + f\left(\frac{\pi}{4} + \arctan(r^{1/n})\right)\right)\right| \leq M$$

for all $r \in [0,1]$, using the bound for f and that the fraction in r is bounded above by $r/(1+r^2)$ (which is bounded by $1/2$, by looking at $(r-1)^2 \geq 0$). In this way, we can apply the dominated convergence theorem to obtain

$$\begin{aligned}
L &= \int_0^1 \lim_{n\to\infty} \frac{r^{1+1/n}}{1+r^{2/n}} \left(f\left(\frac{\pi}{4} - \arctan(r^{1/n})\right) + f\left(\frac{\pi}{4} + \arctan(r^{1/n})\right)\right)\; dr \\
&= \frac{1}{2}\left(f(0) + f\left(\frac{\pi}{2}\right)\right) \int_0^1 r\; dr = \frac{1}{4}\left(f(0) + f\left(\frac{\pi}{2}\right)\right).
\end{aligned}$$

Solution 2, by Michel Bataille.

We show that the required limit is $\frac{f(0)+f(\pi/2)}{4}$. Let

$$I_n = \int_0^{\frac{\pi}{2}} \left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)^{2n} f(x)\, dx = \int_0^{\frac{\pi}{2}} \left(\tan\left(\frac{\pi}{4} - x\right)\right)^{2n} f(x)\, dx.$$

The change of variables $x = \frac{\pi}{4} - \tan^{-1}(y)$ yields

$$I_n = \int_{-1}^1 \frac{y^{2n}}{1+y^2} f\left(\frac{\pi}{4} - \tan^{-1}(y)\right)\, dy.$$

But we have

$$\int_{-1}^0 \frac{y^{2n}}{1+y^2} f\left(\frac{\pi}{4} - \tan^{-1}(y)\right)\, dy = \int_0^1 \frac{u^{2n}}{1+u^2} f\left(\frac{\pi}{4} + \tan^{-1}(u)\right)\, du$$

so that

$$I_n = \int_0^1 y^{2n} g(y)\, dy,$$

where $g(y) = \left(f\left(\frac{\pi}{4} - \tan^{-1}(y)\right) + f\left(\frac{\pi}{4} + \tan^{-1}(y)\right)\right) \cdot \frac{1}{y^2+1}$. It is known that if g is continuous on $[0,1]$, then $\lim\limits_{n\to\infty} n\int_0^1 x^n g(x)\,dx = g(1)$ [for completeness, a quick proof is given at the end]. From this result, it follows that

$$\lim_{n\to\infty} (2n)\cdot I_n = g(1) = \frac{f(0)+f(\pi/2)}{2}$$

and so

$$\lim_{n\to\infty} n\cdot I_n = \frac{f(0)+f(\pi/2)}{4},$$

as claimed.

For the proof of the property used above, let $\epsilon > 0$. Using the continuity of g, we choose $\delta \in (0,1)$ such that $|g(x) - g(1)| \le \epsilon$ whenever $x \in [\delta, 1]$. Then we have

$$\begin{aligned}\left|(n+1)\int_0^1 x^n\cdot g(x)\,dx - g(1)\right| &= \left|(n+1)\int_0^1 x^n\cdot (g(x)-g(1))\,dx\right| \\ &\le (n+1)\int_0^\delta x^n|g(x)-g(1)|\,dx + (n+1)\int_\delta^1 x^n|g(x)-g(1)|\,dx \\ &\le M\cdot\delta^{n+1} + \epsilon\end{aligned}$$

where M denotes the maximum of the continuous function $x \mapsto |g(x) - g(1)|$ on $[0,1]$. Since $0 < \delta < 1$, we deduce $\limsup\limits_{n\to\infty} |(n+1)\int_0^1 x^n\cdot g(x)\,dx - g(1)| \le \epsilon$. Since the latter holds for any positive ϵ, we must have

$$\lim_{n\to\infty}\left((n+1)\int_0^1 x^n\cdot g(x)\,dx - g(1)\right) = 0,$$

and the result follows.

Editor's Comments. This type of problem has its roots in Fourier analysis, where we are interested in limits such as the one in the problem statement. This particular limit picks out half the arithmetic mean of the function's value at the endpoints of the interval $[0, \frac{\pi}{2}]$; more classical Fourier analysis will focus on limits which pick out the function's value at a specific point, like the 'Dirac delta' distribution. All solutions aside from Solution 1 essentially followed Bataille's Solution 2, including the proposer's; À. Plaza's solution used a limit result from the proposer's own book (O. Furdui, *Limits, Series and Fractional Part Integrals*, Springer, Second ed., 2013) to skip a substitution. The techniques in both solutions (i.e. utilizing dominated convergence in a clever way, and separating an integral up into two parts which are handled using the two different functions involved in the integrand) are common techniques in classical analysis. A. Stadler's approach (namely, using 'Big O' notation instead of more precise estimates) is equally successful and is common in analytic number theory.

AUTHORS' INDEX

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(Bold font indicates featured solution.)

Proposers

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