Crux Mathematicorum

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Editorial Board

IN THIS ISSUE / DANS CE NUMÉRO

Crux Mathematicorum

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Crux Mathematicorum with Mathematical Mayhem

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EDITORIAL

Every presentation requires an introduction. In mathematics, we combine it with the motivation for studying whatever it is we are going to talk about next and it is very tempting to provide this motivation in the form of an easily understood problem or puzzle. But what if this puzzle is so interesting, it occupies the audience for the rest of the time? Good news is that, while people might not listen to what you have to say, they will remember you for the good distraction you provided for your own talk.

I have to admit that this happened to me, as an audience member, recently. I will not reveal the names of the conference or the speaker (but if you are reading this, thank you). Here is the problem:

> Prove that for each number
>
> $$n!+2,\ n!+3,\ \ldots,\ n!+n,$$
>
> there exists a prime divisor that does not divide any other number from this set.

How can one *not* get into this? Go ahead, I know you want to: try a couple of small cases, see if there is a pattern, think of some proof ideas, etc. Unfortunately, I did not catch the original source of the problem, so if you know it or have a particularly neat solution, let me know.

So remember: while you would like your audience to get interested in the subject of your talk right from the start, you want to make sure that your original motivation does not overshadow the rest of the presentation. So tread lightly or be ready to face the consequences of all the questions after your talk being about the first 2 minutes of it.

Kseniya Garaschuk

THE CONTEST CORNER

No. 29

Robert Bilinski

The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by the editor by **January 1, 2016**, *although late solutions will also be considered until a solution is published.*

CC141. Alice writes down 100 consecutive natural numbers. Bob multiplies 50 of them: 25 smallest ones and 25 largest ones. He then multiplies the remaining 50 numbers. Can the sum of the two products be equal to $100! = 1 \cdot 2 \cdot \ldots \cdot 100$?

CC142. Roboto writes down a number. Every minute, he increases the existing number by double of the number of its natural divisors (including 1 and itself). For example, if he started with 5, the sequence would be $5, 9, 15, 23, \ldots$. What is the maximum number of perfect squares that appears on the board within 24 hours?

CC143. Summer Camp has attracted 300 students this year. On the first day, the students discovered (as mathematicians would) that the number of triples of students who mutually know each other is greater than the number of pairs of students who know each other. Prove that there is a student who knows at least 5 other students.

CC144. Year 2013 is the first one since Middle Ages that uses 4 consecutive digits in its base 10 representation. How many other years like this will there be before year 10,000?

CC145. Can a natural number be divisible by all numbers between 1 and 500 except for some two consecutive ones? If so, find these two numbers (show all possible cases).

CC141. Arianne choisit 100 nombres naturels consécutifs. Bernard multiplie 50 d'entre eux, les 25 plus petits et les 25 plus gros. Ensuite, il multiplie les 50 autres. Est-ce que la somme de ces deux produits peut être égale à $100! = 1 \cdot 2 \cdot \ldots \cdot 100$?

CC142. Ramses écrit un nombre au tableau. À chaque minute, il en écrit un autre, égal au nombre précédent auquel il ajoute le double de son nombre de diviseurs naturels (incluant 1 et soi-même). À titre d'exemple, si le premier nombre est 5, la suite serait $5, 9, 15, 23, \ldots$. Quel est le nombre maximum de carrés parfaits qui pourraient apparatre au tableau dans les 24 premières heures?

CC143. Un camp d'été a attiré 300 élèves à esprit mathématique. Le premier jour, ils ont constaté que le nombre de triplets d'élèves à connaissance mutuelle dépassait le nombre de couples d'élèves à connaissance mutuelle. Démontrer qu'il existe un élève qui connait au moins 5 autres élèves.

CC144. L'année 2013 est la première depuis le moyen âge dont la représentation en base 10 utilise 4 chiffres consécutifs. Combien d'autres telles années y aura-t-il avant l'année 10 000?

CC145. Est-ce qu'un nombre naturel peut être divisible par tous les entiers de 1 à 500, sauf pour deux entiers consécutifs? Si oui, déterminer ces deux entiers, en fournissant tous les cas possibles.

CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2013: 39(9), p. 392–393.

CC91. A line segment of constant length 1 moves with one end on the x-axis and the other end on the y-axis. The region swept out (that is, the union of all possible placements) is R. Find the equation of the boundary of R.

Originally 2014 Science Atlantic Math Contest, problem 7.

We received one correct solution. We present the solution of David Manes.

The equation of the boundary of R is $x^{2/3} + y^{2/3} = 1$, the equation of an astroid. We will use the following fact: For a family of level curves $F(x, y, t) = C$, with variable parameter t, the boundary of the region R swept out must satisfy the condition $\frac{\partial F(x,y,t)}{\partial t} = 0$.

By symmetry, it suffices to assume that the segment is in the first quadrant. Let t denote the x-coordinate of the point A on the x-axis. Then the y-coordinate of the point B on the y-axis is $\sqrt{1-t^2}$. The equation of line AB is then $F(x, y, t) = 1$, where

$$F(x,y,t) = \frac{x}{t} + \frac{y}{\sqrt{1-t^2}}. \tag{1}$$

Differentiating (1) with respect to t, we obtain

$$-\frac{x}{t^2} + \frac{yt}{\left(\sqrt{1-t^2}\right)^3} = 0.$$

Therefore

$$\frac{x}{t^3} = \frac{y}{\left(\sqrt{1-t^2}\right)^3} = k.$$

Then $x = kt^3$ and $y = k\left(\sqrt{1-t^2}\right)^3$. Substituting these expressions into (1), we get $kt^2 + k\left(1-t^2\right) = 1$, or $k = 1$. Eliminating t gives the claimed equation.

CC92. Each of the positive integers 2013 and 3210 has the following three properties:

1. it is an integer between 1000 and 10000,
2. its four digits are consecutive integers, and
3. it is divisible by 3.

In total, how many positive integers have these three properties?

Originally 2013 Canadian Senior Math Contests, problem A5.

We received six correct submissions and one incorrect submission. We present the solution by Henry Ricardo.

Property 3 holds if and only if the sum of the integer's digits is divisible by 3.

Thus if k, $k+1$, $k+2$, and $k+3$ are four consecutive digits used to construct a number satisfying our requirements, then their sum, $4k+6$, must be divisible by 3. This forces the smallest digit, k, to be a multiple of 3. Eliminating $k=9$, we are left with $k=0$, 3 or 6.

Since a four-digit number cannot begin with 0, there are $3\cdot 3!$ permutations of the digits 0, 1, 2, 3 that form a number satisfying our conditions. Then there are 4! permutations of 3, 4, 5, 6 and 4! permutations of 6, 7, 8, 9 giving us a total of 66 positive integers satisfying all three properties.

CC93. If $x, y, z > 0$ and $xyz = 1$, find the range of all possible values of

$$\frac{x^3+y^3+z^3-x^{-3}-y^{-3}-z^{-3}}{x+y+z-x^{-1}-y^{-1}-z^{-1}}.$$

Originally SMT 2012, problem 11 of Team Test.

We received two correct solutions and one incorrect solution. We present the solution of Šefket Arslanagić, modified slightly by the editor.

Let

$$M = \frac{x^3+y^3+z^3-x^{-3}-y^{-3}-z^{-3}}{x+y+z-x^{-1}-y^{-1}-z^{-1}}.$$

Since $xyz = 1$, we have

$$M = \frac{(x^3-1)(y^3-1)(z^3-1)}{(x-1)(y-1)(z-1)} = (x^2+x+1)(y^2+y+1)(z^2+z+1).$$

By the AM-GM inequality,

$$x^2+x+1 \geq 3x, \quad y^2+y+1 \geq 3y, \quad \text{and} \quad z^2+z+1 \geq 3z,$$

so that $M \geq 27xyz = 27$. Equality occurs in this last inequality if and only if $x = y = z = 1$, which is excluded. The range is thus a subset of $(27, \infty)$. To see that the range is in fact all of $(27, \infty)$, note for instance that if $z = 1$, then

$$M = 3(x^2+x+1)\left(\frac{1}{x^2}+\frac{1}{x}+1\right) = \frac{3(x^2+x+1)^2}{x^2}.$$

The function $f(x) = \frac{3(x^2+x+1)^2}{x^2}$ is continuous on its domain $(0, \infty)$, and $\lim_{x\to 1} f(x) = 27$, while $\lim_{x\to\infty} f(x) = \infty$.

CC94. If $\log_2 x, (1+\log_4 x)$ and $\log_8 4x$ are consecutive terms of a geometric sequence, determine the possible values of x.

Originally 2009 Euclid Contest, problem 9a.

We received three correct submissions and six incorrect solutions (most people here did not properly account for the case when the second and third terms are 0, which is not allowed for a geometric sequence). We present the solution by Paolo Perfetti.

Let a be the common ratio of our geometric sequence and let $b = \log_2 x$. We can rewrite the terms of our sequence as $\log_2 x, 1 + \frac{1}{2}\log_2 x, \frac{2}{3} + \frac{1}{3}\log_2 x$. Then:

$$1 + \frac{1}{2}b = ab, \qquad \frac{2}{3} + \frac{1}{3}b = a^2 b.$$

Solving for (a, b) yields $(2/3, 6)$ and $(0, -2)$. The solution $(0, -2)$ is inadmissable, since the ratio of a geometric sequence cannot be 0. Thus $\log_2 x = 6$ so $x = 64$.

CC95. Positive integers x, y, z satisfy $xy + z = 160$. Determine the smallest possible value of $x + yz$.

Originally American Regions Mathematics League Competition 2013, problem 5 (Team).

We received four correct submissions and two incorrect submissions. We present the solution by Alina Sîntămărian.

The smallest possible value of $x + yz$ is 50.

Let $a = x + yz$. We analyze the following cases.

- $\boxed{y = 1}$ Then $z = 160 - x$ and $a = x + 160 - x = 160$.
- $\boxed{y = 2}$ Because $2x + z = 160$, it follows that $x \leq 79$. Then

  $$a = x + 2(160 - 2x) \implies x = \frac{2 \cdot 160 - a}{3} \leq 79 \implies a \geq 83.$$

  For $x = 79$ and $z = 2$ we have $a = 83$.
- $\boxed{y = 3}$ Because $3x + z = 160$, it follows that $x \leq 53$. Then, from $a = x + 3(160 - 3x)$ we get that $a \geq 56$. For $x = 53$ and $z = 1$ we have $a = 56$.
- $\boxed{y = 4}$ Because $4x + z = 160$, it follows that $x \leq 39$. Then, from $a = x + 4(160 - 4x)$ we get that $a \geq 55$. For $x = 39$ and $z = 4$ we have $a = 55$.
- $\boxed{y = 5}$ Because $5x + z = 160$, it follows that $x \leq 31$. Then, from $a = x + 5(160 - 5x)$ we get that $a \geq 56$. For $x = 31$ and $z = 5$ we have $a = 56$.
- $\boxed{y = 6}$ Because $6x + z = 160$, it follows that $x \leq 26$. Then, from $a = x + 6(160 - 6x)$ we get that $a \geq 50$. For $x = 26$ and $z = 4$, we have $a = 50$.
- $\boxed{y = 7}$ Because $7x + z = 160$, it follows that $x \leq 22$. Then, from $a = x + 7(160 - 7x)$ we get that $a \geq 64$. For $x = 22$ and $z = 6$ we have $a = 64$.

- $\boxed{y=8}$ Because $8x+z=160$, it follows that $x \le 19$. Then, from $a = x + 8(160-8x)$ we get that $a \ge 83$. For $x=19$ and $z=8$ we have $a=83$.
- $\boxed{y=9}$ Because $9x+z=160$, it follows that $x \le 17$. Then, from $a = x + 9(160-9x)$ we get that $a \ge 80$. For $x=17$ and $z=7$ we have $a=80$.

We also analyze the following cases.

- $\boxed{z=1}$ Then $xy = 159 = 3 \cdot 53$. So, we can have

$$\begin{array}{llll} x=1, & y=159 & \Longrightarrow & a=160, \\ x=3, & y=53 & \Longrightarrow & a=56, \\ x=53, & y=3 & & \text{was analyzed,} \\ x=159, & y=1 & & \text{was analyzed.} \end{array}$$

- $\boxed{z=2}$ Then $xy = 158 = 2 \cdot 79$. So, we can have

$$\begin{array}{llll} x=1, & y=158 & \Longrightarrow & a=517, \\ x=2, & y=79 & \Longrightarrow & a=160, \\ x=79, & y=2 & & \text{was analyzed,} \\ x=158, & y=1 & & \text{was analyzed.} \end{array}$$

- $\boxed{z=3}$ Then $xy = 157$. So, we can have

$$\begin{array}{llll} x=1, & y=157 & \Longrightarrow & a=472, \\ x=157, & y=1 & & \text{was analyzed.} \end{array}$$

- $\boxed{z=4}$ Then $xy = 156 = 2^2 \cdot 3 \cdot 13$. So, we can have

$$\begin{array}{llll} x=1, & y=156 & \Longrightarrow & a=625, \\ x=2, & y=78 & \Longrightarrow & a=314, \\ x=3, & y=52 & \Longrightarrow & a=211, \\ x=4, & y=39 & \Longrightarrow & a=160, \\ x=6, & y=26 & \Longrightarrow & a=110, \\ x=12, & y=13 & \Longrightarrow & a=64, \\ x=13, & y=12 & \Longrightarrow & a=61, \\ x=26, & y=6 & & \text{was analyzed,} \\ x=39, & y=4 & & \text{was analyzed,} \\ x=52, & y=3 & & \text{was analyzed,} \\ x=78, & y=2 & & \text{was analyzed,} \\ x=156, & y=1 & & \text{was analyzed.} \end{array}$$

If $y \ge 10$ and $z \ge 5$, then $a = x + yz > 50$. Therefore, the smallest possible value of $x + yz$ is 50.

THE OLYMPIAD CORNER

No. 327

Nicolae Strungaru and Carmen Bruni

The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by the editor by **January 1, 2016**, *although late solutions will also be considered until a solution is published.*

The editor thanks Rolland Gaudet, of l'Université Saint-Boniface in Winnipeg, for translations of the problems.

OC201. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) \in \mathbb{Q}$ and

$$f(x + f(y)^2) = f(x + y)^2.$$

OC202. Let a, b be real numbers such that the equation $x^3 - ax^2 + bx - a = 0$ has three positive real roots . Find the minimum of $\frac{2a^3-3ab+3a}{b+1}$.

OC203. Find all positive integers m and n satisfying $2^n + n = m!$.

OC204. Let ABC be a triangle. Find all points P on segment BC satisfying the following property: if X and Y are the intersections of line PA with the common external tangent lines of the circumcircles of triangles PAB and PAC, then

$$\left(\frac{PA}{XY}\right)^2 + \frac{PB \cdot PC}{AB \cdot AC} = 1.$$

OC205. For each positive integer n determine the maximum number of points in space creating the set A which has the following properties:

1. the coordinates of every point from the set A are integers from the range $[0, n]$;
2. for each pair of different points $(x_1, x_2, x_3), (y_1, y_2, y_3)$ belonging to the set A at least one of the following inequalities $x_1 < y_1, x_2 < y_2, x_3 < y_3$ is satisfied and at least one of the following inequalities $x_1 > y_1, x_2 > y_2, x_3 > y_3$ is satisfied.

OC201. Déterminer toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$ telles que $f(0) \in \mathbb{Q}$ et

$$f(x + f(y)^2) = f(x + y)^2.$$

OC202. Soient a et b des nombres réels tels que l'équation $x^3 - ax^2 + bx - a = 0$ possède trois racines réelles positives. Déterminer le minimum de $\frac{2a^3-3ab+3a}{b+1}$.

OC203. Déterminer tous les entiers positifs m et n tels que $2^n + n = m!$.

OC204. Soit ABC un triangle. Déterminer tous les points P sur le segment BC ayant la propriété suivante : si X et Y sont les intersections de la ligne PA avec les tangentes externes communes des cercles circonscrits des triangles PAB et PAC, alors

$$\left(\frac{PA}{XY}\right)^2 + \frac{PB \cdot PC}{AB \cdot AC} = 1.$$

OC205. Pour tout entier positif n, déterminer le nombre maximum de points dans l'espace formant un ensemble A ayant les propriétés suivantes :

1. les coordonnes de tout point dans l'ensemble A sont des entiers dans $[0, n]$;
2. pour toute paire de points distincts dans A, (x_1, x_2, x_3) et (y_1, y_2, y_3), au moins une des inégalités $x_1 < y_1, x_2 < y_2, x_3 < y_3$ est satisfaite et au moins une des inégalités $x_1 > y_1, x_2 > y_2, x_3 > y_3$ est satisfaite.

OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2013 : 39(9), p. 397–398.

OC141. Find all non-zero polynomials $P(x), Q(x)$ of minimal degree with real coefficients such that for all $x \in \mathbb{R}$ we have :

$$P(x^2) + Q(x) = P(x) + x^5 Q(x)$$

Originally from the Greece National Olympiad 2012 Problem 2.

We received three correct submissions. We present the solution by Titu Zvonaru and Neculai Stanciu.

Isolating for P and Q shows that

$$2\deg(P) = \deg(Q) + 5$$

which shows that the smallest possible degree for P is 3.

If $\deg(P) = 3$, then $\deg(Q) = 1$. Setting $P(x) = ax^3 + bx^2 + cx + d$ and $Q(x) = mx + n$ in the equation yields

$$ax^6 + bx^4 + cx^2 + d + mx + n = ax^3 + bx^2 + cx + d + mx^6 + nx^5$$

and when comparing the coefficient of x^3 yields that $a = 0$, contradicting the fact that $\deg(P) = 3$.

If $\deg(P) = 4$, then $\deg(Q) = 3$. Setting $P(x) = ax^4 + bx^3 + cx^2 + dx + e$ and $Q(x) = mx^3 + nx^2 + px + q$ in the equation yields

$$\begin{aligned} &ax^8 + bx^6 + cx^4 + dx^2 + e + mx^3 + nx^2 + px + q = \\ &ax^4 + bx^3 + cx^2 + dx + e + mx^8 + nx^7 + px^6 + qx^5. \end{aligned}$$

Equating coefficients yields that $m = a$, $n = 0$, $b = p$, $q = 0$, $c = a$, $b = m$, $d = c$ and $d = p$. Hence, we have that

$$P(x) = ax^4 + ax^3 + ax^2 + ax + e \quad \text{and} \quad Q(x) = ax^3 + ax.$$

OC142. Find all functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that

$$f(f(x + y)f(x - y)) = x^2 - yf(y)\,;\ \forall x, y \in \mathbb{R}\,.$$

Originally from the Japan Mathematical Olympiad Problem 2.

We received four correct submissions. We present the solution by Joseph Ling.

It is easy to verify that $f(x) = x$ for all x is a solution to

$$f(f(x+y)f(x-y)) = x^2 - yf(y). \qquad (*)$$

We claim that it is the only solution.

Letting $x = y = 0$, we see that the number $z = f(0)^2$ satisfies $f(z) = 0$. Also,

$$f(0) = f(f(z+0)f(z-0)) = z^2 - 0f(0) = z^2.$$

Now, given any $y \in \mathbb{R}$, we let $x = y + z$. Then $f(x-y) = f(z) = 0$ and the given equation becomes

$$f(0) = (y+z)^2 - yf(y).$$

So,

$$yf(y) = (y+z)^2 - f(0) = (y+z)^2 - z^2 = y(y+2z).$$

It follows that for all $y \neq 0$, $f(y) = y + 2z$. In particular, if $z \neq 0$, then

$$0 = f(z) = z + 2z = 3z \Longrightarrow z = 0,$$

a contradiction. Therefore, $z = 0$. Consequently, $f(y) = y + 0 = y$ for all $y \neq 0$. But we also have $f(0) = z^2 = 0^2 = 0$. This completes the proof.

OC143. Determine all the pairs (p, n) of a prime number p and a positive integer n for which $\frac{n^p+1}{p^n+1}$ is an integer.

Originally from the Asian Pacific Mathematical Olympiad 2012 Problem 3.

We present the solution by Oliver Geupel.

For every prime p, the pair (p, p) is a solution. Moreover, $(2, 4)$ is a solution. We prove that there are no other solutions.

Note that the function $f(x) = \dfrac{\log x}{x}$ is decreasing for $x \geq e$.

The cases $p = 2$ with $n \leq 4$ are easily inspected. For $n \geq 5$ we deduce

$$\frac{\log 2}{2} = \frac{\log 4}{4} > \frac{\log n}{n};$$

whence $n \log 2 > 2 \log n$, so that $0 < \dfrac{n^2+1}{2^n+1} < 1$.

Suppose that (p, n) is a solution with $p \geq 3$. For $n > p$, we have

$$\frac{\log p}{p} > \frac{\log n}{n};$$

whence $0 < \dfrac{n^p+1}{p^n+1} < 1$, a contradiction. Thus

$$1 \leq n \leq p. \qquad (1)$$

Since the integer p^n+1 is even, the number n^p+1 is also even ; whence n is odd. As a consequence, we have the identity $p^n+1=(p+1)(p^{n-1}-p^{n-2}+p^{n-3}-\cdots+1)$. Therefore $p+1$ is a divisor of n^p+1. Similarly, $p+1$ is a divisor of p^p+1. We obtain

$$n^p \equiv -1 \equiv p^p \pmod{p+1}. \tag{2}$$

It follows that the numbers n and $p+1$ are relatively prime. By Euler's Theorem, we obtain $n^{\varphi(p+1)} \equiv 1 \pmod{p+1}$. Applying the same theorem, we also get $p^{\varphi(p+1)} \equiv 1 \pmod{p+1}$. Consequently

$$n^{\varphi(p+1)} \equiv p^{\varphi(p+1)} \pmod{p+1}. \tag{3}$$

Lemma 1 *Let a, b, and m be integers such that* $\gcd(a,m)=\gcd(b,m)=1$ *and suppose that k and ℓ are positive integers such that* $a^k \equiv b^k \pmod m$ *and* $a^\ell \equiv b^\ell \pmod m$. *Then it holds* $a^{\gcd(k,\ell)} \equiv b^{\gcd(k,\ell)} \pmod m$.

The numbers a and b are members of the abelian multiplicative group of congruence classes modulo m which are coprime to m. If, say, $k<\ell$, we obtain $a^{k-\ell} \equiv b^{k-\ell} \pmod m$. By the Euclidean algorithm, we arrive at the result after a finite number of repetitions of this argument. □

From (2) and (3) we deduce by the lemma that

$$n^{\gcd(p,\ \varphi(p+1))} \equiv p^{\gcd(p,\ \varphi(p+1))} \pmod{p+1}.$$

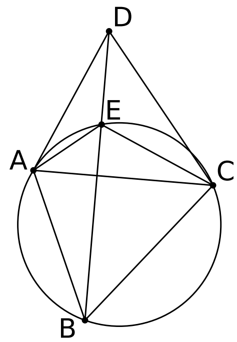
Clearly, $\varphi(p+1)<p$, so that $\gcd(p,\varphi(p+1))=1$ and $n \equiv p \pmod{p+1}$. In view of (1), we conclude $n=p$. The proof is complete.

OC144. Let $ABCD$ be a convex circumscribed quadrilateral such that $\angle ABC+\angle ADC<180^\circ$ and $\angle ABD+\angle ACB=\angle ACD+\angle ADB$. Prove that one of the diagonals of quadrilateral $ABCD$ passes through the midpoint of the other diagonal.

Originally from Romania TST 2012 Day 2 Problem 2.

We present the solution by Oliver Geupel.

We prove the stronger statement that the quadrilateral $ABCD$ is a kite.



Because $\angle ABC + \angle ADC < 180^\circ$, the circle (ABC) meets the diagonal BD at an interior point E. By the inscribed angles theorem and by hypothesis, we have

$$\begin{aligned}\angle EAD = 180^\circ - \angle DEA - \angle ADE &= \angle AEB - \angle ADE = \angle ACB - \angle ADB \\ &= \angle ACD - \angle ABD = \angle ACD - \angle ECA = \angle DCE.\end{aligned}$$

Using the law of sines in triangles AED, CDE, and ABC, we get

$$\frac{AD}{\sin\angle DEA} = \frac{DE}{\sin\angle EAD} = \frac{DE}{\sin\angle DCE} = \frac{CD}{\sin\angle CED}$$

and

$$\begin{aligned}\frac{AB}{\sin\angle DEA} = \frac{AB}{\sin\angle AEB} = \frac{AB}{\sin\angle ACB} \\ = \frac{BC}{\sin\angle BAC} = \frac{BC}{\sin\angle BEC} = \frac{BC}{\sin\angle CED}.\end{aligned}$$

Hence,

$$AB \cdot CD = BC \cdot AD. \tag{1}$$

Since the quadrilateral $ABCD$ is circumscribed, we have

$$AB + CD = BC + AD. \tag{2}$$

From (1) and (2), we deduce that it holds either $AB = BC$ and $CD = AD$ or $AB = AD$ and $BC = CD$. Thus the quadrilateral $ABCD$ is a kite.

OC145. Let $n \geq 2$ be a positive integer. Consider an $n \times n$ grid with all entries 1. Define an operation on a square to be changing the signs of all squares adjacent to it but not the sign of its own. Find all n for which it is possible to find a finite sequence of operations which changes all entries to -1.

Originally from China Western Mathematical Olympiad 2012, Day 2 Problem 3.

There were no solutions submitted.

FOCUS ON...

No. 14

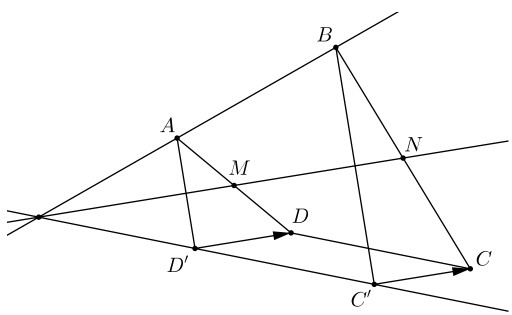
Michel Bataille

Solutions to Exercises from Focus On... No. 6 – 11

From Focus On... No. 6

(a) *Let A, B, C, D be four points in the plane such that $AB = CD$ and M, N be the midpoints of AD, BC, respectively. Show that the angle MN makes with the line AB equals the angle it makes with the line CD.*

Since $AB = CD$, there exists a unique glide reflection g such that $g(A) = D$ and $g(B) = C$ (g may reduce to a reflection). The axis of g is the line MN (since the axis passes through the midpoint of any segment joining a point to its image). It follows that $g = \mathbf{r} \circ \mathbf{t} = \mathbf{t} \circ \mathbf{r}$ where $\mathbf{r}$ denotes the reflection in MN and $\mathbf{t}$ is a translation whose vector $\overrightarrow{u}$, if not $\overrightarrow{0}$, is parallel to MN.



Let $C' = \mathbf{t}^{-1}(C)$ and $D' = \mathbf{t}^{-1}(D)$. We have

$$\mathbf{r}(C') = \mathbf{r} \circ \mathbf{t}^{-1}(C) = (\mathbf{t} \circ \mathbf{r})^{-1}(C) = g^{-1}(C) = B$$

and similarly, $\mathbf{r}(D') = A$. Thus, the line MN is an axis of symmetry of the lines AB and $C'D'$ and, as such, makes the same angle with each of them. The result follows since CD is parallel to $C'D'$.

(b) *If ABC is a triangle, find the axis and the vector of the glide reflection $\mathbf{r}_{AC} \circ \mathbf{r}_{BC} \circ \mathbf{r}_{AB}$ where $\mathbf{r}_{XY}$ denotes the reflection in the line XY.*

The reader is referred to problem **3789**, solution 1 [2013 : 427].

From Focus On... No. 7

(a) *Consider the sums $S_n(m) = \sum\limits_{i=1}^{n} \frac{w_i^m}{D'(w_i)}$ where $D(x) = \prod\limits_{i=1}^{n}(x - w_i)$ and suppose $w_i \neq 0$ for $i = 1, 2, \ldots, n$. Calculate $S_n(-1)$ and $S_n(-2)$.*

Recall the equality $\frac{1}{D(x)} = \sum_{i=1}^{n} \frac{1}{D'(w_i)} \cdot \frac{1}{x-w_i}$. We readily deduce

$$S_n(-1) = \sum_{i=1}^{n} \frac{1}{w_i D'(w_i)} = -\frac{1}{D(0)} = \frac{(-1)^{n+1}}{w_1 \cdot w_2 \cdot \cdots \cdot w_n}.$$

Now, differentiating both sides of the equality, we obtain

$$\frac{D'(x)}{(D(x))^2} = \sum_{i=1}^{n} \frac{1}{(x-w_i)^2 D'(w_i)}$$

so that

$$S_n(-2) = \sum_{i=1}^{n} \frac{1}{w_i^2 D'(w_i)} = \frac{D'(0)}{(D(0))^2}.$$

Since $(D(0))^2 = (w_1 \cdot w_2 \cdot \cdots \cdot w_n)^2$ and $D'(0) = (-1)^{n-1} \sum_{i=1}^{n} \left(\prod_{k=1, k\neq i}^{n} w_k \right)$, we finally get

$$S_n(-2) = \frac{(-1)^{n-1}}{w_1 \cdot w_2 \cdot \cdots \cdot w_n} \cdot \sum_{i=1}^{n} \frac{1}{w_i}.$$

(b) *Using the decomposition of* $\frac{1}{x^n-1}$, *rework problem* ***2657*** [2001 : 336 ; 2002 : 401], *that is prove that*

$$\sum_{n=0}^{2k-1} \tan\left(\frac{(4n-1)\pi + (-1)^n 4\theta}{8k} \right) = \frac{2k}{1 + (-1)^{k+1}\sqrt{2}\sin\theta}.$$

We recall the decomposition

$$\frac{1}{x^n - 1} = \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{\omega^j x - 1}, \tag{1}$$

where $\omega = \exp(-2\pi i/n)$. We shall also make use of the following formula

$$2i\left(\frac{1}{e^{i\alpha}+1} - \frac{1}{e^{i\beta}+1} \right) = \tan\frac{\alpha}{2} - \tan\frac{\beta}{2}, \tag{2}$$

which is easily verified (note that $\tan t = -i \cdot \frac{e^{2it}-1}{e^{2it}+1} = \frac{2i}{e^{2it}+1} - i$).

Returning to the problem, we set $z_1 = -\exp\left(\frac{i(\theta-3\pi/4)}{k}\right)$, $z_2 = -\exp\left(\frac{i(\theta-\pi/4)}{k}\right)$ and first suppose that k is even. Since

$$\frac{1}{1-\sqrt{2}\sin\theta} = \frac{i}{e^{i(\theta-3\pi/4)}-1} - \frac{i}{e^{i(\theta-\pi/4)}-1},$$

(1) yields

$$\frac{2k}{1-\sqrt{2}\sin\theta} = 2ki\left(\frac{1}{z_1^k - 1} - \frac{1}{z_2^k - 1} \right) = 2i\sum_{j=0}^{k-1} \left(\frac{1}{\omega^j z_1 - 1} - \frac{1}{\omega^j z_2 - 1} \right).$$

(Here ω denotes $\exp(-2\pi i/k)$.) But, with the help of (2), we obtain

$$\begin{aligned} 2i\left(\frac{1}{\omega^j z_1 - 1} - \frac{1}{\omega^j z_2 - 1}\right) \\ &= \tan\left(\frac{\theta - \pi/4 - 2\pi j}{2k}\right) + \tan\left(\frac{3\pi/4 - \theta + 2\pi j}{2k}\right) \\ &= \tan\left(\frac{4\theta + \pi(4(2(k-j)) - 1)}{8k}\right) + \tan\left(\frac{\pi(4(2j+1) - 1) - 4\theta}{8k}\right) \end{aligned}$$

and so

$$\frac{2k}{1 - \sqrt{2}\sin\theta} = \sum_{n=0}^{2k-1} \tan\left(\frac{(4n-1)\pi + (-1)^n 4\theta}{8k}\right).$$

The calculation is similar when k is odd. We have $\frac{2k}{1+\sqrt{2}\sin\theta} = 2ki\left(\frac{1}{\overline{z_2}^k - 1} - \frac{1}{\overline{z_1}^k - 1}\right)$. As above, we deduce that

$$\frac{2k}{1 + \sqrt{2}\sin\theta} = \sum_{j=0}^{k-1}\left(\tan\left(\frac{\pi(4(2(k-j)+1) - 1) - 4\theta}{8k}\right) + \tan\left(\frac{4\theta + \pi(4(2j) - 1)}{8k}\right)\right)$$

and the result follows.

(c) *Problem* ***3140*** [2006 : 238, 240 ; 2007 : 243] *required a proof of the inequality* $\prod_{k=1}^{n} a_k^{\frac{1}{p_k}} < 1$ *where* $n \geq 2, a_1, \ldots, a_n > 0$ *and* $p_k = \prod_{j \neq k}(a_j - a_k)$. *Find an alternative to Walther Janous's featured proof.*

We mimic the method developed in the column and omit the details.

Let

$$A(x) = \frac{1}{(x + a_1)(x + a_2) \cdots (x + a_n)}$$

whose decomposition into partial fractions is

$$A(x) = \sum_{i=1}^{n} \frac{1}{p_i} \cdot \frac{1}{x + a_i}.$$

Using $\sum_{i=1}^{n} \frac{1}{p_i} = pp(xA(x)) = 0$, we easily obtain

$$\int_0^\infty A(x)\,dx = -\sum_{i=1}^{n} \frac{1}{p_i}\,\ln(a_i).$$

Since $\int_0^\infty A(x)\,dx > 0$, we see that $\sum_{i=1}^{n} \frac{1}{p_i}\,\ln(a_i)$ must be negative and the desired inequality follows.

From Focus On... No. 8

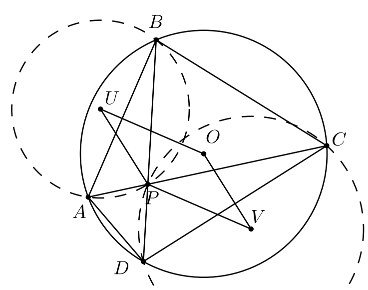
1. *Two circles, Γ with diameter AB, and Δ with centre A, intersect at points C and D. The point M (distinct from C and D) lies on Δ. The lines BM, CM and DM intersect Γ again at N, P and Q, respectively. Show that MN is the geometric mean of NC and ND.*

This is question 2 of **2666**. We keep the notations and figure of question 1 solved in Focus On... No 8. In particular, $\mathbf{I}$ denotes the inversion with centre M exchanging A and R. Since $N = \mathbf{I}(B)$ and $C = \mathbf{I}(P)$, we have

$$NC = \frac{|p|BP}{MB \cdot MP} = \frac{MB \cdot MN \cdot BP}{MB \cdot MP} = MN \cdot \frac{BP}{MP}.$$

In a similar way, $ND = MN \cdot \frac{BQ}{MQ}$. Now, because $MPBQ$ is a parallelogram, we have $BP = MQ$ and $BQ = MP$. It follows that $\frac{NC}{MN} = \frac{MN}{ND}$ and therefore $MN = \sqrt{NC \cdot ND}$.

2. *Let A, B, C, and D be points on a circle with centre O. If AB is not parallel to CD and U, V are the circumcentres of $\Delta APB, \Delta CPD$, prove that $OUPV$ is a parallelogram.*



Let $\mathbf{I}$ denote the inversion with centre P whose power is the power of P with respect to the circle Γ passing through A, B, C, D. Since $\mathbf{I}(A) = C$ and $\mathbf{I}(B) = D$, $\mathbf{I}$ transforms the circle (APB) into the line CD. It follows that PU is perpendicular to CD and so is parallel to the perpendicular bisector OV of CD. Similarly, PV is parallel to OU. Thus, $OUPV$ is a parallelogram (note that O, U, P, V are not collinear since otherwise AB and CD would be parallel).

From Focus On... No. 10

The following limits were to be evaluated in **3604** and in **3642** :

$$\lim_{n\to\infty} \frac{\int_0^1 (x^2 - x - 2)^n \, dx}{\int_0^1 (4x^2 - 2x - 2)^n \, dx} \quad \text{and} \quad \lim_{n\to\infty} \frac{\int_0^1 (2x^2 - 5x - 1)^n \, dx}{\int_0^1 (x^2 - 4x - 1)^n \, dx}.$$

It is easily checked that each of the functions $x \mapsto -x^2+x+2$ and $x \mapsto -4x^2+2x+2$ is positive and attains its maximum on $[0,1]$. From the case (c) of the last paragraph of the column, we deduce

$$\int_0^1 (-x^2+x+2)^n\,dx \;\sim\; \sqrt{\frac{\pi}{n}}\left(2+\frac{1}{4}\right)^{n+\frac{1}{2}} \quad \text{and}$$

$$\int_0^1 (-4x^2+2x+2)^n\,dx \;\sim\; \sqrt{\frac{\pi}{4n}}\left(2+\frac{4}{16}\right)^{n+\frac{1}{2}}$$

as $n \to \infty$. It readily follows that the first required limit is 2.

Each of the functions $x \mapsto -2x^2+5x+1$ and $x \mapsto -x^2+4x+1$ is positive and strictly increasing on $[0,1]$. From the case (a) this time, we obtain

$$\int_0^1 (-2x^2+5x+1)^n\,dx \;\sim\; \frac{(-2+5+1)^{n+1}}{n(2\cdot(-2)+5)} \qquad \text{and}$$

$$\int_0^1 (-x^2+4x+1)^n\,dx \;\sim\; \frac{(1+4-1)^{n+1}}{n(2\cdot(-1)+4)}$$

as $n \to \infty$. Again, the desired limit is 2.

From Focus On... No. 11

1. *Find ρ, α and $\ell > 0$ such that* $\lim\limits_{n\to\infty} \rho^n n^\alpha \sum\limits_{k=1}^{n} \dfrac{5^n}{n\binom{2n-1}{n}} = \ell$.

Let $a_n = \frac{5^n}{n\binom{2n-1}{n}}$. A short calculation gives $\frac{a_{n+1}}{a_n} = \frac{5n}{2(2n+1)}$ and it follows that $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{5}{4} > 1$. From the first of the three results of the column,

$$\sum_{k=1}^{n} a_k \;\sim\; \frac{5/4}{5/4-1}\cdot a_n = 5a_n \quad \text{as} \quad n \to \infty.$$

With the help of Stirling's formula $n! \;\sim\; n^n e^{-n}\sqrt{2\pi n}$, we find

$$n\binom{2n-1}{n} = \frac{n}{2}\cdot\frac{(2n)!}{(n!)^2} \;\sim\; \frac{4^n\cdot\sqrt{n}}{2\sqrt{\pi}}$$

so that $a_n \;\sim\; \left(\frac{5}{4}\right)^n \cdot \frac{2\sqrt{\pi}}{\sqrt{n}}$ and

$$\rho^n n^\alpha \sum_{k=1}^{n} a_k \;\sim\; 10\sqrt{\pi}\left(\frac{5\rho}{4}\right)^n \cdot n^{\alpha-\frac{1}{2}}$$

as $n \to \infty$.

We can now conclude : $\rho^n n^\alpha \sum\limits_{k=1}^{n} a_k$ has a finite nonzero limit as $n \to \infty$ if and only if $\rho = \frac{4}{5}$ and $\alpha = \frac{1}{2}$, in which case the limit is $\ell = 10\sqrt{\pi}$.

2. *Find α for the following sequence to be convergent*

$$\left(\frac{\sum_{k=1}^{n+1} k!\csc(\pi/2^k)}{\sum_{k=1}^{n} k!\csc(\pi/2^k)} - n\alpha\right)_{n\geq 1}.$$

What is its limit in that case ?

Let $a_n = n!\csc(\pi/2^n)$. We easily obtain

$$\frac{a_{n+1}}{a_n} = 2(n+1)\cos(\pi/2^{n+1})$$

and deduce that $\frac{a_{n+1}}{a_n} \sim 2n$ as $n \to \infty$. Furthermore,

$$\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n} - 2n\right) = \lim_{n\to\infty}\left(2\cos(\pi/2^{n+1}) - 2n\left(1-\cos(\pi/2^{n+1})\right)\right) = 2.$$

Note that $\lim\limits_{n\to\infty}\left(2n\left(1-\cos(\pi/2^{n+1})\right)\right) = 0$ since $1-\cos(\pi/2^{n+1}) \sim \frac{\pi^2}{2^{2n+3}}$ as $n \to \infty$.

From the third result proved in the column, we see that the given sequence is convergent when $\alpha = 2$ and its limit then is 2.

A lot of information out of nothing

Mathematician R said the following to mathematicians P and S : "I thought of two natural numbers. They are each greater than 1 and their sum is less than 100. I will secretly tell mathematician P their product and I will secretly tell mathematician S their sum." He did just that and asked mathematicians P and S to guess the numbers. The following dialogue took place :

P : I cannot tell what the numbers are.
S : I knew you couldn't.
P : Then I know what they are.
S : Then so do I.

Can you guess the numbers ?

Originally from article "Many bits out of nothing" by S. Artemov, Y. Gimatov and V. Fedorov, Kvant 1977 (3).

From the Adequality of Fermat to Nonstandard Analysis

Paul Deguire

Pierre de Fermat was a 17th century French mathematician. Although he worked full time as a lawyer at the *parlement* (provincial court) of Toulouse, he found enough time for high-quality mathematical research in many areas. One of his interests was a sort of early differential calculus. His main technique was a concept of approximation that he called *"adégalité"*, which would only be made rigorous centuries later.

He adapted this word from the Latin *"adaequatio"* ("conformity, making equal") which he appears to have taken from Bachet de Méziriac's Latin translation of Diophantus' *Arithmetica* — presumably, the famous copy with margins too small to contain the supposed proof of his "Last Theorem". Diophantus, who wrote in Greek around 250 AD, actually used the word $\pi\alpha\rho\iota\sigma\acute{o}\tau\eta\varsigma$, by which he meant an exact process for finding solutions to number theory problems, known to lie close to a certain value (see [H], pages 95-97, 206-207). Except that it involved small numbers, it had little in common with Fermat's technique.

Fermat used this notion, which we will translate by "adequality", to find things such as extrema or the slopes of tangent lines. This was in the 1630s, three decades before Christian Huygens, presenting the work of Fermat at the *Académie Francaise*, created the expression "infinitely small" (1667), and five decades before Newton and Leibniz wrote about calculus. The terms "function" (Euler) and "derivative" (Lagrange) were not introduced until the 18th century : Fermat referred to "curves" and "slopes of tangent lines," and his optimization problems were not expressed in terms of functions, as we would do today.

Infinitely small quantities had been used before Fermat to calculate areas and volumes. Archimedes, for instance, used them to calculate the volume of the sphere. After the independent invention of analytic geometry by Descartes and Fermat in the early 17th century, the study of curves became an important branch of mathematics, and determining their extrema and their tangent lines became important research topics. Fermat was the first to solve such problems using infinitesimals, and until Weierstrass formulated a rigorous definition of limit in the 19th century, this was the standard technique.

Here's an example of one of Fermat's results using adequality, from a text published in 1636 [T]. Given a segment AB, where should we put the point C such that the rectangle on AC and CB has a maximal area ?

Let the length of AB be a and the length of CB be x ; we want to maximize $x(a-x)$. Suppose that C has been properly located and that our maximum is $b(a-b)$. Fermat knew that Oresme (14th century) and Kepler (early 17th century) had both observed that near a maximum the value of such expressions change very little. Today we would say that, near a maximum, the derivative is very small :

Fermat expressed this by saying that the values were adequal.

Fermat supposed that e is a very small quantity so that $(b+e)(a-(b+e))$ would then be very close to $b(a-b)$. He wrote : $(b+e)(a-(b+e)) \sim b(a-b)$, the symbol $\sim$ meaning that the quantities are adequal. Therefore,

$$ab - b^2 - be + ae - be - e^2 \sim ab - b^2.$$

Eliminating common terms, Fermat was left with

$$ae \sim 2be + e^2.$$

He then divided by e, obtaining

$$a \sim 2b + e$$

and let $e = 0$, obtaining $a = 2b$. Thus the point C is the midpoint of AB, and the rectangle with given perimeter and largest area is a square.

Similar use of infinitely small quantities worked well during the first 200 years of differential calculus, not only to find extrema but to calculate derivatives and their applications.

But even an average modern student would ask : how can you divide by e if e is equal to 0 ? The legitimate objection was made by other mathematicians and by philosophers. But the methods obviously worked, providing fantastic new results not obtainable otherwise. Better yet, as Newton showed, the physical world agreed with the new calculus as well. So the method was adopted, despite its lack of rigour.

Today we would say that Fermat was *essentially* solving

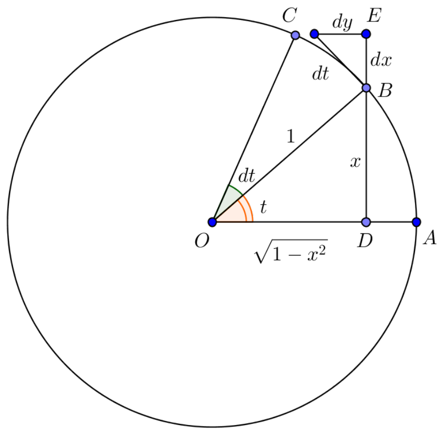
$$\lim_{e\to 0} \frac{f(b+e) - f(b)}{e} = 0$$

where the function f gives the area of the rectangle : that is, setting the derivative equal to 0. We know what a limit is, and we can provide rigorous analytical arguments. But our actual calculation exactly parallels what Fermat did four centuries ago ; and the student who only knows how to grind out the calculation knows no more than Fermat knew at the time.

The next two examples, from Gottfried Wilhelm Leibniz and Guillaume François Antoine, marquis de l'Hospital, give an idea of how the notion of "infinitely small number" was used in the first years of the new calculus.

Leibniz and the differential triangle

In the following diagram the circle is of radius 1, the angle is t, dt is a small increment of t and $x = \sin t$. Leibniz works with the small triangle $\triangle BEC$ whose sides are differentials and therefore infinitely small quantities. Therefore, he can consider the side BC, tangent to the circle at B, to be equal to the little arc dt. We know that this is only a good approximation, but the smaller dt is, the better is the approximation.



All quotients of pairs of sides in this triangle are indeterminate of the form $0/0$. The opponents of the new methods said that these quotients had no meaning. Leibniz had no problem because this infinitely small triangle $\triangle BEC$ is similar to the finite triangle $\triangle ODB$, therefore each quotient of sides of the infinitesimal triangle is equal to a well-defined quotient of sides of a finite triangle. For example, the indeterminate quotient dx/dt is equal to the well-defined finite quotient $\sqrt{1-x^2}/1$. Because $x = \sin t$, then $\sqrt{1-x^2} = \cos t$, and we obtain the now-familiar

$$\frac{d}{dt}\sin t = \cos t.$$

Similarly, $dy = \dfrac{xdx}{\sqrt{1-x^2}}$; using this last expression for dy and the Pythagorean Theorem, $dt^2 = dx^2 + dy^2$, Leibniz found the differential equation

$$\frac{d^2}{dt^2}x(t) = -x(t)$$

from which he was able to find the infinite McLaurin series for $x = \sin t$(see [Ka], pages 321-323). As we can see, he obtained this significant result using only the ordinary laws of geometry and infinitesimals.

Nobody was able then to define properly what an infinitely small quantity was, but the new calculus had so much success in both pure and applied mathematics that nothing stopped its evolution. For most of the next century it would be expanded and applied, although the basic notion of limit was at best a good intuition and the notion of infinitesimal had no solid ground to stand on.

The first differential calculus book

One of the first followers of Leibniz, and himself an important contributor to the new calculus, was Jean Bernoulli. The Marquis de l'Hospital, a French nobleman interested in mathematics, learned the new calculus from Bernoulli and wrote in 1696 the first calculus textbook, entitled *L'analyse des infiniments petits pour* *l'intelligence des lignes courbes* (see [L], page 23). What we now call "l'Hospital rule" – though it appears to have been Bernoulli's discovery – was published here for the first time.

In this text l'Hospital considers differences in variable quantities : what we today would call differentials. For most applications these differences have to be considered to be infinitely small. The rules of derivation are easy to follow (although we wouldn't speak about derivatives for another century). For instance to find the difference of a product xy, l'Hospital simply calculated

$$d(xy) = (x + dx)(y + dy) - xy.$$

Therefore, he had $d(xy) = xdy + ydx + dxdy$. Because $dxdy$ was negligible compared to the other two terms of the right side, he concluded that $d(xy) = xdy + ydx$.

Modern views of infinitely small quantities

In the 19th century, Augustin-Louis Cauchy would define properly the notions of limit and convergence and he had a definition for infinitely small as well that he frequently used. Following the developments of mathematics in the second half of the 19th century, notably the creation of mathematical logic, a mathematical definition of the limit was provided by Weierstrass. The use of quantifiers made it possible to say mathematically what Cauchy was saying in ordinary language. The mathematical definition could be worked with in full rigor, the mathematics were on solid ground again. The use of infinitely small quantities became useless and was somewhat forgotten.

Then, in 1960, Abraham Robinson wrote about a new notion, Nonstandard Analysis, based on an extension of the reals called "hyperreals". This number system obeys the axioms of the real numbers, except the Archimedean axiom ; proving that it is consistent (or at least as consistent as the real number system) requires some subtle mathematical logic. It contains all real numbers, and also some "infinite" numbers so large that you cannot count to them ; and the reciprocals of those infinite numbers are infinitesimal. Infinitesimals lie above all negative reals and below all positive reals, but are not equal to 0. If you can count to (or past) a hyperreal number, then it is "finite" : any finite hyperreal is the sum of a standard real number and an infinitesimal.

Calculus can be done on the nonstandard reals, often quite easily as the notion of convergence is greatly simplified ; and Keisler [K] has even written an elementary calculus textbook using this approach.

However, it should be stressed that any result about the standard reals that can be obtained with nonstandard analysis can be obtained without it as well ; so most mathematicians continue to use classical standard analysis. Nevertheless, nonstandard analysis is sound, and demonstrates that the language of the infinitesimals, as used by Fermat, Leibniz and Cauchy, is fully compatible with mathematical rigour.

Perhaps the final chapter in this story is *synthetic differential geometry* and particularly *smooth infinitesimal analysis*, based on ideas developed by W. Lawvere in the 1960s. This approach to analysis uses infinitesimal elements that are nilpotent (they obey $e^2 = 0$ exactly), and typically avoids introducing infinities. It does, however, use logic without the excluded middle : for an infinitesimal e, the proposition $e = 0$ is neither true nor false ! Unlike nonstandard analysis, smooth infinitesimal analysis has results that are not true classically. For instance, all functions of smooth infinitesimal analysis are differentiable ! A good introduction to this is given by Bell [B].

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The author thanks Robert Dawson for valuable comments and suggestions.

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PROBLEMS

Readers are invited to submit solutions, comments and generalizations to any problem in this section. Moreover, readers are encouraged to submit problem proposals. Please see submission guidelines in side the back cover or online.

To facilitate their consideration, solutions should be received by the editor by **January 1, 2016**, *although late solutions will also be considered until a solution is published.*

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

An asterisk ($\star$) *after a number indicates that a problem was proposed without a solution.*

3981. *Proposed by José Luis Díaz-Barrero.*

Let a, b, c be three positive numbers such that $ab+bc+ca = 6abc$. For all positive integers $n \geq 2$, show that

$$\frac{bc}{a^n(b+c)} + \frac{ca}{b^n(c+a)} + \frac{ab}{c^n(a+b)} \geq 3 \cdot 2^{n-2}.$$

3982. *Proposed by Michel Bataille.*

Let $n \in \mathbb{N}$, $u > 0$ and for $k = 0, 1, \ldots, n-1$, let a_k be such that $0 < a_k \leq \sinh(u)$. Prove that if $x \geq e^u$, then

$$a_{n-1}x^{n-1} - a_{n-2}x^{n-2} + \cdots + (-1)^{n-2}a_1x + (-1)^{n-1}a_0 < \frac{x^n}{2}.$$

3983. *Proposed by Marcel Chiriţă.*

Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$xf(x) - yf(y) = (x^2 - y^2)\max\left(f'(x), f'(y)\right)$$

for all real numbers x, y.

3984. *Proposed by Dragoljub Milošević.*

Let ABC be any right-angled triangle with $\angle C = 90°$. Let w_a be the length of the internal bisector of $\angle A$ from A to the side BC; define w_b similarly. If $[ABC]$ is the area of ABC, prove that

$$w_aw_b \geq 4[ABC](2 - \sqrt{2}).$$

3985. *Proposed by Mihaela Berindeanu.*

Prove that if a, b, c are positive numbers with sum of 18, then

$$\frac{a}{b^2+36} + \frac{b}{c^2+36} + \frac{c}{a^2+36} \geq \frac{1}{4}.$$

3986. *Proposed by George Apostolopoulos.*

Let a, b, c be the lengths of the sides of a triangle ABC with circumradius R. Prove that
$$\frac{1}{(a+b)(b+c)} + \frac{1}{(b+c)(c+a)} + \frac{1}{(c+a)(a+b)} \geq \frac{1}{4R^2}.$$

3987. *Proposed by Michel Bataille.*

Let ABC be a triangle with circumcircle Γ and let A' be the point of Γ diametrically opposite to A. The lines AB and AC intersect the tangent to Γ at A' in B' and C', respectively. Prove that the tangents to Γ at B and C intersect at the centroid of $AB'C'$ if and only if $2\cos A = 3\sin B \sin C$.

3988*. *Proposed by George Apostolopoulos.*

Let a, b, c be positive real numbers. Find the maximum and minimum values of the expression
$$\frac{a}{\sqrt{a^2+3b^2}} + \frac{b}{\sqrt{b^2+3c^2}} + \frac{c}{\sqrt{c^2+3a^2}}.$$

3989. *Proposed by Dragoljub Milošević.*

Let h_a, h_b and h_c be the altitudes, r_a, r_b and r_c be the exradii, r the inradius and R the circumradius of a triangle. Prove that
$$\frac{r_a^2}{h_a} + \frac{r_b^2}{h_b} + \frac{r_c^2}{h_c} \geq 3(2R - r).$$

3990. *Proposed by Ángel Plaza.*

Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 > a_2 > \ldots > a_n$. Prove that
$$(a_1 - a_n)\left(\frac{1}{a_1 - a_2} + \frac{1}{a_2 - a_3} + \cdots + \frac{1}{a_{n-1} - a_n}\right) \geq (n-1)^2.$$
When does equality hold?

. . . .

3981. *Proposé par José Luis Díaz-Barrero.*

Soit a, b, c trois réels strictement positifs tels que $ab + bc + ca = 6abc$. Démontrer que pour tout entier n, $n \geq 2$,
$$\frac{bc}{a^n(b+c)} + \frac{ca}{b^n(c+a)} + \frac{ab}{c^n(a+b)} \geq 3 \cdot 2^{n-2}.$$

3982. *Proposé par Michel Bataille.*

Soit $n \in \mathbb{N}$ et $u > 0$. Pour $k = 0, 1, \ldots, n-1$, soit a_k tel que $0 < a_k \leq \sinh(u)$. Démontrer que si $x \geq e^u$, alors

$$a_{n-1}x^{n-1} - a_{n-2}x^{n-2} + \cdots + (-1)^{n-2}a_1 x + (-1)^{n-1}a_0 < \frac{x^n}{2}.$$

3983. *Proposé par Marcel Chiriţă.*

Déterminer toutes les fonctions dérivables $f : \mathbb{R} \to \mathbb{R}$ telles que

$$xf(x) - yf(y) = (x^2 - y^2) \max\left(f'(x), f'(y)\right)$$

pour tous réels x, y.

3984. *Proposé par Dragoljub Milošević.*

On considère un triangle ABC rectangle en C. Soit w_a la longueur de la bissectrice de l'angle A, du sommet A jusqu'au côté opposé BC. On définit w_b de façon semblable. Si $[ABC]$ représente l'aire du triangle ABC, démontrer que

$$w_a w_b \geq 4[ABC](2 - \sqrt{2}).$$

3985. *Proposé par Mihaela Berindeanu.*

Soit a, b, c trois nombres strictement positifs ayant une somme de 18. Démontrer que

$$\frac{a}{b^2+36} + \frac{b}{c^2+36} + \frac{c}{a^2+36} \geq \frac{1}{4}.$$

3986. *Proposé par George Apostolopoulos.*

Soit a, b, c les longueurs des côtés d'un triangle ABC et soit R le rayon du cercle circonscrit au triangle. Démontrer que

$$\frac{1}{(a+b)(b+c)} + \frac{1}{(b+c)(c+a)} + \frac{1}{(c+a)(a+b)} \geq \frac{1}{4R^2}.$$

3987. *Proposé par Michel Bataille.*

Soit ABC un triangle, Γ son cercle circonscrit et A' le point de Γ qui est diamétralement opposé à A. La tangente à Γ au point A' est coupée par les droites AB et AC aux points respectifs B' et C'. Démontrer que les tangentes à Γ aux points B et C se coupent au centre de gravité du triangle $AB'C'$ si et seulement si $2\cos A = 3 \sin B \sin C$.

3988*. *Proposé par George Apostolopoulos.*

Soit a, b, c des réels strictement positifs. Déterminer la valeur maximale et la valeur minimale de l'expression

$$\frac{a}{\sqrt{a^2+3b^2}}+\frac{b}{\sqrt{b^2+3c^2}}+\frac{c}{\sqrt{c^2+3a^2}}.$$

3989. *Proposé par Dragoljub Milošević.*

Soit h_a, h_b et h_c les hauteurs d'un triangle, r le rayon du cercle inscrit dans le triangle, R le rayon du cercle circonscrit au triangle et r_a, r_b et r_c les rayons des cercles exinscrits du triangle. Démontrer que

$$\frac{r_a^2}{h_a}+\frac{r_b^2}{h_b}+\frac{r_c^2}{h_c}\geq 3(2R-r).$$

3990. *Proposé par Ángel Plaza.*

Soit $a_1, a_2, \ldots, a_n$ des réels strictement positifs tels que $a_1 > a_2 > \ldots > a_n$. Démontrer que

$$(a_1-a_n)\left(\frac{1}{a_1-a_2}+\frac{1}{a_2-a_3}+\cdots+\frac{1}{a_{n-1}-a_n}\right)\geq (n-1)^2.$$

Quand y a-t-il égalité?

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

The editor would like to acknowledge the following authors for the solution of problem 3853: George Apostolopoulos, Phil McCartney, and Daniel Văcaru for a minor generalization of the problem. Sincere apologies for the oversight.

Statements of the problems in this section originally appear in 2013: 39(9), p. 413–417.

3881. *Proposed by Ovidiu Furdui.*

Calculate
$$\sum_{n=2}^{\infty}\left(n^2\ln\left(1-\frac{1}{n^2}\right)+1\right).$$

We received ten correct submissions. We present the solution by Michel Bataille.

Let
$$U_n = n^2\ln\left(1-\frac{1}{n^2}\right)+1.$$

It is readily checked that for all $n \geq 2$
$$U_n = 1 + 2\ln(n-1) + (a_n - a_{n-1}) + (b_n - b_{n-1}) - 2(c_n - c_{n-1})$$

where $a_n = \ln(n),\ b_n = n^2(\ln(n+1)-\ln(n))$ and $c_n = n\ln(n)$. It follows that for all integer $N > 2$:

$$\begin{aligned}\sum_{n=2}^{N} U_n &= N-1+2\ln[(N-1)!] + a_N - a_1 + b_N - b_1 - 2(c_N - c_1)\\ &= N-1+2\ln\left(\frac{N!}{N}\right)+\ln(N)+N^2(\ln(N+1)-\ln(N))-\ln(2)-2N\ln(N)\\ &= 2\ln(N!)-2N\ln(N)-\ln(N)+N+N^2\ln\left(1+\frac{1}{N}\right)-1-\ln(2).\end{aligned}$$

We know that
$$\ln(N!) = N\ln(N) - N + \frac{\ln(N)}{2} + \ln(\sqrt{2\pi}) + o(1),$$

and
$$\ln\left(1+\frac{1}{N}\right) = \frac{1}{N} - \frac{1}{2N^2} + o\left(\frac{1}{N^2}\right)$$

as $N \to \infty$.

Thus,

$$\sum_{n=2}^{N} U_n = 2\ln(\sqrt{2\pi}) - \frac{1}{2} - 1 - \ln(2) + o(1)$$

and we can conclude that

$$\sum_{n=2}^{\infty} U_n = \sum_{n=2}^{\infty} \left(n^2 \ln\left(1 - \frac{1}{n^2}\right) + 1\right) = \ln(\pi) - \frac{3}{2}.$$

3882. *Originally proposed by Mehmet Sahin; corrected version by Arkady Alt.*

Let ABC be a right angle triangle with $\angle CAB = 90^\circ$ and hypotenuse a. Let $[AD]$ be an altitude and let I_1 and I_2 be the incenters of the triangles ABD and ADC, respectively. Let ρ be the radius of the circle through the points B, I_1 and I_2 and let r be the inradius of the triangle ABC. Prove that

$$\rho = \sqrt{\frac{a^2 + 2ar + 2r^2}{2}}$$

and $\min \dfrac{\rho}{r} = \sqrt{3} + \sqrt{6}$.

We received seven correct solutions.

Solution to Part 1, by AN-anduud Problem Solving Group.

Let E and F be the points where the line I_1I_2 meets the sides AB and AC, respectively. From the given conditions, both triangles I_1BD and I_2AD have angles of 45° and $\frac{B}{2}$, so they are similar and

$$k := \frac{DI_1}{DB} = \frac{DI_2}{DA}.$$

It follows that the dilative rotation defined by a rotation through -45° about D followed by the dilatation centred at D with ratio k takes B to I_1 and A to I_2, and therefore, it takes BA to I_1I_2. From this we deduce that $\angle AEF = \angle AFE = 45^\circ$, whence

$$\begin{aligned}\angle BI_1I_2 + \angle I_2CB &= \left(\frac{1}{2}\angle ABC + 135^\circ\right) + \frac{1}{2}\angle ACB \\ &= 135^\circ + \frac{1}{2}\left(\angle ABC + \angle ACB\right) = 180^\circ.\end{aligned}$$

Therefore, I_1BCI_2 is a cyclic quadrilateral with circumradius ρ. If we denote the centre of the circumcircle by O' (and recall that the angle at O' equals twice any inscribed angle subtended by the same chord), we have

$$\angle BO'C = \angle I_1O'C + \angle BO'I_2 - \angle I_1O'I_2 = \angle B + \angle C - 2\psi,$$

where we set $2\psi = \angle I_1O'I_2$. Thus $\angle BO'C = 90^\circ - 2\psi$, and we have

$$a^2 = \rho^2 + \rho^2 - 2\rho^2\cos(90^\circ - 2\psi) = 2\rho^2(1 - \sin 2\psi) = 2\rho^2 - 4\rho^2 \cdot \sin\psi\cos\psi.$$

As a chord of a circle whose radius is ρ, $I_1I_2 = 2\rho\sin\psi$. But it is known that $I_1I_2 = \sqrt{2}r$. Briefly, we denote the inradii of triangles DBA and DAC by r_1 and r_2, so that the similarity of these triangles to ΔABC yields $r_1 = \frac{c}{a}r$ and $r_2 = \frac{b}{a}r$. In the right triangle DI_1I_2 we have $DI_1 = r_1\sqrt{2}$ and $DI_2 = r_2\sqrt{2}$, whence

$$I_1I_2 = \sqrt{2\left(r_1^2 + r_2^2\right)} = \frac{r\sqrt{2}}{a}\sqrt{b^2 + c^2} = \sqrt{2}r.$$

Equating the two expressions for I_1I_2 gives us

$$\sin\psi = \frac{r\sqrt{2}}{2\rho} \quad \text{and} \quad \cos\psi = \sqrt{1 - \sin^2\psi} = \frac{1}{2\rho}\sqrt{4\rho^2 - 2r^2}.$$

Hence,

$$a^2 = 2\rho^2 - \sqrt{8\rho^2r^2 - 4r^4}.$$

Finally, ρ^2 will be the larger root of the resulting quadratic with $x = \rho^2$, so we calculate

$$\rho = \sqrt{\frac{a^2 + 2ar + 2r^2}{2}}.$$

Editor's Comment. Solver Modak derived the formula

$$\rho = \sqrt{\frac{a^2 + bc}{2}},$$

which can easily be seen to be equivalent to the requested form using $2r + a = b + c$ and $b^2 + c^2 = a^2$.

Solution to Part 2, by Prithwijit De.

Dividing ρ by r leads to

$$\frac{\rho}{r} = \sqrt{\frac{a^2 + 2ar + 2r^2}{2r^2}} = \sqrt{\frac{1}{2}\left(\frac{a}{r} + 1\right)^2 + \frac{1}{2}}.$$

Thus $\frac{\rho}{r}$ attains a minimum value when $\frac{r}{a}$ attains a maximum value. Because $\frac{b}{a} = \sin B$ and $\frac{c}{a} = \cos B$, we conclude that

$$\frac{r}{a} = \frac{b + c - a}{2a} = \frac{\sin B + \cos B - 1}{2} = \frac{\sqrt{2}\sin(B + 45^\circ) - 1}{2} \le \frac{\sqrt{2} - 1}{2};$$

equality is achieved when $b = c = \frac{a}{\sqrt{2}}$. Hence $\min\frac{a}{r} = \left(\max\frac{r}{a}\right)^{-1} = 2(\sqrt{2} + 1)$, and it follows that

$$\min\frac{\rho}{r} = \sqrt{9 + 6\sqrt{2}} = \sqrt{(\sqrt{3} + \sqrt{6})^2} = \sqrt{3} + \sqrt{6}.$$

3883. *Proposed by Max A. Alekseyev.*

Let a, b, c, d be positive integers such that $a+b$ and $ad+bc$ are odd. Prove that if $2^a - 3^b > 1$, then $2^a - 3^b$ does not divide $2^c + 3^d$.

No solution was received. We present the solution of the proposer.

Since $2^a - 3^b > 1$ and $a+b$ is odd, we have $a > 2$. Consider two cases.

Case I: If a is even, then b and c are odd.

In this case $2^a \equiv 16 \pmod{24}$ and $3^b \equiv 3 \pmod{24}$, hence $2^a - 3^b \equiv 13 \pmod{24}$. For every prime divisor p of $2^a - 3^b$, we have $3^b \equiv 2^a \pmod{p}$, implying that 3 is a square modulo p, i.e. $p \equiv 1, 11, 13,$ or $23 \pmod{24}$. Since the products of residues 1 and 11 modulo 24 cannot produce the residue 13, there exists a prime divisor $p \equiv 13$ or $23 \pmod{24}$. For such p, the number 3 (and therefore 3^d) is a quadratic residue, whereas -2 (and therefore $(-2)^c$) is not. Hence $-2^c \not\equiv 3^d \pmod{p}$. So p does not divide $2^c + 3^d$ and thus neither does $2^a - 3^b$.

Case II: If a is odd, then b is even and d odd.

This time $2^a - 3^b \equiv 23 \pmod{24}$. Following the same steps as above, we find that 2 is a square modulo p for any prime divisor p of $2^a - 3^b$. We conclude that there has to be a prime divisor $p \equiv 17$ or $23 \pmod{24}$ for which 2^c is a quadratic residue, whereas -3^d is not. Again we obtain that p (and therefore $2^a - 3^b$) does not divide $2^c + 3^d$.

3884. *Proposed by Mihai Bogdan.*

Let a, b, c and d be positive real numbers such that $a+b+c+d = k$, where $k \in (0, 8)$. Prove that:

$$\frac{a}{b^2+1} + \frac{b}{c^2+1} + \frac{c}{d^2+1} + \frac{d}{a^2+1} \geq \frac{k(8-k)}{8}.$$

When does the equality hold?

We received ten correct submissions. We present two different solutions.

Solution 1, by AN-anduud Probem Solving Group and Šefket Arslanagić, done independently.

Applying the AM-GM inequality twice, we have:

$$\begin{aligned}
&\frac{a}{b^2+1} + \frac{b}{c^2+1} + \frac{c}{d^2+1} + \frac{d}{a^2+1} \\
&\quad = \sum \left(a - \frac{ab^2}{b^2+1} \right) \geq \sum \left(a - \frac{ab^2}{2b} \right) = \sum a - \frac{1}{2} \sum ab \\
&\quad = k - \frac{1}{2}(ab+bc+cd+da) = k - \frac{1}{2}(a+c)(b+d) \\
&\quad \geq k - \frac{1}{2} \left(\frac{a+b+c+d}{2} \right)^2 = k - \frac{1}{8}k^2 = \frac{k(8-k)}{8}.
\end{aligned}$$

Equality holds if and only if k=4 and $a = b = c = d = 1$.

Solution 2, by Cao Minh Quang and Oliver Geupel, done independently.

As in Solution 1 above, we have:

$$\begin{aligned}\sum \frac{a}{b^2+1} &\geq \sum a - \frac{1}{2}\sum ab \\ &= k - \frac{1}{8}((a+b+c+d)^2 - (a-b+c-d)^2) \\ &\geq k - \frac{1}{8}(a+b+c+d)^2 = k - \frac{1}{8}k^2 = \frac{k(8-k)}{8}.\end{aligned}$$

Equality holds if and only if k=4 and $a = b = c = d = 1$.

3885. *Proposed by Oai Thanh Dao.*

Let ABC be a triangle and let F be a point that lies on the circumcircle of ABC. Further, let H_a, H_b and H_c denote projections of the orthocenter H onto sides BC, AC and AB, respectively. The three circles AH_aF, BH_bF and CH_cF meet the three sides BC, AC and AB at points A_1, B_1 and C_1, respectively. Prove that the points A_1, B_1 and C_1 are collinear.

We received ten correct submissions. We present two solutions.

Solution 1, by Titu Zvonaru modified by the editor.

Because the points A, H_a, A_1, F lie on a circle and $\angle AH_aA_1 = 90°$, AA_1 must be a diameter of the circle and $\angle AFA_1$ must also equal $90°$. Moreover, if we denote by A' the point where FA_1 again meets the circumcircle of ΔABC, AA' must be a diameter of that circle. We can reverse the argument and define AA' to be a diameter of the circumcircle, and then define A_1 to be the point where $A'F$ intersects the line BC. Similarly, define B' and C' to be diametrically opposite B and C so that

$$A_1 = A'F \cap BC, \quad B_1 = B'F \cap CA, \quad \text{and} \quad C_1 = C'F \cap AB.$$

Applying Pascal's theorem to the cyclic hexagon $AA'FB'BC$ we find that

$$AA' \cap B'B = O, \quad A'F \cap BC = A_1, \quad \text{and} \quad FB' \cap CA = B_1$$

are collinear. Similarly, applying the theorem to the cyclic hexagon $BB'FC'CA$, we deduce that O, B_1, C_1 are also collinear. We conclude that A_1, B_1, and C_1 all lie on a line through O.

Solution 2, by Michel Bataille.

With H the orthocentre, let the line FH meet the circumcircle Γ of ΔABC again at F_1, and let F' be the midpoint of HF_1. Expressing half the power of H with respect to Γ in four ways, we obtain

$$HF \cdot HF' = HA \cdot HH_a = HB \cdot HH_b = HC \cdot HH_c.$$

(Here and in what follows, all distances are signed.) It follows that F' is on the circles AH_aF, BH_bF and CH_cF and, therefore, the respective centres O_a, O_b, O_c of these circles are collinear (on the perpendicular bisector of FF'). Since O_a is the midpoint of AA_1 and A_1 is on BC, the line VW joining the midpoints V and W of CA and AB passes through O_a. The homothety with centre A and factor 2 maps V to C, W to B and O_a to A_1. As a result, we have

$$\frac{A_1C}{BA_1} = \frac{O_aV}{WO_a}.$$

In a similar way, if U is the midpoint of BC, we obtain $\frac{B_1A}{CB_1} = \frac{O_bW}{UO_b}$ and $\frac{C_1B}{AC_1} = \frac{O_cU}{VO_c}$ and so

$$\frac{A_1C}{BA_1} \cdot \frac{B_1A}{CB_1} \cdot \frac{C_1B}{AC_1} = \frac{O_aV}{WO_a} \cdot \frac{O_bW}{UO_b} \cdot \frac{O_cU}{VO_c}.$$

The collinearity of A_1, B_1, C_1 immediately results from the collinearity of O_a, O_b, O_c with the help of Menelaus's theorem and its converse.

Editor's Comments. Bataille pointed out that some care must be taken in the definitions of the points A_1, B_1, C_1: when F equals a vertex of ΔABC or when it is a reflection of H in a side of the triangle, one or more of the circles $AH_aF, \ldots,$ will not be defined. If one allows points at infinity, the alternative definition of the points A_1, B_1, C_1 used in Solution 1 is valid for all positions of F on the circumcircle. De observed that the Miquel point common to the circles AB_1C_1, A_1BC_1, and A_1B_1C lies on the circumcircle of ΔABC, and F is its image under reflection in the line $A_1B_1C_1$. Several readers used elegant arguments to set up the converse of Menelaus's theorem, but failed to prove that one or three of A_1, B_1, C_1 are exterior to ΔABC. (This editor also failed in his attempts to devise a convincing proof. Recall that should zero or two of the points be exterior, then Ceva's theorem, not Menelaus's, would apply.) Note that Bataille's argument in Solution 2 cleverly circumvents the difficulty.

3886. *Proposed by Michel Bataille.*

Let $H_n = \sum_{k=1}^{n} \frac{1}{k}$ be the nth harmonic number and let $H_0 = 0$. Prove that for $n \geq 1$, we have

$$\sum_{k=1}^{n}(-1)^{n-k}\binom{n}{k}2^k H_k = 2H_n - H_{\lfloor n/2 \rfloor}.$$

We received eight correct submissions. We present the solution by José H. Nieto.

Let

$$f(n) = \sum_{k=1}^{n}(-1)^{n-k}\binom{n}{k}2^k H_k.$$

Since $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$, we have

$$\begin{aligned}
f(n) &= \sum_{k=1}^{n}(-1)^{n-k}\binom{n-1}{k-1}2^k H_k + \sum_{k=1}^{n}(-1)^{n-k}\binom{n-1}{k}2^k H_k \\
&= \sum_{k=0}^{n-1}(-1)^{n-1-k}\binom{n-1}{k}2^{k+1}\left(H_k + \frac{1}{k+1}\right) - f(n-1) \\
&= 2f(n-1) + \sum_{k=0}^{n-1}(-1)^{n-1-k}\binom{n-1}{k}\frac{2^{k+1}}{k+1} - f(n-1) \\
&= f(n-1) + \frac{1}{n}\sum_{k=1}^{n}(-1)^{n-k}\binom{n}{k}2^k \\
&= f(n-1) + \frac{1}{n}\left((2-1)^n - (-1)^n\right) \\
&= f(n-1) + \frac{1}{n}\left(1 - (-1)^n\right).
\end{aligned}$$

Since we may write

$$f(n) = f(1) + \sum_{k=2}^{n}(f(k) - f(k-1)),$$

we have

$$\begin{aligned}
f(n) &= 2 + \sum_{k=2}^{n}\frac{1}{k}\left(1-(-1)^k\right) = \sum_{k=1}^{n}\frac{1}{k}\left(1-(-1)^k\right) = \sum_{\substack{1\le k\le n\\ k \text{ odd}}}\frac{2}{k} \\
&= 2H_n - \sum_{\substack{1\le k\le n\\ k \text{ even}}}\frac{2}{k} = 2H_n - H_{\lfloor n/2\rfloor},
\end{aligned}$$

as required.

3887. *Proposed by Dao Hoang Viet.*

Let a, b and c be positive real numbers. Prove that

$$\frac{a^2}{bc(a^2+ab+b^2)} + \frac{b^2}{ac(b^2+bc+c^2)} + \frac{c^2}{ab(a^2+ac+c^2)} \ge \frac{9}{(a+b+c)^2}.$$

We received 16 correct submissions. We present three solutions.

Solution 1, by Arkady Alt and Dragoljub Milošević, done independently.

Since $a^2 + ab + b^2 \ge 3ab$, we have

$$\frac{a^3}{a^2+ab+b^2} = a - \frac{a^2b+ab^2}{a^2+ab+b^2} \ge a - \frac{ab(a+b)}{3ab} = a - \frac{a+b}{3} = \frac{2a-b}{3}.$$

Similarly,
$$\frac{b^3}{b^2+bc+c^2} \geq \frac{2b-c}{3} \quad \text{and} \quad \frac{c^3}{c^2+ca+a^2} \geq \frac{2c-a}{3}.$$

Adding up these three inequalities yields

$$\sum_{cyc} \frac{a^3}{a^2+ab+b^2} \geq \frac{a+b+c}{3}.$$

Hence, by the AM-GM inequality, we have

$$\begin{aligned}
\sum_{cyc} \frac{a^2}{bc(a^2+ab+b^2)} &= \frac{1}{abc} \sum_{cyc} \frac{a^3}{a^2+ab+b^2} \\
&\geq \frac{a+b+c}{3abc} = \frac{(a+b+c)^3}{3abc(a+b+c)^2} \\
&\geq \frac{27abc}{3abc(a+b+c)^2} = \frac{9}{(a+b+c)^2}.
\end{aligned}$$

Equality occurs if and only if $a=b=c$.

Solution 2, by AN-anduud Problem Solving Group.

Applying the AM-GM inequality twice, we get:

$$\begin{aligned}
\sum_{cyc} \frac{a^2}{bc(a^2+ab+b^2)} &= \sum_{cyc} \frac{(a^2+ab+b^2)-b(a+b)}{bc(a^2+ab+b^2)} \\
&= \sum_{cyc} \frac{1}{bc} - \sum_{cycl} \frac{a+b}{c(a^2+ab+b^2)} \\
&= \frac{a+b+c}{abc} - \sum_{cycl} \frac{a+b}{c(a^2+ab+b^2)} \\
&\geq \frac{a+b+c}{abc} - \sum_{cyc} \frac{a+b}{3c \cdot \sqrt[3]{a^2 \cdot ab \cdot b^2}} = \frac{1}{3} \cdot \frac{a+b+c}{abc} \\
&\geq \frac{1}{3} \cdot \frac{a+b+c}{\left(\frac{a+b+c}{3}\right)^3} = \frac{9}{(a+b+c)^2}.
\end{aligned}$$

Equality occurs if and only if $a=b=c$.

Solution 3, by Titu Zvonaru.

The given inequality is equivalent to

$$\sum_{cyc} \frac{a^3}{a^2+ab+b^2} \geq \frac{9abc}{(a+b+c)^2}. \tag{1}$$

Since
$$\sum_{cyc} \frac{a^3 - b^3}{a^2 + ab + b^2} = \sum_{cyc} (a - b) = 0,$$
(1) is equivalent to
$$\sum_{cyc} \frac{a^3 + b^3}{a^2 + ab + b^2} \geq \frac{18abc}{(a + b + c)^3}. \tag{2}$$
Next,
$$\frac{a^3 + b^3}{a^2 + ab + b^2} \geq \frac{a + b}{3}$$
is equivalent, in succession, to
$$\frac{a^2 - ab + b^2}{a^2 + ab + b^2} \geq \frac{1}{3},$$
$$3(a^2 - ab + b^2) \geq a^2 + ab + b^2,$$
or
$$a^2 + b^2 \geq 2ab,$$
which is true. Hence, by the AM-GM inequality we have:
$$\begin{aligned}\sum_{cyc} \frac{a^3 + b^3}{a^2 + ab + b^2} &\geq \sum_{cyc} \frac{a + b}{3} \\ &= \frac{2}{3}(a + b + c) = \frac{2}{3} \cdot \frac{(a + b + c)^3}{(a + b + c)^2} \\ &\geq \frac{2}{3} \cdot \frac{27abc}{(a + b + c)^2} = \frac{18abc}{(a + b + c)^2},\end{aligned}$$
which establishes (2) and completes the proof.

Editor's Comment. As usual, Wagon provided a proof based on the algebraic algorithm FindInstance which took 5 minutes to confirm the result.

3888. *Proposed by Peter Woo.*

My greatly admired high school teacher taught me one foolproof method when solving triangles. Suppose in triangle ABC you are given the measure of $\angle A$ and the lengths of the adjacent sides b and c; then to find the remaining angles in terms of the given quantities, one should use the law of cosines to find the length of the third side and then the law of sines to find measures of $\angle B$ and $\angle C$. Or so I was taught. But after many years, I found a way to solve this problem while avoiding the cosine law and the use of square roots. Can you discover such a way?

There were 10 correct solutions to this problem, split into two approaches.

We feature one of each type.

Solution 1, by Oliver Geupel.

First, applying the law of tangents

$$\tan\frac{\angle B-\angle C}{2}=\frac{b-c}{b+c}\cot\frac{\angle A}{2},$$

we find $\frac{\angle B-\angle C}{2}$. Then, adding and subtracting

$$\frac{\angle B-\angle C}{2}\qquad\text{and}\qquad\frac{\angle B+\angle C}{2}=90^\circ-\frac{\angle A}{2},$$

we compute $\angle B$ and $\angle C$.

Solution 2, by Roy Barbara.

To calculate the angle B (angle C can be calculated similarly), let O denote the projection of C onto the line AB. Consider the coordinate system with origin O that contains the triangle ABC with coordinates $A(\alpha,0)$, $B(\beta,0)$, and $C(0,h)$ with $h>0$. Then $h=b\sin A$. From $\alpha=-b\cos A$ we obtain $\beta=\alpha+c=c-b\cos A$. Finally,

$$\cot B=\frac{\beta}{h}=\frac{c-b\cos A}{b\sin A}.$$

3889. *Proposed by Cristinel Mortici.*

Prove that

$$e^\pi>\left(\frac{e^2+\pi^2}{2e}\right)^e.$$

We received ten correct solutions, and a Mathematica verification. We present two solutions which are representative of all solutions.

Solution 1, by Paolo Perfetti.

More generally we prove that for $x\in\mathbb{R}$,

$$e^{|x|}>\left(\frac{e^2+x^2}{2e}\right)^e\quad\Longleftrightarrow\quad |x|\geq e.$$

The inequality is equivalent to

$$|x|\geq e\ln(e^2+x^2)-e\ln 2-e.$$

Let

$$f(x)=|x|-e\ln(e^2+x^2)+e\ln 2+e.$$

The function f is even, ie. $f(x)=f(-x)$, so it suffices to consider $x\geq 0$. Clearly $f(0)=e(\ln 2-1)<0$, $f(e)=0$ and

$$f'(x) = 1 - \frac{2xe}{e^2 + x^2} = \frac{(e-x)^2}{e^2 + x^2} \geq 0.$$

The result follows.

Solution 2, by C.R. Pranesachar.

We shall prove the more general inequality

$$e^{e+x} > \left(\frac{e^2 + (e+x)^2}{2e}\right)^e \tag{1}$$

for all $x > 0$. The desired inequality follows by substituting $x = \pi - e$, which is positive. To prove (1), we replace x by et, where $t > 0$. Then (1) reduces to

$$e^{e(1+t)} > \left(\frac{2e^2 + 2e^2t + e^2t^2}{2e}\right)^e,$$

for $t > 0$, which is equivalent to $e^t > 1 + t + \frac{t^2}{2}$. This is true for $t > 0$, since the right-hand side is the degree two Maclaurin expansion of e^t, and all of the terms in the Maclaurin series for e^t are positive. This proves (1). Note that (1) becomes equality for $x = 0$, and the inequality sign flips for $x < 0$, since for $t < 0$, we have $e^t < 1 + t + \frac{t^2}{2}$, which may be proven by some simple derivative arguments. We are done.

Editor's Comments. The main ideas of Solution 1, i.e. taking a logarithm, defining a function, and taking its derivative to prove the inequality, were utilized by a large majority of the solvers, though the specific function used varied. The Taylor approximation inequality in Solution 2 is also the main step in the proposer's solution, which looks slightly different and is used as the starting point, as opposed to arising at the end of the solution. Paolo Perfetti commented that this problem showed up in *Mathematical Reflections*, 2015–2, as problem U334.

3890*. *Proposed by Šefket Arslanagić.*

Let $\alpha, \beta, \gamma \in \mathbb{R}$. Prove or disprove that

$$|\sin\alpha| + |\sin\beta| + |\sin\gamma| + |\cos(\alpha + \beta + \gamma)| \leq 1 + \frac{3\sqrt{3}}{2}.$$

We received nine correct solutions. The proposed inequality is false in general, which was pointed out by all the solvers who gave various counterexamples as listed below. We also present a correct version with proof.

Solution 1, various counterexamples.

Let S denote the left side of the given inequality.

a) AN-anduud Problem Solving Group proved that the inequality is reversed if $\alpha = \beta = \frac{\pi}{3} + x$ and $\gamma = \frac{\pi}{3} - x$ for all $x \in (0, \frac{\pi}{3})$.

b) Norman Hodžić and Salem Malikić (jointly) gave, without proof, the counterexample $\alpha = \beta = \gamma = \frac{5\pi}{8}$, which is essentially the same as a) since $\sin\left(\frac{5\pi}{8}\right) = \sin\left(\frac{3\pi}{8}\right)$ and $\cos\left(\frac{15\pi}{8}\right) = \cos\left(\frac{\pi}{8}\right) = -\cos\left(\frac{9\pi}{8}\right)$.

c) David Manes gave the counterexample $\alpha = \beta = \gamma = \frac{7\pi}{18}$.

d) Dragoljub Milošević and Digby Smith (independently) showed that when $\alpha = \beta = \gamma = \frac{5\pi}{12}$, then $S = \frac{3\sqrt{6}+5\sqrt{2}}{4} > 1 + \frac{3\sqrt{3}}{2}$.

e) Roberto de la Cruz, Missouri State University Problem Solving Group and C.R. Pranesachar (independently) showed that when $\alpha = \beta = \gamma = \frac{3\pi}{8}$, then $S = 2\sqrt{2+\sqrt{2}} > 1 + \frac{3\sqrt{3}}{2}$. Missouri State University Problem Solving Group also stated, without proof, the general result that for real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ we have the sharp inequality below:

$$\sum_{i=1}^{n} |\sin \alpha_i| + \left|\cos\left(\sum_{i=1}^{n} \alpha_i\right)\right| \leq \begin{cases} (n+1)\cos\frac{\pi}{2(n+1)} & \text{if } n \text{ is odd,} \\ n+1 & \text{if } n \text{ is even.} \end{cases}$$

Solution 2, a correct version by Roy Barbara.

We prove that

$$|\sin \alpha| + |\sin \beta| + |\sin \gamma| + |\cos(\alpha+\beta+\gamma)| \leq 2\sqrt{2+\sqrt{2}} \tag{1}$$

with equality if and only if for some integers $l, m, n \in \mathbb{Z}$ we have

$$(\alpha, \beta, \gamma) = \left(\tfrac{3\pi}{8} + l\pi, \tfrac{3\pi}{8} + m\pi, \tfrac{3\pi}{8} + n\pi\right) \text{ or } \left(\tfrac{5\pi}{8} + l\pi, \tfrac{5\pi}{8} + m\pi, \tfrac{5\pi}{8} + n\pi\right). \tag{2}$$

(Thus, the upper bound is sharp.)

First, consider the function $g(x) = 3\sin x - \cos 3x$ and $h(x) = 3\sin x + \cos 3x$ for $x \in [0, \pi]$. Using the identities $\cos\left(\frac{\pi}{8}\right) = \sin\left(\frac{3\pi}{8}\right) = \frac{1}{2}\sqrt{2+\sqrt{2}}$ and $\sin\left(\frac{\pi}{8}\right) = \cos\left(\frac{3\pi}{8}\right) = \frac{1}{2}\sqrt{2-\sqrt{2}}$, it is routine to check that $g(x)$ attains its maximum of $2\sqrt{2+\sqrt{2}}$ at $x = \frac{3\pi}{8}$ and $h(x)$ attains its maximum of $2\sqrt{2+\sqrt{2}}$ at $x = \frac{5\pi}{8}$. Hence

$$3\sin x + |\cos 3x| \leq \sqrt{2+\sqrt{2}} \tag{3}$$

for all $x \in [0, \pi]$ with equality if and only if $x = \frac{3\pi}{8}$ or $\frac{5\pi}{8}$.

Next, consider the function f defined on $\mathbb{R}^3$ by

$$f(\alpha, \beta, \gamma) = |\sin \alpha| + |\sin \beta| + |\sin \gamma| + |\cos(\alpha+\beta+\gamma)|.$$

Note that f is periodic with period π relative to each of the variables α, β, γ.

Hence, we could consider only those (α, β, γ) lying in the compact set $K = [0, \pi]^3$, so $f(\alpha, \beta, \gamma) = \sin\alpha + \sin\beta + \sin\gamma + |\cos(\alpha+\beta+\gamma)|$. Since f is continuous on a compact set, it attains its maximum at some point $(a, b, c) \in K$.

We claim that $a = b = c$. It clearly suffices to show that $a = b$. Suppose to the contrary that $a \neq b$. Then we may assume that $a < b$. Set $\theta = \frac{a+b}{2}$ and $\lambda = \frac{b-a}{2}$. Then $(\theta, \theta, c) \in K$, $a = \theta - \lambda, b = \theta + \lambda$ and $0 < \lambda < \pi$. Since $\cos\lambda < 1$ and $\sin\theta \neq 0$, we have

$$\begin{aligned} f(a,b,c) &= f(\theta-\lambda, \theta+\lambda, c) \\ &= \sin(\theta-\lambda) + \sin(\theta+\lambda) + \sin c + |\cos((\theta-\lambda)+(\theta+\lambda)+c)| \\ &= 2\sin\frac{(\theta-\lambda)+(\theta+\lambda)}{2}\cos\frac{(\theta-\lambda)-(\theta+\lambda)}{2} + \sin c + |\cos(2\theta+c)| \\ &= 2\sin\theta\cos\lambda + \sin c + |\cos(2\theta+c)| \\ &< 2\sin\theta + \sin c + |\cos(2\theta+c)| = f(\theta,\theta,c) \end{aligned}$$

contradicting the assumption that $f(a,b,c)$ is a maximum on K.

Hence, $f(\alpha, \beta, \gamma)$ attains its maximum over K when $\alpha = \beta = \gamma$ and this value is the maximum of $3\sin x + |\cos 3x|$ for $x \in [0, \pi]$ which is $2\sqrt{2+\sqrt{2}}$ by (3) and our proof is complete.

Math Quotes

Don't just read it; fight it! Ask your own questions, look for your own examples, discover your own proofs. Is the hypothesis necessary? Is the converse true? What happens in the classical special case? What about the degenerate cases? Where does the proof use the hypothesis?

Halmos, Paul R. in "I Want to be a Mathematician".

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(Bold font indicates featured solution.)

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