Crux Mathematicorum

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Crux Mathematicorum

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Former Editors / Anciens Rédacteurs: G.W. Sands, R.E. Woodrow, Bruce L.R. Shawyer, Shawn Godin

Crux Mathematicorum with Mathematical Mayhem

Former Editors / Anciens Rédacteurs: Bruce L.R. Shawyer, James E. Totten, Václav Linek, Shawn Godin

EDITORIAL

Dear ***Crux*** readers,

Welcome to issue 9 of Volume 39.

First of all, I have some very exciting news!

The CMS office and Michael Doob have been hard at work scanning old ***Crux*** issues and now all the volumes are available online - all the way from the very first issue of Eureka (journal's first name) in March 1975. Now, you can witness first-hand the evolution and the growth ***Crux*** has gone through over the past 39 years; to guide you in your explorations of ***Crux***'s early years, you can consult ***Crux*** Chronology by J. Chris Fisher (Volume 37, issue 2). I strongly recommend flipping through some early issues as you can get easily inspired by the materials and Léo Sauvé's personality present on each page of the journal. Just for fun, here is the very first problem (that is 3889 problems ago):

1. *Proposed by Léo Sauvé, Algonquin College.*
75 cows have in 12 days grazed all the grass in a 60-acre pasture, and 81 cows have in 15 days grazed all the grass in a 72-acre pasture. How many cows can in 18 days graze all the grass in a 96-acre pasture? (Newton)

While you ponder that, I will take care of some typos and omissions of the past 2 issues:

- There was a typo in the problem 3872 and the new corrected version is presented in this issue.
- Anastasios Kotronis should be included in the list of solvers of problem 3780.
- There is a small typo on page 338 of Volume 39: in the solution of problem 3770, in line 2 "$ab + 2a + b + c$" should be "$ab + a + b + c$".

Please do not hesitate to send your corrections to the typos you see to me at `crux-editors@cms.math.ca`. Please note that the new address for submission of problem proposals and numbered problems is `crux-psol@cms.math.ca`.

Since our journal is very audience driven, please email me your comments and feedback on our materials. I always look forward to your emails.

Kseniya Garaschuk

THE CONTEST CORNER

No. 19

Kseniya Garaschuk

The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please email your submissions to `crux-contest@cms.math.ca` *or mail them to the address inside the back cover. Electronic submissions are preferable.*

Submissions of solutions. *Each solution should be contained in a separate file named using the convention* LastName_FirstName_CCProblemNumber *(example Doe_Jane_OC1234.tex). It is preferred that readers submit a LaTeX file and a pdf file for each solution, although other formats are also accepted. Submissions by regular mail are also accepted. Each solution should start on a separate page and name(s) of solver(s) with affiliation, city and country should appear at the start of each solution.*

To facilitate their consideration, solutions should be received by the editor by **1 March 2015***, although late solutions will also be considered until a solution is published.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

CC87. *Correction. In issue 8, we accidentally re-printed CC33 as CC87. This is the corrected version of CC87.*
Let $ABCDE$ be a regular pentagon with each side of length 1. The length of BE is θ and the angle FEA is α, where F is the intersection of AC and BE. Find θ and $\cos\alpha$.

CC91. A line segment of constant length 1 moves with one end on the x-axis and the other end on the y-axis. The region swept out (that is, the union of all possible placements) is R. Find the equation of the boundary of R.

CC92. Each of the positive integers 2013 and 3210 has the following three properties:

1. it is an integer between 1000 and 10000,
2. its four digits are consecutive integers, and
3. it is divisible by 3.

In total, how many positive integers have these three properties?

CC93. If $x, y, z > 0$ and $xyz = 1$, find the range of all possible values of

$$\frac{x^3 + y^3 + z^3 - x^{-3} - y^{-3} - z^{-3}}{x + y + z - x^{-1} - y^{-1} - z^{-1}}.$$

CC94. If $\log_2 x, (1 + \log_4 x)$ and $\log_8 4x$ are consecutive terms of a geometric sequence, determine the possible values of x.

CC95. Positive integers x, y, z satisfy $xy + z = 160$. Determine the smallest possible value of $x + yz$.

. . . .

CC87. *Correction. Dans le numéro 8 de la revue, on a présenté le problème CC33 à la place du problème CC87. Voici le vrai problème CC87.*
Soit $ABCDE$ un pentagone régulier ayant des côtés de longueur 1. Soit θ la longueur de BE, F le point d'intersection de AC et BE et α la mesure de l'angle FEA. Déterminer θ et $\cos\alpha$.

CC91. Un segment de droite de longueur 1 se déplace de manière qu'une de ses extrémités soit toujours sur l'axe des abscisses et l'autre sur l'axe des ordonnées. Soit R la région balayée par le segment (c'est-à-dire la réunion de tous les points sur les positions du segment à mesure qu'il se déplace). Déterminer l'équation de la frontière de R.

CC92. Chacun des entiers 2013 et 3210 satisfait aux trois propriétés suivantes :

1. il est un entier entre 1000 et 10000,
2. ses quatre chiffres sont des entiers consécutifs et
3. il est divisible par 3.

Combien y a-t-il d'entiers positifs qui satisfont à ces trois propriétés ?

CC93. Sachant que $x, y, z > 0$ et $xyz = 1$, déterminer l'étendue de toutes les valeurs possibles de l'expression

$$\frac{x^3 + y^3 + z^3 - x^{-3} - y^{-3} - z^{-3}}{x + y + z - x^{-1} - y^{-1} - z^{-1}}.$$

CC94. Sachant que $\log_2 x, (1 + \log_4 x)$ et $\log_8 4x$ sont des termes consécutifs d'une suite géométrique, déterminer toutes les valeurs possibles de x.

CC95. Les entiers strictement positifs x, y, z vérifient l'équation $xy + z = 160$. Déterminer la plus petite valeur possible de $x + yz$.

CONTEST CORNER SOLUTIONS

CC41. Ace runs with constant speed and Flash runs x times as fast, $x > 1$. Flash gives Ace a head start of y metres, and, at a given signal, they start off in the same direction. Find the distance Flash must run to catch Ace.

Originally problem 7 of 2005 W.J. Blundon Mathematics Contest.

Solved by R. I. Hess ; and D. Văcaru. We present the solution by Daniel Văcaru.

Let v be Ace's speed, then Flash has speed vx. Let t be the amount of time it takes for Flash to catch Ace. When Flash catches up to Ace, Ace is $vt + y$ metres from the start and Flash is xvt metres from the start.

Thus, $vt + y = xvt$. Solving for t, we get $t = \frac{y}{v(x-1)}$. At that time, Flash has run $\frac{xvy}{v(x-1)} = \frac{xy}{x-1}$ metres.

CC42. $\triangle ABC$ has its vertices on a circle of radius r. If the lengths of two of the medians of $\triangle ABC$ are equal to r, determine the side lengths of $\triangle ABC$.

Originally 2012 Canadian Senior Mathematics Contest, problem B3c.

Solved by M. Amengual Covas ; Š. Arslanagić ; M. Bataille ; M. Coiculescu ; R. Hess ; and D. Văcaru. We present the solution by Miguel Amengual Covas.

Let G be the centroid of $\triangle ABC$ and suppose that the two equal medians are the median AD to side BC and the median to side CA. Clearly, then, $\triangle ABC$ is isosceles with $BC = CA$. Thus the median CM to side AB lies along the perpendicular bisector of chord AB and it passes through the circumcentre O of $\triangle ABC$. Therefore, we have

$$AO^2 - OM^2 = AG^2 - GM^2. \tag{1}$$

Let $GM = x$. Since G trisects each median of $\triangle ABC$, we have $OM = OA - MC = r - 3x$ and $AG = \frac{2}{3}AD = \frac{2}{3}r$. When these are substituted into (1), we get $r^2 - (r - 3x)^2 = (\frac{2}{3}r)^2 - x^2$. Solving for x, we obtain $x = \frac{2}{3}r$ (which is not admissible) and $x = \frac{r}{12}$. Hence,

$$AB = 2 \cdot AM = 2\sqrt{r^2 - \left(\frac{3r}{4}\right)^2} = \frac{r\sqrt{7}}{2}$$

and

$$BC = CA = \sqrt{AM^2 + MC^2} = \sqrt{\left(\frac{r\sqrt{7}}{4}\right)^2 + \left(\frac{r}{4}\right)^2} = \frac{r\sqrt{2}}{2}.$$

CC43. A circle has diameter AB. P is a fixed point of AB lying between A and B. A point X, distinct from A and B, lies on the circumference of the circle. Prove that $\tan(\angle AXP) \div \tan(\angle XAP)$ is constant for all values of X.

Originally Question 6 of 2005 APICS Math Competition.

Solved by M. Amengual Covas ; Š. Arslanagić ; M. Bataille ; R. I. Hess ; J. G. Heuver ; and T. Zvonaru. We present the solution of Michel Bataille modified by the editor.

For simplicity, let $\alpha = \angle XAP$ and $\beta = \angle AXP$. Since AB is a diameter, $\angle AXB = 90^\circ$ and hence $\angle BXP = 90^\circ - \beta$. Since triangle AXB is right-angled with right angle at X, $\angle PBX = 90^\circ - \alpha$. We now apply Law of Sines on $\triangle AXP$ and $\triangle PXB$. On $\triangle AXP$ we have $\frac{PA}{\sin\beta} = \frac{PX}{\sin\alpha}$, so

$$\frac{\sin\beta}{\sin\alpha} = \frac{PA}{PX}. \tag{1}$$

On $\triangle PXB$,

$$\frac{PB}{\sin(90^\circ - \beta)} = \frac{PX}{\sin(90^\circ - \alpha)}.$$

Since $\sin(90^\circ - \beta) = \cos\beta$ and $\sin(90^\circ - \alpha) = \cos\alpha$, this implies

$$\frac{\cos\alpha}{\cos\beta} = \frac{PX}{PB}. \tag{2}$$

Equations (1) and (2) imply

$$\frac{\tan(\angle AXP)}{\tan(\angle XAP)} = \frac{\tan\beta}{\tan\alpha} = \frac{\sin\beta}{\cos\beta} \cdot \frac{\cos\alpha}{\sin\alpha} = \frac{PA}{PX}\frac{PX}{PB} = \frac{PA}{PB},$$

which is constant for all values of X.

CC44. Let $a_0 = 1$ and for $n \geq 0$ let $a_{n+1} = a_n - \frac{1}{2}a_n^2$. Find $\lim_{n\to\infty} na_n$, if it exists.

Originally Question 6 on 2009 University of Waterloo Big E Contest.

Solved by M. Bataille ; and D. Văcaru. We present Michel Bataille's solution.

We show that $\lim\limits_{n\to\infty} na_n = 2$.

Since $a_{n+1} - a_n = -\frac{1}{2}a_n^2 < 0$ for all $n \geq 0$, the sequence $\{a_n\}$ is decreasing. It follows that $a_n \leq a_0 = 1$ for all $n \geq 0$. From $a_{n+1} = \frac{a_n}{2}(2 - a_n)$, an easy induction shows that $a_n > 0$ for all $n \geq 0$. Being decreasing and bounded below, the sequence $\{a_n\}$ is convergent.

Let $\ell = \lim\limits_{n\to\infty} a_n$. Since ℓ is also the limit of $\{a_{n+1}\}$, we must have $\ell = \ell - \frac{1}{2}\ell^2$ and so $\ell = 0$. Because $\dfrac{a_{n+1}}{a_n} = 1 - \frac{1}{2}a_n$, we have $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = 1$. Now, we calculate

$$\frac{1}{a_{n+1}} - \frac{1}{a_n} = \frac{a_n - a_{n+1}}{a_n a_{n+1}} = \frac{\frac{1}{2}a_n^2}{a_n a_{n+1}} = \frac{1}{2} \cdot \frac{a_n}{a_{n+1}}$$

and so the sequence $\dfrac{1}{a_{n+1}} - \dfrac{1}{a_n}$ is convergent towards $\dfrac{1}{2}$. The same is true of its Cesàro mean $\{C_n\}$ defined by

$$C_n = \frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{a_n} - \frac{1}{a_{n-1}}\right).$$

But

$$C_n = \frac{1}{n}\left(\frac{1}{a_n} - 1\right) = \frac{1}{na_n} - \frac{1}{n}$$

and so $\displaystyle\lim_{n\to\infty} na_n = \lim_{n\to\infty} \frac{1}{C_n + \frac{1}{n}} = 2.$

CC45. The *baseball sum* of two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ is defined to be $\frac{a+c}{b+d}$. Starting with the rational numbers $\frac{0}{1}$ and $\frac{1}{1}$ as Stage 0, the baseball sum of each consecutive pair of rational numbers in a stage is inserted between the pair to arrive at the next stage. The first few stages of this process are shown below :

$$\begin{array}{lccccccccc}
\text{STAGE 0 :} & \frac{0}{1} & & & & & & & & \frac{1}{1} \\
\text{STAGE 1 :} & \frac{0}{1} & & & & \frac{1}{2} & & & & \frac{1}{1} \\
\text{STAGE 2 :} & \frac{0}{1} & & \frac{1}{3} & & \frac{1}{2} & & \frac{2}{3} & & \frac{1}{1} \\
\text{STAGE 3 :} & \frac{0}{1} & \frac{1}{4} & \frac{1}{3} & \frac{2}{5} & \frac{1}{2} & \frac{3}{5} & \frac{2}{3} & \frac{3}{4} & \frac{1}{1}
\end{array}$$

Prove that :

i) no rational number will be inserted more than once,

ii) no inserted fraction is reducible, and

iii) every rational number between 0 and 1 will be inserted in the pattern at some stage.

Originally 2006 Canadian Open Mathematics Challenge, problem B4 b).

One incorrect solution was received.

THE OLYMPIAD CORNER

No. 317

Nicolae Strungaru

The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please email your submissions to `crux-olympiad@cms.math.ca` *or mail them to the address inside the back cover. Electronic submissions are preferable.*

Submissions of solutions. *Each solution should be contained in a separate file named using the convention* LastName_FirstName_OCProblemNumber *(example Doe_Jane_OC1234.tex). It is preferred that readers submit a LaTeX file and a pdf file for each solution, although other formats are also accepted. Submissions by regular mail are also accepted. Each solution should start on a separate page and name(s) of solver(s) with affiliation, city and country should appear at the start of each solution.*

To facilitate their consideration, solutions should be received by the editor by **1 March 2015**, *although late solutions will also be considered until a solution is published.*

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The editor thanks Rolland Gaudet, de l'Université Saint-Boniface à Winnipeg, for translations of the problems.

OC151. Let ABC be a triangle. The tangent at A to the circumcircle intersects the line BC at P. Let Q and R be the symmetrical of P with respect to the lines AB and AC, respectively. Prove that $BC \perp QR$.

OC152. Find all non-constant polynomials $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ with integer coefficients whose roots are exactly the numbers $a_0, a_1, \ldots, a_{n-1}$ each with multiplicity 1.

OC153. Find all non-decreasing functions from the set of real numbers to itself such that for all real numbers x, y we have

$$f(f(x^2) + y + f(y)) = x^2 + 2f(y)\,.$$

OC154. For $n \in \mathbb{Z}^+$ we denote

$$x_n := \binom{2n}{n}.$$

Prove there exist infinitely many finite sets A, B of positive integers, such that $A \cap B = \emptyset$, and

$$\frac{\prod\limits_{i \in A} x_i}{\prod\limits_{j \in B} x_j} = 2012\,.$$

OC155. There are 42 students taking part in the Team Selection Test. It is known that every student knows exactly 20 other students. Show that we can divide the students into 2 groups or 21 groups such that the number of students in each group is equal and every two students in the same group know each other.

. . . .

OC151. Soit ABC un triangle et soit P le point d'intersection de a ligne BC et de la tangente du cercle circonscrit au point A. Soit Q et R symétriques à P par rapport aux lignes AB et AC respectivement. Démontrer que $BC \perp QR$.

OC152. Déterminer tous les polynômes non constants $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ avec coefficients entiers dont les racines sont exactement les nombres $a_0, a_1, \ldots, a_{n-1}$ avec les mêmes multiplicités.

OC153. Déterminer toutes les fonctions non décroissantes des nombres réels aux nombres réels telles que pour tout x, y réels on a

$$f(f(x^2) + y + f(y)) = x^2 + 2f(y)\,.$$

OC154. Pour $n \in \mathbb{Z}^+$, dénotons

$$x_n := \binom{2n}{n}.$$

Démontrer qu'il existe un nombre infini d'ensembles finis d'entiers positifs A et B, tels que $A \cap B = \emptyset$ et

$$\frac{\prod\limits_{i \in A} x_i}{\prod\limits_{j \in B} x_j} = 2012\,.$$

OC155. Soit 42 étudiants, où on sait que tout étudiant connait exactement 20 autres étudiants. Démontrer qu'il est possible de répartir l'ensemble des étudiants en 2 sous-ensembles ou en 21 sous-ensembles de façon à ce que chaque sous-ensemble ait le même nombre d'étudiants et que tous les étudiants dans un sous-ensemble se connaissent.

OLYMPIAD SOLUTIONS

OC91. Prove that no integer consisting of one 2, one 1 and the rest of digits 0 can be written neither as the sum of two perfect squares nor the sum of two perfect cubes.

Originally question 8 from the 2011 Estonian National Olympiad.

Solved by O. Geupel; D. E. Manes; and T. Zvonaru. We give the solution by Titu Zvonaru.

Since the sum of the digits of n is 3, it follows that n is divisible by 3 but not divisible by 9.

Suppose by contradiction that there exists a, b such that $n = a^2 + b^2$. As the quadratic residues modulo 3 are 0 and 1 and $a^2 + b^2 \equiv 0 \pmod 3$ it follows that

$$a \equiv b \equiv 0 \pmod 3 .$$

Then $a^2 + b^2$ is divisible by 9. But this is a contradiction.

Next assume by contradiction that there exist a, b so that $n = a^3 + b^3$. By Fermat Little theorem, $x^3 \equiv x \pmod 3$ for all integers x, and therefore

$$0 \equiv n \equiv a^3 + b^3 \equiv a + b \pmod 3 .$$

This implies that $b = 3k - a$ for some integer k. Then we have

$$n = a^3 + b^3 = a^3 + (3k-a)^3 = a^3 + 27k^3 - 27k^2a + 9ka^2 - a^3 = 9(3k^3 - 3k^2a + ka^2).$$

This shows that $9|n$ which is a contradiction.

OC92. Let $ABCD$ be a convex quadrilateral. Let P be the intersection of external bisectors of $\angle DAC$ and $\angle DBC$. Prove that $\angle APD = \angle BPC$ if and only if $AD + AC = BC + BD$.

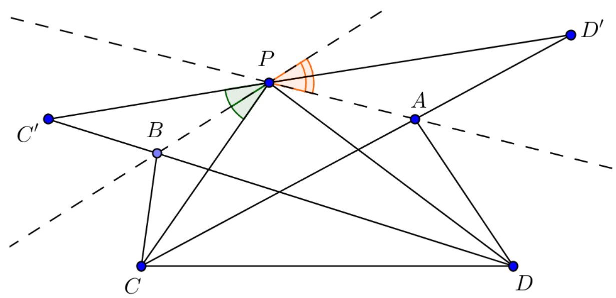
Originally question 4 from the 2011 Italian National Olympiad.

Solved by Š. Arslanagić; O. Geupel; and J. G. Heuver. We give the solution of John G. Heuver.

Suppose $AD + AC = BC + BD$. Let A, B be points on the ellipse with foci C and D. Then the external bisectors of $\angle DAC$ and $\angle DBC$ are known to be tangents to the ellipse at A and B.

Let C' be the reflection of C in PB, and D' be the reflection of D in PA. Then $BC' = BC, AD' = AD$ and hence $DC = DB + BC = CA + AD = CD$.

The triangles $CD'P$ and $C'DP$ are congruent from which it follows that $\angle C'PD = \angle CPD'$. By subtracting $\angle CPD$ from both, we get $\angle C'PC = \angle CPD'$ and thus $\angle APD = \angle BPC$ as required.



Conversely, assume $\angle APD = \angle BPC$ and reflecting PC and PD in PB respectively PA we observe that triangles $C'PD$ and CPD' are congruent by a rotation. Therefore,

$$C'D = C'B + BD = CD' = CA + AD'.$$

Since $C'B = CB$ and $AD' = AD$ we have

$$AD + DC = BC + BD\,,$$

which completes the proof.

OC93. For every positive integer n, determine the maximum number of edges a simple graph with n vertices can have if it contain no cycles of even length.

Originally question 3 from Day 1 Romanian Team Selection Test, Day 1, 2011.

No solution to this problem was received.

OC94. Let $x_1, x_2, \cdots, x_{25}$ be real numbers such that for all $1 \leq i \leq 25$ we have $0 \leq x_i \leq i$. Find the maximum value of

$$x_1^3 + x_2^3 + \cdots + x_{25}^3 - (x_1x_2x_3 + x_2x_3x_4 + \cdots x_{25}x_1x_2)\,.$$

Originally question 4 from the Koreean National Olympiad 2011, Test 2.

There was one incorrect solution received to this problem.

OC95. Can we find three relatively prime integers a, b, c so that the square of each number is divisible by the sum of the other two ?

Originally question 4 from Russia National Olympiad 2011, Grade 9, Day 1.

Solved by David E. Manes.

The answer is yes. Let p, q be two different odd primes. Then $a = p, b = q, c = -(p+1)$ works.

Editor's Comment : There was a typo in the problem ; the three integers were supposed to be positive.

BOOK REVIEWS

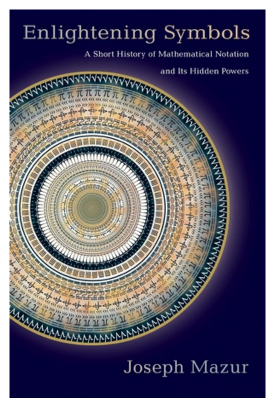
John McLoughlin

Enlightening Symbols : A Short History of Mathematical Notation and Its Hidden Powers by Joseph Mazur
ISBN 9781400850112, available in ePub, PDF, and hardcover
Princeton University Press, 2014, $19.95–29.95 (US)

Reviewed by ***Paul Libbrecht****, Weingarten University of Education, Germany*

Enlightening Symbols is a captivating book on the evolution of the mathematical notations. Compared to classical works on the topics, such as F. Cajori's *History of mathematical notation*, this book is much more of an easy read. It focuses on just a few mathematical notations (the numeric systems, the basic operations, and simple equations) but offers a thorough treatment around them including the historical context in which the mathematical works were written and read, as well as some graphical extracts of the ancient works. This way, one has an idea how the symbols impacted the way of thinking at that time.

Imagine you could use only words to describe and solve polynomial equations in one variable... That is how the Pythagoreans and Euclid wrote and communicated. The book provides this example : "Given a sum of three quantities and also the sums of every pair containing one of those specified quantities, then that specified quantity is equal to the difference between the sums of those pairs and the total sum of the original three quantities". The author compares how almost any college student would be able to solve this using x, y, and z (and a, b, c the indicated sums) and how such a recipe, called *flower of Thymaridas*, was made available at the times of the Pythagoreans so as to solve it.



The book by J. Mazur starts with a fairly long exposition on the various number systems : babylonian, greek, roman, hebrew, aztec, chinese, indian... For a few of them, operations are shown, and this is where one really meets these numbers, being able to speak about their "capabilities" (e.g. how they can be manipulated to perform long additions, either graphically or on abacus). For a while, the hazardous ways through which the indo-arabic numeral system came through Europe are explored. This part is a bit hard to read as it has a few repetitions which circle around the question of original entry ; however, it sets the stage and provides some expressive descriptions of numbers, such as the hands of merchants of Western Europe negotiating with merchants from further east without understanding much of the rest of the language.

After the numbers, the symbols of early algebra are explored from word-based algebra (until our x and y, or a and b) with quite a number of imaginative ways presented through the eyes of Diophantus, R. Bombelli, C. Rudolff, or F. Viète. The link to geometry is omnipresent, and indeed this is how a square or a product was considered, but the expressivity of algebra is being developed, until the regularity of combining products of powers of the unknown (adding the exponents), until the resolution of equations becomes easiest, until... the fundamental theorem of algebra and the introduction of the square root of -1. G. Leibnitz and I. Newton conclude the panaché of notations, all leading to the notations currently in use in much of the western world. Then follows a part where less mathematical assertions are made, and more perception and psychology is explored. J. Mazur describes how the perception of symmetry and other patterns influences our perception of formulae, he describes an experiment by himself as well as several other psychologists or neuroscientists, concluding a potential evidence of a sense of reading mathematical formulae that may be transmitted through evolution.

Unfortunately, the book does not mention the differences of notations across the cultures and languages (such as the various ways of doing the long division, or the usage of j for the root of -1 in electrical engineering). However, this is a rather natural consequence of the book ; notations have evolved differently depending on the usages. Neither does the book offer sufficient material for proposing problems to students. However, most of the references are well documented, with URLs given when possible. This should allow a teacher to go and understand the ancient works so as to propose introductory and exercise materials. Certainly such a continuation of the book would be very interesting to share in a wiki space such as the census of mathematical notations (`http://wiki.math-bridge.org/display/ntns/`).

A tiny note is offered here to the readers who, similarly to me, like to read the electronic versions of the book. Reading on a small mobile device is an option as the eBook version is available (from Google Play Books as ePub and PDF and many others). However, some characters may be missing on the mobile version (such as the inverse psi to indicate a minus in Diophantus times) ; moreover, the ePub version suffers from missing characters in the formulae and words, with which normal mathematicians can easily cope but need some vigilance. The PDF does not have this issue but is non-searchable.

Overall, I recommend this book to anyone thinking about the mathematical notation they use and could use. It may even be a reading for senior college students. For those who think that we should abstain from discussing and varying mathematical notations, the book provides ample illustrations of how deeply notation influences our conceptualization ; the author concludes that "routine and familiarity are the tailwind of conceptions."

Let me conclude with an example of the the author's delightful style. He describes the introduction of the terminology for *complex numbers* with their *real* and *imaginary* parts, which is now standard ; he then editorializes that these names are unfortunate "because they are the names of classes of numbers that are neither imaginary nor complex."

FOCUS ON...

No. 9

Michel Bataille

Solutions to Exercises from Focus On... No. 2, 3, 4 and 5

Starting from Focus On... No. 2, this column included exercises for the reader's enjoyment and practice. In this number, we present solutions to problems proposed in Focus On... No. 2, 3, 4 and 5.

From Focus On... No. 2

Show that for any complex numbers a, b, c*,*

$$|ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)| \le \frac{9}{16}(|a|^2+|b|^2+|c|^2)^2.$$

To bring "the geometry behind the scene" to light, we recall the given hint :

$$ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)=(b-a)(a-c)(c-b)(a+b+c).$$

This identity immediately reduces the problem to proving the inequality

$$3OG\cdot AB\cdot BC\cdot CA \le \frac{9}{16}(OA^2+OB^2+OC^2)^2, \tag{1}$$

where G denotes the isobarycentre of A, B, C.

Writing $OA^2=(\overrightarrow{OG}+\overrightarrow{GA})^2=OG^2+GA^2+2\overrightarrow{OG}\cdot\overrightarrow{GA}$, etc. and taking $\overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC}=\overrightarrow{0}$ into account, we readily obtain

$$OA^2+OB^2+OC^2=3OG^2+GA^2+GB^2+GC^2.$$

Now, with $BC=a, AB=c, CA=b$ as usual, we have

$$GA^2+GB^2+GC^2=\frac{1}{9}(2b^2+2c^2-a^2+2c^2+2a^2-b^2+2a^2+2b^2-c^2)=\frac{1}{3}(a^2+b^2+c^2)$$

and so $OA^2+OB^2+OC^2=3OG^2+\frac{1}{3}(AB^2+BC^2+CA^2)$. To conclude, the AM-GM inequality yields

$$OA^2+OB^2+OC^2 \ge 4\left(3OG^2\cdot\frac{AB^2}{3}\cdot\frac{BC^2}{3}\cdot\frac{CA^2}{3}\right)^{1/4}=4\sqrt{\frac{OG\cdot AB\cdot BC\cdot CA}{3}}$$

and the inequality (1) follows by squaring.

From Focus On... No. 3

For integers m, n *such that* $0\le m\le n$*, prove that the following equality holds :*

$$\sum_{j=0}^{m}\binom{n-j}{m-j}\binom{2n+1}{2j}=2^{2m}\binom{m+n}{2m}.$$

As indicated in the column, we consider the identity $A_{2n}(1, -\frac{Y}{4}, 0, 1) = B_{2n}(1, -\frac{Y}{4}, 0, 1)$, which we write as

$$\sum_{k=0}^{n} \frac{1}{4^k} \binom{2n-k}{k} Y^k = \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n+1}{2k+1} (1+Y)^k.$$

Expanding $(1+Y)^k$ with the binomial theorem transforms the right-hand side into

$$\frac{1}{2^{2n}} \sum_{k=0}^{n} Y^k \left(\sum_{j=k}^{n} \binom{j}{k} \binom{2n+1}{2j+1} \right).$$

A comparison with the left-hand side leads to

$$\sum_{j=k}^{n} \binom{j}{k} \binom{2n+1}{2j+1} = 2^{2(n-k)} \binom{2n-k}{k}$$

for $k = 0, 1, \ldots, n$. This holds in particular for $k = n - m$, giving

$$\sum_{i=0}^{m} \binom{n-m+i}{n-m} \binom{2n+1}{2(n-m+i)+1} = 2^{2m} \binom{m+n}{n-m}.$$

Changing the index from i to $j = m - i$ and with the help of the formula $\binom{N}{p} = \binom{N}{N-p}$ for $0 \leq p \leq N$, we readily obtain the desired equality.

From Focus On... No. 4

(a) *Let ℓ be a tangent to the circumcircle Γ of ΔABC and let $BC = a, CA = b, AB = c, d_a = d(A, \ell), d_b = d(B, \ell), d_c = d(C, \ell)$. Then, one of the numbers $a\sqrt{d_a}, b\sqrt{d_b}, c\sqrt{d_c}$ is the sum of the other two.*

A solution resting on barycentric equations is proposed in the column itself, with a call for a synthetic proof. Actually, a nice application of Ptolemy's Theorem provides such a proof. The key idea is to start with the case when ℓ is tangent to the circumcircle at a vertex of the triangle, say at A (see Figure 1, left).

Let B_1, C_1 be the orthogonal projections of B, C, respectively, on ℓ. Since the chord AB subtends both $\angle BAB_1$ and $\angle ACB$, we have $\sin C = \frac{BB_1}{c}$, hence $\frac{c}{2R} = \frac{BB_1}{c}$, where R is the circumradius. Similarly, $\frac{b}{2R} = \frac{CC_1}{b}$ and so $b\sqrt{d_b} = c\sqrt{d_c} (= \frac{bc}{\sqrt{2R}})$. Since $d_a = 0$, this implies the desired result.

Now, consider the general case where the point of tangency M is different from A, B, C, say on the arc BC not containing A (Figure 1, right). We apply the particular result just obtained to the triangles ΔMBC and ΔMAC. Denoting the projections of A, B, C onto ℓ as A_1, B_1, C_1, respectively, we find $MC \cdot \sqrt{BB_1} = MB \cdot \sqrt{CC_1}$ and $MC \cdot \sqrt{AA_1} = MA \cdot \sqrt{CC_1}$ so that

$$\frac{b\sqrt{d_b}}{b \cdot MB} = \frac{c\sqrt{d_c}}{c \cdot MC} = \frac{a\sqrt{d_a}}{a \cdot MA}.$$

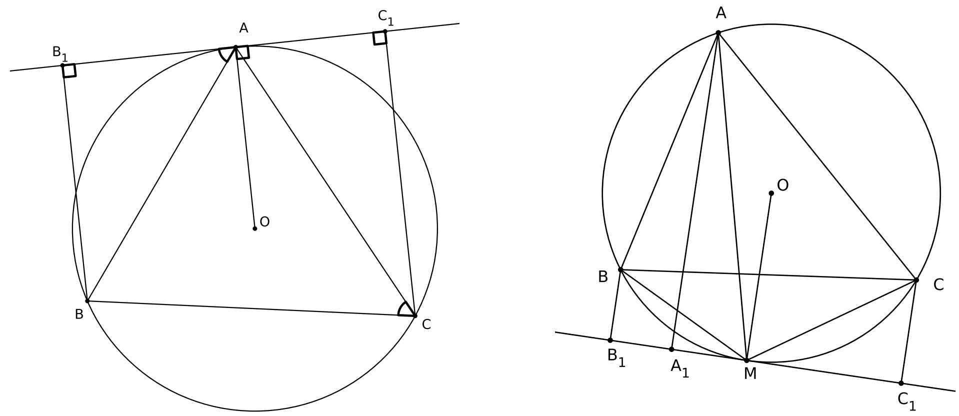
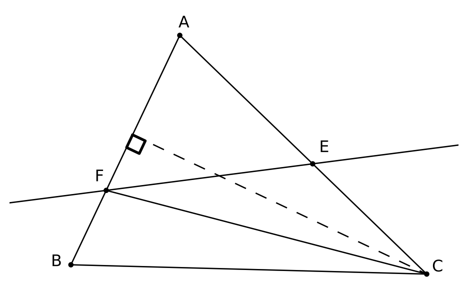


FIGURE 1: the point of tangency is at a vertex of the triangle or not.

Since $a \cdot MA = b \cdot MB + c \cdot MC$ (from Ptolemy's Theorem), we can conclude $a\sqrt{d_a} = b\sqrt{d_b} + c\sqrt{d_c}$.

(b) *Let E and F be points on the sides AC and AB of ΔABC, respectively. Show that $[PBC]$ is the geometric mean of $[PAB]$ and $[PCA]$ for some point P on the line segment EF if and only if $AE \cdot AF \geq 4CE \cdot BF$.*

Let $\beta = \frac{BF}{AF}$ and $\gamma = \frac{CE}{AE}$. Remarking that $\beta = \frac{BF \cdot d(C,BA)}{AF \cdot d(C,BA)} = \frac{[FBC]}{[FCA]}$, the barycentric coordinates of F relatively to (A, B, C) are $(\beta, 1, 0)$. In the same way, the coordinates of E are $(\gamma, 0, 1)$ and so the equation of the line EF is $x = \beta y + \gamma z$.



It follows that $[PBC] = \beta[PCA] + \gamma[PAB]$ for any point P on the line segment EF and therefore

$$\frac{[PBC]^2}{[PCA] \cdot [PAB]} = \beta^2 \rho + \gamma^2 \cdot \frac{1}{\rho} + 2\beta\gamma$$

where $\rho = \dfrac{[PCA]}{[PAB]}$.

A quick study of the function $f:\ x \mapsto f(x) = \beta^2 x + \gamma^2 \cdot \frac{1}{x} + 2\beta\gamma$ shows that f takes all values of the interval $[4\beta\gamma, \infty)$ when x varies in $(0, \infty)$, hence take the value 1 if and only if $4\beta\gamma \leq 1$. The result follows since $[PBC]$ is the geometric mean of $[PAB]$ and $[PCA]$ for some point P of the segment EF if and only if $f(\rho) = 1$ for some positive ρ.

From Focus On... No. 5

(a) *For $x, y, z > 0$, let $f(x,y,z) = (1-x)(1-y)(1-z)$ and*

$$g(x,y,z) = 2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) - 4 - \left(\frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}\right).$$

Show that $(a,b,c) = (\frac{3}{4},\frac{3}{4},\frac{3}{4})$ satisfies the constraint $g(x,y,z) = 0$ and $\partial_i f(a,b,c) = \lambda\partial_i g(a,b,c)$, $i = 1,2,3$ for some λ but $f(a,b,c)$ is not an extremum of f under the constraint.

It is readily seen that for $i = 1,2,3$, $\partial_i f(a,b,c) = -\frac{1}{16}$ and $\partial_i g(a,b,c) = -\frac{80}{9}$, hence $\partial_i f(a,b,c) = \lambda\partial_i g(a,b,c)$ with $\lambda = \frac{9}{16\times 80}$. However, $f(a,b,c) = \frac{1}{64}$ while $g(\frac{1}{4},\frac{1}{4},\frac{3}{4}) = 0$ and $f(\frac{1}{4},\frac{1}{4},\frac{3}{4}) > \frac{1}{64}$, so that $f(a,b,c)$ is not a maximum of f under the constraint.

Also, if $w \in (0,\frac{1}{2})$, then $(w,w,4w(1-w))$ satisfies the constraint and $f(w,w,4w(1-w)) = (1-w)^2(1-2w)^2$. Since $\lim\limits_{w\to 1/2} f(w,w,4w(1-w)) = 0$, $\frac{1}{64}$ is not a minimum under the constraint either.

This example emphasizes the non-sufficiency of the existence of λ in the statement of the Lagrange multipliers theorem as given in the column.

(The interested reader can show that the sharp inequality $f(x,y,z) < 1$ holds for $x,y,z > 0$ satisfying $g(x,y,z) = 0$.)

(b) *Consider the inequality*

$$\frac{1}{1-\left(\frac{x+y}{2}\right)^2} + \frac{1}{1-\left(\frac{y+z}{2}\right)^2} + \frac{1}{1-\left(\frac{z+x}{2}\right)^2} \leq \frac{11}{3}$$

when $x+y+z = 1$ and $x,y,z \geq 0$. Prove this inequality with the method of Lagrange multipliers.

Let $\phi(t) = \frac{1}{1-t^2}, h(t) = \frac{1}{3-t}+\frac{1}{1+t}, K = \{(x,y,z) | x,y,z \geq 0, x+y+z = 1\}$, and let $f(x,y,z)$ denote the left-hand side of the inequality. Whenever $(x,y,z) \in K$, we have

$$\phi\left(\frac{x+y}{2}\right) = \frac{4}{(2+x+y)(2-x-y)} = \frac{4}{(3-z)(1+z)} = \frac{1}{3-z}+\frac{1}{1+z} = h(z)$$

so that so that $f(x,y,z) = h(x)+h(y)+h(z)$.

Assume that f attained its maximum on K at (a,b,c) interior to K. Then we would have $\partial_i f(a,b,c) = \lambda$ for some λ, hence $h'(a) = h'(b) = h'(c)$. Since h' is strictly monotone on $[0,1]$ (easily checked), we must have $a = b = c = \frac{1}{3}$ and $f(a,b,c) = \frac{27}{8}$. However, $\frac{27}{8}$ cannot be the maximum of f on K since $f(1,0,0) = \frac{11}{3} > \frac{27}{8}$ (actually,

$\frac{27}{8}$ is the minimum of f on K, as it follows from the convexity of ϕ on $[0, \frac{1}{2}]$). Thus, the maximum of f on the compact K must be reached on the boundary of K. To conclude, we show that if $x = 0$ and $y, z \geq 0, y + z = 1$ (say), then $f(x, y, z) \leq \frac{11}{3}$. Indeed, in that case, $y^2 + z^2 = 1 - 2yz$ and so

$$f(x, y, z) = \frac{4}{3} + 4\left(\frac{8 - (y^2 + z^2)}{4 - 4(y^2 + z^2) + y^2z^2}\right) = \frac{4}{3} + 4\left(\frac{7 + 2yz}{12 + 8yz + y^2z^2}\right)$$

and the inequality follows from $\frac{7+2yz}{12+8yz+y^2z^2} \leq \frac{7}{12}$ (equivalent to the obvious $0 \leq 32yz + 7y^2z^2$).

PROBLEM SOLVER'S TOOLKIT

No. 8

Gerhard J. Woeginger

The Problem Solver's Toolkit contains short articles on topics of interest to problem solvers at all levels. Occasionally, these pieces will span several issues.

If it is not prime, it must be composite: part 1

There are a number of mathematical problems that ask for a proof that a given integer sequence contains at least one composite number, or that the sequence contains infinitely many composites, or that it entirely consists of composites. For instance, problem 96 in the book "250 Problems in Elementary Number Theory" by Wacław Sierpiński [2] asks: *"Does the sequence* 1*,* 31*,* 331*,* 3331*,* 33331*, ... contain infinitely many composite numbers?"* The answer turns out to be yes: the nth term of this sequence is $\frac{1}{3}(10^n - 7)$, and its 9th term is divisible by 17. A little bit of pondering then shows that for all $k \geq 1$ also its $(16k + 9)$th term is divisible by 17.

In this 2-part article, we will survey several standard solution approaches to this problem type, and we will provide a multitude of illustrating examples. But first let us list a number of useful tools that we are going to apply throughout:

Difference of two powers. $a - b$ divides $a^n - b^n$ for integers a, b, n with $n \geq 1$.

Polynomial divisibility. $a - b$ divides $P(a) - P(b)$, for integers a and b and any polynomial $P(x)$ with integer coefficients.

Little Fermat. $n^{p-1} \equiv 1 \pmod{p}$ for a prime p and an integer n not divisible by p.

Wilson. $(p - 1)! + 1 \equiv 0 \pmod{p}$ for every prime p.

Sophie Germain identity. $m^4 + 4n^4 = (m^2 + 2n^2 + 2mn)(m^2 + 2n^2 - 2mn)$

Fermat's theorem on sums of two squares. Every prime $p = 4k + 1$ can be written as sum of two squares in a unique way (up to the order of the two summands).

1 Proper divisors

There is a very simple and direct approach for proving that a given integer N is composite: exhibit a proper divisor of N. Of course, how to detect the *right* proper divisor for settling a problem remains an art. The following two examples illustrate this approach.

Problem 1 *Show that $n^n + (n+1)^{n+1}$ is composite for infinitely many n.*

Let us analyze the auxiliary sequence n^n starting with 1, 4, 27, 256, 3125, 46656. Modulo 3 these six terms are $1, 1, 0, 1, 2, 0$, and it is not hard to see that this block of length 6 then repeats over and over again. Indeed, Fermat's little theorem implies $n^3 \equiv n \pmod 3$, and this yields $(n+6)^{n+6} \equiv n^{n+6} \equiv n^n \pmod 3$.

Now the solution of the problem has become straightforward: Pick $n = 6k + 4$ so that $n^n \equiv 1 \pmod 3$ and $(n+1)^{n+1} \equiv 2 \pmod 3$, and use 3 as proper divisor.

Problem 2 *Let x and y be integers with $2 \le y < x \le 100$. Prove that there exists a positive integer n for which $x^{2^n} + y^{2^n}$ is composite.*

That's a baffling puzzle, primarily built around two crucial properties of the prime 257. The first crucial property is that $257 = 2^8 + 1$ is a power of 2 plus 1, and the second crucial property is that the only way of writing it as the sum of two squares is $257 = 16^2 + 1^2$. (Since $257 = 4 \cdot 64 + 1$, uniqueness follows from Fermat's theorem on sums of two squares.) Now note that

$$x^{256} - y^{256} = (x^{128} + y^{128})\,(x^{64} + y^{64}) \cdots\cdots (x^2 + y^2)\,(x + y)\,(x - y).$$

By Fermat's little theorem the left hand side is divisible by 257, so that one of the eight factors in the right hand side must be divisible by 257. As the two factors $x + y$ and $x - y$ are too small for this, there exists a positive integer n with $1 \le n \le 7$ such that $x^{2^n} + y^{2^n}$ is divisible by 257. Since x^{2^n} and y^{2^n} are squares and since $x, y > 1$, we conclude that $x^{2^n} + y^{2^n} \neq 257$. Hence this number $x^{2^n} + y^{2^n}$ is composite, as desired.

The reader may try to find the right divisors for the following exercises.

Problem 3 *Prove that there are infinitely many composites of the form*
(a) $10^n + 3$; *(b)* $(4^n + 1)^2 + 4$; *(c)* $n! - 1$.

Problem 4 *Prove that there exist infinitely many integers n for which $2^n + 3^n - 4$ and $2^n + 3^n - 6$ are simultaneously composite.*

Problem 5 *Prove that for integers $n \ge 1$ each of the following numbers is composite: (a)* $11 \cdot 14^n + 1$; *(b)* $19 \cdot 8^n + 17$; *(c)* $\frac{1}{3}(2^{2^{n+1}} + 2^{2^n} + 1)$.

2 Factorizations

Another fundamental approach for proving that a given algebraic expression is composite is to write it as the product of two or more non-trivial factors. Here are two examples in which the factorizations are not straightforward to guess:

Problem 6 *Find all integers $n \ge 1$ for which $A(n) = n^4 + 4^n$ is prime.*

If n is even, then $A(n)$ is divisible by 4 and definitely not prime. If $n = 2k+1$ is odd, then $A(n) = n^4 + 4 \cdot (2^k)^4$. The Sophie Germain identity yields

$$A(n) \;=\; (n^2 + 2^{2k+1} + 2^{k+1}n)\,(n^2 + 2^{2k+1} - 2^{k+1}n).$$

For $n \geq 3$ both factors are greater than 1, so that $A(n)$ is composite. For $n = 1$ we get the prime $A(1) = 5$.

Problem 7 *Let a, b, c be positive integers with $3ab = 2c^2$. Show that $a^3 + b^3 + c^3$ is composite.*

The condition $3ab = 2c^2$ naturally causes one to consider the identity $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$. This then leads to

$$\begin{aligned} a^3 + b^3 + c^3 &= (a+b)^3 - 2c^2(a+b) + c^3 \\ &= (a+b)\left((a+b)^2 - c^2\right) + c^2(c-a-b) \\ &= (a+b-c)\,\left((a+b)(a+b+c) - c^2\right). \end{aligned}$$

The arithmetic-geometric mean inequality yields $a+b \geq 2\sqrt{ab} = 2\sqrt{2c^2/3} > c+1$. Hence both factors are greater than 1, and $a^3 + b^3 + c^3$ is composite.

Here are three exercises to test the reader.

Problem 8 *Find all integers n for which the following expressions are prime: (a) $n^4 + n^2 + 1$; (b) $n^{10} + n^5 + 1$; (c) $4n^3 + 6n^2 + 4n + 1$.*

Problem 9 *Let a, b, c, d be positive integers. Show that (a) $a+b+c+d$ is composite whenever $ab = cd$, (b) $a^2 + b^2 + c^2 + d^2$ is composite whenever $ad = b^2 + bc + c^2$.*

Problem 10 *Find all integers $n \geq 1$ for which $\frac{1}{5}(2^{4n+2} + 1)$ is prime.*

Hints, comments, and references

1. This is problem 19 from the 2012 Baltic Way competition.

2. This is problem 10.8 from the 2009 All-Russian Olympiad.

3. Parts (a) and (b) are problems 98 and 124 in Sierpiński [2].

(a) The terms with $n = 12k+1$ are divisible by 13.

(b) The terms with $n = 28k+1$ are divisible by 29.

(c) By Wilson's theorem, any prime p divides $(p-2)! - 1$.

4. For $n = 6k+2$, the first number is divisible by 3 and the second number by 7.

5. Part (c) is problem 123 in Sierpiński [2]. The integers in the three parts are respectively divisible by one of (a) 3 and 5; (b) 3, 5, 13; (c) 7.

8. The terms in (a) and (b) have $n^2 + n + 1$ as a factor, and the term in (c) has $2n + 1$ as factor.

9. Part (a) is due to Olaf Krafft [1], and part (b) is from the 2nd round of the 2007 Polish Mathematical Olympiad.

(a) Use that $ab = cd$ implies that there are positive integers p, q, r, s with $a = pq$, $b = rs$, $c = pr$, and $d = qs$.

(b) Use that $a^2 + b^2 + c^2 + d^2$ can be rewritten as $(a + b + c + d)(a - b - c + d)$.

10. This is problem 99 in Sierpiński [2]. Use the Sophie Germain identity.

References

[1] O. Krafft (1983). Problem E3005. *American Mathematical Monthly 90*, 1983, 400.

[2] W. Sierpiński (1970). *"250 Problems in Elementary Number Theory"*. Polish Scientific Publishers, Warszawa.

PROBLEMS

Readers are invited to submit solutions, comments and generalizations to any problem in this section. Moreover, readers are encouraged to submit problem proposals. Please email your submissions to `crux-psol@cms.math.ca` *or mail them to the address inside the back cover. Electronic submissions are preferable.*

Submissions of solutions. *Each solution should be contained in a separate file named using the convention* LastName_FirstName_ProblemNumber *(example Doe_Jane_1234.tex). It is preferred that readers submit a LaTeX file and a pdf file for each solution, although other formats are also accepted. Submissions by regular mail are also accepted. Each solution should start on a separate page and name(s) of solver(s) with affiliation, city and country should appear at the start of each solution.*

Submissions of proposals. *Original problems are particularly sought, but other interesting problems are also accepted provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by someone else without permission. Solutions, if known, should be sent with proposals. If a solution is not known, some reason for the existence of a solution should be included by the proposer. Proposal files should be named using the convention* LastName_FirstName_Proposal_Year_number *(example Doe_Jane_Proposal_2014_4.tex, if this was Jane's fourth proposal submitted in 2014).*

To facilitate their consideration, solutions should be received by the editor by **1 March 2015***, although late solutions will also be considered until a solution is published.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

An asterisk ($\star$) after a number indicates that a problem was proposed without a solution.

3872. *Proposed by F. R. Ataev. Correction.*

Let x, y, z be the distances from the vertices of a triangle to its incircle and let r be the inradius of the triangle. Show that the area of the triangle is given by

$$A = \frac{\sqrt{xyz(x+2r)(y+2r)(z+2r)}}{r}\,.$$

3881. *Proposed by Ovidiu Furdui.*

Calculate

$$\sum_{n=2}^{\infty}\left(n^2\ln\left(1-\frac{1}{n^2}\right)+1\right).$$

3882. *Originally proposed by Mehmet Sahin; corrected version by Arkady Alt.*

Let ABC be a right angle triangle with $\angle CAB = 90^\circ$ and hypotenuse a. Let $[AD]$ be an altitude and let I_1 and I_2 be the incenters of the triangles ABD and ADC, respectively. Let ρ be the radius of the circle through the points B, I_1 and I_2 and let r be the inradius of the triangle ABC. Prove that

$$\rho = \sqrt{\frac{a^2 + 2ar + 2r^2}{2}}$$

and $\min \dfrac{\rho}{r} = \sqrt{3} + \sqrt{6}$.

3883. *Proposed by Max A. Alekseyev.*

Let a, b, c, d be positive integers such that $a + b$ and $ad + bc$ are odd. Prove that if $2^a - 3^b > 1$, then $2^a - 3^b$ does not divide $2^c + 3^d$.

3884. *Proposed by Mihai Bogdan.*

Let a, b, c and d be positive real numbers such that $a + b + c + d = k$, where $k \in (0, 8)$. Prove that:

$$\frac{a}{b^2+1} + \frac{b}{c^2+1} + \frac{c}{d^2+1} + \frac{d}{a^2+1} \ge \frac{k(8-k)}{8}.$$

When does the equality hold?

3885. *Proposed by Oai Thanh Dao.*

Let ABC be a triangle and let F be a point that lies on a circumcircle of ABC. Further, let H_a, H_b and H_c denote projections of the orthocenter H onto sides BC, AC and AB, respectively. The three circles AH_aF, BH_bF and CH_cF meet the three sides BC, AC and AB at points A_1, B_1 and C_1, respectively. Prove that the points A_1, B_1 and C_1 are collinear.

3886. *Proposed by Michel Bataille.*

Let $H_n = \sum_{k=1}^{n} \frac{1}{k}$ be the nth harmonic number and let $H_0 = 0$. Prove that for $n \ge 1$, we have

$$\sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} 2^k H_k = 2H_n - H_{\lfloor n/2 \rfloor}.$$

3887. *Proposed by Dao Hoang Viet.*

Let a, b and c be positive real numbers. Prove that

$$\frac{a^2}{bc(a^2+ab+b^2)} + \frac{b^2}{ac(b^2+bc+c^2)} + \frac{c^2}{ab(a^2+ac+c^2)} \ge \frac{9}{(a+b+c)^2}.$$

3888. *Proposed by Peter Woo.*

My greatly admired high school teacher taught me one foolproof method when solving triangles. Suppose in triangle ABC you are given the measure of $\angle A$ and the lengths of the adjacent sides b and c; then to find the remaining angles in terms of the given quantities, one should use the law of cosines to find the length of the third side and then the law of sines to find measures of $\angle B$ and $\angle C$. Or so I was taught. But after many years, I found a way to solve this problems while avoiding the cosine law and the use of square roots. Can you discover such a way?

3889. *Proposed by Cristinel Mortici.*

Prove that

$$e^{\pi} > \left(\frac{e^2+\pi^2}{2e}\right)^e .$$

3890*. *Proposed by Šefket Arslanagić.*

Let $\alpha, \beta, \gamma \in \mathbb{R}$. Prove or disprove that

$$|\sin\alpha| + |\sin\beta| + |\sin\gamma| + |\cos(\alpha+\beta+\gamma)| \le 1 + \frac{3\sqrt{3}}{2}.$$

. . . .

3872. *Proposé par F. R. Ataev. Correction.*

Soit x, y et z les distances des sommets dun triangle jusquau cercle inscrit dans le triangle et soit r le rayon de ce cercle. Montrer que l'aire du triangle est donnée par

$$A = \frac{\sqrt{xyz(x+2r)(y+2r)(z+2r)}}{r} .$$

3881. *Proposé par Ovidiu Furdui.*

Calculer

$$\sum_{n=2}^{\infty} \left(n^2 \ln\left(1 - \frac{1}{n^2}\right) + 1\right).$$

3882. *Proposé par Mehmet Sahin ; correction par Arkady Alt.*

Soit ABC un triangle rectangle en A avec hypoténuse a et soit $[AD]$ une hauteur du triangle. I_1 et I_2 sont les centres des cercles inscrits dans les triangles respectifs ABD et ADC. Soit ρ le rayon du cercle qui passe aux points B, I_1 et I_2 et soit r le rayon du cercle inscrit dans le triangle ABC. Démontrer que

$$\rho = \sqrt{\frac{a^2+2ar+2r^2}{2}}$$

et $\min \dfrac{\rho}{r} = \sqrt{3} + \sqrt{6}$.

3883. *Proposé par Max A. Alekseyev.*

Soit a, b, c, d des entiers supérieurs à 0 tels que $a + b$ et $ad + bc$ soient impairs. Démontrer que si $2^a - 3^b > 1$, alors $2^a - 3^b$ n'est pas un diviseur de $2^c + 3^d$.

3884. *Proposé par Mihai Bogdan.*

Soit a, b, c et d des réels strictement positifs tels que $a + b + c + d = k$, $k \in (0, 8)$. Démontrer que

$$\frac{a}{b^2+1} + \frac{b}{c^2+1} + \frac{c}{d^2+1} + \frac{d}{a^2+1} \geq \frac{k(8-k)}{8}.$$

Quand y a-t-il égalité ?

3885. *Proposé par Oai Thanh Dao.*

Soit ABC un triangle et F un point sur le cercle circonscrit au triangle ABC. Soit H_a, H_b et H_c les projections de l'orthocentre H sur les côtés respectifs BC, AC et AB. Les trois cercles AH_aF, BH_bF et CH_cF coupent les côtés BC, AC et AB aux points respectifs A_1, B_1 et C_1. Démontrer que les points A_1, B_1 et C_1 sont alignés.

3886. *Proposé par Michel Bataille.*

Soit $H_n = \sum_{k=1}^{n} \frac{1}{k}$ le n^{ieme} nombre harmonique et soit $H_0 = 0$. Démontrer que pour tout n, $n \geq 1$, on a

$$\sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} 2^k H_k = 2H_n - H_{\lfloor n/2 \rfloor}.$$

3887. *Proposé par Dao Hoang Viet.*

Soit a, b et c des réels supérieurs à 0. Démontrer que

$$\frac{a^2}{bc(a^2+ab+b^2)} + \frac{b^2}{ac(b^2+bc+c^2)} + \frac{c^2}{ab(a^2+ac+c^2)} \geq \frac{9}{(a+b+c)^2}.$$

3888. *Proposé par Peter Woo.*

À l'école, une enseignante que j'admirais beaucoup nous a montré que pour déterminer la mesure des autres angles d'un triangle, étant donné la longueur de deux côtés et la mesure de l'angle compris entre eux, on détermine d'abord la longueur du troisième côté en utilisant la loi des cosinus (théorème d'Al-Kashi), puis on détermine la mesure des autres angles en utilisant la loi des sinus. Beaucoup plus tard, j'ai découvert une façon plus facile qui évite l'utilisation de la loi des cosinus et les racines carrées. Pouvez-vous découvrir une telle méthode ?

3889. *Proposé par Cristinel Mortici.*

Démontrer que

$$e^{\pi} > \left(\frac{e^2+\pi^2}{2e}\right)^e .$$

3890*. *Proposé par Šefket Arslanagić.*

Soit $\alpha, \beta, \gamma \in \mathbb{R}$. Démontrer ou infirmer que

$$|\sin\alpha| + |\sin\beta| + |\sin\gamma| + |\cos(\alpha+\beta+\gamma)| \leq 1 + \frac{3\sqrt{3}}{2}.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3781. [2012 : 378, 380] *Proposed by Marcel Chiriţiţă.*

Solve the equation

$$3^{1-x} + 3^{\sqrt{3x-2x^2}} = 4\,.$$

Solved by M. Bataille; D. M. Bătineţu-Giurgiu, N. Stanciu and T. Zvonaru; B. D. Beasley; P. Deiermann; N. Hodžić and S. Malikić; O. Kouba; D. Koukakis; C. R. Pranesachar; D. Smith; and the proposer. There were five solutions that were either incorrect or incomplete. We present the solution composed from solutions by several solvers.

Any solution must satisfy $0 \le x \le 3/2$. Two solutions are $x = 0$ and $x = 1$. Note that when $1 < x < 3/2$, then $1 - (3x - 2x^2) = (x - 1)(2x - 1) > 0$, so that

$$3^{1-x} + 3^{\sqrt{3x-2x^2}} < 1 + 3 = 4.$$

Suppose that $4/19 < x < 1$. Then $9(3x - 2x^2) - (2 + x)^2 = (19x - 4)(1 - x) > 0$ and so

$$1 - x + 3(\sqrt{3x - 2x^2} - 1) = 3\sqrt{3x - 2x^2} - (2 + x) > 0.$$

Hence, by the arithmetic-geometric means inequality, we have

$$\begin{aligned} 3^{1-x} + 3^{\sqrt{3x-2x^2}} &= 3^{1-x} + 3^{\sqrt{3x-2x^2}-1} + 3^{\sqrt{3x-2x^2}-1} + 3^{\sqrt{3x-2x^2}-1} \\ &\ge 4\left(3^{1-x+3(\sqrt{3x-2x^2}-1)}\right)^{1/4} > 4. \end{aligned}$$

Now, suppose that $0 < x < 3/11$. Then $(3x - 2x^2) - 9x^2 = x(3 - 11x) > 0$, so that

$$\begin{aligned} 3^{1-x} + 3^{\sqrt{3x-2x^2}} &= 3^{-x} + 3^{-x} + 3^{-x} + 3^{\sqrt{3x-2x^2}} \\ &\ge 4\left(3^{-3x+\sqrt{3x-2x^2}}\right)^{1/4} > 4. \end{aligned}$$

Since $4/19 < 3/11$, we conclude that the equation has no solution in the set $(0, 1) \cup (1, 3/2)$, and so $x = 0$ and $x = 1$ are the only solutions.

Editor's Comments. There are quick arguments for some parts of the domain. Since $3x - 2x^2 > 1$ for $1/2 < x < 1$, it is easy to see that the left side of the equation exceeds 4 on this interval. Since $3^{1-x} + 3^{\sqrt{3x-2x^2}}$ strictly decreases for $3/4 \le x \le 3/2$, the only solution in this interval is $x = 1$.

Deiermann showed more generally that $x = 0$ and $x = 1$ are the only solutions of

$$b^{\sqrt{b-a}(1-x)} + b^{\sqrt{bx-ax^2}} = 1 + b^{\sqrt{b-a}}$$

when $0 < a < b \le 2a$, $1 < b$ and $c - c^{-1} \le b(b-a)^{-1}$, where $c = b^{\sqrt{b-a}}$. He did this by recasting the equation as $h(x) = g(x)$, with $h(x) = \sqrt{bx - ax^2}\ln b$ and $g(x) = \ln(1 + b^{\sqrt{b-a}} - b^{\sqrt{b-a}(1-x)})$. Noting that $h(0) = g(0) = h(b/a) = 0$ and $h(1) = g(1) = \sqrt{b-a}\ln b$, he analyzed the graphs of these two functions to show that they crossed only when $x = 0$ and $x = 1$. The above problem is the case $(a, b) = (2, 3)$.

3782. [2012 : 378, 380] *Proposed by Edward T. H. Wang and Billy Jin.*

For $n \in \mathbb{N}$, let $S = \{1, 2, 3, \ldots, n\}$. For each nonempty $T \subseteq S$ define the "drop" of T by $d(T) = f(T) - g(T)$ where $f(T)$ and $g(T)$ denote the maximum and minimum elements of T, respectively. (e.g., $d(\{2\}) = 0$, $d(\{2, 3, 7\}) = 5$) Evaluate $D_n = \sum d(T)$, the total of the drops of S, where the summation is over all non-empty subsets T of S.

Solved by AN-anduud Problem Solving Group; M. Bataille; D. Bătineţu-Giurgiu, N. Stanciu and T. Zvonaru; P. De; O. Kouba; K. Lau; S. Malikić; Missouri State University Problem Solving Group; C. R. Pranesachar; D. Smith; E. Suppa; I. Uchiha; and the proposers. We present the solution by Itachi Uchiha slightly expanded by the editor.

Note first that each $i \in S$ is the maximum element of 2^{i-1} subsets of S and the minimum element of 2^{n-i} subsets of S. Hence,

$$\begin{aligned} D_n = \sum d(T) &= \sum f(T) - \sum g(T) = \sum_{i=1}^{n} i2^{i-1} - \sum_{i=1}^{n} i2^{n-i} \\ &= \sum_{i=1}^{n} i2^{i-1} - \sum_{i=0}^{n-1}(n-i)2^i = 3\sum_{i=1}^{n} i2^{i-1} - n\sum_{i=0}^{n} 2^i. \end{aligned}$$

For $x \neq 1$, define

$$h(x) = \frac{1 - x^{n+1}}{1 - x} = 1 + x + x^2 + \cdots + x^n. \tag{1}$$

Now we can write

$$D_n = 3h'(2) - nh(2). \tag{2}$$

Since $h'(x) = 1 + 2x + 3x^2 + \cdots + nx^{n-1}$ and $xh'(x) = x + 2x^2 + 3x^3 + \cdots + nx^n$, we have

$$(1 - x)h'(x) = 1 + x + x^2 + \cdots + (n-1)x^{n-1} - nx^n.$$

Or, in other words,

$$h'(x) = \frac{1 - x^n}{(1 - x)^2} - \frac{nx^n}{1 - x}. \tag{3}$$

Now, by substituting (1) and (3) into (2), we conclude that

$$\begin{aligned} D_n &= 3(1-2^n+n2^n)+n(1-2^{n+1}) \\ &= (n-3)2^n+n+3. \end{aligned}$$

Editor's Comment. Using a counting argument, Lau established the formula $D_n = \sum_{k=1}^{n-1} k(n-k)2^{k-1}$*. AN-anduud Problem Solving Group obtained the recurrence formula* $D_n = D_{n-1} + (n-2)2^{n-1} + 1$*. Prithwijit gave the recurrence* $D_n = D_{n-1} + \sum_{k=1}^{n-1} k2^{k-1}$ *without proof.*

3783. [2012 : 378, 380] *Proposed by George Apostolopoulos.*

Let a, b, c be positive real numbers. Prove that

$$(3a^2+2)\frac{a^3+b^3}{a^2+ab+b^2} + (3b^2+2)\frac{b^3+c^3}{b^2+bc+c^2} + (3c^2+2)\frac{c^3+a^3}{c^2+ca+a^2} \ge 10abc\,.$$

Solved by A. Alt; AN-anduud Problem Solving Group; Š. Arslanagić; D. Bailey, E. Campbell, and C. Diminnie; M. Bataille; D. M. Bătineţu-Giurgiu, N. Stanciu and T. Zvonaru (2 solutions); P. De; C. M. Quang; N. Hodžić; O. Kouba; S. Malikić; P. Perfetti; C. R. Pranesachar; D. Smith; D. Văcaru; S. Wagon; and the proposer. We present the solution by AN-anduud Problem Solving Group.

Let L denote the left side of the given inequality. Since

$$3(x^2-xy+y^2)-(x^2+xy+y^2) = 2(x^2-2xy+y^2) = 2(x-y)^2 \ge 0,$$

we have

$$\frac{x^2-xy+y^2}{x^2+xy+y^2} \ge \frac{1}{3}. \tag{1}$$

Using (1), the condition that $ab+bc+ca=3$ and the AM-GM Inequality, we have that

$$\begin{aligned} L &= \sum_{\text{cyclic}} (3a^2+2)(a+b)\left(\frac{a^2-ab+b^2}{a^2+ab+b^2}\right) \\ &\ge \frac{1}{3}\sum_{\text{cyclic}} (3a^2+2)(a+b) \\ &= \sum_{\text{cyclic}} a^3 + \sum_{\text{cyclic}} a^2b + \frac{4}{3}\left(\sum_{\text{cyclic}} a\right)\left(\frac{ab+bc+ca}{3}\right) \\ &\ge 3\sqrt[3]{a^3b^3c^3} + 3\sqrt[3]{(a^2b)(b^2c)(c^2a)} + 4\sqrt[3]{abc}\cdot\sqrt[3]{(ab)(bc)(ca)} \\ &= 3abc+3abc+4abc \\ &= 10abc. \end{aligned}$$

Editor's Comment. Almost all the submitted solutions are similar to the one featured above. Both Alt and Arslanagić gave the counterexample $a=b=c=2$ *to* *disprove the original incorrect version and both gave a variant, with proof, of the original inequality by replacing the left side with*

$$\sum(3a^2+2b^2)\left(\frac{a^3+b^3}{a^2-ab+b^2}\right).$$

Wagon's proof was based on using Mathematica's FindInstance.

Arslanagić, Bailey, Campbell, Diminnie, Bătineţu-Giurgiu, Stanciu, and Zvonaru all gave another variant, with proof, in which the condition is $a+b+c=3$. In addition, Kouba gave a variant, with proof, in which the condition is $abc=1$. Bătineţu-Giurgiu, Stanciu, and Zvonaru gave the following two generalizations :

1. *If $a,b,c,m,n\in(0,\infty)$ such that $ab+bc+ca=3$, then*

$$\sum(ma^2+n)\left(\frac{a^3+b^3}{a^2+ab+b^2}\right)\geq 2(m+n)abc.$$

2. *If $a,b,c,m,n,k\in(0,\infty)$ such that $ab+bc+ca\leq k$, then*

$$\sum(ma^2+n)\left(\frac{a^3+b^3}{a^2+ab+b^2}\right)\geq\left(\frac{2(mk+3n)}{k}\right)abc.$$

3784. [2012 : 378, 380] *Proposed by Constantin Mateescu.*

Let ABC be a triangle with circumradius R, inradius r and semiperimeter s for which we denote $Q=\sum_{\text{cyclic}}\cos\left(\frac{A}{2}\right)$. Prove that

$$s=2Q\left(\sqrt{R^2Q^2-Rr}-2R\right)\ .$$

Solved by A. Alt ; M. Bataille ; K. Lau ; S. Malikić ; C. R. Pranesachar ; P. Y. Woo ; T. Zvonaru ; and the proposer. We present the solution by Kee-Wai Lau, modified slightly by the editor.

The following well-known identities can be found as entries 56, 57, and 58 in *Recent Advances in Geometric Inequalities* by D.S. Mitrinović, J.E. Pečarić, and V. Volenec (Kluwer Academic Publishers, The Netherlands, 1989) :

$$\sum_{\text{cyclic}}\cos^2\left(\frac{A}{2}\right)=\frac{4R+r}{2R},$$

$$\sum_{\text{cyclic}}\cos^2\left(\frac{A}{2}\right)\cos^2\left(\frac{B}{2}\right)=\frac{s^2+(4R+r)^2}{16R^2},$$

$$\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)=\frac{s}{4R}.$$

Hence,

$$Q^2 = \left(\sum_{\text{cyclic}} \cos\left(\frac{A}{2}\right)\right)^2 = \frac{4R+2}{2R} + 2\sum_{\text{cyclic}} \cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right) \qquad (1)$$

and

$$\begin{aligned}(2RQ^2-(4R+r))^2 &= \left(4R\sum_{\text{cyclic}} \cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\right)^2\\ &= 16R^2\left(\sum_{\text{cyclic}} \cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\right)^2\\ &= 16R^2\left(\frac{s^2+(4R+r)^2}{16R^2} + 2\left(\frac{s}{4R}\right)Q\right)\\ &= s^2+(4R+r)^2+8RQs \qquad (2)\end{aligned}$$

Simplifying (2), we obtain

$$\begin{aligned}&4R^2Q^4 - 4RQ^2(4R+r) = s^2+8RQs \quad \text{or}\\ &s^2+8RQs+4RQ^2(4R+r)-4R^2Q^4 = 0. \qquad (3)\end{aligned}$$

The discriminant of the quadratic equation $f(s)=0$ in (3) is

$$\begin{aligned}D &= 64R^2Q^2 - 16RQ^2(4R+r) + 16R^2Q^4\\ &= 16R^2Q^4 - 16rRQ^2\\ &= 16RQ^2(RQ^2-r) \qquad (4)\end{aligned}$$

From (1) it is clear that $Q^2 > 2$. Furthermore, $R \geq 2r$ by Euler's formula. Hence, from (4), $RQ^2 - r > 2R - r > 0$, so $D > 0$. Therefore, $f(s)$ has two real roots given by

$$s = \frac{1}{2}\left(-8RQ \pm \sqrt{16RQ^2(RQ^2-r)}\right) = 2Q\left(\pm\sqrt{R^2Q^2-Rr} - 2R\right).$$

Rejecting the negative root, we finally have $s = 2Q\left(\sqrt{R^2Q^2-Rr} - 2R\right)$, which completes the proof.

3785. [2012 : 378, 380] *Proposed by Václav Konečný.*

Consider an ellipse $\mathcal{E}$ given by the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with $a > b$. Find the coordinates, in the first quadrant, of the point P on $\mathcal{E}$ such that the acute angle θ between the tangent t to $\mathcal{E}$ at P and the line OP is minimized.

Solved by A. Alt; G. Apostolopoulos; Š. Arslanagić; M. Bataille; C. Curtis; O. Geupel; O. Kouba; S. Malikić; Missouri State University Problem Solving Group; D. Smith; I. Uchiha; and the proposer. In addition, three submissions were incorrect and one was incomplete. We present a composite of solutions by Salem Malikić and Itachi Uchiha.

Let (x, y) be the coordinates of a variable point P of the ellipse in the first quadrant ; that is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $x, y > 0$. The line OP has slope $\frac{y}{x}$ while the tangent to the ellipse at P has slope -$\frac{b^2x}{a^2y}$, so the acute angle between these lines satisfies (with the help of the AM-GM inequality)

$$\tan\theta = \frac{\frac{y}{x} - \left(-\frac{b^2x}{a^2y}\right)}{1 - \frac{b^2}{a^2}} = \frac{1}{a^2 - b^2}\left(\frac{a^2y}{x} + \frac{b^2x}{y}\right) \geq \frac{2ab}{a^2 - b^2}.$$

Equality holds if and only if $\frac{a^2y}{x} = \frac{b^2x}{y}$; that is, if and only if $\frac{x^2}{a^2} = \frac{y^2}{b^2}$. But in the equation of the ellipse these equal fractions sum to 1, so they must each equal $\frac{1}{2}$. Since the tangent function is strictly increasing, $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$ is the point where the acute angle θ achieves its minimum.

3786. [2012 : 379, 380] *Proposed by Mehmet Şahin.*

Let ABC be a triangle with medians m_a, m_b and m_c, circumradius R and inradius r. Let P be the point of intersection of the bisector of $\angle A$ and the median from B, Q be the point of intersection of the bisector of $\angle B$ and the median from C, and R be the point of intersection of the bisector of $\angle C$ and the median from A. If $\angle APB = \alpha$, $\angle BQC = \beta$ and $\angle CRA = \gamma$, prove that

$$\frac{m_a m_b m_c \sin\alpha \sin\beta \sin\gamma}{(a + 2b)(b + 2c)(c + 2a)} = \frac{r}{32R}\ .$$

Solved by A. Alt; AN-anduud Problem Solving Group; G. Apostolopoulos; M. Bataille; C. Curtis; J. G. Heuver; D. Koukakis; S. Malikić; C. R. Pranesachar; D. Văcaru; P. Y. Woo; T. Zvonaru; and the proposer. We present a composite solution.

Let M be the midpoint of AC. Because AP bisects $\angle A$ in ΔABM, we have $\frac{BP}{PM} = \frac{AB}{AM} = \frac{c}{b/2}$, whence

$$\frac{BP}{2c} = \frac{PM}{b} = \frac{BP + PM}{2c + b} = \frac{m_b}{b + 2c}.$$

From the sine law applied to ΔABP, $\frac{BP}{c} = \frac{\sin A/2}{\sin\alpha}$, which yields

$$\frac{m_b \sin\alpha}{b + 2c} = \frac{1}{2}\sin\frac{A}{2}.$$

Similarly,
$$\frac{m_c \sin\beta}{c+2a} = \frac{1}{2}\sin\frac{B}{2} \qquad \text{and} \qquad \frac{m_a \sin\gamma}{a+2b} = \frac{1}{2}\sin\frac{C}{2}.$$
By multiplication it follows that
$$\frac{m_a m_b m_c \sin\alpha\sin\beta\sin\gamma}{(a+2b)(b+2c)(c+2a)} = \frac{1}{8}\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$
The desired result follows immediately from the identity
$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{r}{4R},$$
which holds for all triangles ABC.

3787. [2012 : 379, 381] *Proposed by Michel Bataille.*

Let S be a finite set with cardinality $|S| = n \geq 1$ and let k be a positive integer. Calculate
$$\sum_{(A)} |A(1)\cap A(2)\cap\cdots\cap A(k)| \text{ and } \sum_{(A)} |A(1)\cup A(2)\cup\cdots\cup A(k)|,$$
where the summation $\sum_{(A)}$ is over all mappings A from $\{1, 2, \ldots, k\}$ to the power set $\mathcal{P}(S)$.

Solved by AN-anduud Problem Solving Group; O. Geupel; O. Kouba; Missouri State University Problem Solving Group; C. R. Pranesachar; and the proposer. Skidmore College Problem Group provided correct solution but without proof. We present the solution by Oliver Geupel.

Let us introduce convenient notation :
$$f(n,k) = \sum_{(A)} |A(1)\cap A(2)\cap\cdots\cap A(k)|,$$
$$g(n,k) = \sum_{(A)} |A(1)\cup A(2)\cup\cdots\cup A(k)|.$$

We prove that
$$f(n,k) = n\cdot 2^{k(n-1)}, \tag{1}$$
$$g(n,k) = n\cdot 2^{k(n-1)}(2^k-1). \tag{2}$$

Without loss of generality, we may assume that $S = \{1, 2, \ldots, n\}$. We let $\mathcal{F}$ denote the set of all mappings $A : \{1, 2, \ldots, k\} \to \mathcal{P}(S)$. Then for any $A \in \mathcal{F}$, we define a $k\times n$ matrix $M(A) = (a_{ij})$ such that
$$a_{ij} = \begin{cases} 0 & \text{if } j \notin A(i), \\ 1 & \text{if } j \in A(i). \end{cases}$$

Then it is easy to check that this correspondence defines a bijection between $\mathcal{F}$ and the set of all $k \times n$ $(0,1)$-matrices.

Note that for any $j \in S$, we have $j \in A(1) \cap A(2) \cap \cdots \cap A(k)$ if and only if the j-th column of the matrix $M(A)$ is $(1,1,\ldots,1)^T$. This holds for $2^{k(n-1)}$ $(0,1)$-matrices since the entries outside of the j-th column can be chosen arbitrarily from $\{0,1\}$. Hence, every element of S contributes a portion $2^{k(n-1)}$ to $f(n,k)$ and (1) follows.

Next, for any $j \in S$, we have $j \in A(1) \cup A(2) \cup \cdots \cup A(k)$ if and only if the j-th column of $M(A)$ is distinct from $(0,0,\ldots,0)^T$. This holds for $2^{k(n-1)}(2^k-1)$ $(0,1)$-matrices since the entries outside of the j-th column can be chosen arbitrarily from $\{0,1\}$ and exactly one of the 2^k choices for the jth column, namely $(0,0,\ldots,0)^T$, is forbidden. Thus, every element of S contributes a portion $2^{k(n-1)}(2^k-1)$ to $g(n,k)$ and (2) follows.

3788. [2012 : 379, 381] *Proposed by Panagiote Ligouras.*

Let a, b and c be the sides of an acute-angled triangle ABC. Let H be the orthocentre, and let d_a, d_b and d_c be the distances from H to the sides BC, CA and AB, respectively. Prove that

$$\sum_{\text{cyclic}} \sqrt{\frac{1}{a^2b^2} + \frac{1}{b^2c^2} - \frac{1}{c^2a^2}} \leq \frac{9}{4(d_a+d_b+d_c)^2}\ .$$

Solved by A. Alt; G. Apostolopoulos; Š. Arslanagić; M. Bataille; J. G. Heuver; N. Hodžić and Malikić; and the proposer. We present the solution by John G. Heuver modified by the editor.

Let s, r, and R denote the semiperimeter, the inradius, and the circumradius of $\triangle ABC$, respectively. Furthermore, let L denote the summation of the left side of the given inequality, and set $\alpha = \angle A$, $\beta = \angle B$, and $\gamma = \angle C$ for notational convenience.

We shall make use of the following identities or inequalities all of which are either well-known or easy to show :

(a) $c\cos\beta + b\cos\gamma = a$

(b) $abc = 4rsR$

(c) $d_c = 2R\cos\beta\cos\gamma$

(d) $\sum \cos\beta\cos\gamma = \dfrac{s^2+r^2-4R^2}{4R^2}$

(e) $s^2 \leq 4R^2 + 4rR + 3r^2$ (Gerretsen's Inequality)

(f) $2r \leq R$ (Euler's Inequality)

By the Root-Mean-Square Inequality, together with (a) and (b), we have

$$\begin{aligned} L &= \sum_{\text{cyclic}} \sqrt{\frac{c^2+a^2-b^2}{a^2b^2c^2}} = \frac{1}{abc}\sum_{\text{cyclic}}\sqrt{2ca\cos\beta} \\ &= \frac{1}{abc}\sum_{\text{cyclic}}\frac{\sqrt{2ca\cos\beta}+\sqrt{2ab\cos\gamma}}{2} \\ &\le \frac{1}{abc}\sqrt{a(c\cos\beta+b\cos\gamma)} = \frac{1}{abc}\sum_{\text{cyclic}}\sqrt{a^2} \\ &= \frac{2s}{abc} = \frac{2s}{4rsR} = \frac{1}{2rR}. \end{aligned} \tag{1}$$

Next, by (c) and (d), we have

$$\sum_{\text{cyclic}} d_c = 2R\sum_{\text{cyclic}}\cos\beta\cos\gamma = 2R\left(\frac{s^2+r^2-4R^2}{4R^2}\right) = \frac{s^2+r^2-4R^2}{2R}. \tag{2}$$

We now show that

$$\frac{1}{2rR} \le \frac{9R^2}{(s^2+r^2-4R^2)^2} \tag{3}$$

or equivalently that

$$(s^2+r^2-4R^2)^2 \le 18rR^3. \tag{4}$$

By (e) and (f), we have

$$(s^2+r^2-4R^2)^2 \le (r^2+4rR+3r^2)^2 = 16r^2(r+R)^2 \le 16r\left(\frac{R}{2}\right)\left(\frac{3R}{2}\right)^2 = 18rR^3.$$

So (4) holds and (3) follows.

Finally, from (1), (2), and (3), we have

$$L \le \frac{9R^2}{(s^2+r^2-4R^2)^2} = \frac{9}{4}\left(\frac{2R}{s^2+r^2-4R^2}\right)^2 = \frac{9}{4(d_e+d_b+d_c)},$$

and our proof is complete.

3789. [2012 : 379, 381] *Proposed by Michel Bataille.*

Let triangle ABC be inscribed in a circle with centre O and radius R and P be any point in its plane. Let P' be such that $\Delta PBP'$ is directly similar to ΔCOA and P'' be the reflection of P in AC. Prove that

$$P'P'' \ge \frac{2F}{R}$$

where F is the area of ΔABC. For which P does equality hold?

Solved by O. Geupel ; L. Giugiuc ; O. Kouba ; C. R. Pranesachar ; and the proposer. We present four different solutions.

Solution 1 by the proposer.

Let ρ_{MN} denote the reflection whose axis is the line MN. Since $BP = BP'$ and $\angle PBP' = \angle COA = 2\angle CBA$, it follows that $P = \rho_{BC} \circ \rho_{AB}(P')$. As a result, $P'' = g(P')$ where g is the glide reflection $\rho_{AC} \circ \rho_{BC} \circ \rho_{AB}$. Let us determine g.

Let H_1, H_2, H_3 be the feet of the altitudes from A, B, C, respectively and let $C_1 = \rho_{AB}(C)$, $B_1 = \rho_{AC}(B)$, $U = \rho_{AB}(H_1)$, $V = \rho_{AC}(H_1)$. Since $g(C_1) = C$ and $g(B) = B_1$, the axis of g passes through the respective midpoints H_3 and H_2 of C_1C and BB_1. Thus, the axis is the line H_3H_2 when $\angle BAC \neq 90°$. The homothety with centre A that transforms H_1 into the orthocentre H of triangle ABC also transforms the midpoints K, L of H_1U, H_1V to H_3, H_2 respectively. Thus, $KL \| H_3H_2$; since $UV \| KL$, we have that $UV \| H_3H_2$. But $g(U) = V$, so the midpoint of UV is on the axis H_3H_2 of G. Therefore U, V, H_2, H_3 are collinear and so $g = \rho_{UV} \circ \tau_{UV} = \tau_{UV} \circ \rho_{UV}$, where τ_{UV} denotes the translation that takes U to V. (This decomposition remains valid when $\angle BAC = 90°$.)

Since $P'' = g(P')$, we see $P'P'' \geq UV$ with equality if and only if P' and P'' are on the axis UV of g. The calculation of UV is straightforward. Since A, K, H_1, L are on the circle with diameter AH_1, we have $KL = AH_1 \cdot \sin A$, and so

$$UV = 2KL = 2\sin A \cdot \frac{2F}{BC} = \frac{2F}{R}.$$

The desired inequality follows, with equality if and only if P lies on the line H_1H_2, the reflection of UV in AC. Note that the line through H_1 is perpendicular to OC.

Solution 2 by Leonard Giugiuc.

We set the situation in the complex plane with the affixes $A(2a)$, $B(2i)$, $C(-2c)$ with a and c real and $a + c > 0$. Let $D(0)$, the foot of the altitude from A, be at the origin. The centre O of the circle is located at $(a - c) + (1 - ac)i$, as can be seen by computing its distances from A, B and C. Let $P(z)$, with $z = 2x + 2yi$, and $P'(w)$ be two vertices of triangle PBP'. Since the triangles COA and PBP' are directly similar

$$\frac{A - O}{C - O} = \frac{P' - B}{P - B},$$

so that

$$\frac{a + c + (ac - 1)i}{-(a + c) + (ac - 1)i} = \frac{w - 2i}{2[x + (y - 1)i]}.$$

Let $u = a + c$ and $v = ac - 1$. Then $u^2 + v^2 = R^2$, $2F = 4u$ and

$$w = 2i - 2[x + (y - 1)i](u + vi)^2(u^2 + v^2)^{-1}.$$

Noting that the affix of P'' is $\bar{z}$, we compute

$$\begin{aligned}\bar{z} - w &= 2[x - (y + 1)i] + 2[(u^2 + v^2)^{-1}(u + vi)^2(x + (y - 1)i)] \\ &= 4(u^2 + v^2)^{-1}[xu^2 - (y - 1)uv + (-yv^2 - u^2 + xuv)i].\end{aligned}$$

The desired inequality is equivalent to $(u^2+v^2)|\bar{z}-w|^2 \geq (4u)^2$ or

$$\begin{aligned}[x^2u^4 + (y-1)^2u^2v^2 - 2x(y-1)u^3v] + [y^2v^4 + u^4 + x^2u^2v^2 + 2yu^2v^2 - 2xu^3v - 2xyuv^3] \\ \geq u^4 + u^2v^2,\end{aligned}$$

which reduces to $(xu-yv)^2(u^2+v^2) \geq 0$. Thus, the inequality holds with equality occurring if and only if $xu = yv$, *i.e.*, P is on the line that contains D and is perpendicular to CO.

Solution 3 by Omran Kouba.

We situate the problem in the complex plane with O at the origin, $R = 1$ and the vertices A, B, C at the respective points a, b, 1.

Suppose that z, z' and z'' are the respective affixes of P, P' and P''. Then $z' = b + a(z-b)$ and $z'' = 1 + a - a\bar{z}$. (Note that $|z''-1| = |z-1|$ and $|z''-a| = |z-a|$, so that AC right bisects PP''.)

Since

$$\begin{aligned} z'' - z' &= 1 + a - b + ab - a(z+\bar{z}) \\ &= a(\bar{a} + 1 - \bar{a}b + b - 2\text{Re } z), \end{aligned}$$

we have that

$$P'P'' = 2|v - \text{Re } z| = 2\sqrt{(\text{Im } v)^2 + |\text{Re } v - \text{Re } z|^2},$$

where $v = \frac{1}{2}(1 + \bar{a} + b - \bar{a}b)$. Thus

$$P'P'' \geq 2|\text{Im } v|$$

with equality if and only if $\text{Re } z = \text{Re } v$.

We now interpret $\text{Im } v$ and $\text{Re } v$. Recalling the formula $\frac{1}{2}\text{Im } (\bar{z}_1z_2 + \bar{z}_2z_3 + \bar{z}_3z_1)$ for the area of a triangle with vertices at z_1, z_2, z_3, we find that

$$F = \frac{1}{2}|\text{Im } (\bar{a} + a\bar{b} + b)| = \frac{1}{2}|\text{Im } (\bar{a} - \bar{a}b + b)| = |\text{Im } v|.$$

The image of the point B under reflection in AC has the affix $b'' = 1 + a - a\bar{b}$, so that $\frac{1}{2}(b''+b) = \bar{v} + \frac{1}{2}(b - \bar{b})$ is the affix of the midpoint between B and its reflection AC, *i.e.*, the foot D of the perpendicular from B to AC. Thus the equation $\text{Re } z = \text{Re } v = \text{Re } \bar{v}$ represents the line perpendicular to OC that passes through D. (One way to find the reflected image of B is to perform the rotation $z \longrightarrow 1 + (z-1)e^{i\theta}$ where $a = -e^{-2i\theta}$; this rotates CA onto the real axis. Reflect the image of B in the real axis and rotate back.)

In conclusion, we have proved that $P'P'' \geq 2F$ with equality if and only if P belongs to the line through the foot of the altitude to AC that is orthogonal to OC. Since $R = 1$, this solves the problem.

Solution 4 by Oliver Geupel.

We use Cartesian coordinates. Wolog, let the circumcircle of triangle ABC be the unit circle and let AC be parallel to the $y-$axis, so that we have $A = (\cos\alpha, \sin\alpha)$, $B = (\cos\beta, \sin\beta)$ and $C = (\cos\alpha, -\sin\alpha)$. The area F of the triangle is equal to $|\sin\alpha(\cos\alpha - \cos\beta)|$.

Let D, Q and S be, respectively, the midpoint of AC, the midpoint of PP' and the foot of the perpendicular from P to AC. Then $P'P'' = 2QS$. The triangle PBQ is obtained from the triangle COD by the following successive transformations :

(i) Scale triangle COD by a factor r to obtain triangle C_1OD_1 ;

(ii) Rotate triangle C_1OD_1 around O by an angle ϕ to obtain triangle C_2OD_2 ;

(iii) Translate C_2OD_2 by the vector $\overrightarrow{OB} = (\cos\beta, \sin\beta)$ to obtain triangle PBQ.

We compute the coordinates of various points step by step :

$D = (\cos\alpha, 0)$,

$C_1 = (r\cos\alpha, -r\sin\alpha)$,

$D_1 = (r\cos\alpha, 0)$,

$C_2 = (r\cos\alpha\cos\phi + r\sin\alpha\sin\phi, r\cos\alpha\sin\phi - r\sin\alpha\cos\phi)$,

$D_2 = (r\cos\alpha\cos\phi, r\cos\alpha\sin\phi)$,

$P = (r(\cos\alpha\cos\phi + \sin\alpha\sin\phi) + \cos\beta, r(\cos\alpha\sin\phi - \sin\alpha\cos\phi) + \sin\beta)$,

$Q = (r\cos\alpha\cos\phi + \cos\beta, r\cos\alpha\sin\phi + \sin\beta)$,

$S = (\cos\alpha, r(\cos\alpha\sin\phi - \sin\alpha\cos\phi) + \sin\beta)$.

Therefore

$$\begin{aligned}
&(P'P'')^2 = 4QS^2 \\
&= 4(r\cos\phi - \cos\alpha(\cos\alpha - \cos\beta))^2 + 4\sin^2\alpha(\cos\alpha - \cos\beta)^2 \\
&= 4(r\cos\phi - \cos\alpha(\cos\alpha - \cos\beta))^2 + \left(\frac{2F}{R}\right)^2 \geq \left(\frac{2F}{R}\right)^2.
\end{aligned}$$

This proves the required inequality. Equality occurs if and only if

$$r\cos\phi = \cos\alpha(\cos\alpha - \cos\beta),$$

that is when P lies on a line perpendicular to OC.

Editor's Comment. One solver erroneously claimed, without explanation, that the result failed to hold when $A = P = P''$. However, if we have an equilateral triangle inscribed in a unit circle, then $2F = 3 \times (\sqrt{3}/2) < 3 = P'P''$.

3790. [2012 : 379, 381] *Proposed by Ovidiu Furdui.*

Let $a, \alpha \geq 0$ be nonnegative real numbers and let β be a positive number. Determine the limit

$$L(\alpha,\beta) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^\alpha}{(n^2+kn+a)^\beta} \ .$$

Solved by M. Bataille ; O. Geupel ; R. Hess ; A. Kotronis ; O. Kouba ; P. Perfetti ; and the proposer. We present the solution by Oliver Geupel.

We prove that

$$L(\alpha,\beta) = \begin{cases} 0 & \text{if } \alpha+1<2\beta, \\ +\infty & \text{if } \alpha+1>2\beta, \\ \int_0^1 \frac{x^{2\beta-1}}{(1+x)^\beta}\,dx & \text{if } \alpha+1=2\beta. \end{cases}$$

Note that

$$\sum_{k=1}^{n} \frac{k^\alpha}{(n^2+kn+a)^\beta} = n^{\alpha+1-2\beta}\left[\frac{1}{n}\sum_{k=1}^{n}\frac{(\frac{k}{n})^\alpha}{\left(1+\frac{k}{n}+\frac{a}{n^2}\right)^\beta}\right].$$

For positive a and ϵ and sufficiently large n,

$$\frac{1}{n}\sum_{k=1}^{n}\frac{(\frac{k}{n})^\alpha}{\left(1+\frac{k}{n}+\epsilon\right)^\beta} \leq \frac{1}{n}\sum_{k=1}^{n}\frac{(\frac{k}{n})^\alpha}{\left(1+\frac{k}{n}+\frac{a}{n^2}\right)^\beta} \leq \frac{1}{n}\sum_{k=1}^{n}\frac{(\frac{k}{n})^\alpha}{\left(1+\frac{k}{n}\right)^\beta}.$$

Considering Riemann sums, we have, for all $\epsilon > 0$, that

$$\begin{aligned}\int_0^1 \frac{x^\alpha dx}{(1+x+\epsilon)^\beta} &\leq \liminf \frac{1}{n}\sum_{k=1}^{n}\frac{(\frac{k}{n})^\alpha}{\left(1+\frac{k}{n}+\frac{a}{n^2}\right)^\beta} \\ &\leq \limsup \frac{1}{n}\sum_{k=1}^{n}\frac{(\frac{k}{n})^\alpha}{\left(1+\frac{k}{n}+\frac{a}{n^2}\right)^\beta} \leq \int_0^1 \frac{x^\alpha dx}{(1+x)^\beta}.\end{aligned}$$

Therefore

$$L(\alpha,\beta) = \lim_{n\to\infty}(n^{\alpha+1-2\beta})\int_0^1 \frac{x^\alpha dx}{(1+x)^\beta}$$

and the result follows.

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(Bold font indicates featured solution.)

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