Crux Mathematicorum

VOLUME 39, NO. 4 APRIL / AVRIL 2013

IN THIS ISSUE / DANS CE NUMÉRO

Published by
Canadian Mathematical Society
209 - 1725 St. Laurent Blvd.
Ottawa, Ontario, Canada K1G 3V4
FAX: 613–733–8994
email: subscriptions@cms.math.ca

Publié par
Société mathématique du Canada
209 - 1725 boul. St. Laurent
Ottawa (Ontario) Canada K1G 3V4
Téléc : 613–733–8994
email : abonnements@smc.math.ca

THE CONTEST CORNER

No. 14

Shawn Godin

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. Nous préférons les réponses électroniques et demandons aux lecteurs de présenter chaque solution dans un fichier distinct. Il est recommandé de nommer les fichiers de la manière suivante : Nom de famille_Prénom_CCNuméro du problème *(exemple : Tremblay_Julie_CC1234.tex). De préférence, les lecteurs enverront un fichier au format LaTeX et un fichier pdf pour chaque solution, bien que les autres formats (Word, etc.) soient aussi acceptés. Nous invitons les lecteurs à envoyer leurs solutions et réponses aux concours au rédacteur à l'adresse* `crux-contest@smc.math.ca`. *Nous acceptons aussi les contributions par la poste, envoyées à l'adresse figurant en troisième de couverture. Le nom de la personne qui propose une solution doit figurer avec chaque solution, de même que l'établissement qu'elle fréquente, sa ville et son pays ; chaque solution doit également commencer sur une nouvelle page.*

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er août 2014** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

Chaque problème est présenté en anglais et en français, les deux langues officielles du Canada. Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le franais précédera l'anglais. Dans la section Solutions, le problème sera écrit dans la langue de la première solution présentée.

La rédaction souhaite remercier Rolland Gaudet, de Université de Saint-Boniface, Winnipeg, MB, d'avoir traduit les problèmes.

CC66. Les longueurs de chacune des six arêtes dun tétraèdre sont des entiers. Cinq des longueurs d'arêtes sont 14, 20, 40, 52 et 70. Déterminer le nombre de longueurs possibles de la sixième arête.

CC67. Les vingt volumes d'une encyclopédie sont numérotés sur le dos, de 1 à 20. Dix volumes ont une couverture bleue, six volumes ont une couverture rouge et les autres ont une couverture verte. On veut placer les volumes sur une étagère. Déterminer le nombre de façons de placer les livres de manière qu'il n'y ait pas deux livres d'une même couleur l'un à côté de l'autre.

CC68. On considère une famille de droites dans le plan cartésien. Chaque droite de la famille est telle que la somme des inverses de son abscisse à l'origine et de son ordonnée à l'origine est égale à k. Démontrer que toutes les droites de cette famille sont concourantes.

CC69. La suite de Fibonacci est définie par $f_1 = f_2 = 1$ et $f_n = f_{n-1} + f_{n-2}$ $(n \geq 3)$. Un triangle pythagoricien est un triangle rectangle dont les mesures des côtés sont toutes des entiers. Démontrer que pour chaque entier positif k $(k \geq 2)$, f_{2k+1} est la longueur de l'hypoténuse d'un triangle pythagoricien.

CC70. Dans un jeu, on commence avec un tas de N cacahuètes. Deux joueurs, à tour de rôle, mangent un nombre de cacahuètes, ce nombre étant un carré strictement positif $(1, 4, 9, \dots)$. Le joueur qui mange la dernière cacahuète gagne. Quelles sont les valeurs de N pour lesquelles le premier joueur peut toujours gagner ?

. . . .

CC66. The lengths of all six edges of a tetrahedron are integers. The lengths of five of the edges are 14, 20, 40, 52, and 70. Determine the number of possible lengths for the sixth edge.

CC67. The twenty volumes, clearly numbered 1 to 20, of an encyclopedia are to be arranged on a shelf. If ten volumes have blue covers, six have red covers, and the remainder have green covers, determine in how many ways the books can be arranged so that no two books of the same colour are side by side.

CC68. A family of straight lines is determined by the condition that the sum of the reciprocals of the x and y intercepts is a constant k for each line in the family. Show that all members of the family are concurrent.

CC69. The Fibonacci sequence is defined by $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. A Pythagorean triangle is a right-angled triangle with integer side lengths. Prove that f_{2k+1} is the hypotenuse of a Pythagorean triangle for every positive integer k with $k \geq 2$.

CC70. The game of Square Meal is played with a heap of peanuts, initially containing N nuts. The two players take it in turns to eat a positive square number $(1, 4, 9, \dots)$ of nuts. Whoever eats the last nut wins. For which values of N can the first player always win?

CONTEST CORNER SOLUTIONS

CC16. In a magic square, the numbers in each row, the numbers in each column, and the numbers on each diagonal have the same sum. Given the magic square shown with $a, b, c, x, y, z > 0$, determine the product xyz in terms of a, b and c.

$\log a$	$\log b$	$\log x$
p	$\log y$	$\log c$
$\log z$	q	r

(*Originally question A6 from the 2011 Canadian Senior Mathematics Contest.*)

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Richard I. Hess, Rancho Palos Verdes, CA, USA; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We present the solution of Manes.

Let S denote the sum of each row, column, and diagonal. Taking the sum along the first row and the minor diagonal gives

$$\log(abx) = S = \log(xyz),$$
$$xyz = abx, \text{ and} \tag{1}$$
$$yz = ab. \tag{2}$$

Similarly, considering the third column with the main diagonal gives us

$$xc = ay \tag{3}$$

and taking the first column with the second row gives us

$$az = yc. \tag{4}$$

Combining equations (2) and (4) gives us

$$z = \sqrt{bc}. \tag{5}$$

We are now able to solve for the value of xyz.

$$\begin{aligned} xyz &= abx && \text{from (1)}, \\ &= \frac{a^2by}{c} && \text{from (3)}, \\ &= \frac{a^3bz}{c^2} && \text{from (4)}, \\ &= \frac{a^3b^{\frac{3}{2}}}{c^{\frac{3}{2}}} && \text{from (5)}. \end{aligned}$$

CC17. A line with slope m meets the parabola $y = x^2$ at A and B. If the length of segment AB is ℓ what is the equation of that line in terms of ℓ and m? (*Inspired by question # 8 from the 2010 Manitoba Mathematical Competition.*)

Solved by Matei Coiculescu, East Lyme High School, East Lyme, CT, USA; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Richard I. Hess, Rancho Palos Verdes, CA, USA; Mihaï-Ioan Stoënescu, Bischwiller, France; Daniel Văcaru, Piteşti, Romania; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We present the solution of Coiculescu.

Let $y = mx + n$ be the equation of the line intersecting the parabola $y = x^2$. The points of intersection, A and B are:

$$\begin{aligned} A = (x_1, y_1) &= \left(\frac{m - \sqrt{m^2 + 4n}}{2},\ x_1^2 \right), \\ B = (x_2, y_2) &= \left(\frac{m + \sqrt{m^2 + 4n}}{2},\ x_2^2 \right). \end{aligned}$$

The square of the length of the segment AB is

$$\ell^2 = (y_2 - y_1)^2 + (x_2 - x_1)^2.$$

We have that $x_2 - x_1 = \sqrt{m^2 + 4n}$ and $x_2 + x_1 = m$, so we see that:

$$\begin{aligned} l^2 &= (y_2 - y_1)^2 + (x_2 - x_1)^2 \\ &= (x_2 - x_1)^2 (x_2 + x_1)^2 + (x_2 - x_1)^2 \\ &= (m^2 + 4n)(m^2 + 1). \end{aligned}$$

Solving for n yields

$$n = \frac{1}{4} \left(\frac{l^2}{m^2 + 1} - m^2 \right),$$

so the equation of the line is

$$y = mx + \frac{1}{4} \left(\frac{l^2}{m^2 + 1} - m^2 \right).$$

CC18. The left end of a rubber band e meters long is attached to a wall and a slightly sadistic child holds on to the right end. A point-sized ant is located at the left end of the rubber band at time $t = 0$, when it begins walking to the right along the rubber band as the child begins stretching it. The increasingly tired ant walks at a rate of $1/(ln(t + e))$ centimeters per second, while the child uniformly stretches the rubber band at a rate of one meter per second. The rubber band is infinitely stretchable and the ant and child are immortal. Compute the time in seconds, if it exists, at which the ant reaches the right end of the rubber band. (*Originally question # 8 from the 2012 Stanford Math Tournament, team test.*)

Solved by Richard I. Hess, Rancho Palos Verdes, CA, USA.

The initial length of the band is $100e$ and the length as a function of time is $L(t) = 100(t+e)$, both in centimetres. In the time interval dt at time t the ant covers a fraction of the band's total length equal to

$$f(t) = \frac{dt}{100(t+e)\ln(t+e)}\,.$$

Thus, the total fraction of the band covered in time T is

$$F(T) = \int_0^T \frac{dt}{100(t+e)\ln(t+e)} = \int_e^{T+e} \frac{du}{100u\ln u} = \frac{1}{100}[\ln(\ln(T+e)) - \ln(\ln e)]\,.$$

The time at which $F(T) = 1$ is when $\ln(\ln(T+e)) = 100 \Rightarrow \ln(T+e) = e^{100}$, thus the ant reaches the end of the band at $T = e^{e^{100}} - e$ seconds.

CC19. Evaluate

$$\frac{1}{3+\frac{1}{3+\frac{1}{4+\frac{1}{5+\frac{1}{\cdots+\frac{1}{2013}}}}}} + \frac{1}{1+\frac{1}{2+\frac{1}{3+\frac{1}{4+\frac{1}{\cdots+\frac{1}{2013}}}}}}.$$

(*Inspired by question # 6 from the 2005 University of Waterloo, Bernoulli Trials.*)

Solved by Matei Coiculescu, East Lyme High School, East Lyme, CT, USA; ; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Richard I. Hess, Rancho Palos Verdes, CA, USA; and Mihaï-Ioan Stoënescu, Bischwiller, France. We present the solution of Coiculescu and Stoënescu.

Let $a = 2 + \frac{1}{3+\frac{1}{4+\frac{1}{5+\frac{1}{\cdots+\frac{1}{2013}}}}}$. Then we see that,

$$\frac{1}{3+\frac{1}{3+\frac{1}{4+\frac{1}{5+\frac{1}{\cdots+\frac{1}{2013}}}}}} + \frac{1}{1+\frac{1}{2+\frac{1}{3+\frac{1}{4+\frac{1}{\cdots+\frac{1}{2013}}}}}} = \frac{1}{1+a} + \frac{1}{1+\frac{1}{a}} = \frac{1}{1+a} + \frac{a}{1+a} = 1.$$

CC20. When the Math Club advertises an "(M,N) sock hop", this means that the DJ has been instructed that the M^{th} dance after a fast dance must be a slow dance, while the N^{th} dance after a slow dance must be a fast dance. (All dances are slow or fast; the DJ avoids the embarrassing ones where nobody is quite sure what to do.) For some values of M and N this means that the dancing must end early and everybody can start in on the pizza; for other values the dancing can in principle go on forever. For which ordered pairs (M,N) is there no upper bound to the number of dances?

(*Originally question # 7 from the 2008 Science Atlantic Math Competition.*)

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

We regard the sequence of dances as a sequence $\{x_k\}$ of 1s and 0s, with a 1 representing a fast dance and a 0 representing a slow dance. Write $M = 2^r \cdot m$ and $N = 2^s \cdot n$, where m and n are odd. We claim that there is an infinite sequence satisfying the requirements if and only if $r = s$.

We note that the M^{th} dance before a fast dance is a slow dance, since the alternative leads immediately to a contradiction. Likewise, the N^{th} dance before a slow dance is a fast dance.

Suppose that $r < s$, and suppose the sequence of dances is infinite. Choose a positive integer p such that $x_p = 0$. Then we have the following chains of implications:

$$\begin{aligned} x_p = 0 &\Rightarrow x_{p+N} = 1 \Rightarrow x_{p+M+N} = 0 \Rightarrow x_{p+M} = 1 \Rightarrow x_{p+2M} = 0 \\ &\Rightarrow \cdots \Rightarrow x_{p+4M} = 0 \Rightarrow \cdots \Rightarrow x_{p+(2^{s-r}\cdot n)M} = 0 \end{aligned}$$

and

$$\begin{aligned} x_p = 0 &\Rightarrow x_{p+N} = 1 \Rightarrow x_{p+M+N} = 0 \Rightarrow x_{p+M+2N} = 1 \Rightarrow x_{p+2N} = 0 \\ &\Rightarrow \cdots \Rightarrow x_{p+4N} = 0 \Rightarrow \cdots \Rightarrow x_{p+mN} = 1. \end{aligned}$$

But,

$$p + (2^{s-r} \cdot n)M = p + (2^{s-r} \cdot n)(2^r \cdot m) = p + (2^s \cdot n) \cdot m = p + mN,$$

so we have a contradiction. Thus the sequence must be finite.

If instead $r > s$, the sequence again terminates. The proof starts with a positive integer q such that $x_q = 1$ and proceeds as before.

If $r = s = 0$, so that M and N are odd, each of the sequences,

$$1, 0, 1, 0, 1, 0, 1, 0, \cdots \qquad \text{or} \qquad 0, 1, 0, 1, 0, 1, 0, 1, \cdots$$

is an (M, N) sock hop.

If $r = s \geq 1$, we define a sequence as follows. The first 2^r terms of the sequence can be chosen arbitrarily from $\{0, 1\}$. For each $j \in \{1, 2, 3, ..., 2^r\}$, the subsequence $\{x_{j+k\cdot 2^r}\}_{k=1}^{\infty}$ is chosen to be an alternating sequence of 1s and 0s. The resulting sequence is an $(2^r \cdot m, 2^r \cdot n)$ sock hop.

If you know of a mathematics contest at the high school or undergraduate level whose problems you would like to see in *Contest Corner*, please send information about the contest to `crux-contest@cms.math.ca`.

A Taste **O**f **M**athematics
Aime-**T**-**O**n les **M**athématiques
ATOM

ATOM Volume XIII: Quadratics and Complex Numbers
by Edward J. Barbeau

ATOM Volume XII: Transformational Geometry
by Edward J. Barbeau

ATOM Volume XI: Problems for Junior Mathematics Leagues
by Bruce L.R. Shawyer & Bruce B. Watson

ATOM Volume X: Modular Arithmetic
by Naoki Sato

ATOM Volume IX: The CAUT Problems
by Edward J. Barbeau

ATOM Volume VIII: Problems for Mathematics Leagues III
by Peter I. Booth, John McLoughlin and Bruce L.R. Shawyer

ATOM Volume VII: Problems of the Week
by Jim Totten

ATOM Volume VI: More Problems for Mathematics Leagues
by Peter I. Booth, John McLoughlin and Bruce L.R. Shawyer

ATOM Volume V: Combinatorial Explorations
by Richard Hoshino and John McLoughlin

ATOM Volume IV: Inequalities
by Edward J. Barbeau and Bruce L.R. Shawyer

ATOM Volume III: Problems for Mathematics Leagues
by Peter I. Booth, John McLoughlin and Bruce L.R. Shawyer

ATOM Volume II: Algebra Intermediate Methods
by Bruce L.R. Shawyer

ATOM Volume I: Mathematical Olympiads' Correspondence Program (1995-96)
by Edward J. Barbeau

There are currently 13 booklets in the series. For information on tiles in this series and how to order, visit the **ATOM** page on the CMS website:

`http://cms.math.ca/Publications/Books/atom.`

THE OLYMPIAD CORNER

No. 312

Nicolae Strungaru

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. Nous préférons les réponses électroniques et demandons aux lecteurs de présenter chaque solution dans un fichier distinct. Il est recommandé de nommer les fichiers de la manière suivante : Nom de famille_Prénom_OCNuméro du problème *(exemple : Tremblay_Julie_OC1234.tex). De préférence, les lecteurs enverront un fichier au format LaTeX et un fichier pdf pour chaque solution, bien que les autres formats (Word, etc.) soient aussi acceptés. Nous invitons les lecteurs à envoyer leurs solutions et réponses aux concours au rédacteur à l'adresse* `crux-olympiad@smc.math.ca`. *Nous acceptons aussi les contributions par la poste, envoyées à l'adresse figurant en troisième de couverture. Le nom de la personne qui propose une solution doit figurer avec chaque solution, de même que l'établissement qu'elle fréquente, sa ville et son pays ; chaque solution doit également commencer sur une nouvelle page.*

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er août 2014** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

Chaque problème est présenté en anglais et en français, les deux langues officielles du Canada. Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le français précédera l'anglais. Dans la section Solutions, le problème sera écrit dans la langue de la première solution présentée.

La rédaction souhaite remercier Rolland Gaudet, de l'Université Saint-Boniface à Winnipeg, d'avoir traduit les problèmes.

OC126. Soient A, B et C des points sur un cercle Γ, de centre O. Supposons que $\angle ABC > 90°$. Soit D le point d'intersection de la ligne AB avec la perpendiculaire vers AC au point C. Soit ℓ la perpendiculaire de D vers AO. Enfin, soit E le point d'intersection de ℓ avec la ligne AC et soit F le point d'intersection de Γ avec ℓ, se trouvant entre D et E. Démontrer que les cercles circonscrits des triangles BFE et CFD sont tangents à F.

OC127. Soient p et k des entiers positifs tels que p est premier et $k > 1$. Démontrer qu'il existe au plus une paire d'entiers positifs (x, y) tels que $x^k + px = y^k$.

OC128. Soit n un entier positif. Démontrer que l'équation

$$\sqrt{x} + \sqrt{y} = \sqrt{n}$$

a solution (x, y) avec x et y entiers positifs si et seulement si n est divisible par un entier m^2, où $m > 1$ est entier.

OC129. Déterminer toutes les fonctions $f : (0, +\infty) \mapsto (0, +\infty)$ qui sont surjectives et telles que

$$2x \cdot f(f(x)) = f(x) \cdot (x + f(f(x)))$$

pour tout $x \in (0, \infty)$.

OC130. Au début, $n + 1$ monômes sont écrits au tableau: $1, x, x^2, \ldots, x^n$. Chaque minute, chacun de k garçons y écrit la somme de deux des polynômes qui y étaient écrits auparavant. Après m minutes, on observe au tableau les polynômes suivants, entre autres, $S_1 = 1 + x$, $S_2 = 1 + x + x^2$, $S_3 = 1 + x + x^2 + x^3$, ..., $S_n = 1 + x + x^2 + \cdots + x^n$. Démontrer que $m \geq \frac{2n}{k+1}$.

. . . .

OC126. Let A, B and C be points lying on a circle Γ with center O. Assume that $\angle ABC > 90°$. Let D be the point of intersection of the line AB with the perpendicular at C on AC. Let ℓ be the perpendicular from D onto AO. Let E be the point of intersection of ℓ with the line AC, and let F be the point of intersection of Γ with ℓ that lies between D and E. Prove that the circumcircles of triangles BFE and CFD are tangent at F.

OC127. Let p and k be positive integers such that p is prime and $k > 1$. Prove that there is at most one pair (x, y) of positive integers such that $x^k + px = y^k$.

OC128. Let n be a positive integer. Prove that the equation

$$\sqrt{x} + \sqrt{y} = \sqrt{n}$$

has solution (x, y) with x, y positive integers, if and only if n is divisible by some m^2 where $m > 1$ is an integer.

OC129. Find all functions $f : (0, +\infty) \mapsto (0, +\infty)$ which are onto and for which

$$2x \cdot f(f(x)) = f(x) \cdot (x + f(f(x)))$$

for all $x \in (0, \infty)$.

OC130. Initially there are $n+1$ monomials on the blackboard: $1, x, x^2, \ldots, x^n$. Every minute each of k boys simultaneously write on the blackboard the sum of some two polynomials that were written before. After m minutes among others there are the polynomials $S_1 = 1 + x$, $S_2 = 1 + x + x^2$, $S_3 = 1 + x + x^2 + x^3$, ..., $S_n = 1 + x + x^2 + \cdots + x^n$ on the blackboard. Prove that $m \geq \frac{2n}{k+1}$.

OLYMPIAD SOLUTIONS

OC66. Let $n \geq 2$ be a positive integer. Find all functions $f : \mathbb{R} \to \mathbb{R}$ so that

$$f(x - f(y)) = f(x + y^n) + f(f(y) + y^n), \qquad \forall x, y \in \mathbb{R}.$$

(*Originally question 1 from the 2011 China Team Selection, Day 1, Test 2.*)

No Solution to this problem was received.

OC67. A convex 2011-gon is drawn on the board. Peter keeps drawing its diagonals in such a way that each newly drawn diagonal intersects no more than one of the already drawn diagonals. What is the greatest number of diagonals that Peter can draw?
(*Originally question 3 from the 2011 Russia Math Olympiad, Grade 9.*)

Solved by Oliver Geupel, Brühl, NRW, Germany.

The required number of diagonals is 4016. Let D_n be the respective number of diagonals for a convex n-gon. We prove that

$$D_n = 2(n-3)\,. \tag{1}$$

First we prove that D_n does not exceed $2(n-3)$ by Mathematical Induction.

For $n = 3$ there is no diagonal at all. This establishes the base case.

Assume that $D_k \leq 2(k-3)$ for each integer k in the range $3 \leq k < n$. Consider a convex n-gon $P_1P_2 \dots P_n$ and a sequence $\mathcal{S}$ of diagonals such that each newly drawn diagonal intersects no more than one of the already drawn diagonals. Suppose that P_1P_q be the last diagonal in $\mathcal{S}$. Then, P_1P_q intersects no more than one of the diagonals in $\mathcal{S}$, say the diagonal P_rP_s if any.

The sequence $\mathcal{S}$, with P_1P_q and (if applicable) P_rP_s removed, now splits into two subsequences. The first subsequence contains diagonals of the convex q-gon $P_1P_2 \dots P_q$. The other subsequence contains diagonals of the convex $(n+2-q)$-gon $P_1P_qP_{q+1} \dots P_n$. Each of these subsequences has the property that every newly drawn diagonal intersects no more than one of the already drawn diagonals. By the induction hypothesis, the number of diagonals in $\mathcal{S}$ is not greater than $2(q-3) + 2(n+2-q-3) + 2 = 2(n-3)$.

This completes the induction and the proof that $D_n \leq 2(n-3)$.

We finish by explicitly stating a sequence of length $2(n-3)$ with the required property for a convex n-gon $P_1P_2 \dots P_n$. Here it is:

$$P_2P_4,\ P_1P_3,\ P_3P_5,\ P_1P_4,\ P_4P_6,\ P_1P_5,\ \dots,\ P_{n-2}P_n,\ P_1P_{n-1}.$$

This completes the proof of the formula (1).

OC68. Find all integers x, y so that

$$x^3 + x^2 + x = y^2 + y\,.$$

(*Originally question 4 from the 2011 Croatia Team Selection Test, Day 2.*)

Solved by Oliver Geupel, Brühl, NRW, Germany; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. There was one incorrect solution. We give the solution of Wang.

We prove that the only solutions are $(x, y) = (0, 0)$ and $(x, y) = (0, -1)$.

Note first that if $x = 0$ then $y^2+y = 0$ leading to the two solutions mentioned above. So it suffices to show there are no other solutions.

To this end, suppose that (x, y) is a solution where $x \neq 0$. Note that $y^2 + y \geq 0$ for all integers y. As $x^2 + x + 1 > 0$ and $0 \leq y^2 + y = x(x^2 + x + 1)$ it follows that $x > 0$.

The equation can be rewritten as

$$x^3 = y^2 - x^2 + y - x$$

or

$$x^3 = (y - x)(y + x + 1)\,. \qquad (1)$$

We claim that $y - x$ and $y + x + 1$ are relatively prime. If not, let p be a common prime divisor. Then $p \mid x^3$ and hence $p \mid x$. But then, as $p \mid y - x$ it follows that $p \mid y$. Hence $p \mid x$, $p \mid y$ and $p \mid y + x + 1$, which implies $p \mid 1$, a contradiction.

Therefore, it follows from (1) that

$$\begin{aligned} y + x + 1 &= m^3, \\ y - x &= n^3, \text{ and} \\ x &= mn, \end{aligned}$$

for some positive integers m, n with $n < m$. Eliminating y we obtain

$$m^3 - n^3 = 2x + 1 = 2mn + 1$$

or

$$(m - n)(m^2 + mn + n^2) = 2mn + 1\,. \qquad (2)$$

As $m > n$ we have $m - n \geq 1$ and $m^2 + n^2 \geq 2mn$. Hence

$$2mn + 1 = (m - n)(m^2 + mn + n^2) \geq m^2 + mn + n^2 \geq 3mn > 2mn + 1\,,$$

contradicting (2) and our proof is complete.

OC69. Let n be a positive integer and let $P(x,y) = x^n + xy + y^n$. Prove that we cannot find two non-constant polynomials $G(x,y)$ and $H(x,y)$ with real coefficients such that

$$P(x,y) = G(x,y) \cdot H(x,y)\,.$$

(*Originally question 3 from the 2011 Vietnam National Olympiad, Day 2.*)

No solution to this problem was received.

OC70. $\triangle ABC$ is a triangle such that $\angle C$ and $\angle B$ are acute. Let D be a variable point on BC such that $D \neq B, C$ and AD is not perpendicular to BC. Let d be the line passing through D and perpendicular to BC. Assume $d \cap AB = E$, $d \cap AC = F$. Let M, N, P be the incentres of $\triangle AEF$, $\triangle BDE$, $\triangle CDF$. Prove that A, M, N, P are concyclic if and only if d passes through the incentre of $\triangle ABC$.
(*Originally question 2 from the 2011 Vietnam National Olympiad, Day 2.*)

Solved by Oliver Geupel, Brühl, NRW, Germany.

The median from vertex B in ΔABC is also the median from B in ΔFBL. It follows that

$$2(AB^2 + BC^2) - AC^2 = 2(BL^2 + BF^2) - FL^2. \tag{1}$$

Let $\theta = \angle FBL$. Since

$$BL^2 + BF^2 - FL^2 = 2BF \cdot BL \cdot \cos(\theta),$$

and

$$AB^2 + BC^2 = AL^2 + LC^2,$$

(1) can be rewritten as

$$2(AL^2 + LC^2) - (AL + LC)^2 = 4BF \cdot BL \cdot \cos(\theta) + FL^2,$$

or

$$(AL - LC)^2 = 4BF \cdot BL \cdot \cos(\theta) + FL^2.$$

Since $AL - LC = AL - AF = FL$ we finally get

$$4BF \cdot BL \cdot \cos(\theta) = 0.$$

Thus $\theta = 90°$.

BOOK REVIEWS

John McLoughlin

Learning Modern Algebra: From Early Attempts to Prove Fermat's Last Theorem
by Al Cuoco and Joseph J. Rotman
The Mathematical Association of America, 2013
ISBN: 978-1-93951-201-7 (print), 978-1-614444-612-5 (electronic), Hardcover/e-book, 459 + xix pages, US$60.00 (print), US$34.00 (electronic)

Beyond the Quadratic Formula by Ron Irving
The Mathematical Association of America, 2013
ISBN: 978-0-88385-783-0 (print), 978-1-614444-112-0 (electronic), Hardcover/e-book, 228 + xvi pages, US$55.00 (print), US$24.00 (electronic)

Reviewed by ***Edward Barbeau****, University of Toronto, Toronto, ON*

The modern high school syllabus covers very little of what used to be called theory of equations. Students have little opportunity to gain fluency with polynomials and appreciate their rich mathematical structure and history. Both these books redress this. The authors have the dual goal of providing teachers with a solid foundation of algebra that might inform their teaching and guidance of more able students, and of providing students and teachers material suitable for self-study. They succeed admirably, albeit in different ways. The organizing principle behind Irving's book is the solution of polynomial equations while that behind that of Cuoco and Rotman is providing some of the apparatus needed to handle the Fermat equation.

While Irving clearly sets out his propositions, he avoids formal proofs and invites the reader to a hands–on understanding by working through examples and sequences of exercises. Almost every chapter has a large section on the history of the theory of equations that highlights notable mathematicians and their methods. A strength of the book is the author's ability to give a good feel for mathematical developments without burdening the reader with detail. For example, the flavour of such results as the intermediate value theorem, Euler's formula for $e^{i\theta}$, and most proofs of the fundamental theorem of algebra is given without going beyond the elementary scope of the book. Readers who wish to learn more can consult a seventy–item bibliography.

After an introductory chapter that introduces polynomials and their graphs, Irving begins on familiar territory with a complete discussion of quadratic equations. One pleasant feature of this chapter is that he does not merely state (as is often done in school courses) that the graph of a quadratic is a parabola, a curve that students generally have no experience with, but verifies that the graph has the focus-directrix property; he provides a geometric interpretation of completion of the square. The same thorough treatment occurs in subsequent chapters in which he treats the cubic and quartic equations. Not only is there a comprehensive treatment of the several methods of solution, but he goes into significant detail on the discriminant and the character of the roots. A separate chapter is devoted to the basics of the complex field and its geometric realization.

Where do we go from here? Not to Galois theory, as one might expect, but to the issue of polynomials of any degree having a root. The final chapter is masterly. After a historical survey of progress on the quintic equation, the author turns to the fundamental theorem of algebra. He briefly describes a number of approaches to the theorem, but sets his sights on a less common proof that uses induction on the exponent of the highest power of 2 that divides the degree of the polynomial. For this, he develops the basic theory of fields and field extensions, rings of polynomials and the relationship between coefficients and symmetric functions of the roots. The idea is to construct the splitting field, apply the induction hypothesis to the polynomial whose roots are $r + s + mrs$, where r, s run through pairs of roots of the original polynomial, and show that $r + s$ and rs are actually in the base complex field for some pair.

Irving's book can be used by a secondary teacher looking for material that can be handed to a curious student who wants to go beyond the current slender syllabus.

Cuoco and Rotman, on the other hand, have produced an updated version of a standard first modern algebra course at the tertiary level. The definitions, theorems and proofs are laid out more formally, but there are many enhancements to the treatment. They are sensitive to historical developments and the book is punctuated by historical notes. They advise students on how to assimilate material in sections entitled "How to think about it". Side notes in the margin elucidate notation and vocabulary, comment on the treatment and provide context. Unlike the Irving book in which exercises are woven into the treatment, here exercises occur at the end of sections. On occasion, they preview later material or invite readers to follow up some line of enquiry on their own.

In contrast to Irving with his greater emphasis on algorithms, Cuoco and Rotman are more concerned with the structures of algebra. However, they start close to the ground with an opening chapter on early number theory that takes us through some Babylonian problems, Pythagorean triples and Diophantus' method to find them, the Euclidean algorithm and the fundamental theorem of arithmetic. A chapter on induction provides an opportunity to cover diverse applications of the method. A brief treatment of the solution of cubics and quartics leads to an extended section on the complex plane, roots of unity, and Gaussian and Eisenstein integers.

A detailed treatment of modular arithmetic is enriched by the inclusion of the basics of ring theory, public key codes, Conway's method for determining day of the week and patterns in decimal expansions. Finally, we get to the structures of modern algebra: rings, domains, fields, ideals and homomorphisms. The ring of polynomials with its analogies to $\mathbb{Z}$ along with extension fields and a sidelight on straight–edge–and–compass constructions, is treated. After a chapter on algebraic integers and primes, we get to the main business of the book, how to approach Fermat's Last Theorem. After the cubic case is settled, we study the attempt of Kummer using cyclotomic integers and the later introduction of ideals to get around the difficulties when the cyclotomic ring does not admit unique factorization. An epilog gives a brief overview of the work of Abel and Galois on unsolvability by radicals, group theory, as well as elliptic functions and curves and their relevance to Wiles' proof of Fermat's Last Theorem.

Both of these books require substantial commitment by their readers, but the treatment is comprehensive enough that anyone with a solid secondary background will be richly rewarded by increased insight into the history and techniques of mathematics.

PROBLEM SOLVER'S TOOLKIT

No. 5

J. Chris Fisher

The Problem Solver's Toolkit is a new feature in ***Crux Mathematicorum****. It will contain short articles on topics of interest to problem solvers at all levels. Occasionally, these pieces will span several issues.*

Harmonic Sets Part 2: Quadrangles and Quadrilaterals

In part 1 we investigated Desargues's theorem in a projective plane that is represented by the Euclidean plane extended by a line at infinity. We shall now see how that theorem allows us to define harmonic sets. But first we must define the dual notions of quadrangle and quadrilateral.

Four points P, Q, R, S, no three collinear, are the vertices of a *complete quadrangle* of which the six sides are the lines PQ, RS, QR, PS, RP, QS. The intersections of *opposite sides*, namely

$$A = PQ \cap RS, \quad C = QR \cap PS,$$

$$U = RP \cap QS,$$

are called *diagonal points*. See Figure 1.

Four lines p, q, r, s, no three concurrent, are the sides of a *complete quadrilateral* of which the six vertices are the points $p \cap q$, $r \cap s$, $q \cap r$, $p \cap s$, $r \cap p$, $q \cap s$. The joins of *opposite vertices*, namely

$$a = (p \cap q)(r \cap s), \quad c = (q \cap r)(p \cap s),$$

$$u = (r \cap p)(q \cap s),$$

are called *diagonal lines*. See Figure 2.

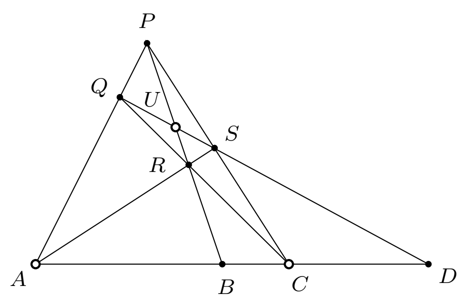


Figure 1: The complete quadrangle $PQRS$.

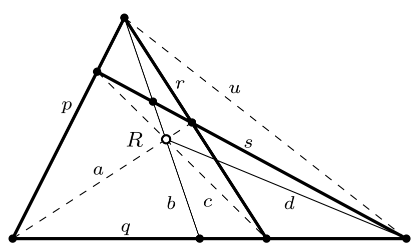


Figure 2: The complete quadrilateral $pqrs$.

Four collinear points A, B, C, D are said to form a *harmonic set* if there is a quadrangle of which two opposite sides pass through A and two other opposite sides through C, while one of the remaining sides passes through B and the other through D. We say that B and D are *harmonic conjugates* with respect to A and

C, a relationship we abbreviate by H(AC, BD). Desargues's theorem then tells us that H(AC, BD) if and only if H(BD, AC): As shown in Figure 3, triangles QBR and PSD are perspective from A, so the points $C = QR \cap PS$ and $U = BR \cap DS$ must be collinear with the new point $V = BQ \cap DP$; thus B and D are diagonal points of the quadrangle $VQUP$ while A and C lie on the remaining sides, whence H(BD, AC), as claimed. Similarly, any classical projective geometry text would show how Desargues's theorem justifies the definition of a harmonic set; in other words, it implies that the harmonic conjugate of B with respect to A and C is uniquely defined and does not depend of the choice of quadrangle. (The details: Given any three collinear points A, B, and C, choose R to be any point not on the line AB, and take $S \neq A, R$ arbitrarily on RA. Then $P = RB \cap SC, Q = AP \cap RC$, and $D = QS \cap AC$; moreover, a different choice of R and S produces the same point D by three applications of Desargues's theorem.)

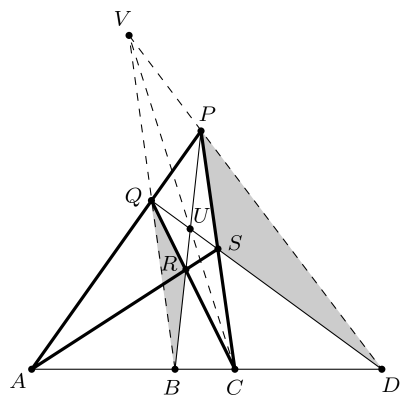


Figure 3: H(AC, BD) if and only if H(BD, AC).

A harmonic set of lines through a point is defined by the dual of the definition of a harmonic set of points. In Figure 2, for example, in the quadrilateral *pqrs* we have H(ac, bd) for the four lines through the point R, which is the intersection of the diagonals a and b. It turns out that for any line ℓ and any point O not on ℓ, the points A, B, C, D form a harmonic set of points on ℓ if and only if the lines OA, OB, OC, OD form a harmonic set of lines through O, as is easily seen by superimposing Figure 1 on Figure 2: let the points in Figure 2 have the labels $P = p \cap r,\ Q = s \cap p,\ R = a \cap c,\ S = s \cap r$, etc. so that A, B, C, D are the points where q intersects the corresponding lines through R. As we have just seen, we could have used any point of the plane not on q to play the role of R in locating the harmonic conjugate of B with respect to A and C. As a consequence, we deduce that if A, B, C, D and A', B', C', D' are two quadruples of collinear points that are situated so that AA', BB', CC', DD' all pass through a point O, then H(AC, BD) implies H$(A'C', B'D')$. We say that these two harmonic sets are related by a *perspectivity with center O*. With the help of Desargues's theorem one can prove that any two harmonic sets are related by a sequence of perspectivities [1, Section

3.5].

Exercise 1. Prove that the two sides of the diagonal triangle of a quadrangle that meet in any diagonal point are harmonic conjugates with respect to the two sides of the quadrangle through that point.

Exercise 2. Prove that if B, C, D and B', C', D' are triples of collinear points on distinct lines that intersect at A while both $\mathrm{H}(AC, BD)$ and $\mathrm{H}(AC', B'D')$, then the lines BB', CC', DD' are concurrent.

Exercise 3. Use the isosceles trapezoid $PQRS$ shown in Figure 4 to verify two familiar examples of harmonic sets.

a. If D is the point at infinity of the line AC, then it is the harmonic conjugate of the midpoint of the segment AC with respect to A and C.
b. When the lines b and d are perpendicular, then they are harmonic conjugates with respect to a and c if and only if they bisect the angles formed by a and c.

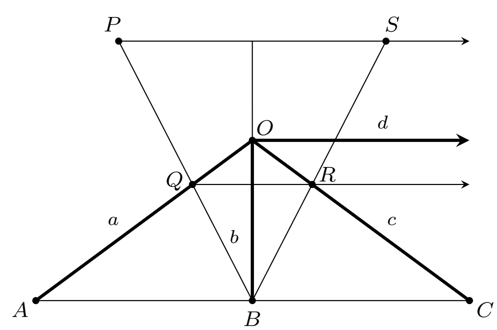


Figure 4: Exercise 3.

Exercise 4. Part 1 of this series explained how a perspective collineation is determined by a center, an axis, and the image of one point different from the center but not on an axis. Prove that if the central collineation has period two (such as a reflection in a point or a line of the Euclidean plane), then any point P is the harmonic conjugate of its image point P' with respect to the center and the point where PP' intersects the axis.

Exercise 5. If PQR is a triangle and $\mathrm{H}(AA', QR)$ and $\mathrm{H}(BB', RP)$, then prove that P and Q are harmonic conjugates with respect to

$$C = AB' \cap BA' \quad \text{and} \quad C' = AB \cap A'B'.$$

The theory of harmonic sets can be derived algebraically using coordinates and cross ratios. For this, one needs the notion of directed distance: on a given line we arbitrarily choose one direction as the positive direction and the other as the negative direction. A segment AB on the line will then be considered positive or negative according as the direction from A to B is the positive or negative direction. Because it will always be clear from the context, the same symbol AB will be used to denote the line AB (extended infinitely in both directions) as well as the distance directed from A to B. Because we will be working in the extended Euclidean plane, when I is the point at infinity of the line AB we define the ratio $\frac{AI}{IB} = -1$, which is consistent with the notion of the ratio $\frac{AP}{PB}$ approaching -1 as P tends to infinity in either direction along the line AB. We can now define the *cross ratio* $(AB;CD)$ of four points on a line of the Euclidean plane to be

$$(AB;CD) = \frac{AC \cdot BD}{AD \cdot BC}.$$

Note that with the points listed in a different order, one might get a different cross ratio. An important theorem in most classical projective geometry texts tells us that cross ratios are preserved by perspectivities; another asserts that when four points lie on a line in the order A, B, C, D, then B and D are harmonic conjugates with respect to A and C if and only if $(AC;BD) = -1$. Likewise, the lines OA, OB, OC, OD form a harmonic set if the cross ratio of their slopes equals -1.

Exercise 6. Repeat Exercise 3 using coordinates for part (a) and slopes for part (b).

References

[1] H.S.M. Coxeter, *Projective Geometry*, 2nd ed. Springer-Verlag, 1987.

The Seebach-Walser Line of a Quadrangle

Rudolf Fritsch and Günter Pickert

Introduction

There are three sorts of centers of gravity for a plane noncrossed polygon: centroid of vertices, centroid of perimeter and centroid of area. In the case of a triangle it is well known that the centroid of vertices and the centroid of area coincide while the centroid of perimeter is the Spieker center (X(10) in [6]), the center of the Spieker circle, the incircle of the medial triangle. The last coincides with the former centroids if and only if the triangle is equilateral.

For quadrangles the situation is much more subtle. A summary of basic information has been given by Karl Seebach [10]. Short proofs for the familiar equivalence, "the centroid of vertices coincides with the centroid of area $\Leftrightarrow$ the quadrangle is a parallelogram", can be found in [2]; a weakening of the requirement of planarity is shown in [9].

The following considerations are inspired by the observation of Hans Walser [13] that the intersection point of the diagonals, the centroid of vertices and the centroid of area for a convex quadrangle, not a parallelogram, are collinear.[1]

For the main lemma of this note we offer three different proofs which confirm a remark of Hanfried Lenz (1916-2013) in [8] that in geometry one may be lucky to find a nice synthetic proof but more often one has to consider tedious different cases while an analytic reasoning gives the desired result at once.

An interesting tool for the synthetic proof is the widely unknown Sum Theorem of van Aubel:

Theorem. *Given points A', B', C' on the sides BC, CA, AB of a triangle ABC respectively — the sides considered as whole lines, not only as segments — such that the cevians AA', BB', CC' concur in a point P, then*

$$\frac{AP}{PA'} = \frac{AB'}{B'C} + \frac{AC'}{C'B}$$

holds for the signed ratios (explained below).

Setup

We consider quadrangles $ABCD$ with vertices A, B, C, D — no three of them on a line — edges $[AB]$, $[BC]$ $[CD]$, $[DA]$ and diagonals AC, BD which intersect in

* This note is an English adaption of the first part of a more comprehensive paper written in German [3, pp. 35-41]. The authors thank Chris Fisher for his helpful comments.

[1] Shortly before submission of this note we realized that Walser's observation has already been stated by Karl Seebach (1912-2007) in 1994 [11]. Since the papers [11] and [13] are difficult to obtain and our approach differs from both we decided to continue with the publication of our thoughts.

a point S; the diagonals may be considered — depending on the situation — as segments or lines. The existence of the intersection point S insures that we deal with plane quadrangles; they may be convex, concave or crossed — only crossed quadrangles with parallel diagonals are excluded. The lengths of the edges are denoted in the usual manner by a, b, c, d while the position vectors of the vertices with respect to a suitably chosen origin are written as $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ (in computations using vector algebra).

We focus on the centroid of vertices $S_{\mathcal{E}}$ and the centroid of area $S_{\mathcal{F}}$ with position vectors $\overrightarrow{s_{\mathcal{E}}}$, $\overrightarrow{s_{\mathcal{F}}}$ respectively[2]. The centroid of vertices $S_{\mathcal{E}}$ may coincide with a vertex of the quadrangle. This is the case for concave quadrangles where one vertex is the centroid of vertices of the triangle formed by the other three vertices.

The Seebach-Walser line

Unlike [13], we see the key to Walser's observation in the following lemma for which we provide three different proofs.

Lemma. *Consider a noncrossed quadrangle. Its centroid of vertices belongs to each segment which connects one vertex to the centroid of vertices of the triangle formed by the three other vertices and divides this segment in the ratio 3 : 1.*

Physical reasoning. We provide each vertex of the quadrangle with the mass "1" so that we have a system of four mass points. Then we concentrate the mass of three vertices in the centroid of vertices of the triangle formed by them provided with the mass "3". By Archimedes' Law of the Lever the center of mass of the whole system lies on the segment connecting the fourth vertex with this centroid dividing it in the ratio 3 : 1.

Vector algebra. In general the centroid of vertices $S_{\mathcal{E}}$ has the vector representation

$$\overrightarrow{s_{\mathcal{E}}} = \frac{1}{4}(\vec{a} + \vec{b} + \vec{c} + \vec{d}).$$

Taking the point $S_{\mathcal{E}}$ as origin we obtain

$$\vec{a} + \vec{b} + \vec{c} + \vec{d} = \vec{0}. \tag{1}$$

For $X \in \{A, B, C, D\}$ let S_X denote the centroid of vertices of the triangle formed by the vertices of the quadrangle different from X and $\overrightarrow{s_X}$ the corresponding position vector. Then we compute

$$\overrightarrow{s_A} = \frac{1}{3}(\vec{b} + \vec{c} + \vec{d}) = -\frac{1}{3}\vec{a}, \ \ldots$$

which proves the assertion.

Synthetic geometry. This needs some effort and a preliminary remark concerning our use of signed distances and ratios[3]. Considering points on a line l, we assume

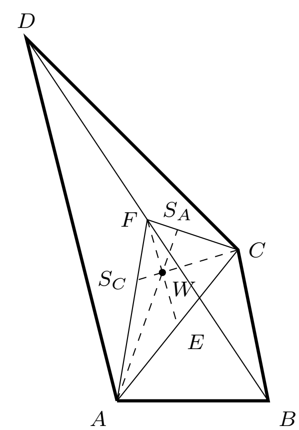
[2] The symbols $\mathcal{E}$ and $\mathcal{F}$ suggest the German words for vertex (Ecke) and area (Fläche).
[3] following the advice of Chris Fisher

this line to be oriented by choosing one of its unit vectors $\vec{u}$ (i.e. $|\vec{u}| = 1$, $\vec{u} = \overrightarrow{UV}$ with $U,\ V \in l$); then the *signed distance* $\overline{'XY}$ of $X,\ Y \in l$ is the scalar such that $\overrightarrow{XY} = \overline{'XY} \cdot \vec{u}$. Thus, $\overline{'YX} = -\overline{'XY}$ and from $\overrightarrow{XY} + \overrightarrow{YZ} = \overrightarrow{XZ}$ it follows $\overline{'XY} + \overline{'YZ} = \overline{'XZ}$. For $A,\ B,\ C \in l$ with $A \neq C$ the *signed ratio* of A rel. B, C is defined as $\overline{'BA}/\overline{'AC} = \overline{'AB}/\overline{'CA}$. In the following XY is written instead of $\overline{'XY}$ if there is no danger of misunderstanding with the line joining X and Y.

Now we show without loss of generality that the point $S_{\mathcal{E}}$ divides the segment $[AS_A]$ internally in the ratio $3 : 1$. First we assume that the midpoint F of the diagonal $[BD]$ does not belong to the diagonal $[AC]$ whose midpoint is denoted by E. The sides of the triangle ACF are divided internally by the points E, S_A, and S_C. We compute the product of the (signed) ratios

$$\frac{AE}{EC} \cdot \frac{CS_A}{S_AF} \cdot \frac{FS_C}{S_CA} = \frac{1}{1} \cdot \frac{2}{1} \cdot \frac{1}{2} = 1.$$

Thus by the theorem of al-Mu'taman-Möbius[4] the lines EF, S_AA, S_CC either concur or are parallel. Since the points A and C lie on different sides of the line FE, but the points C and S_A lie on the same side, the points A and S_A lie on different sides of this line. Therefore the lines EF and S_AA intersect – we have the case of concurrence; let W denote the point of concurrence.



The sides of the triangle AEF are intersected by the line CS_C in the points C, W, S_C. Menelaus' theorem yields

$$\begin{aligned} -1 &= \frac{AC}{CE} \cdot \frac{EW}{WF} \cdot \frac{FS_C}{S_CA} \\ &= -\frac{2}{1} \cdot \frac{EW}{WF} \cdot \frac{1}{2} = -\frac{EW}{WF}, \end{aligned}$$

4 This theorem is often called "converse of Ceva's theorem". The Arabic king of Saragossa Yussuf al-Mu'taman ibn Hūd (b. after 1028, d. 1085) discovered the theorem which later was attributed to the Italian Giovanni Ceva (1647-1734). The German August Ferdinand Möbius (1790-1868) introduced the sign which made the theorem invertible.

so $EW/WF = 1$, hence the point W is the midpoint of the segment $[EF]$ which coincides with the centroid of vertices $S_{\mathcal{E}}$. Thus the point $S_{\mathcal{E}}$ lies on the segment $S_A A$. The division ratio is computed by means of van Aubel's Sum Theorem:

$$\frac{AS_{\mathcal{E}}}{S_{\mathcal{E}}S_A} = \frac{AE}{EC} + \frac{AS_C}{S_C F} = \frac{1}{1} + \frac{2}{1} = 3 : 1.$$

Secondly we have to consider the case where the midpoint F belongs to the diagonal AC; then the quadrangle $ABCD$ is a bisecting quadrangle in the classification of [12]. Now all the relevant points belong to the line AC: the vertices A and C, the midpoints E and F of the diagonals, the centroid S_A of the triangle BCD (since CF is a median of the triangle), as well as the centroid $S_{\mathcal{E}}$ of vertices (the midpoint of the segment $[EF]$). Using the signed distances $e = AE$, $f = AF$ one gets

$$AS_{\mathcal{E}} = \frac{1}{2}(e+f), \;\; AC = 2e, AS_A = f + \frac{1}{3}(2e - f) = \frac{2}{3}(e+f);$$

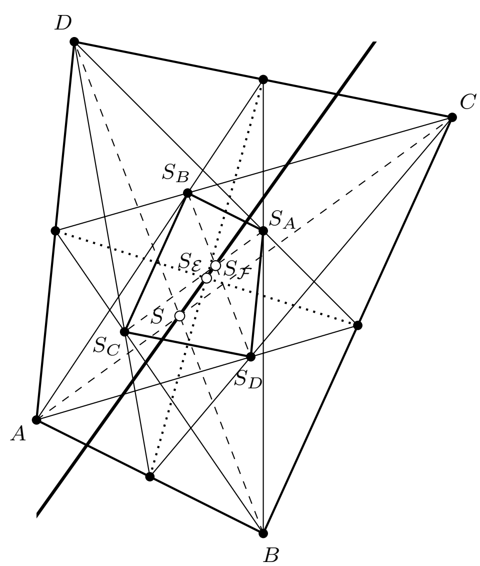
the latter equation holds since $CF = 3S_AF$. Thus

$$S_{\mathcal{E}}S_A = AS_A - AS_{\mathcal{E}} = \frac{1}{6}(e+f) = \frac{1}{3}S_{\mathcal{E}}.$$

This completes the third proof of the lemma.

An easy consequence of the lemma is

Theorem [Seebach 1994, Walser 2012]. *The centroid of vertices of a non-crossed quadrangle lies on the segment connecting the intersection point of the diagonals to the centroid of area and divides this segment in the ratio* 3 : 1. [11], [13]



The lemma implies that the homothety with center $S_{\mathcal{E}}$ and ratio 3:1 maps the quadrangle $ABCD$ to the quadrangle $S_AS_BS_CS_D$ (see also [7]).

Since a homothety preserves lines and incidence, it maps the intersection point S of the diagonals of $ABCD$ to the intersection point of the diagonals of $S_A S_B S_C S_D$. Since the points $S_X,\ X \in \{A, B, C, D\}$ are also the centroids of area of the corresponding triangles the latter intersection point is the centroid $S_{\mathcal{F}}$ of area of the quadrangle $ABCD$. This is evident for convex triangles but holds also for concave ones, see [10]. This finishes the proof of the theorem.

The theorem says that for non-crossed quadrangles which are not parallelograms the points S, $S_{\mathcal{E}}$, $S_{\mathcal{F}}$ lie on a line which we call the *Seebach-Walser line* of the quadrangle (unless somebody informs us that someone else anticipated Seebach's discovery; we keep the name of Walser since he informed us about this fact).

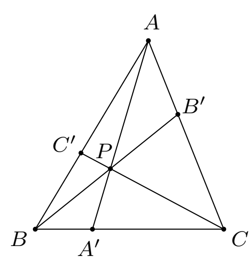
The described homothety exists also for crossed quadrangles so one can take the image of the intersection point S as analogue for the centroid of area for such quadrangles.

Remark on van Aubel's Sum Theorem

Henricus Hubertus van Aubel (* Maastricht, November 20, 1830, † Antwerpen, February 3, 1906) taught mathematics at the Athenäum in Antwerpen. Two theorems in elementary geometry are attributed to him. His Square Theorem will be discussed in a forthcoming essay [5]. We used the Sum Theorem in the synthetic proof of the lemma in the preceding section. Since a correct proof of this theorem is not easy to find in the literature we present one.

Theorem [van Aubel]. *Let A', B' C' points of the sides (considered as full lines, not only as segments) BC, CA, AB of the triangle ABC such that the lines AA', BB', CC' concur in a point P. Then*

$$\frac{AP}{PA'} = \frac{AB'}{B'C} + \frac{AC'}{C'B}.$$



We apply Menelaos' theorem to the triangles ABA' and ACA':

$$\frac{AC'}{C'B} \cdot \frac{BC}{CA'} \cdot \frac{A'P}{PA} = -1,$$
$$\frac{AB'}{B'C} \cdot \frac{CB}{BA'} \cdot \frac{A'P}{PA} = -1,$$

so

$$\frac{AC'}{C'B} = -\frac{CA'}{BC} \cdot \frac{PA}{A'P},$$
$$\frac{AB'}{B'C} = -\frac{BA'}{CB} \cdot \frac{PA}{A'P}.$$

Addition of the latter equations yields the desired result

$$\frac{AC'}{C'B} + \frac{AB'}{B'C} = -\left(\frac{A'C}{CB} + \frac{BA'}{CB}\right) \cdot \frac{PA}{A'P} = -\frac{BC}{CB} \cdot \frac{PA}{A'P} = \frac{PA}{A'P}.$$

Centroid of Vertices = Centroid of Area

The theorem leads to a further simple proof of the fact mentioned in the introduction that a quadrangle whose centroid of vertices coincides with its centroid of area is a parallelogram. Under this assumption the theorem implies $S = S_{\mathcal{E}} = S_{\mathcal{F}}$. We use vector algebra and take this point again as origin. Then the vectors $\vec{a}$, $\vec{c}$ are linearly dependent as well as the vectors $\vec{b}$, $\vec{d}$. Since according to our general assumptions the intersection point S of the diagonals must be different from all the vertices, we have $\vec{a} \neq 0$, $\vec{b} \neq 0$; thus there are scalars r, s such that $\vec{c} = r \cdot \vec{a}$, $\vec{d} = s \cdot \vec{b}$. Using equation (1) we compute

$$\vec{0} = \vec{a} + \vec{b} + \vec{c} + \vec{d} = (1 + r) \cdot \vec{a} + (1 + s) \cdot \vec{b}.$$

Since the vertex B does not lie on the diagonal $AC = AS$ the vectors $\vec{a}$ and $\vec{b}$ are linearly independent. Thus $r = s = -1$ which implies that the quadrangle is a parallelogram.

Preview

We shall continue these thoughts in two further notes. In [4] we discuss the relations of the centroid of perimeter to the centroids of vertices and area. We show the equivalence,

"the centroid of vertices coincides with the centroid of perimeter ⇔
⇔ the quadrangle is a parallelogram"

and give counterexamples to the conjecture that the centroid of area coincides with the centroid of perimeter only for parallelograms. In [5] we connect the centroid of vertices to van Aubel's Square Theorem [1].

References

[1] H. H. van Aubel, *Note concernant les centres de carrés construits sur les côtés d'un polygon quelconque,* Nouvelle Correspondance Mathématique, **4** 1878, pp. 40-44

[2] R. Fritsch, *Zum Flächenschwerpunkt für Vierecke,* Der mathematische und naturwissenschaftliche Unterricht, **65** 2012, pp. 464-465

[3] R. Fritsch, G. Pickert, *Vierecksschwerpunkte I, Die Wurzel* **48**(2) (2014); parts II and III are to appear in issues (3/4) and (5).

[4] R. Fritsch, G. Pickert, *A Quadrangle's Centroid of Perimeter,* to appear in CRUX.

[5] R. Fritsch, G. Pickert, *A Quadrangle's Centroid of Vertices and van Aubel's Square Theorem,* to appear in CRUX.

[6] C. Kimberling, *Encyclopedia of Triangle Centers – ETC,* http://faculty.evansville.edu/ck6/encyclopedia/ETC.html

[7] J. Kratz, *"Das Schwerpunktsviereck" - Eine Ergänzung zum Beitrag von Karl Seebach über Vierecksschwerpunkte,* Didaktik der Mathematik, **22** 1994, pp. 316-317

[8] H. Lenz, *Grundlagen der Elementarmathematik,* München: 1976 Hanser

[9] G. Pickert (communicated by R. Fritsch), *Zu: Zum Flächenschwerpunkt für Vierecke,* Der mathematische und naturwissenschaftliche Unterricht, **66** 2013, pp. 51-52

[10] K. Seebach, *Über Schwerpunkte von Dreiecken, Vierecken und Tetraedern, Teil 1,* Didaktik der Mathematik, **11** 1983, pp. 270-282

[11] K. Seebach, *Nochmals Vierecksschwerpunkte,* Didaktik der Mathematik, **22** 1994, pp. 309-315

[12] M. de Villiers, *Some Adventures in Euclidean Geometry,* Pinetown: 2009, pp. 154-155

[13] H. Walser, *Schwerpunkt,* Mathematikinformation, **57** 2012, pp. 14-22

Rudolf Fritsch
Mathematisches Institut
Ludwig-Maxmilians-Universität München
Munich, Germany
fritsch@math.lmu.de

Günter Pickert
Mathematisches Institut
Justus-Liebig-Universität Giessen
Giessen, Germany

PROBLEMS

Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème présenté dans cette section. Nous préférons les réponses électroniques et demandons aux lecteurs de présenter chaque solution dans un fichier distinct. Il est recommandé de nommer les fichiers de la manière suivante : Nom de famille_Prénom_Numéro du problème *(exemple : Tremblay_Julie_1234.tex). De préférence, les lecteurs enverront un fichier au format LaTeX et un fichier pdf pour chaque solution, bien que les autres formats (Word, etc.) soient aussi acceptés. Nous invitons les lecteurs à envoyer leurs solutions au rédacteur à l'adresse* `crux-redacteurs@smc.math.ca`. *Nous acceptons aussi les contributions par la poste, envoyées à l'adresse figurant en troisième de couverture. Le nom de la personne qui propose une solution doit figurer avec chaque solution, de même que l'tablissement qu'elle fréquente, sa ville et son pays; chaque solution doit également commencer sur une nouvelle page. Un astérisque ($\star$) signale un problème proposé sans solution.*

Nous sommes surtout à la recherche de problèmes originaux, mais d'autres problèmes intéressants peuvent aussi être acceptables pourvu qu'ils ne soient pas trop connus et que leur provenance soit indiquée. Normalement, si l'on connaît l'auteur d'un problème, on ne doit pas le proposer sans lui en demander la permission. Les solutions connues doivent accompagner les problèmes proposés. Si la solution n'est pas connue, la personne qui propose le problème doit tenter de justifier l'existence d'une solution. Il est recommandé de nommer les fichiers de la manière suivante : Nom de famille_Prénom_Proposition_Année_numéro *(exemple : Tremblay_Julie_Proposition_2014_4.tex, s'il s'agit du 4e problème proposé par Julie en 2014).*

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er août 2014***; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

Chaque problème est présenté en anglais et en français, les deux langues officielles du Canada. Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le français précédera l'anglais. Dans la section Solutions, le problème sera écrit dans la langue de la première solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal d'avoir traduit les problèmes.

3831. *Proposé par George Apostolopoulos, Messolonghi, Grèce.*

Soit a, b, c, d quatre nombres réels positifs avec $abcd = 16$. Montrer que

$$\frac{a^3}{b^3} + \frac{b^3}{c^3} + \frac{c^3}{d^3} + \frac{d^3}{a^3} + 4 \geq a + b + c + d\,.$$

3832. *Proposé par Marcel Chiriţă, Bucarest, Roumanie.*

Soit P un point intérieur du triangle équilatéral ABC de côtés mesurant 1 et soit $x = PA$, $y = PB$, $z = PC$. Montrer que

$$(x^2 + y^2 + z^2) + (x^2 + y^2 + z^2 - 1)^2 = 3(x^2y^2 + y^2z^2 + z^2x^2)\,.$$

3833. *Proposé par Angel Plaza, University of Las Palmas de Gran Canaria, Spain.*

Soit x, y, z trois nombres réels positifs. Montrer que

$$\frac{x^2}{z^3(zx+y^2)}+\frac{y^2}{x^3(xy+z^2)}+\frac{z^2}{y^3(yz+x^2)}\geq\frac{3}{2xyz}\,.$$

3834. *Proposé par George Apostolopoulos, Messolonghi, Grèce.*

Soit $ABCD$ un parallélogramme et respectivement E, F des points intérieurs des côtés BC, CD, tels que $\frac{BE}{EC}=\frac{CF}{FD}$. Soit respectivement K et L les points d'intersection des segments AE et AF avec la diagonale BD.

(a) Montrer que $\text{Aire}(\Delta AKL)=\text{Aire}(\Delta BKE)+\text{Aire}(\Delta DLF)$.

(b) Trouver le quotient $\dfrac{\text{Aire}(\Delta ABCD)}{\text{Aire}(\Delta AECF)}$.

3835. *Proposé par Marcel Chiriţă, Bucarest, Roumanie.*

Déterminer les fonctions $f:\mathbb{R}\to\mathbb{R}$, continues en $x=0$, telles que $f(0)=1$ et

$$3f(x)-5f(\alpha x)+2f(\alpha^2x)=x^2+x\,,$$

pour tout $x\in\mathbb{R}$, où $\alpha\in(0,1)$ est donné.

3836. *Proposé par Jung In Lee, École Secondaire Scientifique de Séoul, Séoul, République de Corée.*

Trouver tous les triplets (a,b,c) d'entiers positifs satisfaisant

$$a!+b^b=c!$$

3837. *Proposé par Arkady Alt, San José, CA, É-U.*

Soit $(u_n)_{n\geq 0}$ une suite définie récursivement par

$$u_{n+1}=\frac{u_n+u_{n-1}+u_{n-2}+u_{n-3}}{4},$$

pour $n\geq 3$. Calculer $\lim\limits_{n\to\infty}u_n$ en termes de u_0,u_1,u_2,u_3.

3838. *Proposé par Jung In Lee, École Secondaire Scientifique de Séoul, Séoul, République de Corée.*

Montrer qu'il n'existe aucun triplet (a,b,c) d'entiers positifs distincts satisfaisant les conditions

- $a+b$ divise c^2, $b+c$ divise a^2, $c+a$ divise b^2, et
- le nombre de facteurs premiers distincts de abc est au plus 2.

3839. *Proposé par Peter Y. Woo, Université Biola, La Mirada, CA, É-U.*

Soit ΔABC un triangle acutangle, et P un point quelconque du plan. Soit AD, BE, CF les hauteurs de ΔABC. Soit respectivement D', E', F' les centres des cercles circonscrits de ΔPAD, ΔPBE, ΔPCF. Montrer que D', E', F' sont colinéaires.

3840$\star$. *Proposé par Šefket Arslanagić, Université de Sarajevo, Sarajevo, Bosnie et Herzégovine.*

Vérifier la véracité de l'inégalité

$$a^3c + ab^3 + bc^3 \geq a^2b^2 + b^2c^2 + c^2a^2\,,$$

où $a, b, c > 0$.

. . . .

3831. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let a, b, c, d be positive real numbers with $abcd = 16$. Prove that

$$\frac{a^3}{b^3} + \frac{b^3}{c^3} + \frac{c^3}{d^3} + \frac{d^3}{a^3} + 4 \geq a + b + c + d\,.$$

3832. *Proposed by Marcel Chiriţă, Bucharest, Romania.*

Let P be a point inside the equilateral triangle ABC with side length equal to 1, and let $x = PA$, $y = PB$, $z = PC$. Prove that:

$$(x^2 + y^2 + z^2) + (x^2 + y^2 + z^2 - 1)^2 = 3(x^2y^2 + y^2z^2 + z^2x^2)\,.$$

3833. *Proposed by Angel Plaza, University of Las Palmas de Gran Canaria, Spain.*

Let x, y, z be positive real numbers. Prove that

$$\frac{x^2}{z^3(zx + y^2)} + \frac{y^2}{x^3(xy + z^2)} + \frac{z^2}{y^3(yz + x^2)} \geq \frac{3}{2xyz}\,.$$

3834. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let $ABCD$ be a parallelogram and E, F be interior points of the sides BC, CD, respectively, such that $\frac{BE}{EC} = \frac{CF}{FD}$. The line segments AE and AF meet the diagonal BD at the points K and L respectively.

(a) Prove that $\text{Area}(\Delta AKL) = \text{Area}(\Delta BKE) + \text{Area}(\Delta DLF)$.

(b) Find the ratio $\dfrac{\text{Area}(\Delta ABCD)}{\text{Area}(\Delta AECF)}$.

3835. *Proposed by Marcel Chiriţă, Bucharest, Romania.*

Determine the functions $f : \mathbb{R} \to \mathbb{R}$, continuous at $x = 0$, for which $f(0) = 1$ and

$$3f(x) - 5f(\alpha x) + 2f(\alpha^2 x) = x^2 + x\,,$$

for all $x \in \mathbb{R}$, where $\alpha \in (0, 1)$ is fixed.

3836. *Proposed by Jung In Lee, Seoul Science High School, Seoul, Republic of Korea.*

Determine all triplets (a, b, c) of positive integers that satisfy

$$a! + b^b = c!$$

3837. *Proposed by Arkady Alt, San Jose, CA, USA.*

Let $(u_n)_{n\ge 0}$ be a sequence defined recursively by

$$u_{n+1} = \frac{u_n + u_{n-1} + u_{n-2} + u_{n-3}}{4},$$

for $n \ge 3$. Determine $\lim\limits_{n\to\infty} u_n$ in terms of u_0, u_1, u_2, u_3.

3838. *Proposed by Jung In Lee, Seoul Science High School, Seoul, Republic of Korea.*

Prove that there are no triplets (a, b, c) of distinct positive integers that satisfy the conditions:

- $a + b$ divides c^2, $b + c$ divides a^2, $c + a$ divides b^2, and
- the number of distinct prime factors of abc is at most 2.

3839. *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Let ΔABC be an acute triangle, and P any point on the plane. Let AD, BE, CF be the altitudes of ΔABC. Let D', E', F' be the circumcentres of ΔPAD, ΔPBE, ΔPCF respectively. Prove that D', E', F' are collinear.

3840★. *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Prove or disprove

$$a^3c + ab^3 + bc^3 \ge a^2b^2 + b^2c^2 + c^2a^2\,,$$

where $a, b, c > 0$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3731. [2012 : 149, 151] *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$a^{n+1}+b^{n+1}+c^{n+1} \geq (a^2+b^2+c^2)^n,$$

for all nonnegative integers n.

Solution by Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, TX, USA.

For $n=0$ or $n=1$ it is easy to check that equality holds, so the statement is true.

Suppose $n \geq 2$. First, notice that the function $f(x)=x^n$ is convex on $(0,\infty)$, as $f''(x)=n(n-1)x^{n-2}>0$ for $x>0$. Since $a,b,c>0$ and $a+b+c=1$, by Jensen's inequality we have

$$f(ax+by+cz) \leq af(x)+bf(y)+cf(z)$$

for all $x,y,z>0$, with equality if and only if $x=y=z$. Then, it follows that

$$\begin{aligned} a^{n+1}+b^{n+1}+c^{n+1} &= af(a)+bf(b)+cf(c) \\ &\geq f(a\cdot a+b\cdot b+c\cdot c) \\ &= \left(a^2+b^2+c^2\right)^n, \end{aligned}$$

with equality attained if and only if $a=b=c=\frac{1}{3}$, which concludes the proof.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina(2 solutions); MICHEL BATAILLE, Rouen, France; D. M. BĂTINEŢU-GIURGIU, Bucharest, Romania and NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; MARIAN DINCĂ, Bucharest, Romania; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College, Saratoga Springs, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; ITACHI UCHIHA, Hong Kong, China; DANIEL VĂCARU, Piteşti, Romania; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; TITU ZVONARU, Cománeşti, Romania; and the proposer.

3732. [2012 : 149, 151] *Proposed by Ataev Farrukh Rakhimjanovich, Westminster International University, Tashkent, Uzbekistan.*

A circle of radius 1 is rolling on the x-axis in the first quadrant towards the parabola with equation $y = x^2$. Find the coordinates of the point of contact when the circle hits the parabola.

I. Solution by Missouri State University Problem Solving Group, Springfield, MO, USA.

If the contact point on the parabola is (t, t^2), the slope of the common tangent there is $2t$, so the downward-pointing unit normal is

$$\frac{1}{\sqrt{4t^2+1}}\,(2t,\ -1)\,,$$

and the centre of the rolling unit circle, when tangent to the parabola, has y-coordinate $t^2 - \frac{1}{\sqrt{4t^2+1}}$. Hence,

$$\begin{aligned} t^2 - \frac{1}{\sqrt{4t^2+1}} &= 1 \\ t^2 - 1 &= \frac{1}{\sqrt{4t^2+1}} \\ (t^2-1)^2 &= \frac{1}{4t^2+1} \\ 4t^6 - 7t^4 + 2t^2 &= t^2(4t^4 - 7t^2 + 2) = 0. \end{aligned}$$

Because we must have $t > 1$ for the y-coordinate of the contact point to exceed that of the centre, the only valid root is

$$t = \frac{\sqrt{14+2\sqrt{17}}}{4},$$

so the point of contact is

$$\left(\frac{\sqrt{14+2\sqrt{17}}}{4},\ \frac{7+\sqrt{17}}{8}\right).$$

II. Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.

Let $T = (x, y)$ be the point where the circle touches the parabola, and C be the centre of the circle. Then $TC = 1$. If the slope of TC is $-\tan\theta$ then

$$y = 1 + \sin\theta = x^2. \qquad (1)$$

The tangent to the parabola at T has slope $2x$; because it is perpendicular to TC, we must have

$$2x = \cot\theta. \qquad (2)$$

Combining equations (1) and (2), we get

$$1+\sin\theta = \frac{1}{4}\cot^2\theta = \frac{1-\sin^2\theta}{4\sin^2\theta},$$

or, because $\sin\theta \neq -1$,

$$4\sin^2\theta = 1-\sin\theta.$$

Because we want $\sin\theta > 0$, we want the root $\sin\theta = \dfrac{-1+\sqrt{17}}{8}$, whence $y = \dfrac{7+\sqrt{17}}{8}$ and $x = \sqrt{\dfrac{7+\sqrt{17}}{8}}$. Thus,

$$T = \left(\sqrt{\frac{7+\sqrt{17}}{8}},\ \frac{7+\sqrt{17}}{8}\right).$$

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, EMILIO FERNANDEZ, and LUZ RONCAL, Logroño, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College, Saratoga Springs, NY, USA; DIGBY SMITH, Mount Royal University, Calgary, AB; MIHAÏ STOËNESCU, Bischwiller, France; EDMUND SWYLAN, Riga, Latvia; ITACHI UCHIHA, Hong Kong, China; TITU ZVONARU, Comăneşti, Romania; and the proposer.

The Skidmore College Problem Group observed that the rolling unit circle is tangent to the parabola again at $\left(\frac{\sqrt{14-2\sqrt{17}}}{4}, \frac{7-\sqrt{17}}{8}\right)$*, the point obtained using an upward pointing unit normal in the first of the featured solutions (and the smaller positive root of the resulting quartic).*

3733. [2012 : 149, 151] *Proposed by Angel Dorito, Geld, ON.*

Suppose that f and g are different, nonconstant polynomials of degree at most 2 so that $f(x)-g(x) = f(g(x)) - g(f(x))$ for all real numbers x. Prove that exactly one of the two functions, f or g, must be linear and find all possible values of its slope.

Solution by Oliver Geupel, Brühl, NRW, Germany; and Missouri State University Problem Solving Group, Springfield, MO, USA; (independently).

Let $f(x) = a_2x^2 + a_1x + a_0$ and $g(x) = b_2x^2 + b_1x + b_0$. It is straightforward to check that $f(x) - g(x) - f(g(x)) + g(f(x)) = 0$ identically if and only if

$$0 = a_2b_2(b_2 - a_2) \quad (1)$$
$$0 = 2a_2b_2(b_1 - a_1) \quad (2)$$
$$0 = a_2b_1^2 - a_1^2b_2 + 2a_2b_2(b_0 - a_0) + a_1b_2 - a_2b_1 + b_2 - a_2 \quad (3)$$
$$0 = 2a_2b_0b_1 - 2a_0a_1b_2 + b_1 - a_1 \quad (4)$$
$$0 = a_2b_0^2 - a_0^2b_2 + a_1b_0 - a_0b_1. \quad (5)$$

From (1), we have that $a_2 = b_2 = 0$, $a_2 = b_2 \neq 0$ or that exactly one of a_2 and b_2 vanishes.

If $a_2 = b_2 = 0$, then by (4) and the condition that the polynomials be nonconstant, $a_1 = b_1 \neq 0$. But then, by (5), $a_0 = b_0$, and $f = g$ which is not permitted.

If $a_2 = b_2 \neq 0$, then by (2) and (3) successively, $a_1 = b_1$ and $a_0 = b_0$ which again is not permitted.

In the third instance, there is no loss of generality in assuming that $a_2 = 0$ and $b_2 \neq 0$, whence the desired result. In addition, by (3), we must have that a_1 is a root of the polynomial $t^2 - t - 1$, namely $\frac{1}{2}(1 \pm \sqrt{5})$.

Let τ be such a root. An example of a pair of polynomials satisfying the conditions of the problem is given by $f(x) = \tau x$ and $g(x) = x^2 + \tau x$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College, Saratoga Springs, NY, USA; DIGBY SMITH, Mount Royal University, Calgary, AB; EDMUND SWYLAN, Riga, Latvia; TITU ZVONARU, Comăneşti, Romania; and the proposer. Some solutions lacked examples confirming the possible values of a_1 and one had an incorrect example.

3734. [2012 : 149, 151] *Proposed by Nguyen Thanh Binh, Hanoi, Vietnam.*

Given a pair of triangles ABC and DEF with a point D' on BC.

(a) Describe how to locate points E' on CA and F' on AB such that $\Delta D'E'F'$ is directly similar to ΔDEF.

(b) For which point D' on BC is the area of $\Delta D'E'F'$ (from part (a)) a minimum?

Solution by Michel Bataille, Rouen, France.

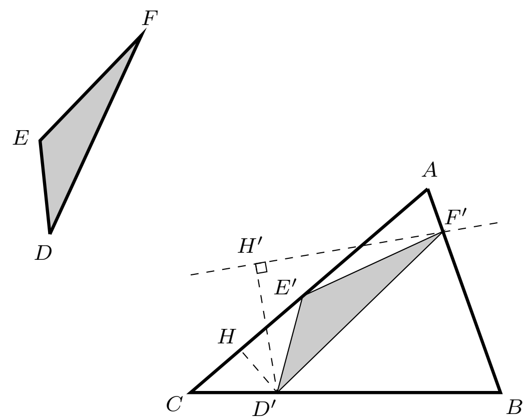
(a) Let σ be the spiral similarity with centre D', ratio $\frac{DF}{DE}$ and directed angle $\angle(\overrightarrow{DE}, \overrightarrow{DF})$. Triangle $D'E'F'$ is suitable if and only if E' is on AC, F' is on AB and $F' = \sigma(E')$; that is, if and only if F' is on AB and on the image of the line AC under σ, with $E' = \sigma^{-1}(F')$. Thus, the construction is as follows:

(1) Draw the orthogonal projection H of D' onto AC and $H' = \sigma(H)$.

(2) Draw the perpendicular to $D'H'$ at H' to meet AB at F'.

(3) Draw $E' = \sigma^{-1}(F')$. (See the accompanying figure.)

The construction fails if $D'H'$ is perpendicular to AB (then $\sigma(AC)$ is parallel to AB). Clearly this occurs if and only if $\angle(\overrightarrow{DE},\overrightarrow{DF}) = 180^\circ - \angle(\overrightarrow{AB},\overrightarrow{AC})$. If such is the case, there is no suitable triangle except if $\frac{d(D',AC)}{d(D',AB)} = \frac{DE}{DF}$, in which case any point of AB is a suitable F'.



(b) Let $\mathcal{T}$ denote the family of all triangles $D'E'F'$ constructed as in (a). Note that for such a triangle, the area $[D'E'F']$ is $\frac{1}{2}D'E' \cdot D'F' \sin(\angle E'D'F')$, or, since $\angle E'D'F' = \angle EDF$ and $\frac{D'F'}{D'E'} = \frac{DF}{DE}$,

$$[D'E'F'] = \frac{1}{2} \cdot D'E'^2 \cdot \frac{DF}{DE} \cdot \sin(\angle EDF).$$

Thus, $[D'E'F']$ is minimum when $D'E'$ is minimum.

Any two triangles of $\mathcal{T}$ are directly similar (since they are both directly similar to ΔDEF). We know that the centre of the spiral similarity transforming one (say $D'E'F'$) into the other (say $D_1E_1F_1$) is a point O independent of the chosen triangles, obtained as the second point of intersection of the circumcircles of $BD'F'$ and BD_1F_1. (See, for example, I.M. Yaglom, *Geometric Transformations II*, Random House, 1968, p. 68-9).

Now, with the same notation, we have $\frac{OD_1}{OD'} = \frac{D_1E_1}{D'E'}$, and so $D_1E_1 \geq D'E'$ if and only if $OD_1 \geq OD'$. Thus, $D'E'$ and the area $[D'E'F']$ are minimal when OD' is minimal, and this occurs when D' is the orthogonal projection of O onto BC.

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Geupel, Woo, and the proposer all showed implicitly that the point O of the featured solution is the Miquel point *of the triangle $D'E'F'$ inscribed in ΔABC—the point common to the circles $AE'F'$, $BD'F'$, and $CE'D'$. Using O as pivot, imagine the lines OD', OE', and OF' rotated, as if they formed a rigid structure, about the point O to a position where the given lines intersect BC, CA, and AB respectively in points D_1, E_1, and F_1; these points are the vertices of a triangle similar to $\Delta D'E'F'$, which also has O as its Miquel point. All triangles of $\mathcal{T}$ can be constructed in this way. This result is part of the solution to problem 1755 [1992: 176; 1993 : 152-153]. The complete story can be found in Roger A. Johnson,* Advanced Euclidean Geometry*, Dover, 1960, paragraphs 184-190.*

3735. [2012 : 150, 151] *Proposed by Yakub N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan.*

Let A, B, C, and D be points on line ℓ in this order, and let M be a point not on ℓ such that $\angle AMB = \angle CMD$. Prove that

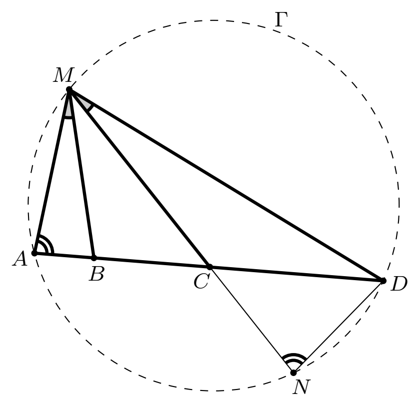
$$\frac{\sin\angle BMC}{\sin\angle AMD} > \frac{|BC|}{|AD|}.$$

Solution by George Apostolopoulos, Messolonghi, Greece.

First, notice that

$$\frac{|BC|}{|AD|} = \frac{[MBC]}{[MAD]} = \frac{|MB|\cdot|MC|\cdot\sin\angle BMC}{|MA|\cdot|MD|\cdot\sin\angle AMD}.$$

Therefore, it suffices to prove that $|MA|\cdot|MD| > |MB|\cdot|MC|$.



Let Γ be the circumcircle of $\triangle MAD$. Since MC is inside Γ, we can extend the ray MC beyond C to intersect Γ at a point N. Now, we have $\angle MAD = \angle MND$ and since $\angle AMB = \angle CMD$, the triangles MBA and MDN are similar. Hence $|MA|\cdot|MD| = |MB|\cdot|MN| > |MB|\cdot|MC|$ which completes the proof.

Also solved by MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; MARIAN DINCĂ, Bucharest, Romania; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; EDMUND SWYLAN, Riga, Latvia; DANIEL VĂCARU, Piteşti, Romania; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3736. [2012 : 150, 151] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $f : [0, \infty) \to [0, \infty)$ be a bounded continuous function. Evaluate

$$\lim_{n\to\infty} \frac{1}{n} \sqrt[n]{\int_0^\infty \ln^n(1 + e^{nf(x)})e^{-x}\,dx}\,.$$

Solution.

Let

$$I_n = \frac{1}{n}\left(\int_0^\infty \ln^n(1 + e^{nf(x)})e^{-x}dx\right)^{1/n}.$$

We show that $\lim\limits_{n\to\infty} I_n = \|f\|_\infty \equiv \sup\{f(x) : x \geq 0\}$.

Observe that

$$nf(x) \leq \ln(1 + e^{nf(x)}) \leq \ln(2e^{nf(x)}) = \ln 2 + nf(x) \leq \ln 2 + n\|f(x)\|_\infty.$$

Therefore

$$I_n \leq \frac{1}{n}(\ln 2 + n\|f\|_\infty)\left(\int_0^\infty e^{-x}dx\right)^{1/n} = \frac{\ln 2}{n} + \|f\|_\infty,$$

whence $\limsup\limits_{n\to\infty} I_n \leq \|f\|_\infty$.

On the other hand,

$$I_n \geq \frac{1}{n}\left(\int_0^\infty n^n f(x)^n e^{-x}dx\right)^{1/n} = \left(\int_0^\infty f(x)^n e^{-x}dx\right)^{1/n}.$$

Let $\epsilon > 0$ and select an interval $[a, b] \subseteq [0, \infty)$ for which $f(x) > \|f\|_\infty - \epsilon$ for $x \in [a, b]$. Then

$$\begin{aligned}\left(\int_0^\infty f(x)^n e^{-x}dx\right)^{1/n} &\geq \left(\int_a^b f(x)^n e^{-x}dx\right)^{1/n} \geq (\|f\|_\infty - \epsilon)\left(\int_a^b e^{-x}dx\right)^{1/n}\\ &= (\|f\|_\infty - \epsilon)[e^{-a} - e^{-b}]^{1/n},\end{aligned}$$

so that $\liminf\limits_{n\to\infty}\left(\int_0^\infty f(x)^n e^{-x}dx\right)^{1/n} \geq \|f\|_\infty - \epsilon$ for each $\epsilon > 0$. It follows that $\liminf\limits_{n\to\infty} I_n \geq \|f\|_\infty$ and the desired result follows.

The solution is based on those of the following solvers: MICHEL BATAILLE, Rouen, France; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; KEE-WAI LAU, Hong Kong, China; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; and the proposer. Two additional solutions submitted were flawed.

3737. [2012 : 150, 152] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Four nonnegative real numbers a, b, c, d are given. Find the greatest number k such that the following inequality is valid.

$$\frac{1}{a^3+b^3}+\frac{1}{b^3+c^3}+\frac{1}{c^3+d^3}+\frac{1}{a^3+c^3}+\frac{1}{b^3+d^3}+\frac{1}{a^3+d^3}\geq\frac{k}{(a+b+c+d)^3}.$$

Solution by the proposer, modified by the editor.

Setting $a=0$, $b=c=d=m>0$, the given inequality becomes

$$\frac{3}{m^3}+\frac{3}{2m^3}\geq\frac{k}{(3m)^3}$$

or $\frac{9}{2}\geq\frac{k}{27}$ so $k\leq\frac{243}{2}$. We now prove that $k=\frac{243}{2}$.

By symmetry, we may also assume, without loss of generality, that $d=\min\{a,b,c,d\}$. Let $x=a+\frac{d}{3}$, $y=b+\frac{d}{3}$, and $z=c+\frac{d}{3}$. Then

$$x^3=\left(a+\frac{d}{3}\right)^3\geq a^3+3a^2\left(\frac{d}{3}\right)=a^3+a^2d\geq a^3+d^3.$$

Similarly, $y^3\geq b^3+d^3$ and $z^3\geq c^3+d^3$. Furthermore,

$$a^3+b^3\leq\left(a+\frac{d}{3}\right)^3+\left(b+\frac{d}{3}\right)^3=x^3+y^3\,.$$

Similarly $b^3+c^3\leq y^3+z^3$ and $c^3+a^3\leq z^3+x^3$. Since $x+y+z=a+b+c+d$ it then suffices to show that

$$\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}+\frac{1}{x^3+y^3}+\frac{1}{y^3+z^3}+\frac{1}{z^3+x^3}\geq\frac{243}{2(x+y+z)^3}\,. \tag{1}$$

By the AM-GM Inequality we have

$$\begin{aligned}\frac{1}{x^3}+\frac{1}{y^3}+\frac{2}{x^3+y^3}&\geq 3\left(\frac{2}{x^3y^3(x^3+y^3)}\right)^{1/3}\\&=3\left(\frac{2}{x^3y^3(x+y)(x^2-xy+y^2)}\right)^{1/3}\end{aligned} \tag{2}$$

and

$$\begin{aligned}x^3y^3(x^2-xy+y^2)&=(xy)^3(x^2-xy+y^3)\\&\leq\left(\frac{3xy+x^2-xy+y^2}{4}\right)^4=\frac{(x+y)^8}{256}\,.\end{aligned} \tag{3}$$

It follows from (2) and (3) that

$$\frac{1}{x^3}+\frac{1}{y^3}+\frac{2}{x^3+y^3}\geq 3\left(\frac{2\cdot 256}{(x+y)^9}\right)^{1/3}=\frac{24}{(x+y)^3}\,. \tag{4}$$

Similarly,

$$\frac{1}{y^3} + \frac{1}{z^3} + \frac{2}{y^3+z^3} \geq \frac{24}{(y+z)^3} \tag{5}$$

and

$$\frac{1}{z^3} + \frac{1}{x^3} + \frac{2}{z^3+x^3} \geq \frac{24}{(z+x)^3}\,. \tag{6}$$

Denote the left hand side of (1) by L. We have by adding (4), (5) and (6) that

$$L \geq 12\left(\frac{1}{(x+y)^3} + \frac{1}{(y+z)^3} + \frac{1}{(z+x)^3}\right) \geq \frac{36}{(x+y)(y+z)(z+x)}\,. \tag{7}$$

Finally,

$$(x+y)(y+z)(z+x) \leq \left(\frac{2(x+y+z)}{3}\right)^3 = \frac{8(x+y+z)^3}{27}\,. \tag{8}$$

Substituting (8) into (7) then yields

$$L \geq \frac{36\cdot 27}{8(x+y+z)^3} = \frac{243}{2(x+y+z)^3}$$

establishing (1) and the proof is complete.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; and PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.

3738. [2012 : 150, 152] *Proposed by Michel Bataille, Rouen, France.*

Let triangle ABC be inscribed in circle Γ and let A' and B' be the feet of the altitudes from A and B, respectively. Let the circle with diameter BA' intersect BB' a second time at M and Γ at P. Prove that A, M, P are collinear.

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA, modified by the editor to exploit directed angles.

We prove that the result holds for any point A' on the line BC, not just for when A' is the foot of the altitude. To avoid the need for special cases we use directed angles; thus $\angle XYZ$ denotes the angle through which the line YX must be rotated about the point Y in the positive direction in order to coincide with YZ. Thus in the circle with diameter BA',

$$\angle BPM = \angle BA'M$$

whether or not the points P and A' are located on the same side of the chord BM. Since BA' is the diameter of the circle, $BM \perp MA'$; but BM indicates the same line as the altitude BB', which is also perpendicular to AC. Consequently, $MA'||AC$ and

$$\angle BA'M = \angle BCA.$$

Finally, in circle Γ,

$$\angle BCA = \angle BPA.$$

We conclude that $\angle BPM = \angle BPA$, which forces P, M, and A to lie on the same line, as desired.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece (2 solutions); ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; CRISTÓBAL SÁNCHEZ–RUBIO, I.B. Penyagolosa, Castellón, Spain; MIHAÏ STOËNESCU, Bischwiller, France; EDMUND SWYLAN, Riga, Latvia; ITACHI UCHIHA, Hong Kong, China; JACQUES VERNIN, Marseille, France; HUONG VU, Washington and Lee University, Lexington, VA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

Although most of the submitted solutions were similar to the featured solution, only Woo observed that it was not necessary to require A' to be the foot of the altitude from A.

3739. [2012 : 150, 152] *Proposed by Cristinel Mortici, Valahia University of Târgovişte, Romania.*

Let $f, g : \mathbb{R} \to \mathbb{R}$ be strictly monotone functions. Prove that:

(i) If $f \circ f + g \circ g$ is continuous, then f and g are continuous.

(ii) If $f \circ g + g \circ f$ is continuous, then f and g are continuous.

Solution adapted from that of the proposer.

We begin with two lemmas.

Lemma 1. Suppose that u and v are both increasing or both decreasing functions from $\mathbb{R}$ to $\mathbb{R}$ and that $u + v$ is continuous. Then u and v are both continuous.

Proof. For any $a \in \mathbb{R}$, we have that

$$\begin{aligned} 0 &= \lim_{x\to a^+} (u+v)(x) - \lim_{x\to a^-} (u+v)(x) \\ &= \left[\lim_{x\to a^+} u(x) - \lim_{x\to a^-} u(x)\right] + \left[\lim_{x\to a^+} v(x) - \lim_{x\to a^-} v(x)\right]. \end{aligned}$$

Since the terms in square brackets are either both nonnegative or both nonpositive, they must vanish and the result follows. □

Lemma 2. Suppose that u and v are strictly monotone functions from $\mathbb{R}$ to $\mathbb{R}$ and that $u \circ v$ is continuous. Then v is continuous.

Proof. Assume that u and v are both increasing; the other cases are similarly handled. If v were not continuous at b, then there would exist numbers r and s for which, whenever $x < b < y$, $v(x) \le r < s \le v(y)$. Then

$$u(v(x)) \le u(r) < u(s) \le u(v(y))$$

and we would be led to a contradiction. □

In the situation of the problem, we note that $f \circ f$ and $g \circ g$ have the same monotonicity as do $f \circ g$ and $g \circ f$ and deduce from (i) or (ii) that the corresponding pair of functions is continuous. In either case, it follows that both f and g are continuous.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; EDMUND SWYLAN, Riga, Latvia; and the proposer. Note that in Lemma 2, it may happen that the function u could have a discontinuity outside of the range of v.

3740. [2012 : 150, 152] *Proposed by Yunus Tuncbilek, Ataturk High School of Science, Istanbul, Turkey.*

Let R, r, r_a, r_b, r_c represent the circumradius, inradius and exradii, respectively, of ΔABC. Find the largest k that satisfies

$$r_a^2 + r_b^2 + r_c^2 + (1+4k)r^2 \geq (7+k)R^2.$$

Composite of solutions by Michel Bataille, Rouen, France; and Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; modified by the editor.

We claim that the largest such k is $k = 1$. For the proof, we show that the inequality holds for $k = 1$ and fails for all larger k. We use two known results:

$$r_a^2 + r_b^2 + r_c^2 = (4R+r)^2 - 2s^2, \tag{1}$$

where s is the semiperimeter, and

$$4R^2 + 4Rr + 3r^2 \geq s^2, \tag{2}$$

known as Gerretsen's inequality. [See [1] and [2].]

By (1), the inequality in the problem statement is equivalent to

$$(4R+r)^2 - 2s^2 + (1+4k)\,r^2 \geq (7+k)\,R^2,$$

and thus to

$$9R^2 + 8Rr + 2r^2 - 2s^2 \geq k\left(R^2 - 4r^2\right). \tag{3}$$

If $k = 1$, (3) is equivalent to

$$8R^2 + 8Rr + 6r^2 - 2s^2 \geq 0,$$

which is equivalent to (2). Hence the inequality holds for $k = 1$.

To show that there is no larger k for which the inequality always holds, let $t \in (0,1)$ and $a = t$, $b = c = 1$. The semiperimeter of this triangle is $s = \dfrac{2+t}{2}$, its area is $F = \dfrac{t}{4}\sqrt{4-t^2}$, its circumradius is $R = \dfrac{abc}{4F} = \dfrac{1}{\sqrt{4-t^2}}$, and its inradius is $r = \dfrac{F}{s} = \dfrac{t\sqrt{4-t^2}}{2\,(2+t)}$. For this triangle, therefore, inequality (3) is successively equivalent to

$$\begin{aligned} \frac{9}{4-t^2}+\frac{4t}{2+t}+\frac{t^2\left(4-t^2\right)}{2\left(2+t\right)^2}-\frac{\left(2+t\right)^2}{2} &\geq k\left[\frac{1}{4-t^2}-\frac{t^2\left(4-t^2\right)}{\left(2+t\right)^2}\right] \\ \frac{\left(1+t\right)^2\left(1-t\right)^2}{\left(2+t\right)\left(2-t\right)} &\geq k\cdot\frac{\left(1-t\right)^2\left(1+2t-t^2\right)}{\left(2+t\right)\left(2-t\right)} \\ \frac{\left(1+t\right)^2}{1+2t-t^2} &\geq k. \qquad (4) \end{aligned}$$

Fix $k>1$ and let $t\to 0$. Since $\dfrac{\left(1+t\right)^2}{1+2t-t^2}\to 1$, (4) eventually fails.

References

[1] O. Bottema, R.Z. Djordjević, R.R. Janic, D.S. Mitrinović, P.M. Vasić, *Geometric Inequalities*, Wolters-Noordhoff Publishing, 1969, p. 50.

[2] K. W. Feuerbach, *Eigenschaften einiger merkwürdigen Punkte des geradlinigen Dreiecks*, 1822, p. 5.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; KEE-WAI LAU, Hong Kong, China; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. Four incomplete solutions were received.