Crux Mathematicorum

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EDITORIAL

Shawn Godin

Hello ***Crux Mathematicorum*** readers and welcome to Volume 39! The last volume was one of change and transition as we made some layout adjustments and developed a few new regular features. This next volume will see fewer changes as we focus on continuing the features introduced in Volume 38.

One change that we will be making is to the instructions in *The Contest Corner*, *The Olympiad Corner* and the *Problems* sections. Currently, much of the correspondence received by the editor is electronic, which is preferred. Most readers send in proposals and solutions as a file. Currently, the editor's "routine" is based upon a time when all things came through regular mail. The editor is in the process of bringing his process more or less up to date. As such, there are some things that *Crux* readers can do to make life easier for the editor.

- Send separate files for each item. Since there are many problem editors, if you solve several problems they are probably going to be sent to several different people for evaluation. If you do not use separate files, the editor has to spend time splitting your file into several files.
- Please put your name and personal information on everything. This just makes things easier to identify. Quite a few readers have it set up so that their name and personal information is in the header or footer of every page and this works great.
- PDF is the desired format for everything. If you did your solution in LaTeX also sending the source for that would be helpful. Other formats are acceptable, but if you send something in Microsoft Word, the editor will then have to convert it to PDF, so if you can do it, please do so.
- We have introduced a file naming convention to make organization easier for the editor, for example, if I sent in a solution to problem **3803**, I would send in the files **Godin_Shawn_3803.tex** and **Godin_Shawn_3803.pdf**.

If you could follow these guidelines it would make things run smoother, thank you.

Mathematical Mayhem and *Skoliad* are finishing up and will be discontinued in *Crux*. The last set of *Mayhem* solutions will appear in the next issue. In this issue, the last set of *Skoliad* solutions appear. I want to take a moment to thank Lily and Mogens for the great job they have done over the years. It has been a pleasure working with them, and many have benefitted from all the work they put in to making *Skoliad* a great column. They will be missed.

One change that is coming is that I will be stepping down as Editor-in-Chief at the end of the school year in June. My time in the role has been very enjoyable, but I just cannot keep up the pace that is needed for the job. The CMS is in the process of looking for a new Editor-in-Chief, as well as other members of the *Crux* board. If you are interested in being involved with the publication you can email your intentions, or questions, directly to `CRUX-EIC-2014@cms.math.ca`.

Shawn Godin

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SKOLIAD No. 143

Lily Yen and Mogens Hansen

In this final Skoliad column we present the solutions to the Mathematics Association of Quebec Contest, Secondary level, 2011, given in Skoliad 137 at [2011:481–483].

1. Otto likes palindromes (numbers that read the same forwards and backwards) so much that he has constructed this alphametic:

$$\mathrm{AMQMA} \times 6 = \mathrm{LUCIE}\,.$$

Find the values of the eight digits.

(Recall that an *alphametic* is a small mathematical puzzle consisting of an equation in which the digits have been replaced by letters. The task is to identify the value of each letter in such a way that the equation comes out true. Different letters have different values, different digits are represented by different letters, and no number begins with a zero. For example, the alphametic PAPA + PAPA = MAMAN has the solution P = 7, A = 5, M = 1, and N = 0, yielding 7575 + 7575 = 15150.)

Solution by Emily Yang, student, Dr. Charles Best Secondary School, Coquitlam, BC.

Since LUCIE only has five digits, AMQMA $< 100\,000 \div 6 \approx 16666.7$, so A = 1. (The leading digit cannot be zero.)

The ones' digit of a product depends only on the ones' digits of the factors, so E = A $\times$ 6 = 6.

If M = 0, then AMQMA $\times$ 6 = 10Q01 $\times$ 6 =???06, so I = 0, but M and I cannot both be zero. Moreover, M $\neq$ 1 since A = 1, and M $\neq$ 6 since E = 6. If M = 2, then AMQMA $\times$ 6 = 12Q21 $\times$ 6 =???26, so I = 2, but M and I cannot both be 2. If M = 4, then AMQMA $\times$ 6 = 14Q41 $\times$ 6 =???46, so I = 4, but M and I cannot both be 4. If M $\geq$ 7, then AMQMA $\times$ 6 $\geq 17\,000 \times 6 = 102\,000$, which has too many digits. This leaves just two possibilities: either M = 3 or M = 5.

If M = 3, then AMQMA $\times$ 6 = 13Q31 $\times$ 6 =???86, so I = 8. If Q = 0, then AMQMA $\times$ 6 = 13031 $\times$ 6 = 78186, but U and I cannot both be 8. If Q = 2, then AMQMA $\times$ 6 = 13231 $\times$ 6 = 79386, but C and M cannot both be 3. If Q $\geq$ 4, then AMQMA $\times$ 6 $\geq$ 13431 $\times$ 6 = 83586, so L = 8, but I and L cannot both be 8. This eliminates all possible values for Q, so M $\neq$ 3.

Thus M = 5. Trying the available values for Q yields: 15051 $\times$ 6 = 90306 (but U and I cannot both be zero), 15251 $\times$ 6 = 91506 (but U and A cannot both be 1), 15351 $\times$ 6 = 92106 (but C and A cannot both be 1), 15451 $\times$ 6 = 92706 (works out), 15751 $\times$ 6 = 94506 (but C and M cannot both be 5), 15851 $\times$ 6 = 95106 (but U and M cannot both be 5), and $15951 \times 6 = 95706$ (but L and Q cannot both be 9).

Thus a single solution exists: $15451 \times 6 = 92706$.

Also solved by GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; KELLY HO, student, École Banting Middle School, Coquitlam, BC; and DOUGLAS ZHU, student, Meadowridge School, Maple Ridge, BC.

2. Anik is going 108 km/h on the highway. On her way to a math contest, she passes a train that travels beside the highway in the same direction as Anik. She notices that it takes her exactly 77 seconds to pass the train from the rear to the front. Upon arrival, she finds that she has forgotten her calculator and turns back. She again passes the train, which still travels at the same speed. This time it takes her seven seconds to pass from the front of the train to the rear. How long is the train?

Solution by Lucy Yuan, student, New Westminster Secondary School, New Westminster, BC.

Say the train is going at x km/h. Because Anik is traveling in the same direction as the train, her speed relative to the train would be

$$\frac{(108-x)\,\text{km}}{1\,\text{hour}} = \frac{(108-x)\,\text{km}}{3600\,\text{seconds}}\,.$$

Now, when traveling in the same direction as the train, it takes Anik 77 seconds to pass the train, so if ℓ is the length (in km) of the train, then

$$\frac{108-x}{3600} \cdot 77 = \ell\,.$$

When Anik passes the train in the opposite direction, her speed relative to the train is

$$\frac{(108+x)\,\text{km}}{1\,\text{hour}} = \frac{(108+x)\,\text{km}}{3600\,\text{seconds}}\,.$$

At this speed it takes Anik 7 seconds to pass, so

$$\frac{108+x}{3600} \cdot 7 = \ell\,.$$

Thus

$$\frac{108-x}{3600} \cdot 77 = \frac{108+x}{3600} \cdot 7\,,$$

so, solving for x we get

$$\begin{aligned}
(108-x)\cdot 77 &= (108+x)\cdot 7\\
11(108-x) &= 108+x\\
1188-11x &= 108+x\\
1080 &= 12x\\
x &= 90.
\end{aligned}$$

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Therefore
$$\ell = \frac{108+x}{3600}\cdot 7 = \frac{108+90}{3600}\cdot 7 = 0.385,$$
so the length of the train is 0.385 km or 385 m.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and SAMUEL HUANG, student, Moscrop Secondary School, Burnaby, BC.

3. At an intersection, the traffic light is red for 30 seconds and green for 30 seconds. (Ignore the yellow light.) How long do you have to wait, on average, at the intersection? Justify your answer.

Solution by Douglas Zhu, student, Meadowridge School, Maple Ridge, BC.

If the traffic light is green when you arrive at the intersection, the (average) wait is obviously 0 seconds.

If the traffic light is red when you arrive, you may have to wait t seconds or $30-t$ seconds ($0 \le t \le 15$), and these two possibilities are equally likely (for each value of t). Thus, for each value of t, the average wait is $\frac{t+(30-t)}{2} = \frac{30}{2} = 15$ seconds. Since this average is independent of t, the average wait, if the light is red upon arrival, is 15 seconds.

When you arrive at the intersection, the probability that the light is green is $\frac{1}{2}$ and the probability that the light is red is $\frac{1}{2}$. Therefore the average wait is $0\cdot\frac{1}{2} + 15\cdot\frac{1}{2} = \frac{15}{2} = 7.5$ seconds.

Also solved by GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; and RICHARD I. HESS, Rancho Palos Verdes, CA, USA.

That you must wait an average of 15 *seconds if you arrive at a red light is either intuitively clear or somewhat technical to prove. The problem is that the probability of waiting* exactly t *seconds is infinitely small, regards of the value of* t*. Since* t *can take infinitely many values, you are now adding infinitely many infinitely small values. To accomplish such a feat is the task of calculus, which is beyond Skoliad.*

4. How many integers from 0 to 999 (inclusive) do not contain the digit 7? What is the sum of these numbers?

Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

If you allow leading zeroes, the task is to count the number of three-digit numbers that do not use the digit 7. Each digit has nine possibilities: 0, 1, 2, 3, 4, 5, 6, 8, and 9. Therefore $9^3 = 729$ such numbers exist.

Consider all three-digit numbers with ones' digit 5. They have the form $ab5$, where a and b are (possibly identical) digits other than 7. Thus there are nine possibilities for a and nine for b, so $9^2 = 81$ numbers (without 7) have ones' digit 5.

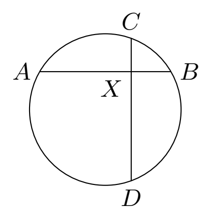
This is of course true for any digit, not just for 5, and it is true for the tens' digit and the hundreds' digit as well: the digit d occurs as ones' digit 81 times, as tens' digit 81 times, and as hundreds' digit 81 times. Therefore the sum of all such numbers is

$$81 \cdot 111 \cdot (0 + 1 + 2 + 3 + 4 + 5 + 6 + 8 + 9) = 81 \cdot 111 \cdot 38 = 341\,658\,.$$

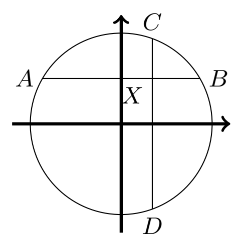
Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and SZERA PINTER, student, Moscrop Secondary School, Burnaby, BC.

5. In a circle with radius r, the two chords AB and CD intersect at a right angle at X. Show that

$$|XA|^2 + |XB|^2 + |XC|^2 + |XD|^2 = 4r^2\,.$$



Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA.



Impose a coordinate system such that $(0, 0)$ is the centre of the circle and the chords are parallel to the axes. Let (x, y) be the coordinates of X, and let (a, y) be the coordinates of A. Since A is on the circle, $a^2 + y^2 = r^2$, so $a = -\sqrt{r^2 - y^2}$. Hence $|XA| = x - a = x + \sqrt{r^2 - y^2}$.

Similarly,

$$\begin{aligned} |XB| &= \sqrt{r^2 - y^2} - x, \\ |XC| &= \sqrt{r^2 - x^2} - y, \text{ and} \\ |XD| &= y + \sqrt{r^2 - x^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} |XA|^2 &= \left(x + \sqrt{r^2 - y^2}\right)^2 = x^2 + 2x\sqrt{r^2 - y^2} + \left(\sqrt{r^2 - y^2}\right)^2 \\ &= x^2 + 2x\sqrt{r^2 - y^2} + r^2 - y^2, \end{aligned}$$

and similarly

$$\begin{aligned} |XB|^2 &= r^2 - y^2 - 2x\sqrt{r^2 - y^2} + x^2, \\ |XC|^2 &= r^2 - x^2 - 2y\sqrt{r^2 - x^2} + y^2, \text{ and} \\ |XD|^2 &= y^2 + 2y\sqrt{r^2 - x^2} + r^2 - x^2. \end{aligned}$$

Thus

$$|XA|^2 + |XB|^2 = 2x^2 - 2y^2 + 2r^2$$

and

$$|XC|^2 + |XD|^2 = 2r^2 - 2x^2 + 2y^2,$$

so

$$|XA|^2 + |XB|^2 + |XC|^2 + |XD|^2 = 4r^2$$

as required.

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6. The cubes $173^3 = 5\,177\,717$, $192^3 = 7\,077\,888$ and $1309^3 = 2\,242\,946\,629$ are examples of a whole number N that contains as many different digits as its cube, N^3. If N^3 contains fewer different digits than N, then N is said to be *deficient*. For example, $13\,798$ has five different digits, while its cube, $2\,626\,929\,525\,592$, has four (2, 5, 6, and 9), so $13\,798$ is deficient. Show that there are infinitely many deficient whole numbers.

Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA.

If $N = 1\,023\,456\,789$, then

$$N^3 = 1\,072\,033\,936\,267\,303\,561\,560\,897\,069\,.$$

Since N contains all ten digits while N^3 does not contain the digit 4, N is deficient.

If k is a positive integer, then $N \cdot 10^k$ clearly still contains all ten digits, while $(N \cdot 10^k)^3 = N^3 \cdot 10^{3k}$ consists of the same digits as N^3 followed by $3k$ zeroes. Therefore $(N \cdot 10^k)^3$ does not contain the digit 4, so $(N \cdot 10^k)^3$ is deficient for every positive integer k. Thus infinitely many deficient integers exist.

Our solver must have used a tool more powerful than a pocket calculator to find the cube of his ten-digit number N. A more modest example will do. Indeed, the problem states that $192^3 = 7\,077\,888$ from which follows that $1920^3 = 7\,077\,888\,000$. Since 1920 contains four different digits while 1920^3 only contains three, 1920 is deficient. Our solver's idea of looking at $1920 \cdot 10^k$ now yields an infinitude of deficient integers.

7. If x, y, and z are real numbers such that

$$x = \sqrt{11 - 2yz}, \quad y = \sqrt{12 - 2xz}, \quad \text{and} \quad z = \sqrt{13 - 2xy}$$

what is the value of $x + y + z$?

Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA.

Note that

$$\begin{aligned}(x+y+z)^2 &= (x+y+z)(x+y+z)\\ &= x^2 + xy + xz + yx + y^2 + yz + zx + zy + z^2\\ &= x^2 + y^2 + z^2 + 2xy + 2xz + 2yz.\end{aligned}$$

Now, $x^2 = 11 - 2yz$, $y^2 = 12 - 2xz$, and $z^2 = 13 - 2xy$. Therefore,

$$\begin{aligned}(x+y+z)^2 &= 11 - 2yz + 12 - 2xz + 13 - 2xy + 2xy + 2xz + 2yz\\ &= 36,\end{aligned}$$

so $x + y + z = \pm 6$. Since x, y, and z all equal square roots, they are nonnegative, so their sum cannot be negative. Thus $x + y + z = 6$.

This issue's prize for the best solutions goes to Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany. We thank our readers for the solutions it has been our privilege to receive, edit, and publish during our tenure as Skoliad editors.

THE CONTEST CORNER

No. 11

Shawn Godin

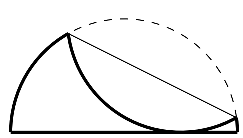
The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Electronic submissions are preferable, with each solution contained in a separate file. Files should be named using the convention LastName_FirstName_CCProblemNumber *(example Doe_Jane_CC1234.tex). It is preferred that readers submit a LaTeX file* and *a pdf file for each solution, although other formats, such as Microsoft Word, are also accepted. Readers are invited to email solutions and contests to the editor at* `crux-contest@cms.math.ca`. *Submissions by regular mail are also accepted and should be sent to the address inside the back cover. Name(s) of solver(s) with affiliation, city, and country should appear on* each *solution, and each solution should start on a separate page.*

To facilitate their consideration, solutions to the problems should be received by the editor by **1 May 2014**, *although solutions received after this date will also be considered until the time when a solution is published.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the Solutions section, the problem will be stated in the language of the primary featured solution.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

CC51. A semicircular piece of paper with radius 2 is creased and folded along a chord so that the arc is tangent to the diameter as shown in the diagram. If the contact point of the arc divides the diameter in the ratio 3 : 1, determine the length of the crease.



CC52. There are some marbles in a bowl. Alphonse, Beryl and Colleen each take turns removing one or two marbles from the bowl, with Alphonse going first, then Beryl, then Colleen, then Alphonse again, and so on. The player who takes the last marble from the bowl is the loser, and the other two players are the winners. If the game starts with N marbles in the bowl, for what values of N can Beryl and Colleen work together and force Alphonse to lose?

CC53. Determine an infinite family of quadruples (a, b, c, d) of positive integers, each of which is a solution to $a^4 + b^5 + c^6 = d^7$.

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CC54. Let k, l, m, n be positive integers such that $k+l+m \geq n$. Prove the following relation for binomial coefficients

$$\sum_{p+q+r=n} \binom{k}{p}\binom{l}{q}\binom{m}{r} = \binom{k+l+m}{n}.$$

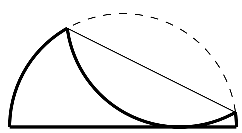
The summation in the left-hand side runs over all ordered partitions of n into three integers p, q, r such that $0 \leq p \leq k$, $0 \leq q \leq l$, $0 \leq r \leq m$.

CC55. If α, β, γ are the roots of $x^3 - x - 1 = 0$, compute

$$\frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma}.$$

. . . .

CC51. Un morceau de papier de forme semi-circulaire, de rayon 2, est plié le long d'une corde de manière que l'arc soit tangent au diamètre, comme dans la figure. Sachant que le point de contact de l'arc et du diamètre divise le diamètre dans un rapport de 3 : 1, déterminer la longueur du pli.



CC52. Un bol contient un nombre de billes. Alain, Bianca et Carla enlèvent tour à tour une ou deux billes du bol, en commençant par Alain, suivi de Bianca, puis de Carla, puis d'Alain et ainsi de suite. Le joueur qui enlève la dernière bille perd la joute, tandis que les deux autres sont gagnants. S'il y a N billes dans le bol au départ, pour quelles valeurs de N Bianca et Carla peuvent-elles travailler ensemble pour qu'Alain soit toujours perdant ?

CC53. Déterminer une famille infinie de quadruplets (a, b, c, d) d'entiers strictement positifs, chaque quadruplet étant une solution de l'équation $a^4 + b^5 + c^6 = d^7$.

CC54. Soit k, l, m et n des entiers strictement positifs tels que $k+l+m \geq n$. Démontrer la relation suivante qui comporte des coefficients binômiaux :

$$\sum_{p+q+r=n} \binom{k}{p}\binom{l}{q}\binom{m}{r} = \binom{k+l+m}{n}$$

La sommation du membre de gauche se fait sur toutes les partitions ordonné de n en trois entiers p, q, r de manière que $0 \leq p \leq k$, $0 \leq q \leq l$ et $0 \leq r \leq m$.

CC55. Soit α, β, γ les racines de l'équation $x^3 - x - 1 = 0$. Évaluer l'expression

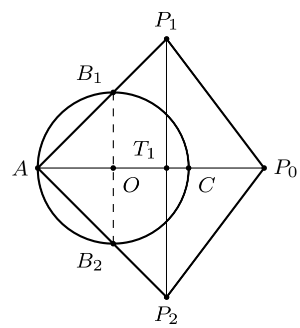
$$\frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma}.$$

CONTEST SOLUTIONS

CC1. A circle has centre O, diameter AC, and radius 1. A chord is drawn from A to an arbitrary point B (different from A) on the circle and extended to the point P with $BP = 1$. Thus P can take many positions. Let S be the set of points P. Determine whether or not there is a circle on which all points of S lie.
(*Originally question B3 part c) from the 2010 Sun Life Financial Canadian Open Mathematics Challenge.*)

Solved by George Apostolopoulos, Messolonghi, Greece; and Titu Zvonaru, Comăneşti, Romania. One incorrect solution was received. We give the solution of Zvonaru.

Let P_0 be the symmetric of O with respect to C, then P_0 is the position of P when B is located at C. Let B_1B_2 be a diameter of the given circle such that $B_1B_2 \perp AC$ and let P_1 and P_2 be the locations of P when B is located at B_1 and B_2 respectively. Let T_1 be the point of intersection of AP_0 and P_1P_2.



We obtain:

$$AB_1 = \sqrt{2}, \quad AP_1 = 1 + \sqrt{2},$$
$$AT_1 = P_1T_1 = \frac{1+\sqrt{2}}{\sqrt{2}} = \frac{2+\sqrt{2}}{2}.$$

Hence

$$T_1P_0 = AP_0 - AT_1 = 3 - \frac{2+\sqrt{2}}{2} = \frac{4-\sqrt{2}}{2}$$

and by the Pythagorean theorem we deduce that

$$P_0P_1^2 = P_1T_1^2 + T_1P_0^2 = \left(\frac{2+\sqrt{2}}{2}\right)^2 + \left(\frac{4-\sqrt{2}}{2}\right)^2$$
$$= \frac{6+4\sqrt{2}+18-8\sqrt{2}}{4} = 6 - \sqrt{2}.$$

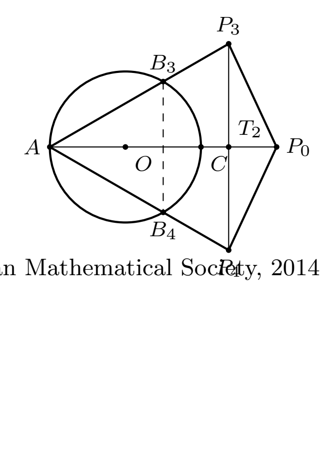
If R_1 is the circumradius of $\Delta P_0P_1P_2$, then

$$R_1 = \frac{P_1P_2 \cdot P_0P_1 \cdot P_2P_0}{4 \cdot \text{Area}(\Delta P_0P_1P_2)} = \frac{P_1P_2 \cdot P_1P_0^2}{4\left(P_1P_2 \cdot \frac{T_1P_0}{2}\right)} = \frac{6-\sqrt{2}}{4-\sqrt{2}} = \frac{11+\sqrt{2}}{7}.$$

Similarly, we let B_3 and B_4 be on the circle, so that

$$\angle B_3AC = \angle B_4AC = 30^\circ;$$

let P_3 and P_4 be the corresponding locations of P, and let T_2 be the point of intersection of P_3P_4 and AP_0.



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The triangle AP_3P_4 is equilateral and we have:

$$\begin{aligned} AB_3 &= \sqrt{3}, \quad AP_3 = 1+\sqrt{3}, \\ AT_2 &= AP_3 \cdot \frac{\sqrt{3}}{2} = \frac{3+\sqrt{3}}{2}, \\ T_2P_0 &= 3 - \frac{3+\sqrt{3}}{2} = \frac{3-\sqrt{3}}{2}\,. \end{aligned}$$

From the Law of Cosines applied to ΔAP_0P_3, we obtain

$$\begin{aligned} P_0P_3^2 &= AP_0^2 + AP_3^2 - 2 \cdot AP_0 \cdot AP_3 \cdot \cos 30^\circ \\ &= 9 + 4 + 2\sqrt{3} - 2 \cdot \frac{\sqrt{3}}{2} \cdot 3 \cdot (1+\sqrt{3}) \\ &= 4 - \sqrt{3}\,. \end{aligned}$$

If R_2 is the circumradius of $\Delta P_0P_1P_2$, then it results that

$$R_2 = \frac{P_3P_4 \cdot P_3P_0^2}{4\left(P_3P_4 \cdot \frac{T_2P_0}{2}\right)} = \frac{4-\sqrt{3}}{3-\sqrt{3}} = \frac{9+\sqrt{3}}{6}\,.$$

Since $R_1 \neq R_2$, it follows that there is not a circle on which all points of S lie.

CC2. Let $f : \mathbb{R}^2 \to \mathbb{R}$. Suppose that f is continuous and that $\int_0^1 f(a+tu)dt = 0$ for every point $a \in \mathbb{R}^2$ and every vector $u \in \mathbb{R}^2$ with $\|u\| = 1$. Show that f is constant.
(*Originally question # 4 from the 2012 University of Waterloo Big E Contest.*)

One incorrect solution was received. The problem remains open

CC3. All three sides of a right triangle are integers. Prove that the area of the triangle: is also an integer; is divisible by 3; and is even.
(*Originally question # 10 from the 2010 Manitoba Mathematical Competition.*)

Solved by George Apostolopoulos, Messolonghi, Greece; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Daniel Văcaru, Piteşti, Romania; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; Haley Williams, student, Auburn University at Montgomery, Montgomery, AL, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Apostolopoulos modified by the editor.

Suppose the legs of the triangle have lengths a and b, and the hypotenuse has length c. The area of the triangle is $\frac{ab}{2}$, so in order to prove the area is even and divisible by 3, it suffices to show $4|ab$ and $3|ab$.

In this light, let $d \in \{3,4\}$. Observe that for any integer x,

$$x^2 \equiv 0, 1 \pmod{d} \tag{1}$$

Thus, if both a and b are not divisible by d, then $a^2 \equiv 1 \pmod{d}$ and $b^2 \equiv 1 \pmod{d}$, and hence

$$c^2 = a^2 + b^2 \equiv 2 \pmod{d},$$

which contradicts (1). Therefore, for $d \in \{3, 4\}$, $d|a$ or $d|b$, implying $d|ab$.

CC4. Suppose that $n \geq 3$. A sequence $a_1, a_2, a_3, \ldots, a_n$ of n integers, the first m of which are equal to -1 and the remaining $p = n - m$ of which are equal to 1, is called an MP sequence. Consider all of the products $a_i a_j a_k$ (with $i < j < k$) that can be calculated using the terms from an MP sequence $a_1, a_2, a_3, \ldots, a_n$. Determine the number of pairs (m, p) with $1 \leq m \leq p \leq 1000$ and $m + p \geq 3$ for which exactly half of these products are equal to 1.
(*Originally question B3 part b) from the 2011 Canadian Senior Mathematics Contest.*)

Solved together by Billy Jin, Waterloo Collegiate Institute, Waterloo, ON and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. One incorrect solution was received.

Call an MP sequence *balanced* if it satisfies the specified condition. For a given MP sequence let M and P denote the number of triples (a_i, a_j, a_k) where $i < j < k$ such that $a_i a_j a_k = -1$ or 1, respectively. Then the condition $M = P$ is successively equivalent to:

$$\binom{m}{3} + \binom{m}{1}\binom{p}{2} = \binom{p}{3} + \binom{p}{1}\binom{m}{2}$$
$$\frac{m(m-1)(m-2)}{6} + \frac{mp(p-1)}{2} = \frac{p(p-1)(p-2)}{6} + \frac{pm(m-1)}{2}$$
$$3mp(m-p) = m(m-1)(m-2) - p(p-1)(p-2)$$
$$3mp(m-p) = m^3 - p^3 - 3(m^2 - p^2) + 2(m-p) \qquad (1)$$

Hence by setting $m = p = c$ where $c \in \mathbb{N}$ and $1 < c \leq 1000$, we get 999 balanced MP sequences.

Suppose now that $m \neq p$. Then dividing both sides of (1) by $m - p$ yields

$$3mp = m^2 + mp + p^2 - 3(m+p) + 2$$
$$p^2 - (2m+3)p + (m^2 - 3m + 2) = 0 \qquad (2)$$

Since p is an integer, the discriminant, D, of (2) must be a perfect square.

Now, $D = (2m+3)^2 - 4(m^2 - 3m + 2) = 24m + 1$, so

$$24m + 1 = k^2 \text{ for some } k \in \mathbb{N} \text{ with } k \geq 5 \qquad (3)$$

Note that $k^2 \equiv 1 \pmod 8$ (if and only if k is odd) and $k^2 \equiv 1 \pmod 3$ (if and only if $k \equiv 1$ or $2 \pmod 3$). Hence any odd $k \geq 5$ such that $k \not\equiv 0 \pmod 3$ would satisfy (3) for some m which would yield an admissible pair (m, p) provided the condition $1 \leq m \leq p \leq 1000$ holds. To satisfy this condition we solve (2) and choose $p = \frac{2m+3+\sqrt{D}}{2} = m + \frac{k+3}{2} > m$.

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Since $p \leq 1000$ we have $\frac{k^2-1}{24} + \frac{k+3}{2} \leq 1000$ which implies $k^2 + 12k + 35 \leq 24000$ or $(k+6)^2 \leq 24001$, so $k+6 \leq 154$ or $k \leq 148$. Hence it remains to count the number, q, of odd integers k with $5 \leq k \leq 148$ which are not divisible by 3. Among the 72 odd integers in the set $\{5, 6, \cdots, 148\}$, exactly 24 of them are divisible by 3 since $k = 3 + 6d \leq 147$ implies $d \leq 24$. Hence, $q = 72 - 24 = 48$.

Therefore, the required number is $999 + 48 = 1047$.

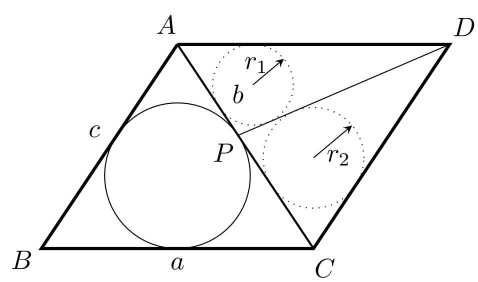
CC5. Let $ABCD$ be a parallelogram. We draw in the diagonal AC. A circle is drawn inside ΔABC tangent to all three sides and touches side AC at a point P. Draw in the line DP. A circle of radius r_1 is drawn inside ΔDAP tangent to all three sides and a circle of radius r_2 is drawn inside ΔDCP tangent to all three sides. Prove that

$$\frac{r_1}{r_2} = \frac{AP}{PC}.$$

(*Originally question C3 part b) from the 2012 Sun Life Financial Canadian Open Mathematics Challenge.*)

Solved by Daniel Văcaru, Piteşti, Romania; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Zvonaru.

We denote $a = BC$, $b = AC$, $c = AB$.



It is well known that

$$AP = \frac{b+c-a}{2}, \quad \text{and} \quad PC = \frac{a+b-c}{2}.$$

Since

$$r_1 = \frac{\text{area}(\Delta APD)}{\frac{AP+PD+DA}{2}}, \quad r_2 = \frac{\text{area}(\Delta PCD)}{\frac{PC+CD+DP}{2}}, \quad \text{and} \quad \frac{\text{area}(\Delta APD)}{\text{area}(\Delta PCD)} = \frac{AP}{PC}.$$

Thus

$$\frac{r_1}{r_2} = \frac{AP(PC+CD+PD)}{PC(AP+PD+AD)} = \frac{AP}{PC} \times \frac{\frac{a+b-c}{2} + c + PD}{\frac{b+c-a}{2} + a + PD} = \frac{AP}{PC} \times \frac{\frac{a+b+c}{2} + PD}{\frac{a+b+c}{2} + PD},$$

and therefore $\frac{r_1}{r_2} = \frac{AP}{PC}$.

THE OLYMPIAD CORNER

No. 309

Nicolae Strungaru

The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Electronic submissions are preferable, with each solution contained in a separate file. Files should be named using the convention LastName_FirstName_OCProblemNumber *(example Doe_Jane_OC1234.tex). It is preferred that readers submit a LATEX file and a pdf file for each solution, although other formats, such as Microsoft Word, are also accepted. Readers are invited to email solutions and contests to the editor at* `crux-olympiad@cms.math.ca`. *Submissions by regular mail are also accepted and should be sent to the address inside the back cover. Name(s) of solver(s) with affiliation, city, and country should appear on each solution, and each solution should start on a separate page.*

To facilitate their consideration, solutions to the problems should be received by the editor by **1 May 2014**, *although solutions received after this date will also be considered until the time when a solution is published.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Rolland Gaudet, of l'Université Saint-Boniface in Winnipeg, for translations of the problems.

OC111. Let x, y and z be positive real numbers. Show that

$$x^2 + xy^2 + xyz^2 \geq 4xyz - 4\,.$$

OC112. Find all pairs of natural numbers (a, b) such that

$$\gcd(a, b) + 9\,\mathrm{lcm}(a, b) + 9(a + b) = 7ab\,.$$

OC113. Prove that among any n vertices of a regular $(2n-1)$–gon, where $n \geq 3$, we can find 3 which form an isosceles triangle.

OC114. Let ABC be a scalene triangle. Its incircle touches BC, AC, AB at D, E, F respectively. Let L, M, N be the symmetric points of D, E, F with respect to EF, FD, and DE, respectively. The line AL intersects BC at P, the line BM intersects CA at Q, and the line CN intersects AB at R. Prove that P, Q, R are collinear.

OC115. Find the smallest positive integer n for which there exists a positive integer k such that the last 2012 decimal digits of n^k are all 1's.

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OC111. Soit x, y et z trois nombres réels positifs. Démontrez que

$$x^2 + xy^2 + xyz^2 \geq 4xyz - 4\,.$$

OC112. Déterminer toutes les paires de nombres naturels (a, b) tels que

$$\text{pgcd}(a, b) + 9\,\text{ppcm}(a, b) + 9(a + b) = 7ab\,.$$

OC113. Démontrer que parmi n'importe quels n sommets d'un polygone régulier à $(2n - 1)$ sommets, où $n \geq 3$, on peut en tirer 3 qui forment un triangle isocèle.

OC114. Soit ABC un triangle scalène. Son cercle inscrit touche BC, AC et AB aux points D, E et F respectivement. Soient L, M et N les points symétriques à D, E et F par rapport à EF, FD et DE respectivement. La ligne AL intersecte BC en P, la ligne BM intersecte CA en Q et la ligne CN intersecte AB en R. Démontrer que P, Q et R sont colinéaires.

OC115. Déterminer le plus petit entier positif n pour lequel il existe un entier positif k tel que les 2012 dernières positions décimales de n^k sont toutes 1.

OLYMPIAD SOLUTIONS

OC51. Determine all pairs (a, b) of nonnegative integers so that $a^b + b$ divides $a^{2b} + 2b$. Note, for this problem $0^0 = 1$.
(*Originally question 4 from the second day of Austrian Mathematical Olympiad.*)

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; Victor Pambuccian, Arizona State University West, Phoenix, AZ, USA; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON and Titu Zvonaru, Cománeşti, Romania. We give the solution of Wang.

The only solutions are $(a, 0), (0, b)$ for $a, b \geq 0$ and $(a, b) = (2, 1)$. Since $(a, 0)$ and $(0, b)$ clearly satisfy the equation, it remains to show that if $a \geq 1$ and $b \geq 1$ then $(a, b) = (2, 1)$.

Suppose that (a, b) is a solution where $a, b \geq 1$. If $a = 1$ then the condition becomes $1 + b \mid 1 + 2b$ which is impossible since $1 + b < 1 + 2b < 2(1 + b)$. Hence we have $a \geq 2$.

As $a^b + b \mid a^{2b} + 2b$ we have

$$\begin{aligned} a^b + b &= \gcd(a^b + b, a^{2b} + 2b) = \gcd(a^b + b, a^b(a^b + b) - (a^{2b} + 2b)) \\ &= \gcd(a^b + b, ba^b - 2b) = \gcd(a^b + b, b(a^b + b) - (ba^b - 2b)) \\ &= \gcd(a^b + b, b^2 + 2b) \end{aligned}$$

Thus

$$a^b + b \le b^2 + 2b\,.$$

Hence

$$a^b \le b^2 + b\,. \tag{1}$$

We claim that (1) cannot hold if $a \ge 3$ by proving by induction that

$$3^b > b^2 + b$$

for all $b \ge 1$. This is clear when $b = 1$.

Suppose that $3^b > b^2 + b$ for some $b \ge 1$. Then

$$\begin{aligned} 3^{b+1} > 3(b^2 + b) &= b^2 + 2b + b^2 + b + b^2 \\ &\ge b^2 + 2b + 1 + b + 1 = (b+1)^2 + (b+1), \end{aligned}$$

completing the induction.

It remains to consider the case when $a = 2$. Using the same induction argument, it can be proved easily that

$$2^b > b^2 + b$$

for all $b \ge 5$, which contradicts (1). Thus we only need to check the cases $(a, b) \in \{(2,1), (2,2), (2,3), (2,4)\}$, and a direct computation shows that only $(2,1)$ yields a solution.

This completes the proof.

OC52. Let d, d' be two divisors of n with $d' > d$. Prove that

$$d' > d + \frac{d^2}{n}\,.$$

(*Originally question 1 from the Russia National Olympiad 2011: Grade 11.*)

Solved by Michel Bataille, Rouen, France ; Chip Curtis, Missouri Southern State University, Joplin, MO, USA ; Marian Dincă, Bucharest, Romania ; Oliver Geupel, Brühl, NRW, Germany ; Victor Pambuccian, Arizona State University West, Phoenix, AZ, USA ; Kim Uyen Truong, California State University, Fullerton, CA, USA ; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the solution of Truong.

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Since d and d' are divisors of n there exist some a, b so that

$$ad' = bd = n\,.$$

As $d' > d$ we have $a < b$ and hence $a \leq b - 1$. Thus

$$\begin{aligned} d' > d + \frac{d^2}{n} &\Leftrightarrow \frac{n}{a} > \frac{n}{b} + \frac{n}{b^2} \\ &\Leftrightarrow \frac{1}{a} > \frac{1}{b} + \frac{1}{b^2} \\ &\Leftrightarrow b^2 > ab + a = a(b+1)\,. \end{aligned}$$

This last inequality holds as

$$a(b+1) \leq (b-1)(b+1) = b^2 - 1 < b^2\,.$$

Thus,

$$d' > d + \frac{d^2}{n}\,.$$

OC53. Find all the polynomials $P(x) \in \mathbb{R}[x]$ so that $P(a) \in \mathbb{Z}$ implies $a \in \mathbb{Z}$.
(*Originally question 4 from the 2011 Singapore National Olympiad.*)

Chip Curtis, Missouri Southern State University, Joplin, MO, USA ; Oliver Geupel, Brühl, NRW, Germany and Victor Pambuccian, Arizona State University West, Phoenix, AZ, USA. We give the solution of Curtis.

Case 1: $\deg(P) = 0$. Then, there exists a constant c so that $P(x) = c$ for all real x. Then this polynomial has the desired property if and only if c is not an integer. [If c is not an integer, then P has the desired property by vacuity].

Case 2: $\deg(P) = 1$. Then $P(x) = ax + b$ for some real numbers a $(\neq 0)$ and b. As the range of P is the real numbers, for each integer m there is an integer x_m so that $P(x_m) = m$. Then

$$ax_m + b = m \quad \text{and} ax_{m-1} + b = m - 1,$$

which implies that

$$x_{m-1} = \frac{(m-1) - b}{a} \quad \text{and} x_m = \frac{m-b}{a}.$$

Subtracting, we get that

$$\frac{1}{a} = x_m - x_{m-1} \in \mathbb{Z}\,.$$

Thus $a = \frac{1}{k}$ with k an integer. Then

$$x_m = \frac{m-b}{a} = km - kb \Rightarrow kb = x_m - km \in \mathbb{Z}\,.$$

Let $l = kb \in \mathbb{Z}$, then we get

$$P(x) = \frac{x+l}{k}\,.$$

It is easy to see that this polynomial has the desired property.

Case 3: $\deg(P) \geq 2$. We claim there is no such polynomial in this case.

Since for any polynomial P which satisfies the condition, $-P$ also satisfies the condition, without loss of generality we can assume that the leading coefficient of P is positive. We then have $\lim\limits_{x\to\infty} P'(x) = \infty$, and so there exists some x_0 so that $P'(x) \geq 2$ for all $x \geq x_0$.

Moreover, as $\lim\limits_{x\to\infty} P(x) = \infty$, by the Intermediate Value Theorem there exists some M so that the interval

$$[M, \infty) \subset P((x_0, \infty))\,.$$

Thus, for every $m > M$ there exist two distinct integers $u, v > x_0$ so that

$$P(u) = m\,;\; P(v) = m+1\,.$$

By the Mean Value Theorem, there exists some c_m between u, v so that

$$P'(c_m) = \frac{P(u) - P(v)}{u - v} = \frac{1}{u-v} < 1\,.$$

But as $c_m > x_0$ we also have $P'(c_m) \geq 2$, a contradiction. Thus there is no polynomial of degree 2 or higher satisfying this condition.

In summary, a polynomial P satisfies this condition if and only if either P is a nonintegral constant or $P(x) = \frac{x+l}{k}$ for some integers l, k with $k \neq 0$.

OC54. Given four points in the plane so that the incircles of the four triangles formed by three of the four points are equal, prove that the four triangles are equal. (*Originally question 5 from the 2011 Japan National Olympiad.*)

Solved by Oliver Geupel, Brühl, NRW, Germany. No other solution was received.

We show that the four points are the vertices of a rectangle.

Let the four points be A_1, A_2, A_3, and A_4. Let a_{ik} denote the length of the segment A_iA_k. Let I_1, I_2, I_3, and I_4 denote the incentres of the triangles $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, and $A_1A_2A_3$, respectively. If the convex hull $\mathcal{C}$ of the four points is a triangle, say, triangle $A_1A_2A_3$, then A_4 is an interior point of that triangle, which implies that the inradius of triangle $A_1A_2A_3$ is greater than the inradius of triangle $A_1A_2A_4$, a contradiction. Hence, $\mathcal{C}$ is a quadrilateral, say, the convex quadrilateral $A_1A_2A_3A_4$. Let P and Q be the orthogonal projections of I_3 and I_4 on the line A_1A_2. Because triangles $A_1A_2A_4$ and $A_1A_2A_3$ have congruent incircles, the quadrilateral PQI_4I_3 is a rectangle.

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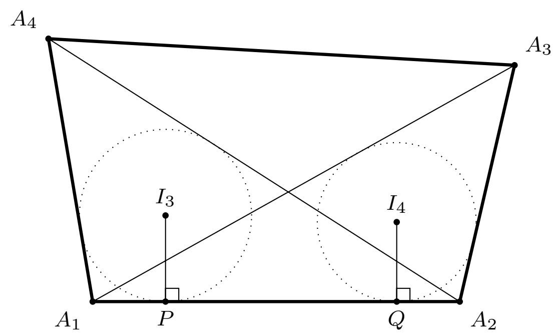

We obtain

$$\begin{aligned} I_3I_4 = PQ &= A_1A_2 - A_1P - A_2Q \\ &= a_{12} - \frac{a_{12} + a_{14} - a_{24}}{2} - \frac{a_{12} + a_{23} - a_{13}}{2} \\ &= \frac{a_{13} + a_{24} - a_{14} - a_{23}}{2}. \end{aligned}$$

Similarly,

$$I_1I_2 = \frac{a_{13} + a_{24} - a_{14} - a_{23}}{2},$$

thus $I_1I_2 = I_3I_4$. Analogously, $I_4I_1 = I_2I_3$. Hence, the quadrilateral $I_1I_2I_3I_4$ is a parallelogram.

We obtain $A_1A_2 \parallel I_3I_4 \parallel I_1I_2 \parallel A_3A_4$ and similarly $A_1A_4 \parallel A_2A_3$. Thus, the quadrilateral $A_1A_2A_3A_4$ is a parallelogram. Therefore,

$$\begin{aligned} a_{12} + a_{14} + a_{24} &= \frac{2[A_1A_2A_4]}{I_3P} = \frac{2[A_1A_2A_3]}{I_4Q} \\ &= a_{12} + a_{23} + a_{13} = a_{12} + a_{14} + a_{13}. \end{aligned}$$

Consequently, $A_2A_4 = a_{24} = a_{13} = A_1A_3$.

This shows that the quadrilateral $A_1A_2A_3A_4$ is a rectangle.

OC55. Let d be a positive integer. Show that for every integer S there exists an integer $n > 0$ and a sequence $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$, where for any $k, \epsilon_k = 1$ or $\epsilon_k = -1$, such that

$$S = \epsilon_1(1 + d)^2 + \epsilon_2(1 + 2d)^2 + \epsilon_3(1 + 3d)^2 + \cdots + \epsilon_n(1 + nd)^2 .$$

(*Originally question 5 from 2011 Canadian Mathematical Olympiad.*)

Solved by Oliver Geupel, Brühl, NRW, Germany. No other solution was received.

Let S be any integer. We are going to show that S can be written in the required form.

Let m be an integer such that $m \geq 4d^3$ and that the number

$$S_0 = -\sum_{k=1}^{m}(1+kd)^2\,,$$

has the same parity as S. Define now

$$S_k = S_{k-1} + 2(1+2kd^2)^2$$

for $k = 1, 2, 3, \ldots, 2d^2-1$. We have

$$S_k \equiv S_{k-1} + 2 \pmod{4d^2}\,.$$

Hence, the sequence

$$S_0, S_1, S_2, \ldots, S_{2d^2-1}$$

includes a number that has the same remainder modulo $4d^2$ as S. Let S' be that number.

Then, S' can be written in the required form and there is an integer q such that

$$S = S' + 4d^2q\,.$$

Observe that for each integer k we have

$$(1+kd)^2 - (1+(k+1)d)^2 - (1+(k+2)d)^2 + (1+(k+3)d)^2 = 4d^2\,.$$

Therefore, for each $q \geq 0$ we have

$$4d^2q = \sum_{j=0}^{q-1}\Big((1+(m+4j+1)d)^2 - (1+(m+4j+2)d)^2 \\ - (1+(m+4j+3)d)^2 + (1+(m+4j+4)d)^2\Big).$$

Similarly, for $q < 0$ we have

$$4d^2q = \sum_{j=0}^{-q-1}\Big(-(1+(m+4j+1)d)^2 + (1+(m+4j+2)d)^2 \\ + (1+(m+4j+3)d)^2 - (1+(m+4j+4)d)^2\Big).$$

Consequently, S can be written in the desired form.

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BOOK REVIEWS

Amar Sodhi

The Universe in Zero Words:
The Story of Mathematics as Told Through Equations
by Dana MacKenzie
Princeton University Press, 2012
ISBN: 978-0-69116-016-0, 223 pp., Softcover, US$19.95
ISBN: 978-0-69115-282-0, 223 pp., Hardcover, US$27.95
Reviewed by ***Brenda Davison****, Simon Fraser University, Burnaby, BC*

My recommendation to you after reading "The Universe in Zero Words" by Dana Mackenzie is — buy and read this book!

The subtitle "*The story of mathematics as told through equations*" tells us how and what the author intended to do and succeeded in doing — giving a brief and selective history of mathematics by choosing to focus on twenty–four equations. These twenty–four equations are evenly divided into 4 parts, where each part is representative of a time period.

Roughly these time periods are: antiquity, the age of exploration (1500 – 1700), the Promethean age (1700 – 1900), and in our own time (1900 to present). A very nice feature of the book is that certain concepts are traced through from one period to the next. For example, infinity in antiquity comes up in Zeno's paradoxes, infinity in the age of exploration appears in the development of calculus, infinity in the promethean age appears in Fourier series and in our time, we see infinity conceptualized more precisely in the work of Cantor and the continuum hypothesis.

By choosing to discuss mathematics throughout its entire history, the author out of necessity had to make hard choices, and it would be hard to argue that anything that was chosen should not have been. The absence of the Schrödinger equation is notable but quantum mechanics is discussed by considering the Dirac wave equation.

The varying level of difficulty of material in a historical survey such as this can be a formidable obstacle to an entertaining and inspiring presentation. This issue is handled brilliantly. The author is straightforward and clear in all of his presentations, and he is very good at giving real world analogies to illustrate difficult concepts. For example, when discussing the Dirac wave equation (nicely tied back to the discussion of quaternions in the preceding time period), the author uses the Feynman plate trick to illustrate how you can rotate one thing by 360 degrees (a plate on your open palm standing in for space) and have something else rotate by 180 degrees (your arm standing in for the wave function and causing the electron to go from "spin–up" to "spin–down"). The reader with a bit of familiarity with physics will get more from the latter part of the book.

The final equation of the book — the Black–Scholes equation — was an inspired choice. This equation underlies the methods that Wall Street uses to price a variety of financial products (derivatives). The spectacular market failure of Lehman Brothers that we have recently seen makes this topic very relevant, and it allows the author the opportunity to discuss a second example (the first having been weather forecasting) where mathematical modeling of the world is extraordinarily difficult and subject to chaos.

Interspersed throughout the book are amusing anecdotes that cleverly complement the more serious material and give some colour and depth to the people portrayed. I learned that a vuvuzela is a type of plastic horn at the same time I learned that it is topologically equivalent to a coffee cup or doughnut.

The book is beautifully printed in full colour. The accompanying artwork has been carefully and intelligently chosen and adds tremendously to the beauty and interest of the book.

A final word on beauty: I was thrilled that this idea — that mathematics is beautiful — was emphasized throughout. A great equation is surprising, concise, consequential, and universal. And the sum of these equations is itself an object of beauty. I reiterate: buy and read this book.

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FOCUS ON ...

No. 5

Michel Bataille

Inequalities *via* Lagrange Multipliers

Introduction

Problems requiring a proof of an inequality are frequent and often difficult. For constrained inequalities, the method of Lagrange multipliers is generally put aside at first, its application being considered as delicate. Without advocating any systematic use, we propose a few examples where, with some care, the method leads to a simple solution. We will restrict ourselves to three real variables and one equality constraint, in which case the theorem to be used is as follows: let f, g be continuously differentiable functions on an open subset U of $\mathbb{R}^3$ and $(a, b, c) \in F = \{(x, y, z) \in U : g(x, y, z) = 0\}$. If $f(a, b, c) \leq f(x, y, z)$ [or $f(a, b, c) \geq f(x, y, z)$] for all $(x, y, z) \in F$ and $(\partial_1 g(a, b, c), \partial_2 g(a, b, c), \partial_3 g(a, b, c)) \neq (0, 0, 0)$, then for some real number λ, we have $\partial_i f(a, b, c) = \lambda \partial_i g(a, b, c)$ for $i = 1, 2, 3$ (for a proof, see [1] for example). Here ∂_i denotes the partial derivative with respect to the i^{th} variable.

A maximum on the boundary

Our first example, Bilkent University Problem of the Month in November 2007, asked for the maximal value of $x^4y + y^4z + z^4x$ when x, y, z are nonnegative real numbers satisfying $x + y + z = 5$. Note that the continuous function

$$f : (x, y, z) \mapsto f(x, y, z) = x^4y + y^4z + z^4x$$

attains its maximum M on the compact set

$$K = \{(x, y, z) \in \mathbb{R}^3 \mid x, y, z \geq 0 \quad \text{and} \quad x + y + z = 5\}$$

and we are to find the value of M. First, we examine what occurs when (x, y, z) is on the boundary of K, say $z = 0,\ x + y = 5$. Then,

$$\begin{aligned} f(x, y, z) &= f(x, 5 - x, 0) = x^4(5 - x) \\ &= 4^4 \cdot \frac{x}{4} \cdot \frac{x}{4} \cdot \frac{x}{4} \cdot \frac{x}{4}(5 - x) \leq 4^4 \cdot \left(\frac{1}{5}(4 \times \frac{x}{4} + 5 - x)\right)^5 = 4^4 \end{aligned}$$

with equality when $x = 4$. Thus, the maximum value of f on the boundary of K is 256 and $M \geq 256$.

Now, assume that M is attained at an interior point (a, b, c) of K. Since the interior of K is open and the constraint is $x + y + z - 5 = 0$, there would exist a Lagrange multiplier λ such that

$$\partial_1 f(a, b, c) = \partial_2 f(a, b, c) = \partial_3 f(a, b, c) = \lambda$$

that is,

$$4a^3b + c^4 = \lambda,\ 4b^3c + a^4 = \lambda,\ 4c^3a + b^4 = \lambda.$$

Therefore, on the one hand, we would have $\lambda(a+b+c) = 5f(a,b,c)$ that is, $\lambda = M$ and on the other hand,

$$3\lambda = a^4 + b^4 + c^4 + 4a^3b + 4b^3c + 4c^3a < (a+b+c)^4 = 5^4.$$

As a result, $M < \frac{5^4}{3}$, in contradiction with $M \geq 256$. Thus, M is attained on the boundary of K and so $M = 4^4 = 256$. [For an alternative solution, see [2]].

A maximum in the interior

A slightly different example is provided by problem **3032** [2005 : 174, 177; 2006 : 190]:

Prove that

$$\frac{1}{1-ab} + \frac{1}{1-bc} + \frac{1}{1-ca} \leq \frac{9}{2}$$

whenever a, b, c are nonnegative real numbers such that $a^2+b^2+c^2 = 1$.

The featured solution by the proposer Vasile Cîrtoaje rests upon clever manipulations and identities. The method of Lagrange multipliers allows a different approach.

Let

$$f(x,y,z) = \frac{1}{1-xy} + \frac{1}{1-yz} + \frac{1}{1-zx}$$

and

$$K = \{(x,y,z) \in \mathbb{R}^3 : x,y,z \geq 0\ ,\ x^2+y^2+z^2 = 1\}.$$

The continuous function f reaches its maximum M on the compact set K and $M \geq f(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}) = \frac{9}{2}$. Moreover, if $(0,y,z) \in K$, then $f(0,y,z) = 2 + \frac{1}{1-yz} \leq 4 < M$. It follows that M is attained at (a,b,c), an interior point of K. We show that $a = b = c$, from which the result follows at once.

Since $f(x,y,z)$ is invariant under any permutation of x,y,z, we may suppose that $a \leq b \leq c$ so that $ab \leq ac \leq bc$. Since $bc < 1/3$ implies $f(a,b,c) < 3 \times \frac{3}{2} \leq M$, we must have $bc \geq 1/3$ and so $ab \leq 1/3, ab+ac \leq 2/3$ (note that $ab+ac+bc \leq a^2+b^2+c^2 = 1$). Now, there exists a Lagrange multiplier λ such that

$$\frac{b}{(1-ab)^2} + \frac{c}{(1-ac)^2} = 2\lambda a; \qquad \frac{a}{(1-ab)^2} + \frac{c}{(1-bc)^2} = 2\lambda b;$$
$$\frac{b}{(1-bc)^2} + \frac{a}{(1-ac)^2} = 2\lambda c;$$

from which we readily deduce

$$\frac{c^2-b^2}{bc(1-bc)^2} = a^2(\phi(ac) - \phi(ab)) \tag{1}$$

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where the function ϕ is defined by $\phi(t) = \frac{1}{t(1-t)^2}$. It follows that

$$\phi(ab) \leq \phi(ac). \tag{2}$$

It is easily checked that ϕ is decreasing on $(0, 1/3]$ and increasing on $[1/3, 1)$ and that $\phi(2/3 - t) < \phi(t)$ for $t \in (0, 1/3)$. These properties and (2) forbid $ac > 1/3$, hence $ab \leq ac \leq 1/3$ and so $\phi(ab) \geq \phi(ac)$ and $b = c$ (by (1)). Now, from

$$\lambda = \frac{b}{a(1-ab)^2} = \frac{a}{2b(1-ab)^2} + \frac{1}{2(1-b^2)^2}$$

and using $a^2 = 1 - b^2 - c^2 = 1 - 2b^2$, we deduce

$$(4b^2 - 1)(1 - b^2)^2 = \frac{1}{\phi(ab)} \leq \frac{1}{\phi(1/3)} = \frac{4}{27}.$$

But a quick study of the function $t \mapsto \psi(t) = (4t - 1)(1 - t)^2$ shows that $\psi(t) \geq \psi(1/3) = \frac{4}{27}$ for $t \in [\frac{1}{3}, \frac{1}{2})$ with equality only for $t = 1/3$. It follows that $b^2 = 1/3$ and so $a = b = c = 1/\sqrt{3}$.

Two exercises

(a) In reference to problem **2843** [2003 : 463; 2004 : 250], for $x, y, z > 0$, let

$$f(x, y, z) = (1 - x)(1 - y)(1 - z)$$

and

$$g(x, y, z) = 2\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) - 4 - \left(\frac{x}{yz} + \frac{y}{zx} + \frac{z}{xy}\right).$$

Show that $(a, b, c) = (\frac{3}{4}, \frac{3}{4}, \frac{3}{4})$ satisfies the constraint $g(x, y, z) = 0$ and $\partial_i f(a, b, c) = \lambda \partial_i g(a, b, c)$ $(i = 1, 2, 3)$ for some λ but $f(a, b, c)$ is not an extremum of f under the constraint.

(b) A part of problem **2787** [2002 : 460; 2003 : 477] was the inequality

$$\frac{1}{1 - \left(\frac{x+y}{2}\right)^2} + \frac{1}{1 - \left(\frac{y+z}{2}\right)^2} + \frac{1}{1 - \left(\frac{z+x}{2}\right)^2} \leq \frac{11}{3}$$

when $x + y + z = 1$ and $x, y, z \geq 0$. Prove this inequality with the method of Lagrange multipliers. (Hint : under the constraint, the left-hand side of the inequality is $h(x) + h(y) + h(z)$ where $h(t) = \frac{1}{3-t} + \frac{1}{1+t}$ and the derivative h' is strictly monotone.)

References

[1] J.E. Marsden, A.J. Tromba, *Vector Calculus*, Freeman, 1981, p. 217.

[2] Bilkent University Problem of the Month, November 2007(solution), `www.fen.bilkent.edu.tr/~cvmath/Problem/0711a.pdf`

PROBLEM OF THE MONTH

No. 4

Ross Honsberger

This column is dedicated to the memory of former ***CRUX with MAYHEM*** *Editor-in-Chief Jim Totten. Jim shared his love of mathematics with his students, with readers of CRUX with MAYHEM, and, through his work on mathematics contests and outreach programs, with many others. The "Problem of the Month" features a problem and solution that we know Jim would have liked.*

This problem comes from the 2004 Canadian Open Mathematics Challenge, a contest for high school students.

> *In a* sumac *sequence, $t_1, t_2, \ldots, t_m$, each term is an integer greater than or equal to* 0*. Also, each term, starting with the third, is the difference between the preceding two terms:*
>
> $$t_{n+2} = t_n - t_{n+1}, n \geq 1.$$
>
> *The sequence terminates at t_m if t_m is greater than t_{m-1}. For example,*
>
> $$120, 71, 49, 22, 27$$
>
> *is a sumac sequence of length* 5*.*
>
> *(a) In the set of sumac sequences that have $t_1 = 150$, find the positive integer B such that the sumac sequence $150, B, \ldots,$ has maximum length.*
>
> *(b) Let m be a positive integer greater than or equal to* 5*. Determine the number of sumac sequences of length m in which $t_m \leq 2000$ and in which no term is divisible by* 5*.*

Part (a) of this problem appeared on the 1980 Michigan Mathematics Prize Competition [1984 : 42; 1985 : 10-12] and its solution is given in my *More Mathematical Morsels* (Dolciani Series, Vol. 10, MAA, 1991, pages 191-194). The outstanding problem poser and solver Murray Klamkin (University of Alberta) found that, for arbitrary first term A, the value of B that gives the sequence of maximum length is either $[\tau A]$ or $[\tau A] + 1$, where $\tau = \dfrac{1+\sqrt{5}}{2}$, is the golden ratio, and the square brackets indicate the integer part of its argument. However, our interest here is in part (b).

The key to the part (b) is the observation that a sumac sequence is a general Fibonacci sequence in disguise: $120, 71, 49, 22, 27$ is just the Fibonacci sequence $27, 22, 49, 71, 120$ written backwards.

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In order for a sumac sequence to terminate, its last term x must exceed its second-last term y:

$$t_1, t_2, \ldots, y, x,$$

where $x > y$. A Fibonacci sequence that begins $x, y, \ldots$, where $x > y$,

$$x, y, x+y, x+2y, 2x+3y, 3x+5y, \ldots,$$

being of infinite length, can be truncated and reversed at any term to yield a sumac sequence of every length $m \geq 2$. It appears, then, that our task is to count the Fibonacci sequences that

(i) begin x, y,

(ii) have $x > y$,

(iii) have $x \leq 2000$, and

(iv) have no term divisible by 5.

Since each such Fibonacci sequence provides a sumac sequence of every length, the number of sumac sequences of length m that are generated by these Fibonacci sequences is the same for every $m \geq 2$. It is easy to find additional Fibonacci sequences which yield acceptable sumac sequences of lengths 2, 3, and 4. For example, reversing $17, 2, 19, 21, 40, \ldots$, yields the sumac sequence $2, 17$ of length 2, the sequence $19, 2, 17$ of length 3, and $21, 19, 2, 17$ of length 4, none of which has a term that is divisible by 5. The sequence $40, 21, 19, 2, 17$, of length 5, is not acceptable since 40 is divisible by 5. As we shall see, however, the acceptable sumac sequences of length $m \geq 5$ are generated *exclusively* by the Fibonacci sequences we are proposing to count. Mercifully, then, the number of acceptable sumac sequences of length $m \geq 5$ is the same for every m.

Clearly no term of such a Fibonacci sequence can end in 0 or 5. Now, the last digits of x and y set the last digits of the entire sequence. For example, if x ends in 7 and y in 2, then the last digits of the Fibonacci sequence are $7, 2, 9, 1, 0, 1, 1, 2, \ldots$. The presence of the 0 implies that every such sequence of length at least 5 contains a multiple of 5 and therefore, when reversed, fails to yield an acceptable sumac sequence of any length $m \geq 5$. Thus the pair $(7, 2)$ is not an acceptable pair for the final digits (a, b) of x and y.

Let us proceed to determine the acceptable pairs (a, b). As noted above, neither a nor b can itself be 0 or 5. Now, we mustn't assume, just because x is greater than y, that the last digit of x must be bigger than the last digit of y. I am embarrassed to confess that I thoughtlessly fell into this trap the first time I tried the problem and consequently failed to count almost half the sequences. There is nothing for it but to check all 64 of the *ordered* pairs (a, b) where $a, b \in \{1, 2, 3, 4, 6, 7, 8, 9\}$.

This threatens to be a long, tedious undertaking, but our salvation lies in the fact that if the n^{th} term f_n of a Fibonacci sequence is divisible by 5, then so is the term f_{n-5}:

$$f_n = f_{n-1} + f_{n-2} = 2f_{n-2} + f_{n-3} = 3f_{n-3} + 2f_{n-4} = 5f_{n-4} + 3f_{n-5};$$

since 3 and 5 are relatively prime, 5 divides f_n implies 5 divides f_{n-5}.

Hence, if any term f_n is divisible by 5, it follows that, proceeding toward the beginning of the sequence in steps of 5 terms, one of the first five terms will also be divisible by 5. Therefore we need only check the first five terms of a sequence of last digits in order to discover whether a pair (a, b) is acceptable.

At this point, it is evident that there is a one–to–one correspondence between the Fibonacci sequences that we are in the process of counting and the acceptable sumac sequences, *in view of their length being at least 5*, thus confirming our earlier claim that the acceptable sumac sequences are generated exclusively by our Fibonacci sequences:

> since no term of a sumac sequence is divisible by 5, in particular none of its last 5 terms (which are guaranteed by $m \geq 5$), none of the first 5 terms of the corresponding Fibonacci sequence is divisible by 5; therefore no term whatsoever of the Fibonacci sequence is divisible by 5, and consequently the Fibonacci sequence generates an acceptable sumac sequence of every length $m \geq 5$.

Not needing to check more than the first 5 terms of a sequence of last digits, it doesn't take very long to find that there are 16 acceptable ordered pairs (a, b):

$$\begin{gathered}(9,7),(9,2),(8,9),(3,9),(8,4),(6,8),(1,8),(7,6),\\(7,1),(4,7),(6,3),(2,6),(4,2),(3,4),(1,3),(2,1).\end{gathered}$$

Consider the pair $(9, 7)$. The largest $x \leq 2000$ that ends in 9 is 1999 and it occurs in each of the 200 $(9, 7)$–pairs:

$$(1999, 1997), (1999, 1987), (1999, 1977), \ldots, (1999, 7)$$

Each of these pairs gives a Fibonacci sequence that begins with 1999, and hence a sumac sequence that ends in 1999, of every length at least 5 (in fact, greater than or equal to 2). For example, the pair $(1999, 1997)$ yields the Fibonacci sequence

$$1999, 1997, 3996, 5993, 9989, \ldots$$

and hence the sumac sequence of length 5

$$9989, 5993, 3996, 1997, 1999.$$

Thus the largest x, 1999, gives rise to a total of 200 sumac sequences of length at least 5.

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The next largest value of x is 1989 and it occurs in the 199 $(9, 7)$–pairs

$$(1989, 1987), (1989, 1977), \ldots, (1989, 7).$$

Similarly for $x = 1979, 1969, \ldots, 19, 9$, which occur, respectively, in $198, 197, \ldots, 2, 1$ pairs. Altogether, then, the pair $(9, 7)$ gives rise to a grand total of

$$1 + 2 + \cdots + 200 = \frac{200(201)}{2} = 20\ 100$$

sequences.

Similarly, whenever $a > b$, the pair (a, b) yields 20 100 sequences. However, when $a < b$, there are slightly fewer sequences.

Consider the pair $(4, 7)$. The greatest x is 1994, and it pairs with each smaller positive integer that ends in 7. Since 4 is less than 7, these smaller integers start at 1987, giving 199 pairs. Again, $x = 1984$ pairs with 198 smaller positive integers ending in 7, starting at 1977, and so on for a total of

$$199 + 198 + \cdots + 1 + 0 = 19\ 900$$

(for $x = 4$ there are 0 smaller positive integers that end in 7).

Since there are eight ordered pairs (a, b) with $a > b$ and eight with $a < b$, the grand total of acceptable sequences is

$$8(20\ 100) + 8(19\ 900) = 8(40\ 000) = 320\ 000.$$

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PROBLEMS

Readers are invited to submit solutions, comments and generalizations to any problem in this section. Electronic submissions are preferable, with each solution contained in a separate file. Solution files should be named using the convention LastName_FirstName_ProblemNumber *(example Doe_Jane_1234.tex). It is preferred that readers submit a LaTeX file and a pdf file for each solution, although other formats, such as Microsoft Word, are also accepted. Readers are invited to email solutions to the editor at* `crux-editors@cms.math.ca`*. Submissions by regular mail are also accepted and should be sent to the address inside the back cover. Name(s) of solver(s) with affiliation, city, and country should appear on each solution, and each solution should start on a separate page. An asterisk (⋆) after a number indicates that a problem was proposed without a solution.*

Original problems are particularly sought, but other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by someone else without permission. Solutions, if known, should be sent with proposals. If a solution is not known, some reason for the existence of a solution should be included by the proposer. Proposal files should be named using the convention LastName_FirstName_Proposal_Year_number *(example Doe_Jane_Proposal_2014_4.tex, if this was Jane's fourth proposal submitted in 2014).*

To facilitate their consideration, solutions to the problems should be received by the editor by **1 May 2014**, *although solutions received after this date will also be considered until the time when a solution is published.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

3783. *Correction. Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let a, b, c be positive real numbers such that $ab + bc + ca = 3$. Prove that

$$(3a^2+2)\frac{a^3+b^3}{a^2+ab+b^2}+(3b^2+2)\frac{b^3+c^3}{b^2+bc+c^2}+(3c^2+2)\frac{c^3+a^3}{c^2+ca+a^2}\geq 10abc\,.$$

3792. *Correction. Proposed by Marcel Chiriţă, Bucharest, Romania.*

Solve the following system

$$\begin{aligned} 2^x+2^y&=12\\ 3^x+3^y&=36 \end{aligned}$$

for $x, y \in \mathbb{R}$.

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3801. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Triangle ABC is isosceles with $AB = AC$ and $\angle A = 100^\circ$. Let D be the point on AB such that $\angle BCD = 10^\circ$ and let E be the point on BC such that $EC = AC$. Determine the point K on CD such that triangles KAD and KCE have equal areas.

3802. *Proposed by Marcel Chiriţă, Bucharest, Romania.*

Solve the following system

$$\begin{aligned} \sqrt{2x+1} + \sqrt{3y+1} + \sqrt{4z+1} &= 15 \\ 3^{2x+\sqrt{3y+1}} + 3^{3y+\sqrt{4z+1}} + 3^{4z+\sqrt{2x+1}} &= 3^{30} \end{aligned}$$

for $x, y, z \in \mathbb{R}$.

3803. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let a, b, and c be positive real numbers. Prove that

$$\sqrt{a^2+ca} + \sqrt{b^2+ab} + \sqrt{c^2+bc} \le \sqrt{2}(a+b+c)\,.$$

3804. *Proposed by Václav Konečný, Big Rapids, MI, USA.*

Let $ABCD$ be a convex quadrilateral. Construct, using only compass and straight-edge, the line parallel to one side of the quadrilateral which bisects its area.

3805. *Proposed by Mehmet Şahin, Ankara, Turkey.*

Let ABC be a triangle with incentre I. Let A' be on ray IA beyond A such that $A'A = BC$. Let B' and C' be similarly defined, such that $B'B = CA$ and $C'C = AB$. Prove that

$$\frac{[A'B'C']}{[ABC]} \ge (1+\sqrt{3})^2,$$

where $[\cdot]$ denotes the area.

3806. *Proposed by Michel Bataille, Rouen, France.*

Let triangle ABC with angles $\alpha, \beta, \gamma \ne 90^\circ$ be inscribed in a circle with centre O and radius R, and let U, V, W be the centres of the hyperbolas with parameter R, focus O and associated directrices BC, CA, AB, respectively. Prove that

$$[UVW] \times [ABC] = R^4(\cos^2\alpha + \cos^2\beta + \cos^2\gamma),$$

where $[\cdot]$ denotes the area.

3807. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let ABC be a triangle with incentre I through which an arbitrary line passes meeting sides AB and AC at the points D and E respectively. Show that

$$\frac{1}{r} \geq \frac{1}{AD} + \frac{1}{AE}$$

where r denotes the inradius of ABC.

3808. *Proposed by Mehmet Şahin, Ankara, Turkey.*

Let ABC be a triangle with sides a, b, c, angle bisectors AT_a, BT_b, CT_c and medians $m_a = AM_a, m_b = BM_b, m_c = CM_c$. These lines define a new triangle with vertices

$$A' = BT_b \cap CM_c \quad B' = CT_c \cap AM_a \quad \text{and} \quad C' = AT_a \cap BM_b,$$

with angles α at A', β at B', and γ at C'. Prove that

$$\frac{m_a m_b m_c \sin\alpha \sin\beta \sin\gamma}{(a+2b)(b+2c)(c+2a)} = \frac{r}{32R},$$

where R is the circumradius and r is the inradius of ABC.

3809. *Proposed by Michel Bataille, Rouen, France.*

For positive real numbers x, y, let

$$G(x,y) = \sqrt{xy}, \quad A(x,y) = \frac{x+y}{2}, \quad Q(x,y) = \sqrt{\frac{x^2+y^2}{2}}\,.$$

Prove that

$$G(x^x, y^y) \geq (Q(x,y))^{A(x,y)}\,.$$

3810. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $k > 0$ be a positive real number. Find the value of

$$\int_0^1 \int_0^1 \left\{ \frac{x^k}{y} \right\} dxdy,$$

where $\{a\} = a - \lfloor a \rfloor$ denotes the fractional part of a.

. . . .

3783. *Correction. Proposé par George Apostolopoulos, Messolonghi, Grèce.*

Soit a, b, c trois nombres réels positifs avec $ab + bc + ca = 3$. Montrer que

$$(3a^2+2)\frac{a^3+b^3}{a^2+ab+b^2} + (3b^2+2)\frac{b^3+c^3}{b^2+bc+c^2} + (3c^2+2)\frac{c^3+a^3}{c^2+ca+a^2} \geq 10abc\,.$$

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3792. *Correction. Proposé par Marcel Chiriţă, Bucarest, Roumanie.*

Résoudre le système suivant

$$\begin{aligned} 2^x + 2^y &= 12 \\ 3^x + 3^y &= 36 \end{aligned}$$

for $x, y \in \mathbb{R}$.

3801. *Proposé par George Apostolopoulos, Messolonghi, Grèce.*

Soit ABC un triangle isocèle avec $AB = AC$ et $\angle A = 100°$. Soit D le point sur AB tel que $\angle BCD = 10°$ et soit E le point sur BC tel que $EC = AC$. Déterminer le point K sur CD tel que les aires des triangles KAD et KCE soient égales.

3802. *Proposé par Marcel Chiriţă, Bucarest, Roumanie.*

Résoudre le système suivant

$$\begin{aligned} \sqrt{2x+1} + \sqrt{3y+1} + \sqrt{4z+1} &= 15 \\ 3^{2x+\sqrt{3y+1}} + 3^{3y+\sqrt{4z+1}} + 3^{4z+\sqrt{2x+1}} &= 3^{30} \end{aligned}$$

pour $x, y, z \in \mathbb{R}$.

3803. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit a, b et c trois nombres réels positifs. Montrer que

$$\sqrt{a^2 + ca} + \sqrt{b^2 + ab} + \sqrt{c^2 + bc} \leq \sqrt{2}(a + b + c)\,.$$

3804. *Proposé par Václav Konečný, Big Rapids, MI, É-U.*

Soit $ABCD$ un quadrilatère convexe. En utilisant uniquement la règle et le compas, construire la droite parallèle à un côté du quadrilatère qui divise son aire en deux moitiés.

3805. *Proposé par Mehmet Şahin, Ankara, Turquie.*

Soit ABC un triangle et I son centre de gravité. Soit A' sur le rayon IA au-delà de A de sorte que $A'A = BC$. Soit B' et C' définis de manière analogue, de sorte que $B'B = CA$ et $C'C = AB$. Montrer que

$$\frac{[A'B'C']}{[ABC]} \geq (1 + \sqrt{3})^2,$$

où $[\cdot]$ désigne la surface.

3806. *Proposé par Michel Bataille, Rouen, France.*

Soit un triangle ABC, d'angles $\alpha, \beta, \gamma \neq 90°$, inscrit dans un cercle de centre O et de rayon R, et soit respectivement U, V, W les centres des hyperboles de paramètre R, de foyer O et de directrices associées BC, CA, AB. Montrer que

$$[UVW] \times [ABC] = R^4(\cos^2\alpha + \cos^2\beta + \cos^2\gamma),$$

où $[\cdot]$ désigne la surface.

3807. *Proposé par George Apostolopoulos, Messolonghi, Grèce.*

Soit ABC un triangle avec I comme centre du cercle inscrit par lequel passe une droite arbitraire coupant respectivement les côtés AB et AC aux points D et E. Montrer que

$$\frac{1}{r} \geq \frac{1}{AD} + \frac{1}{AE}$$

où r dénote le rayon du cercle inscrit de ABC.

3808. *Proposé par Mehmet Şahin, Ankara, Turquie.*

Soit ABC un triangle de côtés a, b, c, de bissectrices AT_a, BT_b, CT_c, et de médianes $m_a = AM_a, m_b = BM_b, m_c = CM_c$. Ces droites définissent un nouveau triangle de sommets

$$A' = BT_b \cap CM_c \quad B' = CT_c \cap AM_a \quad \text{et} \quad C' = AT_a \cap BM_b,$$

d'angles α en A', β en B' et γ en C'. Montrer que

$$\frac{m_a m_b m_c \sin\alpha \sin\beta \sin\gamma}{(a+2b)(b+2c)(c+2a)} = \frac{r}{32R},$$

où R est le rayon du cercle circonscrit de ABC et r celui de son cercle inscrit.

3809. *Proposé par Michel Bataille, Rouen, France.*

Soit

$$G(x,y) = \sqrt{xy}, \quad A(x,y) = \frac{x+y}{2}, \quad Q(x,y) = \sqrt{\frac{x^2+y^2}{2}}$$

où x et y sont des nombres réels positifs. Montrer que

$$G(x^x, y^y) \geq (Q(x,y))^{A(x,y)}.$$

3810. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Soit $k > 0$ un nombre réel positif. Trouver la valeur de

$$\int_0^1 \int_0^1 \left\{ \frac{x^k}{y} \right\} dxdy,$$

où $\{a\} = a - \lfloor a \rfloor$ désigne la partie fractionnaire de a.

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SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

1464. [1989 : 207 ; 1990 : 282-284] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $A'B'C'$ be a triangle inscribed in a triangle ABC, so that $A' \in BC$, $B' \in CA$, $C' \in AB$.

(a) Prove that

$$\frac{BA'}{A'C} = \frac{CB'}{B'A} = \frac{AC'}{C'A} \tag{1}$$

if and only if the centroids G, G' of the two triangles coincide.

(b) Prove that if (1) holds, and either the circumcenters O, O' or the orthocenters H, H' of the triangles coincide, then ΔABC is equilateral.

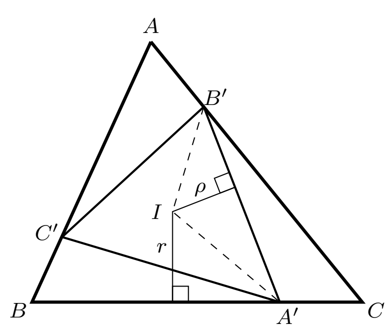
(c)★ If (1) holds and the incenters I, I' of the triangles coincide, characterize ΔABC.

Solution by C.R. Pranesachar, Indian Institute of Science, Bangalore, India.

(c)★ We will show that, as in part (b), (1) together with $I = I'$ imply that ΔABC (as well as $\Delta A'B'C'$) must be equilateral. Let

$$\frac{BA'}{A'C} = \frac{CB'}{B'A} = \frac{AC'}{C'A} = w$$

be the common ratio into which the sides are divided by A', B', and C'. Let, further, $BC = a, CA = b, AB = c$, area $[ABC] = F$, semiperimeter of triangle $ABC = s$, its inradius $= r$, and inradius of triangle $A'B'C' = \rho$.



By the Cosine formula we have

$$\begin{aligned} A'B'^2 &= A'C^2 + CB'^2 - 2 \cdot A'C \cdot CB' \cdot \cos C \\ &= \frac{a^2}{(1+w)^2} + \frac{b^2w^2}{(1+w)^2} - \frac{2abw}{(1+w)^2} \cdot \frac{(a^2+b^2-c^2)}{2ab} \\ &= \frac{1}{(1+w)^2} \left(b^2(w^2-w) + c^2w + a^2(1-w)\right). \end{aligned}$$

Also

$$\begin{aligned} [IA'CB'] = [IA'C] + [ICB'] &= \frac{1}{2}r \cdot A'C + \frac{1}{2}r \cdot CB' \\ &= \frac{1}{2} \cdot \frac{F}{s} \cdot \frac{(a+bw)}{(1+w)} = \frac{(a+bw)F}{(a+b+c)(1+w)}; \end{aligned}$$

and

$$[B'A'C] = \frac{1}{2} \cdot A'C \cdot CB' \cdot \sin C = \frac{1}{2} \frac{abw}{(1+w)^2} \cdot \frac{2F}{ab} = \frac{wF}{(1+w)^2}.$$

Hence

$$\begin{aligned} [IA'B'] &= [IA'CB'] - [B'A'C] \\ &= \frac{(a+bw)F}{(a+b+c)(1+w)} - \frac{wF}{(1+w)^2} = \frac{F}{(1+w)^2} \cdot \frac{(bw^2-cw+a)}{(a+b+c)}. \end{aligned}$$

Now the inradius ρ of triangle $A'B'C'$ is given by

$$\rho = 2 \cdot \frac{[IA'B']}{A'B'};$$

so

$$\rho^2 = 4 \cdot \frac{[IA'B']^2}{A'B'^2} = \frac{4F^2}{(1+w)^2(a+b+c)^2} \cdot \frac{(bw^2-cw+a)^2}{(b^2(w^2-w)+c^2w+a^2(1-w))}.$$

We have two further expressions for ρ^2 obtained by cyclically permuting a, b, c in the last expression. Canceling the common factor $4F^2/((1+w)^2(a+b+c)^2)$, we have

$$\begin{aligned} \frac{(aw^2-bw+c)^2}{(a^2(w^2-w)+b^2w+c^2(1-w))} &= \frac{(bw^2-cw+a)^2}{(b^2(w^2-w)+c^2w+a^2(1-w))} \\ &= \frac{(cw^2-aw+b)^2}{(c^2(w^2-w)+a^2w+b^2(1-w))}. \end{aligned}$$

From the equality of the second and third quotients we get (after cross-multiplying and removing the nonzero factor $w(w^2-w+1)(a+b+c)$),

$$\begin{aligned} (b^3+c^3-b^2c+bc^2-ab^2-ac^2)w^2 - 2ca(c-a)w& \\ - (a^3+b^3+a^2b-ab^2-a^2c-b^2c) = 0.& \quad (2) \end{aligned}$$

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Similarly from the equality of the first and third quotients we get

$$(c^3 + a^3 - c^2a + ca^2 - bc^2 - ba^2)w^2 - 2ab(a-b)w \\ - (b^3 + c^3 + b^2c - bc^2 - b^2a - c^2a) = 0. \quad (3)$$

We know that if two quadratic equations $P_1t^2 + Q_1t + R_1 = 0$ and $P_2t^2 + Q_2t + R_2 = 0$ have a common root then

$$(R_1P_2 - R_2P_1)^2 = (Q_1R_2 - Q_2R_1)(P_1Q_2 - P_2Q_1).$$

Thus, eliminating w from the quadratic equations (2) and (3), we get on factoring

$$(a+b+c)(a^2+b^2+c^2-bc-ca-ab) \\ \times (a^3+b^3+c^3+abc)(a^3+b^3+c^3-2ab^2-2ac^2+abc)^2 = 0.$$

Since the first and third factors on the left are positive, one of the remaining factors must be zero. From the Cauchy-Schwarz inequality applied to the vectors (a,b,c) and (b,c,a), we know that $a^2+b^2+c^2-bc-ca-ab = 0$ implies $a = b = c$. Finally, if $a^3+b^3+c^3-2ab^2-2ac^2+abc = 0$, we observe that had we chosen other pairs of equations in the previous argument we would have cyclically permuted the roles of a, b, c in equations (2) and (3), resulting in $a^3+b^3+c^3-2bc^2-2ba^2+abc = 0$ and $a^3+b^3+c^3-2ca^2-2cb^2+abc = 0$. From these three equations we deduce that

$$a(b^2+c^2) = b(c^2+a^2) = c(a^2+b^2),$$

from which we again conclude that $a = b = c$.

No other solutions have been received.

Pranesachar used MAPLE for his computations, but the results can be verified easily enough by hand. He remarked that because his calculations used squares of distances, his argument shows further that I could not even be an excentre of triangle $A'B'C'$. He went on to prove that even if the ratio w were allowed to be negative (in which case A', B', and C' lie outside triangle ABC), the incentres of the two triangles would coincide only if the triangles were equilateral. Because the proof of this claim is similar to the above solution, we will not reproduce it here. This result is in contrast with Problem 1492(b) [1999 : 508-510], where he with his coauthor B.V. Venkatachala showed that there exist nonequilateral triangles for which the incentre of ΔABC coincides with an excentre of $\Delta A'B'C'$ when condition (1) is replaced by the requirement that $AA' = BB' = CC'$.

3701. [2012 : 23, 25] *Proposed by R. F. Stöckli, Buenos Aires, Argentina.*

Let S be the set of all real continuous functions f defined on the closed interval $[0,1]$ with $f(0) = f(1) = 0$. Find all numbers $0 < r < 1$ such that for every f belonging to S there exists $c = c(f)$ and $d = d(f)$ belonging to $[0,1]$ such that $d - c = r$ and $f(c) = f(d)$.

I. Solution by George Apostolopoulos, Messolonghi, Greece.

The values of r are reciprocals of positive integers exceeding 1.

For integers $n \geq 2$, let $f(x)$ satisfy the conditions of the problem and let

$$g(x) = f\left(x + \frac{1}{n}\right) - f(x)$$

for $0 \le x \le (n-1)/n$. Suppose that $g(x)$ never vanishes. Then $g(x)$ must be either always positive or always negative. But then

$$\begin{aligned} 0 < \sum_{i=0}^{n-1} \left| g\left(\frac{i}{m}\right) \right| &= \sum_{i=0}^{n-1} \left| f\left(\frac{i+1}{n}\right) - f\left(\frac{i}{n}\right) \right| \\ &= \left| \sum_{i=0}^{n-1} f\left(\frac{i+1}{n}\right) - f\left(\frac{i}{n}\right) \right| = |f(1) - f(0)| = 0, \end{aligned}$$

a contradiction. Therefore $f(x+1/n) = f(x)$ for some $x \in [0, 1-1/n]$ and $r = 1/n$ satisfies the condition of the problem.

However, we give a counterexample for any other value of $r \in (0,1)$. Suppose that $k = 1 - r\lfloor 1/r \rfloor$. Since $0 < 1/r - \lfloor 1/r \rfloor < 1$, it follows that

$$0 < k = 1 - r\lfloor 1/r \rfloor < r.$$

Define $f(0) = 0$, $f(k) = -\lfloor 1/r \rfloor$ and $f(r) = 1$. Extend f linearly to the intervals $[0,k]$ and $[k,r]$. For $0 \le x \le 1-r$, let $f(x+r) = f(x)+1$. It is clear that $f(x+r)$ never equals $f(x)$. In addition,

$$f(1) = f\left(k + \left\lfloor \frac{1}{r} \right\rfloor r\right) = f(k) + \left\lfloor \frac{1}{r} \right\rfloor = 0.$$

II. Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

First let $r = 1/n$ for a positive integer n exceeding 1. Define $g(x)$ as in the foregoing solution. Then for $f(x)$ satisfying the conditions of the problem, $g(0) + g(1/n) + g(2/n) + \cdots + g((n-1)/n) = f(1) - f(0) = 0$. Hence either all of the $g(i/n)$ must vanish, or there are indices $i < j$ for which $g(i/n)$ and $g(j/n)$ have opposite signs. In the latter case, since $f(x)$ is continuous, there exists $c \in [i/n, j/n]$ for which $f(c+1/n) - f(c) = 0$.

On the other hand, let $r \in (0,1)$ not be the reciprocal of an integer. For $0 \le x \le 1$, define

$$f(x) = \left(\sin \frac{\pi x}{r}\right)^2 - x\left(\sin \frac{\pi}{r}\right)^2.$$

Then $f(x)$ is continuous, $f(1) = f(0) = 0$ and

$$\begin{aligned} f(x+r) &= \left(\sin \frac{\pi(x+r)}{r}\right)^2 - (x+r)\left(\sin \frac{\pi}{r}\right)^2 \\ &= \left(\sin\left(\frac{\pi x}{r} + \pi\right)\right)^2 - (x+r)\left(\sin \frac{\pi}{r}\right)^2 \\ &= \left(\sin\left(\frac{\pi x}{r}\right)\right)^2 - x\left(\sin \frac{\pi}{r}\right)^2 - r\left(\sin \frac{\pi}{r}\right)^2 = f(x) - r\left(\sin \frac{\pi}{r}\right)^2. \end{aligned}$$

Since $\sin(\pi/r) \ne 0$, $f(x+r) \ne f(x)$ for all $x \in [0, 1-r]$.

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Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; and the proposer. Geupel provided a piecewise linear example for the case $1/r$ not an integer. Bataille gave the same example as in the second solution. There was one incorrect solution.

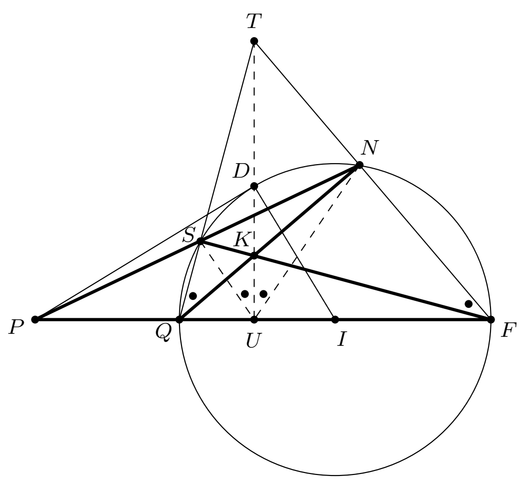
3702. [2012 : 23, 25] *Proposed by Nguyen Thanh Binh, Hanoi, Vietnam.*

Let the incircle (with centre I) of triangle ABC touch the sides BC at D, CA at E, and AB at F. Define M and P to be the points where BC intersects the lines IE and IF, N and Q to be the second intersections of the incircle with IE and IF, and R and S to be the second intersections of the incircle with MQ and PN. If H and K are the points where NQ intersects ER and FS, prove that $\angle KDH = \angle BAC$.

I. Solution by the proposer.

We will prove that DK is perpendicular to IF. Since IF is, by definition, also perpendicular to AB, we will be able to deduce that $KD||BA$. A similar argument will give us $DH||AC$, so that the angles KDH and BAC have corresponding sides parallel and are therefore equal as claimed.

Consider the portion of the given configuration shown in the accompanying figure : the diameter FQ lies along the line PI, while the secant SN also passes through P. The line PD is tangent at D to the circle with centre I, and the chords SF and QN intersect at K. Define the further points $T = QS \cap NF$ and $U = TK \cap QF$. We are to prove that TK is perpendicular to QF at U and passes through D.



Note that because QF is a diameter of a circle through S and N, K is the orthocentre of triangle QFT; it follows that TK defines the third altitude, whence $TU \perp QF$. It remains to show that D lies on TK. To that end we note that the quadrilateral $SQUK$ (with right angles at S and U) is cyclic, so that $\angle SQK = \angle SUK$. Similarly, the cyclic quadrilateral $FNKU$ gives us $\angle KFN = \angle KUN$. We have, therefore,

$$\begin{aligned}\angle SUN &= \angle SUK + \angle KUN = \angle SQK + \angle KFN \\ &= \angle SQN + \angle SFN = 2\angle SQN = \angle SIN.\end{aligned}$$

Thus, $SUIN$ is a cyclic quadrilateral and, therefore, $PS \cdot PN = PU \cdot PI$. But $PD^2 = PS \cdot PN$, whence $PD^2 = PU \cdot PI$. Since PDI is a right-angled triangle, we deduce that DU is perpendicular to PI. In other words, D lies on the perpendicular to PI through U, as do T and K. We conclude that the lines DK and $PI(= IF)$ are perpendicular, as desired. A similar argument using M in place of P finishes the proof.

II. Composite of solutions by Václav Konečný, Big Rapids, MI, USA ; and by Peter Y. Woo, Biola University, La Mirada, CA, USA.

Starting with the cyclic quadrilateral $FQSN$ whose side FQ is the diameter of the incircle Γ, point K is the intersection of its diagonals while the external point P is the intersection of two of its sides. The tangent to Γ from P touches it at D and, therefore, DK is the polar of P ; furthermore, the point where DK meets Γ again, call it D', must be the point where the second tangent from P touches Γ. Consequently, PI is the perpendicular bisector of DD' and, since F is also on this line, we conclude that $DK \perp IF$. But $IF \perp BA$, so $KD||BA$. Similarly, H lies on the polar of M and therefore $DH \perp IE$, or $DH||AC$. Thus, $\angle KDH = \angle BAC$, as desired.

Also solved by MICHEL BATAILLE, Rouen, France ; OLIVER GEUPEL, Brühl, NRW, Germany ; and EDMUND SWYLAN, Riga, Latvia.

The projective theorem used in solution II applies to any cyclic quadrilateral $FQSN$, whether or not one of its sides is a diameter. Specifically, if $FQSN$ is a convex quadrilateral inscribed in a circle with centre I, and the points K, P, and D are related to the quadrilateral as in the problem, then $DK \perp PI$. As far as this editor can see, none of the submitted solutions could be easily modified to provide an elementary Euclidean proof of this more general result.

3703. [2012 : 24, 26] *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.*

Let a, b, and c be the sides, and r the inradius of a triangle ABC. Prove that

$$\frac{a\sqrt[3]{abc}}{bc(a^2+bc)} + \frac{b\sqrt[3]{abc}}{ac(b^2+ca)} + \frac{c\sqrt[3]{abc}}{ab(c^2+ab)} \le \frac{1}{8r^2}\,.$$

Solution by John G. Heuver, Grande Prairie, AB.

Let H denote the summation on the left side of the given inequality. Then

$$H = \frac{\sqrt[3]{abc}}{abc}\left(\frac{a^2}{a^2+bc} + \frac{b^2}{b^2+ca} + \frac{c^2}{c^2+ab}\right)\,. \tag{1}$$

In the solution to ***Crux*** problem **3374** [2009 : 414–415] Peter Y. Woo proved that

$$\frac{a^2}{a^2+bc} + \frac{b^2}{b^2+ca} + \frac{c^2}{c^2+ab} \le \frac{a+b+c}{2\sqrt[3]{abc}}\,. \tag{2}$$

Let s and R denote the semi-perimeter and circumradius of triangle ABC, respectively. It is well known that $abc = 4rsR$ which, when combined with (1) and (2) then yields

$$H \le \frac{a+b+c}{2abc} = \frac{2s}{8rsR} = \frac{1}{4rR} \le \frac{1}{8r^2}$$

since $2r \le R$ by Euler's Inequality.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina(two solutions); RADOUAN BOUKHARFANE, Polytechnique de Montréal, PQ; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC(two solutions); PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; EDMUND SWYLAN, Riga, Latvia; and the proposer. There was also one incomplete solution.

3704. [2012 : 24, 26] *Proposed by Richard McIntosh, University of Regina, Regina, SK.*

Let $p \equiv 1 \pmod 3$ be a prime and let n be an integer satisfying $n^2+n+1 \equiv 0 \pmod p$. Prove that $(n+1)^p \equiv n^p + 1 \pmod{p^3}$.

Composite of similar solution by George Apostolopoulos, Messolonghi, Greece; and Michel Bataille, Rouen, France.

Let $Z[x]$ denote the ring of all polynomials in x with integer coefficients. Since $p \mid n^2+n+1$ implies $p^2 \mid (n^2+n+1)^2$, it suffices to prove that

$$p(x^2+x+1)^2 \mid (x+1)^p - x^p - 1$$

in $Z[x]$.

Let $f(x) = x^2+x+1$ and $g(x) = (x+1)^p - x^p - 1$. Since $p \mid \binom{p}{k}$ for all $k = 1, 2, \ldots, p-1$ and $g(x) = \sum_{k=1}^{p-1} \binom{p}{k} x^{p-k}$, clearly $p \mid g(x)$. It now suffices to show that the two complex roots, ω and $\overline{\omega}$, of $f(x)$, where $\omega = \frac{-1+\sqrt{3}i}{2}$, are multiple roots of $g(x)$.

Since $\omega^2+\omega+1 = 0$, $\omega^3 = 1$ and $p \equiv 1 \pmod 3$ we have $\omega^p = \omega$. Hence,

$$g(\omega) = (\omega+1)^p - \omega^p - 1 = (-\omega^2)^p - \omega^p - 1 = -\omega^2 - \omega - 1 = 0\,.$$

Furthermore, $g'(x) = p(x+1)^{p-1} - px^{p-1}$ so

$$g'(\omega) = p((\omega+1)^{p-1} - \omega^{p-1}) = p((-\omega^2)^{p-1} - \omega^{p-1}) = p(1-1) = 0\,.$$

Therefore, both ω and $\overline{\omega}$ are multiple roots of $g(x)$; that is, $(f(x))^2 \mid g(x)$ and our proof is complete.

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer. There was also a partially incorrect solution which claims that the condition $p \equiv 1 \pmod 3$ follows from the other given condition. The example $n = 1$ and $p = 3$ shows that the claim is false. However, it is interesting to note that using Fermat's little theorem one can easily show that if $(n+1)^p \equiv n^p + 1 \pmod p$ then $p \equiv 1 \pmod 3$ or $p = 3$.

3705. [2012 : 24, 26] *Proposed by Michel Bataille, Rouen, France.*

Let ABC be a scalene triangle and G its centre of gravity. Let the perpendicular to BC through G meet the internal bisector of $\angle BAC$ at A'.

(a) Show that G and the orthogonal projections of A' onto the lines AB and AC are collinear.

(b) If B' and C' are defined similarly to A', prove that

$$\frac{GA' \cdot GB'}{CA \cdot CB} + \frac{GB' \cdot GC'}{AB \cdot AC} + \frac{GC' \cdot GA'}{BC \cdot BA} = \frac{1}{9}\,.$$

Similar solutions by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; and by Peter Y. Woo, Biola University, La Mirada, CA, USA.

(a) Define A_2 to be the point where AA' again intersects the circumcircle of triangle ABC. Denote the midpoint of side BC by A_1, and note that $A_1A_2 \perp BC$. Because GA' has been defined to be perpendicular to BC, we have $A_1A_2 || GA'$, whence triangles $AA'G$ and AA_2A_1 are homothetic with ratio of magnitude $\frac{2}{3}$. If A_1, B_1, C_1 are the feet of perpendiculars dropped from A_2 to BC, CA, AB, then these points lie on a line, namely the Simson line of A_2 with respect to ΔABC. If D and E are the orthogonal projections of A' onto the lines AC and AB, then the dilatation with centre A and ratio $\frac{2}{3}$ that shrinks ΔAA_2A_1 to $\Delta AA'G$ takes B_1 to D and C_1 to E; consequently, D, E, G are collinear.

(b) From the right triangle BA_2A_1 we see that $A_1A_2 = \frac{BC}{2}\tan\frac{A}{2}$; from the dilatation of part (a) we have $GA' = \frac{2}{3}A_1A_2$. Thus,

$$\frac{GA'}{BC} = \frac{1}{3}\tan\frac{A}{2}.$$

Multiplying this by the analogous quotient with respect to the point B' and side CA, we have

$$\frac{GA' \cdot GB'}{CA \cdot CB} = \frac{1}{9}\tan\frac{A}{2}\tan\frac{B}{2}.$$

Continuing in this manner, we deduce that

$$\frac{GA' \cdot GB'}{CA \cdot CB} + \frac{GB' \cdot GC'}{AB \cdot AC} + \frac{GC' \cdot GA'}{BC \cdot BA} = \frac{1}{9}\left(\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2}\right).$$

The sum in parentheses on the right is known to be 1 (this is just the tangent-of-a-sum identity applied to three angles that sum to $90°$), which completes the proof of part (b).

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; OLIVER GEUPEL, Brühl, NRW, Germany; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; EDMUND SWYLAN, Riga, Latvia;

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TITU ZVONARU, Comăneşti, Romania; and the proposer. There was one incomplete submission.

Konečný observed that the role of G can be replaced by any point M (except A) on the median AA_1; specifically, the featured solution to part (a) shows that if A' were defined to be the intersection of the bisector of $\angle BAC$ with the perpendicular to BC through M, then M is collinear with the orthogonal projections of A' onto the lines AB and AC.

3706. [2012 : 24, 26] *Proposed by Vo Quoc Ba Can, Can Tho University of Medicine and Pharmacy, Can Tho, Vietnam.*

Prove that for all positive real numbers a, b, c, and d which satisfy $a, b, c \geq 1$ and $abcd = 1$,

$$\sum_{\text{cyclic}} \frac{1}{(a^2 - a + 1)^2} \leq 4 .$$

I. Solution by Michel Bataille, Rouen, France.

Since $2(x^2 - x + 1)^2 - (x^4 + 1) = (x - 1)^4 \geq 0$ for all real x, it suffices to show that

$$\frac{1}{a^4 + 1} + \frac{1}{b^4 + 1} + \frac{1}{c^4 + 1} + \frac{(abc)^4}{(abc)^4 + 1} \leq 2$$

or, equivalently,

$$\frac{1}{a^4 + 1} + \frac{1}{b^4 + 1} + \frac{1}{c^4 + 1} \leq 1 + \frac{1}{(abc)^4 + 1}.$$

Thus, we have to establish that

$$f(u) + f(v) + f(w) \leq f(u + v + w)$$

where $u = \ln a$, $v = \ln b$, $w = \ln c$, and

$$f(x) = \frac{1}{e^{4x} + 1} - \frac{1}{2}.$$

It is straightforward to check that $f(0) = 0$, $f'(x) < 0$ and $f''(x) > 0$, so that $f(x)$ is negative, decreasing and convex on $(0, \infty)$. Thus $f'(x)$ increases to 0. For each fixed $y \geq 0$, let $\phi(x) = f(x+y) - f(x) - f(y)$. Then $\phi(0) = 0$ and the derivative $\phi'(x) = f'(x + y) - f'(x)$ is positive on $(0, \infty)$, so that $f(x + y) \geq f(x) + f(y)$. Therefore

$$f(u + v + w) \geq f(u + v) + f(w) \geq f(u) + f(v) + f(w)$$

and the desired inequality follows.

II. Solution by the proposer.

First we establish that, for $x, y \geq 1$,

$$\frac{1}{g(x)^2} + \frac{1}{g(y)^2} \leq 1 + \frac{1}{g(xy)^2},$$

where $g(t) = t^2 - t + 1$. The difference between the two sides of this inequality is

$$\left(1 - \frac{1}{g(x)^2}\right)\left(1 - \frac{1}{g(y)^2}\right) + \frac{1}{g(xy)^2} - \frac{1}{g(x)^2 g(y)^2},$$

a fraction whose denominator is $g(x)^2 g(y)^2 g(xy)^2$ and whose numerator is

$$\begin{aligned}
&(g(x)-1)(g(x)+1)(g(y)-1)(g(y)+1)g(xy)^2 \\
&\qquad + (g(x)g(y) - g(xy))(g(x)g(y) + g(xy)) \\
&\quad = x(x-1)(x^2-x+2)y(y-1)(y^2-y+2)(x^2y^2-xy+1)^2 \\
&\qquad - (x-1)(y-1)(x+y)(2x^2y^2 - xy(x+y) + x^2 + y^2 - x - y + 2). \\
&\quad = (x-1)(y-1)[xy(x^2-x+2)(y^2-y+2)(x^2y^2-xy+1)^2 \\
&\qquad - (x+y)(2x^2y^2 - xy(x+y) + x^2 + y^2 - x - y + 2)].
\end{aligned}$$

Since

$$(x^2 - x + 2)(y^2 - y + 2)(x^2y^2 - xy + 1) \geq 4$$

and

$$4xy = 2(x+y) + (2x-1)(2y-1) - 1 \geq 2(x+y),$$

this numerator is not less than

$$\begin{aligned}
&(x-1)(y-1)(x+y) \\
&\times[2(x^2y^2 - xy + 1) - 2x^2y^2 + xy(x+y) - (x+y)^2 + 2xy + (x+y) - 2] \\
&\qquad = (x-1)(y-1)(x+y)[xy(x+y) - (x+y)^2 + (x+y)] \\
&\qquad = (x-1)^2(y-1)^2(x+y)^2 \geq 0.
\end{aligned}$$

Hence, for $a, b, c \geq 1$ and $d = (abc)^{-1}$,

$$\begin{aligned}
\frac{1}{g(a)^2} + \frac{1}{g(b)^2} + \frac{1}{g(c)^2} &\leq 1 + \frac{1}{g(ab)^2} + \frac{1}{g(c)^2} \\
&= 2 + \frac{1}{g(abc)^2} = 2 + \frac{1}{g(1/d)^2} \\
&= 2 + \frac{d^4}{g(d)^2} \leq 4 - \frac{1}{g(d)^2},
\end{aligned}$$

which is the desired result. The final inequality is due to the fact that

$$2 - \frac{1 + d^4}{g(d)^2} = \frac{(1-d)^4}{g(d)^2} \geq 0.$$

III. Solution by Kee-Wai Lau, Hong Kong, China.

We prove that

$$\sum_{k=1}^{n} \frac{1}{(x_k^2 - x_k + 1)^2} \leq n$$

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subject to the restrictions $x_1, x_2, \cdots, x_{n-1} \geq 1$ and $x_1 x_2 \cdots x_n = 1$. This holds for $n = 2$.

Denote the left side by $f_n(x_1, x_2, \cdots, x_{n-1})$ with x_n dependent on the remaining variables. Assume as an induction hypothesis that

$$f_n(x_1, x_2, \cdots, x_{n-1}) \leq n$$

subject to the stated conditions. Assume that $x_1, x_2, \cdots, x_n \geq 1$, that $x_1 x_2 \cdots x_n x_{n+1} = 1$ and $f_{n+1}(x_1, \cdots, x_n)$ is the analogous left side of the inequality. Then

$$f_{n+1}(x_1, x_2, \cdots, x_{n-1}, 1) = f_n(x_1, x_2, \cdots, x_{n-1}) + 1 \leq n + 1.$$

It remains to show that

$$f_{n+1}(x_1, x_2, \cdots, x_{n-1}, x_n) \leq f_{n+1}(x_1, x_2, \cdots, x_{n-1}, 1)$$

when $x_1, \cdots, x_n \geq 1$. Now

$$\frac{\partial f_{n+1}}{\partial x_n} = \frac{2(1 - 2x_n)}{(x_n^2 - x_n + 1)^3} + \frac{2}{x_n} g(x_1 x_2 \cdots x_n)$$

where $g(t) = t^4(2-t)(t^2-t+1)^{-3}$. When $x_1 x_2 \cdots x_n \geq 2$, then $g(x_1, x_2, \cdots x_n) \leq 0$ and $\frac{\partial f_{n+1}}{\partial x_n} \leq 0$.

Now suppose that $1 \leq x_1 x_2 \cdots x_n \leq 2$. Since

$$g'(t) = t^3(t-1)(t^2 - t - 8)(t^2 - t + 1)^{-4} < 0$$

for $1 < t < 2$, then $g(t)$ is decreasing. Since $1 \leq x_n \leq x_1 x_2 \cdots x_n \leq 2$, it follows that $g(x_n) \geq g(x_1 x_2 \cdots x_n)$ and

$$\begin{aligned}\frac{\partial f_{n+1}}{\partial x_n} &\leq \frac{2(1 - 2x_n)}{(x_n^2 - x_n + 1)^3} + \frac{2x_n^3(2 - x_n)}{(x_n^2 - x_n + 1)^3} \\ &= \frac{-2(x_n + 1)(x_n - 1)^3}{(x_n^2 - x_n + 1)^3} < 0\end{aligned}$$

for $x_n > 1$, so that $f_n(x_1, \cdots, x_n)$ is a decreasing function of x_n and the desired result follows.

IV. Solution by Haohao Wang and Yanping Xia, Southeast Missouri State University, Cape Girardeau, MO, USA.

Define $f_n(x_1, x_2, \cdots, x_{n-1})$ as in the foregoing solution with $x_i \geq 1$ $(1 \leq i \leq n-1)$ and $x_n = (x_1 x_2 \cdots x_{n-1})^{-1}$. We have to show that f_n is maximized at $(1, 1, \cdots, 1)$, where it assumes the value n.

A critical value f_n occurs in the interior of its domain if and only if

$$0 = \frac{\partial f_n}{\partial x_i} = \frac{2}{x_i}[h(x_n) - h(x_i)]$$

or $h(x_n) = h(x_i)$ for $1 \le i \le n-1$, where $h(t) = t(2t-1)(t^2-t+1)^{-3}$. Since, for $t > 1$, $h'(t) = -(t-1)(8t^2+t-1)(t^2-t+1)^{-4} < 0$, h is strictly decreasing on $[1,\infty)$ so the condition for an extremum is $x_1 = x_2 = \cdots = x_n$. This implies that $h(x_1) = h(x_1^{-(n-1)})$. However, $h(t) > h(t^{-1})$ when $t > 1$, so that $h(x_1) > h(x_1^{n-1}) > h(x_1^{-(n-1)})$ from which we see that there are no extrema in the interior of the domain of f.

Consider the values of f_n on the boundary of its domain, say where $x_{n-1} = 1$. Then $f_n(x_1, x_2, \cdots, x_{n-2}, 1) = f_{n-1}(x_1, x_2, \cdots, x_{n-2}) + 1$ with $x_1, x_2, \cdots, x_{n-2} \ge 1$ and $x_1 x_2 \cdots x_{n-1} = 1$. By descent, we eventually need to prove the result for $n = 2$. But this follows from

$$\frac{1}{(x_1^2 - x_1 + 1)^2} + \frac{1}{(x_1^{-2} - x_1^{-1} + 1)^2} = 2 - \frac{(1-x_1)^4}{(x_1^2 - x_1 + 1)^2},$$

with equality if and only if $x_1 = x_2 = 1$. The desired result follows.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; RADOUAN BOUKHARFANE, Polytechnique de Montréal, PQ; and OLIVER GEUPEL, Brühl, NRW, Germany. Geupel pointed out that the problem was previously posed by the proposer in Vietnamese and reproduced his solution. He refers us to Problem 1.45 on page 38f in the e-paper toanhocmuonmaumain.pdf, available on the MathLinks forum website www.mathlinks.ro/viewtopic.php?t=197674. There was one incorrect solution.

It is straightforward to adapt the first two solutions to establish generalizations of the last two.

3707. [2012 : 24, 26] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let k and m be positive integers. Prove that

$$\int_0^\infty \frac{\sin^{2k} x}{(\pi^2 - x^2)((2\pi)^2 - x^2)\cdots((k\pi)^2 - x^2)} dx = 0\,.$$

I. Solution by Michel Bataille, Rouen, France.

Let I be the given integral. Then I $= \int_0^\infty f(x)dx$ where $f(x) = 0$ when x is a positive multiple of π and $f(x) = (\sin^{2k} x)(\prod_{j=1}^k ((j\pi)^2 - x^2)^{-1}$ otherwise for $x \ge 0$. For $1 \le j \le k$,

$$\frac{\sin^{2k} x}{j^2\pi^2 - x^2} = \frac{\sin^{2k-1} x}{j\pi + x} \cdot \frac{\sin(j\pi - x)}{j\pi - x} \cdot (-1)^{j-1}$$

and $\lim_{x \to j\pi} (\sin(j\pi - x))(j\pi - x)^{-1} = 1$, so that $\lim_{x \to j\pi} f(x) = 0$ and $f(x)$ is continuous on $[0,\infty)$. Thus $\int_0^{(k+1)\pi} f(x)dx$ exists. Since also $|f(x)| \le (\prod_{j=1}^k |(j\pi)^2 - x^2)|^{-1}$ for $x \ge (k+1)\pi$ and $\int_{(k+1)\pi}^\infty (\prod_{j=1}^k |(j\pi)^2 - x^2|)^{-1} dx < \infty$, the integral I exists.

We have the partial fraction decomposition

$$\frac{1}{\prod_{j=1}^k (j\pi)^2 - x^2} = a_j \left(\frac{1}{j\pi + x} + \frac{1}{j\pi - x} \right),$$

so that it suffices to prove that

$$\int_0^\infty \left(\frac{\sin^{2k} x}{j\pi + x} + \frac{\sin^{2k} x}{j\pi - x} \right) dx = 0.$$

Recall that $\int_0^\infty (\sin^{2k} x) x^{-1} dx$ exists. Then

$$\begin{aligned}\int_0^\infty \left(\frac{\sin^{2k} x}{j\pi + x} + \frac{\sin^{2k} x}{j\pi - x} \right) dx &= \int_0^\infty \frac{\sin^{2k} x}{j\pi + x} dx + \int_0^{j\pi} \frac{\sin^{2k} x}{j\pi - x} dx - \int_{j\pi}^\infty \frac{\sin^{2k} x}{x - j\pi} dx \\ &= \int_{j\pi}^\infty \frac{\sin^{2k} u}{u} du - \int_{j\pi}^0 \frac{\sin^{2k} v}{v} dv - \int_0^\infty \frac{\sin^{2k} w}{w} dw,\end{aligned}$$

using the respective substitutions $x = u - j\pi$, $x = j\pi - v$, $x = j\pi + w$. Thus

$$\int_0^\infty \left(\frac{\sin^{2k} x}{j\pi + x} + \frac{\sin^{2k} x}{j\pi - x} \right) dx = \int_{j\pi}^\infty \frac{\sin^{2k} x}{x} dx + \int_0^{j\pi} \frac{\sin^{2k} x}{x} dx - \int_0^\infty \frac{\sin^{2k} x}{x} dx = 0$$

as desired.

II. Solution by Oliver Geupel, Brühl, NRW, Germany.

We prove the more general statement, that for positive integers k and m,

$$\int_0^1 \frac{|\sin x|^m}{(\pi^2 - x^2)((2\pi)^2 - x^2) \cdots ((k\pi)^2 - x^2)} dx = 0.$$

For each positive integer k, let $q_k(x) = \prod_{j=1}^k (x - j\pi)(x + j\pi)$. Then

$$\frac{1}{q_k(x)} = \sum_{j=1}^k a_j \left(\frac{1}{x - j\pi} - \frac{1}{x + j\pi} \right)$$

where $a_j = 1/q_k'(jx)$.

We establish, by induction on k, the identity

$$\sum_{n=0}^\infty \frac{1}{q_k(x + nx)} = \sum_{n=0}^{k-1} \left(\sum_{j=n+1}^k a_j \right) \left(\frac{1}{x + n\pi} + \frac{1}{x - (n+1)\pi} \right).$$

The case $k = 1$ is covered by the equation

$$\sum_{n=0}^\infty \frac{1}{q_1(x + n\pi)} = a_1 \sum_{n=0}^\infty \left(\frac{1}{x + (n-1)\pi} - \frac{1}{x + (n+1)\pi} \right) = a_1 \left(\frac{1}{x - \pi} + \frac{1}{x} \right).$$

Suppose that the identity has been established when the degree of $q(x)$ is less than $2k$. Then

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{q_k(x+n\pi)} &= \sum_{n=0}^{\infty}\frac{1}{q_{k-1}(x+n\pi)}+\sum_{n=0}^{\infty}a_k\left(\frac{1}{x+n\pi-k\pi}-\frac{1}{x+nx+k\pi}\right)\\ &= \sum_{n=0}^{k-2}\left(\sum_{j=n+1}^{k-1}a_j\right)\left(\frac{1}{x+n\pi}+\frac{1}{x-(n+1)\pi}\right)+a_k\sum_{j=-k}^{k-1}\frac{1}{x+j\pi}\\ &= \sum_{n=0}^{k-1}\left(\sum_{j=n+1}^{k}a_j\right)\left(\frac{1}{x+n\pi}+\frac{1}{x-(n+1)\pi}\right),\end{aligned}$$

as desired.

For the result, we have that

$$\begin{aligned}\int_0^{\infty}\frac{|\sin x|^m}{q_k(x)}dx &= \sum_{n=0}^{\infty}\int_{n\pi}^{(n+1)\pi}\frac{|\sin x|^m}{q_k(x)}dx = \sum_{n=0}^{\infty}\int_0^{\pi}\frac{|\sin(x+n\pi)|^m}{q_k(x+n\pi)}dx\\ &= \int_0^{\pi}|\sin x|^m\sum_{n=0}^{\infty}\frac{1}{q_k(x+n\pi)}dx = \int_0^{\pi}f(x)dx,\end{aligned}$$

where

$$f(x)=|\sin x|^m\sum_{n=0}^{k-1}\left(\sum_{j=n+1}^{k}a_j\right)\left(\frac{1}{x+n\pi}+\frac{1}{x-(n+1)\pi}\right).$$

Note that $f(x)=-f(\pi-x)$. Therefore,

$$\int_0^{\pi}f(x)dx=-\int_0^{\pi}f(x-\pi)dx=-\int_{\pi}^{0}f(u)(-du)=-\int_0^{\pi}f(x)dx,$$

whence $\int_0^{\pi}f(x)=0$ and the conclusion of the problem follows.

Also solved by PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and the proposer. Perfetti set $\sin x=\frac{1}{2i}(e^{ix}-e^{-ix})$ *and used contour integration to solve the problem.*

3708. [2012 : 24, 26] *Proposed by Václav Konečný, Big Rapids, MI, USA.*

Construct the isosceles trapezoid with three equal sides a, with a straightedge and a compass alone, provided that its base $b=AB$ and the angle α ($0^\circ<\alpha<90^\circ$) at the base are given.

Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

Construct rays with initial points A and B that intersect at a point C such that $\angle CAB=\angle CBA=\alpha$. Let the bisector of $\angle CAB$ intersect BC at D, and the bisector of $\angle CBA$ intersect AC at E. Construct segment DE. Then $ABDE$ is the desired isosceles trapezoid. *Proof*: Since $\angle CED=\angle CAB=\alpha$, we have $\angle AED=180^\circ-\alpha$. Since $\angle EAD=\alpha/2$,

$$\angle EDA=180^\circ-\frac{\alpha}{2}-(180^\circ-\alpha)=\frac{\alpha}{2}.$$

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It follows that triangle ADE is isosceles with base angles at A and D. Thus, $AE = ED$. By symmetry $BD = DE$, so that $AE = ED = DB$.

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; EDMUND SWYLAN, Riga, Latvia; TITU ZVONARU, Comăneşti, Romania; and the proposer. There was one incorrect submission.

Most of the constructions used the bisectors of the base angles as in the featured solution. For the solutions of Konečný and of Zvonaru, they constructed the length a after having determined that $a = \frac{b}{1+2\cos\alpha} = \frac{bt}{t+b}$, *where* $t = AC$ *(in the notation of the featured solution). Bataille located the reference I.M. Yaglom,* Geometric Transformations *I, MAA (1962) pages 132-133, where a more challenging variant of our problem can be found : Given* ΔABC *(not necessarily isosceles), construct* D *on side* BC *and* E *on side* AC *such that* $AE = ED = EB$.

3709. [2012 : 25, 27] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let a, b, and c be nonnegative real numbers, k and $l \geq 0$ and define

$$\frac{a+b}{2} - \sqrt{ab} \;=\; k^2\,,\; \frac{a+b+c}{3} - \sqrt[3]{abc} \;=\; l^2\,.$$

Prove that

$$\max(a,b,c) \;\geq\; \min(a,b,c) + \frac{3}{2}(k-l)^2\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany, expanded by the editor.

We set $M = \max(a,b,c)$, $m = \min(a,b,c)$ and assume, without loss of generality, that $a \leq b$.

Note that

$$k^2 = \frac{2}{3}(b-a) - \frac{1}{6}(b + 6\sqrt{ab} - 7a) \leq \frac{2}{3}(M-m)$$

since $0 \leq b - a \leq M$ and $b + 6\sqrt{ab} - 7a \geq 0$. Hence

$$k - \sqrt{\frac{2}{3}(M-m)} \leq 0 \leq l\,. \tag{1}$$

By Schur's inequality we have for $x, y, x \geq 0$, that

$$x^2(x^2-y^2)(x^2-z^2) + y^2(y^2-z^2)(y^2-x^2) + z^2(z^2-x^2)(z^2-y^2) \geq 0$$

which together with the AM-GM inequality then yields

$$x^6 + y^6 + z^6 + 3x^2y^2z^2 \geq \sum_{\text{cyclic}} (x^4y^2 + x^2y^4) \geq 2(x^3y^3 + y^3z^3 + z^3x^3)\,. \tag{2}$$

Setting $x^6 = a$, $y^6 = b$ and $z^6 = c$, then (2) becomes

$$a + b + c + 3\sqrt[3]{abc} \geq 2(\sqrt{ab} + \sqrt{bc} + \sqrt{ca})$$

or

$$\frac{a+b+c}{3} \geq \frac{2}{3}(\sqrt{ab} + \sqrt{bc} + \sqrt{ca}) - \sqrt[3]{abc}\,.$$

Hence,

$$l^2 = \frac{a+b+c}{3} - \sqrt[3]{abc} \le \frac{2}{3}\left(a+b+c-\sqrt{ab}-\sqrt{bc}-\sqrt{ca}\right) . \qquad (3)$$

We now prove the inequality below :

$$2(a+b+c-\sqrt{ab}-\sqrt{bc}-\sqrt{ca}) \le 3k^2 + 2(M-m) . \qquad (4)$$

To this end we consider three cases separately :

Case (i). If $a \le b \le c$, then $M - m = c - a$, so (4) becomes

$$2\left(a+b+c-\sqrt{ab}-\sqrt{bc}-\sqrt{ca}\right) \le \frac{3(a+b)}{2} - 3\sqrt{ab} + 2(c-a) .$$

The last inequality above, after straightforward computations, reduces to

$$5a + 2\sqrt{ab} + b \le 4\sqrt{bc} + 4\sqrt{ca}$$

which is true since $4a \le 4\sqrt{ca}$ and $a + 2\sqrt{ab} + b \le 4\sqrt{bc}$.

Case (ii). If $a \le c \le b$, then $M - m = b - a$ and after simplifications (4) reduces to

$$5a + 2\sqrt{ab} + 4c \le 4\sqrt{ca} + 4\sqrt{bc} + 3b$$

which is true since $5a = 4a+a \le 4\sqrt{ca}+\sqrt{bc}$, $2\sqrt{ab} \le 2b$ and $4c = 3c+c \le 3\sqrt{bc}+b$.

Case (iii). If $c \le a \le b$, then $M - m = b - c$ and (4) reduces to

$$a + 2\sqrt{ab} + 8c \le 4\sqrt{bc} + 4\sqrt{ca} + 3b$$

which is true since $a + 2\sqrt{ab} \le b + 2b = 3b$ and $8c \le 4\sqrt{bc} + 4\sqrt{ca}$.

This completes the proof of (4).

From (3) and (4) we deduce

$$l^2 \le k^2 + \frac{2}{3}(M-m) \le \left(k + \sqrt{\frac{2}{3}(M-m)}\right)^2$$

so

$$l \le k + \sqrt{\frac{2}{3}(M-m)} . \qquad (5)$$

From (1) and (5) we obtain

$$|k - l| \le \sqrt{\frac{2}{3}(M-m)}$$

and the desired conclusion follows.

Also solved by the proposer.

3710. [2012 : 23, 25] *Proposed by Billy Jin, Waterloo Collegiate Institute and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Given $n \in \mathbb{N}$, show that there exists a $k \in \mathbb{N}$ such that for all $m \geq k$, there exists a sequence of m consecutive natural numbers which contains exactly n primes.

Solution by Victor Pambuccian, Arizona State University West, Phoenix, AZ, USA.

We claim that the statement of the problem holds with $k = p_n$ where p_n represents the n^{th} prime.

To prove the claim, let $m \geq k$ be an integer, and let $f_m(x)$ be the number of primes in the set $\{x, x+1, \ldots, x+m-1\}$. Then

$$f_m(x+1) = f_m(x) + \epsilon_m(x), \tag{1}$$

where $\epsilon_m(x) \in \{-1, 0, 1\}$. Then, since $m \geq p_n$, we must have $f_m(1) \geq n$ and since the numbers $(m+1)!+2, (m+1)!+3, \ldots, (m+1)!+m+1$ are all composite, we must have $f_m((m+1)!+2) = 0$. Thus, from (1), the list $f_m(1), f_m(2), \ldots, f_m((m+1)!+2)$ contains all integers from 0 to $f_m(1) \geq n$ (and possibly some greater than $f_m(1)$), hence there is a number, y, between 1 and $(m+1)!+2$ such that $f_m(y) = n$ and therefore there are exactly n primes in the list $y, y+1, \ldots, y+m-1$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; RADOUAN BOUKHARFANE, Polytechnique de Montréal, PQ; OLIVER GEUPEL, Brühl, NRW, Germany; ALBERT STADLER, Herrliberg, Switzerland; and the proposers.

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