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This month's "free sample" is:

3689. *Proposé par Ivaylo Kortezov, Institut de Mathématiques et Informatique, Académie des Sciences, Sofia, Bulgarie.*

Dans un groupe de n personnes, chacune possède un livre différent. Disons qu'une paire de personnes opère un *échange* si elles s'échangent leur livre présentement en leur possession. Trouver le plus petit nombre possible d'échanges $E(n)$, de sorte que chaque paire de personnes a procédé à au moins un échange et que finalement chaque personne se retrouve avec son livre de départ.

. . . .

3689. *Proposed by Ivaylo Kortezov, Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia, Bulgaria.*

In a group of n people, each one has a different book. We say that a pair of people performs a *swap* if they exchange the books they currently have. Find the least possible number $E(n)$ of swaps such that each pair of people has performed at least one swap and at the end each person has the book he or she had at the start.

545 Solutions: 3589–3600

560 YEAR END FINALE

562 Index to Volume 37, 2011

SKOLIAD No. 137

Lily Yen and Mogens Hansen

Please send your solutions to problems in this Skoliad by **December 15, 2012**. A copy of ***CRUX with Mayhem*** will be sent to one pre-university reader who sends in solutions before the deadline. The decision of the editors is final.

Our contest this month is the Mathematics Association of Quebec Contest, Secondary level, 2011. Our thanks go to Denis Lavigne, Royal Military College Saint-Jean, Quebec, for providing us with this contest and for permission to publish it.

Concours de l'Association mathématique du Québec, 2011 Ordre secondaire Dure : 3 heures

1. Otto aime tellement les palindromes (les nombres qui demeurent les mêmes lorsqu'on inverse l'ordre de leurs chiffres) qu'il a concocté l'alphamétique suivant :

$$\text{AMQMA} \times \mathbf{6} = \text{LUCIE}\,.$$

Trouver les valeurs des huit chiffres.

(N. B. Un alphamétique est un petit casse-tête mathématique qui consiste en une équation où les chiffres sont remplacés par des lettres. Le résoudre consiste à trouver quelle lettre correspond à quel chiffre pour que l'équation soit vraie. Dans le problème, le même chiffre ne peut être représenté par deux lettres différentes et une lettre représente toujours le même chiffre. Bien entendu, un nombre ne doit jamais commencer par zéro. Par exemple, l'alphamétique PAPA + PAPA = MAMAN a pour solution $\boldsymbol{P = 7}$, $\boldsymbol{A = 5}$, $\boldsymbol{M = 1}$ et $\boldsymbol{N = 0}$. Ainsi, en remplaçant les lettres par les chiffres, on a bien $\mathbf{7575 + 7575 = 15150}$.)

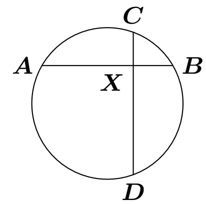
2. Anik roule toujours à **108** km/h sur l'autoroute. En se rendant à un concours de mathématique, elle dépasse un train qui longe l'autoroute et qui se dirige dans le même sens qu'elle. Elle remarque qu'elle met exactement **77** secondes à le dépasser, i.e. franchir la distance qui sépare la queue du train et sa tête. Arrivée à destination, elle réalise qu'elle a oublié sa calculatrice, alors elle rebrousse chemin. Elle recroise alors le train, qui roule à la même vitesse, et cette fois, elle met exactement sept secondes à parcourir le train de la tête à la queue. Quelle est la longueur du train ?

3. À un coin de rue, le feu de circulation reste vert pendant **30** secondes et rouge pendant **30** secondes (on suppose le temps du feu jaune inclus à même le temps du feu vert). Combien de temps perd-on, en moyenne, à attendre à ce coin de rue ? Justifier.

4. Combien y a-t-il de nombres entiers entre **0** et **999** (inclusivement) dont l'écriture décimale ne contient aucun **7** ? Quelle est la somme de ces nombres ?

5. Dans un cercle de rayon r, deux cordes AB et CD se coupent perpendiculairement en X. Montrer que

$$|XA|^2 + |XB|^2 + |XC|^2 + |XD|^2 = 4r^2\,.$$



6. $173^3 = 5\,177\,717$, $192^3 = 7\,077\,888$ et $1309^3 = 2\,242\,946\,629$ sont trois exemples d'entiers N dont le cube compte le même nombre de chiffres différents que N lui-même. Mais existe-t-il des entiers qui contiennent plus de chiffres différents que leur cube ? Oui : le nombre **13 798** compte cinq chiffres différents tandis que son cube, **2 626 929 525 592** n'en compte que quatre (**2** ; **5** ; **6** et **9**). On dira d'un tel nombre (quand il contient plus de chiffres différents que son cube) qu'il est *déficient*. Montrer qu'il y a une infinité d'entiers déficients.

7. Sachant que le système d'équations

$$x = \sqrt{11 - 2yz}, \quad y = \sqrt{12 - 2xz}, \quad \text{et} \quad z = \sqrt{13 - 2xy}$$

possède des solutions réelles, que vaut $x + y + z$?

Mathematics Association of Quebec Contest, 2011
Secondary level
3 hours allowed

1. Otto likes palindromes (numbers that read the same forwards and backwards) so much that he has constructed this alphametic:

$$\text{AMQMA} \times 6 = \text{LUCIE}\,.$$

Find the values of the eight digits.

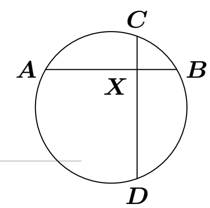
(Recall that an *alphametic* is a small mathematical puzzle consisting of an equation in which the digits have been replaced by letters. The task is to identify the value of each letter in such a way that the equation comes out true. Different letters have different values, different digits are represented by different letters, and no number begins with a zero. For example, the alphametic PAPA + PAPA = MAMAN has the solution P = **7**, A = **5**, M = **1**, and N = **0**, yielding $7575 + 7575 = 15150$.)

2. Anik is going **108** km/h on the highway. On her way to a math contest, she passes a train that travels beside the highway in the same direction as Anik. She notices that it takes her exactly **77** seconds to pass the train from the rear to the front. Upon arrival, she finds that she has forgotten her calculator and turns back. She again passes the train, which still travels at the same speed. This time it takes her seven seconds to pass from the front of the train to the rear. How long is the train?

3. At an intersection, the traffic light is red for **30** seconds and green for **30** seconds. (Ignore the yellow light.) How long do you have to wait, on average, at the intersection? Justify your answer.

4. How many integers from **0** to **999** (inclusive) do not contain the digit **7**? What is the sum of these numbers?

5. In a circle with radius r, the two chords AB and CD intersect at a right angle at X. Show that



$$|XA|^2 + |XB|^2 + |XC|^2 + |XD|^2 = 4r^2\,.$$

6. The cubes $173^3 = 5\,177\,717$, $192^3 = 7\,077\,888$ and $1309^3 = 2\,242\,946\,629$ are examples of a whole number N that contains as many different digits as its cube, N^3. If N^3 contains fewer different digits than N, then N is said to be *deficient*. For example, **13 798** has five different digits, while its cube, **2 626 929 525 592**, has four (**2**, **5**, **6**, and **9**), so **13 798** is deficient. Show that there are infinitely many deficient whole numbers.

7. If x, y, and z are real numbers such that

$$x = \sqrt{11 - 2yz}, \quad y = \sqrt{12 - 2xz}, \quad \text{and} \quad z = \sqrt{13 - 2xy}$$

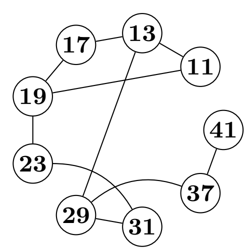
what is the value of $x + y + z$?

Next follow solutions to the Baden-Württemberg Mathematics Contest, 2010, given in Skoliad 134 at [2011:259–260].

1. Sonja has nine cards on which the nine smallest two-digit prime numbers are printed. She wants to order these cards in such a way that neighbouring cards always differ by a power of **2**. In how many ways can Sonja order her cards?

Solution by Elisa Kuan, student, Meadowridge School, Maple Ridge, BC.

The first nine two-digit primes are **11**, **13**, **17**, **19**, **23**, **29**, **31**, **37**, and **41**. The relevant powers of **2** are **1**, **2**, **4**, **8**, **16**, and **32**. In the figure, those primes that differ by a power of **2** are connected. Since **41** is only connected to one other prime, any arrangement of these primes must begin (or end) **41**, **37**, **29**.



If the arrangement then continues with **31**, it must go on as **41**, **37**, **29**, $\boxed{31}$, **23**, **19**. At **19** you again have a choice: **41**, **37**, **29**, **31**, **23**, $\boxed{19}$, **17**, **13**, **11** or **41**, **37**, **29**, **31**, **23**, $\boxed{19}$, **11**, **13**, **17**. Both of these arrangements work out (that is, they use all nine primes).

If the arrangement instead has **13** following **29**, you immediately have the choice: **11** or **17**. In either case, you must go on to **19**, so now you have **41**, **37**, **29**, **13**, **11**, **19** or **41**, **37**, **29**, **13**, **17**, **19**. If you here go to **17** or **11** (whichever is available), then **23** and **31** become stranded. If, on the other hand, you continue as **41**, **37**, **29**, **13**, **11/17**, $\boxed{19}$, **23**, **31**, then **11** or **17** becomes stranded.

Thus, the only possible arrangements are:

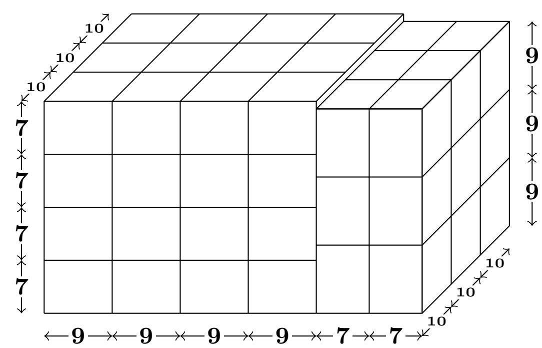
41, **37**, **29**, **31**, **23**, **19**, **17**, **13**, **11**;
41, **37**, **29**, **31**, **23**, **19**, **11**, **13**, **17**;

and these reversed, so Sonja can arrange her cards in four ways.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; and SZERA PINTER, student, Moscrop Secondary School, Burnaby, BC.

2. A **50** cm by **30** cm by **28** cm box contains wooden blocks that all measure **10** cm by **9** cm by **7** cm. At most how many blocks can fit in the box? Explain how to fit that many blocks into the box.

Solution by Jay Chau, student, Burnaby Mountain Secondary School, Burnaby, BC.



The volume of the box is $\mathbf{50 \cdot 30 \cdot 28}\,\text{cm}^{\mathbf{3}} = \mathbf{42\,000}\,\text{cm}^{\mathbf{3}}$, and the volume of each block is $\mathbf{10 \cdot 9 \cdot 7}\,\text{cm}^{\mathbf{3}} = \mathbf{630}\,\text{cm}^{\mathbf{3}}$, so there is room for at most $\lfloor \frac{\mathbf{42\,000}}{\mathbf{630}} \rfloor = \lfloor \frac{\mathbf{200}}{\mathbf{3}} \rfloor = \lfloor \mathbf{66}\frac{\mathbf{2}}{\mathbf{3}} \rfloor = \mathbf{66}$ blocks.

The figure shows that fitting **66** blocks is indeed possible.

Also solved by GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

You may well wonder, how our solver found a way to squeeze all **66** *blocks into the box. As a first try, you might line the* **10***-side of the blocks against the* **50***-side of the box and the* **7***-side against the* **28***-side since both work out perfectly. However, this leaves room for only* $\lfloor \frac{\mathbf{30}}{\mathbf{9}} \rfloor = \mathbf{3}$ *layers, so only* $\mathbf{5 \cdot 4 \cdot 3 = 60}$ *blocks. The trouble is that the* **9***-side of the blocks does not fit perfectly against any side of the box. The* **50***-side has the most wiggle room, so fitting as many* **9***'s as convenient could provide a better fit. Trying out different numbers of* **9***'s, you will find that* $\mathbf{4 \cdot 9 + 2 \cdot 7 = 50}$*, which leads to our solver's solution.*

3. Five distinct positive numbers are given. Forming all possible sums of two of these numbers you obtain seven different sums. Show that the sum of the five original numbers is divisible by $\mathbf{5}$.

Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

Suppose the five distinct numbers are a, b, c, d, and e, and that $a < b < c < d < e$. Then

$$a + b < a + c < a + d < a + e < b + e < c + e < d + e\,.$$

This is already seven distinct sums, so the remaining three sums, $b + c$, $b + d$, and $c + d$ must be in the seven sums listed above.

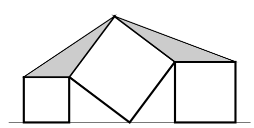
Since $a + d < b + d < b + e$, it follows from the long inequality above that $b + d = a + e$. Since $a + e = b + d < c + d < c + e$, it similarly follows that $c + d = b + e$. Finally, since $a + c < b + c < b + d = a + e$, it follows that $b + c = a + d$. That is, $b + d = a + e$, $c + d = b + e$, and $b + c = a + d$.

If you add the second and third of these equations and subtract the first, you get that $(c + d) + (b + c) - (b + d) = (b + e) + (a + d) - (a + e)$, so $2c = b + d$. On the other hand, $a + e = b + d$, so $a + e = 2c$. Thus $a + b + c + d + e = (a + e) + (b + d) + c = 2c + 2c + c = 5c$, which clearly is divisible by 5.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

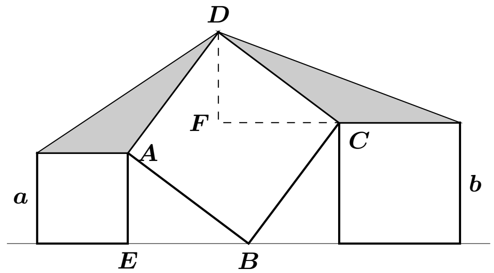
As our solver clearly assumes (as does every other submitted solution), the five numbers should be integers. *The problem should have made this clear; we apologise for the omission.*

4. Three squares are arranged as in the figure. Show that the two shaded triangles have the same area.



Solution by Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

Let a and b denote the side lengths of the two outer squares, label the vertices as in the figure, and extend the top side of the right-hand square until it meets the vertical line through D.

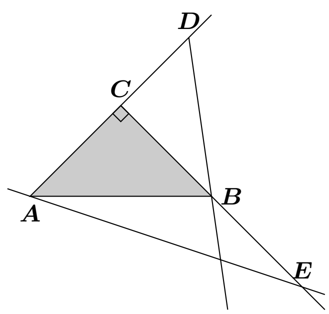


Since the sides of $\triangle ABE$ are parallel to the the corresponding sides of $\triangle DCF$ and $|AB| = |DC|$, it follows that $\triangle ABE \cong \triangle DCF$. Hence the distance from D to the base line is $a + b$.

If you now use the horizontal side of each shaded triangle as the base, then the left-hand shaded triangle has base a and height b, while the right-hand shaded triangle has base b and height a. Therefore the area of either shaded triangle is $\frac{1}{2}ab$.

Also solved by GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

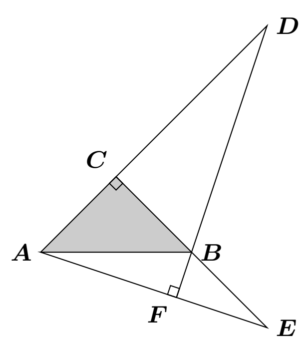
5. Triangle $\triangle ABC$ is isosceles and $\angle ACB = 90°$. The point D is on the line AC beyond C, and the point E is on the line CB beyond B. Show that $|CD| = |CE|$ if line BD is perpendicular to line AE.



Solution by Rowena Ho, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

In the figure in the statement of the problem, BD does not seem perpendicular to AE, so move D farther out. Moreover, let F denote the intersection between BD and AE.

Now similar triangles do the work: $\angle EBF = \angle DBC$ and $\angle BFE = 90° = \angle BCD$, so $\triangle BEF \sim \triangle BDC$. Also, $\triangle BEF$ and $\triangle AEC$ are both right-angled and share an angle, so they are also similar. Thus $\triangle AEC \sim \triangle BEF \sim \triangle BDC$, so $\frac{|CE|}{|AC|} = \frac{|CD|}{|BC|}$.



Since $\triangle ABC$ is given to be isosceles, $|AC| = |BC|$. Therefore, $|CE| = |CD|$.

Also solved by LISA CHEN, student, Moscrop Secondary School, Burnaby, BC; LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; and JUSTINE HANSEN, student, Burnaby North Secondary School, Burnaby, BC.

6. The product of three positive integers is three times as large as their sum. Find all such triples.

Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

Let a, b, and c denote the three positive integers. Since the order of the integers is not relevant, you may assume that $a \le b \le c$. The problem states that $abc = 3(a+b+c)$. Therefore, $ab = 3(\frac{a}{c} + \frac{b}{c} + 1) \le 3(1+1+1) = 9$, because $\frac{a}{c} \le 1$ and $\frac{b}{c} \le 1$.

Only a few pairs of integers, (a,b), such that $1 \le a \le b$ satisfy that $ab \le 9$, namely $(1,1)$, $(1,2)$, $(1,3)$, $(1,4)$, $(1,5)$, $(1,6)$, $(1,7)$, $(1,8)$, $(1,9)$, $(2,2)$, $(2,3)$, $(2,4)$, and $(3,3)$. Since $abc = 3a+3b+3c$, $(ab-3)c = 3a+3b$, so $c = \frac{3a+3b}{ab-3}$. For each possible pair, (a,b), you may now calculate c:

a	b	c	
1	1	-3	but c is positive
1	2	-9	but c is positive
1	3	impossible	
1	4	15	
1	5	9	
1	6	7	
1	7	6	but $c \ge b$

a	b	c	
1	8	$\frac{27}{5}$	but c is an integer
1	9	5	but $c \ge b$
2	2	12	
2	3	5	
2	4	$\frac{18}{5}$	but c is an integer
3	3	3	

Thus, the only solutions are $(1,4,15)$, $(1,5,9)$, $(1,6,7)$, $(2,2,12)$, $(2,3,5)$, and $(3,3,3)$.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

This issue's prize for the best solutions goes to Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

We hope that you will enjoy our featured contest, and we look forward to receiving your solutions at crux-skoliad@cms.math.ca or the postal address listed inside the back cover.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

Mayhem Year End Wrap Up

Shawn Godin

Hello ***Mayhem*** readers. Another "year" has ended and it marks the end of another chapter in the story of ***Mathematical Mayhem***. ***Mayhem*** was created "by students, for students" in the fall of 1988 and was published 5 times a year for 8 years. After losing its funding, ***Mayhem*** joined ***Crux*** in 1997, volume 23. Now, with times changing, it is time for the journal to make yet another change.

Mayhem will revert to a stand alone journal that appears 5 times a year and follows the school year (issues in September, November, January, March and May). The difference is that ***Mayhem*** will exist on-line. The plan is to expand the journal beyond just problems to include columns, articles, interactive material and possibly even videos. We are working hard toward the relaunch of ***Mayhem*** which should occur some time in early 2013. Keep your eye on the CMS web site and the ***Crux*** Facebook page for updates.

At this point I need to thank the ***Mayhem*** staff for their help preparing the material for each issue. The problems editors ANN ARDEN, NICOLE DIOTTE, MONIKA KHBEIS and DAPHNE SHANI who sift through all the submitted solutions and prepare the featured solutions, your work is very much appreciated. Also to my assistant editor LYNN MILLER, for your work behind the scenes and preparing solutions, I want to thank you from the bottom of my heart. You are always there when I need a little extra help.

I wish all the best to our readers. I look forward to receiving your problem proposals and solutions. Keep your eye open for the new ***Mayhem***.

Shawn Godin

Mayhem Problems

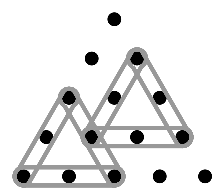
*Please send your solutions to the problems in this edition by **15 November 2012**. Solutions received after this date will only be considered if there is time before publication of the solutions.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Rolland Gaudet, Université de Saint-Boniface, Winnipeg, MB, for translating the problems from English into French.

M513. *Proposé par l'Équipe de Mayhem.*

Un grillage triangulaire équilatéral consiste de chevilles espacées d'un centimètre l'une de l'autre, tel qu'indiqué au schéma. Des bandes élastiques sont placées autours des chevilles de façon à former des triangles équilatéraux ; deux tels triangles équilatéraux à deux centimètres de côté sont illustrés au schéma. Combien de triangles équilatéraux différents sont possibles ?



M514. *Proposé par Dragoljub Milošević, Gornji Milanovac, Serbie.*

Le nonagone $ABCDEFGHI$ est régulier. Démontrer que $AE - AC = AB$.

M515. *Proposé par Titu Zvonaru, Cománeşti, Roumanie.*

Sans utiliser les techniques du calcul différentiel, déterminer les valeurs minimales et maximales de

$$\frac{2x}{x^2 + 2x + 2}$$

où x est un nombre réel.

M516. *Proposé par Syd Bulman-Fleming et Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Démontrer que pour tout k entier non nul il existe au moins quatre paires ordonnées d'entiers, (x, y), telles que

$$\frac{y^2 - 1}{x^2 - 1} = k^2 - 1.$$

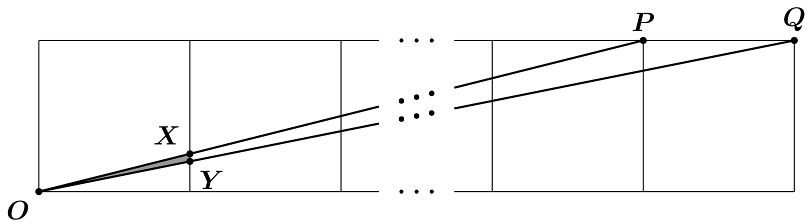
M517. *Proposé par Šefket Arslanagić, Université de Sarajevo, Sarajevo, Bosnie et Herzégovine.*

Déterminer toutes les solutions réelles de l'équation

$$3\sqrt{x + y} + 2\sqrt{8 - x} + \sqrt{6 - y} = 14.$$

M518. *Sélectionné à partir de concours mathématiques.*

Un nombre de carrés unitaires sont placés sur une ligne, tel qu'indiqué au schéma ci-bas.

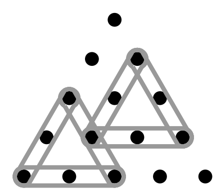


Soit O le coin inférieur à gauche du premier carré et soient P et Q les coins supérieurs à droite des $2011^{\text{ième}}$ et $2012^{\text{ième}}$ carrés respectivement. Lorsque P et Q sont reliés à O, ils intersectent le côté droit du premier carré à X et Y respectivement. Déterminer la surface du triangle OXY.

. . . .

M513. *Proposed by the Mayhem Staff.*

An equilateral triangular grid is formed by removable pegs that are one centimetre apart as shown in the diagram. Elastic bands may be attached to pegs to form equilateral triangles, two different equilateral triangles two centimetres on each side are shown in the diagram. How many different equilateral triangles are possible?



M514. *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Nonagon $ABCDEFGHI$ is regular. Prove that $AE - AC = AB$.

M515. *Proposed by Titu Zvonaru, Comăneşti, Romania.*

Without using calculus, determine the minimum and maximum values of

$$\frac{2x}{x^2 + 2x + 2}$$

where x is a real number.

M516. *Proposed by Syd Bulman-Fleming and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Show that for any given nonzero integer k there exists at least four distinct ordered pairs (x, y) of integers such that

$$\frac{y^2 - 1}{x^2 - 1} = k^2 - 1.$$

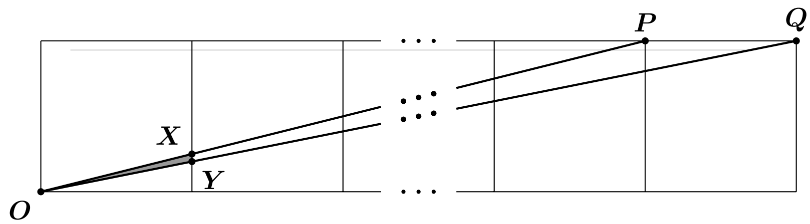
M517. *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Find all real solutions of the equation

$$3\sqrt{x+y} + 2\sqrt{8-x} + \sqrt{6-y} = 14.$$

M518. *Selected from a mathematics competition.*

A number of unit squares are placed in a line as shown in the diagram below.



Let O be the bottom left corner of the first square and let P and Q be the top right corners of the 2011^{th} and 2012^{th} squares respectively. When P and Q are connected to O they intersect the right side of the first square at X and Y respectively. Determine the area of triangle OXY.

Mayhem Solutions

M476. *Proposed by the Mayhem Staff*

Define $s(n)$ to be the sum of the digits of the positive integer n. For example, $s(2011) = 2 + 0 + 1 + 1 = 4$. Determine the number of four-digit positive integers n with $s(n) = 4$.

Solution by David E. Manes, SUNY at Oneonta, Oneonta, NY, USA.

There are five ways to write 4 as the sum of positive integers; namely 4, $3+1$, $2+2$, $2+1+1$, and $1+1+1+1$. If $s(n) = 4+0+0+0$, then $n = 4000$ is the only such integer. If $s(n) = 3+1$, then there are six possible values for n; namely $n = 3100$, 3010, 3001, 1300, 1030, or 1003. If $s(n) = 2+2$, then the three possible values for n are $n = 2200$, 2020, or 2002. If $s(n) = 2+1+1$, then the nine possible values for n are $n = 2011$, 2101, 2110, 1021, 1012, 1102, 1120, 1210, or 1201. Finally, if $s(n) = 1+1+1+1$, then $n = 1111$ is the only such integer. Hence, there are 20 positive integers with $s(n) = 4$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; ALICIA GÓMEZ GÓMEZ, Club Mathématique de l'Instituto de Ecuación Secundaria No. 1, Requena-Valencia, *Spain; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; JAIME PIQUERAS GARCÍA, Club Mathématique de l'Instituto de Ecuación Secundaria No. 1, Requena-Valencia, Spain; and BRUNO SALGUEIRO FANEGO, Viveiro, Spain.*

M477. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania*

Let m be an integer parameter such that the equation $x^2-mx+m+8=0$ has one integer root. Determine the value of the parameter m.

Solution by George Apostolopoulos, Messolonghi, Greece.

If there is only one root then the discriminant must be zero, so

$$\begin{aligned}(-m)^2-4(m+8)=0 \Leftrightarrow & \quad (m-2)^2=36\\ \Leftrightarrow & \quad m-2=\pm 6\\ \Leftrightarrow & \quad m=8 \text{ or } m=-4.\end{aligned}$$

Also solved by DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA. Two incorrect solutions were submitted.

M478. *Proposed by the Mayhem Staff*

Consider the set of points (x,y) in the plane such that

$$x^2+y^2-22x-4y+100=0\,.$$

Let P be the point in this set for which $\dfrac{y}{x}$ is the largest. Determine the distance of P from the origin.

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.

We recognize the given set of points as the circle with centre at $(11,2)$ and radius 5 and whose parametric equations are

$$x=11+5\cos\theta,\quad y=2+5\sin\theta \tag{1}$$

Denote $\tan\frac{\theta}{2}$ by t. Since $\sin\theta=\dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$ and $\cos\theta=\dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$, equations (1) are equivalent to

$$x=\frac{6t^2+16}{1+t^2},\quad y=\frac{2t^2+10t+2}{1+t^2}$$

Hence, $\frac{y}{x}=\frac{t^2+5t+1}{3t^2+8}$, the derivative $\frac{d(\frac{y}{x})}{dt}$ is $\frac{5(-3t^2+2t+8)}{(3t^2+8)^2}$, and the critical values are solutions of $-3t^2+2t+8=0$ or $t=-\frac{4}{3},2$. The value $t=-\frac{4}{3}$ corresponds to a minimum, and $t=2$ corresponds to a maximum. We find x and y for $t=2$ to be $x=8,\ y=6$, so that the distance of P from the origin is

$$\sqrt{x^2+y^2}=\sqrt{8^2+6^2}=10$$

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; and PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy. Three incorrect solutions were submitted.

M479. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania*

Let $A = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot 2011 = 2011!$.

(a) Determine the largest positive integer n for which 3^n divides exactly into A.

(b) Determine the number of zeroes at the end of the base 10 representation of A.

Solution by David E. Manes, SUNY at Oneonta, Oneonta, NY, USA.

(a) The exponent n of the highest power of 3 that divides exactly into $2011!$ is given by $n = \sum_{k=1}^{\infty} \lfloor \frac{2011}{3^k} \rfloor$. One calculates $\lfloor \frac{2011}{3} \rfloor = 670,$ $\lfloor \frac{2011}{3^2} \rfloor = 223,$ $\lfloor \frac{2011}{3^3} \rfloor = 74,$ $\lfloor \frac{2011}{3^4} \rfloor = 24,$ $\lfloor \frac{2011}{3^5} \rfloor = 8,$ $\lfloor \frac{2011}{3^6} \rfloor = 2,$ and if $k \geq 7$, then $\lfloor \frac{2011}{3^k} \rfloor = 0$. Therefore, $n = 670 + 223 + 74 + 24 + 8 + 2 = 1001$.

(b) The number of zeroes with which the decimal representation of $2011!$ terminates is equal to the exponent, m, of the highest power of 10 that divides $2011!$. Furthermore, m is also the exponent of the highest power of 5 that divides $2011!$, that is, $m = \sum_{k=1}^{\infty} \lfloor \frac{2011}{5^k} \rfloor = 402+80+16+3 = 501$. Hence, the base 10 representation of $2011!$ ends in 501 zeroes.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; and BRUNO SALGUEIRO FANEGO, Viveiro, Spain.

M480. *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia*

Let x, y, and k be positive numbers such that $x^2 + y^2 = k$. Determine the minimum possible value of $x^6 + y^6$ in terms of k.

Solution by Bruno Salgueiro Fanego, Viveiro, Spain.

Note that $x^6 + y^6 = (x^2)^3 + (y^2)^3 = (x^2 + y^2)(x^4 - x^2y^2 + y^4) = k((x^2 + y^2)^2 - 3x^2y^2) = k(k^2 - 3x^2y^2)$ attains its minimum possible value if and only if $-3x^2y^2$ attains its minimum value, or equivalently, if and only if x^2y^2 attains its maximum value. By the arithmetic mean-geometric mean inequality, $x^2y^2 \leq \left(\frac{x^2 + y^2}{2}\right)^2 = \frac{k^2}{4}$ with equality if and only if $x = y$, that is x^2y^2 attains its maximum value if and only if $x = y = \sqrt{\frac{k}{2}}$, so the minimum possible value of $x^6 + y^6$ is $x^6 + y^6 = k\left(k^2 - 3\frac{k^2}{4}\right) = \frac{k^3}{4}$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania;

M481. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON*

Suppose that a, b, and x are real numbers with $ab \neq 0$ and $a+b \neq 0$. If $\frac{\sin^4 x}{a} + \frac{\cos^4 x}{b} = \frac{1}{a+b}$, determine the value of $\frac{\sin^6 x}{a^3} + \frac{\cos^6 x}{b^3}$ in terms of a and b.

Solution by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.

From $\frac{\sin^4 x}{a} + \frac{\cos^4 x}{b} = \frac{1}{a+b}$, it follows

$$\begin{aligned}
& \frac{\sin^4 x}{a} + \frac{\cos^4 x}{b} - \frac{1}{a+b} = 0 \\
\Rightarrow\quad & b(a+b)\sin^4 x + a(a+b)\cos^4 x - ab = 0 \\
\Rightarrow\quad & b^2\sin^4 x + a^2\cos^4 x + ab(\sin^4 x + \cos^4 x - 1) = 0 \\
\Rightarrow\quad & b^2\sin^4 x + a^2\cos^4 x + ab[(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x - 1] = 0 \\
\Rightarrow\quad & b^2\sin^4 x + a^2\cos^4 x + ab(1 - 2\sin^2 x\cos^2 x - 1) = 0 \\
\Rightarrow\quad & b^2\sin^4 x + a^2\cos^4 x - 2ab\sin^2 x\cos^2 x = 0 \\
\Rightarrow\quad & (b\sin^2 x - a\cos^2 x)^2 = 0.
\end{aligned}$$

Therefore we have

$$\frac{\sin^4 x}{a} + \frac{\cos^4 x}{b} = \frac{1}{a+b} \Rightarrow (b\sin^2 x - a\cos^2 x)^2 = 0,$$

and consequently, $\frac{\sin^2 x}{a} = \frac{\cos^2 x}{b} = \frac{1}{a+b}$. Finally we obtain that

$$\begin{aligned}
\frac{\sin^6 x}{a^3} + \frac{\cos^6 x}{b^3} &= \left(\frac{1}{a+b}\right)^3 + \left(\frac{1}{a+b}\right)^3 \\
&= \frac{2}{(a+b)^3}.
\end{aligned}$$

Also solved by by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; and BRUNO SALGUEIRO FANEGO, Viveiro, Spain.

THE OLYMPIAD CORNER

No. 298

R.E. Woodrow and Nicolae Strungaru

The solutions to the problems are due to the editors by **1 December 2012**.

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editors thank Jean-Marc Terrier of the University of Montreal for translations of the problems.

OC51. Déterminer toutes les paires (a, b) d'entiers non négatifs tels que $a^b + b$ divise $a^{2b} + 2b$. Noter que, pour ce problème, $0^0 = 1$.

OC52. Soit d, d' deux diviseurs de n avec $d' > d$. Montrer que

$$d' > d + \frac{d^2}{n}\,.$$

OC53.Trouver tous les polynômes $P(x) \in \mathbb{R}[x]$ tels que $P(a) \in \mathbb{Z}$ implique $a \in \mathbb{Z}$.

OC54. On donne quatre points dans le plan tels que les cercles inscrits des quatre triangles formés par trois des quatre points sont égaux. Montrer que les quatre triangles sont égaux.

OC55. Soit d un entier positif. Montrer que, pour tout entier S, il existe un entier $n > 0$ et une suite $\epsilon_1, \epsilon_2, ..., \epsilon_n$, où pour tout $k, \epsilon_k = 1$ ou $\epsilon_k = -1$, tels que

$$S = \epsilon_1(1 + d)^2 + \epsilon_2(1 + 2d)^2 + \epsilon_3(1 + 3d)^2 + \ + \epsilon_n(1 + nd)^2\,.$$

OC56. On suppose que $f : \mathbb{N} \to \mathbb{N}$ est une fonction telle que pour tout $a, b \in \mathbb{N}$, l'expression $af(a) + bf(b) + 2ab$ est un carré parfait. Montrer que $f(a) = a$ pour tout $a \in \mathbb{N}$.

OC57. Soit ABC un triangle et A', B', C' les points milieu respectifs de BC, CA, AB. Soit P et P' deux points dans le plan tels que $PA = P'A'$, $PB = P'B'$, $PC = P'C'$. Montrer que tous les PP' passent par un même point.

OC58. Trouver le plus petit n pour lequel il existe des polynômes $f_1(x), f_2(x), \ldots, f_n(x) \in \mathbb{Q}[x]$ tels que

$$f_1^2(x) + f_2^2(x) + \cdots + f_n^2(x) = x^2 + 7\,.$$

OC59. Soit n un entier positif impair tel que $\phi(n)$ et $\phi(n+1)$ sont des puissances de deux. Montrer que $n+1$ est une puissance de deux ou $n=5$.

OC60. On écrit les nombres $1, 2, \ldots, 20$ au tableau noir. Une opération consiste à choisir deux nombres a, b tels que $b \geq a+2$, effacer a et b et les remplacer par $a+1$ et $b-1$. Trouver le nombre maximal d'opérations possibles.

. . . .

OC51. Determine all pairs (a, b) of nonnegative integers so that a^b+b divides $a^{2b}+2b$. Note, for this problem $0^0=1$.

OC52. Let d, d' be two divisors of n with $d' > d$. Prove that

$$d' > d + \frac{d^2}{n}\,.$$

OC53. Find all the polynomials $P(x) \in \mathbb{R}[x]$ so that $P(a) \in \mathbb{Z}$ implies $a \in \mathbb{Z}$.

OC54. Given four points in the plane so that the incircles of the four triangles formed by three of the four points are equal, prove that the four triangles are equal.

OC55. Let d be a positive integer. Show that for every integer S there exists an integer $n > 0$ and a sequence $\epsilon_1, \epsilon_2, ..., \epsilon_n$, where for any $k, \epsilon_k = 1$ or $\epsilon_k = -1$, such that

$$S = \epsilon_1(1+d)^2 + \epsilon_2(1+2d)^2 + \epsilon_3(1+3d)^2 + \ + \epsilon_n(1+nd)^2\,.$$

OC56. Suppose that $f : \mathbb{N} \to \mathbb{N}$ is a function so that for all $a, b \in \mathbb{N}$ the expression $af(a)+bf(b)+2ab$ is a perfect square. Prove that $f(a)=a$ for all $a \in \mathbb{N}$.

OC57. Let ABC be a triangle and A', B',C' be the midpoints of BC, CA, AB respectively. Let P and P' be points in a plane such that $PA = P'A'$, $PB = P'B'$, $PC = P'C'$. Prove that all PP' pass through a fixed point.

OC58. Find the smallest n for which there exists polynomials $f_1(x), f_2(x), \ldots, f_n(x) \in \mathbb{Q}[x]$ such that

$$f_1^2(x) + f_2^2(x) + \cdots + f_n^2(x) = x^2 + 7\,.$$

OC59. Let n be an odd positive integer such that both $\phi(n)$ and $\phi(n+1)$ are powers of two. Prove $n+1$ is a power of two or $n=5$.

OC60. On a blackboard we write the numbers $1, 2, ..., 20$. A move consists of selecting two numbers a, b from the blackboard so that $b \geq a+2$, erasing a and b and writing instead $a+1$ and $b-1$. Find the maximum number of possible moves.

In this number of the *Corner* we will complete the files of solutions from the readers to problems given in the *Corner*, and also my time as editor of the *Corner*. Thanks to those who contributed solutions to problems discussed in 2011 numbers:

Arkady Alt	David Manes
Miguel Amengual Covas	Norvald Midttun
George Apostolopoulos	Soohyun Park
Mohammed Aassila	Paolo Perfetti
Ricardo Barroso Campos	Henry Ricardo
Michel Bataille	Bruce Shawyer
Chip Curtis	D.J. Smeenk
Prithwijit De	Gheorge Ghita Stanciu
José Luis Díaz-Barrero	Edward T.H.Wang
Gesine Geupel	Dexter Wei
Oliver Geupel	Konstantine Zelator
Geoffrey A. Kandall	Kaiming Zhao
Giulio Loddi	Titu Zvonaru

I've enjoyed my association with ***Crux Mathematicorum*** and the *Corner* since January 1988. I would like to express my thanks to all those over nearly a quarter century who have supported the *Corner* and ***Crux Mathematicorum*** by sending in problem sets, comments, and solutions. I would also like to express my thanks to Joanne Canape who has been transcribing my scribbles into LaTeX manuscripts over the last decades. With the next volume of ***Crux Mathematicorum*** there will be a new team and new features to draw your interest and support.

R.E. Woodrow

First, we look at the readers' solutions to problems from the Croatian Mathematical Competition 2007, National Competition, given at [2010: 435–436], that we started in the last issue.

4^{th} Grade

2. Sequence $(a_n)_{n\geq 0}$ is defined recursively by

$$a_0 = 3,$$
$$a_n = 2 + a_0 \cdot a_1 \cdot \ldots \cdot a_{n-1}, \quad n \geq 1.$$

(a) Prove that any two terms of the sequence are relatively prime positive integers.

(b) Determine a_{2007}.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solutions of Manes.

For part (a), assume there exist two integers $m > n > 0$ such that $\gcd(a_m, a_n) = d > 1$. Then $a_0 a_1 \ldots a_{m-1} = a_m - 2$. Since the term a_n is one of the factors on the left side of this equation, it follows that d divides 2. But each a_n $(n \geq 0)$ is odd, therefore $d = 2$ is impossible. Hence, $d = 1$.

For part (b), we begin by showing inductively that the terms a_n of the sequence are given by the Fermat numbers $2^{(2^n)} + 1$. Note that $a_0 = 3 = 2^{(2^0)} + 1$. Assume that $k \geq 0$ is a nonnegative integer and

$$a_k = 2 + a_0 a_1 \cdots a_{k-1} = 2^{(2^k)} + 1$$

or

$$a_k - 2 = a_0 a_1 \cdots a_{k-1} = 2^{(2^k)} - 1.$$

Then

$$\begin{aligned} a_{k+1} &= (a_0 a_1 \cdots a_{k-1}) a_k + 2 = (2^{(2^k)} - 1)(2^{(2^k)} + 1) + 2 \\ &= (2^{(2^{k+1})} - 1) + 2 = 2^{(2^{k+1})} + 1. \end{aligned}$$

Therefore, by induction, $a_n = 2^{(2^n)} + 1$ for each $n \geq 0$. Hence,

$$a_{2007} = 2^{(2^{2007})} + 1.$$

4. In acute triangle ABC let A_1, B_1 and C_1 be the midpoints of sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$, respectively. The radius of its circumscribed circle, with centre O, is 1. Prove that

$$\frac{1}{|OA_1|} + \frac{1}{|OB_1|} + \frac{1}{|OC_1|} \geq 6.$$

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution by Zvonaru.

Since it is known that $OA_1 = R \cos A$, we have to prove that

$$\frac{1}{\cos A} + \frac{1}{\cos B} + \frac{1}{\cos C} \geq 6.$$

Let $f : \left(0, \frac{\pi}{2}\right) \to R$ be the function $f(x) = \frac{1}{\cos x}$. We have

$$f'(x) = \frac{\sin x}{\cos^2 x};$$

$$f''(x) = \frac{\cos^3 x - \sin x(-2\cos x \sin x)}{\cos^4 x} = \frac{\cos^2 x + 2\sin^2 x}{\cos^3 x} > 0,$$

hence f is a convex function.

Applying Jensen's Inequality we get

$$f(A)+f(B)+f(C)\geq 3f\left(\frac{A+B+C}{3}\right),$$

that is

$$\frac{1}{\cos A}+\frac{1}{\cos B}+\frac{1}{\cos C}\geq\frac{3}{\cos\frac{\pi}{3}}=6.$$

The equality holds if and only if $A=B=C$, that is $\triangle ABC$ is equilateral.

Next we move to readers' solutions to problems of the 51st National Mathematical Olympiad in Slovenia, Selection Examinations for the IMO 2007, given at [2010: 436–437].

First Selection Examination, December 2006

1. Show that the inequality

$$(1+a^2)(1+b^2)\ \geq\ a(1+b^2)+b(1+a^2)$$

holds for any pair of real numbers a and b.

Solved by George Apostolopoulos, Messolonghi, Greece; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; Henry Ricardo, Tappan, NY, USA; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Perfetti.

We know that

$$(x-y)^2\geq 0\implies x^2+y^2\geq 2|x||y|$$

thus

$$\begin{aligned}(1+a^2)(1+b^2)&=(1+a^2)(1+b^2)/2+(1+a^2)(1+b^2)/2\\&\geq|a|(1+b^2)+|b|(1+a^2)\geq a(1+b^2)+b(1+a^2)\end{aligned}$$

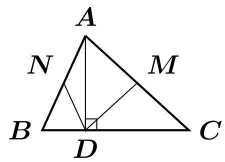
since $|x|\geq x$.

2. Prove that any triangle can be decomposed into n isosceles triangles for every positive integer $n\geq 4$.

Solved by Titu Zvonaru, Comăneşti, Romania.

We will prove three claims.

Claim 1. Any triangle can be decomposed into **4** isosceles triangles.

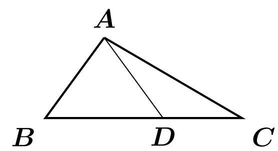


Let ABC be a triangle. Suppose that $\angle BAC$ is the greatest angle of $\triangle ABC$. Denoting by D the projection of A onto BC, it follows that D lies between B and C (since the angles $\angle ABC$ and $\angle ACB$ are acute).

Let M, N be the midpoints of the sides AC and AB, respectively.

Since in a right-angled triangle the median to the hypotenuse is half of the hypotenuse, it follows that the triangles BDN, NDA, DCM and DMA are isosceles.

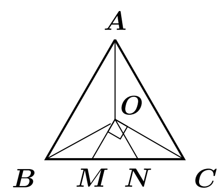
Claim 2. Any triangle can be decomposed into **5** isosceles triangles.



(a) If $\triangle ABC$ is not equilateral, then we may suppose that $AB < BC$.

We take a point D on the side BC such that $AB = BD$. We obtain an isosceles triangle ABD and for $\triangle ADC$ we can apply claim 1.

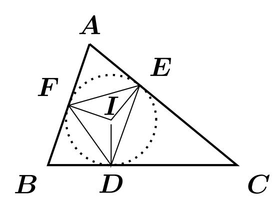
(b) Suppose that ABC is an equilateral triangle.



Let O be the circumcentre of $\triangle ABC$. The perpendicular through O to OC meets BC at M, and let N be the midpoint of MC.

Since $\angle BOC = 120°$, it follows that $\angle BOM = 30° = \angle OBM$. We deduce that the triangles OAB, OAC, OBM, OMN and ONC are isosceles.

Claim 3. Any triangle can be decomposed into **6** isosceles triangles.



Let I be the incentre of $\triangle ABC$. The incircle is tangent to the sides BC, CA, AB at the points D, E and F, respectively.

We have the following **6** isosceles triangles: AFE, IFE, BDF, IDF, CDE and IDE.

Using Claim 1, we see that if we can decompose a triangle into k isosceles triangles, we can decompose this triangle into $k + 3$ isosceles triangles, and so on.

Using Claims 1, 2 and 3 we deduce that any triangle can be decomposed into $n \geq 4$ isosceles triangles.

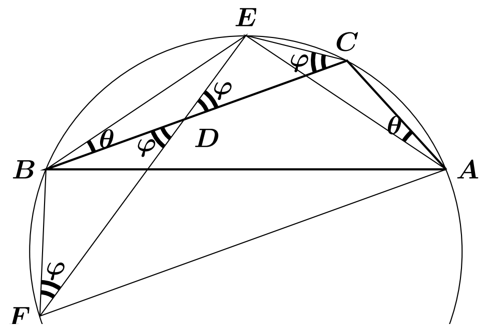
Comment. If ABC is an acute-angled triangle, then it can also be decomposed into **3** isosceles triangles (namely AOB, BOC, COA, where O is the circumcentre).

3. Let ABC be a triangle with $|AC| < |BC|$ and denote its circumcircle by $\mathcal{K}$. Let E be the midpoint of the arc AB that contains the point C and let D be a point on the segment BC, such that $|BD| = |AC|$. The line DE meets the circle $\mathcal{K}$ again in F. Prove that A, B, C and F are the vertices of an isosceles trapezoid.

Solved by George Apostolopoulos, Messolonghi, Greece; Oliver Geupel, Brühl, NRW, Germany; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Zelator.

Since E is the midpoint of the arc AB [that contains C], we have

$$|\overline{BE}| = |\overline{EA}|. \qquad (1)$$



Also by hypothesis,

$$|\overline{BD}| = |\overline{AC}|. \qquad (2)$$

And obviously,

$$\angle EBC = \angle EAC = \theta. \qquad (3)$$

From (1), (2), and (3), it follows that the triangles BDE and AEC are congruent. Therefore $|\overline{ED}| = |\overline{EC}|$ and so triangle DEC is isosceles.

$$\angle EDC = \varphi = \angle ECD. \qquad (4)$$

Furthermore,

$$\angle EDC = \varphi = \angle BDF \qquad (5)$$

and

$$\angle BFE = \varphi = \angle BCE. \qquad (6)$$

From (4), (5), and (6); it follows that the triangle FBD is isosceles with

$$|\overline{BD}| = |\overline{BF}|. \qquad (7)$$

Hence, from (7) and (2) $\Rightarrow$ $|\overline{BF}| = |\overline{CA}|$, which further implies that $\angle BCF = \angle CFA$, which in turn implies that $\overline{BC}$ and $\overline{FA}$ are parallel.

We have $\{|\overline{BF}| = |\overline{CA}|$ and $\overline{BC} \| \overline{FA}\}$ $\Rightarrow$ $BCAF$ is an isosceles trapezoid.

Second Selection Examination, February 2007

1. Every point in the plane with positive integer coordinates (x, y) such that $x \leq 19$ and $y \leq 4$ is colored green, red or blue. Prove that there exists a rectangle with sides parallel to the coordinate axes and with vertices of the same colour.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Zvonaru.

We rephrase the problem as follows: "Each square of a 4×19 chessboard is coloured green, red or blue. Prove that there exists a rectangle whose four corner squares are all coloured the same."

There are $\mathbf{19}$ columns each containing $\mathbf{4}$ squares. By the Pigeon-hole Principle, each column must contain some pair of squares of the same colour (we have $\mathbf{4}$ squares per column and only $\mathbf{3}$ colours).

In order for a rectangle to have its four corners coloured the same colour, there must be two different columns in which squares of the same colour are placed in the same two rows.

There are $\binom{4}{2}$ ways to place a pair of green squares, $\binom{4}{2}$ ways to place a pair of red squares and $\binom{4}{2}$ ways to place a pair of blue squares. Thus, there are $3 \cdot \binom{4}{2} = 3 \cdot 6 = 18$ ways altogether. Since there are $\mathbf{19}$ columns in the board, there must be at least $\mathbf{2}$ different columns in which a pair of squares of the same colour are placed in the same way.

It follows that there exists a rectangle whose four corner squares are coloured with the same colour.

Comment. In a similar way, we can solve the problem with a $(t+1) \times (\frac{t^2(t+1)}{2}+1)$ chessboard and t colours.

2. The circles $\mathcal{K}_1$ and $\mathcal{K}_2$ of different radii meet at A_1 and A_2. Let t be the common tangent of the two circles, such that the distance from t to A_1 is shorter than the distance from t to A_2. Let B_1 and B_2 be the points in which t touches $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively.

Let $\mathcal{K}_3$ and $\mathcal{K}_4$ be the circles with radii $|A_1B_1|$ and $|A_1B_2|$ and the centre A_1. The circles $\mathcal{K}_1$ and $\mathcal{K}_3$ meet again at C_1, while the circles $\mathcal{K}_2$ and $\mathcal{K}_4$ meet again at C_2. Denote the intersection of the lines B_1C_1 and B_2C_2 by D and let E be the intersection of B_1C_1 and $\mathcal{K}_4$ which lies on the same side of the line B_2C_2 as C_1.

Show that A_1D is perpendicular to EC_2.

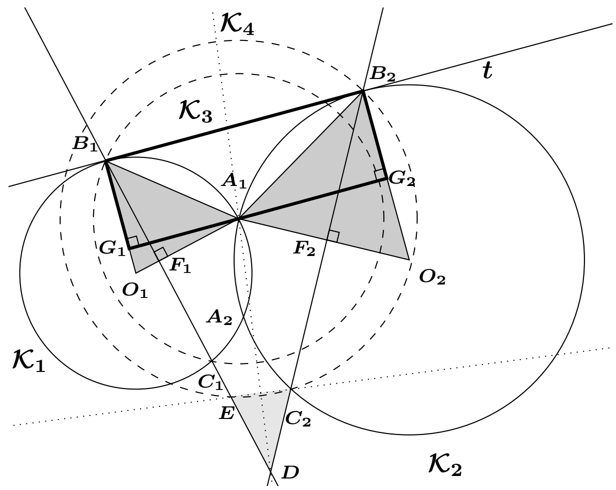
Solved by Oliver Geupel, Brühl, NRW, Germany.

Let O_1 and O_2 be the centres of $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. Let F_1 and G_1 be the feet of the altitudes from B_1 and A_1 in triangle $A_1B_1O_1$, respectively. Let F_2 and G_2 be the feet of the altitudes from B_2 and A_1 in triangle $A_1B_2O_2$, respectively.

By $A_1G_1 \perp B_1G_1$ and $A_2G_2 \perp B_2G_2$, the quadrilateral $B_1B_2G_2G_1$ is a rectangle. Hence, we see that $B_1G_1 = B_2G_2$. Also, the triangles $A_1B_1O_1$ and $A_1B_2O_2$ are isosceles with $O_1A_1 = O_1B_1$ and $O_2A_1 = O_2B_2$, which implies that $A_1F_1 = B_1G_1$ and $A_1F_2 = B_2G_2$. Therefore,

$$A_1F_1 = A_1F_2,$$

that is, the lines B_1E and B_2C_2 have the same distance from the centre A_1 of $\mathcal{K}_4$. Hence, the two lines are symmetric with respect to the axis A_1D. Thus, the triangle DC_2E is isosceles with $DC_2 = DE$ and with the line A_1D as its axis of symmetry. Consequently, A_1D is perpendicular to EC_2.



3. Find a positive integer n such that $n^2 - 1$ has exactly 10 positive divisors. Show that $n^2 - 4$ cannot have exactly 10 positive divisors for any positive integer n.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solutio by Manes.

If n is even, then $n^2 - 1 = (n+1)(n-1)$ and $\gcd(n+1, n-1) = 1$. Since $\tau(n)$, the number of positive divisors of n, is multiplicative, it follows that $\tau(n^2-1) = \tau(n+1)\tau(n-1) = 10 = 5 \cdot 2$. Unique factorization implies one such factorization is $\tau(n+1) = 2$ and $\tau(n-1) = 5$. If $n - 1 = 3^4$, then $\tau(n-1) = 5$. Therefore, $n = 82$ and $\tau(n+1) = \tau(83) = 2$. Accordingly, if $n = 82$, then $n^2 - 1 = 6723 = 3^4 \cdot 83$ and $\tau(n^2-1) = 10$. The solution $n = 82$ is not unique since $n = 2400$ also satisfies the property that $\tau(n^2-1) = 10$.

Note that if $\tau(m) = 10$, then either $m = p^4 q$ or $m = p^9$ for some distinct primes p and q. Thus, if $\tau(n^2 - 4) = 10$, then either $n^2 - 4 = p^4 q$ or $n^2 - 4 = p^9$. If $n^2 - 4 = (n+2)(n-2) = p^9$, then unique factorization implies $n + 2 = p^r$ and $n - 2 = p^s$ where $r > s$ and $r + s = 9$. Subtracting the two equations, it follows that $p^r - p^s = 4$, hence $p = 2$. Therefore, $2^{r-2} - 2^{s-2} = 1$ implies $r - 2 = 1$ and $s - 2 = 0$, a contradiction.

If $n^2 - 4 = (n+2)(n-2) = p^4 q$, then either $n+2 = p^r$ and $n-2 = p^s q$ or $n+2 = p^r q$ and $n-2 = p^s$. If $n+2 = p^r$ and $n-2 = p^s q$, then $r+s = 4$ and $r > s$. Subtracting the two equations, one obtains $p^r - p^s q = 4$, again implying that $p = 2$. Therefore, $2^r - 2^s q = 4$ or $2^{r-2} = 2^{s-2} q + 1$, whence $r = 2$ or $s = 2$, each of which is a contradiction since $r + s = 4$ and $r > s$. On the other hand, if $n + 2 = p^r q$ and $n - 2 = p^s$, then $r + s = 4$. Solving for n in each equation, it follows that $n = p^r q - 2 = p^s + 2$ or $p^r q - p^s = 4$, whence $p = 2$. Therefore, $2^r q - 2^s = 4$ so that $2^{r-2} q = 2^{s-2} + 1$. But $r + s = 4$

implies $(r-2)+(s-2)=0$ or $r=s=2$. Hence, $q=2$, a contradiction since p and q are distinct primes. Consequently, all of these contradictions imply that $\tau(n^2-4)\neq 10$ for any positive integer n.

Third Selection Examination, March 2007

2. Let

$$x = 0.a_1a_2a_3a_4\ldots \quad \text{and} \quad y = 0.b_1b_2b_3b_4\ldots$$

be the decimal representations of two positive real numbers. The equality $b_n = a_{2^n}$ holds for all positive integers n. Given that x is a rational number, show that y is rational, too.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

(i) If the decimal representation for x terminates, then so does that for y, and y is rational.

(ii) Otherwise, the decimal representation for x eventually repeats with some smallest period $t\in\mathbb{N}$. Thus, there is a positive integer k_0 such that for $k\geq k_0$, $a_{k+t}=a_k$. Hence, for $j\in\{0,1,\ldots,t-1\}$, there exists $r_j\in\mathbb{N}\cup\{0\}$ such that for $k\geq k_0$, $a_k=r_j$ if $k\equiv j \bmod t$. Thus, $b_n=a_{2^n}=r_i$ if $2^n\equiv i \bmod t$.

(a) If $\gcd(2,t)=1$, that is, if t is odd, let α be the order of $2 \bmod t$. Then $2^\alpha\equiv 1 \bmod t$. Suppose $2^n\equiv i \bmod t$. Then $2^{n+\alpha}=2^n\cdot 2^\alpha\equiv 2^n\equiv i \bmod t$, so that $b_{n+\alpha}=a_{2^{n+\alpha}}=r_i=a_{2^n}=b_n$. Hence, α is a period for $\{b_n\}$, so y is rational.

(b) If $t=2^s\cdot u$, with u odd, let α be the order of $2 \bmod u$. Then $2^\alpha\equiv 1 \bmod u$. For $n\geq s$, 2^s divides $2^{n+\alpha}-2^n$, so that $2^{n+\alpha}=2^n\cdot 2^\alpha\equiv 2^n \bmod t$. As before, this implies that y is rational.

3. Let $ABCD$ be a trapezoid with AB parallel to CD and $|AB|>|CD|$. Let E and F be the points on segments AB and CD, respectively, such that $\frac{|AE|}{|EB|}=\frac{|DF|}{|FC|}$. Let K and L be two points on the segment EF such that

$$\angle AKB = \angle DCB \quad \text{and} \quad \angle CLD = \angle CBA.$$

Show that K, L, B and C are concyclic.

Solved by Oliver Geupel, Brühl, NRW, Germany.

Let the perpendicular to BC at B intersect the perpendicular bisector of AB at O, and let the perpendicular to BC at C intersect the perpendicular bisector of CD at P. The line OP intersects AB and CD at points G and H, respectively, such that $|BG|:|EG|=|CH|:|FH|$. Hence, the lines BC, EF, and GH have a common intersection S. From $\angle AKB=180^\circ-\angle ABC$ we see that the line BC is tangent to the circle (ABK). Thus, O is the centre of (ABK). Analogously, P is the centre of the circle (CDL). Hence, (ABK) and (CDL) are homothetic with respect to the centre of homothety S.

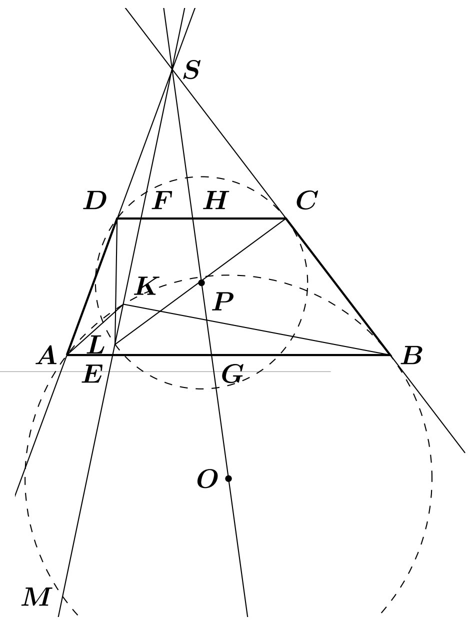
If M is the second point of intersection of the line SK and the circle (ABK), then (B, C), (O, P), (G, H), (E, F), and (M, L) are pairs of corresponding points under this homothety. Therefore, $\angle KMB = \angle SMB = \angle SLC$. Furthermore, $\angle KMB = \angle KBC$, because the line BC is tangent to the circle $(ABKM)$. Thus, $\angle KBC = \angle SLC$. If the point K is between S and L then

$$\angle KBC = \angle KLC. \qquad (1)$$

Otherwise L is between S and K, and we conclude

$$\angle KBC = 180^\circ - \angle KLC. \qquad (2)$$

Either of (1) and (2) implies that K, L, B, and C are concyclic.



Next, we look at solutions submitted to problems of the Correspondence Mathematical Competition in Slovakia 2006/7 First Round, First Set, given at [2010: 438–439].

1. There are some pigeons and some sparrows sitting on a fence. Five sparrows flew away and there remained two pigeons for each sparrow. Then 25 pigeons flew away and there remained three sparrows for each pigeon. Find the initial numbers of sparrows and pigeons.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Curtis.

Let p and s be the initial numbers of pigeons and sparrows, respectively. Then p and s satisfy the system

$$\begin{cases} p &= 2(s-5) \\ s-5 &= 3(p-25). \end{cases}$$

Substituting the first equation into the second, solving for s, and then substituting back gives

$$\begin{cases} p &= 30 \\ s &= 20. \end{cases}$$

3. We have eight cubes with digits $1, 2, 3, 4, 5, 6, 7, 9$ (each cube has one digit written on one of its faces). In how many ways can we create four two-digit primes from the cubes?

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Comăneşti, Romania. We give the solution by Curtis.

We claim that there are four ways. We assume that the order in which the four primes occur is immaterial.

Note first that none of the four primes can end in 2, 4, 5, or 6. Hence these digits must be the first digits of the four primes and 1, 3, 7, and 9 must be the second digits of the primes. The prime ending in 1 must be 41 or 61. The prime ending in 3 must be 23, 43, or 53. The prime ending in 7 must be 47 or 67. The prime ending in 9 must be 29 or 59.

(i) Suppose that 41 is one of the primes. Then 47 is not one of the primes, so 67 must be. The other two primes can then be either 23 and 59 or 29 and 53.

(ii) Suppose that 61 is one of the primes. Then 67 is not one of the primes, so 47 must be. As before, the other primes can then be either 23 and 59 or 29 and 53.

Hence, there are four possible sets of primes:

$$\{41, 67, 23, 59\}, \{41, 67, 29, 53\}, \{61, 47, 23, 59\}, \{61, 47, 29, 53\}.$$

4. A nine-member committee was formed to select a chief of the KMS. There are three candidates for the chief. Each member of the committee orders the candidates and gives 3 points to the first one, 2 points to the second one and 1 point to the last one. After summing the points of the candidates it turned out that no two candidates have the same number of points, hence the order of the candidates is clear. Someone noticed that if every member of the committee selected only one candidate, the resulting order of candidates would be reversed. How many points did the candidates get?

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Comăneşti, Romania. We give the write-up of Zvonaru.

Let C_1, C_2, C_3 be the candidates, with the final order $C_1 - C_2 - C_3$. We denote by $s(C_1)$, $s(C_2)$, $s(C_3)$ the sum of the points of the candidates C_1, C_2, C_3, respectively.

For $i = 1, 2, 3$, we denote

x_i — the number of first place received by C_i
y_i — the number of second place received by C_i
z_i — the number of third place received by C_i.

We have

$$s(C_i) = 3x_i + 2y_i + z_i, \quad i = 1, 2, 3$$

with $x_i + y_i + z_i = 9$, $i = 1, 2, 3$ and $x_1 + x_2 + x_3 = 9$, $y_1 + y_2 + y_3 = 9$, $z_1 + z_2 + z_3 = 9$.

The sum of the points is 54. Since $s(C_1) > s(C_2) > s(C_3)$, we deduce that $s(C_3) < 18$.

Since $x_1 < x_2 < x_3$, it results that $x_3 \geq 4$;

- if $x_3 \geq 6$, then $s(C_3) \geq 18$, a contradiction.
- if $x_3 = 5$, then $y_3 + z_3 = 4$ and $s(C_3) = 15 + 4 + y_3 > 18$, also a contradiction.

It follows that $x_3 = 4$, and from $x_1 + x_2 = 5$ and $x_1 < x_2$ we deduce that $x_1 = 2$, $x_2 = 3$.

We have

$$\begin{aligned} s(C_1) &= 6 + y_1 + z_1 + y_1 = 13 + y_1 \\ s(C_2) &= 9 + y_2 + z_2 + y_2 = 15 + y_2 \\ s(C_3) &= 12 + y_3 + z_3 + y_3 = 17 + y_3 \end{aligned}$$

Since $s(C_3) < 18$ we get $y_3 = 0$ and $y_1 + y_2 = 9$.

Because $s(C_1) > s(C_2) > s(C_3)$ we obtain $y_1 > y_2 + 2$ and $y_2 > y_3 + 2 = 2$.

It is easy to see that the only possibility is $y_2 = 3$, $y_1 = 6$, hence $s(C_1) = 19$, $s(C_2) = 18$, $n(C_3) = 17$.

5.

(a) Find all positive integers n such that both of the numbers $2^n - 1$ and $2^n + 1$ are primes.

(b) Find all primes p such that both of the numbers $4p^2 + 1$ and $6p^2 + 1$ are primes.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give the write-up of Wang.

(a) The only such integer is $n = 2$.

Let $f(n) = 2^n + 1$ and $g(n) = 2^n - 1$. Clearly $g(1) = 1$ which is not a prime. Next, $f(2) = 5$ and $g(2) = 3$ are both primes. Now suppose $n \geq 3$. If n is odd, then $n = 2k + 1$ where $k \geq 1$ so $f(n) = 2^{2k+1} + 1 = 2(4^k) + 1 \equiv 2(1) + 1 \equiv 0 \pmod 3$. Since $f(n) > 3$ and is divisible by 3, it is a composite. If n is even, then $n = 2k$ where $k \geq 1$ so $g(n) = 2^{2k} - 1 = 4^k - 1 \equiv 1 - 1 \equiv 0 \pmod 3$. Since $g(n) > 3$ and is divisible by 3, it is a composite. This completes the proof.

(b) This is the same as Problem #3 of the Finnish Math Olympiad, 2006, (Final Round) the solution of which has appeared in *Crux* 36(7), 2010; p. 447.

6.

Find all positive integers n such that $n + 200$ and $n - 269$ are cubes of integers.

Solved by George Apostolopoulos, Messolonghi, Greece; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution by Geupel.

We prove that the unique solution is $n = 1997$. Suppose that a positive integer n has the desired property. Let $a^3 = n + 200$, $b^3 = n - 269$, and $d = a - b$. We have $d > 0$ and

$$0 = a^3 - (a - d)^3 - 469 = 3da^2 - 3d^2a + d^3 - 469, \qquad (1)$$

a quadratic equation in a with the discriminant

$$9d^4 - 12d(d^3 - 469) = 3d(1876 - d^3) \geq 0.$$

Hence $1 \leq d \leq 12$. Moreover, the integer d is a divisor of $a^3 - b^3 = 469 = 7 \cdot 67$; hence $d \in \{1,\ 7\}$. If $d = 7$, then the equation (1) becomes $0 = 21(a^2 - 7a - 6)$, which has no integer solution, a contradiction. Consequently $d = 1$. Equation (1) becomes $0 = 3(a + 12)(a - 13)$; thus $a = 13$, $b = 12$, and $n = 13^3 - 200 = 1997$.

[*Ed.: Note that instead of (1) we note that* $a^3 - b^3 = (a - b)(a^2 + ab + b^2) = 469 = 7 \times 67$*, which leads to four choices for* $a - b$*, from which the solution follows.*]

7. There were 33 children at a camp. Every child answered two questions: "How many other children have the same first name as you?" and "How many other children at camp have the same family name as you?". Among the answers each of the numbers from 0 to 10 occurred at least once. Show that there were at least two children with the same first name and the same family name.

(Mathematical Contests 1997–1998, 1.18 Russia, 29/95)

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Oliver Geupel, Brühl, NRW, Germany. We give the solution by Geupel.

If the number k occurs among the answers, then there is a name that occurs exactly $k + 1$ times. Hence, among the first and family names of the 33 children, there are names $N_1,\ N_2,\ \ldots,\ N_{11}$ that occur exactly $1,\ 2,\ \ldots,\ 11$ times, respectively. Since $1 + 2 + \cdots + 11 = 2 \cdot 33$, the names $N_1,\ N_2,\ \ldots,\ N_{11}$ cover all the first and family names of the group. By symmetry there is no loss of generality in assuming that N_{11} is a *first* name occurring 11 times. Then among the 11 children with first name N_{11} at most 10 distinct family names occur. By the Pigeonhole Principle, two children among these 11 children have the same family name. The proof is complete.

9. Find all triples of integers x, y, z satisfying

$$2^x + 3^y = z^2.$$

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Soohyun Park, Student, University of Toronto Schools, Toronto, ON; Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the solution of Wang.

We show that the only solutions are $(x, y, z) = (0, 1, \pm 2), (3, 0, \pm 3)$ and $(4, 2, \pm 5)$.

Note first that $z \neq 0$ since the left side is always positive. Thus, $z^2 \geq 1$. If at least one of x and y is negative then the left side cannot be an integer, a contradiction. Hence, we may assume that x and y are nonnegative. We may also assume for the time being that z is positive. We first establish a lemma.

Lemma. If p is a prime, then the only solutions in nonnegative integers u and v to the equation $p^u + 1 = v^2$ is $(u, v) = (3, 3)$ if $p = 2$, and $(u, v) = (1, 2)$ if $p = 3$. If $p \neq 2, 3$, then there are no solutions.

Proof. Clearly $u \neq 0$ since 2 is not a square. Also, $v \neq 0$. We have $p^u = v^2 - 1 = (v - 1)(v + 1)$ so $v - 1 = p^a$ and $v + 1 = p^b$ for some integers a and b such that $0 < a < b$ with $a + b = u$. Then $p^b - p^a = 2$ or $p^a(p^{b-a} - 1) = 2$. If $p \neq 2$, then $a = 0$ and $p^{b-a} - 1 = 2$ so $p^b = 3$ which implies that $p = 3$ and $b = 1$. It follows that $u = a + b = 1$ and $v = 2$. If $p = 2$, then we have either (i) $2^a = 1$, $2^{b-a} - 1 = 2$; or (ii) $2^a = 2$, $2^{b-a} - 1 = 1$. In case (i) we get $a = 0$ and $2^b = 3$ which is impossible. In case (ii) we have $a = 1$ and $2^{b-1} - 1 = 1$ so $b = 2$. It follows that $u = a + b = 3$ and $v = 1 + 2 = 3$ so $(u, v) = (3, 3)$ and the lemma is established.

Back to the given problem, note that if $x = 0$, then the equation becomes $3^y + 1 = z^2$. Hence by the lemma above the only solution is $(x, y, z) = (0, 1, 2)$. Similarly, if $y = 0$, then the equation becomes $2^x + 1 = z^2$, so by the lemma again, the only solution is $(x, y, z) = (3, 0, 3)$. Now we assume $x \geq 1$ and $y \geq 1$. If $x = 1$ then we have $2 + 3^y = z^2$ which is impossible since $\bmod 3$, $2 + 3^y \equiv 2$ while $z^2 \equiv 0$ or 1. Thus, $x \geq 2$. Since $2^x + 3^y$ is odd so is z. Then $z^2 \equiv 1 \pmod 4$. But $\bmod 4$, $2^x \equiv 0$ so $3^y \equiv (-1)^y \equiv 1$. Thus, y is even. Let $y = 2t$ where $t \geq 1$. Then we have $2^x = z^2 - 3^{2t} = (z - 3^t)(z + 3^t)$. Hence $z - 3^t = 2^c$ and $z + 3^t = 2^d$ where $0 \leq c < d$ such that $c + d = x$. Subtracting the two equations above we get $2 \cdot 3^t = 2^d - 2^c$ so $3^t = 2^{c-1}(2^{d-c} - 1)$. Since $2 \cdot 3^t$ is even, $2^c \neq 1$ so $c \geq 1$. Therefore, $2^{c-1} = 1$ and $2^{d-c} - 1 = 3$. This yields $c = 1$ and $2^{d-1} - 1 = 3^t$. We now show that the only solution to this last equation is $d = 3$ and $t = 1$. Since $t \geq 1$ we have $2^{d-1} \geq 4$ so $d \geq 3$. Thus, $2^{d-1} \equiv 0 \pmod 4$ which implies that $2^{d-1} - 1 \equiv -1 \pmod 4 \Rightarrow 3^t \equiv (-1)^t \equiv -1 \pmod 4$ so t is odd. Hence, we have $2^{d-1} = 3^t + 1 = (3 + 1)(3^{t-1} - 3^{t-2} + \cdots - 3 + 1)$. But $3^{t-1} - 3^{t-2} + \cdots - 3 + 1$, is the sum of an odd number of odd integers so it is odd. Since it divides 2 it must be 1. It then follows that $2^{d-1} = 4$ so $d = 3$ which implies $t = 1$. Finally, $y = 2t = 2$, $z = 3 + 2^c = 5$ from which $x = 4$ follows. This yields the third and last solution $(4, 2, 5)$.

12. We are given an acute triangle ABC with circumcentre O. Let T be the circumcentre of AOC. Let M be the midpoint of AC. The points D and E lie on the lines AB and CB respectively in such a way that the angles MDB and MEB are equal to the angle ABC. Prove that the lines BT and DE are perpendicular.

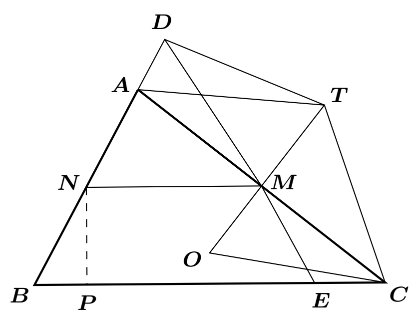
Solved by Titu Zvonaru, Comăneşti, Romania.

As usual we write $a = BC$, $b = CA$, $c = AB$. Let R be the circumradius of $\triangle ABC$, and let N be the midpoint of AB.

We denote $TA = R_1$, and we suppose that $a \geq b \geq c$.

Since $MN \| BE$ and $\angle MEB = \angle NBE$, the quadrilateral $BEMN$ is an isosceles trapezoid with

$$MN = \frac{a}{2}, \quad ME = NB = \frac{c}{2}.$$



Denoting by P the projection of N onto BC, we deduce that $BP = \frac{c}{2}\cos B$, hence $BE = \frac{a}{2} + c\cos B$. Applying the Law of Sines we have

$$\begin{aligned} EC &= \frac{a}{2} - c\cos B \\ &= R(\sin A - 2\sin C\cos B) = R[\sin A - \sin(C+B) - \sin(C-B)] \\ &= R[\sin A - \sin A + \sin(B-C)] = R\sin(B-C). \end{aligned}$$

Similarly we obtain

$$BD = \frac{c}{2} + a\cos B; \quad AD = R\sin(A-B).$$

Since $\angle TCE = \angle TCO + \angle OCE = B + 90^\circ - A = 90^\circ - (A-B)$ by the Law of Cosines we have

$$\begin{aligned} TE^2 &= TC^2 + EC^2 - 2TC \cdot EC\cos\angle TCE \\ &= R_1^2 + EC^2 - 2RR_1\sin(B-C)\sin(A-B) \end{aligned}$$

and similarly

$$TD^2 = R_1^2 + AD^2 - 2RR_1\sin(A-B)\sin(B-C),$$

hence

$$TE^2 - TD^2 = EC^2 - AD^2. \tag{1}$$

On the other hand we deduce that

$$\begin{aligned} EC^2 - AD^2 &= (a - BE)^2 - (BD - c)^2 \\ &= a(a - 2BE) - c(c - 2BD) + BE^2 - BD^2 \\ &= a(a - a - 2c\cos B) - c(c - c - 2a\cos B) + BE^2 - BD^2, \end{aligned}$$

hence

$$BE^2 - BD^2 = EC^2 - AD^2. \qquad (2)$$

By (1) and (2) it results that the lines BT and DE are perpendicular.

13. A line passing through the centroid T of the triangle ABC meets the side AB at P and the side CA at Q. Prove that

$$4 \cdot PB \cdot QC \ \leq \ PA \cdot QA.$$

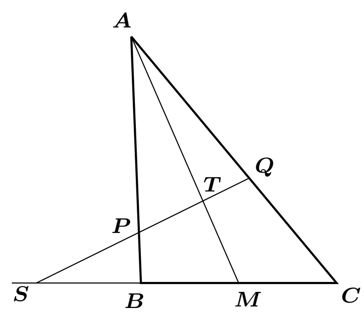
(R.B. Manfrino: Inequalities, 111/3.29, Spain 1998)

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Comǎneşti, Romania. We give the version of Zvonaru.

Let M be the midpoint of BC. We denote $a = BC$, $x = \dfrac{AP}{PB}$, $y = \dfrac{AQ}{QC}$. We will prove that

$$xy = x + y \qquad (1)$$

If $PQ \| BC$, then $x = y = \dfrac{AT}{TM} = 2$ and (1) is true.



Suppose that PQ meets BC at the point S such that B lies between S and C. By Menelaus' Theorem applied in $\triangle ABC$, we have

$$\frac{BS}{SC} \cdot \frac{CQ}{QA} \cdot \frac{AP}{PB} = -1,$$

that is

$$\frac{SB}{SC} = \frac{y}{x} \Leftrightarrow \frac{SB}{SB + a} = \frac{y}{x}$$

hence $SB = \dfrac{ay}{x - y}$.

Now we apply Menelaus' Theorem in $\triangle ABM$ to get

$$\frac{SB}{SM} \cdot \frac{TM}{TA} \cdot \frac{PA}{PB} = 1 \Leftrightarrow \frac{\frac{ay}{x-y}}{\frac{ay}{x-y} + \frac{a}{2}} \cdot \frac{1}{2} \cdot x = 1$$

$$\Leftrightarrow \frac{ay}{x - y} \cdot \frac{2(x - y)}{a(2y + x - y)} \cdot \frac{x}{2} = 1$$

hence $xy = x + y$.

Using (1) and AM-GM Inequality we obtain

$$xy = x + y \geq 2\sqrt{xy} \Rightarrow (xy)^2 \geq 4xy \Rightarrow xy \geq 4,$$

that is

$$\frac{PA}{PB} \cdot \frac{QA}{QC} \geq 4.$$

14. Integers x and y greater than 1 satisfy the relation $2x^2 - 1 = y^{15}$.

(a) Prove that x is divisible by five.

(b) Are there such integers x and y greater than 1 satisfying $2x^2 - 1 = y^{15}$? Could you find all such numbers?

(Russia 2004/05)

Solved by David E. Manes, SUNY at Oneonta, Oneonta, NY, USA.

(a) Note that if integers $x, y > 1$ satisfy $2x^2 - 1 = y^{15}$ then $2x^2 - 1 = (y^3)^5$. Let $z = y^3 > 1$ and consider the Diophantine equation $2x^2 - 1 = z^5$. Then z is odd and

$$2x^2 = z^5 + 1 = (z+1)(z^4 - z^3 + z^2 - z + 1).$$

Let $d = \gcd(z+1, z^4 - z^3 + z^2 - z + 1)$. Then d divides $z+1$ and

$$d \mid (z^4 - z^3 + z^2 - z + 1) = (z^3 - 2z^2 + 3z - 4)(z+1) + 5.$$

Hence, d divides 5 and so $d = 1$ or 5. If $d = 5$, then $5 \mid 2x^2$ whence 5 divides x and we are done. If $d = 1$, then $z + 1$ and $z^4 - z^3 + z^2 - z + 1$ are relatively prime and their product is twice a square. Since $z^4 - z^3 + z^2 - z + 1$ is odd, it must be a square. However, for $y > 1$,

$$(2z^2 - z)^2 < 4(z^4 - z^3 + z^2 - z + 1) < (2z^2 - z + 1)^2.$$

Therefore, $z^4 - z^3 + z^2 - z + 1$ is not a square, a contradiction. Accordingly if $2x^2 - 1 = y^{15}$ and $x, y > 1$, then 5 divides x.

(b) Assume integers $x, y > 1$ exist so that $2x^2 - 1 = y^{15}$. Then integers $x, z = y^5 > 1$ exist such that $2x^2 - 1 = z^3$, By a result of Cohn, the only solution of this equation with $x, y > 1$ is $x = 78$, $z = 23$ (cf. Theorem 2, p. 27, J.H.E. Cohn, *The Diophantine equations* $x^3 = Ny^2 \pm 1$, Quart. J. Math. Oxford, 42(1991), 27–30). Since 23 is not a fifth power of any integer, it follows that the only solution of $2x^2 - 1 = y^{15}$ is $x = y = 1$.

Next we give readers' solutions to the Latvian School Mathematical Olympiad, Grade 11, given at [2010: 440].

1. For a positive integer n:

(a) can the sums of digits of n and $n + 2007$ be equal?

(b) can the sums of digits of n and $n + 199$ be equal?

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comănești, Romania. We give the write-up of Curtis.

For a positive integer n, let $d(n)$ denote the sum of the digits of n. Let m and n be positive integers. Write

$$m = \sum_{k=0}^{r} a_k \cdot 10^k \quad \text{and} \quad n = \sum_{k=0}^{r} b_k \cdot 10^k,$$

where the a_k and b_k are nonnegative integers. If $r = 0$ and $a_0 + b_0 < 10$, then $d(m+n) = d[(a_0+b_0)] = a_0+b_0 = d(m)+d(n)$. If $r = 0$ and $a_0+b_0 \geq 10$, that is if a carry is required, then $d(m + n) = d[1 \cdot 10 + (a_0 + b_0 - 10)] = a_0 + b_0 - 9 = d(m) + d(n) - 9$. The same occurs each time a carry is required, so $d(m + n) = d(m) + d(n)$ minus 9 times the number of carries required. Hence,

$$9 \mid [d(m + n) - (d(m) + d(n))].$$

(a) If $n = 3$, then $d(n) = 3$ and $d(n + 2007) = d(2010) = 3 = d(n)$, so the answer to the first question is 'yes'.

(b) If $d(n) = d(n+199)$, then $9 \mid [d(n+199) - d(n) - d(199)]$ implies that 9 divides $d(199)$. But $d(199) = 19$, a contradiction. Hence, the answer to the second question is 'no'.

2. Do there exist three quadratic trinomials such that each of them has at least one root, but the sum of any two quadratic trinomials doesn't have any roots?

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Comănești, Romania. We give the solution of Zvonaru.

The answer is YES.

Let a, b, c be three real numbers such that $a \neq b \neq c \neq a$.

$$\begin{array}{rcll}
(x-a)^2 &=& 0 & \text{has the root } a \\
(x-b)^2 &=& 0 & \text{has the root } b \\
(x-c)^2 &=& 0 & \text{has the root } c \\
(x-a)^2 + (x-b)^2 &=& 0 & \text{has no real roots} \\
(x-a)^2 + (x-c)^2 &=& 0 & \text{has no real roots} \\
(x-b)^2 + (x-c)^2 &=& 0 & \text{has no real roots.}
\end{array}$$

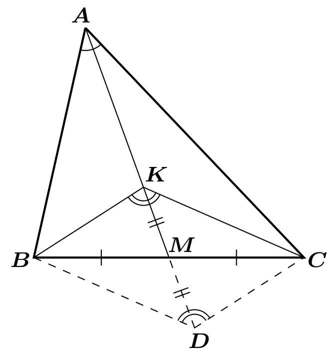
4. In triangle ABC a point K lies on median AM and $\angle BAC + \angle BKC = 180^\circ$. Prove that $AB \cdot KC = AC \cdot KB$.

Solved by George Apostolopoulos, Messolonghi, Greece; and Titu Zvonaru, Comănești, Romania. We give the solution of Apostolopoulos.

We assume $\angle BAC < 90°$. On AM take $MD = KM$. Then $KBDC$ is a parallelogram, since the diagonals bisect each other. So $\angle BKC = \angle BDC$ and $\angle BAC + \angle BDC = 180°$ hence $ABDC$ is cyclic and $\angle ABD + \angle ACD = 180°$. Thus

$$\frac{[\triangle ABD]}{[\triangle ACD]} = \frac{AB \cdot BD}{AC \cdot CD},$$

where $[\triangle ABD]$, as usual, represents the area of $\triangle ABC$. Also $[\triangle ABD] = [\triangle ACD]$ so $AB \cdot BD = AC \cdot CD$, but $BD = KC$ and $CD = KB$ namely $AB \cdot KC = AC \cdot KB$.



5. For a sequence of real numbers $a_1, a_2, a_3, \ldots$ we have $a_{11} = 4$, $a_{22} = 2$ and $a_{33} = 1$. In addition, the relation

$$\frac{a_{n+3} - a_{n+2}}{a_n - a_{n+1}} = \frac{a_{n+3} + a_{n+2}}{a_n + a_{n+1}}$$

holds for each n. Prove that:

(a) $a_i \neq 0$ for each i,

(b) the sequence is periodic,

(c) $a_1^k + a_2^k + \cdots + a_{100}^k$ is a square of an integer for any arbitrary positive integer k.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania. We give the version of Zvonaru.

(a) The given relation is equivalent to

$$\begin{aligned} & a_{n+3}a_n + a_{n+3}a_{n+1} - a_n a_{n+2} - a_{n+1}a_{n+2} \\ & \quad = a_{n+3}a_n - a_{n+1}a_{n+3} + a_n a_{n+2} - a_{n+1}a_{n+2} \\ & \quad \Leftrightarrow a_n a_{n+2} = a_{n+1}a_{n+3}. \end{aligned} \tag{1}$$

By the same relation we deduce that $a_n \neq a_{n+1}$ for each n. Using (1) we have

$$a_i a_{i+2} = a_{i+1}a_{i+3} = a_{i+2}a_{i+4}.$$

Suppose that $a_i = 0$; it results that $a_{i+1} \neq 0$ and from $a_{i+1}a_{i+3} = 0$ we get $a_{i+3} = 0$. Since $a_{i+2} \neq a_{i+3} \neq a_{i+4}$, we obtain a contradiction $a_{i+2}a_{i+4} = 0$.

(b) By (1) we have

$$a_n a_{n+2} = a_{n+1}a_{n+3} \; ; \qquad a_{n+1}a_{n+3} = a_{n+2}a_{n+4},$$

hence $a_n a_{n+2} = a_{n+2} a_{n+4}$.

Since $a_{n+2} \neq 0$, it results that $a_n = a_{n+4}$, that is the sequence is periodic.

(c) We have

$$a_1 = a_5 = a_9 = \ldots = a_{33} = 1$$
$$a_2 = a_6 = a_{10} = \ldots = a_{22} = 2$$
$$a_3 = a_7 = a_{11} = 4$$

and from $a_1 a_3 = a_2 a_4$ we obtain $a_4 = 2$.

It follows that

$$\begin{aligned} a_1^k + \cdots + a_{100}^k &= 25(a_1^k + a_2^k + a_3^k + a_4^k) = 25(1 + 2^k + 4^k + 2^k) \\ &= 25((2^k)^2 + 2 \cdot 2^k + 1), \end{aligned}$$

hence $a_1^k + \cdots + a_{100}^k = [5(2^k + 1)]^2$.

Next we look at solutions to the Latvian Mathematical Olympiad Grade 12, given at [2010: 440–441].

1. What can be the values of nonnegative real numbers a and b, if it is known that equations $x^2 + a^2 x + b^3 = 0$ and $x^2 + b^2 x + a^3 = 0$ have a common real root?

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We use the solution of Perfetti.

Answer: $a = b = 0$, $a = b \geq 4$.

Proof. The solutions of the two equations are

$$x_{1,2} = \frac{-a^2 \pm \sqrt{a^4 - 4b^3}}{2}, \qquad x_{3,4} = \frac{-b^2 \pm \sqrt{b^4 - 4a^3}}{2}$$

They are real if and only if

$$\{a^4 \geq 4b^3 \wedge b^4 \geq 4a^3\} \Rightarrow (ab)^3(ab) \geq 16(ab)^3 \Rightarrow \{ab = 0 \vee ab \geq 16\}$$

Moreover we have

$$\{a^4 \geq 4b^3 \ \wedge \ b^4 \geq 4a^3\} \Longrightarrow \left\{a \geq (4b^3)^{1/4} \ \wedge \ a \leq \left(\frac{b^4}{4}\right)^{1/3}\right\}$$

The graphs of the functions $(4b^3)^{1/4}$ and $\left(\frac{b^4}{4}\right)^{1/3}$ show that $a = b = 0$ or $a, b \geq 4$.

If $a = b = 0$ the roots are all equal to zero. If $a = b \geq 4$ the two parabola coincide. If $a = b = 4$ they have one root with multiplicity two. If $a = b > 4$ the have two distinct roots.

Thus let us suppose $a \neq b$ and because of the symmetry with respect to the change $(a, b) \to (b, a)$ we may suppose $b > a$.

Moreover we note that

$$b^4 - 4a^3 > a^4 - 4b^3$$

so x_3 is smaller than any other root and therefore we can have only $x_4 = x_1$ or $x_4 = x_2$.

First case $x_4 = x_1$.

$$-b^2 + \sqrt{b^4 - 4a^3} = -a^2 - \sqrt{a^4 - 4b^3}, \qquad b^2 - a^2 = \sqrt{b^4 - 4a^3} + \sqrt{a^4 - 4b^3}$$

or squaring

$$4a^3 + 4b^3 - 2a^2b^2 = \sqrt{b^4 - 4a^3}\sqrt{a^4 - 4b^3}$$

The left-hand side may be both negative and positive. For the values which make it positive we can square again and obtain

$$4(4a^6 - 8a^3b^3 - 4a^5b^2 + 4b^6 - 4b^5a^2 + 4b^7 + 4a^7) = 0 \qquad (1)$$

By AGM we have

$$(a^7 + a^7 + a^7 + a^7 + a^7 + b^7 + b^7)/7 \geq a^5b^2, \qquad (a^6 + b^6) \geq a^3b^3$$

with equality in both cases if and only if $a = b$. If follows that (1) is never zero unless $a = b$.

Second case. $x_4 = x_2$.

$$-b^2 + \sqrt{b^4 - 4a^3} = -a^2 + \sqrt{a^4 - 4b^3}, \qquad b^2 - a^2 = \sqrt{b^4 - 4a^3} - \sqrt{a^4 - 4b^3}$$

or squaring

$$4a^3 + 4b^3 - 2a^2b^2 = -\sqrt{b^4 - 4a^3}\sqrt{a^4 - 4b^3}$$

For the values which make the left hand side negative we can square again and obtain as above

$$4(4a^6 - 8a^3b^3 - 4a^5b^2 + 4b^6 - 4b^5a^2 + 4b^7 + 4a^7) = 0$$

The consequences are the same.

The conclusions are that if $a = b = 0$ the four solutions all coincide. If $a = b \geq 4$ the solutions coincide pairwise. For any other values of (a, b) the solutions are all distinct.

3. Solve the system of equations

$$\begin{cases} \sin^2 x + \cos^2 y = y^2 \\ \sin^2 y + \cos^2 x = x^2 \end{cases}$$

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA, modified by the editor.

From the two equations, we find that

$$\sin^2 x - \sin^2 y = y^2 - 1 = 1 - x^2 = \frac{1}{2}(y^2 - x^2),$$

from which it follows that

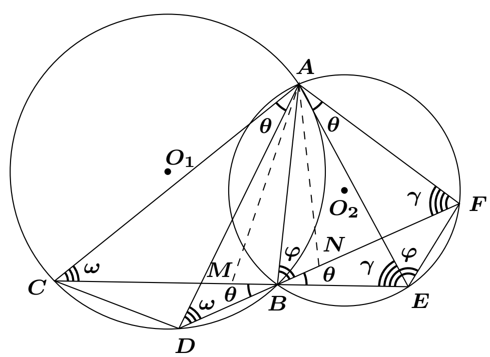
$$2\sin^2 x + x^2 = 2\sin^2 y + y^2. \qquad (2)$$

The function $2\sin^2 t + t^2$ is an even function of t that is strictly increasing for $0 \le t \le \frac{\pi}{2}$. Since, from the original equations, we have that $|x|$ and $|y|$ do not exceed $\sqrt{2} < \frac{\pi}{2}$, it follows that from (1) $|x| = |y|$.

On the other hand, when this condition holds, we must have $x^2 = y^2 = 1$, since $x^2 + y^2 = 2$. So the equations are satisfied when $(x, y) \in \{(1, 1), (1, -1), (-1, 1), (-1, -1)\}$.

4. Two circles w_1 and w_2 intersect in points A and B. Line t_1 is drawn through point B with other intersection point with w_1 being C and other intersection point with w_2 being E. Line t_2 is drawn through point B with other intersection point with w_1 being D and other intersection point with w_2 being F. Point B lies between C and E and between D and F. Midpoints of segments CE and DF are denoted by M and N. Prove that triangles ACD, AEF and AMN are similar.

Solved by Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comǎneşti, Romania. We give Zelator's write-up.



Looking at the figure and the cyclic quadrilaterals $ACDB$ and $ABEF$, the following statements are clear:

$$\angle CAD = \angle CBD = \angle EBF = \angle EAF = \theta, \qquad (1)$$

$$\angle ABF = \angle AEF = \angle ACD = \varphi, \qquad (2)$$

$$\angle ACB = \angle ADB = \omega, \qquad (3)$$

$$\angle BFA = \angle BEA = \gamma. \tag{4}$$

From (1) and (2) we have $\angle CAD = \angle EAF = \theta$ and $\angle ACD = \angle AEF = \varphi$ which proves that the triangles ACD and AEF are similar (two-angle criterion).

From (3) and (4), it follows that triangles ACE and ADF are similar (two-angle criterion) (also note that $\angle CAE = \theta + \angle DAE$ and $\angle DAF = \angle DAE + \theta$). The sides CE and DF lie across the congruent angles $\angle CAE$ and $\angle DAF$. Thus, since N and M are the midpoints of CE and DF, and the triangles ACE and ADF are similar, it follows that

$$\begin{aligned} \angle DAM &= \angle CAN, \\ \angle DAN + \angle NAM &= \angle CAD + \angle DAN, \\ \angle NAM &= \angle CAD = \theta \quad \text{by (1).} \end{aligned} \tag{5}$$

And since $\angle EBF = \theta$, it follows by (5) that

$$\angle EBF = \theta = \angle CAD \tag{6}$$

It follows from (6) that $ANBM$ is a cyclic quadrilateral. Thus $\angle ANM = \angle ABM = \varphi$ and by (2) we see that

$$\angle ANM = \angle ACD = \varphi. \tag{7}$$

Clearly (6) and (7) imply that the triangles ACD and ANM are similar (two-angle criterion). And since we have already shown that ACD and AEF are similar, we conclude that the triangles ACD, AEF, and ANM are similar.

5. The set of all positive integers has been split in several parts in such a way that each integer belongs exactly to one part and each of the parts contains infinitely many integers. Can this be done so that one part contains a multiple of any positive integer? Give the answer if

(a) there are a finite number of parts,

(b) there are an infinite number of parts.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Oliver Geupel, Brühl, NRW, Germany. We give the solution of Geupel.

We prove that the answer is "Yes" in both cases. Let A_1 and A_2 denote the set of even numbers and the set of odd numbers, respectively. Then, A_1 contains the double of each positive integer, which completes the proof for part (a). Let $p_1 = 2,\ p_2 = 3,\ p_3 = 5,\ \dots$ be the sequence of prime numbers. For $k = 1,\ 2,\ 3,\ \dots$, let B_k denote the set of positive integers which are divisible by p_k but not divisible by any of the numbers $p_1,\ p_2,\ \dots,\ p_{k-1}$. Then, $\{B_1,\ B_2,\ B_3,\ \dots\}$ is an infinite partition of the set of positive integers and B_1 is the set of even numbers, which completes the proof for part (b).

While this was obvious, the following question is more interesting: Is it true that, for each partition of the set of positive integers, one part contains a multiple of each positive integer?

We prove that the answer is "Yes" for partitions into a finite number of parts. Suppose the contrary. Then there exists a partition $\{A_1,\ A_2,\ \ldots,\ A_n\}$ such that for each A_k there is a number a_k with the property that no multiple of a_k is contained in A_k. The number $a = a_1a_2\cdots a_n$ is in some A_k, say in A_n. By $a = (a_1a_2\cdots a_{n-1})a_n$, it follows that a multiple of a_n is contained in A_n. This is a contradiction, which completes the proof.

However, the answer is "No" if infinitely many parts can occur. A counterexample is the partition $\{B_1,\ B_2,\ B_3,\ \ldots\}$ where B_k is the set of positive integers which have exactly k distinct prime divisors.

Next we turn to solutions from our readers to problems of the Finnish National High School Mathematics Competition, Final Round, given at [2010: 441–442].

1. Show that when a prime number is divided by 30, the remainder is either a prime number or 1. Is a similar claim true when the divisor is 60 or 90?

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Cománeşti, Romania. We give the solution by Manes.

Note that if n is a composite number less than 30, then $\gcd(30, n) \neq 1$. Let p be a prime number and assume that the remainder r when p is divided by 30 is neither 1 nor a prime. Then $p = 30m + r$ for some integers m and r with $0 < r < 30$. Therefore, $\gcd(30, r) \neq 1$ implies there is a prime q such that q divides both of the integers 30 and r. Hence, q divides p and this implies $q = p$, a contradiction that proves the result.

However, no such claim can be made when the divisor is 60 or 90 since the prime 229 divided by 60 or 90 leaves a remainder of 49.

2. Determine the number of real roots of the equation

$$x^8 - x^7 + 2x^6 - 2x^5 + 3x^4 - 3x^3 + 4x^2 - 4x + \frac{5}{2} \;=\; 0.$$

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Cománeşti, Romania. We give the write-up of Curtis.

Let $P(x)$ be the given polynomial. Then

$$\begin{aligned} P(x) &= \frac{1}{2}[2x^7(x-1) + 4x^5(x-1) + 6x^3(x-1) + 8x(x-1) + 5] \\ &= \frac{1}{2}[2x(x-1)(x^6 + 2x^4 + 3x^2 + 4) + 5] > 0 \text{ for } x > 1. \end{aligned}$$

Also,

$$P(-x) = \frac{1}{2}(2x^8 + 2x^7 + 4x^6 + 4x^5 + 6x^4 + 6x^3 + 8x^2 + 8x + 5).$$

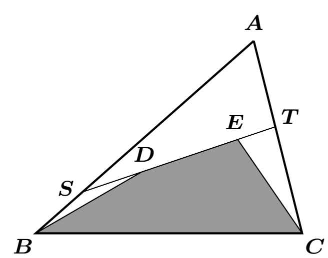
Thus, $P(-x) > 0$ for $x > 0$, so that $P(x) > 0$ for $x < 0$. Hence, any real zero of $P(x)$ lies in the interval $[0, 1]$. But, neither $P(0)$ nor $P(1)$ equals 0, and for $0 < x < 1$, $-\frac{1}{2} < 2x(x - 1) < 0$ and $0 < x^6 + 2x^4 + 3x^2 + 4 < 10$. This implies that for $0 < x < 1$,

$$0 < P(x) < 5.$$

Thus, $P(x)$ has no real zeros.

3. There are five points in the plane, no three of which are collinear. Show that some four of these points are the vertices of a convex quadrilateral.

Solved by Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Cománeşti, Romania. We give the solution by Geupel.



The convex hull of the given points is a convex n-gon where $n \in \{3,\ 4,\ 5\}$.

If $n = 4$ then we are done.

If $n = 5$ then any four of the five points are the vertices of a convex quadrilateral.

Finally suppose $n = 3$. The convex hull is a triangle, say ABC, and the other two points, say D and E, are in the interior of that triangle. Because no three of the points A, B, C, D, and E are collinear, the line DE does not pass through a vertex of the triangle. Hence, the line DE meets two sides of the triangle, say AB and AC at inner points S and T, respectively. Assume $DS < ES$. Then

$$\angle CBD < \angle CBA < 180^\circ, \qquad \angle BCE < \angle BCA < 180^\circ,$$

$$\angle BDE < \angle SDE = 180^\circ, \qquad \angle CED < \angle TED = 180^\circ.$$

Consequently, $BCED$ is a convex quadrilateral. This completes the proof.

5. Show that there exists a polynomial $P(x)$ with integer coefficients such that the equation $P(x) = 0$ has no integer solutions but for each positive integer n there is an $x \in \mathbb{Z}$ such that $n \mid P(x)$.

Solved by David E. Manes, SUNY at Oneonta, Oneonta, NY, USA.

Let $P(x) = 6x^2 + 5x + 1 = (3x + 1)(2x + 1)$. Then $P(x) = 0$ has no integer roots. Let n be an arbitrary positive integer and write $n = 2^r \cdot s$, where r is an integer ≥ 0 and s is odd. Since $\gcd(2^r, 3) = 1 = \gcd(s, 2)$, it follows that there exist positive integers u and v such that

$$3u \equiv -1 \pmod{2^r} \quad \text{and} \quad 2v \equiv -1 \pmod{s}\,.$$

Moreover, $\gcd(2^r, s) = 1$ so that the Chinese Remainder Theorem implies there is an integer m such that

$$3m \equiv -1 \pmod{2^r \cdot s} \quad \text{and} \quad 2m \equiv -1 \pmod{2^r \cdot s}\,.$$

Therefore, $P(m) = (3m + 1)(2m + 1) = 0 \bmod n$, whence $n \mid P(m)$.

To complete this number of the *Corner* we turn to the solutions to problems of the IX Olimpiada Matemático de Centramérica y el Cariba 2007, given at [2010: 442–443].

1. The OMCC is an annual mathematical competition. The ninth olympiad takes place in the year 2007. Which positive integers n divide the year in which the n^{th} olympiad takes place?

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Zelator.

Since the OMCC is an annual event and the 9^{th} olympiad takes place in 2007, it follows that the *first* olympiad took place in the year $2007 - 8 = 1999$. Thus the n^{th}olympiad takes placed in the year $1999 + (n - 1) = 1998 + n$. So if $n \mid (1998 + n)$ we have $1998 + n = kn$, for some positive integer k, so that $1998 = (k - 1) \cdot n$, whence $n \mid 1998$. So, the positive integers n which divide the year in which the n^{th} olympiad takes place are precisely the positive divisors of 1998. Now, $1998 = 2(999) = 2 \cdot 9 \cdot (111) = 2 \cdot 9 \cdot 3 \cdot 37 = 2 \cdot 3^3 \cdot 37$. The number of divisors of 1998 is $\tau(1998) = (1 + 1) \cdot (3 + 1) \cdot (1 + 1) = 16$. The sixteen divisors of 1998 are the positive integers:

$$n = 1, 2, 2 \cdot 3, 2 \cdot 3^2, 2 \cdot 3^3, 2 \cdot 37, 3, 3^2, 3^3, 37, 3 \cdot 37,$$

$$3^2 \cdot 37, 3^3 \cdot 37, 2 \cdot 3^3 \cdot 37, 2 \cdot 3^2 \cdot 37, 2 \cdot 3 \cdot 37.$$

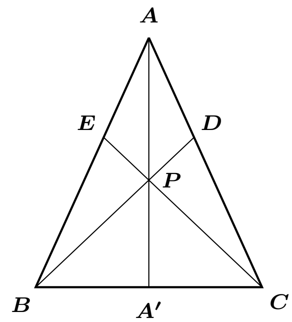
We list these sixteen positive integers in increasing order.

$$n = 1, 2, 3, 6, 9, 18, 27, 37, 54, 74, 111, 222, 333, 666, 999, 1998.$$

2. Let ABC be a triangle, D and E points on the sides AC and AB, respectively, such that the lines BD, CE and the angle bisector of angle A concur in an interior point P of the triangle. Prove that there is a circle tangent to the four sides of the quadrilateral $ADPE$ if and only if $AB = AC$.

Solved by Titu Zvonaru, Comăneşti, Romania.

Let A' be the point of intersection of BC with the angle bisector of angle A.



(i) Suppose that $AB = AC$; then A' is the midpoint of BC. By Ceva's Theorem we obtain

$$\frac{AE}{EB} \cdot \frac{BA'}{A'C} \cdot \frac{CD}{DA} = 1$$

hence

$$\frac{AE}{AB - AE} = \frac{DA}{AC - DA}$$

$$\Leftrightarrow AE \cdot AC - AE \cdot DA = AB \cdot DA - AE \cdot DA,$$

that is $AE = AD$.

It results that $\triangle AEP$ and $\triangle ADP$ are congruent (side-angle-side) hence $AE + DP = AD + EP$, which means that there is a circle tangent to the four sides of quadrilateral $ADPE$.

(ii) Suppose that there is a circle tangent to the four sides of quadrilateral $ADPE$. Then the centre of this circle lies on the bisector of angle $\angle EAD$, that is on AP. We deduce that AP is the bisector of $\angle EPD$, and the triangles AEP and ADP are congruent (angle-side-angle), hence $AE = AD$. Because A' lies on the bisector AP of $\angle BAC$, $\frac{AB}{AC} = \frac{BA'}{A'C}$. Inserting the last two equalities into Ceva's theorem gives us

$$\frac{EA}{EB} \cdot \frac{A'B}{A'C} \cdot \frac{DC}{DA} = 1$$

$$\Rightarrow \frac{AB}{AC} = \frac{EB}{DC} \Rightarrow \frac{AB}{AC} = \frac{AB - AE}{AC - AE}$$

$$\Rightarrow AB \cdot AC - AB \cdot AE = AB \cdot AC - AC \cdot AE,$$

hence $AB = AC$.

3. Let S be a finite set of integers. For any two integers $p, q \in S$, with $p \neq q$, there are integers a, b, c in S, not necessarily distinct and with $a \neq 0$, such that the polynomial $F(x) = ax^2 + bx + c$ satisfies $F(p) = F(q) = 0$. Determine the maximum number of elements set S can have.

Solved by Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Cománeşti, Romania. We give Geupel's write-up.

We prove that $\max|S| = 3$.

A valid set with three elements is $S = \{-1,\ 0,\ 1\}$.

We show by contradiction that $|S| < 4$.

Suppose S is such that $|S| \geq 4$. Then, there is a member $p \in S$ such that $|p| > 1$. Let $T = \{-1,\ 0,\ 1,\ p\}$.

If $p > 1$ then $-a(p+1) \notin T$ for each $a \in T \setminus \{0\}$. Hence, there are no $a \neq 0,\ b,\ c \in T$ such that

$$ax^2 + bx + c \ = \ a(x-1)(x-p) \ = \ ax^2 - a(p+1)x + ap.$$

If $p < -1$ then $-a(p-1) \notin T$ for each $a \in T \setminus \{0\}$. Hence, there are no $a \neq 0,\ b,\ c \in T$ such that

$$ax^2 + bx + c \ = \ a(x+1)(x-p) \ = \ ax^2 - a(p-1)x - ap.$$

Therefore $S \neq T$. Thus, there are distinct members $p,\ q \in S$ such that $|p| > 1$ and $|q| > 1$. Let p and q be the elements of S with the greatest absolute values. Then, $apq \notin S$ for each $a \in S \setminus \{0\}$. Hence, there are no $a \neq 0,\ b,\ c \in S$ such that

$$ax^2 + bx + c \ = \ a(x-p)(x-q) \ = \ ax^2 - a(p+q)x + apq,$$

a contradiction.

This completes the proof that $|S| < 4$.

4. The inhabitants of a certain island speak a language in which every word can be written with the following letters: a, b, c, d, e, f, g. A word is said to *produce* another one if the second word can be formed from the first one applying any of the following rules as many times as needed:

(i) Replace a letter by two letters according to one of the substitutions:

$$a \to bc,\ b \to cd,\ c \to de,\ d \to ef,\ e \to fg,\ f \to ga,\ g \to ab.$$

(ii) If only one letter is between two letters that are the same, these two letters can be eliminated. Example: $dfd \to f$

As another example, $cafed$ produces $bfed$, since $cafed \to cbcfed \to bfed.$

Prove that every word on this island *produces* any other word.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Comăneşti, Romania. We give the write-up of Curtis.

We note first that each letter produces each letter as seen by the following progression:

$$\begin{aligned} a \ &\to \ bc \to cdc \to d \to ef \to fgf \to g \to ab \to bcb \to c \to de \to efe \\ &\to \ f \to ga \to aba \to b \to cd \to ded \to e \to fg \to gag \to a. \end{aligned}$$

Suppose word M has m letters and word N has n letters and we wish to produce N from M. By the above, we can assume that each letter of M and N is 'a'. If $m = n$, we are done. From

$$a \to bc \to aa$$

and

$$aa \to abc \to cbc \to b \to a,$$

we see that if $m < n$, we can expand using $a \to aa$ repeatedly: if $m > n$, we can condense using $aa \to a$ repeatedly. Hence we can produce N from M.

5. Given two non-negative integers m and n, with $m > n$, we say that m ends in n if one can erase some consecutive digits from left of m to obtain n. For example, 329 ends only in 9 and in 29. Determine how many three-digit numbers end with the product of their digits.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania. We give the write-up of Zvonaru.

Let $\overline{abc}$ be the three-digit numbers, with $a \neq 0$.
(i) If $c = abc$, then we have $c = 0$ or $ab = 1$. It results that the numbers $\overline{ab0}$ (with $a = 1, 2, \ldots, 9$ and $b = 0, 1, 2, \ldots, 9)$ and $\overline{11c}$ (with $c = 1, 2, \ldots, 9$) end in the product of their digits.

In this case we have $90 + 9 = 99$ numbers with the desired property.
(ii) If $\overline{bc} = abc$, then we have successively

$$\overline{bc} = abc \Leftrightarrow 10b + c = abc \Leftrightarrow b = \frac{c}{ac - 10}$$
$$\Leftrightarrow ab = \frac{ac}{ac - 10} \Leftrightarrow ab = 1 + \frac{10}{ac - 10}.$$

Since ab is an integer, we obtain

$$ac = 11, \qquad ab = 11 \tag{1}$$
$$ac = 12, \qquad ab = 6 \tag{2}$$
$$ac = 15, \qquad ab = 3 \tag{3}$$
$$ac = 20, \qquad ab = 2 \tag{4}$$

The systems (1) and (4) have no solution.

From (2) we deduce the following possibilities:

$$(a = 2, b = 3, c = 6)(a = 3, b = 2, c = 4)(a = 6, b = 1, c = 2)$$

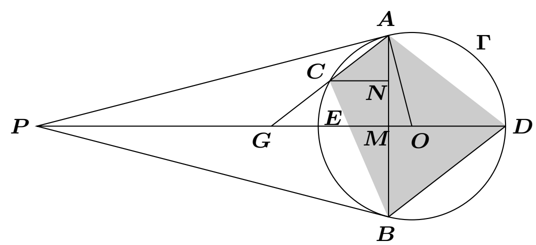
and from (3) we deduce the solution $(a = 3, b = 1, c = 5)$.

It results that the numbers 236, 324, 612 and 315 end in the product of their digits.

As a conclusion, there are $99 + 4 = 103$ three-digit numbers which end in the product of their digits.

6. Let A and B be points on the circle Γ such that the lines PA and PB are tangent to Γ for an exterior point P. Let M be the midpoint of AB. The perpendicular bisector of AM intersects Γ at C which is interior to $\triangle ABP$, the line AC intersects the line PM at G, and the line PM intersects Γ at D, which is exterior to the triangle $\triangle ABP$. If BD is parallel to AC, prove that G is the point in which the medians of $\triangle ABP$ concur.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Cománeşti, Romania. We give the write-up of Zvonaru.



Let O be the centre of Γ, E be the intersection of PD with Γ and N be the midpoint of AM.

We denote $PE = a$, $PD = b$; it results that $OA = \frac{b-a}{2}$ and $PO = \frac{a+b}{2}$.

By the power of the point P with respect to the circle Γ we obtain $PA^2 = ab$, and by similitude in $\triangle APO$ we have $AP^2 = PM \cdot PO$, hence $PM = \frac{2ab}{a+b}$ and $MD = b - \frac{2ab}{a+b} = \frac{b(b-a)}{a+b}$.

By the Pythagorean theorem we deduce

$$AM^2 = AP^2 - PM^2 = ab - \frac{4a^2b^2}{(a+b)^2} = \frac{ab(b-a)^2}{(a+b)^2},$$

$$AD^2 = AM^2 + MD^2 = \frac{ab(b-a)^2}{(a+b)^2} + \frac{b^2(b-a)^2}{(a+b)^2} = \frac{(ab+b^2)(b-a)^2}{(a+b)^2}.$$

Since $AC \| BD$, the cyclic quadrilateral $ACBD$ is an isosceles trapezoid; because $AD = BD = BC = 2AC$, we have

$$\begin{aligned} AB^2 &= \left(\frac{BD+AC}{2}\right)^2 + AD^2 - \left(\frac{BD-AC}{2}\right)^2 \\ &\Leftrightarrow 4AM^2 = AD^2 + AD \cdot AC \\ &\Leftrightarrow 4AM^2 = \frac{3AD^2}{2} \\ &\Leftrightarrow \frac{4ab(b-a)^2}{(a+b)^2} = \frac{3(b-a)^2 \cdot b(a+b)}{(a+b)^2} \\ &\Leftrightarrow 8a = 3a + 3b, \end{aligned}$$

hence

$$5a = 3b \tag{1}$$

Since $AG \| BD$ and $AM = MB$ we deduce that $GM = MD$. Using (1) we have

$$GM = \frac{b(b-a)}{a+b} = \frac{\frac{5a}{3}(\frac{5a}{3} - a)}{\frac{51}{3} + a} = \frac{5a}{12}, \tag{2}$$

$$PM = \frac{2ab}{a+b} = \frac{5a}{4}. \tag{3}$$

By (2) and (3) it follows that $PM = 3GM$; since PM is a median in $\triangle ABP$, we deduce that G is the centroid of $\triangle ABP$.

THE END

BOOK REVIEWS

Amar Sodhi

Loving + Hating Mathematics: Challenging the Myths of Mathematical Life
by Reuben Hersh and Vera John-Steiner
Princeton University Press, 2011
ISBN: 978-0-691-14247-0 , Hardcover, 416 + x pp., US$24.95
Reviewed by ***Georg Gunther****, Grenfell Campus (MUN), Brook, NL*

There is an anecdote about the brilliant, but notoriously absent-minded, mathematician Norbert Wiener. After moving to a new city he got lost on the way home from the university and stopped a little girl on the street to ask her if she knew where the Wieners lived. She took his hand and said "Yes Daddy, come with me."

Wiener was a child prodigy who received his Harvard PhD in mathematics at age 18. He exemplified the stereotype of a mathematician: "someone who solved the Rubik's cube at eight, took calculus at fourteen, and was tackling serious mathematics at sixteen. And he's a guy". [Haas and Henle, Notices of the AMS, September 2007].

How true to the real thing are these stereotypes? The book "Loving + Hating Mathematics: Challenging the Myths of Mathematical Life" challenges four commonly accepted myths about mathematicians. (1) mathematicians are different from other people, lacking emotional complexity; (2) mathematics is a solitary pursuit; (3) mathematics is a 'young man's game'; and (4) mathematics is an effective filter for higher education.

This is not a book about mathematics; instead, it is about mathematicians and the kinds of people they are. The authors both have impeccable credentials; both have received prestigious awards for previous books. They are colleagues at the University of New Mexico, where they hold the rank of Professor Emeritus.

In nine chapters, the authors lead the reader through a voyage that traces the lives of mathematicians from childhood and early years as students, through their productive years into maturity and old age. Each chapter begins with a simple question. For example, Chapter 1 asks: "*How does a child first begin to become a mathematician? Is it a predisposition or some special gift? What makes it possible, finally, to commit one's life to this risky, forbidding pursuit*"? This leads into the second chapter which explores the culture of the mathematical community.

One common trait of mathematicians is their ability to focus, often for many years, upon one particular result. This can lead to obsessive and compulsive behaviors swinging in extreme situations into paranoia, sociopathy and other forms of madness. The British philosopher, A. N. Whitehead, described the pursuit of mathematics as a 'divine madness of the human spirit'; in Chapter 3, the authors explore the boundaries that separate this 'divine madness' from more destructive forms of mental illness. Among the examples given is the tragic case of the logician Kurt Gödel, who in his later years suffered from the delusion that any food not prepared by his wife was poisoned. When she was hospitalized for a while he refused to eat anything at all: he subsequently died of malnutrition, weighing a mere 68 pounds.

The next two chapters explore the importance of professional collaborations and supportive social networks. There is a subtle irony in this: of all forms of intellectual activity, doing mathematics is perhaps the one that most isolates its practitioners from the society that supports them; yet without fruitful collaboration, without colleagues who speak the same language, without a vital and vibrant community of peers, mathematics would wither and die.

After a thought-provoking chapter that focuses on the complex issues of gender and age, the book concludes with two chapters dealing with various aspects of mathematical pedagogy.

Chapter 8 looks in detail at two contrasting approaches to teaching at the post-secondary level. The first of these was promoted by R .L. Moore who taught at the University of Texas for 49 years, producing 50 PhD students and 1678 doctoral descendants. His method was to recruit students early in their careers, so that he could exercise total draconian control over their mathematical education, to the extent that he forbade them to read or talk about mathematics outside of his classes. This pedagogical approach is compared to that of Clarence Stephens, who eventually became Chair of the Mathematics Department at SUNY (Potsdam), where he championed a student-centered approach to teaching that has been phenomenally successful. One measure of this success lies in the number of students who graduate with a major in mathematics; whereas nationally, this number lies at roughly 1% of all graduates, at Potsdam, it lies at approximately 20%.

Finally, the authors consider the perplexing and troubling question of why so many students who graduate from high school do so hating mathematics and feeling hopelessly inadequate about their skill levels. The problem is magnified when, in steadily increasing numbers, these young people enter college where they run into an impenetrable barrier in the form of a mathematics filter that makes no sense. Why, after all, should a student aiming for admission to medical school be required to study calculus?

The book is well researched, and written in a style that is both informal and informative. At times, the prose is almost too episodic; in many instances, the reader is left looking for more depth and detail about some particular point or individual. However, this potential flaw is mitigated by the presence of an extensive bibliography at the end of each chapter.

Mathematicians will enjoy this book. They will be able to relate on a personal level to many of the issues that are raised; as well, they will be able to re-acquaint themselves on a more human level with many of the individuals whose names have been long familiar for their mathematical contributions. Non-mathematicians will find this fascinating reading. By experiencing the human side of mathematics, they may come to appreciate in new ways a discipline that lives in their recollection as a seemingly impenetrable tangle of abstraction and complexity.

The Beauty of Fractals: Six Different Views
edited by Denny Gulick and Jon Scott
Mathematics Association of America, 2010
Electronic ISBN: 9780883859711, 95 + x pp., downloadable pdf, US$25
Reviewed by ***Daryl Hepting****, University of Regina, Regina, SK*

The Beauty of Fractals: Six Different Views comes out of a special session on chaos and fractals at the Annual General Meeting of the MAA. The editors, Denny Gulick and Jon Scott, together have organized a series of summer workshops and several contributed paper sessions at Joint Mathematics Meetings on these topics. They have also worked to advance the cause of visual thinking in various ways, not the least of which is this book on the beauty of fractals.

I commend the editors and authors for assembling this volume. Being a graduate student when *The Science of Fractal Images* and The *Algorithmic Beauty of Plants* were first published, I have many fond memories of the topics presented here. No matter how many times I approach these topics, I never tire of them. This book is sure to inspire many in their own explorations of fractals.

The preface rightly mentions the books of Mandelbrot, Barnsley, and Devaney as important in making the mathematics of fractals accessible to a wide audience. It is, therefore, exciting to see a contribution from Robert Devaney in this volume. More interesting still, is the fact that his chapter deals with Tom Stoppard's play *Arcadia* that offers "teachers of both mathematics and the humanities to join forces in a unique and rewarding way." Devaney and the other contributors do indeed provide a "taste of the breadth" of fractals; both in terms of topics and also of the disciplines involved. This small volume is certainly not exhaustive, but it provides many entry points. For those interested in the study of fractals, this book provides a good introduction and each chapter provides references that allow the reader to go further.

There are many wonderful illustrations throughout the book. I believe that images like the ones found in this book are an important way to introduce the concepts of fractals and to build an appreciation of their beauty that comes from the underlying mathematics. My personal wish would have seen more time spent on the algorithms or software tools used to generate the images seen throughout the book so that interested readers could easily move to hands-on exploration with the aid of a computer. Anne Burns' beautifully reproduced colour images in Chapter 1 (*Mathscapes - Fractal Scenery*) are included with little detail about how those final images were pulled together.

Without a clear path to recreating the images seen in the book, there is still much to recommend it. Each of the 6 chapters presents an interesting perspective, with solid mathematical foundations, that is valuable on its own.

RECURRING CRUX CONFIGURATIONS 4

J. Chris Fisher

Bicentric Quadrilaterals

A convex quadrilateral is called **bicentric** (or **chord-tangent**) if it has both a circumcircle (passing through all four vertices) and an incircle (tangent to all four sides). One sees the terminology *inscribable* (or inscriptible) for cyclic, and *circumscribable* (or circumscriptible) for containing an incircle, but in English these terms are often confused, as we explained before in **CRUX with MAYHEM** [1997: 530-531; 2011: 243-244]; it is therefore wise to avoid them when the context is not clear. It is standard practice to assume a cyclic polygon to be convex; thus, $ABCD$ is cyclic if its vertices are arranged around a circle in the order in which they appear in the label.

We begin with a brief look at Léo Sauvé's essay, "On Circumscribable Quadrilaterals" [1976 : 63-67], which deals with sets of four lines that are tangent to the same circle. Four lines in general position intersect in six points and form what is called a complete quadrilateral; four of those points are vertices of a convex quadrangle $ABCD$, while the opposite sides of that quadrangle intersect at $E = AB \cap CD$ and $F = AD \cap BC$. It is possible for a complete quadrilateral to have an incircle (as on the left of Figure 1) or an excircle (on the right). Both these configurations are composed of a pair of triangles ABF and AED having a common angle A and sides BF and ED opposite A that have a common point C. Howard Grossman [5] tied together earlier results of Pitot, Durrande, Urquhart, and others into two main theorems that were extended by Sokolowsky [1976 : 163-170]:

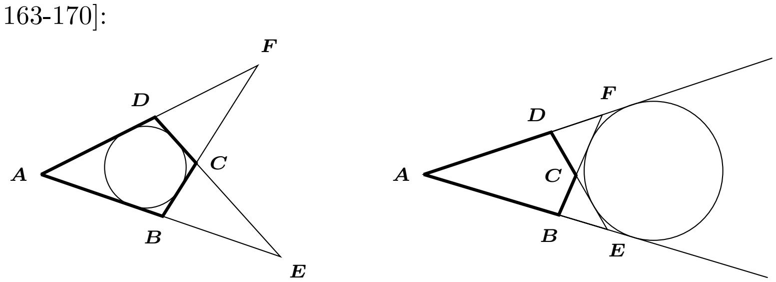


Figure 1: Circles internally tangent (left) and externally tangent (right) to the four sides of a complete quadrilateral.

Theorem 1. *If one of the following four properties holds, then they all hold and triangles ABF and AED have a common incircle.*
(a) $AB - BC = AD - DC$, (b) $AE - EC = AF - FC$,
(c) $DE - EB = BF - FD$, (d) $AB - BF + FA = AE - ED + DA$.

Theorem 2. *If one of the following four properties holds, then they all hold and triangles ABF and AED have a common excircle.*

(a) $AB + BC = AD + DC$,
(b) $AE + EC = AF + FC$,
(c) $DE + EB = BF + FD$,
(d) ΔABF and ΔAED have equal perimeters.

Because bicentric quadrilaterals have an incircle by definition, Theorem 2 will be relevant in what follows only to Problem 777. Of course, a quadrangle requires more than the properties of Theorem 1 to be bicentric—the condition that $ABCD$ be cyclic requires the further property that each pair of its opposite angles sums to 180°. A quite different characterization of bicentric quadrilaterals came from Euler: if a circle of radius r is placed in the interior of a circle of radius R so that the distance x between their centres satisfies

$$\frac{1}{(R-x)^2} + \frac{1}{(R+x)^2} = \frac{1}{r^2}$$

or, equivalently,

$$x^2 = R^2 + r^2 - \sqrt{r^2(4R^2 + r^2)},$$

then every point of the outer circle is the vertex of a bicentric quadrilateral that is inscribed in the outer circle and circumscribed about the inner circle. As an immediate consequence of Euler's formula we deduce that for any bicentric quadrilateral with circumradius R and inradius r,

$$R^2 \geq 2r^2,$$

with equality if and only if the quadrilateral is a square.

Useful formulas involving the semiperimeter s; consecutive sides a, b, c, d; area F; and diagonals e, f of a bicentric quadrilateral can be found in standard references:

$$s = a + c = b + d, \quad F = \sqrt{abcd} = rs, \quad ac + bd = ef,$$

$$R = \frac{\sqrt{(ab+cd)(ac+bd)(ad+bc)}}{4F}.$$

Problem 777. [1982 : 246; 1984 : 20-22] (Proposed by O. Bottema) Let $ABCD$ be a convex quadrilateral that is not a rectangle. If two of
(i) $ac + bd = ef$,
(ii) s equals the sum of two sides of the quadrilateral, or
(iii) $F = \sqrt{abcd} = rs$
hold, then the third holds also. In other words, for a nonrectangular convex quadrilateral $ABCD$, two of the following three statements implies the third:
(i) $ABCD$ is cyclic.
(ii) $ABCD$ has either an incircle or an excircle.
(iii) The area of $ABCD$ is $\sqrt{abcd}$.

Problem 1203. [1987 : 14; 1988 : 91-93] (Proposed by Milen N. Naydenov) Bicentric quadrilaterals satisfy

a) $2\sqrt{F} \leq s \leq r + \sqrt{r^2 + 4R^2}$;

b) $6F \ \leq \ ab + ac + ad + bc + bd + cd \ \leq \ 4r^2 + 4R^2 + 4r\sqrt{r^2 + 4R^2}$;

c) $8sr^2 \ \leq \ abc + abd + acd + bcd \ \leq \ 2r\left(r + \sqrt{r^2 + 4R^2}\right)^2$;

d) $4Fr^2 \ \leq \ abcd \ \leq \ \frac{16}{9}r^2\left(r^2 + 4R^2\right)$.

Equality holds on the left in each part and on the right in part (d) if and only if the bicentric quadrilateral is a square; it holds on the right of (a), (b), and (c) if and only if at least one pair of opposite angles of the quadrilateral are right angles.

The featured proof by Murray Klamkin showed, among other things, that the left-hand inequalities in parts (a), (b), and (c) are immediate consequences of the AM-GM inequality, while the right-hand inequality of part (a) appeared earlier in [1]. Along the way he proved a result that is interesting in its own right, namely

$$ef = 2r\left(r + \sqrt{r^2 + 4R^2}\right). \qquad (1)$$

Problem 1376. [1988 : 235; 1989 : 287-288] (Proposed by G.R. Veldkamp) The diagonals e and f of a bicentric quadrilateral satisfy

$$\frac{ef}{4r^2} - \frac{4R^2}{ef} = 1.$$

This equation appears (with a typo) on page 49 of [3]; it is an immediate consequence of formula (1).

Problem 1983. [1994 : 259; 1995 : 257-258] (Proposed by K.R.S. Sastry) If the line through I parallel to the side AB of the bicentric quadrilateral $ABCD$ meets AD in A' and BC in B', then $A'B' = \frac{s}{2}$.

Problem 2027. [1995 : 90; 1996 : 94-95] (Proposed by D.J. Smeenk), and
Problem 3338. [2008 : 239, 242; 2009 : 241-242] (Proposed by Toshio Seimiya) Let EF be the diameter of the circumcircle of bicentric quadrilateral $ABCD$ that is perpendicular to the diagonal BD, labeled with E on the same side of BD as A. Let BD intersect EF at M and AC at X. Then

$$\frac{AX}{XC} = \frac{EM}{MF} = \frac{AI^2}{IC^2}.$$

The first equality is Problem 2027, which (with a new proof) appeared again as Problem 3211 [2007 : 43, 46; 2008 : 61-62]; the second equality is Problem 3338. See also Problem 3598, whose solution appears later in this issue, for yet another result involving the quantities AI^2, BI^2, CI^2, DI^2.

Problem 2194. [1996 : 362; 1997 : 530-532] (Proposed by Christopher J. Bradley) Find an integer-sided triangle ABC and a transversal LMN with N on side AB, M on side AC, and L on the extension of side BC such that $BCMN$

is an integer-sided bicentric quadrilateral, and also the segments ML, and CL are of integer length.

Michael Lambrou began his featured solution to 2194 with a rational-sided triangle ABC and chose the transversal to be the line tangent to the incircle of ΔABC for which $\angle AMN = \angle B$. Lambrou and Richard Hess each provided further examples in which $\angle AMN = 90°$, while B and L are reflections of M and A in the bisector of the exterior angle at N of ΔAMN.

Problem 2203. [1997 : 46; 1998 : 112-113] (Proposed by Miguel Amengual Covas; reworded here) Given a cyclic quadrilateral $PQRS$, the tangents to its circumcircle at its vertices form a bicentric quadrilateral if and only if $PR \perp QS$.

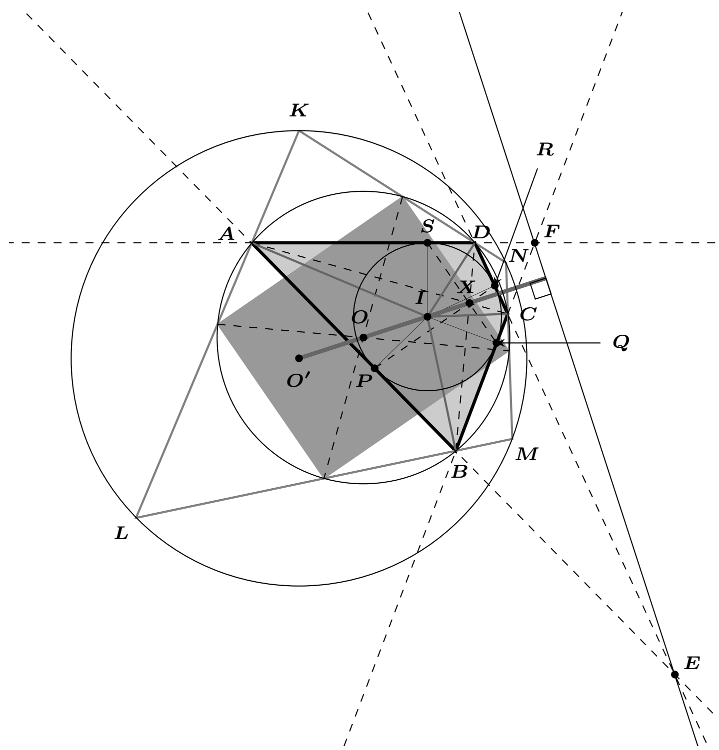


Figure 2: Problems 2203 and 2209.

Two proofs were provided to this problem even though the result is readily found in the literature such as [2, Section 39, pages 188-191] where much more is provided, including the following consequence of Brianchon's Theorem (called Newton's theorem in [4, Sections 1274 and 1275, pp. 563-565]):

> *If the sides of a quadrangle $ABCD$ are tangent to an ellipse at the points P, Q, R, and S, then AC, BD, PR, and QS are concurrent* (at X in Figure 2).

Problem 2209. [1997 : 47; 1998 : 112-113] (Proposed by Miguel Amengual

Covas; reworded here) Denote by $\boldsymbol{I}$ the intersection point of the diagonals of a cyclic quadrilateral $\boldsymbol{KLMN}$. The feet $\boldsymbol{A, B, C, D}$ of the perpendiculars from $\boldsymbol{I}$ onto the sides of $\boldsymbol{KLMN}$ are the vertices of a quadrangle that has an incircle with centre $\boldsymbol{I}$; moreover, if the diagonals $\boldsymbol{KM}$ and $\boldsymbol{LN}$ are perpendicular then $\boldsymbol{ABCD}$ is also cyclic (and therefore bicentric) and its circumcircle passes also through the midpoints of the sides of the given quadrilateral $\boldsymbol{KLMN}$. Conversely, if $\boldsymbol{ABCD}$ is a bicentric quadrilateral, then the lines that are perpendicular at $\boldsymbol{A, B, C}$, and $\boldsymbol{D}$ to the lines from the incentre $\boldsymbol{I}$ form the sides of a cyclic quadrilateral whose perpendicular diagonals meet at $\boldsymbol{I}$.

The closely related **Problem 1836** [1993 : 113; 1994 : 84-85] (proposed by Jisho Kotani) defined $\boldsymbol{ABCD}$ from the cyclic quadrilateral $\boldsymbol{KLMN}$ as in Problem 2209; the result there is that the sum of the areas of the lunes inside the circles with diameters $\boldsymbol{KL, LM, MN, NK}$ that lie outside the circumcircle equals the area of the quadrangle $\boldsymbol{KLMN}$ if and only if the diagonals $\boldsymbol{KM}$ and $\boldsymbol{LN}$ are perpendicular (and $\boldsymbol{ABCD}$ is therefore bicentric).

Of course, when $\boldsymbol{ABCD}$ is bicentric the midpoints of the sides of $\boldsymbol{KLMN}$ are the vertices of a rectangle whose diagonals pass through the circumcentre $\boldsymbol{O}$ of $\boldsymbol{ABCD}$. This result together with all of Problem 2209 forms part of Theorem 159 in Section 749 (pages 321-322) of [4], where further properties and references are provided. For example,

- $\boldsymbol{O}$ is the midpoint between the incentre $\boldsymbol{I = KM \cap LN}$ and the centre $\boldsymbol{O'}$ of the circumcircle of $\boldsymbol{KLMN}$;
- the line $\boldsymbol{OI}$ also contains the point $\boldsymbol{X = AC \cap BD = PR \cap QS}$ where the diagonals of $\boldsymbol{ABCD}$ and of $\boldsymbol{PQRS}$ meet;
- it follows from the featured solution that the diagonal $\boldsymbol{LIN}$ passes through the intersection of $\boldsymbol{AD}$ with $\boldsymbol{BC}$, while $\boldsymbol{KIM}$ passes through $\boldsymbol{AB \cap CD}$; moreover, the line joining those two points is perpendicular to the line $\boldsymbol{O'OIX}$. (This last result was proved as part of Michel Bataille's solution to Problem 3256 [2007 : 298, 300; 2008 : 312-313], proposed by Václav Konečný.)

Returning to the quadrilateral $\boldsymbol{ABCD}$, we have

Theorem. A quadrilateral $\boldsymbol{ABCD}$ that has an incircle tangent to the sides $\boldsymbol{AB, BC, CD}$, and $\boldsymbol{DA}$ at the points $\boldsymbol{P, Q, R}$, and $\boldsymbol{S}$, respectively, is cyclic if and only if $\boldsymbol{AP \times CR = BQ \times DS}$.

The "if" part was Problem 2 on Peru's Olimpiada Nacional Escolar de Matematica 2009, Level 3 [2010 : 374; 2011 : 381-383]; the converse is a simple exercise in trigonometry. Note how this theorem is related to Problem 2209: the product condition is equivalent to saying that the line $\boldsymbol{PR}$ divides the opposite sides $\boldsymbol{AB}$ and $\boldsymbol{DC}$ is the same ratio, namely $\boldsymbol{AP : PB = DR : RC}$, while $\boldsymbol{QS}$ divides the opposite sides $\boldsymbol{AD}$ and $\boldsymbol{BC}$ in the same ratio. (Note that $\boldsymbol{PR}$ and $\boldsymbol{QS}$ bisect the angles at $\boldsymbol{X}$ formed by the diagonals $\boldsymbol{AC}$ and $\boldsymbol{BD}$.)

Constructions. We conclude with two constructions taken from the article "A Duality for Bicentric Quadrilaterals", by Michel Bataille [2009 : 310-312]. The first is based on the characterization given above in Problem 2203: *A bicentric* *quadrilateral is uniquely determined by three of its sides tangent to a given circle.* If the points of tangency are $\boldsymbol{P}, \boldsymbol{Q}$, and $\boldsymbol{R}$, then the fourth side is the tangent to the circle at the point where it meets the perpendicular to $\boldsymbol{PR}$ through $\boldsymbol{Q}$. (Bataille includes yet another proof of 2203.)

The second construction can be thought of as a dual of the first: *A bicentric quadrilateral is uniquely determined by three of its vertices.* It is a consequence of the following extension of Problems 2027 and 3338:

> Let $\boldsymbol{ABCD}$ be a convex quadrilateral inscribed in a circle with diagonals $\boldsymbol{AC}$ and $\boldsymbol{BD}$ meeting at $\boldsymbol{X}$. Then $\boldsymbol{ABCD}$ is bicentric if and only if both (i) $\boldsymbol{B}$ and $\boldsymbol{D}$ are on the same side of the perpendicular bisector of $\boldsymbol{AC}$, and (ii) $\boldsymbol{BX} = \left(\cos^2 \frac{\boldsymbol{B}}{2}\right) \boldsymbol{BD}$.

Figure 3 shows how to construct $\boldsymbol{D}$ using $\boldsymbol{K}$ on the bisector $\boldsymbol{BV}$ of $\angle \boldsymbol{ABC}$: $\boldsymbol{K}$ is the point where the perpendicular to $\boldsymbol{BC}$ through $\boldsymbol{C}$ meets $\boldsymbol{BV}$, $\boldsymbol{C'}$ is the point of $\boldsymbol{BC}$ on the perpendicular to $\boldsymbol{BV}$ at $\boldsymbol{K}$, and $\boldsymbol{D}$ is the first point of the circumcircle of $\boldsymbol{ABC}$ on the line through $\boldsymbol{C'}$ parallel to $\boldsymbol{AC}$.

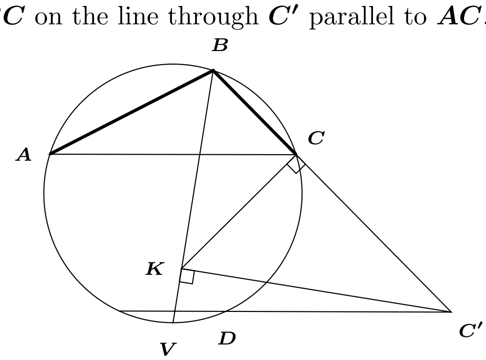


Figure 3: Construction of the bicentric quadrangle $\boldsymbol{ABCD}$ given the vertices $\boldsymbol{A}, \boldsymbol{B}$, and $\boldsymbol{C}$.

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That old root flipping trick of Andrey Andreyevich Markov

Gerhard J. Woeginger

A straightforward fact

All the mathematics in this article is based on the following straightforward fact.

> **Fact.** *Let p and q be positive integers. If the equation $x^2 - px + q = 0$ has a positive integer root, then also its second root is a positive integer.*

Why do we call this fact straightforward? Well, it easily follows from Vieta's formulas $x_1 + x_2 = p$ and $x_1 x_2 = q$ for the roots x_1 and x_2 of a quadratic equation. As p and x_1 are integers, also $x_2 = p - x_1$ is integer. And as q and x_1 both are positive, also $x_2 = q/x_1$ is positive. So the fact can be proved in two lines and indeed is simple. Then what's the reason for spending several pages on it? Well, the fact turns out to be surprisingly useful in the analysis of certain Diophantine equations. This article will illustrate this usefulness by a number of examples and also discuss some of its background.

Short history lesson

The Russian mathematician Andrey Andreyevich Markov (1856–1922) received his Master's degree in 1880 from the university of St Petersburg. His supervisor was Pafnuty Lvovich Chebyshev (1821–1894), and the title of his thesis was *"About binary quadratic forms with positive determinant"*. Among many other contributions, the thesis contained as a side result the complete analysis of the following Diophantine equation:

$$a^2 + b^2 + c^2 \ = \ 3abc. \tag{2}$$

This equation is nowadays called the *Markov equation*. Let us take a closer look at it. For a solution triple (a, b, c) of (2) over the positive integers, we consider the following quadratic equation in x.

$$x^2 - (3bc)x + (b^2 + c^2) \ = \ 0. \tag{3}$$

The quadratic equation (3) has two roots: one of them of course is a, and the other one is — at this place we apply our straightforward fact — the positive integer $a' = 3bc - a$. Unless the two roots coincide, we have found a new solution triple (a', b, c) for equation (2). Symmetric observations for b and c yield the following: whenever (a, b, c) is a positive integer solution of (2), then also the triples (a', b, c), (a, b', c), and (a, b, c') are positive integer solutions where

$$a' = 3bc - a, \qquad b' = 3ac - b, \qquad c' = 3ab - c.$$

The weight of triple (a, b, c) is defined as the maximum of a, b, c. How does the weight of the new triples relate to the weight of the old triple? Let us assume for the moment that a, b, c are pairwise distinct and satisfy $a > b > c$. We then derive

$$\begin{aligned}(a-b)(b-a') &= a^2 - b^2 + 3b^2c - 3abc \\ &= a^2 - b^2 + 3b^2c - (a^2 + b^2 + c^2) \\ &= 3b^2c - 2b^2 - c^2 = 2b^2(c-1) + c(b^2 - c) > 0.\end{aligned}$$

This implies $a' < b < a$, and analogous arguments lead to $b' > a$ and $c' > a$. Hence by flipping the largest coordinate from a into a' we decrease the weight of the triple, and by flipping one of the smaller coordinates we increase the weight of the triple. Now let us repeatedly flip the largest coordinate, so that the weight of the resulting triples keeps decreasing. Since the weight cannot decrease below zero, we must eventually get stuck with a triple whose coordinates are not pairwise distinct anymore. By symmetry we may assume that the root flipping process terminates with $b = c$. Then (2) becomes $a^2 = (3a-2)b^2$, which implies that b divides a. Thus $a = kb$, and substituting this into the equation and rewriting yields $k(3b - k) = 2$. Hence k must be a divisor of 2: if $k = 1$ then $b = c = 1$ and $a = 1$; if $k = 2$ then $b = c = 1$ and $a = 2$.

Let us summarize our findings. For every positive integer solution (a, b, c) of the Markov equation (2) with pairwise distinct coordinates, root flipping produces three neighbor solutions. One of these neighbors has smaller weight, and is called the predecessor of (a, b, c). The other two neighbors have larger weight, and are called the successors of (a, b, c). If we start in an arbitrary solution and follow the chain of predecessors, we eventually arrive in one of the special solutions $(1, 1, 1)$ or $(2, 1, 1)$. Vice versa, by starting in $(1, 1, 1)$ and by repeately moving to successors, we can reach every possible solution of equation (2). Figure 1 lists the first few solutions of the equation.

Three shiny examples

Next, we want to discuss three concrete problems from mathematical competitions where that old root flipping trick of Andrey Andreyevich Markov serves as a crucial

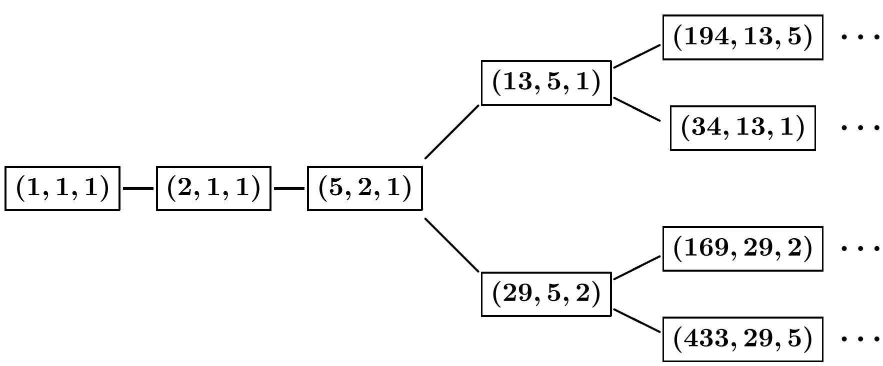

Figure 1: Some solutions of the Markov equation.

tool. Our first problem was posed as problem B4 on the 1978 William Lowell Putnam Mathematics Competition.

Problem 1 *Prove that for every positive integer N, the equation*

$$a^2 + b^2 + c^2 + d^2 \;=\; abc + abd + acd + bcd \tag{4}$$

has a solution in integers $a, b, c, d \geq N$.

Consider an arbitrary positive integer solution (a, b, c, d), and assume without loss of generality that a with $a \leq b, c, d$ is the smallest coordinate. For applying Markov's root flipping trick, we introduce the quadratic equation

$$x^2 - (bc + bd + cd)x + (b^2 + c^2 + d^2 - bcd) \;=\; 0. \tag{5}$$

One root of this equation (5) is a itself, and the other root is $a' = bc+bd+cd-a$. Since we selected a as the smallest coordinate in the quadruple, the second root satisfies

$$a' \;\geq\; 3a^2 - a \;>\; a.$$

Hence flipping the smallest coordinate will automatically increase the value of that coordinate! The rest is easy. We start from the trivial solution $(1, 1, 1, 1)$, and repeatedly flip the smallest coordinate. Eventually this must bring all coordinates above any fixed bound N. This fully settles the Putnam question.

It is instructive to see the actual sequence of solutions. Starting from $(1, 1, 1, 1)$ we get successively $(2, 1, 1, 1)$, $(2, 4, 1, 1)$, $(2, 4, 13, 1)$, $(2, 4, 13, 85)$. One can show — and we encourage the reader to do so — that from this point onwards the four coordinates are always pairwise distinct. Furthermore if the coordinates are ordered as $a < b < c < d$, then $2a \leq b$, $2b \leq c$, $2c \leq d$ and $2d \leq a'$.

In the year 1988, Markov's old root flipping trick made its first appearance at the International Mathematical Olympiad. Only eleven students managed to find the solution to the following problem.

Problem 2 *Let a and b be two positive integers such that $q = \dfrac{a^2 + b^2}{ab + 1}$ is integer. Show that q is a perfect square.*

We choose a pair (a, b) with the smallest possible sum $a + b$ among all pairs of positive integers that yield this ratio q and hence satisfy

$$a^2 + b^2 - abq - q \;=\; 0. \tag{6}$$

Without loss of generality we assume $a \geq b$. The next step is already routine for us: in order to apply Markov's root flipping trick, we introduce the quadratic equation

$$x^2 - bq \cdot x + (b^2 - q) \;=\; 0. \tag{7}$$

The first root of (7) is a, and the second root is $a' = bq - a$. Clearly a' is integer, but is it positive or negative? For settling this question we will distinguish three cases on the sign of $b^2 - q$.

In the first case we deal with $b^2 - q > 0$. Then the root $a' = (b^2 - q)/a$ is indeed positive. Furthermore $a \geq b$ implies $a' = (b^2 - q)/a < (a^2 - q)/a < a$. The resulting contradiction $a' + b < a + b$ shows that this first case cannot occur.

In the second case we deal with $b^2 - q < 0$. Then (6) yields $a(a - bq) = q - b^2 > 0$, which implies $a > bq$. Since (6) also yields $q = b^2 + a(a - bq) > a > bq$, we get the contradiction $1 > b$. Hence also the second case cannot occur.

All in all, the only remaining possibility is the third case with $b^2 - q = 0$. But then $q = b^2$ indeed is a perfect square, and we have arrived at the desired conclusion.

Our next problem is the central piece of a slightly more difficult question from the 2007 International Mathematical Olympiad.

Problem 3 *Let a and b be positive integers such that $4ab - 1$ divides $(a - b)^2$. Show that $a = b$.*

Once again Markov's root flipping trick applies. Suppose that for some integer $q \geq 0$ there exists a pair (a, b) of positive integers with

$$a^2 - 2ab + b^2 \;=\; (4ab - 1)q. \tag{8}$$

Among all such pairs (a, b) we pick one that minimizes the value $\max\{a, b\}$, and without loss of generality we furthermore assume $a \geq b$. The quadratic equation

$$x^2 - (2b + 4bq) \cdot x + (b^2 + q) \;=\; 0.$$

has a and $a' = (b^2 + q)/a$ as positive integer roots. Since we considered a pair with smallest possible value $\max\{a, b\}$, we conclude $a' \geq a$ and hence $b^2 + q \geq a^2$. By plugging this into (8) we derive

$$(a - b)^2 \;=\; (4ab - 1)q \;\geq\; q \;\geq\; a^2 - b^2,$$

which implies $b \geq a$. Since we started from the assumption $a \geq b$, we arrive at the desired conclusion $a = b$. This implies $q = 0$ and completes the proof.

Homework exercises

We challenge the reader to settle the following five problems along the lines indicated above.

Problem 4 *Show that there are infinitely many quadruples of positive integers (a, b, c, d) that satisfy the equation*

$$a^2 + b^2 + c^2 + d^2 \;=\; abcd.$$

One possible approach to this problem first guesses a solution with $a = b = c = d$ and then repeatedly applies Markov's root flipping trick.

Problem 5 *For $q \geq 4$, show that the following equation has no solution over the integers:*

$$a^2 + b^2 + c^2 \;=\; q \cdot abc.$$

Note that for $q = 3$ the given equation coincides with the Markov equation (2) that we have discussed in detail. For $q \geq 4$ one can actually recycle the machinery from the $q = 3$ case: every solution triple with pairwise distinct coordinates generates another solution triple with strictly smaller weight. But this time the final step and the conclusion of the argument are different, as the chain of predecessors does never terminate! This leads to an infinite descent argument and shows that there are no solutions.

Problem 6 *Determine all positive integers q for which the following equation has a solution (a, b) over the positive integers.*

(a) $a^2 + b^2 + 1 = q \cdot ab$

(b) $a^2 + b^2 + 6 = q \cdot ab$

(c) $a^2 + b^2 - 1 = q \cdot ab$

In parts (a) and (b) you will detect all possible values for q by setting $a = b = 1$. For showing that these are the only possible values, you should evoke the root flipping trick. Part (c) is a trick question, and you may want to understand what's going on for $a = q$ and $b = q^2 - 1$.

Problem 7 *Determine all positive integers q for which the following equation has a solution (a, b) over the positive integers.*

(a) $a^2 + b^2 = q \cdot (ab - 1)$

(b) $a^2 + b^2 + ab = q \cdot (ab - 1)$

Part (a) of this problem is from the 2002 USA team selection test for the International Mathematical Olympiad, and part (b) is from the 2004 Romanian team selection test for the International Mathematical Olympiad. It is not difficult to guess that the answer to part (a) is $q = 5$, and that the answer to part (b) is $q = 4$ and $q = 7$. Both parts can be attacked by Markov's trick, and in both parts the root flipping process gets stuck as soon as $a' \geq a \geq b$ holds (where as usually a denotes the old root and a' denotes the new root). All effort then goes into understanding and characterizing these terminal situations.

Problem 8 *Show that there are infinitely many pairs of positive integers (a, b) for which $\dfrac{a+1}{b} + \dfrac{b+1}{a}$ is a positive integer.*

This problem is taken from the second round of the 2007 British Mathematical Olympiad. The pair $(1, 1)$ yields the integer value 4, and then once again Markov's root flipping trick does the job.

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PROBLEMS

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1 decembre 2012**. *Une étoile* (⋆) *après le numéro indique que le problème a été soumis sans solution.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

3670. *Correction. Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Calculer

$$\int_0^1 \int_0^1 \int_0^1 \frac{dxdydz}{x+y+z}.$$

3688. *Proposé par Arkady Alt, San José, CA, É-U.*

Soit $T_n(x)$ le polynôme de Chebychev de première espèce défini par la récurrence $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ pour $n \geq 1$ et les conditions initiales $T_0(x) = 1$ et $T_1(x) = x$. Trouver tous les entiers positifs n tels que

$$T_n(x) \;\geq\; (2^{n-2}+1)x^n - 2^{n-2}x^{n-1}\,, \quad x \in [1, \infty)\,.$$

[*Ed. : Noter que le problème* **3585** *a été originellement imprimé avec la fausse inégalité.*]

3689. *Proposé par Ivaylo Kortezov, Institut de Mathématiques et Informatique, Académie des Sciences, Sofia, Bulgarie.*

Dans un groupe de n personnes, chacune possède un livre différent. Disons qu'une paire de personnes opère un *échange* si elles s'échangent leur livre présentement en leur possession. Trouver le plus petit nombre possible d'échanges $E(n)$, de sorte que chaque paire de personnes a procédé à au moins un échange et que finalement chaque personne se retrouve avec son livre de départ.

3690. *Proposé par Michel Bataille, Rouen, France.*

Soit a, b et c trois nombres réels positifs distincts avec $a+b+c=1$. Montrer que

$$(5x^2 - 6xy + 5y^2)(a^3 + b^3 + c^3) + 12(x^2 - 3xy + y^2)abc > (x-y)^2$$

pour tous les nombres réels x et y.

3691. *Proposé par Pham Kim Hung, étudiant, Université de Stanford, Palo Alto, CA, É-U.*

Soit a, b et c trois nombres réels non négatifs tels que $a+b+c=3$. Montrer que

$$\frac{a^2b}{4-bc}+\frac{b^2c}{4-ca}+\frac{c^2a}{4-ab}\ \leq\ 1\,.$$

3692. *Proposé par Nguyen Thanh Binh, Hanoï, Vietnam.*

Etant donné un point arbitraire M dans le plan d'un triangle ABC, on définit D, E et F comme les deuxièmes points où le cercle circonscrit coupe respectivement les droites AM, BM, et CM. Si O_1, O_2 et O_3 sont les centres respectifs des cercles BCM, CAM et ABM, montrer que DO_1, EO_2 et FO_3 sont concourants.

3693. *Proposé par Michel Bataille, Rouen, France.*

Etant donné $k \in \left(\frac{1}{4}, 0\right)$, soit $\{a_n\}_{n=0}^{\infty}$ la suite définie par $a_0=2$, $a_1=1$ et la récursion $a_{n+2}=a_{n+1}+ka_n$. Evaluer

$$\sum_{n=1}^{\infty}\frac{a_n}{n^2}\,.$$

3694. *Proposé par Pham Van Thuan, Université de Science des Hanoï, Hanoï, Vietnam.*

Soit x, y et z trois nombres réels non négatifs tels que $x^2+y^2+z^2=1$. Montrer que

$$\sqrt{1-\left(\frac{x+y}{2}\right)^2}+\sqrt{1-\left(\frac{y+z}{2}\right)^2}+\sqrt{1-\left(\frac{x+z}{2}\right)^2}\ \geq\ \sqrt{6}\,.$$

3695. *Proposé par Michel Bataille, Rouen, France.*

Soit a un nombre réel positif. Trouver toutes les fonctions strictement monotones $f:(0,\infty)\to(0,\infty)$ telles que

$$(x+a)f(x+y)=af(yf(x))$$

pour tous les x, y positifs.

3696. *Proposé par Nguyen Thanh Binh, Hanoï, Vietnam.*

Supposons que le cercle inscrit du triangle ABC touche les côtés BC en D, CA en E, et AB en F. Construire avec la règle et le compas les trois cercles mutuellement tangents qui sont tangents intérieurement au cercle inscrit, un en D, un en $E4$ et un en F.

3697. *Proposé par Michel Bataille, Rouen, France.*

Pour un entier positif n, montrer que

$$\left(\tan\frac{\pi}{7}\right)^{6n}+\left(\tan\frac{2\pi}{7}\right)^{6n}+\left(\tan\frac{3\pi}{7}\right)^{6n}$$

est un entier et trouver la plus haute puissance de 7 qui divise cet entier.

3698. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Trouver la valeur de

$$\lim_{n\to\infty}\int_0^1 \sqrt[n]{1+n^n x^{n^2}}\,dx\,.$$

3699. *Proposé par Mehmet Şahin, Ankara, Turquie.*

Dans un triangle ABC, soit I le centre de son cercle inscrit, ρ_a, ρ_b et ρ_c les rayons respectifs des cercles inscrits. Montrer que

$$\frac{1}{\rho_a}+\frac{1}{\rho_b}+\frac{1}{\rho_c}\;\geq\;\frac{18\tan(75^\circ)}{a+b+c}\,.$$

3700. *Proposé par Michel Bataille, Rouen, France.*

Soit ABC un triangle et $a=BC$, $b=CA$, $c=AB$. Si l'on a que

$$aPA^2+cPB^2+bPC^2=cPA^2+bPB^2+aPC^2=bPA^2+aPB^2+cPC^2$$

pour un certain point P, montrer que ABC est équilatéral.

. . . .

3670. *Correction. Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Calculate

$$\int_0^1\int_0^1\int_0^1\frac{dxdydz}{x+y+z}.$$

3688. *Proposed by Arkady Alt, San Jose, CA, USA.*

Let $T_n(x)$ be the Chebyshev polynomial of the first kind defined by the recurrence $T_{n+1}(x)=2xT_n(x)-T_{n-1}(x)$ for $n\geq 1$ and the initial conditions $T_0(x)=1$ and $T_1(x)=x$. Find all positive integers n such that

$$T_n(x)\;\geq\;(2^{n-2}+1)x^n-2^{n-2}x^{n-1}\,,\quad x\in[1,\infty)\,.$$

[*Ed.: Note problem* **3585** *was originally printed with the wrong inequality.*]

3689. *Proposed by Ivaylo Kortezov, Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia, Bulgaria.*

In a group of n people, each one has a different book. We say that a pair of people performs a *swap* if they exchange the books they currently have. Find the least possible number $E(n)$ of swaps such that each pair of people has performed at least one swap and at the end each person has the book he or she had at the start.

3690. *Proposed by Michel Bataille, Rouen, France.*

Let a, b, and c be three distinct positive real numbers with $a+b+c=1$. Show that

$$(5x^2-6xy+5y^2)(a^3+b^3+c^3)+12(x^2-3xy+y^2)abc>(x-y)^2$$

for all real numbers x and y.

3691. *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let a, b, and c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$\frac{a^2b}{4-bc}+\frac{b^2c}{4-ca}+\frac{c^2a}{4-ab}\ \leq\ 1\,.$$

3692. *Proposed by Nguyen Thanh Binh, Hanoi, Vietnam.*

For an arbitrary point M in the plane of triangle ABC define D, E, and F to be the second points where the circumcircle meets the lines AM, BM, and CM, respectively. If O_1, O_2, and O_3 are the respective centres of the circles BCM, CAM, and ABM, prove that DO_1, EO_2, and FO_3 are concurrent.

3693. *Proposed by Michel Bataille, Rouen, France.*

Given $k\in\left(\frac{1}{4},0\right)$, let $\{a_n\}_{n=0}^{\infty}$ be the sequence defined by $a_0=2$, $a_1=1$ and the recursion $a_{n+2}=a_{n+1}+ka_n$. Evaluate

$$\sum_{n=1}^{\infty}\frac{a_n}{n^2}\,.$$

3694. *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let x, y, and z be nonnegative real numbers such that $x^2+y^2+z^2=1$. Prove that

$$\sqrt{1-\left(\frac{x+y}{2}\right)^2}+\sqrt{1-\left(\frac{y+z}{2}\right)^2}+\sqrt{1-\left(\frac{x+z}{2}\right)^2}\ \geq\ \sqrt{6}\,.$$

3695. *Proposed by Michel Bataille, Rouen, France.*

Let a be a positive real number. Find all strictly monotone functions $f : (0, \infty) \to (0, \infty)$ such that

$$(x + a)f(x + y) = af(yf(x))$$

for all positive x,y.

3696. *Proposed by Nguyen Thanh Binh, Hanoi, Vietnam.*

Let the incircle of triangle ABC touch the sides BC at D, CA at E, and AB at F. Construct by ruler and compass the three mutually tangent circles that are internally tangent to the incircle, one at D, one at E, and one at F.

3697. *Proposed by Michel Bataille, Rouen, France.*

For positive integer n, prove that

$$\left(\tan\frac{\pi}{7}\right)^{6n} + \left(\tan\frac{2\pi}{7}\right)^{6n} + \left(\tan\frac{3\pi}{7}\right)^{6n}$$

is an integer and find the highest power of 7 dividing this integer.

3698. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Find the value of

$$\lim_{n\to\infty}\int_0^1 \sqrt[n]{1 + n^n x^{n^2}}\,dx\,.$$

3699. *Proposed by Mehmet Şahin, Ankara, Turkey.*

Let ABC denote a triangle, I its incenter, and ρ_a, ρ_b, and ρ_c the inradii of IBC, ICA, and IAB, respectively. Prove that

$$\frac{1}{\rho_a} + \frac{1}{\rho_b} + \frac{1}{\rho_c} \;\geq\; \frac{18\tan(75^\circ)}{a + b + c}\,.$$

3700. *Proposed by Michel Bataille, Rouen, France.*

Let ABC be a triangle and $a = BC$, $b = CA$, $c = AB$. Given that

$$aPA^2 + cPB^2 + bPC^2 = cPA^2 + bPB^2 + aPC^2 = bPA^2 + aPB^2 + cPC^2$$

for some point P, show that $\triangle ABC$ is equilateral.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3589. [2010 : 548, 550] *Proposed by Václav Konečný, Big Rapids, MI, USA.*

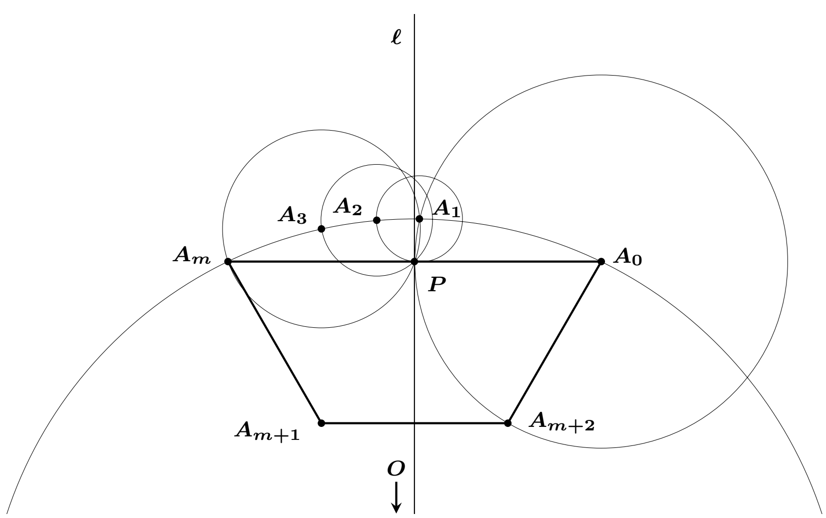
Find all integers $n > 6$ for which there exists a convex n-gon with an interior point P such that $PA_i = A_iA_{i+1}$ for each i, where indices are taken modulo n.

Solution by Roy Barbara, Lebanese University, Fanar, Lebanon.

We will see how to construct a suitable n-gon for $n \geq 6$; the accompanying figure shows the construction for $n = 7$.

Lemma. For $m \geq 3$ there exists a convex $(m+1)$-gon $A_0A_1A_2 \cdots A_m$ such that if P is the midpoint of A_0A_m, then we have $PA_i = A_iA_{i+1}$ for $0 \leq i \leq m-1$; furthermore, the angles PA_mA_{m-1} and A_1A_0P will both be at most $60°$.

Suppose the lemma has been proved and fix $n \geq 6$. Set $m = n - 3$ (so that $m \geq 3$) and let $A_0A_1A_2 \cdots A_m$ be as in the lemma. Orient the polygon so that A_mA_0 is a horizontal line segment whose midpoint is P, and attach the bottom half of the regular hexagon $A_mA_{m+1}A_{m+2}A_0$ that has A_mA_0 as a main diagonal; specifically, let A_{m+1} and A_{m+2} be the vertices of equilateral triangles erected below segments A_mP and PA_0 (as in the figure). Note that $PA_i = A_iA_{i+1}$ for $i = m, m+1$, and $m+2$, whence the resulting convex $(m+3)$-gon together with the point P satisfy the conditions $PA_i = A_iA_{i+1}$, as required.



Proof of the Lemma. Draw a horizontal segment A_0B of length 2 and define P to be its midpoint. (At the end of the proof the figure will be adjusted so that $A_m = B$.) Let O be a point below A_0B on its perpendicular bisector ℓ, "sufficiently far" from P (the meaning of which will soon be specified). Let Γ denote the circle centred at O that passes through A_0 and B, and let γ denote the open arc A_0B above the segment A_0B. Let A_1 be the point of γ for which $A_0A_1 = 1\ (= PA_0)$. Note that as O tends to infinity away from P along ℓ, the limit position of the arc γ is the segment A_0B, so that the limit position of A_1 is P. Consequently, we may situate O sufficiently far from P so that

$$PA_1 < \frac{1}{2^m}. \tag{1}$$

Note further that

$$\text{a point } X \in \Gamma \text{ is on } \gamma \text{ if and only if } PX < 1. \tag{2}$$

For $2 \leq i \leq m$, we define A_i recursively to be the unique point of Γ counterclockwise from A_i for which $A_iA_{i-1} = PA_{i-1}$. *Claim*: $A_1, A_2, \cdots, A_m$ all lie on γ. This holds for A_1 by its definition; we prove the claim for the remaining points. For $i = 2, \cdots, m$, the triangle inequality applied to $PA_{i-1}A_i$ implies that $PA_i \leq PA_{i-1} + A_{i-1}A_i$. Since $PA_{i-1} = A_{i-1}A_i$ (by our construction), we get $PA_i \leq 2 \cdot PA_{i-1}$ for all i. By induction we deduce that

$$PA_i \leq 2^{i-1} \cdot PA_1 \quad 2 \leq i \leq m. \tag{3}$$

Using (3) and (1) we obtain for $i \leq m$,

$$PA_i \leq 2^{i-1} \cdot PA_1 \leq 2^{m-1} \cdot PA_1 < 2^{m-1}\frac{1}{2^{m-1}} < 1.$$

By virtue of (2) we conclude that the points A_i all lie on γ as claimed.

To conclude the proof, it remains to adjust the position of O so that $A_m = B$. If we let O move along ℓ toward P, the circle Γ as well as the points $A_1, \cdots, A_m$ are subjected to a continuous motion. For some position of $O \in \ell$ the point A_m will coincide with the point B: The limit of A_3 is B when O reaches P (because then $A_0A_1A_2A_3$ would form the top half of a regular hexagon); when $m > 3$, our construction established that A_m lies on the arc γ between B and A_3, so that A_m would necessarily reach B before O reaches P. The construction is complete for that position of O for which $A_m = B$. Convexity is guaranteed because all $m + 1$ vertices of the upper portion of the polygon lie on the arc of a circle.

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; ALBERT STADLER, Herrliberg, Switzerland; EDMUND SWYLAN, Riga, Latvia; and the proposer. There was one incorrect submission.

It would be nice if some reader could provide simple answers to two questions that arise naturally from this problem: It is clear that there are infinitely many different solution n-gons when $n > 6$; is the regular hexagon the only convex 6-gon that satisfies the conditions of the problem, namely, there exists an interior point P such that $PA_i = A_iA_{i+1}$ for all i? Are there any such n-gons for $n < 6$?

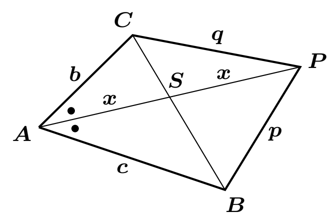
3590. [2010 : 548, 551] *Proposed by G.W. Indika Amarasinghe, University of Kelaniya, Kelaniya, Sri Lanka.*

Let $ABPC$ be a quadrilateral such that BC bisects the segment AP and AP bisects $\angle BAC$. Let $a = BC$, $b = AC$, $c = AB$, $p = BP$, and $q = PC$. Prove that

$$\frac{p^2}{c} + \frac{q^2}{b} = b + c\,.$$

Composite of nearly identical solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; and Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina.

Denote by S the intersection point of AP and BC, and by x the length of the equal segments $AS = SP$. Applying Stewart's theorem to triangles PCB and ABC and their respective cevians PS and AS, we get



$$p^2 \cdot SC + q^2 \cdot BS = a(BS \cdot SC + x^2) = b^2 \cdot BS + c^2 \cdot SC. \qquad (1)$$

Because AS bisects $\angle BAC$, we have $BS = \dfrac{ac}{b+c}$ and $SC = \dfrac{ab}{b+c}$. Inserting these values of BS and SC into (1), we deduce that

$$\frac{p^2ab}{b+c} + \frac{q^2ac}{b+c} = \frac{b^2ac}{b+c} + \frac{c^2ab}{b+c}.$$

Finally, multiply both sides of the last equality by $\dfrac{b+c}{abc}$ to get the desired result.

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; D. KIPP JOHNSON, Valley Catholic School, Beaverton, OR, USA; DAG JONSSON, Uppsala, Sweden; KEE-WAI LAU, Hong Kong, China; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; SCOTT PAULEY, NATALYA WEIR, and ANDREW WELTER, students, Southeast Missouri State University, Cape Girardeau, MO, USA; BOB SERKEY, Tucson, AZ, USA; EDMUND SWYLAN, Riga, Latvia; HAOHAO WANG and YANPING XIA, Southeast Missouri State University, Cape Girardeau, MO, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3591. [2010 : 548, 551] *Proposed by Michel Bataille, Rouen, France.*

Let $\mathcal{E}$ be an ellipse with centre O. At exactly four points P of $\mathcal{E}$, the tangent to $\mathcal{E}$ makes a 45° angle with OP. What is the eccentricity of $\mathcal{E}$?

Solution by Václav Konečný, Big Rapids, MI, USA.

We choose coordinates so that $\mathcal{E}$ has equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; its eccentricity is $e=\sqrt{1-\frac{b^2}{a^2}}$. Because of the symmetry of $\mathcal{E}$ about the axes, there will be exactly one point P in each quadrant where the tangent makes a 45° angle with OP, so we will restrict our attention to the first quadrant. If P has coordinates (x_1,y_1), OP will have slope $m=\frac{y_1}{x_1}$. Moreover, the tangent at P will have equation $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$, and its slope will be

$$m_t=-\frac{b^2x_1}{a^2y_1}=(e^2-1)\frac{1}{m}.$$

If θ is the acute angle at P between OP and the tangent, then

$$\tan\theta=\frac{m-m_t}{1+mm_t}=\frac{1}{e^2}\left(m+(1-e^2)\frac{1}{m}\right).$$

The condition $\theta=45^\circ$ yields $m^2-e^2m+(1-e^2)=0$, whence $m=\dfrac{e^2\pm\sqrt{e^4-4(1-e^2)}}{2}$. This last equation has a unique solution if and only if $e^4+4e^2-4=0$; because the eccentricity of an ellipse must lie between 0 and 1, we conclude that $e=\sqrt{2(\sqrt{2}-1)}$ (which is about $.91$). It is of interest to note that $m=\frac{e^2}{2}=\sqrt{2}-1$, so that OP makes an angle of 22.5° with the x-axis and, consequently, so does the tangent at P.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GERHARDT HINKLE, Student, Central High School, Springfield, MO, USA; VÁCLAV KONEČNÝ, Big Rapids, MI, USA(a second solution); MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Note that as a point P moves about any ellipse centred at O from one vertex to the next, the smaller angle that OP makes with the tangent at P decreases from 90° to some minimum value, and then increases back to 90°. When the ellipse is a circle (whose eccentricity is $e=0$), that angle is constant at 90°; as e increases to 1 the ellipse flattens to a line segment, while that minimum angle decreases to zero. Our featured solution shows how to determine the invertible function that relates the eccentricity of an ellipse to the minimum angle between OP and the tangent at P.

3592★. [2010 : 549, 551] *Proposed by Faruk Zejnulahi and Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Let a, b, and c be nonnegative real numbers such that $a+b+c=3$. Prove or disprove that

$$\frac{19}{20}\;\le\;\frac{1}{1+a+b^2}+\frac{1}{1+b+c^2}+\frac{1}{1+c+a^2}\;\le\;\frac{27}{20}\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

We show that both inequalities are false.

Let

$$f(a,b,c) = \frac{1}{1+a+b^2} + \frac{1}{1+b+c^2} + \frac{1}{1+c+a^2}.$$

Then

$$f\left(0,\frac{3}{2},\frac{3}{2}\right) = \frac{4}{13} + \frac{4}{19} + \frac{2}{5} < \frac{13}{40} + \frac{9}{40} + \frac{16}{40} = \frac{38}{40} = \frac{19}{20}$$

which invalidates the left inequality.

A counterexample for the right inequality is given by

$$f(0,0.1,2.9) = \frac{1}{1.01} + \frac{1}{9.51} + \frac{1}{3.9} > 0.99 + 0.1051 + 0.2563 = 1.3514 > \frac{27}{20}.$$

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; D. KIPP JOHNSON, Valley Catholic School, Beaverton, OR, USA; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; C. R. PRANESACHAR, Department of Mathematics, Indian Institute of Science, Bangalore, India; and STAN WAGON, Macalester College, St. Paul, MN, USA.

Johnson set $a = \frac{3}{2} - x$, $b = \frac{3}{2} + x$ *and* $c = 0$ *with* $-\frac{3}{2} \le x \le \frac{3}{2}$ *and considered the resulting function* $f(x)$. *Using a TI84+ graphing calculator he found the minimum and maximum values of* $f(x)$ *to be* $f(-0.0815361) \approx 0.91692 < \frac{19}{20}$, *and* $f(1.441528) \approx 1.3532 > \frac{27}{20}$, *respectively. Using the Maximize and Minimize commands of the symbolic algebra, Mathematica, as he did in the past, Wagon found that these values obtained by Johnson are in fact the true extrema of the function* $f(a,b,c)$, *and furthermore, they are the two real roots of the following polynomial of degree 8:*

$$1753490160x^8 - 7242829056x^7 + 11338524468x^6 - 8267726008x^5$$
$$+ 2802538373x^4 - 513184720x^3 + 132643611x^2 - 8858430x + 2068173.$$

In an 8-page solution, Pranesachar, with the help of MAPLE, obtained the same polynomial and actually gave a detailed and complete proof.

3593. [2010 : 549, 551] *Proposed by Daryl Tingley, University of New Brunswick, Fredericton, NB.*

Show that for all nonnegative integers n the rightmost nonzero digit of $(4 \cdot 5^n)!$ is 4. Furthermore, show that if $n \ge k \ge 0$, then the string of $k+1$ consecutive digits with this digit 4 at the right is independent of n.

Solution by the proposer, modified and expanded by the editor.

We first define $P_n = \dfrac{(4 \cdot 5^n)!}{10^{5^n - 1}}$. By a well known formula, the largest integer t such that $5^t \mid (4 \cdot 5^n)!$ is given by

$$E_5\left((4 \cdot 5^n)!\right) = \sum_{i=1}^{\infty} \left\lfloor \frac{4 \cdot 5^n}{5^i} \right\rfloor = 4(5^{n-1} + 5^{n-2} + \cdots + 5 + 1) = 5^n - 1.$$

Since clearly $E_5\left((4 \cdot 5^n)!\right) \le E_2\left((4 \cdot 5^n)!\right)$, $10^{5^n-1} \mid (4 \cdot 5^n)!$ so P_n is an integer. Note that the second part of the problem can now be restated as follows: show that if $n \ge k \ge 0$ then $P_n \equiv P_k \pmod{10^{k+1}}$.

We also note in passing that

$$\begin{aligned} E_2\left((4\cdot 5^n)!\right) &= \sum_{i=1}^{\infty}\left\lfloor \frac{4\cdot 5^n}{2^i}\right\rfloor = \sum_{i=1}^{\infty}\left\lfloor 4\cdot\left(\frac{5}{2}\right)^i 5^{n-i}\right\rfloor \\ &\geq \sum_{i=1}^{\infty}\left\lfloor 2^{i+2}\cdot 5^{n-i}\right\rfloor = \sum_{i=1}^{\infty} 2^2\cdot 5^n\cdot\left(\frac{2}{5}\right)^i = 4\left(5^n\right)\left(\frac{2}{3}\right) \\ &= 2\cdot 5^n + \left(\frac{2}{3}\right)\cdot 5^n > 2\cdot 5^n + 1 \end{aligned}$$

so $2^{2\cdot 5^n+1} | P_n$ and in particular, $2^{k+1} | P_n$ if $0 \leq k \leq n$. This fact will be used later on in the proof.

Next define $R_n = \dfrac{(4\cdot 5^n)!}{5^{5^n-1}}$ and let $S = \{1, 2, \cdots, 4\cdot 5^n\}$. Fix k where $k+1 \leq n$. Let $A = \{m \in S :\ 5 \mid m\}$ and $B = \{m \in S :\ 5 \nmid m\}$. Then $(4\cdot 5^n)! = A_n B_n$ where $A_n = \prod_{m\in A} m$ and $B_n = \prod_{m\in B} m$. For each $m \in A$ factor out one copy of 5 to obtain

$$A_n = 5^{4\cdot 5^{n-1}}\left(4\cdot 5^{n-1}\right). \qquad (1)$$

On the other hand, for each $m \in B$, we have $m = q\cdot 5^{k+1} + r$ where $0 \leq q \leq 4\cdot 5^{n-k-1} - 1$ and $1 \leq r < 5^{k+1}$ such that $(r, 5) = 1$. Hence,

$$B_n = \prod_{q=0}^{4\cdot 5^{n-k-1}-1}\left(\prod_{r=1,\ (r,5)=1}^{5^{k+1}} q\cdot 5^{k+1} + r\right) \qquad (2)$$

Now recall the following generalization of Wilson's Theorem by Gauss: [*Ed.: cf. e.g., Elementary Number Theory and its Applications by Kenneth H. Rosen; 4th ed., page 204.*]

Theorem. For any integer $m \geq 1$,

$$\prod_{1\leq k\leq m,\ (k,m)=1} k \equiv \begin{cases} -1 \pmod{m} & \text{if } m = 4, p^a, 2p^a \\ 1 \pmod{m} & \text{otherwise} \end{cases}$$

where p is an odd prime and a is any positive integer. By the theorem above we have

$$\prod_{r=1,\ (r,5)=1}^{5^{k+1}}\left(q\cdot 5^{k+1} + r\right) \equiv -1 \pmod{5^{k+1}}$$

so (2) becomes

$$B_n \equiv \prod_{q=0}^{4\cdot 5^{n-k-1}-1}(-1) = (-1)^{4\cdot 5^{n-k-1}} = 1 \pmod{5^{k+1}}. \qquad (3)$$

From (1) and (3) we have

$$R_n = \frac{A_n B_n}{5^{5^n-1}} \equiv \frac{5^{4\cdot 5^{n-1}}\left(4\cdot 5^{n-1}\right)!}{5^{5^n-1}} = \frac{\left(4\cdot 5^{n-1}\right)!}{5^{5^{n-1}-1}} = R_{n-1} \ \left(\text{mod } 5^{k+1}\right).$$

Simple induction yields $R_n \equiv R_k \ \left(\text{mod } 5^{k+1}\right)$ for $k \leq n$. Furthermore, for $k+1 \leq n$ we have, since $\left(2, 5^{k+1}\right) = 1$, by Euler's Theorem on totient function ϕ that

$$\begin{aligned} 2^{5^n-1} &= \frac{1}{2}\cdot 2^{4\cdot 5^{n-1}}\cdot 2^{5^{n-1}} = \frac{1}{2}\left(2^{4\cdot 5^k}\right)^{5^{n-k-1}}\cdot 2^{5^{n-1}} \\ &= \frac{1}{2}\left(2^{\phi\left(5^{k+1}\right)}\right)^{5^{n-k-1}}\cdot 2^{5^{n-1}} \equiv \frac{1}{2}(1)\cdot 2^{5^{n-1}} = 2^{5^{n-1}-1} \ \left(\text{mod } 5^{k+1}\right). \end{aligned}$$

Inductively, we then have $2^{5^n-1} \equiv 2^{5^k-1} \ \left(\text{mod } 5^{k+1}\right)$ for $k \leq n$. So,

$$2^{5^k-1}P_n \equiv 2^{5^n-1}P_n = R_n \equiv R_k = 2^{5^k-1}P_k \ \left(\text{mod } 5^{k+1}\right)$$

for $k \leq n$. Since $\left(2^{5^k-1}, 5^{k+1}\right) = 1$, it follows that $P_n \equiv P_k \ \left(\text{mod } 5^{k+1}\right)$. Since both P_n and P_k are divisible by 2^{k+1} as mentioned and proved above, we have $P_n \equiv P_k \ \left(\text{mod } 2^{k+1}\right)$. Since $\left(5^{k+1}, 2^{k+1}\right) = 1$, it follows that $P_n \equiv P_k$ $\left(\text{mod } 10^{k+1}\right)$.

Finally, $P_0 = 4! \equiv 4 \ (\text{mod } 10)$ so the rightmost nonzero digit of $(4\cdot 5^n)!$ is 4 for all $n \geq 0$, and our proof is complete.

Also solved by ALBERT STADLER, Herrliberg, Switzerland. A partial solution for $k \leq 1$ *was submitted by OLIVER GEUPEL, Brühl, NRW, Germany.*

3594. [2010 : 549, 551] *Proposed by Michel Bataille, Rouen, France.*

Let x, y, z be three indeterminates and $A = (y-z)(y+x)(x+z)$, $B = (z-x)(z+y)(y+x)$, $C = (x-y)(x+z)(z+y)$. Find all polynomials P, Q, $R \in \mathbb{C}[x,y,z]$ such that

$$\frac{x^2P + y^2Q + z^2R}{xP + yQ + zR} = \frac{x^2A + y^2B + z^2C}{xA + yB + zC}.$$

Solution by the proposer.

Let $N = x^2A + y^2B + z^2C$, $D = xA + yB + zC$ and $F = \frac{N}{D}$. Since N and D both vanish when $x = y$ or $y = z$ or $z = x$, they are both divisible by $(x-y)(y-z)(z-x)$. Thus, F is not in its lowest form and we simplify it.

Denoting by S the polynomial $xy + yz + zx$, we have

$$\begin{aligned}D &= x(y-z)(x^2+S) + y(z-x)(y^2+S) + z(x-y)(z^2+S)\\ &= x^3(y-z) + y^3(z-x) + z^3(x-y)\\ &\qquad\qquad + S\cdot[x(y-z)+y(z-x)+z(x-y)]\\ &= x^3(y-z) + y^3(z-x) + z^3(x-y)\\ &= (z-y)(y-x)(x-z)(x+y+z)\end{aligned}$$

and similarly

$$\begin{aligned}N &= x^4(y-x) + y^4(z-x) + z^4(x-y)\\ &\qquad\qquad + S\cdot[x^2(y-z)+y^2(z-x)+z^2(x-y)]\\ &= (z-y)(y-x)(x-z)(x^2+y^2+z^2+S)\\ &\qquad\qquad + S\cdot(z-y)(y-x)(x-z)\\ &= (z-y)(y-x)(x-z)(x+y+z)^2.\end{aligned}$$

[*Ed.: By induction, we see that for every integer* $n \geq 2$, $x^n(y-z)+y^n(z-x)+z^n(x-y) = (z-y)(y-x)(x-z)\sum x^a y^b z^c$, *where the sum is extended to all triples* (a,b,c) *of nonnegative integers for which* $a+b+c = n-2$.] From these results, we deduce that $F = x+y+z$.

The problem reduces to finding P, Q, R such that

$$x^2P + y^2Q + z^2R = (x+y+z)(xP+yQ+zR),$$

that is,

$$xy(P+Q) + yz(Q+R) + zx(R+P) = 0. \tag{1}$$

Thus, z must divide $P+Q$ and we can set $P+Q = 2zW$. Similarly $Q+R = 2xU$ and $R+P = 2yV$ for some polynomials U, V, W. Substituting in (1), we see that $U+V+W = 0$. Solving for P, Q, R we easily find

$$P = -xU+yV+zW, \qquad Q = xU-yV+zW, \qquad R = xU+yV-zW.$$

We can select polynomials K_1, K_2, K_3 (in many ways) such that $U = K_2 - K_3$, $V = K_3 - K_1$, $W = K_1 - K_2$. We obtain

$$\begin{aligned}P &= (z-y)K_1 - (z+x)K_2 + (x+y)K_3,\\ Q &= (y+z)K_1 + (x-z)K_2 - (x+y)K_3,\\ R &= -(y+z)K_1 + (z+x)K_2 + (y-x)K_3.\end{aligned}$$

Conversely, if K_1, K_2, K_3 are arbitrary polynomials in x, y, z, then (1) is satisfied and we have the general form of the solution.

Note that the obvious solutions (A,B,C) is obtained with $K_1 = -x^2$, $K_2 = -y^2$, $K_3 = -z^2$.

There were no other solutions. However, Albert Stadler, Herrliberg, Switzerland noted that

$$(x^2P + y^2Q + z^2R)(xA + yB + zC) - (xP + yQ + zR)(x^2A + y^2B + z^2C)$$
$$= (x - y)(y - z)(z - x)(x + y + z)[x(y + z)P + y(z + x)Q + z(x + y)R].$$

The expression in square brackets must vanish and Stadler isolated some properties of the polynomials P, Q, R.

3595. [2010 : 549, 551-552] *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

Let a, b, n be positive integers satisfying $a < b$ and $n < a + b$, and so that

exactly $\frac{1}{n}$ of the integers a^2, $a^2 + 1$, $a^2 + 2$, $\ldots$, b^2 are squares. (1)

Do the following:

(a) Prove that also exactly $\frac{1}{n}$ of the consecutive integers $(n - a)^2$, $(n - a)^2 + 1$, $(n - a)^2 + 2$, $\ldots$, b^2 are squares.

(b) Exactly $\frac{1}{n}$ of the integers 1, 2, $\ldots$, n^2 are squares, and also exactly $\frac{1}{n}$ of the integers $(n - 1)^2 = n^2 - 2n + 1$, $n^2 - 2n + 2$, $\ldots$, n^2 are squares. Thus, for every integer $n \geq 3$, the values $a = 1$, $b = n$ and $a = n - 1$, $b = n$ always satisfy (1). For which integers $n \geq 3$ are these the only solutions of (1)?

Solution by Kathleen E. Lewis, University of the Gambia, Brikama, Gambia.

(a) If $a < n - a$, the list of integers a^2, $a^2 + 1$, $\ldots$, $(n - a)^2 - 1$ contains $(n - a)^2 - a^2 = n^2 - 2an$ integers, of which $(n - a) - a = n - 2a$, or $\frac{1}{n}$ of the total, are squares. Therefore if $\frac{1}{n}$ of the list a^2, $a^2 + 1$, $\ldots$, b^2 consists of squares, then it follows that $\frac{1}{n}$ of the difference of the lists, $(n - a)^2$, $(n - a)^2 + 1$, $\ldots$, b^2 must consist of squares. In the same way, if $n - a < a$, the sequence $(n - a)^2$, $(n - a)^2 + 1$, $\ldots$, $a^2 - 1$ contains $a^2 - (n - a)^2 = 2an - n^2$ numbers of which $a - (n - a) = 2a - n$, which is $\frac{1}{n}$ of them, are squares, so once again, the list $(n - a)^2$, $(n - a)^2 + 1$, $\ldots$, b^2 must have $\frac{1}{n}$ of its elements being squares.

(b) The list a^2, $a^2 + 1$, $\ldots$, b^2 contains $b^2 - a^2 + 1$ numbers, of which $b - a + 1$ are squares, so the proportion of squares is $\frac{b - a + 1}{b^2 - a^2 + 1}$. Setting this equal to $\frac{1}{n}$ gives $b^2 - a^2 + 1 = (b - a + 1)n$, which can be rearranged to $(b - a)(b + a - n) = n - 1$. Setting one of the factors on the left equal to 1 and the other equal to $n - 1$ yields the trivial solutions $(a, b) = (1, n)$ and $(a, b) = (n - 1, n)$. If $n - 1$ is prime, these are the only solutions. Also, in the case that $n - 1$ is a power of 2, they are the only solutions. This is because the only nontrivial factorization of a power of 2 consists of two even numbers. But if $n-1$ is even, n is odd and $b-a$ and $b+a-n$ have opposite parity.

Suppose that $n-1$ is odd and not a prime. Then it can be written as a nontrivial product of two odd integers. Since $b-a$ and $b+a-n$ have the same parity, they can be set equal to these integers to get integer values of a and b.

On the other hand, if $n-1$ is even but not a power of 2, it can be written as the product of an even and an odd integer, both exceeding 1. Since n is odd, $b-a$ and $b+a-n$ have opposite parity and we can once again get a nontrivial solution for a and b.

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; JOEL SCHLOSBERG, Bayside, NY, USA; and the proposer.

3596. [2010 : 549, 552] *Proposed by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.*

Let x, y and z be positive real numbers. Prove that

$$\sum_{\text{cyclic}} \frac{x(y+z)}{(x+2y+2z)^2} \le \sum_{\text{cyclic}} \frac{(x+y)(x+y+2z)}{(3x+3y+4z)^2}\,.$$

Composite of similar solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; and D. Kipp Johnson, Valley Catholic School, Beaverton, OR, USA, expanded slightly by the editor.

Since both summations are homogeneous we may assume that $x+y+z=1$. Then the given inequality is equivalent, in succession, to

$$\sum_{\text{cyclic}} \left(\frac{(y+z)(y+z+2x)}{(3y+3z+4x)^2} - \frac{x(y+z)}{(x+2y+2z)^2}\right) \ge 0$$

$$\sum_{\text{cyclic}} \left(\frac{(1-x)(1+x)}{(3+x)^2} - \frac{x(1-x)}{(2-x)^2}\right) \ge 0$$

$$\sum_{\text{cyclic}} \frac{(1-x)\left((1+x)(2-x)^2 - x(3+x)^2\right)}{(3+x)^2(2-x)^2} \ge 0$$

$$\sum_{\text{cyclic}} \left(\frac{(1-x)(4-9x-9x^2)}{(3+x)^2(2-x)^2} + \frac{27(3x-1)}{250}\right) \ge 0 \quad \text{as} \quad \sum_{\text{cyclic}} (3x-1) = 0$$

$$\sum_{\text{cyclic}} \left(\frac{9x^3-13x+4}{(3+x)^2(2-x)^2} + \frac{27(3x-1)}{250}\right) \ge 0$$

$$\sum_{\text{cyclic}} \frac{(3x-1)(250(3x^2+x-4)+27(3+x)^2(2-x)^2)}{250(3+x)^2(2-x)^2} \ge 0$$

$$\sum_{\text{cyclic}} \frac{(3x-1)(27x^4+54x^3+453x^2-74x-28)}{250(3+x)^2(2-x)^2} \geq 0$$

$$\sum_{\text{cyclic}} \frac{(3x-1)^2(9x^3+21x^2+158x+28)}{250(3+x)^2(2-x)^2} \geq 0$$

The last inequality above is clearly true. Clearly the equality holds if and only if $x = y = z$.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; KEE-WAI LAU, Hong Kong, China; JOEL SCHLOSBERG, Bayside, NY, USA; and the proposer.

3597. [2010 : 550, 552] *Proposed by Johan Gunardi, student, SMPK 4 BPK PENABUR, Jakarta, Indonesia.*

One hundred students take an exam consisting of 50 true or false questions. Prove that there exist three students whose answers coincide for at least 13 questions.

Solution by D. Kipp Johnson, Valley Catholic School, Beaverton, OR, USA.

Suppose that for each question i, $1 \leq i \leq 50$, a_i students answer 'TRUE' and $100 - a_i$ students answer 'FALSE'. The number of sets of three students whose answers agree for question i is then

$$\binom{a_i}{3} + \binom{100-a_i}{3},$$

with the usual convention that $\binom{n}{k} = 0$ when $n < k$.

Now for any integer x, $1 \leq x \leq 100$,

$$\binom{x}{3} + \binom{100-x}{3} = 49(x^2-100x+3300) = 49((x-50)^2+800) \geq 39200$$

Thus the average number of coinciding answers among all sets of three students is

$$\frac{\sum_{i=1}^{50}\left(\binom{a_i}{3} + \binom{100-a_i}{3}\right)}{\binom{100}{3}} \geq \frac{(50)(39200)}{161700} \approx 12.12$$

However, the actual number of coinciding answers for any set of three students is an integer, so at least one set of three students must have coinciding answers for at least 13 questions.

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; and the proposer.

3598. [2010 : 550, 552] *Proposed by Zhang Yun, High School attached to Xi′ An Jiao Tong University, Xi′ An City, Shan Xi, China.*

The quadrilateral $ABCD$ has both a circumscribed circle and an inscribed circle, the latter with centre I. Put $a = AB$, $b = BC$, $c = CD$, and $d = DA$. Prove that

$$\frac{IB^2}{ab} + \frac{IC^2}{bc} + \frac{ID^2}{cd} + \frac{IA^2}{da} \;=\; 2\,.$$

I. Solution by Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina.

Let $\angle DAB = \alpha$ and $\angle ABC = \beta$. Since $ABCD$ is cyclic, $\angle BCD = 180^\circ - \alpha$ and $\angle CDA = 180^\circ - \beta$. Also, since I is the centre of the inscribed circle, AI bisects $\angle BAD$, so $\angle BAI = \angle DAI = \frac{\alpha}{2}$. Similar results hold for angles at other vertices.

Using The Law of Sines, we have

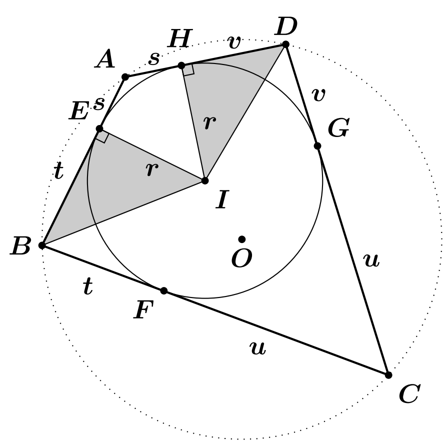
$$\frac{IB^2}{ab} = \frac{IB}{a}\cdot\frac{IB}{b} = \frac{\sin\frac{\alpha}{2}}{\sin\frac{\alpha+\beta}{2}}\cdot\frac{\cos\frac{\alpha}{2}}{\cos\frac{\beta-\alpha}{2}}$$
$$\frac{IA^2}{da} = \frac{IA}{d}\cdot\frac{IA}{a} = \frac{\cos\frac{\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\cdot\frac{\sin\frac{\beta}{2}}{\sin\frac{\alpha+\beta}{2}}$$

Summing the last two equalities, we have

$$\frac{IB^2}{ab} + \frac{IA^2}{da} = \frac{\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} + \sin\frac{\beta}{2}\cos\frac{\beta}{2}}{\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}} = \frac{\sin\alpha + \sin\beta}{\sin\alpha+\sin\beta} = 1.$$

Since for similar reasons we have $\frac{IB^2}{ab} + \frac{IA^2}{da} = 1$, then the desired proposition holds.

II. Solution by Oliver Geupel, Brühl, NRW, Germany.



Let E, F, G, and H denote the feet of the perpendiculars from the point I to the lines AB, BC, CD, and DA, respectively. Since I is the centre of the inscribed circle, then $EI = HI$, $AE = AH$, $BE = BF$, $CF = CG$, and $DG = DH$. Let these lengths be denoted r, s, t, u, and v, respectively. Since the quadrilateral is cyclic, we have $\angle HDI = 90^\circ - \angle EBI = \angle BIE$. Hence, the right triangles BEI, and IHD are similar, which yields $\frac{r}{t} = \frac{v}{r}$. Analogously, we have $\frac{r}{s} = \frac{u}{r}$. Therefore

$$r^2 = su = tv.$$

Since

$$\begin{aligned} IB^2 \cdot c + IC^2 \cdot a &= (r^2+t^2)(u+v) + (r^2+u^2)(s+t) \\ &= t^2(u+v) + u^2(s+t) + r^2(s+t+u+v) \\ &= t^2(u+v) + u^2(s+t) + stv + stu + tuv + suv \\ &= t^2(u+v) + u^2(s+t) + st(u+v) + uv(s+t) \\ &= (u+v)t(s+t) + (s+t)u(u+v) \\ &= (s+t)(t+u)(u+v) \\ &= abc, \end{aligned}$$

then

$$\frac{IB^2}{ab} + \frac{IC^2}{bc} = 1.$$

Analogously,

$$\frac{ID^2}{cd} + \frac{IA^2}{da} = 1,$$

hence the proposition.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; KEE-WAI LAU, Hong Kong, China; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; JOEL SCHLOSBERG, Bayside, NY, USA; EDMUND SWYLAN, Riga, Latvia; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3599 ★. [2010 : 550, 552] *Proposed by Cristinel Mortici, Valahia University of Târgovişte, Romania.*

Let m and n be positive integers such that $2^m - 3^n \geq n$. Prove that

$$2^m - 3^n \geq m\,.$$

Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

The claim is false, as can be seen by taking $(m,n) = (2,1)$. The inequality $2^m - 3^n \geq n$ becomes $1 \geq 1$, whereas the inequality $2^m - 3^n \geq m$ becomes $1 \geq 2$.

The following modification of the problem does have a solution.

Suppose m and n are positive integers such that

$$2^{m-1} - 3^n \geq n. \tag{1}$$

Then

$$2^m - 3^n \geq m + n. \tag{2}$$

For the proof, fix n, and suppose (1) holds. Then $m \geq 1 + \log_2(3^n + n)$. Set

$$f_n(m) = 2^m - m - 3^n.$$

Then

$$f_n'(m) = 2^m \cdot \ln 2 - 1,$$

which is greater than 0 for $m \geq 1$. Hence $f_n(m)$ is increasing on $[1, \infty)$. Therefore,

$$\begin{aligned} f_n(m) &\geq f_n(1 + \log_2(3^n + n)) \\ &= 2^{1+\log_2(3^n+n) - [1+\log_2(3^n+n)] - 3^n} \\ &= 2(3^n + n) - 1 - \log_2(3^n + n) - 3^n \\ &= (3^n + 2n - 1) - \log_2(3^n + n). \end{aligned}$$

For $t > 0$, define

$$g(t) = t - \log_2 t.$$

Since

$$g'(t) = 1 - \tfrac{1}{t \ln 2},$$

we have $g'(t) > 0$ if and only if $t > \frac{1}{\ln 2}$. In particular, $g'(t) > 0$ for $t \geq 2$. Hence, $g(t)$ is increasing on $[2, \infty)$, so $g(t) \geq g(2) = 1$ for $t \geq 2$. Also, $g(1) = 1$. Thus, $g(k) \geq 1$ for all positive integers k. We now have

$$\begin{aligned} f_n(m) &\geq g(3^n + n) + (n - 1) \\ &\geq n. \end{aligned}$$

This implies that

$$2^m - 3^n \geq m + n,$$

as claimed.

OLIVER GEUPEL, Brühl, NRW, Germany also pointed out the claim was false, citing the same counterexample as Curtis.

3600. [2010 : 550, 552] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $k \geq 1$ be a nonnegative integer. Prove that

$$\sum_{n_1, n_2, \ldots, n_k = 1}^{\infty} \frac{1}{(n_1 + n_2 + \cdots + n_k)!} = (-1)^{k-1} \left(e \sum_{j=0}^{k-1} \frac{(-1)^j}{j!} - 1 \right).$$

Solution by Michel Bataille, Rouen, France.

For positive integer m, the cardinality of the set of all $k-$tuples of positive integers $(n_1, n_2, \ldots, n_k)$ with $n_1 + n_2 + \cdots + n_k = m$ is $\binom{m-1}{k-1}$, this number being 0 if $k > m$. Let S denote the left side. Since all terms in the sum are nonnegative,

$$S = \sum_{m=1}^{\infty} \sum_{n_1+n_2+\cdots+n_k=m} \frac{1}{(n_1+n_2+\cdots+n_k)!} = \sum_{m=1}^{\infty} \binom{m-1}{k-1} \frac{1}{m!},$$

that is

$$S = \frac{1}{(k-1)!} \sum_{m=1}^{\infty} (m-1)(m-2)\ldots(m-k+1)\frac{1}{m!} = \frac{1}{(k-1)!} f^{(k-1)}(1),$$

where, for positive x,

$$f(x) = \frac{e^x}{x} - \frac{1}{x} = \sum_{m=1}^{\infty} \frac{x^{m-1}}{m!}.$$

By Leibniz' formula,

$$f^{(k-1)}(x) = \sum_{j=0}^{k-1} \binom{k-1}{j} e^x \cdot (-1)^j j! x^{-(j+1)} - (-1)^{k-1}(k-1)! x^{-k}.$$

Hence

$$\begin{aligned} S &= \frac{1}{(k-1)!} \left(\sum_{j=0}^{k-1} \binom{k-1}{j} e \cdot (-1)^j j! - (-1)^{k-1}(k-1)! \right) \\ &= e \sum_{j=0}^{k-1} \frac{(-1)^j}{(k-1-j)!} - (-1)^{k-1} \\ &= (-1)^{k-1} \left[e \sum_{j=0}^{k-1} \frac{(-1)^{j-k+1}}{(k-1-j)!} - 1 \right] \\ &= (-1)^{k-1} \left[e \sum_{j=0}^{k-1} \frac{(-1)^j}{j!} - 1 \right]. \end{aligned}$$

Also solved by DON KRUG, Northern Kentucky University, Highland Heights, KY, USA; and the proposer. The proposer showed that the sum was equal to $S_{k-1}(1)/(k-1)!$*, where*

$$S_{k-1}(x) = \sum_{j=k}^{\infty} \frac{x^j}{(j-k)!} = \int_0^x t^{k-1} e^t dt.$$

Integration by parts yields the recurrence relation $S_{k-1}(1) = e - (k-1)S_{k-2}(1)$*, from which the desired result follows. Krug followed a similar strategy.*

YEAR END FINALE

We have made it! It has been an interesting year here at ***CRUX with MAYHEM***. There were delays getting me appointed, changes in personnel, technical difficulties, and more delays renewing the contract between the Ottawa Carleton District School Board, my employer, and the CMS. At the same time we made some changes to ***CRUX with MAYHEM*** and developed a plan for the journal in the future. The journal saw some changes this year, with the introduction of a new *Olympiad Corner* format, and there will be more changes to come.

We have worked out a plan for ***CRUX with MAYHEM*** that we will be implementing in volume 39. As a result, volume 38 will be streamlined as we prepare for the changes that will take place the following year. In volume 38 we will drop from 8 to 6 issues.

Also, ***Mathematical Mayhem*** and *Skoliad* will be separating from ***Crux Mathematicorum***. The solutions to all problems from these sections that have appeared in print in ***CRUX with MAYHEM*** will appear in volume 38, but there will be no new problems. ***Mathematical Mayhem*** (with *Skoliad*) will carry on separately and be expanded. To fill the void, a new problems column will be created so there will be more problems for you to solve. The solutions to all problems from the regular problems section that have appeared in volume 37 will appear in volume 38 and we will continue to publish 100 problems per volume. Also the solutions to the new *Olympiad Corner* problems will start to appear.

As Vazz pointed out at the end of the last volume, he had cleared the backlog of articles, which left us with few for this year. You may notice that we ended 8 pages short, which was due, in part, to the shortage of articles. To help us get caught up, we are planning not to publish any articles in the next volume. The hope is that articles will return in volume 39 on a more regular basis.

We will work though volume 38 as quickly as possible, and let you know when we have a firm idea of what ***Crux Mathematicorum*** will look like for volume 39 and onward. Our hope is that the publication schedule will be back on track by the end of volume 39.

I would like to thank our readers for their patience in this time of change and delays. The kind words and encouragement that you have sent to me are greatly appreciated. I will continue to work hard to make ***Crux Mathematicorum*** the best it can be.

I am not alone putting the journal together, there are a number of people that I must thank. First, I must thank VÁCLAV (VAZZ) LINEK. Since I started this position, Vazz was always there to help me learn the ropes. Through numerous emails and a few phone calls he helped me quickly figure out all that I needed to do. Even though the number of times I contact him has greatly dropped off, when I need him, he is always there. All the best to you in the future Vazz and thanks.

Next, I must thank the editorial staff. I thank my associate editor JEFF HOOPER, whose detailed proofreading picks up many of the things I have missed. I thank CHRIS FISHER for his work as problems editor and for all his other contributions like his history of ***CRUX with MAYHEM*** and his current column *Recurring Crux Configurations*. His dedication and expertise are greatly appreciated. I thank EDWARD WANG for his work as problems editor as well as his continued contribution of nice problems and solutions to the ***MAYHEM*** section. Our readers greatly benefit from his contributions. I thank NICOLAE STRUNGARU for his dual role as problem editor and new editor to the *Olympiad Corner*. His willingness to take over the column, while continuing his old role until it could be filled, has helped to fill the void left when Robert stepped down last year. I also thank CHRIS GRANDISON for his precise work as a problems editor.

This year IAN AFFLECK, ROBERT CRAIGEN, and COSMIN POHOATA have left the editorial board. I thank them for their time with us, their work has been greatly appreciated.

I want to thank EDWARD BARBEAU, RICHARD GUY and BILL SANDS. When I took over, I approached these gentlemen for some "unofficial" help. They were a great help to me proofreading and working my way through some of the backlog of problems from our files. I am thrilled that Ed has agreed to join the board as a problems editor and that Bill has agreed to become the Editor at Large. I also want to welcome ANNA KUCZYNSKA to the board as problems editor and thank her for the great job she has done so far.

I must thank JOANNE CANAPE and ROBERT WOODROW at the University of Calgary, for continuing to send those last sets of solutions for the *Olympiad Corner*. Your dedication

to this column and ***CRUX with MAYHEM*** for over 20 years is greatly appreciated and will be sorely missed.

I thank JEAN-MARC TERRIER and ROLLAND GAUDET for providing French translations. The translations are always done quickly and occasionally there are comments on how to improve the English part as well!

I thank ROBERT DAWSON for taking over as the articles editor and for providing us with such nice articles this year. I thank AMAR SODHI for keeping me in good supply of interesting book reviews (I have purchased and read several of these books already).

I thank LILY YEN and MOGENS LEMVIG HANSEN for their work as ***Skoliad*** Editors and the great job they do. I thank the ***MAYHEM*** staff ANN ARDEN, NICOLE DIOTTE, MONIKA KHBEIS and DAPHNE SHANI for their help preparing solutions. I especially thank my assistant editor LYNN MILLER for her dedication and extra help behind the scenes. I thank IAN VANDERBURGH for his time doing *The Problem of the Month*, it was a pleasure working with you and your column will be missed.

I thank the staff at the CMS head office in Ottawa. In particular I thank JOHAN RUDNICK, DENISE CHARRON, and STEVE LA ROCQUE. Their behind the scenes support of me at ***CRUX with MAYHEM*** is greatly appreciated. I thank TAMI EHRLICH and the staff at Thistle Printing for taking the files that I send them and turning them into the journal you have in your hands.

I must also thank the OTTAWA DISTRICT SCHOOL BOARD for partnering with the CMS to allow me to become the Editor-in-Chief. In particular I would like to thank director JENNIFER ADAMS, superintendent PINO BUFFONE, Human Resources Officer JENNIFER BALDELLI and principal KEVIN GILMORE for their support and encouragement.

I look forward to working with all of you through the next $\boldsymbol{n}$ volumes.

I also want to thank my wife JULIE and my sons SAMUEL and SIMON for their support during this past year. You guys are my rock (and roll!).

Finally, I sincerely thank all of the readers of ***CRUX***. Without your problems, solutions, articles and comments there is no ***CRUX***. I look forward to your contributions, letters and email for the next volume.

Shawn Godin

Crux Mathematicorum

with Mathematical Mayhem

Crux Mathematicorum

Mathematical Mayhem

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b. Paid Circulation *(By Mail and Outside the Mail)*	(1)	Mailed Outside-County Paid Subscriptions Stated on PS Form 3541(Include paid distribution above nominal rate, advertiser's proof copies, and exchange copies)	142	138
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	(3)	Paid Distribution Outside the Mails Including Sales Through Dealers and Carriers, Street Vendors, Counter Sales, and Other Paid Distribution Outside USPS®	508	329
	(4)	Paid Distribution by Other Classes of Mail Through the USPS (e.g. First-Class Mail®)	0	0
c. Total Paid Distribution *(Sum of 15b (1), (2), (3), and (4))*			650	467
d. Free or Nominal Rate Distribution *(By Mail and Outside the Mail)*	(1)	Free or Nominal Rate Outside-County Copies included on PS Form 3541	0	0
	(2)	Free or Nominal Rate In-County Copies Included on PS Form 3541	0	0
	(3)	Free or Nominal Rate Copies Mailed at Other Classes Through the USPS (e.g. First-Class Mail)	0	0
	(4)	Free or Nominal Rate Distribution Outside the Mail *(Carriers or other means)*	0	0
e. Total Free or Nominal Rate Distribution *(Sum of 15d (1), (2), (3) and (4))*			0	0
f. Total Distribution *(Sum of 15c and 15e)* ▶			650	467
g. Copies not Distributed *(See Instructions to Publishers #4 (page #3))* ▶			138	163
h. Total *(Sum of 15f and g)* ▶			788	630
i. Percent Paid *(15c divided by 15f times 100)* ▶			100	100

16. Publication of Statement of Ownership

☒ If the publication is a general publication, publication of this statement is required. Will be printed in the next issue of this publication.

☐ Publication not required.

17. Signature and Title of Editor, Publisher, Business Manager, or Owner	Date
Adele D'Ambrosio, Circulation Supervisor	September 29, 2011

I certify that all information furnished on this form is true and complete. I understand that anyone who furnishes false or misleading information on this form or who omits material or information requested on the form may be subject to criminal sanctions (including fines and imprisonment) and/or civil sanctions (including civil penalties).