

SKOLIAD No. 131

Lily Yen and Mogens Hansen

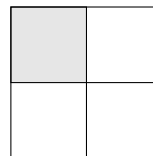
Please send your solutions to problems in this Skoliad by **October 15, 2011**. A copy of *CRUX with Mayhem* will be sent to one pre-university reader who sends in solutions before the deadline. The decision of the editors is final.

Our contest this month is the National Bank of New Zealand Junior Mathematics Competition, 2010. Our thanks go to Warren Palmer, Otago University, Otago, New Zealand for providing us with this contest and for permission to publish it.

La rédaction souhaite remercier Rolland Gaudet, de Collège universitaire de Saint-Boniface, Winnipeg, MB, d'avoir traduit ce concours.

Concours mathématique de la Banque Nationale de la Nouvelle Zélande, 2010 Durée : 1 heure

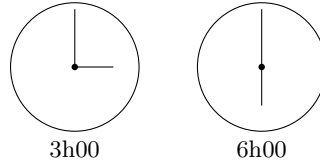
1. Raymonde tient un séminaire à son lieu de travail. Elle désire créer un anneau ininterrompu de tables identiques en formes de polygones réguliers. (Dans un polygone régulier, les côtés sont de longueurs égales et les angles sont de mesures égales. Les carrés et les triangles équilatéraux sont réguliers.) Chaque table doit avoir deux côtés complets qui coïncident avec les côtés d'autres tables, tel qu'illustré à droite dans le cas du carré ombré. Raymonde a l'intention d'étaler du matériel à l'intérieur de l'anneau, de façon à ce que ce matériel soit visible par chacun.



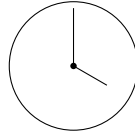
(Si vous ne pouvez pas nommer une forme, simplement en fournir le nombre de côtés. Par exemple, si vous croyez que la forme a **235** côtés, mais n'en connaissez pas le nom, simplement l'appeler un **235**-gone; noter que ceci fait partie de la réponse à aucune des questions ci-bas.)

1. En premier lieu, Raymonde décide d'utiliser des tables carrées identiques. Quel est le nombre minimal de tables carrées à placer les unes contre les autres, de façon à ce qu'il y ait un espace vide au centre ?
2. Si Raymonde utilise le nombre minimal de tables carrées, quelle est la forme de l'espace vide au centre ?
3. Raymonde considère maintenant utiliser des tables en formes d'octagones (huit côtés chacune).
 - (a) Quel est le nombre minimal de tables octogonales que doit utiliser Raymonde afin qu'il y ait un espace vide au centre de l'enclos ?
 - (b) Quel est le nom de la forme de l'espace vide au centre ? Si vous n'en connaissez pas le nom, il suffit d'en donner le nombre de côtés.
4. À part les carrés et les octogones, y a-t-il d'autres formes de tables possibles ? Si oui, les nommer. Sinon, indiquer qu'il n'y en a pas.

2. Une horloge analogue affiche le temps à l'aide de deux aiguilles. Chaque heure, l'aiguille des minutes tourne par **360** degrés, tandis que l'aiguille des heures (qui est plus courte que l'aiguille des minutes) tourne par **360** degrés sur une période de 12 heures. Deux exemples suivent.



1. Dessiner une horloge qui affiche 9h00. Assurez-vous que l'aiguille des heures est plus courte que celle des minutes.
2. Quel est l'angle entre les deux aiguilles à 3h00 et aussi à 9h00 ?
3. Quel temps est affiché par l'horloge qui suit, à l'heure et à la minute près ?



4. Quel est l'angle entre les aiguilles aux moments suivants ?
 - (a) 1h00.
 - (b) 2h00.
 - (c) 1h30.
 5. À quel moment, à la minute près, entre 7h00 et 8h00, les aiguilles sont-elles à la même position ?
- 3.** Un entier à six chiffres "*abcdef*" est formé en utilisant les chiffres **1**, **2**, **3**, **4**, **5**, et **6**, une et une seule fois chacun, de façon à ce que "*abcdef*" est un multiple de **6**, "*abcde*" est un multiple de **5**, "*abcd*" est un multiple de **4**, "*abc*" est un multiple de **3** et "*ab*" est un multiple de **2**.

1. Déterminer une solution "*abcdef*." Montrer votre travail.
 2. La solution que vous avez obtenue est-elle unique (la seule possible) ? Si oui, expliquer brièvement pourquoi. Sinon, donner une deuxième solution.
- 4.** Un rectangle 3×2 est divisé en six carrés égaux, chacun contenant une bibitte. Lorsqu'une cloche sonne, chaque bibitte saute soit horizontalement soit verticalement pour atterrir dans un carré voisin; les bibittes ne peuvent pas sauter diagonalement; aussi, elles doivent rester à l'intérieur du rectangle; on ne peut pas savoir d'avance où les bibittes sauteront; enfin, toute bibitte doit changer de carré, aucune ne pouvant rester au même carré qu'avant.

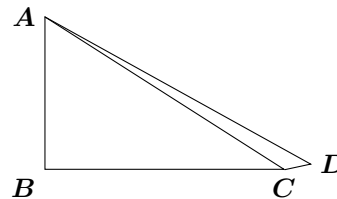
Comme exemple de façon de représenter ceci, le sextuplet **(1, 1, 1, 1, 1, 1)** dénote une situation où chaque carré contient une bibitte, que ce soit au départ ou que ce soit à un moment plus tard, comme ça pourrait bien se produire. Deux

bibittes pourraient bien se retrouver dans le même carré. Le sextuplet $(2, 2, 1, 0, 0, 1)$ représente la situation donnée par le diagramme à droite; il y a plusieurs successions de sauts pouvant donner ceci. Le premier chiffre représente un coin, le second un carré au milieu d'un côté, et ainsi de suite.

2	2	1
0	0	1

1. Quel est le nombre moyen de bibittes par carré du rectangle 3×2 , sans savoir où les bibittes sauteront ?
2. À partir de la situation initiale d'une bibitte par carré, est-ce possible que trois bibittes se retrouvent dans le même carré, après un seul son de cloche? Si vous croyez que oui, écrire un sextuplet ordonné comme les deux ci-haut pour indiquer comment ceci pourrait se produire. Sinon, expliquer brièvement pourquoi.
3. Pour un rectangle 3×2 , avec une situation initiale d'une bibitte par carré, il n'est certainement pas possible que quatre bibittes se retrouvent dans le même carré, après un seul son de cloche. Donner la taille du plus petit rectangle pour lequel aurait été possible.
4. À partir d'une situation initiale avec une bibitte dans chaque carré, il est impossible d'avoir cinq bibittes dans un carré, après un seul son de cloche, quel que soit la taille du rectangle. Dans quelques mots, expliquer pourquoi.
5. Pour le cas 3×2 , avec une bibitte dans chaque carré au départ, combien de sextuplets non uniques, comme $(1, 1, 1, 1, 1, 1)$, sont possibles après un seul son de cloche? Vous n'avez pas besoin d'en fournir la liste, bien que vous pourriez le faire.

5. Pania et Rangî font leur entraînement physique en courant une fois par semaine autour des deux enclos situés à la ferme de leur père, à Kakanui; ils vont de A à B à C à D , puis de retour à A . (Voir le schéma.) En ligne directe de A à C , la distance est de **6250** mètres. AB est plus court que BC .



1. Si $\triangle ABC$ est rectangle en ratio $3 : 4 : 5$ avec l'angle rectangle à B , déterminer les longueurs de ses côtés.
2. Si $\triangle ABC$ est rectangle en ratio $3 : 4 : 5$, avec l'angle rectangle à B , déterminer la mesure de l'angle $\angle CAB$, à une décimale près.
3. L'angle à B est effectivement un angle rectangle, et AB et BC sont de longueurs entières en mètres, mais, cette fois-ci, les côtés ne sont pas en ratio $3 : 4 : 5$. Déterminer les valeurs possibles de AB et BC .
4. L'angle à D n'est pas rectangle, mais égale plutôt 40° . CD est de longueur 600 mètres. Utiliser cette information pour déterminer la longueur de AD .

Indication : Dans tout triangle XYZ , les règles suivantes tiennent :

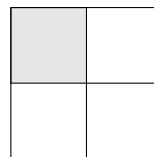
$$\text{loi de sinus : } \frac{x}{\sin X} = \frac{y}{\sin Y} = \frac{z}{\sin Z},$$

$$\text{loi de cosinus : } x^2 = y^2 + z^2 - 2yz \cos X,$$

où le côté x est opposé à l'angle X , le côté y est opposé à l'angle Y et le côté z est opposé à l'angle Z .

National Bank of New Zealand Junior Mathematics Competition, 2010 One hour allowed

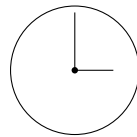
1. Rebecca is holding a seminar at the place at which she works. She wants to create an unbroken ring of tables, using a set of identical tables shaped like regular polygons. (In a regular polygon, all sides have the same length, and all angles are equal. Squares and equilateral triangles are regular.) Each table must have two sides which completely coincide with the sides of other tables, such as the shaded square table seen to the right. Rebecca plans to put items on display inside the ring where everyone can see them.



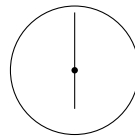
(If you cannot name a shape in this question, just give the number of sides. For example, if you think the shape has **235** sides, but don't know the name, just call it a **235-gon**—that isn't an answer to any of the parts.)

1. Rebecca first decides to use identical square tables. What is the minimum number of square tables placed beside each other so that there is an empty space in the middle?
 2. If Rebecca uses the minimal number of square tables, what shape is left bare in the middle?
 3. Rebecca considers using octagon (eight sides) shaped tables.
 - (a) What is the minimum number of octagonal tables which Rebecca must have in order for there to be a bare space in the middle so that the tables form an enclosure?
 - (b) What is the name given to the bare shape in the middle? If you can't name it, giving the number of sides will be sufficient.
 4. Apart from squares and octagons, are there any other shaped tables possible? If there are any, name one. If there isn't, say so.

2. An analogue clock displays the time with the use of two hands. Every hour the minute hand rotates **360** degrees, while the hour hand (which is shorter than the minute hand) rotates **360** degrees over a **12**-hour period. Two example times are shown below:

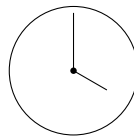


3 o'clock



6 o'clock

1. Draw a clock face which shows 9 o'clock. Make sure the hour hand is shorter than the minute hand.
2. What is the angle between the two hands at both 3 o'clock and 9 o'clock?
3. What time to the closest hour (and minute) does the following clock face show?



4. What is the angle between the two hands at the following times?
 - (a) 1 o'clock.
 - (b) 2 o'clock.
 - (c) Half past one.
 5. At what time (to the nearest minute) between 7 and 8 o'clock do the hands meet?
- 3.** A six-digit number "*abcdef*" is formed using each of the digits **1**, **2**, **3**, **4**, **5**, and **6** once and only once so that "*abcdef*" is a multiple of **6**, "*abcde*" is a multiple of **5**, "*abcd*" is a multiple of **4**, "*abc*" is a multiple of **3**, and "*ab*" is a multiple of **2**.

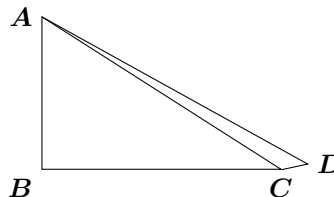
1. Find a solution for "*abcdef*." Show key working.
 2. Is the solution you found unique (the only possible one)? If it is, briefly explain why. If it isn't, give another solution.
- 4.** A 3×2 rectangle is divided up into six equal squares, each containing a bug. When a bell rings, the bugs jump either horizontally or vertically (they cannot jump diagonally and they stay within the rectangle) into a square adjacent to their previous square in any direction, although you cannot know in advance which exact square they will jump into. Every bug changes square; no bug stays put.
- As an example, the ordered sextuplet $(1, 1, 1, 1, 1, 1)$ (where this represents the result, not the movement) represents the situation where every bug jumped so that each square still had one bug in it (it could happen). Alternatively, two

bugs could also land in the same square. An example (not the only way this could happen) of this might be represented by $(2, 2, 1, 0, 0, 1)$ —see the diagram to the right. The first number in the sextuplet represents a corner square, the second represents a square on the middle of a side, and so on.

2	2	1
0	0	1

1. What is the average number of bugs per square in the 3 by 2 rectangle no matter how the bugs jump?
2. From the initial situation of one bug in every square, is it possible for three bugs to end up in the same square if the bell rings only once? If you think it is, write an ordered sextuplet like the two above where this could happen. If you think it can't happen, briefly explain why not.
3. From the initial situation of one bug in every square, it is certainly not possible in a 3×2 rectangle for four bugs to end up in the same square if the bell rings only once. Write down the dimensions of the smallest rectangle for which it would be possible.
4. From the initial situation of one bug in every square, five bugs can never end up in the same square if the bell rings only once, no matter the size of the rectangle. In a few words, explain why not.
5. In the 3×2 case, how many non-unique sextuplets (like $(1, 1, 1, 1, 1, 1)$) are possible from the initial situation of one bug in every square, if the bell rings only once? You do not have to list them, although you might like to.

5. Pania and Rangi exercise weekly by running around two paddocks on their father's farm near Kakanui from A to B to C to D and then back to A (see the diagram). In a direct line from A to C , the distance is **6250** m. AB is shorter than BC .



1. If $\triangle ABC$ is a right angled triangle in the ratio of $3 : 4 : 5$, with B at the right angle, find the lengths of the sides.
2. If $\triangle ABC$ is a right angled triangle in the ratio of $3 : 4 : 5$, with B at the right angle, find the size of $\angle CAB$ to one decimal place.
3. The angle at B is in fact a right angle, and AB and BC are whole metres in length, but the sides are not in the ratio of $3 : 4 : 5$. Find possible lengths for AB and BC .
4. The angle at D is not a right angle but is 40° , and CD is **600** m. Use this information to find the length of AD .

Hint: In any triangle XYZ , the following rules apply:

$$\text{Sine Law: } \frac{x}{\sin X} = \frac{y}{\sin Y} = \frac{z}{\sin Z},$$

$$\text{Cosine Law: } x^2 = y^2 + z^2 - 2yz \cos X,$$

where side x is opposite to angle X , side y is opposite to angle Y , and side z is opposite to angle Z .

Next follow solutions to the Baden-Württemberg Mathematics Contest, 2009, given in Skoliad 125 at [2010:194–196].

1. Find all natural numbers n such that the sum of n and the digit sum of n is **2010**.

Solution by Natalia Desy, student, SMA Xaverius 1, Palembang, Indonesia.

Consider the n -digit number $abcd$; that is, $n = 1000a + 100b + 10c + d$. Then the digit sum of n is $a + b + c + d$, and the condition is that $1000a + 100b + 10c + d + a + b + c + d = 2010$, so $1001a + 101b + 11c + 2d = 2010$. Since all variables represent digits, $a = 1$ or $a = 2$.

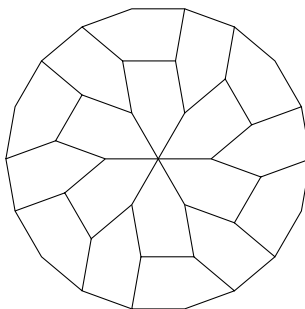
If $a = 2$, the condition is that $2002 + 101b + 11c + 2d = 2010$, so $101b + 11c + 2d = 8$. Since all variables represent digits, b and c must both be zero, and thus $d = 4$. Hence $n = 2004$.

If $a = 1$, the condition is that $1001 + 101b + 11c + 2d = 2010$, so $101b + 11c + 2d = 1009$. Again, all variable represent digits, so $11c + 2d \leq 11 \cdot 9 + 2 \cdot 9 = 117$, so $101b \geq 892$, so $b = 9$. With $a = 1$ and $b = 9$, the condition is that $1001 + 909 + 11c + 2d = 2010$, so $11c + 2d = 100$. Since d is a digit, $2d \leq 18$, so $11c \geq 82$, so $c \geq 8$ because c also is a digit. If $c = 9$, then $2d = 100 - 11c = 1$, which is impossible. If $c = 8$, then $2d = 100 - 11c = 12$, so $d = 6$. Hence $n = 1986$.

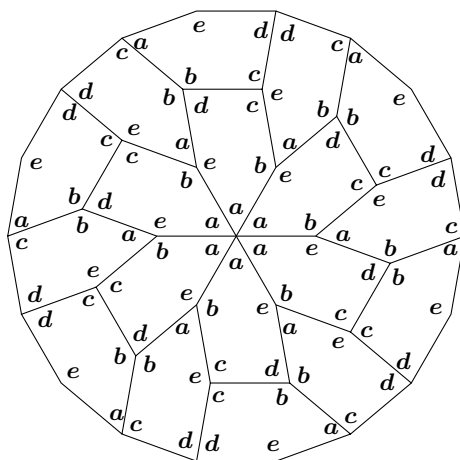
Only two natural numbers satisfy the condition, namely **2004** and **1986**.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; and RICHARD I. HESS, Rancho Palos Verdes, CA, USA.

2. A regular 18-gon can be cut into congruent pentagons as in the figure below. Determine the interior angles of such a pentagon.



Solution by Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC.



Label the angles of the congruent pentagons as in the figure. In the centre of the figure, you can see that $6a = 360^\circ$, so $a = 60^\circ$.

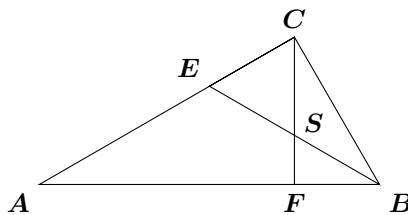
The angle sum of an 18-gon is $(18 - 2) \cdot 180^\circ = 2880^\circ$, so each interior angle in a regular 18-gon is $\frac{1}{18} \cdot 2880^\circ = 160^\circ$. In the figure, the interior angles of the 18-gon are e , $a + c$, and $2d$. Thus $e = 160^\circ$, $a + c = 160^\circ$, and $2d = 160^\circ$, so $c = 100^\circ$ and $d = 80^\circ$.

In the figure you will also find that $2b + d = 360^\circ$, so $2b = 280^\circ$, so $b = 140^\circ$.

The angles of the pentagon are $(a, b, c, d, e) = (60^\circ, 140^\circ, 100^\circ, 80^\circ, 160^\circ)$.

Also solved by VINCENT CHUNG, student, Burnaby North Secondary School, Burnaby, BC; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

3. In the figure on the right, $\triangle ABE$ is isosceles with base AB , $\angle BAC = 30^\circ$, and $\angle ACB = \angle AFC = 90^\circ$. Find the ratio of the area of $\triangle ESC$ to the area of $\triangle ABC$.



Solution by Vincent Chung, student, Burnaby North Secondary School, Burnaby, BC.

Since $\angle BAC = 30^\circ$ and $\angle ACB = 90^\circ$, $\angle ABC = 60^\circ$; that is $\triangle ABC$ is a 30° - 60° - 90° triangle. Assume without loss of generality that $AB = 2$, $AC = \sqrt{3}$, and $BC = 1$. Then the area of $\triangle ABC$ is $\frac{\sqrt{3} \cdot 1}{2} = \frac{\sqrt{3}}{2}$.

Since $\triangle ABE$ is isosceles and $\angle BAC = 30^\circ$, $\angle ABE = 30^\circ$ and $\angle AEB = 120^\circ$. Thus $\angle CES = 60^\circ$. Since $\angle ABE = 30^\circ$ and $\angle AFC = 90^\circ$,

$\angle BSF = 60^\circ$. Thus $\angle CSE = 60^\circ$, and $\triangle ESC$ is equilateral.

Since $\angle ABC = 60^\circ$ and $\angle ABE = 30^\circ$, $\angle CBE = 30^\circ$. As $\angle ACB = 90^\circ$, it follows that $\angle BEC = 60^\circ$, and $\triangle BCE$ is a 30° - 60° - 90° triangle. Since $BC = 1$, $BE = \frac{2}{\sqrt{3}}$ and $CE = \frac{1}{\sqrt{3}}$.

Since $\triangle ESC$ is equilateral with side $\frac{1}{\sqrt{3}}$, its height is

$$\sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 - \left(\frac{1}{2\sqrt{3}}\right)^2} = \sqrt{\frac{1}{3} - \frac{1}{12}} = \sqrt{\frac{1}{4}} = \frac{1}{2}.$$

Therefore the area of $\triangle ESC$ is $\frac{\frac{1}{\sqrt{3}} \cdot \frac{1}{2}}{2} = \frac{\sqrt{3}}{12}$.

The ratio of the area of $\triangle ESC$ to the area of $\triangle ABC$ is, then, $\frac{\sqrt{3}}{12} : \frac{\sqrt{3}}{2} = \frac{1}{12} : \frac{1}{2} = 1 : 6$.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; and KENRICK TSE, student, Point Grey Secondary, Vancouver, BC.

4. Given two nonzero numbers z_1 and z_2 , let z_n be $\frac{z_{n-1}}{z_{n-2}}$ for $n > 2$. Then z_1, z_2, z_3, \dots form a sequence. Prove that if you multiply any **2009** consecutive terms of the sequence, then the product is itself a member of the sequence.

Solution by Kenrick Tse, student, Point Grey Secondary, Vancouver, BC.

Using the recursive definition of z_n ,

$$\begin{aligned} z_3 &= \frac{z_2}{z_1}, & z_4 &= \frac{z_3}{z_2} = \frac{z_2/z_1}{z_2} = \frac{1}{z_1}, \\ z_5 &= \frac{z_4}{z_3} = \frac{1/z_1}{z_2/z_1} = \frac{1}{z_2}, & z_6 &= \frac{z_5}{z_4} = \frac{1/z_2}{1/z_1} = \frac{z_1}{z_2}, \\ z_7 &= \frac{z_6}{z_5} = \frac{z_1/z_2}{1/z_2} = z_1, & z_8 &= \frac{z_7}{z_6} = \frac{z_1}{z_1/z_2} = z_2, \dots \end{aligned}$$

Since $z_7 = z_1$ and $z_8 = z_2$, the sequence will repeat with period six: $z_1, z_2, \frac{z_2}{z_1}, \frac{1}{z_1}, \frac{1}{z_2}, \frac{z_1}{z_2}, z_1, z_2, \frac{z_2}{z_1}, \dots$. Note that the reciprocal of each of the first six terms is itself one of the first six terms. Therefore, whenever z_n is a member of the sequence, then so is its reciprocal, $\frac{1}{z_n}$.

The product of the first six terms is **1**. Therefore the product of any six consecutive terms is **1**. Since **2010** = **6** · **335**, it follows that the product of any **2010** terms is **1**. Therefore the product of any **2009** consecutive terms is the reciprocal of the next term, but this reciprocal is itself a member of the sequence as noted above.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; JONATHAN FENG, student, Burnaby North Secondary School, Burnaby, BC; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

5. Let $\triangle ABC$ be an isosceles triangle such that $\angle ACB = 90^\circ$. A circle with centre C cuts AC at D and BC at E . Draw the line AE . The perpendicular to AE through C cuts the line AB at F , and the perpendicular to AE through D cuts the line AB at G . Show that the length of BF equals the length of GF .

Solution by Kenrick Tse, student, Point Grey Secondary, Vancouver, BC.

Impose a coordinate system such that $C = (0, 0)$, $A = (0, 1)$, $B = (1, 0)$, $D = (0, r)$, and $E = (r, 0)$, where r is the radius of the circle. Then the line AB has the equation $y = 1 - x$.

The slope of the line AE is $-\frac{1}{r}$. Since AE is perpendicular to CF , the slope of CF must then be r . (The product of the slopes of perpendicular lines is -1 .) Therefore the equation of CF is $y = rx$. Now, F is the intersection point of AB and CF , so it is the intersection of $y = 1 - x$ and $y = rx$. Solving these two equations simultaneously yields that $rx = 1 - x$, so $x = \frac{1}{r+1}$ and, thus, $y = rx = \frac{r}{r+1}$. That is, $F = \left(\frac{1}{r+1}, \frac{r}{r+1}\right)$.

Similarly, the slope of DG is also r , so the equation of DG is $y = rx + r$. Intersection with AB , that is $y = 1 - x$, yields that $rx + r = 1 - x$, so $x = \frac{1-r}{r+1}$ and, thus, $y = 1 - x = \frac{r+1}{r+1} - \frac{1-r}{r+1} = \frac{2r}{r+1}$. That is, $G = \left(\frac{1-r}{r+1}, \frac{2r}{r+1}\right)$.

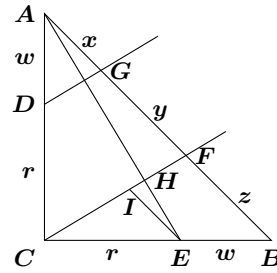
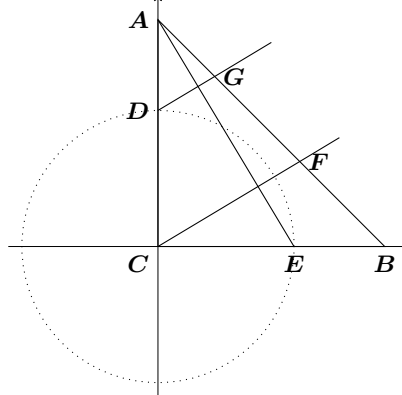
You can now calculate the distance from B to F using the Pythagorean Theorem: $|BF|^2 = \left(\frac{1}{r+1} - 1\right)^2 + \left(\frac{r}{r+1} - 0\right)^2 = \left(\frac{-r}{r+1}\right)^2 + \left(\frac{r}{r+1}\right)^2 = \frac{2r^2}{(r+1)^2}$, so $|BF| = \frac{r}{r+1}\sqrt{2}$. Likewise, $|GF|^2 = \left(\frac{1-r}{r+1} - \frac{1}{r+1}\right)^2 + \left(\frac{2r}{r+1} - \frac{r}{r+1}\right)^2 = \left(\frac{-r}{r+1}\right)^2 + \left(\frac{r}{r+1}\right)^2 = \frac{2r^2}{(r+1)^2}$, so $|GF|$ is also $\frac{r}{r+1}\sqrt{2}$. Thus $|BF| = |GF|$.

Also solved by GEOFFREY A. KANDALL, Hamden, CT, USA, who uses this geometric approach:

Let H be the point of intersection between CF and AE , and let the line through E parallel to AB meet CF at I . Since $|CD| = |CE|$ and $\triangle ABC$ is isosceles, you can label the lengths as in the figure. The problem now is to show that $y = z$.

Since $CH \perp AE$, $\triangle ACE \sim \triangle AHC$. Therefore $\frac{|AC|}{|AE|} = \frac{|AH|}{|AC|}$, so $(r+w)^2 = |AC|^2 = |AE| \cdot |AH|$. Again, since $\triangle ACE \sim \triangle CHE$, $\frac{|CE|}{|AE|} = \frac{|HE|}{|CE|}$, so $r^2 = |CE|^2 = |AE| \cdot |HE|$. Hence $\frac{(r+w)^2}{r^2} = \frac{|AE| \cdot |AH|}{|AE| \cdot |HE|} = \frac{|AH|}{|HE|}$.

Since $EI \parallel AF$, $\triangle HAF \sim \triangle HEI$, so $\frac{|AF|}{|IE|} = \frac{|AH|}{|HE|} = \frac{(r+w)^2}{r^2}$. Moreover, $\triangle CIE \sim \triangle CFB$, so $\frac{|EI|}{|FB|} = \frac{|CE|}{|CB|} = \frac{r}{r+w}$. Therefore $\frac{x+y}{z} = \frac{|AF|}{|FB|} = \frac{|AF|}{|IE|} \cdot \frac{|IE|}{|FB|} = \frac{(r+w)^2}{r^2} \cdot \frac{r}{r+w} = \frac{r+w}{r}$.



Since $DG \parallel CF$, $\triangle ADG \sim \triangle ACF$, so $\frac{x+y}{x} = \frac{w+r}{w}$, so $1 + \frac{y}{x} = 1 + \frac{r}{w}$, so $\frac{y}{x} = \frac{r}{w}$, so $\frac{x}{y} = \frac{w}{r}$, so $\frac{x}{y} + 1 = \frac{w}{r} + 1$, so $\frac{x+y}{y} = \frac{w+r}{r}$. The previous paragraph shows that $\frac{x+y}{z} = \frac{r+w}{r}$. Thus $\frac{x+y}{y} = \frac{x+y}{z}$, and it follows that $y = z$.

We leave it to the reader to judge whether this geometric argument is preferable to the analytic geometry of the first solution.

6. A gaming machine randomly selects a divisor of 2009^{2010} and displays its ones digit. Which digit should you gamble on?

Solution by Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

Since $2009 = 7^2 \cdot 41$, $2009^{2010} = 7^{4020} \cdot 41^{2010}$. Thus any divisor of 2009^{2010} has the form $7^a \cdot 41^b$, where a and b are integers such that $0 \leq a \leq 4020$ and $0 \leq b \leq 2010$.

Note that the ones digit of a product depends only on the ones digits of the factors, and use $x \equiv y$ to mean that x and y have the same ones digit. Then $41^b \equiv 1$ for all 2011 possible values of b . Moreover, $7^a \cdot 41^b \equiv 7^a \cdot 1 = 7^a$, so you just have to study the powers of 7: $7^0 = 1$, $7^1 = 7$, $7^2 = 49 \equiv 9$, $7^3 = 7^2 \cdot 7 \equiv 9 \cdot 7 = 63 \equiv 3$, and $7^4 = 7^3 \cdot 7 \equiv 3 \cdot 7 = 21 \equiv 1$.

Since the ones digit reached 1 again, the sequence of ones digits now repeats with period four. That is,

$$\begin{aligned} 7^a &\equiv 3 && \text{if } a = 3, 7, 10, \dots, 4019 \\ 7^a &\equiv 9 && \text{if } a = 2, 6, 9, \dots, 4018 \\ 7^a &\equiv 7 && \text{if } a = 1, 5, 8, \dots, 4017 \\ 7^a &\equiv 1 && \text{if } a = 0, 4, 7, \dots, 4016, 4020. \end{aligned}$$

Thus 7^a has ones digit 1 for 1006 values of a , while 7^a only has ones digit 7, 9, or 3 for 1005 values of a each.

Hence the ones digit 1 occurs slightly more often among the divisors of 2009^{2010} than any other digit, so you should gamble on the digit 1.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA.

This issue's prize of one copy of *CruX Mathematicorum* for the best solutions goes to Kenrick Tse, student, Point Grey Secondary, Vancouver, BC.

We look forward to receiving our readers' solutions to our featured contest.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as a **Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of *Crua Mathematicorum with Mathematical Mayhem*.

The interim Mayhem Editor is Shawn Godin (Cairine Wilson Secondary School, Orleans, ON). The Assistant Mayhem Editor is Lynn Miller (Cairine Wilson Secondary School, Orleans, ON). The other staff member is Monika Khbeis (Our Lady of Mt. Carmel Secondary School, Mississauga, ON).

Mayhem Problems

Veillez nous transmettre vos solutions aux problèmes du présent numéro avant le 15 septembre 2010. Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

M476. *Proposé par l'Équipe de Mayhem.*

On définit comme $s(n)$ la somme des chiffres de l'entier positif n . Par exemple, $s(2011) = 2 + 0 + 1 + 1 = 4$. Trouver le nombre d'entiers positifs de quatre chiffres n avec $s(n) = 4$.

M477. *Proposé par Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie.*

Soit m un paramètre entier tel que l'équation $x^2 - mx + m + 8 = 0$ ait une racine entière. Trouver la valeur du paramètre m .

M478. *Proposé par l'Équipe de Mayhem.*

On considère l'ensemble des points (x, y) du plan tels que

$$x^2 + y^2 - 22x - 4y + 100 = 0.$$

Soit P le point de cet ensemble pour lequel $\frac{y}{x}$ est maximal. Déterminer la distance de P à l'origine.

M479. *Proposé par Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie.*

Soit $A = 1 \cdot 2 \cdot 3 \cdot \dots \cdot 2011 = 2011!$.

- Trouver le plus grand entier positif n pour lequel 3^n est un diviseur de A .
- Trouver le nombre de zéros en queue de la représentation de A en base 10 .

M480. *Proposé par Dragoljub Milošević, Gornji Milanovac, Serbie.*

Soit x , y et k trois nombres positifs tels que $x^2 + y^2 = k$. Trouver la valeur minimale possible de $x^6 + y^6$ en fonction de k .

M481. *Proposé par Edward T.H. Wang, Université Wilfrid Laurier, Waterloo, ON.*

On suppose que a , b et x sont des nombres réels avec $ab \neq 0$ et $a + b \neq 0$. Si $\frac{\sin^4 x}{a} + \frac{\cos^4 x}{b} = \frac{1}{a+b}$, trouver la valeur de $\frac{\sin^6 x}{a^3} + \frac{\cos^6 x}{b^3}$ en fonction de a et b .

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M476. *Proposed by the Mayhem Staff*

Define $s(n)$ to be the sum of the digits of the positive integer n . For example, $s(2011) = 2 + 0 + 1 + 1 = 4$. Determine the number of four-digit positive integers n with $s(n) = 4$.

M477. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania*

Let m be an integer parameter such that the equation $x^2 - mx + m + 8 = 0$ has one integer root. Determine the value of the parameter m .

M478. *Proposed by the Mayhem Staff*

Consider the set of points (x, y) in the plane such that

$$x^2 + y^2 - 22x - 4y + 100 = 0.$$

Let P be the point in this set for which $\frac{y}{x}$ is the largest. Determine the distance of P from the origin.

M479. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania*

Let $A = 1 \cdot 2 \cdot 3 \cdot \dots \cdot 2011 = 2011!$.

- Determine the largest positive integer n for which 3^n divides exactly into A .
- Determine the number of zeroes at the end of the base 10 representation of A .

M480. *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia*

Let x , y , and k be positive numbers such that $x^2 + y^2 = k$. Determine the minimum possible value of $x^6 + y^6$ in terms of k .

M481. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON*

Suppose that a , b , and x are real numbers with $ab \neq 0$ and $a + b \neq 0$. If $\frac{\sin^4 x}{a} + \frac{\cos^4 x}{b} = \frac{1}{a+b}$, determine the value of $\frac{\sin^6 x}{a^3} + \frac{\cos^6 x}{b^3}$ in terms of a and b .

Mayhem Solutions

M438. *Proposed by the Mayhem Staff.*

Find all pairs of real numbers (x, y) such that

$$x^2 + (y^2 - y - 2)^2 = 0.$$

Solution by Allen Zhu, Conestoga High School, Berwyn, PA, USA.

For $x^2 + (y^2 - y - 2)^2 = 0$ to be true with $x, y \in \mathbb{R}$, both $x = 0$ and $y^2 - y - 2 = (y - 2)(y + 1) = 0$ must hold, so the solutions are: $(0, 2)$, $(0, -1)$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ADAMAS AQSA F.S., student, SMA Kharisma Bangsa, Indonesia; JACLYN CHANG, student, University of Calgary, Calgary, AB; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; ANTONIO LEDESMA LÓPEZ, Instituto de Educación Secundaria No. 1, Requena-Valencia, Spain; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA.

M439. *Proposed by Eric Schmutz, Drexel University, Philadelphia, PA, USA.*

Determine the positive integer x for which $\frac{1}{\log_2 x} + \frac{1}{\log_5 x} = \frac{1}{100}$.

Solution by Gusnadi Wiyoga, student, SMPN 8, Yogyakarta, Indonesia.

Note that since $\log_a b \cdot \log_b a = 1$ then $\log_a b = \frac{1}{\log_b a}$. Consequently, we have that $\frac{1}{\log_2 x} = \log_x 2$ and $\frac{1}{\log_5 x} = \log_x 5$. Thus,

$$\begin{aligned}\log_x 2 + \log_x 5 &= \frac{1}{100} \\ \log_x 10 &= \frac{1}{100} \\ x^{\frac{1}{100}} &= 10 \\ x &= 10^{100}\end{aligned}$$

So, the positive integer x is 10^{100} .

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; G.C. GREUBEL, Newport News, VA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; YOUNGHUAN JUNG, The Woodlands School, Mississauga, ON; MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; DRAGOLJUB MILOŠEVIĆ, Gornji Milanovac, Serbia; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; RICARD PEIRÓ, IES “Abastos”, Valencia, Spain; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; ALEJANDRO S. CONCEPCIÓN RODRÍGUEZ, student, University of Las Palmas de Gran Canaria; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and ALLEN ZHU, Conestoga High School, Berwyn, PA, USA.

M441. *Proposed by Katherine Tsuji and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

What is the maximum number of non-attacking kings that can be placed on an $n \times n$ chessboard? (A “king” is a chess piece that can move horizontally, vertically, or diagonally from one square to an adjacent square.)

Solution by Bruno Salgueiro Fanego, Viveiro, Spain.

We will denote by (i, j) the square situated on row i and column j , where $1 \leq i, j \leq n$.

Case I: If n is even

In each of the n rows, we can place at most $\frac{n}{2}$ kings, because between any two neighbouring kings there must exist at least one free square. Analogously, in each of the n columns we can place at most $\frac{n}{2}$ kings too. If we place a king at the squares (i, j) , where i, j are odd numbers then we have $\frac{n}{2}$ kings in each column and in each row that are non-attacking. Thus, in the $n \times n$ chessboard with n even, we can place a maximum of $\frac{n}{2} \cdot \frac{n}{2} = \left(\frac{n}{2}\right)^2$ kings.

Case II: If n is odd

In each of the n rows we can place a maximum number of $\frac{n+1}{2}$ non-attacking kings. Similar to above, if we want to have the maximum number of non-attacking kings,

then we must place a king at the squares (i, j) , where i, j are odd numbers giving $\frac{n+1}{2}$ kings in each column and in each row that are non-attacking. Thus, in the $n \times n$ chessboard with n odd, we can place a maximum of $\frac{n+1}{2} \cdot \frac{n+1}{2} = \left(\frac{n+1}{2}\right)^2$ kings.

Also solved by JACLYN CHANG, student, University of Calgary, Calgary, AB; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; ANTONIO LEDESMA LÓPEZ, Instituto de Educación Secundaria No. 1, Requena-Valencia, Spain; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; ALLEN ZHU, Conestoga High School, Berwyn, PA, USA; and the proposers.

M443. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Let $\lfloor x \rfloor$ denote the greatest integer not exceeding x . For example, $\lfloor 3.1 \rfloor = 3$ and $\lfloor -1.4 \rfloor = -2$. Let $\{x\}$ denote the fractional part of the real number x (that is, $\{x\} = x - \lfloor x \rfloor$). For example, $\{3.1\} = 0.1$ and $\{-1.4\} = 0.6$. Find all positive real numbers x such that

$$\left\{ \frac{2x+3}{x+2} \right\} + \left\lfloor \frac{2x+1}{x+1} \right\rfloor = \frac{14}{9}.$$

Solution by Adamas Aqsa F.S., student, SMA Kharisma Bangsa, Indonesia.

First, note that $\left\lfloor \frac{2x+3}{x+2} \right\rfloor = \left\lfloor 2 - \frac{1}{x+2} \right\rfloor$. Similarly, $\left\lfloor \frac{2x+1}{x+1} \right\rfloor = \left\lfloor 2 - \frac{1}{x+1} \right\rfloor$. Since x is a positive real number, $0 < \frac{1}{x+1} < 1$ and $0 < \frac{1}{x+2} < 1$, which implies that $1 < 2 - \frac{1}{x+2} < 2$. This means that $\left\lfloor 2 - \frac{1}{x+2} \right\rfloor = 1$. Also, as $1 < 2 - \frac{1}{x+1} < 2$, then that means that $\left\lfloor 2 - \frac{1}{x+1} \right\rfloor = 1$. Now, assembling the values we obtained plus the definition of $\{n\}$, we have

$$\begin{aligned} \left\{ \frac{2x+3}{x+2} \right\} + \left\lfloor \frac{2x+1}{x+1} \right\rfloor &= \frac{14}{9} \\ \frac{2x+3}{x+2} - \left\lfloor \frac{2x+3}{x+2} \right\rfloor + \left\lfloor \frac{2x+1}{x+1} \right\rfloor &= \frac{14}{9} \\ \frac{2x+3}{x+2} - 1 + 1 &= \frac{14}{9} \\ \frac{2x+3}{x+2} &= \frac{14}{9} \\ 9(2x+3) &= 14(x+2) \\ 4x &= 1 \\ x &= \frac{1}{4} \end{aligned}$$

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain;

SAMUEL GÓMEZ MORENO, *Universidad de Jaén, Jaén, Spain*; RICARD PEIRÓ, *IES “Abastos”, Valencia, Spain*; PAOLO PERFETTI, *Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy*; NECULAI STANCIU, *George Emil Palade Secondary School, Buzău, Romania*; EDWARD T.H. WANG, *Wilfrid Laurier University, Waterloo, ON*; GUSNADI WIYOGA, *student, SMPN 8, Yogyakarta, Indonesia*; KONSTANTINE ZELATOR, *University of Pittsburgh, Pittsburgh, PA, USA*; and ALLEN ZHU, *Conestoga High School, Berwyn, PA, USA*.

M444. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let \mathbf{a} and \mathbf{b} be real numbers. Prove that

$$\sqrt{\mathbf{a}^2 + \mathbf{b}^2 + 6\mathbf{a} - 2\mathbf{b} + 10} + \sqrt{\mathbf{a}^2 + \mathbf{b}^2 - 6\mathbf{a} + 2\mathbf{b} + 10} \geq 2\sqrt{10}.$$

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.

Let $\mathbf{A} = (-3, 1)$, $\mathbf{B} = (3, -1)$ and $\mathbf{P} = (\mathbf{a}, \mathbf{b})$. Then

$$\sqrt{\mathbf{a}^2 + \mathbf{b}^2 + 6\mathbf{a} - 2\mathbf{b} + 10} = \sqrt{(\mathbf{a} + 3)^2 + (\mathbf{b} - 1)^2} = \mathbf{PA},$$

$$\sqrt{\mathbf{a}^2 + \mathbf{b}^2 - 6\mathbf{a} + 2\mathbf{b} + 10} = \sqrt{(\mathbf{a} - 3)^2 + (\mathbf{b} + 1)^2} = \mathbf{PB},$$

$$\mathbf{AB} = \sqrt{(-6)^2 + 2^2} = \sqrt{40} = 2\sqrt{10}.$$

Hence the given inequality follows from the triangle inequality,

$$\mathbf{PA} + \mathbf{PB} \geq \mathbf{AB}$$

$$\sqrt{\mathbf{a}^2 + \mathbf{b}^2 + 6\mathbf{a} - 2\mathbf{b} + 10} + \sqrt{\mathbf{a}^2 + \mathbf{b}^2 - 6\mathbf{a} + 2\mathbf{b} + 10} \geq 2\sqrt{10}.$$

Note that equality holds if and only if \mathbf{P} is on the line segment, \mathbf{l} , connecting \mathbf{A} and \mathbf{B} . Since the slope of \mathbf{l} is $-\frac{1}{3}$, its equation is $\mathbf{y} = -\frac{1}{3}\mathbf{x}$. Hence, equality holds if and only if $\mathbf{a} = -3\mathbf{b}$, where $-1 \leq \mathbf{b} \leq 1$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; SHAMIL ASGARLI, *student, Burnaby South Secondary School, Burnaby, BC*; RICHARD I. HESS, *Rancho Palos Verdes, CA, USA*; BRUNO SALGUEIRO FANEGO, *Viveiro, Spain*; MARÍA ASCENSIÓN LÓPEZ CHAMORRO, *I.B. Leopoldo Cano, Valladolid, Spain*; SAMUEL GÓMEZ MORENO, *Universidad de Jaén, Jaén, Spain*; PEDRO HENRIQUE O. PANTOJA, *student, UFRN, Brazil*; PAOLO PERFETTI, *Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy*; RICARD PEIRÓ, *IES “Abastos”, Valencia, Spain*; CAO MINH QUANG, *Nguyen Binh Khiem High School, Vinh Long, Vietnam*; NECULAI STANCIU, *George Emil Palade Secondary School, Buzău, Romania*; GUSNADI WIYOGA, *student, SMPN 8, Yogyakarta, Indonesia*; ALLEN ZHU, *Conestoga High School, Berwyn, PA, USA*; and the proposer.

Problem of the Month

Ian VanderBurgh

Last month, we talked about averages and looked at a couple of related problems. This month, we'll continue by looking at two more problems on this topic.

Problem 1 (2010 Sun Life Financial Canadian Open Mathematics Challenge) On a calculus exam, the average of those who studied was 90% and the average of those who did not study was 40%. If the average of the entire class was 85%, what percentage of the class did not study?

Solution 1 to Problem 1. Let x be the number of people who studied for the exam and let y be the number of people who did not study. We assume without loss of generality that the exam was out of 100 marks.

Since the average of those who studied was 90%, then those who studied obtained a total of $90x$ marks. Since the average of those who did not study was 40%, then those who did not study obtained a total of $40y$ marks. Since the overall average was 85%, then $\frac{90x+40y}{x+y} = 85$. Therefore, $90x + 40y = 85x + 85y$ or $5x = 45y$ or $x = 9y$.

Therefore, $x : y = 9 : 1 = 90 : 10$. This means that 10% of the class did not study for the exam. \square

This is a good solution, but doesn't take advantage of what we looked at last month related to weighted averages. Let's try this approach.

Solution 2 to Problem 1. The combined average (85%) splits the two partial averages (40% and 90%) in the ratio $45 : 5$ or $9 : 1$. This means that the number of people in the two categories must be in the inverse ratio, or $1 : 9$. This is the ratio of the number of students who did not study to the number of students who did study. This ratio is equivalent to $10 : 90$.

Therefore, the percentage of the class that did not study is **10%**. \square

That was much easier, wasn't it? Here is a second problem for this month involving averages.

Problem 2 (2010 Cayley Contest) Connie has a number of gold bars, all of different weights. She gives the 24 lightest bars, which weigh 45% of the total weight, to Brennan. She gives the 13 heaviest bars, which weigh 26% of the total weight, to Maya. She gives the rest of the bars to Blair. How many bars did Blair receive?

- (A) 14 (B) 15 (C) 16 (D) 17 (E) 18

This problem is one of my favourites from the past couple of years. One reason that I like this problem is that it's not at all obvious that there is enough information to solve the problem. In fact, a couple of people involved in the contest

creation process were convinced that there was something missing! However, there is enough information to solve Problem 2. Give it a try before reading on!

Solution to Problem 2. Connie gives 24 bars that account for 45% of the total weight to Brennan. Thus, each of these 24 bars accounts for an average of $\frac{45}{24}\% = \frac{15}{8}\% = 1.875\%$ of the total weight.

Connie gives 13 bars that account for 26% of the total weight to Maya. Thus, each of these 13 bars accounts for an average of $\frac{26}{13}\% = 2\%$ of the total weight.

Since each of the bars that she gives to Blair is heavier than each of the bars given to Brennan (which were the 24 lightest bars) and is lighter than each of the bars given to Maya (which were the 13 heaviest bars), then the average weight of the bars given to Blair must be larger than 1.875% and smaller than 2%.

Note that the bars given to Blair account for $100\% - 45\% - 26\% = 29\%$ of the total weight. If there were 14 bars accounting for 29% of the total weight, the average weight would be $\frac{29}{14}\% \approx 2.07\%$, which is too large. Thus, there must be more than 14 bars accounting for 29% of the total weight.

If there were 15 bars accounting for 29% of the total weight, the average weight would be $\frac{29}{15}\% \approx 1.93\%$, which is in the correct range. If there were 16 bars accounting for 29% of the total weight, the average weight would be $\frac{29}{16}\% \approx 1.81\%$, which is too small. The same would be true if there were 17 or 18 bars.

Therefore, Blair must have received 15 bars. \square

When we read this problem for the first time, the fact that averages might enter in is not clear. But averages are useful in a pretty natural way, and perhaps in a “real life” way too. Often, using an average is a great way of estimating the size of objects in a collection, and that’s exactly what we’ve done here. We don’t know the actual sizes of the gold bars, but we can estimate and compare by using averages.

As one final note on this problem, can you find the piece of information in this problem that seemed important, but was never used?

THE OLYMPIAD CORNER

No. 292

R.E. Woodrow

We begin the section of solutions from our readers with the file of solutions to problems of the Thai Mathematical Olympiad Examinations 2006, Selected problems, given at [2010 : 83–84].

1. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function satisfying

$$f(x^2 + x + 3) + 2f(x^2 - 3x + 5) = 6x^2 - 10x + 17$$

for all real x . Find $f(85)$.

Solved by Arkady Alt, San Jose, CA, USA; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; and Titu Zvonaru, Comănești, Romania. We use an edited version of the solution of Alt.

Let $p(x) = x^2 + x + 3$, $q(x) = x^2 - 3x + 5$. Since

$$p(1-x) = (1-x)^2 + (1-x) + 3 = x^2 - 3x + 5 = q(x),$$

$q(1-x) = p(1-(1-x)) = p(x)$ and

$$6(1-x)^2 - 10(1-x) + 17 = 6x^2 - 2x + 13$$

then

$$6x^2 - 2x + 13 = f(p(1-x)) + 2f(q(1-x)) = f(q(x)) + 2f(p(x))$$

and from the system of equations

$$\begin{cases} f(p(x)) + 2f(q(x)) = 6x^2 - 10x + 17 \\ 2f(p(x)) + f(q(x)) = 6x^2 - 2x + 13 \end{cases}$$

we obtain

$$\begin{aligned} 3f(p(x)) &= 2(2f(p(x)) + f(q(x))) - (f(p(x)) + 2f(q(x))) \\ &= 2(6x^2 - 2x + 13) - (6x^2 - 10x + 17) \\ &= 6x^2 + 6x + 9 \iff f(p(x)) = 2p(x) - 3. \end{aligned}$$

Since $p(x) = x^2 + x + 3 \geq \frac{11}{4}$ then for any $y \geq \frac{11}{4}$ there is x such that $p(x) = y$ and, therefore, for any $y \geq \frac{11}{4}$ we have

$$f(y) = f(p(x)) = 2p(x) - 3 = 2y - 3.$$

Hence, $f(85) = 167$.

2. Evaluate

$$\sum_{k=84}^{8000} \binom{k}{84} \binom{8084-k}{84}.$$

Solved by Arkady Alt, San Jose, CA, USA; Oliver Geupel, Brühl, NRW, Germany; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Wang's combinatorial argument.

More generally, we consider the sum

$$S(n, d) = \sum_{k=d}^{n-d} \binom{k}{d} \binom{n-k}{d}$$

where $n, d \in \mathbf{N}$ such that $d \leq \lfloor \frac{n}{2} \rfloor$. We use a 2-way counting argument to show that $S(n, d) = \binom{n+1}{2d+1}$. Let $T = \{0, 1, 2, \dots, n\}$. Then $|T| = n + 1$ so $\binom{n+1}{2d+1}$ is the number of $(2d + 1)$ -subsets of T . On the other hand, each $(2d + 1)$ -subset of T is completely determined by first choosing a positive integer k to be the *middle* number, $d \leq k \leq n - d$ and then select any d numbers from the k -subset $\{0, 1, 2, \dots, k - 1\}$ and any d numbers from the $(n - k)$ -subset $\{k + 1, k + 2, \dots, n\}$. This procedure which is possible since $d \leq k$ and $d \leq n - k$ would yield *all* the $(2d + 1)$ -subsets of T . Hence

$$S(n, d) = \sum_{k=d}^{n-d} \binom{k}{d} \binom{n-k}{d} = \binom{n+1}{2d+1}$$

follows. In particular, the value of the given summation is $S(8084, 84) = \binom{8085}{169}$.

3. Find all integers such n that $n^2 + 59n + 881$ is a perfect square.

Solved by David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comănești, Romania. We give the solution by Zonvaru.

If $n^2 + 59n + 881$ is a perfect square, then $4(n^2 + 59n + 881)$ is a perfect square. We have

$$\begin{aligned} k^2 &= 4n^2 + 4 \cdot 59n + 3524 \\ \Leftrightarrow k^2 &= 4n^2 + 4 \cdot 59n + 59^2 + 3524 - 59^2 \\ \Leftrightarrow k^2 &= (2n + 59)^2 + 43 \Leftrightarrow (k - 2n - 59)(k + 2n + 59) = 43. \end{aligned}$$

Since k, n are integers and 43 is prime, we have the following possibilities

$$\begin{aligned} \text{(i)} \quad & \begin{cases} k - 2n - 59 = 1 \\ k + 2n + 59 = 43 \end{cases} \Leftrightarrow k = 22, n = -19; \\ \text{(ii)} \quad & \begin{cases} k - 2n - 59 = -1 \\ k + 2n + 59 = -43 \end{cases} \Leftrightarrow k = -22, n = -40; \end{aligned}$$

$$(iii) \begin{cases} k - 2n - 59 = 43 \\ k + 2n + 59 = 1 \end{cases} \Leftrightarrow k = 22, n = -40;$$

$$(iv) \begin{cases} k - 2n - 59 = -43 \\ k + 2n + 59 = -1 \end{cases} \Leftrightarrow k = -22, n = -19.$$

For $n = -19$ and $n = -40$, $n^2 + 59n + 881 = 11^2$.

It results that $n^2 + 59n + 881$ is a perfect square for $n = -19$ and $n = -40$.

4. Find the least positive integer n such that

$$\sqrt{3}z^{n+1} - z^n - 1 = 0$$

has a complex root z with $|z| = 1$.

Solved by Arkady Alt, San Jose, CA, USA.

Let $z = \cos \varphi + i \sin \varphi$, $\varphi \in [0, 2\pi)$. Since

$$\begin{aligned} 1 + \cos n\varphi + i \sin n\varphi &= 2 \cos^2 \frac{n\varphi}{2} + 2i \cos \frac{n\varphi}{2} \sin \frac{n\varphi}{2} \\ &= 2 \cos \frac{n\varphi}{2} \left(\cos \frac{n\varphi}{2} + i \sin \frac{n\varphi}{2} \right) \end{aligned}$$

then

$$\begin{aligned} \sqrt{3}z^{n+1} &= 1 + \cos n\varphi + i \sin n\varphi \\ \Leftrightarrow \sqrt{3}z^{n+1} &= 2 \cos \frac{n\varphi}{2} \left(\cos \frac{n\varphi}{2} + i \sin \frac{n\varphi}{2} \right) \end{aligned}$$

yields

$$\begin{aligned} \sqrt{3}|z^{n+1}| &= 2 \left| \cos \frac{n\varphi}{2} \left(\cos \frac{n\varphi}{2} + i \sin \frac{n\varphi}{2} \right) \right| \\ \Leftrightarrow \sqrt{3}|z|^{n+1} &= 2 \left| \cos \frac{n\varphi}{2} \right| \Leftrightarrow \frac{\sqrt{3}}{2} = \left| \cos \frac{n\varphi}{2} \right| \\ \Leftrightarrow \frac{\sqrt{3}}{2} &= \left| \cos \frac{n\varphi}{2} \right| \Leftrightarrow \frac{3}{2} = 2 \cos^2 \frac{n\varphi}{2} \Leftrightarrow \cos n\varphi = \frac{1}{2}. \end{aligned}$$

If $\cos n\varphi = \frac{1}{2}$ then $\sin n\varphi = \pm \frac{\sqrt{3}}{2}$ and $z^n = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

Since $z^n + 1 = \frac{3}{2} \pm \frac{i\sqrt{3}}{2}$, $z^{n+1} = z \cdot z^n = z \left(\frac{1}{2} \pm \frac{i\sqrt{3}}{2} \right)$ then

$$\begin{aligned} \sqrt{3}z^{n+1} = z^n + 1 &\Leftrightarrow \sqrt{3}z \left(\frac{1}{2} \pm \frac{i\sqrt{3}}{2} \right) = \frac{3}{2} \pm \frac{i\sqrt{3}}{2} \\ \Leftrightarrow z \left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}i \right) &= \left(\frac{\sqrt{3}}{2} \pm \frac{1}{2}i \right) \\ \Leftrightarrow z \left(\cos \left(\pm \frac{\pi}{3} \right) + i \sin \left(\pm \frac{\pi}{3} \right) \right) &= \cos \left(\pm \frac{\pi}{6} \right) + i \sin \left(\pm \frac{\pi}{6} \right) \\ \Leftrightarrow z &= \cos \left(\mp \frac{\pi}{6} \right) + i \sin \left(\mp \frac{\pi}{6} \right). \end{aligned}$$

Hence, $\pm\frac{\pi}{3} = \mp\frac{n\pi}{6} + 2k\pi \iff n = \pm 12k - 2, k \in \mathbb{Z}$ and, therefore, the smallest positive integer n satisfying this equation is $n = 10$. So, a necessary condition is $n \geq 10$.

Let $z = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$ and $n = 10$ then

$$z^{10} = \cos\frac{10\pi}{6} + i\sin\frac{10\pi}{6} = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - \frac{i\sqrt{3}}{2},$$

$$z^{11} = \cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6} = \frac{\sqrt{3}}{2} - \frac{1}{2}i,$$

$$\text{and } \sqrt{3}z^{n+1} = \frac{3}{2} - \frac{\sqrt{3}}{2}i = 1 + \frac{1}{2} - \frac{\sqrt{3}}{2}i = 1 + z^n.$$

Thus, the least positive integer n such that $\sqrt{3}z^{n+1} - z^n - 1 = 0$ has a complex root with $|z| = 1$ is **10**.

5. Let p_k denote the k^{th} prime number. Find the remainder when

$$\sum_{k=2}^{2550} p_k^{p_k^4-1}$$

is divided by **2550**.

Solved by Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA.

First we factor **2550** into prime powers: $2550 = (255) \cdot (10) = 5 \cdot 51 \cdot 10 = 5 \cdot 3 \cdot 17 \cdot 2 \cdot 5$. So, we easily see that

$$2550 = 2 \cdot 3 \cdot 5^2 \cdot 17. \quad (1)$$

First, note that $p_2 = 3$, $p_3 = 5$, and $p_7 = 17$. We will first find the congruence classes the three integers $p_2^{p_2^4-1}$, $p_3^{p_3^4-1}$, and $p_7^{p_7^4-1}$; belong to modulo **2550**. We start with $p_2^{p_2^4-1} = 3^{3^4-1} = 3^{80}$.

Clearly

$$3^{80} \equiv 0 \pmod{3} \quad \text{and} \quad 3^{80} \equiv 1 \pmod{2}. \quad (2)$$

By Fermat's Little Theorem, $3^{16} \equiv 1 \pmod{17}$; and so

$$3^{80} = (3^{16})^5 \equiv 1^5 \equiv 1 \pmod{17}. \quad (3)$$

Consider 3^{80} modulo $5^2 = 25$. First

$$3^8 \equiv (3^4)^2 \equiv (81)^2 \equiv (6)^2 \equiv 36 \equiv 11 \pmod{25}.$$

So that,

$$\begin{aligned} 3^{80} &= (3^8)^{10} \equiv (11)^{10} \equiv (11^2)^5 \equiv (121)^5 \equiv (-4)^5 \\ &\equiv (-4)^4 \cdot (-4) \equiv (256)(-4) \equiv 6 \cdot (-4) \\ &\equiv -24 \equiv 1 \pmod{25} \end{aligned} \quad (4)$$

Altogether we have, from (2), (3), (4), that

$$\left\{ \begin{array}{l} 3^{80} \equiv 1 \pmod{2}, 3^{80} \equiv 1 \pmod{17}, \\ 3^{80} \equiv 1 \pmod{5^2} \text{ and } 3^{80} \equiv 0 \pmod{3} \end{array} \right\} \quad (5)$$

Since $2, 17, 5^2$ are pairwise relatively prime; (5) shows that $3^{80} - 1 \equiv 0 \pmod{2 \cdot 5^2 \cdot 17}$; $3^{80} \equiv 1 \pmod{2 \cdot 5^2 \cdot 17}$. Thus, $3^{80} = 2 \cdot 5^2 \cdot 17 \cdot k + 1$, for some positive integer k . Since $k = 3 \cdot k' + r$ where $r \in \{0, 1, 2\}$ and $k' \in \mathbb{Z}^+$, and since $3^{80} \equiv 0 \pmod{3}$ and $42 \cdot 5^2 \cdot 17 \equiv 2 \cdot 1 \cdot 2 \equiv 4 \equiv 1 \pmod{3}$ we see that

$$\begin{aligned} 3^{80} &= 2 \cdot 3 \cdot 5^2 \cdot 17 \cdot k' + 2 \cdot 5^2 \cdot 17 \cdot r + 1 \\ &\equiv r + 1 \equiv 0 \pmod{3} \end{aligned}$$

so $r = 2$. Thus,

$$\begin{aligned} 3^{80} &= 2 \cdot 3 \cdot 5^2 \cdot 17 \cdot k' + 2 \cdot 5^2 \cdot 17 \cdot 2 + 1 \\ &= 2550 \cdot k' + 1700 + 1 = 2550 \cdot k' + 1701 \end{aligned}$$

we have shown that $3^{80} = 1701 \pmod{2550}$

$$p_2 = 3, \quad p_2^{p_2^4 - 1} \equiv 1701 \pmod{2550}. \quad (6)$$

Next, consider $p_3^{p_3^4 - 1} = 5^{5^4 - 1} = 5^{624} = 5^{(5^2 - 1)(5^2 + 1)} = 5^{24 \cdot 26}$. Clearly

$$\begin{aligned} 5^{624} &= 1 \pmod{3} \text{ (since } 5^2 \equiv 1 \pmod{3}\text{)}; \\ 5^{624} &\equiv 0 \pmod{5^2}; \\ 5^{624} &\equiv 1 \pmod{2}; \text{ and} \\ 5^{624} &= 5^{24 \cdot 26} = 5^{8 \cdot 2 \cdot 3 \cdot 13} = (5^{16})^{39} \equiv 1^{39} \equiv 1 \pmod{17} \end{aligned}$$

by Fermat's Theorem.

We see that $5^{624} \equiv 1 \pmod{3}$, $5^{624} \equiv 1 \pmod{2}$, $5^{624} \equiv 1 \pmod{17}$ which implies that $5^{624} \equiv 1 \pmod{2 \cdot 3 \cdot 17}$ so $5^{624} = 2 \cdot 3 \cdot 17 \cdot l + 1$ for some positive integer l . Observe that $2 \cdot 3 \cdot 17 \equiv 6(-8) \equiv -48 \equiv -(-2) \equiv 2 \pmod{25}$. And so, $5^{624} = 2 \cdot 3 \cdot 17 \cdot l + 1 \equiv 2l + 1 \pmod{25}$. But $5^{624} \equiv 0 \pmod{25}$; and so we must have $2l + 1 \equiv 0 \pmod{25} \Leftrightarrow l \equiv 12 \pmod{25}$; $l = 25 \cdot L + 12$; for some $L \in \mathbb{Z}^+$. Thus,

$$5^{624} = 2 \cdot 3 \cdot 17 \cdot (25L + 12) + 1 = \underbrace{2 \cdot 3 \cdot 5^2 \cdot 17 \cdot L}_{2550} + 2 \cdot 3 \cdot 17 \cdot 12 + 1.$$

And so, $5^{624} \equiv 2 \cdot 3 \cdot 17 \cdot 12 + 1 \equiv 1225 \pmod{2550}$

$$p_3 = 5, \quad p_3^{p_3^4 - 1} \equiv 1225 \pmod{2550}. \quad (7)$$

Next consider $p_7 = 17$. We have,

$$p_7^{p_7^4 - 1} = 17^{17^4 - 1} = 17^{(17^2 - 1)(17^2 + 1)} = 17^{16 \cdot 18 \cdot 290}.$$

And

$$\begin{aligned} 17^{16 \cdot 18 \cdot 290} &\equiv 1 \pmod{2} & \text{and} \\ 17^{16 \cdot 18 \cdot 290} &\equiv 1 \pmod{3} & \text{(since } 17^2 \equiv 1 \pmod{3}\text{)} \end{aligned}$$

$$\text{And also, } 17^{16 \cdot 18 \cdot 290} \equiv 0 \pmod{17}$$

Consider $17^{16 \cdot 18 \cdot 290}$ modulo $5^2 = 25$. Observe that, $17^2 = 289 \equiv 275 + 14 \equiv 14 \pmod{25}$; or equivalently, $17^2 \equiv -11 \pmod{25}$

$$\begin{aligned} 17^{16} &\equiv (17^2)^8 \equiv (-11)^8 \equiv [(-11)^2]^4 \equiv (121)^4 \\ &\equiv (-4)^4 \equiv 256 \equiv 1 \pmod{25}. \end{aligned}$$

And thus $17^{16 \cdot 18 \cdot 290} \equiv [(17^{16})^{18 \cdot 290}] \equiv 1^{18 \cdot 290} \equiv 1 \pmod{25}$. We have $17^{16 \cdot 18 \cdot 290} \equiv 1 \pmod{2}$, $17^{16 \cdot 18 \cdot 290} \equiv 1 \pmod{3}$, and $17^{16 \cdot 18 \cdot 290} \equiv 1 \pmod{5^2}$, which implies that $17^{16 \cdot 18 \cdot 290} \equiv 1 \pmod{2 \cdot 3 \cdot 5^2}$; and so $17^{16 \cdot 18 \cdot 290} = 2 \cdot 3 \cdot 5^2 \cdot m + 1$; for some $m \in \mathbb{Z}^+$. And since $2 \cdot 3 \cdot 5^2 \equiv 2 \cdot 3 \cdot 25 \equiv 2 \cdot 3 \cdot 8 \equiv 2 \cdot 3 \cdot 8 \equiv 48 \equiv -3 \pmod{17}$. We see that we must have $2 \cdot 3 \cdot 5^2 \cdot m + 1 \equiv 0 \pmod{17}$; $-3m + 1 \equiv 0 \pmod{17}$; $3m \equiv 1 \pmod{17}$; $m \equiv 6 \pmod{17}$. So that $m = 6 + 17 \cdot M$; for some $M \in \mathbb{Z}^+$. Altogether,

$$\begin{aligned} 17^{16 \cdot 18 \cdot 290} &= 2 \cdot 3 \cdot 5^2 \cdot (6 + 17M) + 1 \\ &= \underbrace{2 \cdot 3 \cdot 5^2 \cdot 17 \cdot M}_{2550} + 2 \cdot 3 \cdot 5^2 \cdot 6 + 1 = 2550M + 901 \end{aligned}$$

We have shown that,

$$p_7 = 17, \quad p_7^{p_7^4 - 1} \equiv 901 \pmod{2550} \quad (8)$$

We now come to the last part of the problem by considering $p_k^{p_k^4 - 1}$; where $k \geq 2$ and $k \neq 2, 3, 7$. In other words, $p_k \neq 3, 5, 17$; and $p_k > 2$.

Since p_k is odd, we have $p_k^2 \equiv 1 \pmod{8}$; from which it follows (just write $p_k^2 = 8\lambda + 1$; and square both sides) that

$$p_k^4 \equiv 1 \pmod{16}. \quad (9)$$

By Fermat's Little Theorem, we also have (since $p_k \neq 5$),

$$p_k^4 \equiv 1 \pmod{5}. \quad (10)$$

From (9) and (10) it follows that

$$\begin{aligned} p_k^4 - 1 &\equiv 0 \pmod{16 \cdot 5}; \\ p_k^4 - 1 &= 80t, \quad \text{for some positive integer } t. \end{aligned} \quad (11)$$

Thus, $p_k^{p_k^4 - 1} = p_k^{80t}$, which implies

$$\left\{ \begin{array}{l} p_k^{80t} \equiv 1 \pmod{2}, \quad p_k^{80t} \equiv 1 \pmod{3}, \\ \text{and (by Fermat's Little Theorem since } 80 \text{ is divisible by } 16) \\ p_k^{80t} \equiv 1 \pmod{17}. \end{array} \right\} \quad (12)$$

Now, consider p_k^{80t} modulo **25**. Since p_k is not divisible by **5**; we have $p_k = 5 \cdot q + r$; where q is a positive integer and $r = 1, 2, 3$, or 4 . Consider $p_k^{20} = (5q + r)^{20}$. Since both **20** and $5q$ are divisible by **5**; it is clear that in the binomial expansion $(5q + r)^{20}$ every term, except for the last one; is divisible by **25**; thus $p_k^{20} = (5q + r)^{20} \equiv r^{20} \pmod{25}$

When $r = 1, r^{20} \equiv 1 \pmod{25}$

When $r = 2, r^{20} \equiv 2^{20} \equiv (2^6)^3 \cdot 2^2 \equiv (64)^3 \cdot 2^2 \equiv (-11)^3 \cdot 2^2 \equiv (-11)^2 \cdot (-11) \cdot 2^2 \equiv (121)(-11)(2^2) \equiv (-4)(-11) \cdot 4 \equiv (-16)(-11) \equiv 9(-11) \equiv -99 \equiv 1 \pmod{25}$.

When $r = 3; p^{20} \equiv 3^{20} \equiv (3^4)^5 \equiv (81)^5 \equiv (-6)^5 \equiv (-6)^2 \cdot (-6)^2 \cdot (-6) \equiv (36)(36) \cdot (-6) \equiv (11)(11) \cdot (-6) \equiv (121)(-6) \equiv (-4)(-6) \equiv 24 \equiv -1 \pmod{25}$.

When $r = 4; r^{20} \equiv (4^4)^5 \equiv (256)^5 \equiv 1^5 \equiv 1 \pmod{25}$. We see that in all cases; $r^{20} \equiv \pm 1 \pmod{25}$. Therefore $p_k^{20} = (5q + r)^{20} \equiv r^{20} \equiv \pm 1 \pmod{25}$. And so,

$$p_k^{80} \equiv (\pm 1)^4 \equiv 1 \pmod{25} \quad (13)$$

$p_k^{80t} \equiv 1 \pmod{25}$. From (13) and (12), it is clear that for $k \neq 2, 3, 7$

$$\begin{aligned} p_k^{80t} = p_k^{p_k^4 - 1} &\equiv 1 \pmod{2 \cdot 3 \cdot 5^2 \cdot 17} \\ &\equiv 1 \pmod{2550} \end{aligned} \quad (14)$$

Thus, in the sum $\sum_{k=2}^{2550} p_k^{p_k^4 - 1}$; every term with $k \neq 2, 3, 7$; is congruent to **1** modulo **2550** by (14). There are $(2550 - 2) + 1 - 3 = 2550 - 4$ such terms.

We have

$$\begin{aligned} \sum_{k=2}^{2550} p_k^{p_k^4 - 1} &\equiv p_2^{p_2^4 - 1} + p_3^{p_3^4 - 1} + p_7^{p_7^4 - 1} + (2550 - 4) \cdot 1 \\ &\equiv 1701 + 1225 + 901 + \left(\begin{array}{c} 0 \pmod{2550} \\ \widehat{2550} \end{array} - 4 \right) \cdot 1 \text{ by (6), (7), and (8)} \\ &\equiv 3823 \equiv 2550 + 1273 \equiv 1273 \pmod{2550} \end{aligned}$$

Conclusion: The remainder is **1273**.

7. A triangle has perimeter $2s$, inradius r , and the distance from its incenter to the vertices are s_a, s_b and s_c . Prove that

$$\frac{3}{4} + \frac{r}{s_a} + \frac{r}{s_b} + \frac{r}{s_c} \leq \frac{s^2}{12r^2}.$$

Solved by Arkady Alt, San Jose, CA, USA; George Apostolopoulos, Messolonghi, Greece; and Titu Zvonaru, Comănești, Romania. We give the write-up by Zvonaru.

We have $s_a = \frac{r}{\sin \frac{A}{2}}$, $s_b = \frac{r}{\sin \frac{B}{2}}$, $s_c = \frac{r}{\sin \frac{C}{2}}$, and we have to prove that $\frac{3}{4} + \sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} \leq \frac{s^2}{12r^2}$. The last inequality follows by known inequalities

$$\sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} \leq \frac{3}{2} \quad (\text{item 2.9 in [1]})$$

and $27r^2 \leq s^2$ (Item 5.11 in [1]).

[1] O. Bottema, *Geometric Inequalities*, Groningen, 1969.

9. Find all primes p such that $\frac{2^{p-1}-1}{p}$ is a perfect square.

Solved by David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give Mane's version.

The only such primes are $p = 3$ and $p = 7$. Assume that $\frac{2^{p-1}-1}{p} = n^2$ for some integer n . Then

$$2^{p-1} - 1 = \left(2^{\frac{p-1}{2}} + 1\right) \left(2^{\frac{p-1}{2}} - 1\right) = pn^2.$$

The prime factorization of pn^2 is $p^e p_1^{2e_1} p_2^{2e_2} \dots p_r^{2e_r}$, where the exponent e and the primes p, p_1, p_2, \dots, p_r are all odd integers. Then $\gcd\left(2^{\frac{p-1}{2}} + 1, 2^{\frac{p-1}{2}} - 1\right) = 1$ implies that one of the terms $2^{\frac{p-1}{2}} + 1$ or $2^{\frac{p-1}{2}} - 1$ is equal to $p^e a^2$ and the other is equal to b^2 for some integers a and b . Assume $2^{\frac{p-1}{2}} + 1 = b^2$ and let $b = 2x + 1$ so that $2^{\frac{p-1}{2}} + 1 = 4x^2 + 4x + 1$. Therefore $2^{\frac{p-1}{2}} = 4x(x + 1)$. The product $x(x + 1)$ is a power of 2 only if $x = 1$, and so $2^{\frac{p-1}{2}} = 2^3$. Therefore, $\frac{p-1}{2} = 3$ or $p = 7$. If $2^{\frac{p-1}{2}} - 1 = b^2 = 4x^2 + 4x + 1$, then $2^{\frac{p-1}{2}} = 2(2x^2 + 2x + 1)$. Since $2x^2 + 2x + 1$ is odd, it follows that it can be a power of 2 only if $2x^2 + 2x + 1 = 1$ when $x = 0$. Therefore $2^{\frac{p-1}{2}} - 1 = 1$ so that $\frac{p-1}{2} = 1$ or $p = 3$. Accordingly, the only values of p for which $\frac{2^{p-1}-1}{p}$ is a perfect square are $p = 3$ and $p = 7$.

Next we turn to reader's solutions to problems of the 14th Turkish Mathematical Olympiad 2006 given at [2010: 84–85].

5. Let A_1, B_1 and C_1 be the feet of the altitudes belonging to the vertices A, B and C , in acute triangle ABC , respectively, and let O_A, O_B and O_C be the incenters of the triangles AB_1C_1, BC_1A_1 and CA_1B_1 , respectively. Let T_A, T_B and T_C be the points of tangency of the incircle of the triangle ABC to the sides BC, CA and AB , respectively. Show $T_A O_C T_B O_A T_C O_B$ is a regular hexagon.

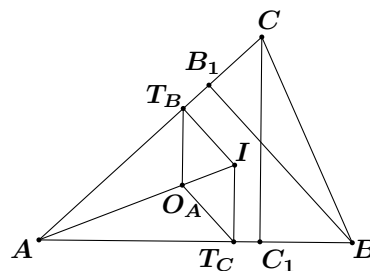
Solved by Titu Zvonaru, Comănești, Romania.

Let I be the incentre of triangle ABC and r be the inradius. The triangles AB_1C_1 and ABC are similar, with

$$\frac{AC_1}{AC} = \frac{AB_1}{AB} = \frac{B_1C_1}{BC} = \cos A.$$

The points A, O_A, I are collinear, and we have

$$AI = \frac{r}{\sin \frac{A}{2}}, \quad AO_A = \frac{r}{\sin \frac{A}{2}} \cos A.$$



We deduce that

$$O_AI = \frac{r}{\sin \frac{A}{2}} - \frac{r}{\sin \frac{A}{2}} \cos A = \frac{r \cdot 2 \sin^2 \frac{A}{2}}{\sin \frac{A}{2}},$$

hence $O_AI = 2r \sin \frac{A}{2}$. In $\triangle IO_AT_C$ we have $IT_C = r$, $O_AI = 2r \sin \frac{A}{2}$ and $\angle O_AIT_C = 90^\circ - \frac{A}{2}$. By the Law of Cosines, we obtain:

$$\begin{aligned} O_AT_C^2 &= r^2 + 4r^2 \sin^2 \frac{A}{2} - 2r \cdot 2r \sin \frac{A}{2} \cos \left(90^\circ - \frac{A}{2} \right) \\ &= r^2 + 4r^2 \sin^2 \frac{A}{2} - 4r^2 \sin^2 \frac{A}{2} = r^2. \end{aligned}$$

It results that $O_AT_C = r$ and all sides of $T_AO_C T_B O_A T_C O_B$ are equal to r .

But this hexagon *has no equal angles*; From the isosceles triangle $O_AT_C I$ we have

$$\angle IO_AT_C = \angle O_AIT_C = 90^\circ - \frac{A}{2}.$$

It follows that $\angle T_BO_AT_C = 180^\circ - A = B + C$, and similarly $\angle T_CO_B T_A = A + C$, $\angle T_AO_C T_B = A + B$.

We also obtain that

$$\angle O_AT_C I = 180^\circ - 2 \left(90^\circ - \frac{A}{2} \right) = A,$$

hence $\angle O_AT_C O_B = A + B$, $\angle O_B T_A O_C = B + C$, $\angle O_C T_B O_A = A + C$.

Next we turn to solutions to problems of the Turkish Team Selection Test for IMO 2007 given at [2010: 85].

1. An airline company is planning to run two-way flights between some of the six cities A, B, C, D, E and F . Determine the number of ways these flights can be arranged so that it is possible to travel between any two of these six cities using only the flights of this company.

Solved by Oliver Geupel, Brühl, NRW, Germany.

For positive integers n_1, \dots, n_p let d_{n_1, \dots, n_p} denote the number of labeled graphs consisting of p disjoint connected subgraphs with n_1, \dots, n_p vertices. We are to determine d_6 . The result will be $d_6 = 26704$.

To begin with, note that $d_1 = d_2 = 1$, $d_3 = 4$.

The total number of labeled graphs with 4 vertices is

$$\begin{aligned} 64 &= 2^{\binom{4}{2}} = d_{1,1,1,1} + d_{2,1,1} + d_{2,2} + d_{3,1} + d_4 \\ &= d_1^4 + \binom{4}{2} d_2 d_1^2 + \frac{1}{2} \binom{4}{2} d_2^2 + \binom{4}{3} d_3 d_1 + d_4 \\ &= 1 + 6 + 3 + 16 + d_4 = 26 + d_4; \end{aligned}$$

hence $d_4 = 38$.

The total number of labeled graphs with 5 vertices is

$$\begin{aligned} 1024 &= 2^{\binom{5}{2}} = d_{1,1,1,1,1} + d_{2,1,1,1} + d_{2,2,1} + d_{3,1,1} + d_{3,2} + d_{4,1} + d_5 \\ &= 1 + \binom{5}{2} + \frac{1}{2} \binom{5}{2} \binom{3}{2} + \binom{5}{3} d_3 + \binom{5}{3} d_3 + \binom{5}{4} d_4 + d_5 \\ &= 1 + 10 + 15 + 40 + 40 + 190 + d_5 = 296 + d_5; \end{aligned}$$

thus $d_5 = 728$.

Finally, the total number of labeled graphs with 6 vertices is

$$\begin{aligned} 32768 &= 2^{\binom{6}{2}} = d_{1,1,1,1,1,1} + d_{2,1,1,1,1} + d_{2,2,1,1} + d_{2,2,2} + d_{3,1,1,1} + d_{3,2,1} \\ &\quad + d_{3,3} + d_{4,1,1} + d_{4,2} + d_{5,1} + d_6 \\ &= 1 + \binom{6}{2} + \frac{1}{2} \binom{6}{2} \binom{4}{2} + \frac{1}{6} \binom{6}{2} \binom{4}{2} + \binom{6}{3} d_3 + \binom{6}{3} \binom{3}{2} d_3 \\ &\quad + \frac{1}{2} \binom{6}{3} d_3^2 + \binom{6}{4} d_4 + \binom{6}{4} d_4 + \binom{6}{5} d_5 + d_6 \\ &= 6064 + d_6; \end{aligned}$$

consequently $d_6 = 26704$.

Remark. The problem is well-known. A different approach is given in Herbert S. Wilf's book *Generating Functionology*, Second edition, Academic Press, 1994, page 87, formula (3.10.2).

3. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$\frac{1}{ab + 2c^2 + 2c} + \frac{1}{bc + 2a^2 + 2a} + \frac{1}{ca + 2b^2 + 2b} \geq \frac{1}{ab + bc + ca}.$$

Solved by Arkady Alt, San Jose, CA, USA; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; and Michel Bataille, Rouen, France. We give the version of Díaz-Barrero.

Multiplying numerator and denominator of the RHS by $ab + bc + ca$ yields

$$\frac{1}{ab + 2c^2 + 2c} + \frac{1}{bc + 2a^2 + 2a} + \frac{1}{ca + 2b^2 + 2b} \geq \frac{ab + bc + ca}{(ab + bc + ca)^2}$$

Now we claim that

$$\frac{1}{ab + 2c^2 + 2c} \geq \frac{ab}{(ab + bc + ca)^2}$$

Indeed, the preceding inequality is equivalent to

$$(ab + bc + ca)^2 \geq ab(ab + 2c^2 + 2c)$$

which after some algebraic computations becomes

$$a^2b^2 + b^2c^2 + c^2a^2 + 2abc(a + b + c) \geq ab(ab + 2c^2 + 2c),$$

and taking into account the constraint, we get

$$a^2b^2 + b^2c^2 + c^2a^2 + 2abc \geq ab(ab + 2c^2 + 2c)$$

Canceling terms, we obtain $b^2c^2 + c^2a^2 \geq 2abc^2$ or equivalently

$$\frac{b^2c^2 + c^2a^2}{2} \geq \sqrt{a^2b^2c^4} = abc^2$$

which holds on account of AM-GM inequality. Likewise,

$$\frac{1}{bc + 2a^2 + 2a} \geq \frac{bc}{(ab + bc + ca)^2}$$

and

$$\frac{1}{ca + 2b^2 + 2b} \geq \frac{ca}{(ab + bc + ca)^2}.$$

Adding the preceding three inequalities the statement follows. Equality holds when $a = b = c = 1/3$, and we are done.

Now we turn to solutions to the Estonian Team Selection Contest 2007 given at [2010: 149].

2. Let D be the foot of the altitude of triangle ABC drawn from vertex A . Let E, F be the points symmetric to D with respect to the lines AB, AC , respectively. Let triangles BDE, CDF have inradii r_1, r_2 and circumradii R_1, R_2 , respectively. If S_K denotes the area of figure K , prove that

$$|S_{ABD} - S_{ACD}| \geq |r_1R_1 - r_2R_2|.$$

Solved by Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Comănești, Romania. We give Geupel's solution.

Without loss of generality let $AD = 1$. Let denote $\varphi_1 = \angle BAD$ and $\varphi_2 = \angle CAD$, where $\varphi_1, \varphi_2 \in [0, \pi/2)$. It holds $BD = BE = \tan \varphi_1$ and $DE = 2 \sin \varphi_1$. Hence $S_{ABD} = \frac{1}{2} \tan \varphi_1$ and

$$r_1 R_1 = \frac{DE \cdot BD \cdot BE}{2(DE + BD + BE)} = \frac{\sin \varphi_1 \tan^2 \varphi_1}{2(\sin \varphi_1 + \tan \varphi_1)} = \frac{1}{2} \left(\frac{1}{\cos \varphi_1} - 1 \right).$$

Similarly, $S_{ACD} = \frac{1}{2} \tan \varphi_2$ and $r_2 R_2 = \frac{1}{2} \left(\frac{1}{\cos \varphi_2} - 1 \right)$. We therefore have to prove that $|\tan \varphi_1 - \tan \varphi_2| \geq \left| \frac{1}{\cos \varphi_1} - \frac{1}{\cos \varphi_2} \right|$. Since $\tan x$ and $\frac{1}{\cos x}$ are both increasing for $0 < x \leq \pi/2$, the terms $\tan \varphi_1 - \tan \varphi_2$ and $\frac{1}{\cos \varphi_1} - \frac{1}{\cos \varphi_2}$ have the same sign.

It therefore suffices to prove

$$\tan \varphi_1 - \tan \varphi_2 \geq \frac{1}{\cos \varphi_1} - \frac{1}{\cos \varphi_2} \quad (1)$$

under the hypothesis $\varphi_1 \geq \varphi_2$. But the function $f(x) = \frac{1}{\cos x} - \tan x$ is decreasing for $0 \leq x < \pi/2$ as can be seen from the derivative $f'(x) = \frac{\sin x - 1}{\cos^2 x}$. This implies (1) and the proof is complete.

3. Let n be a natural number, $n \geq 2$. Prove that if $\frac{b^n - 1}{b - 1}$ is a prime power for some positive integer b then n is prime.

Solved by Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA.

Assume that,

$$\left\{ \begin{array}{l} \frac{b^n - 1}{b - 1} = p^k; \quad b^n - 1 = p^k \cdot (b - 1) \\ \text{where } p \text{ is a prime number,} \\ n, b, k \text{ positive integers such} \\ \text{that } n \geq 2 \text{ and } b \geq 2 \end{array} \right\} \quad (1)$$

First, we will treat the case in which p equals 2. This is done in Case 1. In Case 2, $p \geq 3$; the proof splits into three subcases.

Case 1. $p = 2$.

We have, by (1),

$$\left\{ \begin{array}{l} \frac{b^n - 1}{b - 1} = 2^k; \text{ or equivalently} \\ b^{n-1} + b^{n-2} + \dots + b + 1 = 2^k. \end{array} \right\} \quad (2)$$

It follows from (2) that b must be odd and n even; since there are n terms, with each term being an odd number; in the sum $b^{n-1} + b^{n-2} + \dots + b + 1$. Thus

$$\left\{ \begin{array}{l} n = 2l \\ \text{where } l \text{ is a positive integer.} \end{array} \right\} \quad (3)$$

By (2) and (3) we have,

$$b^{2l} - 1 = 2^k \cdot (b - 1) \Leftrightarrow (b^l - 1)(b^l + 1) = 2^k \cdot (b - 1). \quad (4)$$

Since b is odd; the numbers $b^l - 1$ and $b^l + 1$ are consecutive even integers; their greatest common divisor is 2 :

$$(b^l - 1, b^l + 1) = 2. \quad (5)$$

If $l = 1$, then $n = 2 \cdot 1 = 2$, which is a prime (and $b = 2^k - 1$).

If $l \geq 2$; then

$$b^l - 1 = (b - 1) \cdot (b^{l-1} + b^{l-2} + \dots + b + 1). \quad (6)$$

By (4) and (6) we obtain,

$$\begin{aligned} (b - 1)(b^{l-1} + b^{l-2} + \dots + b + 1) \cdot (b^l + 1) &= 2^k(b - 1); \\ (b^{l-1} + b^{l-2} + \dots + b + 1) \cdot (b^l + 1) &= 2^k, \end{aligned} \quad (7)$$

since b is greater than 1 ; in fact $b \geq 3$, since b is odd; and $l \geq 2$.

Each of the factors on the lefthand side of (7) is greater than 2 . Thus (7) implies that

$$\left\{ \begin{array}{l} b^{l-1} + b^{l-2} + \dots + b + 1 = 2^{k_1} \\ \text{and } b^l + 1 = 2^{k_2}; \text{ with } k_1, k_2 \text{ positive.} \end{array} \right\} \quad (8)$$

Also,

$$b^l - 1 = (b - 1)(b^{l-1} + b^{l-2} + \dots + b + 1) = 2^{k_1} \cdot (b - 1). \quad (9)$$

Since $k_1 \geq 2$, $k_2 \geq 2$; (8) and (9) show that 4 must be a common divisor of $b^l - 1$ and $b^l + 1$, contrary to (5).

Case 2. p is an odd prime, $p \geq 3$.

Recall from number theory that if p is an odd prime and a an integer not divisible by p , then the order of a modulo p is the least positive integer k such that $a^k \equiv 1 \pmod{p}$. When $a \equiv 1 \pmod{p}$; then obviously the order of a modulo p equals 1 . Otherwise the order of a is ≥ 2 . The following Lemma is well-known in number theory and easily provable by using the division algorithm.

Lemma 1. Let p be an odd prime; and b a positive integer not divisible by p , such that $b \not\equiv 1 \pmod{p}$. If m is a positive integer, such that $b^m \equiv 1 \pmod{p}$, then m is divisible by d , $d \mid m$, where d is the order of b modulo p ; $d \geq 2$.

Since by Fermat's (Little) Theorem, $b^{p-1} \equiv 1 \pmod{p}$. We have the following corollary of Lemma 2.

Lemma 2. Let p be an odd prime; $b \in \mathbb{Z}^+$, $b \not\equiv 0, 1 \pmod{p}$. Then the order of b modulo p is a divisor $d \geq 2$ of $p - 1$.

Back to the problem.

Subcase 2a. Assume that $b \equiv 1 \pmod{p}$.

We will prove that this case is impossible, it leads to a contradiction regardless of whether n is a prime or not.

From $b \equiv 1 \pmod{p}$ and (1) we have

$$\left\{ \begin{array}{l} b^{n-1} + b^{n-2} + \dots + b + 1 = p^k = \frac{b^n - 1}{b - 1} \\ \text{and } b = p \cdot t + 1, \text{ for some positive integer } t. \end{array} \right\} \quad (10)$$

Since $b \equiv 1 \pmod{p}$; $b^{n-1} \equiv b^{n-2} \equiv \dots \equiv b \equiv 1 \pmod{p}$. And so by (10) we have,

$$\underbrace{1 + 1 + \dots + 1}_{n \text{ terms}} \equiv 0 \pmod{p};$$

$$n \equiv 0 \pmod{p} \text{ and so } n \geq 3. \quad (11)$$

Clearly, since $n \geq 3$ and $b = p \cdot t + 1$, we have

$$b^{n-1} + b^{n-2} + \dots + b^2 + b + 1 \geq b^2 + b + 1 = (p \cdot t + 1)^2 + (p \cdot t + 1) + 1 > p^2,$$

which shows that we must have $k \geq 3$ in (11). Now,

$$b^n - 1 = p^k \cdot (b - 1) \Rightarrow (p \cdot t + 1)^n - 1 = p^k \cdot (pt).$$

Expanding with the binomial expansion yields

$$(pt)^n + \binom{n}{1}(pt)^{n-1} + \dots + \binom{n}{n-1}(pt) + \overbrace{1-1}^0 = p^{k+1} \cdot t. \quad (12)$$

Let p^f be the highest power of p dividing n (see (11)); and p^e be the highest power of p dividing t . Then $n = n_1 \cdot p^f$, $t = t_1 \cdot p^e$, $f \geq 1$; and $e \geq 0$ and $n_1 \cdot t_1 \not\equiv 0 \pmod{p}$. The highest power p dividing the left hand side of (12); is the highest power of p dividing the term $\binom{n}{n-1} p \cdot t = n \cdot p \cdot t = n_1 \cdot t_1 \cdot p^{f+e+1}$. The right hand side is $p^{k+1} \cdot t = p^{k+1+e}$. Thus, we must have $1 + f + e = k + 1 + e$; and so $f = k$; $n = n_1 \cdot p^k$. If we look at the term $(pt)^n$ (left hand side of (12)); $(pt)^n = p^n \cdot t^n = p^{n_1 \cdot p^k} \cdot t^{n_1 \cdot p^k} > p^{k+1} \cdot t$, since $t \geq 1$ and $n_1 p^k \geq p^k > k + 1$, in view of $p \geq 3$ and $k \geq 3$.

We have a contradiction to (12).

In subcases 2b and 2c below; we argue by *contradiction*. We assume that $b \not\equiv 1 \pmod{p}$ and n to be a composite number ≥ 4 , and we show that this leads to a *contradiction*. Observe that a composite number is either a prime power with exponent at least 2; or otherwise it has two distinct prime bases in its prime factorization.

Subcase 2b. Assume that $b \not\equiv 1 \pmod{p}$; and $n \geq 4$ has at least two prime bases in its prime factorization into prime powers.

This then implies that we can write,

$$\left\{ \begin{array}{l} n = n_1 \cdot n_2 \geq 4 \text{ (actually 6)} \\ \text{with } 1 < n_1, n_2 < n; \text{ where } n_1, n_2 \\ \text{are relatively prime positive integers; } (n_1, n_2) = 1. \end{array} \right\} \quad (13)$$

In other words, n can be written as a product of two relatively prime proper positive divisors. We have,

$$\begin{aligned} b^{n_1 n_2} - 1 &= p^k \cdot (b - 1); \\ (b^{n_1})^{n_2} - 1 &= p^k \cdot (b - 1); (b^{n_2} - 1)[(b^{n_1})^{n_2 - 1} + \dots + b^{n_1} + 1] = p^k \cdot (b - 1); \\ (b - 1) \cdot (b^{n_2 - 1} + \dots + b + 1)[(b^{n_1})^{n_2 - 1} + \dots + b^{n_1} + 1] &= p^k \cdot (b - 1); \\ (b^{n_1 - 1} + b + 1) \cdot [(b^{n_1})^{n_2} + \dots + b^{n_1} + 1] &= p^k \end{aligned} \quad (14)$$

Each factor on the left hand side of (14) is greater than 2 . Clearly then (14) implies that each factor must be a power of p , so that,

$$\left\{ \begin{array}{l} b^{n_1 - 1} + \dots + b + 1 = p^{k_1} \\ k_1 \text{ a positive integer.} \end{array} \right\} \quad (15)$$

Similarly, by factoring

$$b^n - 1 = b^{n_1 n_2} - 1 = (b^{n_2})^{n_1} - 1 = (b^{n_2} - 1)[(b^{n_2})^{n_1 - 1} + \dots + b^{n_2} + 1]$$

and using similar reasoning; we obtain,

$$\left\{ \begin{array}{l} b^{n_2 - 1} + \dots + b + 1 = p^{k_2} \\ \text{for some positive integer } k_2 \end{array} \right\} \quad (16)$$

From (15) and (16) we get

$$\left\{ \begin{array}{l} b^{n_1} - 1 = p^{k_1} \cdot (b - 1) \text{ and} \\ b^{n_2} - 1 = p^{k_2} \cdot (b - 1) \end{array} \right\} \quad (17)$$

Thus, (17) $\Rightarrow (b^{n_1} \equiv 1 \pmod{p})$ and $(b^{n_2} \equiv 1 \pmod{p})$. By Lemma 1, both n_1 and n_2 must be divisible by the order ρ of b modulo p ; since $b \not\equiv 1 \pmod{p}$; we have $\rho \geq 2$. And $2 \leq \rho \mid n_1$ and $\rho \mid n_2$, a contradiction of $(n_1, n_2) = 1$ in (13).

Subcase 2c. $n = q^j$, $j \geq 2$, where q is a prime number.

Since $j \geq 2$, $b^n - 1$ is divisible by $b^{q^2} - 1$; and hence by $b^q - 1$ as well. Indeed,

$$b^{q^j} - 1 = b^{q^2 \cdot q^{j-2}} - 1 = (b^{q^2})^{q^{j-2}} - 1$$

$$\begin{aligned} b^n - 1 &= (b^{q^2})^{q^{j-2}} - 1 \\ &= \left\{ \begin{array}{l} (b^{q^2} - 1), \text{ if } j = 2 \\ (b^{q^2} - 1) \cdot [(b^{q^2})^{q^{j-2}-1} + \dots + (b^{q^2}) + 1], \text{ if } j \geq 3. \end{array} \right\} \end{aligned} \quad (18)$$

In either ($j = 2$ or $j \geq 3$) case, from $b^n - 1 = p^k \cdot (b - 1)$; after the cancelation of the factor $b - 1$ from both sides of the last equation; it follows that $b^{q-1} + b^{q-2} + \dots + b + 1$ must be a power of p . Indeed, since

$$\begin{aligned} b^{q^2} - 1 &= (b^q)^q - 1 = (b^q - 1)[(b^q)^{q-1} + \dots + b^q + 1] \\ &= (b - 1)(b^{q-1} + \dots + b + 1)[(b^q)^{q-1} + \dots + b^q + 1], \end{aligned}$$

thus, $b^{q-1} + \dots + b + 1 = p^\lambda$, for some positive integer λ . And hence

$$b^q - 1 = p^\lambda \cdot (b - 1) \Rightarrow b^q \equiv 1 \pmod{p}. \quad (19)$$

By Lemma 1, (19) implies that the order d of b modulo p must divide q : $2 \leq d \mid q$. But q is a prime and therefore it follows that $d = q$. The order of b modulo p must equal the prime q . Hence, by Lemma 2 it follows that

$$p - 1 \equiv 0 \pmod{q}. \quad (20)$$

Recall from (18) that $b^{q^2} - 1$ is a factor of $b^n - 1$.

$$b^n - 1 = (b^{q^2} - 1) \cdot N, \quad N \text{ a positive integer and } b^n - 1 = p^k(p - 1)$$

Thus,

$$\begin{aligned} (b^{q^2} - 1) \cdot N &= p^k \cdot (b - 1); \\ ((b^q)^q - 1) \cdot N &= p^k \cdot (b - 1); \\ \underbrace{(b^q - 1)}_{(b-1)(b^{q-1}+\dots+b+1)} [(b^q)^{q-1} + \dots + b^q + 1] &= p^k \cdot (b - 1); \end{aligned}$$

So

$$(b^{q-1} + \dots + b + 1)[(b^q)^{q-1} + \dots + b^q + 1] = p^k. \quad (21)$$

So each factor (on the left hand side of (21) must be a power of p : In particular

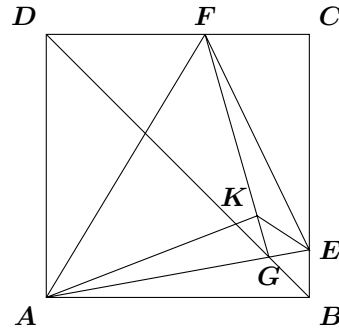
$$(b^q)^{q-1} + \dots + b^q + 1 = p^w, \quad w \geq 1. \quad (22)$$

By (19) and (22) we deduce that,

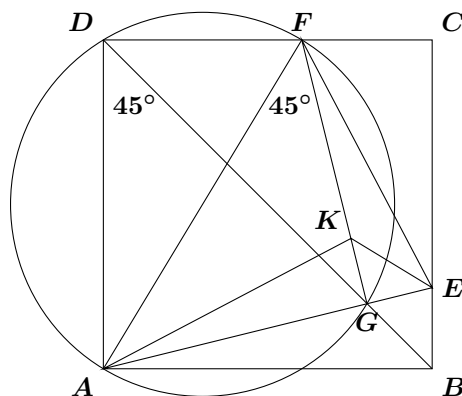
$$\underbrace{1 + 1 + \dots + 1}_q \equiv 0 \pmod{p} \Rightarrow q \equiv 0 \pmod{p}; \quad (23)$$

and since p and q are both primes; (23) implies $p = q$. But (20) then implies $p - 1 \equiv 0 \pmod{p}$, an impossibility. \square

4. In square $ABCD$ the points E and F are chosen in the interior of sides BC and CD , respectively. The line drawn from F perpendicular to AE passes through the intersection point G of AE and BD . A point K is chosen on FG such that $|AK| = |EF|$. Find $\angle EKF$.



Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; Bruce Sawyer, Memorial University of Newfoundland, St. John's, NL; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comănești, Romania. We give the solution of Amengual Covas.



With right angles at D and G , $AGFD$ is a cyclic quadrilateral, and in its circumcircle,

$$\angle GFA = \angle GDA = \angle BDA = 45^\circ$$

Thus, right-triangle AGF is isosceles with $AG = GF$.

Since we also have $AK = EF$, right triangles AGK and FGE are congruent (side-angle-side) with $KG = GE$.

Hence $\triangle KGE$ is isosceles right-angled. Thus, $\angle EKF = 180^\circ - \angle GKE = 180^\circ - 45^\circ = 135^\circ$.

5. Find all continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that for all reals $x, y \in \mathbb{R}$,

$$f(x + f(y)) = y + f(x + 1).$$

Solved by Michel Bataille, Rouen, France.

The functions $x \mapsto x + 1$ and $x \mapsto -x + 1$ are solutions (readily checked). We show that there are no other solutions.

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function satisfying the given functional equation that we will denote by (E) . Taking $x = -1$ in (E) yields $f(f(y) - 1) = y + f(0)$ for all $y \in \mathbb{R}$. An immediate consequence is $f(y) = f(y') \implies y = y'$ and f is injective.

Letting $y \rightarrow 0$ and using the continuity of f gives $f(x + f(0)) = f(x + 1)$, hence $f(0) = 1$ and so $f(f(y) - 1) = y + 1$ for all $y \in \mathbb{R}$. Using (E) , it follows that

$$f(x + y) = f((x - 1) + (y + 1)) = f(x - 1 + f(f(y) - 1)) = f(y) - 1 + f(x).$$

Thus, the continuous function $g : x \mapsto f(x) - 1$ satisfies the Cauchy functional equation $g(x + y) = g(x) + g(y)$. It is known that g must be a linear function $x \mapsto ax$ and so f must be $x \mapsto ax + 1$. Returning to (E) , we must have $a(x + ay + 1) + 1 = y + a(x + 1) + 1$ for all x, y that is, $a^2y = y$ for all y . Hence $a = 1$ or $a = -1$ and the conclusion follows.

Next we turn to solutions from our readers to problems of the Russian Mathematical Olympiad 2007, 10th grade, given at [2010: 150–151].

2. (*A. Khrabrov*) Given a polynomial $P(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$, let $m = \min\{a_0, a_0 + a_1, \dots, a_0 + a_1 + \dots + a_n\}$. Prove that $P(x) \geq mx^n$ for all $x \geq 1$.

Solved by Oliver Geupel, Brühl, NRW, Germany.

By the definition of m , it holds

$$a_0 - m \geq 0, \quad a_0 + a_1 - m \geq 0, \quad \dots, \quad a_0 + a_1 + \dots + a_n - m \geq 0.$$

Hence, we have for $x \geq 1$

$$\begin{aligned} P(x) &= a_0x^n + a_1x^{n-1} + \dots + a_n \\ &= a_0(x^n - x^{n-1}) + (a_0 + a_1)(x^{n-1} - x^{n-2}) \\ &\quad + (a_0 + a_1 + a_2)(x^{n-2} - x^{n-3}) \\ &\quad + \dots + (a_0 + a_1 + \dots + a_{n-1})(x - 1) + (a_0 + a_1 + \dots + a_n) \\ &= (a_0 - m)(x^n - x^{n-1}) + (a_0 + a_1 - m)(x^{n-1} - x^{n-2}) + \dots \\ &\quad + (a_0 + a_1 + \dots + a_n - m)(x - 1) + (a_0 + a_1 + \dots + a_n - m) \\ &\quad + mx^n \\ &\geq mx^n. \end{aligned}$$

3. (*V. Astakhov*) In an acute triangle ABC , BB_1 is a bisector. Point K is chosen on the smaller arc BC of the circumcircle, such that B_1K and AC are perpendicular. Point L is chosen on line AC such that BL and AK are also perpendicular. Line BB_1 meets the smaller arc AC at point T . Prove that points K, L, T are collinear.

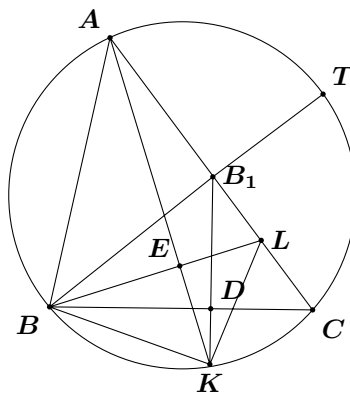
Solved by Oliver Geupel, Brühl, NRW, Germany.

Fixing the typo in the problem, we substitute the hypothesis $B_1K \perp AC$ by the condition $B_1K \perp BC$. Let D be the point of intersection of the lines BC and B_1K , and let the lines AK and BL intersect at the point E . Since the inscribed angles $\angle CBK = \angle DBK$ and $\angle CAK = \angle LAE$ have equal size, the right triangles DBK and EAL are similar.

Hence,

$$\angle BKB_1 = \angle BKD = \angle ALE = \angle BLB_1,$$

that is, the points B, B_1, K , and L are concyclic.



Thus,

$$\angle BB_1D = \angle BB_1K = \angle BLK = \angle KLE,$$

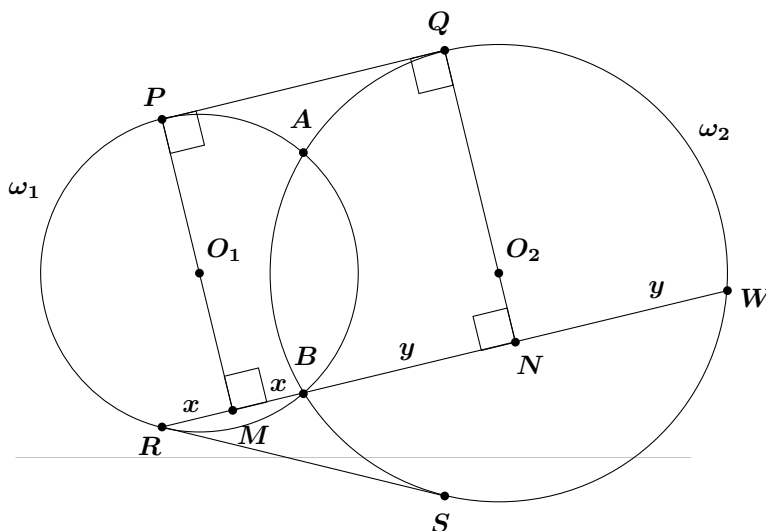
which implies that the right triangles BB_1D and KLE are similar. Therefore,

$$\angle TKA = \angle TBA = \angle TBC = \angle B_1BD = \angle LKE = \angle LKA.$$

Consequently, the point L is on the line TK , which completes the proof.

6. (*S. Berlov*) Two circles ω_1 and ω_2 intersect at points A and B . Let PQ and RS be the segments of common tangents to these circles (points P and R lie on ω_1 , while points Q and S lie on ω_2). Ray RB intersects ω_2 again at point W . If $RB \parallel PQ$, find the ratio RP/BW .

Solved by Geoffrey A. Kandall, Hamden, CT, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comănești, Romania. We give Kandall's solution.



Let O_1 (O_2) be the centre of ω_1 (ω_2). Extend PO_1 (QO_2) to meet RW at M (N). Since $\angle O_1PQ$ and $\angle O_2QP$ are right angles and $RW \parallel PQ$, it follows that $\angle O_1MN$ and $\angle O_2NM$ are right angles (so $PQNM$ is a rectangle) and M (N) is the midpoint of RB (BW). Let $RB = 2x$, $BW = 2y$.

Then $RS = PQ = MN = x + y$. Since $RS^2 = RW \cdot RB$, we have $(x + y)^2 = 2(x + y) \cdot 2x$. Dividing by $x + y$, we obtain $x + y = 4x$, that is $y = 3x$.

Therefore,
$$\frac{RB}{BW} = \frac{x}{y} = \frac{1}{3}.$$

BOOK REVIEWS

Amar Sodhi

The Calculus of Friendship

by Steven Strogatz

Princeton University Press, 2009

ISBN 978-0-691-13493-2 hardcover, 166 + xii pp. US\$19.95

Reviewed by **Georg Gunther**, *Sir Wilfred Grenfell College (MUN), Corner Brook, NL*

New books and new cars have this in common: one cannot wait to open them. Their lure is irresistible. If chemists could ever be bothered to synthesize the smell of a new car, undoubtedly *L'eau d'Auto* would soon become the fragrance of choice.

Cracking open the cover of a new book carries a similar appeal: there is a promise of things to come, an anticipation of words to be read, a sense of excitement. And the magic happens when the book delivers on this promise, when the words capture and enthrall.

The Calculus of Friendship is such a book. It is a remarkable story, based upon a thirty-year correspondence between the author and his former high-school mathematics teacher. The letters, many of which are reproduced in the book, are all about calculus. It is the passion and love for this discipline that connects these two men as they move through their respective lives, and it is calculus, with its unparalleled ability to provide deep and profound metaphors, that provides the connecting context. “*Some books are to be tasted*”, wrote Francis Bacon, “*others to be swallowed, and some few to be chewed and digested*”. This book certainly falls into Bacon’s third category.

The author, Steven Strogatz, is the Jacob Gould Schurman Professor of Applied Mathematics at Cornell University. He has contributed widely to the study of synchronization in dynamical systems. In addition to writing numerous mathematical papers and books, he is the author of the best selling *Sync: the emerging science of spontaneous order* (2003).

The book being reviewed here is written with a deceptive simplicity and grace of style that draws the reader effortlessly from page to page. It provides a rare glimpse into the mind and heart of a top-notch mathematician. Always fascinating, at times deeply profound, it escapes being maudlin by the sheer simplicity of its prose, the elegance of its mathematics and the, at times, brutal self-appraisal of the author.

There is a great deal of fascinating and exciting mathematics in this book. Interesting problems are tossed back and forth between former teacher and former student in a wonderful point and counterpoint of question and answer, leading to yet more questions, more avenues of investigation. Reading these letters, one is struck with the exuberance of the intellectual exercises being posed, with the thrill of the exploration, and the sheer fun of doing mathematics.

The book is superbly organized. The successive chapters follow the chronology of two lives, from 1974 to the present; each explores one of the mathematical topics raised in the course of the correspondence between these two men. In addition to including the actual text of the original letters, the author provides some of the mathematical background, as well as placing these discussions within a personal context.

The mathematical content of this book is far from trivial. It covers diverse topics, from chase problems, discussion of irrationality, subtleties of infinite series, explorations of randomness, all the way to chaos theory and questions about bifurcation. This book is well worth reading for its mathematical content alone.

However, this book is so much more than simply a book about calculus. It is a testimonial to the bonds of friendship and to the complex and ever evolving connections between teacher and student. In mathematics, fixed-point theorems play a central role. The major focus of Strogatz's mathematical work is the science of synchrony: how spontaneous order can arise within inherently chaotic systems. In *The Calculus of Friendship*, the author explores one more facet of this theme: *"like calculus itself, this book is an exploration of change. It's about the transformation that takes place in a student's heart, as he and his teacher reverse roles, as they age, as they are buffeted by life itself. Through all these changes, they are bound together by a love of calculus. For them it is more than a science. It is a game they love playing together – so often the basis of friendship between men – a constant while all around them is flux"*.

Towards the end of this book, Steven Strogatz acknowledges that his former mentor taught him *"something profoundly mathematical, about how to live"*; the lesson is to *"balance the inevitable against the unforeseeable, the two sides of change in this world. The orderly and the chaotic. The changes that calculus can tame, and the ones it cannot. He confronts them all, and not, like Zeno, with his mind alone but also with his heart"*.

"The Calculus of Friendship" is recommended for all. Practicing mathematicians will find much that might be new to them, as well as meeting up with many old friends. Students will get glimpses into numerous directions that will compel their interest. Non-mathematicians can skip the hard stuff, the mathematical equations, and find a different set of valuable truths in the vastly more complicated rules that lie at the heart of human relationships.

*Cru*x Chronology

J. Chris Fisher

1 Highlights

March 1975, *Eureka*.

*Cru*x *Mathematicorum* with *Mathematical Mayhem* began life as the journal *Eureka*. Early in 1975, six members of the Carleton-Ottawa Mathematics Association (COMA) met privately and decided to launch *Eureka*

to provide a forum for the exchange of mathematical information, especially interesting problems and solutions, among the members of the mathematical community in the Ottawa region, students and teachers alike. [1975 : 1]

Three of the six were from Algonquin College: Léo Sauv e, who served as the first editor (until 1986); Fred G.B. Maskell, COMA’s secretary-treasurer who became the first managing editor (through 1984); and H.G. Dworschak. The other three were Viktors Linis of the University of Ottawa, R. Duff Butterill of the Ottawa Board of Education, and Richard J. Semple of Carleton University.

While Dworschak and Linis provided some support and numerous problems, it was L eo and Fred who provided the the energy and dedication required to turn their modest local venture into a journal that within two years developed an international following, a following that included some of the world’s finest mathematicians. In the words of his friend and colleague Kenneth S. Williams [1987 : 240-242], “L eo’s dedication and hard work, his broad knowledge and love of mathematics, his careful eye for detail, all enabled *Cru*x to grow from a four-page problem sheet to the international mathematical problem-solving journal that it is today.” Many of *Cru*x’s faithful contributors became L eo’s friends; their tributes to him can be found in the issue dedicated to him [1986 : 163-168].

March 1978, *Cru*x *Mathematicorum*.

Beginning with volume 4 number 3, the name of the journal changed to *Cru*x *Mathematicorum*. After 32 issues had been published under the name *Eureka*, it was discovered that there was a journal *Eureka* published once a year by the Cambridge University Mathematical Society. Sauv e chose the new name: it is an idiomatic Latin phrase meaning a puzzle or problem for mathematicians [1978 : 89-90].

January 1979, the Olympiad Corner.

Murray S. Klamkin initiated the Olympiad Corner [1979 : 12] to “provide, on a continuing basis, information about mathematical contests taking place in Canada, the U.S.A., and internationally.” It would also provide “practice sets of problems on which interested students could test and sharpen their mathematical skills and thereby possibly qualify to participate in some Olympiad.” Klamkin served as editor of the first 80 columns, through December, 1986.

December 1984, Fred.

Frederick G. B. Maskell (1904-1985) steps down as managing editor [1984 : 340], and died a few months later [1985 : 14, 34]. He was replaced by Kenneth S. Williams.

October 1 1985, the CMS.

For nearly eleven years *CruX* was published by Algonquin College and sponsored by the Carleton-Ottawa Mathematics Association. In 1979 The Canadian Mathematical Olympiad Committee and the Carleton University Mathematics Department added their support, joined later by the University of Ottawa Mathematics Department. In March, 1985, the Canadian Mathematical Society was asked to assume responsibility for its publication; that organization called for suggestions concerning the future of the journal and began the search for a new editor. [1985 : 100] On October 1, 1985, *CruX* became an official publication of the Canadian Mathematical Society. [1985 : 234, 236] The October 1985 issue was dedicated to Philip Kileen, President of Algonquin College, who strongly supported *CruX* during its first eleven years. For all future editors, the mathematics department of the current editor's university lent support to the journal.

February 1986, G.W. (Bill) Sands.

Starting with Volume 12, number 2, Bill Sands (University Calgary) became the second editor (through 1995) . [1986 : 17, 37] He announced [1986 : 65] that the September issue would be dedicated to Léo Sauvé.

January 1987 Robert Woodrow.

The Olympiad Corner's Murray Klamkin was replaced by Robert E. Woodrow [1986 : 263; 1987 : 2-3, 34], but Klamkin continued making valuable contributions to *CruX* until his death in 2004.

June 19, 1987, Léo.

Léo Sauvé died (Dec. 12, 1921 - June 19, 1987). [1987 : 240-242] Not only was Léo the consummate scholar, but he injected into each page of *CruX* a spicy liveliness that will probably never be matched. When a problem failed to attract solutions to his liking, he would write his own, assuming the guise of his contributor persona, Gali Salvatore (Salvatore = Sauvé). If there happened to be a gap he could not fill, he would attach an editorial comment in which he criticized Salvatore's proof; see [1984 : 31] where he attacked the solution (his own!) to problem 783, complaining that the whole argument rested on a formula that came without a proof "presumably because he (or she: is Gali a man's name or a woman's?) felt it was 'easy.' " Edith Orr (= editor) was another persona who was able to make comments that an editor could not; her poetry was sometimes so racy that it would not have been allowed to pass through the mail had the postal inspectors thought to look closely at a math journal! See her comment on lambs [1983 : 211-212], which Léo, wearing his editor's hat, dismissed as "scrofulous."

January 1988, Colour.

Starting with volume 14, each issue came with a coloured front and back cover; although it continued to be printed on $8\frac{1}{2} \times 11$ three-holed paper and

stapled together, it began to take on a more professional look.

January 1989, Kenneth Williams.

Kenneth Williams steps down from his position of managing editor and his role as technical editor. [1988 : 300-301] His duties were taken over by Graham P. Wright, the executive director of the Canadian Mathematical Society, together with members of his staff; Wright served (except for a short period between 1999 and 2000 when the position was filled by Robert Quackenbush) until his retirement in May, 2009.

January 1991, The Editorial Board.

After five years of performing all the editorial duties with only occasional help from others, Sands organized the first formal editorial board. Robert Woodrow was promoted to joint editor, and six others agreed to form the board. The board members will be listed later.

January 1995, New format and Skoliad.

Starting with volume 21, the appearance was changed to its current 10-inch format with purple covers. Also, the Skoliad Corner was inaugurated with its simpler “Pre-Olympiad” problem sets, with Robert Woodrow as its first editor [1995 : 5]. The name was suggested by Richard Guy, who had searched a map of the Mount Olympus area, finding the mountain Scollis, and then making a portmanteau of this with *scholar* and *Olympiad*. At about 1/3 the height of Mount Olympus, Mount Scollis seems the appropriate metaphor for a junior Olympiad. It was later learned that *skolion*, an unrelated ancient Greek word, referred to songs sung by invited guests at banquets in ancient Greece extolling the virtues of the gods or heroic men! It seems to have come from the word for crooked, which seems appropriate for a problem section: short, diverse, and slightly twisted entertainments.

January 1996, Bruce L.R. Shawyer.

Bruce Shawyer (Memorial University of Newfoundland) became the third editor starting with volume 22 [1995 : 354; 1996 : 1]; he served until December 2002. Colin Bartholomew, also of Memorial University, served as assistant editor for one year, after which Clayton Halfyard took over that position from 1997 through 2002. With the new editor, the journal went from 10 issues of 36 pages (360 pages per volume) to 8 issues of 48 pages (384 pages per volume). Although the January and June issues were dropped, there were 24 extra pages per year, allowing increased efficiency of printing and a decrease in mailing costs. The journal went on-line for subscribers later that year [1996 : 289].

The Academy Corner.

Shawyer produced the Academy Corner during his tenure as editor. It dealt with problem solving at the undergraduate level. [1996 : 28] It ended with column 49 when he stepped down in 2002. [2002 : 480]

February 1997, *Cruz Mathematicorum with Mathematical Mayhem.*

Mathematical Mayhem had been founded in 1988 by Ravi Vakil and Patrick Surry, two Canadian IMO alumni, as a journal of high-school and college level

mathematics written by and for students. [1996 : 337; 1997 : 1-2, 30-31] When it amalgamated with *CruX* in volume 23, it brought additional high-school level material and gained, in return, a wider exposure. It was agreed that the new journal would continue the volume numbering of *CruX* as well as maintain its general external appearance. The number of pages per issue jumped from 48 to 64. The then current *Mayhem* editor Naoki Sato and assistant editor Cyrus Hsai continued in their positions for four years, through December 2000. Other editors and staff are listed below.

January 2002, français.

The statement of all problems would henceforth appear in both English and French. Jean-Marc Terrier has been translating from the start; other translators, serving for various periods, have been Hidemitsu Sayeki, Martin Goldstein, and Rolland Gaudet.

January 2003, Jim Totten.

James Edward Totten (University College of the Cariboo, renamed Thompson Rivers University in 2005) became the fourth editor starting with volume 29. [2002 : 287-288; 2003 : 1] His plan was to step down in June of 2008, after serving the final six months as co-editor, but he died on March 9, 2008 in his 61st year (born August 9, 1947). His Assistant editor was Bruce Crofoot.

January 2006, *Mayhem* on-line.

The *Mayhem* portion of the journal became open to the public on the internet starting with volume 32. Currently, all issues of *CruX with Mayhem* become free to the public after five years, while the *Mayhem* portion is always available for free.

January 2008, Václav (Vazz) Linek.

Vazz Linek (University of Winnipeg) became co-editor of the journal, then became the journal's fifth editor starting in July. [2007 : 449; 2008 : 193-194] His assistant editor is Jeff Hooper.

September 2009.

Johan Rudnick became the executive director of the Canadian Mathematical Society and, thereby, the new managing editor of *CruX*.

2 *CruX* Editors

Editors-in-chief

Léo Sauvé	March 1975 through January 1986
G.W. Sands	February 1986 through December 1990
G.W. Sands and Robert Woodrow	January 1991 through December 1995
Bruce L.R. Shawyer	January 1996 through December 2002
James Totten	January 2003 through May 2008
Václav (Vazz) Linek and James Totten	January 2008 through May 2008
Vazz Linek	June 2008 through March 2011
Shawn Godin	April 2011 to present

Managing Editors

Frederick G.B. Maskell	March 1975 through November 1984
Kenneth S. Williams	December 1984 through December 1988
Graham P. Wright	January 1989 through September 1999
Robert Quackenbush	October 1999 through December 2000
Graham P. Wright	January 2001 through August 2009
Johan Rudnick	September 2009 to present

Olympiad Corner

Murray S. Klamkin	1979 through 1986
Robert E. Woodrow	1987 through 2010

***CruX* Editorial Board**

J. Chris Fisher	1991 to present
Richard Guy	1991 through 2003
Denis Hanson (articles)	1991 through September 1999
Andy Liu (book reviews)	1991 through 1998
Richard Nowakowski	1991 through 1994
Edward T.H. Wang	1993 through 2010
Rod De Peiza	1994
Jim Totten	1994 through 2002 (when he became editor-in-chief)
Catherine Baker	1995 through 1999
Loki Jörgenson	1998 through 2002*
Alan Law (book reviews)	1999 through 2001
Bruce Gilligan (articles)	2000 through 2007
Iliya Bluskov	2000 through 2009
John Grant McLoughlin	2002-2009 (book review editor until 2008)
Richard (Rick) Brewster	2003 through 2005
Bruce Shawyer (editor-at-large)	2003 to present
Maria Torres	2006 through 2009
James Currie (articles)	2008 through 2010
Amar Sodhi (book reviews)	2009 to present
Nicolae Strungaru	2009 to present
Jonatan Aronsson	2010
Dzung Minh Ha	2010
Robert Craigen	2011 to present
Robert Dawson (articles)	2011 to present
Chris Grandison	2011 to present
Cosmin Pohoata	2011 to present

* Although he was responsible for the on-line edition starting in 1996, he was a board member for only five years.

Skoliad Corner

Robert Woodrow	1995 through May 2001
Shawn Godin	September 2001 through 2004
Robert Bilinski	2005 through 2008
Václav Linek	February 2009
Lily Yen and Mogens Hansen	March 2009 to present

3 *Mathematical Mayhem*

Mayhem editors

Before joining *Cruæ*: Patrick Surry, Ravi Vakil, Philip Jong, Jeff Higham, J.P. Grossman, Andre Chang, and Naoki Sato.

After the amalgamation:

Naoki Sato	1997 through 2000
Shawn Godin	2001 through 2006
Jeff Hooper	2007
Ian VanderBurgh	2008 through 2010

Mayhem assistant editors

Cyrus Hsai	1997 through 2000
Chris Cappadocia	2001 through 2002
John Grant McLoughlin	2003 through 2005
Jeff Hooper	2006
Ian VanderBurgh	2007
Lynn Miller	2011 to present

Mayhem editorial staff (various terms)

Richard Hoshino, Wai Ling Yee, Adrian Chan, Jimmy Chui, David Savitt, Donny Cheung, Paul Ottaway, Larry Rice, Dan MacKinnon, Ron Lancaster, Eric Robert, Monika Khbeis, Mark Bredin.

4 Special Issues and Articles

Special Issues

3:4 April 1977, The Gauss bicentennial Issue [1976 : 131, 161]

3:10 December 1977, Special Morley Issue; see also [1978 : 33-34, 132; 1978 : 304-305]

12:7 September 1986, Issue dedicated to Léo Sauvé [1986 : 65]

27:2 March 2001, Murray Klamkin 80th Birthday Issue [2001 : 65-85]

31:5 September 2005, Issue dedicated to Murray S. Klamkin [2004 : 361]

35:5 September 2009, Issue dedicated to James Edward Totten [2008 : 193]

Encyclopedic Articles

Léo Sauvé, The Celebrated Butterfly Problem. [1976 : 2-5]

Léo Sauvé, The Steiner-Lehmus Theorem, and

Charles W. Trigg, A Bibliography of the Steiner-Lehmus Theorem. [1976 : 19-24, 191-193]

5 Contributors and Friends

Honours and Awards

Murray Klamkin (University of Alberta), *M.A.A. Distinguished Service Award* [1988 : 33]; *David Hilbert International Award* [1992 : 224]

Marcin E. Kuczma (University of Warsaw), *David Hilbert International Award* [1992 : 224]

Ronald Dunkley (University of Waterloo), *Order of Canada* [1996 : 106]
 Andy Liu (University of Alberta), *David Hilbert International Award* [1996 : 201];
Outstanding University Professor [1999 : 65-66]
 Bruce Shawyer (Memorial University of Newfoundland), *Adrien Pouliot Award*
 [1998 : 19]
 Francisco Bellot Rosado (Institute Emilio Ferrari), *Erdős Prize* [2000 : 320]

Contributor Profiles

R. Robinson Rowe [1977 : 92,184,248; 1978 : 6,63,129,189]
 Kestrajū Satyanarayana [1981 : 294]
 Jack Garfunkel [1990 : 318]
 Leon Bankoff [1995 : 292]
 Jordi Dou [2002 : 56; 2006 : 65]
 K.R.S. Sastry [2006 : 2]
 Toshio Seimiya [2000 : 114; 2006 : 129]
 Christopher J. Bradley [2006 : 257]
 D.J. Smeenk [2006 : 353]
 Michel Bataille [2007 : 1]
 Richard K. Guy [2007 : 65]
 Walther Janous [2007 : 385]
 Peter Y. Woo [2008 : 1]
 Arkady Alt [2010 : 65]
 John G. Heuver [2010 : 193]

Death Notices

Richard J. Sempel, Carleton University (1930-1977) [1977 : 188]
 R. Robinson Rowe (1896-1978) [1978: 152]
 Herman Nyon, Paramaribo, Surinam (? - 1982) [1982 : 268]
 Viktors Linis, University of Ottawa (1916-1983) [1983 : 192]
 Kestrajū Satyanarayana (1897-1985) [1985 : 268]
 Geoffrey James Butler, University of Alberta (1944-1986) [1986 : 203]
 Samuel L. Greitzer, Rutgers University (1905-1988) [1988 : 161-162]
 Charles Trigg, San Diego, CA (1898-1989) [1989 : 224]
 Jakob T. Groenman, Arnhem, The Netherlands [1989 : 96]
 W.J. Blundon, Memorial University of Newfoundland (1916-1990) [1990 : 160]
 Jack Garfunkel (1910-1990) [1991 : 64]
 Oene Bottema, Delft, The Netherlands (1901-1992) [1993 : 32]
 Peter Joseph O'Halloran, University of Canberra (1931-1994) [1994 : 248-249]
 Pál Erdős (1913-1996) [1996 : 339]
 Leon Bankoff, Los Angeles, CA (1908-1997) [1997 : 145]
 Jessie Lei, University of Toronto (1980-2000) [2000 : 1-2]
 Herta Freitag, Roanoke, VA (1908-2000) [2000 : 535]
 H.S.M. Coxeter, University of Toronto (1907-2003) [2003 : 232]
 Murray S. Klamkin, University of Alberta (1921-2004) [2004 : 361]
 Robert Barrington Leigh, University of Toronto (1986-2006) [2006 : 453]
 Jordi Dou (1911-2007) [2007 : 457]
 James Edward Totten, Thompson Rivers University (1947-2008) [2008 : 194]

PROBLEMS

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1er septembre 2011**. Une étoile (★) après le numéro indique que le problème a été soumis sans solution.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

3613. *Proposé par Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie.*

Résoudre le système d'équations

$$\frac{x(y+1)}{x-1} = 7, \frac{y(z+1)}{y-1} = 5, \text{ et } \frac{z(x+1)}{z-1} = 12.$$

où x, y et z sont des entiers positifs.

3614. *Proposé par Neven Jurič, Zagreb, Croatie.*

À partir des décimales consécutives de $\frac{1}{7}$ on obtient l'ensemble des points $A(1, 4)$, $B(4, 2)$, $C(2, 8)$, ... dans le plan. Montrer que tous ces points appartiennent à une même ellipse. Calculer l'aire de cette ellipse.

3615. *Proposé par Pham Van Thuan, Université de Science de Hanoï, Hanoï, Vietnam.*

Montrer que si $x, y, z \geq 0$ et $x + y + z = 1$, alors

$$\frac{xy}{\sqrt{z+xy}} + \frac{yz}{\sqrt{x+yz}} + \frac{zx}{\sqrt{y+zx}} \leq \frac{1}{2}.$$

3616. *Proposé par Dinu Ovidiu Gabriel, Valcea, Roumanie.*

Calculer

$$L = \lim_{n \rightarrow \infty} n^{2k} \left[\frac{\arctan(n^k)}{n^k} - \frac{\arctan(n^k + 1)}{n^k + 1} \right],$$

où $k \in \mathbb{R}$.

3617. *Proposé par Michel Bataille, Rouen, France.*

Soit r un nombre rationnel positif. Montrer que si r^r est rationnel, alors r est un entier.

3618. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Soit $\alpha > 3$ un nombre réel. Trouver la valeur de

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{n}{(n+m)^{\alpha}}.$$

3619. *Proposé par Pham Kim Hung, étudiant, Université de Stanford, Palo Alto, CA, É-U.*

Soit a, b et c trois nombres réels non négatifs tels que $a + b + c = 3$. Montrer que

$$(a^2b - c)(b^2c - a)(c^2a - b) \leq 4(ab + bc + ca - 3a^2b^2c^2).$$

3620. *Proposé par John G. Heuver, Grande Prairie, AB.*

Soit P un point intérieur du tétraèdre $ABCD$ et désignons par A', B', C' et D' les points d'intersection des droites AP, BP, CP et DP avec les faces correspondantes opposées. On a alors

$$\begin{aligned} \frac{AP}{PA'} \frac{BP}{PB'} \frac{CP}{PC'} \frac{DP}{PD'} &= 3 + 2 \left(\frac{AP}{PA'} + \frac{BP}{PB'} + \frac{CP}{PC'} + \frac{DP}{PD'} \right) \\ &+ \frac{AP}{PA'} \frac{BP}{PB'} + \frac{AP}{PA'} \frac{CP}{PC'} + \frac{AP}{PA'} \frac{DP}{PD'} \\ &+ \frac{BP}{PB'} \frac{CP}{PC'} + \frac{BP}{PB'} \frac{DP}{PD'} + \frac{CP}{PC'} \frac{DP}{PD'}. \end{aligned}$$

3621. *Proposé par Titu Zvonaru, Comănești, Roumanie.*

Soit a, b et c trois nombres réels non négatifs avec $a + b + c = 1$. Montrer que

$$\frac{27}{128} [(a-b)^2 + (b-c)^2 + (c-a)^2] + \frac{4}{1+a} + \frac{4}{1+b} + \frac{4}{1+c} \leq \frac{3}{ab + bc + ca}.$$

3622★. *Proposé par George Tsapakidis, Agrinio, Grèce.*

On donne un quadrilatère $ABCD$.

- Trouver des conditions nécessaires et suffisantes sur les côtés et les angles de $ABCD$ pour qu'il existe un point intérieur P tel que deux droites perpendiculaires issues de P divisent le quadrilatère $ABCD$ en quatre quadrilatères d'aire égale.
- Déterminer P .

3623. *Proposé par Michel Bataille, Rouen, France.*

Soit z_1, z_2, z_3, z_4 quatre nombres complexes distincts mais de même module, $\alpha = |(z_3 - z_2)(z_3 - z_4)|$, $\beta = |(z_1 - z_2)(z_1 - z_4)|$ et

$$u(\epsilon) = \frac{\alpha(z_1 - z_4) + \epsilon\beta(z_3 - z_4)}{\alpha(z_1 - z_2) + \epsilon\beta(z_3 - z_2)}.$$

Montrer que $u(+1)$ ou $u(-1)$ est un nombre réel.

3624. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Calculer la somme

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(1 - \frac{1}{2} + \frac{1}{3} - \dots + \frac{(-1)^{n+1}}{n} \right).$$

3625. *Proposé par Pham Van Thuan, Université de Science de Hanoi, Hanoi, Vietnam.*

Soit a, b et c trois nombres réels positifs. Montrer que

$$\sqrt{\frac{a}{a+b}} + \sqrt{\frac{b}{b+c}} + \sqrt{\frac{c}{c+a}} \leq 2\sqrt{1 + \frac{abc}{(a+b)(b+c)(c+a)}}.$$

.....

3613. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Solve the system of equations

$$\frac{x(y+1)}{x-1} = 7, \frac{y(z+1)}{y-1} = 5, \text{ and } \frac{z(x+1)}{z-1} = 12.$$

Where x, y , and z are positive integers.

3614. *Proposed by Neven Jurič, Zagreb, Croatia.*

Taking consecutive decimal digits of $\frac{1}{7}$ the set of points $A(1, 4), B(4, 2), C(2, 8), \dots$ in the plane is obtained. Prove that all these points belong to the same ellipse. Compute the area of the ellipse.

3615. *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Prove that if $x, y, z \geq 0$ and $x + y + z = 1$, then

$$\frac{xy}{\sqrt{z+xy}} + \frac{yz}{\sqrt{x+yz}} + \frac{zx}{\sqrt{y+zx}} \leq \frac{1}{2}.$$

3616. *Proposed by Dinu Ovidiu Gabriel, Valcea, Romania.*

Compute

$$L = \lim_{n \rightarrow \infty} n^{2k} \left[\frac{\arctan(n^k)}{n^k} - \frac{\arctan(n^k + 1)}{n^k + 1} \right],$$

where $k \in \mathbb{R}$.

3617. *Proposed by Michel Bataille, Rouen, France.*

Let r be a positive rational number. Show that if r^r is rational, then r is an integer.

3618. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $\alpha > 3$ be a real number. Find the value of

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{n}{(n+m)^\alpha}.$$

3619. *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let a , b , and c be nonnegative real numbers such that $a + b + c = 3$. Prove that

$$(a^2b - c)(b^2c - a)(c^2a - b) \leq 4(ab + bc + ca - 3a^2b^2c^2).$$

3620. *Proposed by John G. Heuver, Grande Prairie, AB.*

Let P be an interior point in tetrahedron $ABCD$ and let AP , BP , CP , and DP meet the corresponding opposite faces in A' , B' , C' , and D' then

$$\begin{aligned} \frac{AP}{PA'} \frac{BP}{PB'} \frac{CP}{PC'} \frac{DP}{PD'} &= 3 + 2 \left(\frac{AP}{PA'} + \frac{BP}{PB'} + \frac{CP}{PC'} + \frac{DP}{PD'} \right) \\ &+ \frac{AP}{PA'} \frac{BP}{PB'} + \frac{AP}{PA'} \frac{CP}{PC'} + \frac{AP}{PA'} \frac{DP}{PD'} \\ &+ \frac{BP}{PB'} \frac{CP}{PC'} + \frac{BP}{PB'} \frac{DP}{PD'} + \frac{CP}{PC'} \frac{DP}{PD'}. \end{aligned}$$

3621. *Proposed by Titu Zvonaru, Comănești, Romania.*

Let a , b , and c be nonnegative real numbers with $a + b + c = 1$. Prove that

$$\frac{27}{128} [(a-b)^2 + (b-c)^2 + (c-a)^2] + \frac{4}{1+a} + \frac{4}{1+b} + \frac{4}{1+c} \leq \frac{3}{ab + bc + ca}.$$

3622★. Proposed by George Tsapakidis, Agrinio, Greece.

Let $ABCD$ be a quadrilateral.

- (a) Find sufficient and necessary condition on the sides and angles of $ABCD$, so that there is an inner point P such that two perpendicular lines through P divide the quadrilateral $ABCD$ into four quadrilaterals of equal area.
- (b) Determine P .

3623. Proposed by Michel Bataille, Rouen, France.

Let z_1, z_2, z_3, z_4 be distinct complex numbers with the same modulus, $\alpha = |(z_3 - z_2)(z_3 - z_4)|$, $\beta = |(z_1 - z_2)(z_1 - z_4)|$ and

$$u(\epsilon) = \frac{\alpha(z_1 - z_4) + \epsilon\beta(z_3 - z_4)}{\alpha(z_1 - z_2) + \epsilon\beta(z_3 - z_2)}.$$

Prove that $u(+1)$ or $u(-1)$ is a real number.

3624. Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.

Calculate the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(1 - \frac{1}{2} + \frac{1}{3} - \dots + \frac{(-1)^{n+1}}{n} \right).$$

3625. Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.

Let a, b , and c be positive real numbers. Prove that

$$\sqrt{\frac{a}{a+b}} + \sqrt{\frac{b}{b+c}} + \sqrt{\frac{c}{c+a}} \leq 2\sqrt{1 + \frac{abc}{(a+b)(b+c)(c+a)}}.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3514. [2010 : 107, 109] *Proposed by Michel Bataille, Rouen, France.*

Let m be a positive real number and a, b, c real numbers such that

$$a(a - b) + b(b - c) + c(c - a) = m.$$

What is the range of $ab(a - b) + bc(b - c) + ca(c - a)$?

Solution by Albert Stadler, Herrliberg, Switzerland.

We set

$$x = b - a, \quad y = c - b, \quad z = a - c. \tag{1}$$

Then

$$\begin{aligned} x + y + z &= 0, \\ -xy - yz - zx &= a(a - b) + b(b - c) + c(c - a), \\ xyz &= ab(a - b) + bc(b - c) + ca(c - a), \end{aligned}$$

and so $(w - x)(w - y)(w - z) = w^3 - mw - xyz$.

Thus, the range consists of all real numbers n for which $w^3 - mw - n$ has three real roots (if the roots x, y, z are real, then (1) can be solved for the corresponding real numbers a, b, c).

A cubic equation has three real roots if and only if its discriminant D is not positive. Here the discriminant is $D = \left(\frac{-m}{3}\right)^3 + \left(\frac{-n}{2}\right)^2$, so $D \leq 0$ holds if and only if $|n| \leq 2\left(\frac{m}{3}\right)^{3/2}$.

Therefore, the range is the interval $\left[-2\left(\frac{m}{3}\right)^{3/2}, 2\left(\frac{m}{3}\right)^{3/2}\right]$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; OLIVER GEUPEL, Brühl, NRW, Germany; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. One incomplete solution was submitted.

3515. [2010 : 107, 110] *Proposed by José Luis Díaz-Barrero and Josep Rubió-Masagú, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let x , y , and z be positive real numbers. Prove that

$$\left(\frac{\{x\}^2}{y} + \frac{\lfloor x \rfloor^2}{z}\right) + \left(\frac{\{y\}^2}{z} + \frac{\lfloor y \rfloor^2}{x}\right) + \left(\frac{\{z\}^2}{x} + \frac{\lfloor z \rfloor^2}{y}\right) \geq \frac{x^2 + y^2 + z^2}{x + y + z},$$

where $\lfloor a \rfloor$ is the greatest integer not exceeding a , and $\{a\} = a - \lfloor a \rfloor$.

Solution by Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina.

Note first that for all positive reals a , b , and c we have by the Cauchy–Schwarz Inequality that

$$\left(\frac{\{c\}^2}{a} + \frac{\lfloor c \rfloor^2}{b}\right)(a + b) \geq (\{c\} + \lfloor c \rfloor)^2 = c^2,$$

which implies that

$$\frac{\{c\}^2}{a} + \frac{\lfloor c \rfloor^2}{b} \geq \frac{c^2}{a + b} > \frac{c^2}{a + b + c}.$$

Hence,

$$\begin{aligned} & \left(\frac{\{x\}^2}{y} + \frac{\lfloor x \rfloor^2}{z}\right) + \left(\frac{\{y\}^2}{z} + \frac{\lfloor y \rfloor^2}{x}\right) + \left(\frac{\{z\}^2}{x} + \frac{\lfloor z \rfloor^2}{y}\right) \\ & > \frac{x^2}{x + y + z} + \frac{y^2}{x + y + z} + \frac{z^2}{x + y + z} = \frac{x^2 + y^2 + z^2}{x + y + z}. \end{aligned}$$

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; OLEH FAYNSHTEYN, Leipzig, Germany; OLIVER GEUPEL, Brühl, NRW, Germany; JOHAN GUNARDI, student, SMPK 4 BPK PENABUR, Jakarta, Indonesia; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ALBERT STADLER, Herliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposers.

The featured solution shows that the inequality is strict.

The proofs given by most solvers are similar to the one featured above. Faynshteyn obtained the stronger result in which the right side of the inequality is replaced by $\frac{3}{2} \cdot \frac{x^2 + y^2 + z^2}{x + y + z}$ by proving that $\frac{x^2}{y + z} + \frac{y^2}{z + x} + \frac{z^2}{x + y} \geq \frac{3}{2} \cdot \frac{x^2 + y^2 + z^2}{x + y + z}$. He also claimed the following generalization without proof:

$$\sum_{\text{cyclic}} \left(\frac{\{x\}^{2n}}{y} + \frac{\lfloor x \rfloor^{2n}}{z}\right) \geq \frac{3}{2^{2n-1}} \cdot \frac{x^{2n} + y^{2n} + z^{2n}}{x + y + z}.$$

3516. [2010 : 107, 110] *Proposed by János Bodnár, Budapest, Hungary.*

Let P and Q be interior points of triangle ABC . Let AA' , BB' , and CC' be three concurrent cevians through P . The line through A' parallel to AQ intersects the lines BQ and CQ at points L and M' , respectively. The line through B' parallel to BQ intersects the lines CQ and AQ at points M and N' , respectively. The line through C' parallel to CQ intersects the lines AQ and BQ at points N and L' , respectively.

Is it true that triangles LMN and $L'M'N'$ have the same area?

Solution by Shailesh Shirali, Rishi Valley School, India.

The point P plays no role in the result: A' , B' , and C' can be any points on the lines BC , CA , and AB , respectively; furthermore, Q can be any point except a vertex in the plane of $\triangle ABC$. But even these points play no essential role. The result is a special case of a known theorem; for triangles LMN and $L'M'N'$ to have the same area, all that is required is that the six vertices satisfy

$$L'L \parallel MN', \quad LM' \parallel N'N, \quad \text{and} \quad M'M \parallel NL',$$

which they do by definition. In terms of neutral letters,

Theorem. *If A, B, C, D, E, F are six points in an affine plane such that $AB \parallel DE$, $BC \parallel EF$, and $CD \parallel FA$, then triangles ACE and BDF have equal areas and the same orientation.*

It is easier to find a proof for the theorem than a reference. We use vectors with an arbitrary point taken to be the origin. Since $AB \parallel DE$ we have $(\vec{A} - \vec{B}) \times (\vec{D} - \vec{E}) = \mathbf{0}$. Similarly, $(\vec{B} - \vec{C}) \times (\vec{E} - \vec{F}) = \mathbf{0}$ and $(\vec{C} - \vec{D}) \times (\vec{F} - \vec{A}) = \mathbf{0}$. Expanding these cross products we get,

$$\begin{aligned} \vec{A} \times \vec{D} - \vec{B} \times \vec{D} - \vec{A} \times \vec{E} + \vec{B} \times \vec{E} &= \mathbf{0}, \\ \vec{B} \times \vec{E} - \vec{C} \times \vec{E} - \vec{B} \times \vec{F} + \vec{C} \times \vec{F} &= \mathbf{0}, \\ \vec{C} \times \vec{F} - \vec{D} \times \vec{F} - \vec{C} \times \vec{A} + \vec{D} \times \vec{A} &= \mathbf{0}. \end{aligned}$$

Subtracting the second relation from the sum of the first and third we get,

$$\vec{A} \times \vec{C} + \vec{C} \times \vec{E} + \vec{E} \times \vec{A} = \vec{B} \times \vec{D} + \vec{D} \times \vec{F} + \vec{F} \times \vec{B}.$$

The final equation tells us that $\triangle ACE$ and $\triangle BDF$ have equal areas and the same orientation.

Also solved by Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

3517. [2010 : 108, 110] *Proposed by Václav Konečný, Big Rapids, MI, USA.*

Is there a scalene triangle ABC for which there exists a point P in the plane of ABC such that, for each line ℓ through P , the sum of the squares of the distances of A , B , and C to ℓ is constant?

Solutions by George Apostolopoulos, Messolonghi, Greece; Roy Barbara, Lebanese University, Fanar, Lebanon; Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany; and the proposer.

Yes, there are scalene triangles with the required property. In Cartesian coordinates we take P to be the origin. The line through P with normal unit vector $(\cos \phi, \sin \phi)$ is described by the equation $x \cos \phi + y \sin \phi = 0$, whence the square of the distance from a point $X = (x_0, y_0)$ to that line is

$$d_X = (x_0 \cos \phi + y_0 \sin \phi)^2.$$

The examples given by our five correspondents are listed alphabetically by last name; for each example the vertices are given in the first three columns and their squared distances to $x \cos \phi + y \sin \phi = 0$ in the next three, followed in the final column by the constant sum of those three squared distances.

A	B	C	d_A	d_B	d_C	sum
$(0, 5)$	$(3, 0)$	$(4, 0)$	$25 \sin^2 \phi$	$9 \cos^2 \phi$	$16 \cos^2 \phi$	25
$(1, 0)$	$(2, 0)$	$(0, \sqrt{5})$	$\cos^2 \phi$	$4 \cos^2 \phi$	$5 \sin^2 \phi$	5
$(1, -2)$	$(2, 2)$	$(2, -1)$	$(\cos \phi - 2 \sin \phi)^2$	$(2 \cos \phi + 2 \sin \phi)^2$	$(2 \cos \phi - \sin \phi)^2$	9
$(1, 0)$	$(0, \frac{1}{2})$	$(0, \frac{\sqrt{3}}{2})$	$\cos^2 \phi$	$\frac{1}{4} \sin^2 \phi$	$\frac{3}{4} \sin^2 \phi$	1
$(0, 5)$	$(-3, 0)$	$(4, 0)$	$25 \sin^2 \phi$	$9 \cos^2 \phi$	$16 \cos^2 \phi$	25

Also solved by ALBERT STADLER, Herrliberg, Switzerland. There was one incorrect submission.

*Stadler also took P to be the origin, but he used complex numbers \mathbf{a} , \mathbf{b} , and \mathbf{c} to represent the vertices A, B , and C . He proved that the sum of the squares of the distances from A, B , and C to a line through $\mathbf{0}$ is constant if and only if $\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{c}^2 = \mathbf{0}$; in other words, given vertices corresponding to the complex numbers \mathbf{a} and \mathbf{b} , the third vertex can be either square root of $-(\mathbf{a}^2 + \mathbf{b}^2)$. This, of course, produces all triangles, even degenerate, with the required property with respect to the origin. The proposer was motivated by problem **3291** [2007 : 485, 487; 2008 : 492-494], which applied to isosceles triangles.*

3518. [2010 : 108, 110] *Proposed by Yakub N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan.*

Prove that if n and i are integers with $0 \leq i \leq n$, then

$$0 < 1 - \frac{C_i}{2^{2i+1}} - \{10^{(2i+1)n} \cdot \sqrt{10^{2n} - 1}\} < \frac{1}{10^{2n}},$$

where $\{x\}$ denotes the fractional part of the real number x and C_i is the i^{th} Catalan number, $C_i = \frac{1}{i+1} \binom{2i}{i}$, $i \geq 0$.

Solution by Oliver Geupel, Brühl, NRW, Germany.

The case $n = 0$ is immediate; so assume that $n > 0$. We will use the standard formula $C_i = (-1)^i \binom{1/2}{i+1} \cdot 2^{2i+1}$ and the inequality $C_i < 2^{2i}$.

Let $s = 10^{2n(i+1)} - \sum_{k=1}^i C_{k-1} \cdot \frac{10^{2n(i+1-k)}}{2^{2k-1}}$. It is well-known that the Catalan number C_{k-1} is an integer for each integer $k > 0$. Moreover, for $1 \leq k \leq i \leq n$, it holds that $2k-1 < 2n \leq 2n(i+1-k)$; hence $\frac{10^{2n(i+1-k)}}{2^{2k-1}}$ is an integer. Consequently,

$$s \in \mathbb{Z}. \quad (1)$$

Let $t = \frac{1}{10^{2n}} \sum_{k=i+2}^{\infty} \frac{C_{k-1}}{2^{2k-1}} \cdot \left(\frac{1}{10^{2n}}\right)^{k-i-2}$. By the binomial series, we have that

$$\begin{aligned} 0 < t &< \frac{1}{10^{2n}} \sum_{k=1}^{\infty} \frac{C_{k-1}}{2^{2k-1}} = -\frac{1}{10^{2n}} \sum_{k=1}^{\infty} (-1)^k \binom{1/2}{k} \\ &= \frac{1 - \sqrt{1-1}}{10^{2n}} = \frac{1}{10^{2n}}. \end{aligned} \quad (2)$$

Moreover,

$$0 < \frac{C_i}{2^{2i+1}} + t < \frac{2^{2i}}{2^{2i+1}} + \frac{1}{2} = 1. \quad (3)$$

Let $u = 10^{(2i+1)n} \cdot \sqrt{10^{2n} - 1}$. Using the Binomial series, we obtain

$$\begin{aligned} u &= 10^{2n(i+1)} \cdot \sqrt{1 - \frac{1}{10^{2n}}} = 10^{2n(i+1)} \sum_{k=0}^{\infty} (-1)^k \binom{1/2}{k} \frac{1}{10^{2nk}} \\ &= \sum_{k=0}^i (-1)^k \binom{1/2}{k} 10^{2n(i+1-k)} - (-1)^i \binom{1/2}{i+1} \\ &\quad - \sum_{k=i+2}^{\infty} (-1)^{k-1} \binom{1/2}{k} \frac{1}{10^{2n(k-i-1)}} \\ &= s - \frac{C_i}{2^{2i+1}} - t. \end{aligned}$$

By (1) and (3), we obtain $\{u\} = 1 - \frac{C_i}{2^{2i+1}} - t$. The conclusion follows immediately from (2).

Also solved by ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

3519. [2010 : 108, 110] *Proposed by Nguyen Duy Khanh, student, Hanoi University of Science, Hanoi, Vietnam.*

Two triangles ABC and $A'B'C'$ have areas S and S' , respectively. Let w_a, w_b, w_c be the lengths of the internal angle bisectors of ABC to the sides BC, AC, AB , respectively, and define w'_a, w'_b, w'_c similarly. Prove or disprove that $w_a w'_a + w_b w'_b + w_c w'_c \geq 3\sqrt{3SS'}$.

Solution by Oliver Geupel, Brühl, NRW, Germany.

We show that the inequality is not valid in general.

As usual, let $a = BC, b = CA, c = AB$. Let ABC be a triangle with $a = 3$ and $c = b + 1 > 2$. Using a well-known formula for the length of an angle bisector, we obtain

$$w_a = \sqrt{bc \left(1 - \frac{a^2}{(b+c)^2} \right)} = \frac{2\sqrt{(b-1)b(b+1)(b+2)}}{2b+1},$$

and similarly

$$w_b = \frac{2\sqrt{6(b+1)(b+2)}}{b+4}, \quad w_c = \frac{2\sqrt{3b(b+2)}}{b+3}.$$

Let $A'B'C'$ be a triangle with $B'C' = c, C'A' = b, A'B' = a$. We have $w'_a = w_c, w'_b = w_b, w'_c = w_a$. If p denotes the common semiperimeter of both triangles, then we obtain

$$S = S' = \sqrt{p(p-a)(p-b)(p-c)} = \sqrt{2(b-1)(b+2)}.$$

We will prove that for sufficiently large b ,

$$w_a w'_a + w_b w'_b + w_c w'_c < 3\sqrt{3SS'}.$$

By substituting the relations above, we obtain

$$\begin{aligned} \frac{8b(b+2)\sqrt{3(b-1)(b+1)}}{(b+3)(2b+1)} + \frac{24(b+1)(b+2)}{(b+4)^2} \\ < 3\sqrt{6(b-1)(b+2)}. \end{aligned} \quad (1)$$

As $b \rightarrow \infty$, the left side of (1) is asymptotic to $\frac{8\sqrt{3}}{2} \cdot b = 4\sqrt{3}b$, while the right side of (1) is asymptotic to $3\sqrt{6}b$. Since $(4\sqrt{3})^2 = 48 < 54 = (3\sqrt{6})^2$, the right side eventually exceeds the left side. Indeed, with a calculator we find that this occurs when $b = 13$.

This completes the proof.

No other solutions were received.

Geupel remarked that a related inequality appears in CRUX Problem 2029 [1995 : 91, 129-32], namely

$$w_b w_c + w_c w_a + w_a w_b \geq 3\sqrt{3S},$$

which is true.

3520. [2010 : 108, 110] *Proposed by Ricardo Barroso Campos, University of Seville, Seville, Spain.*

Construct a triangle ABC such that the line through the incentre and the circumcentre is parallel to the side AB .

I. *Solution by Michel Bataille, Rouen, France.*

Construct $\triangle ABC$ given its circumcentre O , circumradius R , and $\angle C$. Suppose that we have constructed a scalene triangle ABC such that the line through O and the incentre I is parallel to AB . Since I is interior to the triangle, O must be on the same side of AB as C . For C' the midpoint of AB it follows that $\angle C'OA = \angle BCA (= \angle C)$. Observing that $OC' = d(O, AB) = d(I, AB) = r$ (the inradius) and $OA = R$ (the circumradius), we deduce that $\frac{r}{R} = \cos C$; from Euler's inequality $2r < R$, so that $\angle C$ is necessarily between 60° and 90° . Moreover, if M is the point where the ray $[OC')$ intersects the circumcircle Γ , MC is the angle bisector of $\angle C$ and, from a familiar theorem, $MB = MA = MI$. This analysis of the figure leads to the following three-step construction:

- Construct an angle equal to the given $\angle C$ with vertex at O , and the point A on one of its legs such that $OA = R$. Let C' be the orthogonal projection of A onto the other leg, and let M be the point where the circle with centre O and radius OA intersects that leg. Denote the circle by Γ and the point where Γ intersects AC' by B .
- Since $MO = AO$, when $90^\circ > \angle MOA > 60^\circ$ we have $MA > MO$, so that the circle with centre M , radius MA , intersects the line through O parallel to AC' in two points. Let I be one of those points.
- Let C be the second point of intersection of the line MI with Γ . Then MC is the angle bisector of $\angle BCA$ (since M is the midpoint of the arc AB of Γ that does not contain C), and I is the incentre of $\triangle ABC$ (since it is the point on the angle bisector for which $MI = MA = MB$). The triangle ABC satisfies the requirement.

The construction shows that there is at most one triangle with $OI \parallel AB$ for the given O, R , and $\angle C$, and it exists if and only if $60^\circ < \angle C < 90^\circ$.

II. *Composite of solutions by Mohammed Aassila, Strasbourg, France; John G. Hewer, Grande Prairie, AB; and Ricard Peiró, IES "Abastos", Valencia, Spain.*

Construct $\triangle ABC$ given O, I , and r . Should such a triangle exist, we let F be the orthogonal projection of I onto AB ; then $IF = r$. Consider the right triangle OIF ; since by Euler's formula $OI^2 = R^2 - 2Rr$, we deduce that $OF^2 = R^2 - 2Rr + r^2$, whence $OF = R - r$. It follows that if the point P is at a distance r on the line OF beyond F , $OP = R$. The construction then proceeds in four steps:

- Draw the circle with centre I and radius r (which will become the incircle). Construct the perpendicular to IO at I , and denote by F one of the points

where it intersects the circle. Construct the line perpendicular to IF at F ; call it ℓ . (The line ℓ will become AB .)

- Draw the circle with centre F and radius $r = FI$; denote by P the point where it meets the line OP on the other side of F from O .
- Draw the circle with centre O and radius OP . (This will be the circumcircle because $OP = R$.) Call the points A and B where it meets ℓ .
- Define E to be the second point where the circle with centre A and radius AF intersects the incircle, and define C to be the second point where AE intersects the circumcircle. ABC is the required triangle. (Because by construction, AF is tangent to the incircle and $AE = AF$, it follows that AE must also be tangent to the incircle. BC is also tangent to the incircle because we constructed the length R so that $OI^2 = R^2 - 2Rr$.)

The construction shows that the required triangle exists and is unique for any given pair of points O and I , and positive length r .

Also solved by MOHAMMED AASSILA, Strasbourg, France (a second solution); GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon (2 solutions); FRANCISCO JAVIER GARCÍA CAPITÁN, IES Álvarez Cubero, Priego de Córdoba, Spain; OLIVER GEUPEL, Brühl, NRW, Germany; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; SHAILESH SHIRALI, Rishi Valley School, India; D.J. SMEENK, Zaltbommel, the Netherlands; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was one incomplete submission.

Because the problem called for the construction of just one example of the required triangle without specifying what we are given, our solvers provided a wide variety of constructions, several of which were as simple as our featured solutions. Most were based on properties of triangles for which $OI \parallel AB$; such properties have been the subject of numerous problems in *CRUX with MAYHEM* such as 659 [1982 : 215-216], which was based on Problem 758 in *Mathematics Magazine* 43:5 (Nov. 1970) pages 285-286. Here are three of them:

- (1) Construct the triangle given $\angle A$ and side length c ; Smeenk and Woo both used the property $\cos A + \cos B = 1$.
- (2) Construct the triangle given side lengths a and b ; Aassila, Apostolopoulos, Garcia Capitán, and Shirali all obtained a formula for c in terms of a and b :

$$c = \frac{ab + \sqrt{a^4 - a^2b^2 + b^4}}{a + b}.$$

(For example, one can apply the cosine law to the formula used in (1).) Shirali observed parenthetically that this formula implies that there are no integer-sided triangles with the required property (because the expression under the radical sign has an integer square root only if $a = b = 0$).

- (3) Construct the triangle given R and the distance d_A from O to the side BC ; Geupel noted that Carnot's theorem, $d_A + d_B + d_C = R + r$ [Nathan Altshiller Court, *College Geometry*, page 83, Section 146] becomes simpler when $d_C = r$ —the circle whose diameter is OC contains the projections of O onto BC and AC , which are easily constructed using the distances d_A and $R - d_A$.

3523. [2010 : 109, 111] *Proposed by Slavko Simic, Mathematical Institute SANU, Belgrade, Serbia.*

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuously differentiable function. Solve the functional equation

$$f(x) + f(y) - 2f\left(\frac{x+y}{2}\right) = p(x-y)(f'(x) - f'(y)),$$

where p is a real parameter independent of x, y .

Partial solution by the proposer, modified by the editor.

We show that if f has a continuous second derivative, then

(i) $f(x) = ax + b$ if $p \neq \frac{1}{4}$,

(ii) $f(x) = ax^3 + bx^2 + cx + d$ if $p = \frac{1}{4}$,

where a, b, c , and d are arbitrary constants.

Differentiating the given equation with respect to x , we obtain

$$f'(x) - f'\left(\frac{x+y}{2}\right) = p(f'(x) - f'(y) + (x-y)f''(x)). \quad (1)$$

Next, differentiating (1) with respect to y , we obtain

$$\frac{1}{2}f''\left(\frac{x+y}{2}\right) = p(f''(x) + f''(y)). \quad (2)$$

Setting $y = x$ in (2) we then have $(2p - \frac{1}{2})f''(x) = 0$.

If $p \neq \frac{1}{4}$, then $f''(x) = 0$, from which (i) follows.

If $p = \frac{1}{4}$, then (2) becomes

$$f''\left(\frac{x+y}{2}\right) = \frac{1}{2}(f''(x) + f''(y)).$$

That is,

$$g\left(\frac{x+y}{2}\right) = \frac{1}{2}(g(x) + g(y)), \quad (3)$$

where $g = f''$, which is a variant of Cauchy's equation also known as Jensen's equation. It is a well-known fact in the theory of functional equations that the only solution of (3) is $g(x) = \alpha x + \beta$ for some constants α and β . From this, (ii) follows immediately.

Also solved by MICHEL BATAILLE, Rouen, France, under the same stronger assumption on f'' .

Three other solutions were submitted which were either incomplete or partially incorrect. The problem remains open if we assume only that $f \in C^1(-\infty, \infty)$.

That the only solution of (3) above is $g(x) = \alpha x + \beta$ for some constants α, β when g is continuous or monotone can be found in Functional Equations and How to Solve Them, by C.G. Small, Springer Problem Books in Mathematics, 2007. In case (i) both sides of the given equation vanish, while in case (ii) both sides equal $\frac{3}{4}a(x+y)(x-y)^2 + \frac{1}{2}b(x-y)^2$.

3524. [2010 : 109, 111] *Proposed by Titu Zvonaru, Comănești, Romania.*

Let a_1, a_2, \dots, a_{n+1} be positive real numbers satisfying the condition $a_{n+1} = \min\{a_1, a_2, \dots, a_{n+1}\}$. Prove that

$$\begin{aligned} a_1^{n+1} + a_2^{n+1} + \dots + a_{n+1}^{n+1} - (n+1)a_1 a_2 \cdots a_{n+1} \\ \geq (n+1)a_{n+1}[(a_1 - a_{n+1})^n + (a_2 - a_{n+1})^n + \dots + (a_n - a_{n+1})^n \\ - n(a_1 - a_{n+1})(a_2 - a_{n+1}) \cdots (a_n - a_{n+1})]. \end{aligned}$$

Solution by George Apostolopoulos, Messolonghi, Greece, expanded by the editor.

Let $t = a_{n+1}$ and let $x_i = a_i - a_{n+1}$ for $i = 1, 2, \dots, n$, then we have that $t > 0$ and $x_i \geq 0$ for all i . We can now write the inequality as

$$\begin{aligned} \sum_{i=1}^n (t + x_i)^{n+1} + t^{n+1} - (n+1)t \prod_{i=1}^n (t + x_i) \\ \geq (n+1)t \left(\sum_{i=1}^n x_i^n - n \prod_{i=1}^n x_i \right). \end{aligned} \quad (1)$$

Consider (1) as a polynomial of t . Then, it suffices to show that, for each degree of t , the coefficient of the LHS is not less than the coefficient of the RHS. The coefficients of t^0 and t^{n+1} follow immediately. The coefficient of t^1 is $(n+1)(\sum_{i=1}^n x_i^n - n \prod_{i=1}^n x_i)$ for both sides of the inequality. The remaining coefficients, t^{i+1} for $i = 1, 2, \dots, n-1$, are equal to 0 for the RHS so it now suffices to show that the remaining coefficients of the LHS are non-negative, that is

$$\sum_{j=1}^n \binom{n+1}{i+1} x_j^{n-i} - (n+1) \sum_{1 \leq j_1 < j_2 < \dots < j_{n-i} \leq n} \prod_{k=1}^{n-i} x_{j_k} \geq 0. \quad (2)$$

By the AM-GM Inequality, we have that $\prod_{k=1}^{n-i} x_{j_k} \leq \frac{1}{n-i} \sum_{k=1}^{n-i} x_{j_k}^{n-i}$. Thus,

$$(n+1) \sum_{1 \leq j_1 < j_2 < \dots < j_{n-i} \leq n} \prod_{k=1}^{n-i} x_{j_k} \leq \frac{n+1}{n-i} \binom{n-1}{i} \sum_{i=1}^n x_j^{n-i}. \quad (3)$$

From (2) and (3), it suffices to show that

$$\binom{n+1}{i+1} \geq \frac{n+1}{n-i} \binom{n-1}{i}.$$

From the definition of binomial coefficients, we have that

$$\binom{n+1}{i+1} = \frac{n+1}{n-i} \frac{n}{i+1} \binom{n-1}{i} \geq \frac{n+1}{n-i} \binom{n-1}{i}$$

and we are done.

Also solved by Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and the proposer.

3525. [2010 : 109, 111] *Proposed by an anonymous proposer.*

Let $0 \leq a, b \leq 1$. Prove that $\frac{a+b}{1+ab} \leq (\sqrt{a} + \sqrt{b} - \sqrt{ab})^2$.

Solution by George Apostolopoulos, Messolonghi, Greece.

For convenience, take $\alpha = \sqrt{a}$ and $\beta = \sqrt{b}$. Then we need to show that for all $0 \leq \alpha, \beta \leq 1$ we have

$$\frac{\alpha^2 + \beta^2}{1 + \alpha^2\beta^2} \leq (\alpha + \beta - \alpha\beta)^2,$$

or equivalently

$$(\alpha + \beta - \alpha\beta)^2 (1 + \alpha^2\beta^2) - (\alpha^2 + \beta^2) \geq 0.$$

This can be factored as

$$\alpha\beta(1 - \alpha)(1 - \beta)[\alpha^2\beta^2 - \alpha\beta(\alpha + \beta) - \alpha\beta + 2] \geq 0.$$

—Since $\alpha\beta(1 - \alpha)(1 - \beta) \geq 0$, it suffices to show that —————

$$\alpha^2\beta^2 - \alpha\beta(\alpha + \beta) - \alpha\beta + 2 \geq 0.$$

But

$$\alpha^2\beta^2 - \alpha\beta(\alpha + \beta) - \alpha\beta + 2 \geq \alpha^2\beta^2 - 3\alpha\beta + 2 = (1 - \alpha\beta)(2\alpha\beta) \geq 0.$$

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OLIVER GEUPEL, Brühl, NRW, Germany; KEE-WAI LAU, Hong Kong, China; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; ALBERT STADLER, Herrliberg, Switzerland; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; PETER Y. WOO, Biola University, La Mirada, CA, USA; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and the proposer.

3526.[2010 : 109, 111] *Proposed by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Let a , b , and c be positive real numbers. Prove that

$$\sum_{\text{cyclic}} \frac{a}{\sqrt{a^2 + 2(b+c)^2}} \geq 1.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

By Hölder inequality we have

$$\left(\sum_{\text{cyclic}} \frac{a}{\sqrt{a^2 + 2(b+c)^2}} \right)^2 \left(\sum_{\text{cyclic}} a(a^2 + 2(b+c)^2) \right) \geq (a+b+c)^3.$$

Thus we only need to show that

$$(a+b+c)^3 \geq \sum_{\text{cyclic}} a(a^2 + 2(b+c)^2),$$

or equivalently that

$$a^2b + a^2c + ab^2 + ac^2 + b^2c + bc^2 \geq 6abc.$$

—But this is immediate from the AM-GM inequality. This completes the proof.
Equality holds if and only if $a = b = c$

Also solved by ARKADY ALT, *San Jose, CA, USA*; GEORGE APOSTOLOPOULOS, *Messolonghi, Greece(2 solutions)*; ŠEFKET ARSLANAGIĆ, *University of Sarajevo, Sarajevo, Bosnia and Herzegovina*; MICHEL BATAILLE, *Rouen, France*; CHIP CURTIS, *Missouri Southern State University, Joplin, MO, USA*; PEDRO HENRIQUE O. PANTOJA, *student, UFRN, Brazil*; PAOLO PERFETTI, *Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy*; ALBERT STADLER, *Herrliberg, Switzerland*; PETER Y. WOO, *Biola University, La Mirada, CA, USA*; and the proposer.

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