SKOLIAD No. 127

Lily Yen and Mogens Hansen

Please send your solutions to problems in this Skoliad by **15 January, 2011**. A copy of ***CRUX with Mayhem*** will be sent to one pre-university reader who sends in solutions before the deadline. The decision of the editors is final.

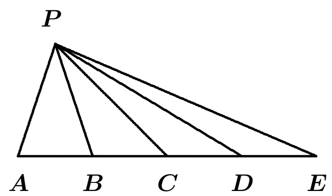
Our contest this month is the British Columbia Secondary School Mathematics Contest 2010, Junior Final Round, Part B. Our thanks go to Clint Lee, Okanagan College, BC, for permission to publish it. We also thank Rolland Gaudet, University College of Saint Boniface, Winnipeg, MB, for translating this contest into French.

Concours mathématique de la Colombie-Britannique
Niveau secondaire, ronde finale junior 2010

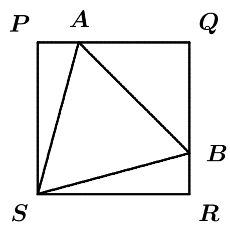
1a. Déterminer la somme de tous les entiers positifs inférieurs à 2010 pour lesquels l'entier en position unité est soit '3' soit '8'.

1b. Deux bidons, X et Y, contiennent chacun de l'eau. De X Thomas verse autant d'eau dans Y que ce que Y contient déja. Ensuite, de Y, il verse autant d'eau dans X que ce que X contient déja. Enfin, il verse de X dans Y autant d'eau que Y contient déja. Chaque bidon contient maintenant 24 unités d'eau. Déterminer le nombre d'unités d'eau dans chaque bidon au départ.

2. La surface du $\triangle APE$ illustré dans la figure est 12. Étant donné que $|AB| = |BC| = |CD| = |DE|$, déterminer la somme des surfaces de tous les triangles qui apparaissent dans la figure.

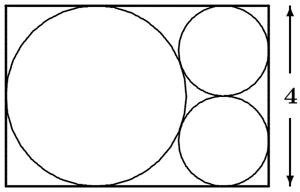


3. Étant donné que $PQRS$ est un carré et que ABS est un triangle équilatéral (voir la figure), déterminer le rapport entre la surface du $\triangle APS$ et la surface du $\triangle ABQ$.



4. Déterminer cinq entiers distincts tels que les sommes des paires d'entiers distincts sont les nombres 0, 1, 2, 4, 7, 8, 9, 10, 11, et 12.

5. Un rectangle contient trois cercles, comme indiqué au diagramme, tous tangents au rectangle et les uns aux autres. La hauteur du rectangle est 4. Déterminer la largeur du rectangle.

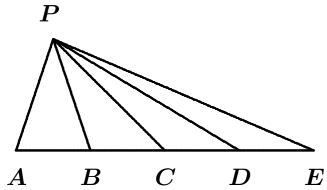


British Columbia Secondary School Mathematics Contest
Junior Final Round, 2010

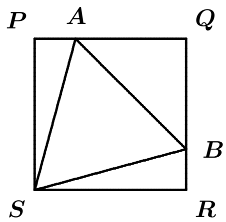
1a. Find the sum of all positive integers less than 2010 for which the ones digit is either a '3' or an '8'.

1b. Two cans, X and Y, both contain some water. From X Tim pours as much water into Y as Y already contains. Then, from Y he pours as much water into X as X already contains. Finally, he pours from X into Y as much water as Y already contains. Each can now contains 24 units of water. Determine the number of units of water in each can at the beginning.

2. The area of $\triangle APE$ shown in the diagram is 12. Given that $|AB| = |BC| = |CD| = |DE|$, determine the sum of the areas of all the triangles that appear in the diagram.

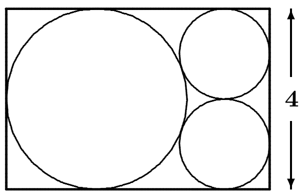


3. Given that $PQRS$ is a square and that ABS is an equilateral triangle (see the diagram), find the ratio of the area of $\triangle APS$ to the area of $\triangle ABQ$.



4. Find the five distinct integers for which the sums of each distinct pair of integers are the numbers 0, 1, 2, 4, 7, 8, 9, 10, 11, and 12.

5. A rectangle contains three circles, as in the diagram, all tangent to the rectangle and to each other. The height of the rectangle is 4. Determine the width of the rectangle.



Next follow solutions to the Niels Henrik Abel Mathematics Contest, 2008–2009, Second Round, given in Skoliad 121 at [2009 : 481–483].

1. Arrange the positive odd numbers in a triangular diagram as shown. What is the sum of the numbers in the first seven rows?

			1
		3	5
	7	9	11
13	15	$\dots$	$\dots$

Solution by Szera Pinter, student, Moscrop Secondary School, Burnaby, BC.

The right-hand column of numbers is 1, 5, 11, 19, 29, 41, 55 (add 4, add 6, add 8, etc.). Therefore the numbers to add are 1, 3, $\dots$, 55. That is a total of $\frac{56}{2}$ or 28 numbers. Now pair 1 with 55, pair 3 with 53, etc. You then have 14 pairs, and each pair has sum 56. Hence the sum of all the numbers is 14×56, or 784.

Also solved by ELLEN CHEN, WEN-TING FAN, VICKY LIAO, JUSTIN MIAO (all students at Burnaby North Secondary School, Burnaby, BC), and LISA WANG, student, Port Moody Secondary School, Port Moody, BC (in collaboration); LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; EVEREST SHI, student, Burnaby North Secondary School, Burnaby, BC; and ALISON TAM, student, Burnaby South Secondary School, Burnaby, BC.

Our solvers found a variety of solutions. Some simply listed all seven rows and added the numbers. Some noticed that the first few row sums are consecutive cubes, 1^3, 2^3, 3^3, $\dots$, and then added those. Of course one ought to prove *that the pattern of cubes continues.*

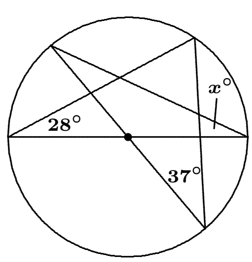
2. A large basket contains many eggs. If you remove the eggs two at a time, a single egg remains in the basket. The same happens if you remove the eggs three at a time, or four or five or six at a time, but if you remove the eggs seven at a time, you empty the basket completely. At least how many eggs were there in the basket?

Solution by Everest Shi, student, Burnaby North Secondary School, Burnaby, BC.

The number of eggs must be one more than a common multiple of 2, 3, 4, 5, and 6. Now $\operatorname{lcm}(2, 3, 4, 5, 6) = 60$, so the number of eggs must be one more than a multiple of 60. That is, the number of eggs is 61, or 121, or 181, etc. None of these listed are divisible by seven, but trial and error quickly finds that the number of eggs is 301.

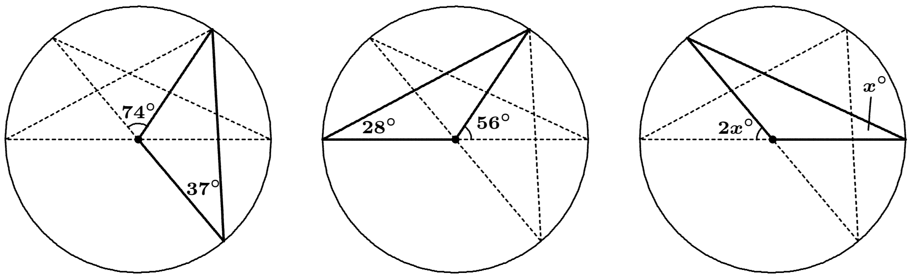
Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; and ALISON TAM, student, Burnaby South Secondary School, Burnaby, BC;

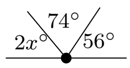
3. Two of the angles in a five-pointed star are $28°$ and $37°$, as shown. All the vertices lie on a circle, and the indicated point is the centre of the circle. Find the measure of angle x.



Solution by Szera Pinter, student, Moscrop Secondary School, Burnaby, BC.

Connect the centre to upper right prong of the star as in the leftmost diagram below. Clearly the triangle with solid sides is isosceles, so the angle at the top is also 37°. Therefore, the last angle in the triangle is 106°, whence the indicated angle is indeed 74°.





Similarly you may find the angles indicated in the other two diagrams. Superimposing the three diagrams yields the diagram on the right, which shows that $2x^\circ + 74^\circ + 56^\circ = 180^\circ$, so $x^\circ = 25^\circ$.

Also solved by ELLEN CHEN, WEN-TING FAN, VICKY LIAO, JUSTIN MIAO (all students at Burnaby North Secondary School, Burnaby, BC), and LISA WANG, student, Port Moody Secondary School, Port Moody, BC (in collaboration); VINCENT CHUNG, student, Burnaby North Secondary School, Burnaby, BC; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; EVEREST SHI, student, Burnaby North Secondary School, Burnaby, BC; and ALISON TAM, student, Burnaby South Secondary School, Burnaby, BC.

4. Sir Lancelot and five other knights sit at a round table. All six knights manage to make enemies of both their neighbours. In how many ways can the six knights sit around the table if Sir Lancelot is to keep his seat and no one sits next to either of his new enemies?

Say the knights originally sit like this $(5, L, 1, 2, 3, 4)$, where L represents Sir Lancelot and the first and last seats are understood to be neighbours. When the six knights sit down again, Sir Lancelot's neighbours must be two of $\{2,3,4\}$. You then have three choices for the left-hand neighbour and two choices for the right-hand neighbour; that is a total of six choices.

For example, the first three knights may sit like this $(2, L, 3, \_, \_, \_)$. Of the remaining knights, 4 and 5 are enemies, so they must be separated by 1; that is $(2, L, 3, \_, 1, \_)$. Now $(2, L, 3, 5, 1, 4)$ is the only way to complete the seating plan.

If the first three knights sit like this $(2, L, 4, \_, \_, \_)$, then only one of the remaining knights is still friendly with 2, and only one is friendly with 4, so the knights must sit like this $(2, L, 4, 1, 3, 5)$.

The remaining four choices for Sir Lancelot's neighbours similarly yield only one seating arrangement each, namely $(3, L, 2, 4, 1, 5)$, $(4, L, 2, 5, 3, 1)$, $(3, L, 4, 2, 5, 1)$, and $(4, L, 3, 1, 5, 2)$. Thus, the six knights can sit in six ways.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; and EVEREST SHI, student, Burnaby North Secondary School, Burnaby, BC.

5. The arithmetic mean, A, of two real numbers, x and y, is $\frac{1}{2}(x+y)$ and their geometric mean, G, is $\sqrt{xy}$. Find $\frac{y}{x}$ if $3A = 5G$.

Solution by Natalia Desy, student, SMA Xaverius 1, Palembang, Indonesia.

We are given that $\frac{3(x+y)}{2} = 5\sqrt{xy}$. Therefore, $3(x+y) = 10\sqrt{xy}$. Squaring yields that $9(x^2+2xy+y^2) = 100xy$, so $9x^2 - 82xy + 9y^2 = 0$, or $(9x-y)(x-9y) = 0$, hence $y = 9x$ or $x = 9y$. Therefore, $\frac{y}{x}$ is 9 or $\frac{1}{9}$.

Also solved by EVEREST SHI, student, Burnaby North Secondary School, Burnaby, BC

6. A function, f, is such that for each positive integer n,

$$f(n+1) = \frac{f(n)}{1+af(n)}\,,$$

where a is a real number, $f(1) = 1$, and $f(9) = \frac{1}{2009}$. Find a.

Solution by Everest Shi, student, Burnaby North Secondary School, Burnaby, BC.

The equation for $f(n+1)$ with $n = 1$ yields $f(2) = \frac{f(1)}{1+af(1)} = \frac{1}{1+a}$. Likewise, $f(3) = \frac{f(2)}{1+af(2)} = \frac{1/(1+a)}{1+a/(1+a)} = \frac{1}{1\cdot(1+a)+a} = \frac{1}{1+2a}$. Again, $f(4) = \frac{1/(1+2a)}{1+a/(1+2a)} = \frac{1}{1\cdot(1+2a)+a} = \frac{1}{1+3a}$. The pattern that emerges is that

$$f(n+1) \;=\; \frac{1}{1+na}\,.$$

Hence, $\frac{1}{2009} = f(9) = \frac{1}{1+8a}$, so $2009 = 1+8a$ and $a = 251$.

Also solved by ELLEN CHEN, WEN-TING FAN, VICKY LIAO, JUSTIN MIAO (all students at Burnaby North Secondary School, Burnaby, BC), and LISA WANG, student, Port Moody Secondary School, Port Moody, BC (in collaboration); and NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia.

One ought, of course, to prove *the pattern for $f(n+1)$. Fortunately that is not difficult, especially if you rewrite the equation for $f(n+1)$ to reduce the number of occurrences of the complicated part:*

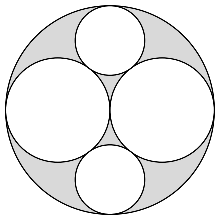
$$f(n+1) \;=\; \frac{1}{\frac{1}{f(n)}+a}\,.$$

Clearly our solver's pattern holds for $f(1)$. If the pattern holds for $f(n)$, then

$$f(n+1) \;=\; \frac{1}{\frac{1}{f(n)}+1} \;=\; \frac{1}{1+(n-1)a+a} \;=\; \frac{1}{1+na}\,,$$

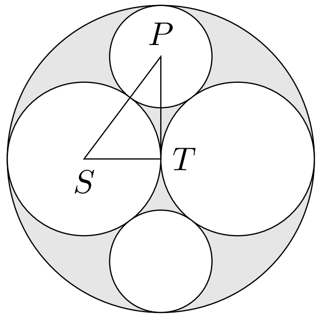
so the pattern holds for $f(n+1)$. Thus, by induction, the pattern holds for all values of n.

7. The large circle has radius $\frac{30}{\sqrt{\pi}}$. The medium circles are tangent to one another at the centre of the large circle. Moreover, the medium circles are tangent to the small circles, and the large circle is tangent to all the circles it contains. Find the area of the shaded region.



Solution by Natalia Desy, student, SMA Xaverius 1, Palembang, Indonesia.

Let P, S, and T be the centres of the upper small circle, left medium circle, and the large circle, respectively. Let r and R be the radii of the small and medium circles, respectively. Then the radius of the large circle is $2R$, so $R = \frac{15}{\sqrt{\pi}}$. Moreover, $|PT| = 2R - r$, $|PS| = R + r$, and $|ST| = R$. By the Pythagorean Theorem, $(R+r)^2 = R^2 + (2R-r)^2$, so $R^2 + 2Rr + r^2 = R^2 + 4R^2 - 4Rr + r^2$, so $6Rr = 4R^2$. Since $R \neq 0$, we have that $r = \frac{2}{3}R = \frac{2}{3} \cdot \frac{15}{\sqrt{\pi}} = \frac{10}{\sqrt{\pi}}$.



Hence, the area of the large circle is $\pi\left(\frac{30}{\sqrt{\pi}}\right)^2 = 900$, the area of each medium circle is $\pi\left(\frac{15}{\sqrt{\pi}}\right)^2 = 225$, and the area of each small circle is $\pi\left(\frac{10}{\sqrt{\pi}}\right)^2 = 100$. The shaded area is thus $900 - 2 \cdot 225 - 2 \cdot 100 = 250$.

Also solved by ELLEN CHEN, WEN-TING FAN, VICKY LIAO, JUSTIN MIAO (all students at Burnaby North Secondary School, Burnaby, BC), and LISA WANG, student, Port Moody Secondary School, Port Moody, BC (in collaboration); LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; EVEREST SHI, student, Burnaby North Secondary School, Burnaby, BC; and ALISON TAM, student, Burnaby South Secondary School, Burnaby, BC.

8. Find the sum of all positive integers, n, such that $2009 + n^2$ is the square of a positive integer.

Solution by Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC

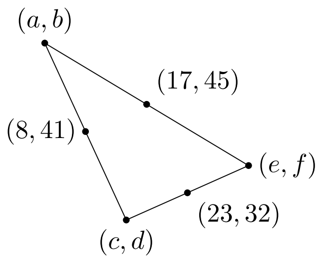
If $2009 + n^2 = x^2$, then $2009 = x^2 - n^2 = (x-n)(x+n)$. But since $2009 = 7^2 \cdot 41$, it can then only be written as a product of two integers in three ways: $1 \cdot 2009$, $7 \cdot 287$, and $41 \cdot 49$. Since both x and n are positive, $x+n$ is larger than $x - n$. Therefore, you must solve $\{x - n = 1,\ x + n = 2009\}$, $\{x - n = 7,\ x + n = 287\}$, and $\{x - n = 41,\ x + n = 49\}$. These give that n equals 1004, 140, and 4, respectively. The sum of these is 1148.

Also solved by ELLEN CHEN, WEN-TING FAN, VICKY LIAO, JUSTIN MIAO (all students *at Burnaby North Secondary School, Burnaby, BC), and LISA WANG, student, Port Moody Secondary School, Port Moody, BC (in collaboration); NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; and EVEREST SHI, student, Burnaby North Secondary School, Burnaby, BC.*

Factoring (into polynomials or into primes) is a very common trick in questions that revolve around integers.

9. The points $(23, 32)$, $(8, 41)$, and $(17, 45)$ are the midpoints of the sides of a triangle. Find the largest possible value of $x + y$ where (x, y) is a vertex of the triangle.

Solution by Everest Shi, student, Burnaby North Secondary School, Burnaby, BC.



By the Midpoint Formula, $a + c = 2 \cdot 8 = 16$, $b + d = 2 \cdot 41 = 82$, $a + e = 34$, $b + f = 90$, $c + e = 46$, and $d + f = 64$. Therefore,

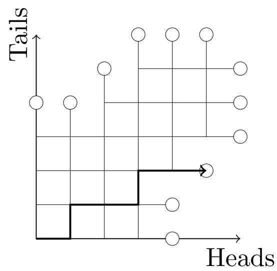
$$(a + c) + (a + e) - (c + e) = 16 + 34 - 46\,,$$

so $2a = 4$ and $a = 2$. Likewise, $b = 54$, $c = 14$, $d = 28$, $e = 32$, and $f = 36$. Thus, the three vertices are $(2, 54)$, $(14, 28)$, and $(32, 36)$. The largest coordinate sum is $32 + 36 = 68$.

No other solutions were submitted.

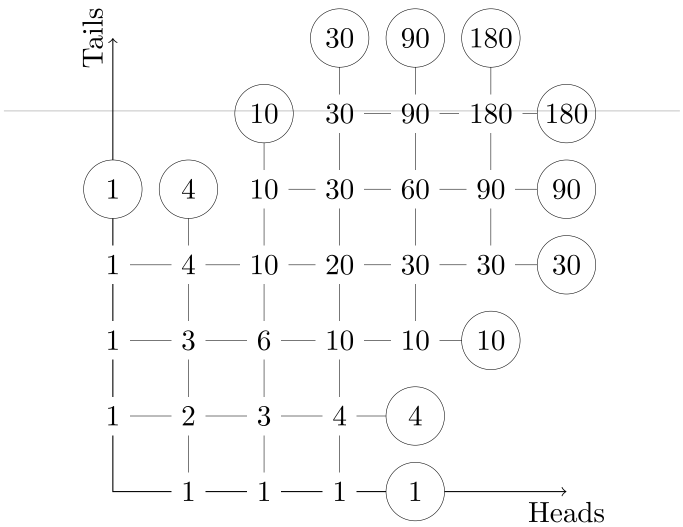
10. Kari and Mons are tossing a coin. Each time they toss heads, Kari earns a point, and each time they toss tails, Mons earns a point. The person who first reaches six points or who has at least four points and leads by at least three points wins. How many different games are possible; that is, how many different sequences of coin tosses end in a win?

Solution by Ellen Chen, Wen-Ting Fan, Vicky Liao, Justin Miao (all students at Burnaby North Secondary School, Burnaby, BC), and Lisa Wang, student, Port Moody Secondary School, Port Moody, BC (in collaboration)



If you graph the number of heads and tails seen so far in the usual coordinate system, then a game is the same as a walk starting at $(0, 0)$ and only moving up or to the right at each step, and which ends in one of the circled positions in the diagram. For example, the walk shown represents the game (H, T, H, H, T, H, H) that ends because Kari with five points has a three point lead over Mons' two points.

The diagram on the next page shows the number of walks that arrive at each point. For example, to arrive at the point $(3, 2)$, you must come from either the point $(3, 1)$ or the point $(2, 2)$. Since you can arrive at those two points in 4 and 6 ways, respectively, you can arrive at the point $(3, 2)$ in 10 ways.



The number of possible games is then the sum of the circled numbers, which is 630.

Our solvers' technique for counting walks in a lattice is often very useful.

This issue's prize of one copy of ***CRUX with MAYHEM*** for the best solutions goes to Everest Shi, student, Burnaby North Secondary School, Burnaby, BC. We hope that our readers will continue to share their joy of mathematics.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff member is Monika Khbeis (Our Lady of Mt. Carmel Secondary School, Mississauga, ON).

Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **1 février 2011**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

M451. *Proposé par l'Équipe de Mayhem.*

Le carré $ABCD$ a un côté de longueur 6. Le point P, à l'intérieur du carré, est tel que $AP = DP = 5$. Trouver la longueur de PC.

M452. *Proposé par Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie.*

(a) On suppose que $x = \sqrt{a + \sqrt{a + \sqrt{a + \cdots}}}$ pour un certain nombre réel $a > 0$. Montrer que $x^2 - a = x$.

(b) Trouver l'entier égal à

$$\frac{\sqrt{30 + \sqrt{30 + \sqrt{30 + \cdots}}}}{\sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}}} - \sqrt{42 + \sqrt{42 + \sqrt{42 + \cdots}}}.$$

M453. *Proposé par Yakub N. Aliyev, Université de Qafqaz, Khyrdalan, Azerbaïdjan.*

Soit $ABCD$ un parallélogramme. On prolonge les côtés AB et AD jusqu'aux points E et F de sorte que E, C et F soient alignés. On a vu dans le problème M447 que $BE \cdot DF = AB \cdot AD$. Montrer que

$$\sqrt{AE + AF} \geq \sqrt{AB} + \sqrt{AD}.$$

M454. *Proposé par Pedro Henrique O. Pantoja, étudiant, UFRN, Brésil.*

Déterminer tous les nombres réels x tels que $16^x + 1 = 2^x + 8^x$.

M455. *Proposé par Gheorghe Ghiță, Collège National "M. Eminescu", Buzău, Roumanie.*

On suppose que n est un entier positif.

(a) Si l'entier positif d est un diviseur de chacun des entiers $n^2 + n + 1$ et $2n^3 + 3n^2 + 3n - 1$, montrer que d est aussi un diviseur de $n^2 + n - 1$.

(b) Montrer que la fraction $\dfrac{n^2 + n + 1}{2n^3 + 3n^2 + 3n - 1}$ est irréductible.

M456. *Proposé par Mihály Bencze, Brasov, Romania.*

Soit f et g deux fonctions à valeurs réelles telles que g soit une fonction impaire, $f(x) \le g(x)$ pour tout x réel, et $f(x+y) \le f(x) + f(y)$ pour tout x et y réels. Montrer que f est une fonction impaire.

. . . .

M451. *Proposed by the Mayhem Staff.*

Square $ABCD$ has side length 6. Point P is inside the square so that $AP = DP = 5$. Determine the length of PC.

M452. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

(a) Suppose that $x = \sqrt{a + \sqrt{a + \sqrt{a + \cdots}}}$ for some real number $a > 0$. Prove that $x^2 - a = x$.

(b) Determine the integer equal to

$$\frac{\sqrt{30 + \sqrt{30 + \sqrt{30 + \cdots}}}}{\sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}}} - \sqrt{42 + \sqrt{42 + \sqrt{42 + \cdots}}}\,.$$

M453. *Proposed by Yakub N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan.*

Let $ABCD$ be a parallelogram. Sides AB and AD are extended to points E and F so that E, C, and F lie on a straight line. In problem M447, we saw that $BE \cdot DF = AB \cdot AD$. Prove that

$$\sqrt{AE + AF} \;\ge\; \sqrt{AB} + \sqrt{AD}\,.$$

M454. *Proposed by Pedro Henrique O. Pantoja, student, UFRN, Brazil.*

Determine all real numbers x with $16^x + 1 = 2^x + 8^x$.

M455. *Proposed by Gheorghe Ghiţă, M. Eminescu National College, Buzău, Romania.*

Suppose that n is a positive integer.

(a) If the positive integer d is a divisor of each of the integers n^2+n+1 and $2n^3+3n^2+3n-1$, prove that d is also a divisor of n^2+n-1.

(b) Prove that the fraction $\dfrac{n^2+n+1}{2n^3+3n^2+3n-1}$ is irreducible.

M456. *Proposed by Mihály Bencze, Brasov, Romania.*

Let f and g be real-valued functions with g an odd function, $f(x) \le g(x)$ for all real numbers x, and $f(x+y) \le f(x)+f(y)$ for all real numbers x and y. Prove that f is an odd function.

Mayhem Solutions

M413. *Proposed by the Mayhem Staff.*

Determine the number of three-digit positive integers whose digits have a product of 36.

Solution by Chris Pickett and Daniel Thompson, students, Auburn University, Montgomery, AL, USA.

The only single digit divisors of 36 are 1, 2, 3, 4, 6, and 9, and thus any three-digit integer whose digits have a product of 36 must use only these digits. Ignoring the order in which the digits are multiplied, the only ways to multiply three of the digits from the set $\{1, 2, 3, 4, 6, 9\}$ to get 36 are

$$36 = 9\cdot 4\cdot 1 = 9\cdot 2\cdot 2 = 6\cdot 6\cdot 1 = 6\cdot 3\cdot 2 = 4\cdot 3\cdot 3 .$$

The factorizations $9\cdot 4\cdot 1$ and $6\cdot 3\cdot 2$ each give 6 different three-digit positive integers whose digits have a product of 36. (If a, b, c are different, then the arrangements of a, b, c are abc, acb, bac, bca, cab, cba.)

The factorizations $9\cdot 2\cdot 2$, $6\cdot 6\cdot 1$, and $4\cdot 3\cdot 3$ each give 3 different three-digit positive integers whose digits have a product of 36. (If a and b are different, the arrangements of a, a, and b are aab, aba, and baa.)

Thus, there are $2(6)+3(3)=21$ different three-digit positive integers whose digits have a product of 36.

Also solved by SZÉP GYUSZI, Dimitrie Leonida Technological High School, Petrosani, Romania; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; MITEA MARIANA, No. 2 Secondary School, Cugir, Romania; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; JOCHEM VAN GAALEN, student, Medway High School, Arva, ON; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON. There were five incorrect solutions submitted.

M414. *Proposed by the Mayhem Staff.*

The positive integers that can be expressed as the sum of 21 consecutive (not necessarily positive) integers are listed in increasing order. Determine the 21^{st} integer in this list.

Solution by Jochem van Gaalen, student, Medway High School, Arva, ON.

Let the eleventh term (that is, the middle term) in the sequence of 21 integers be n. Then the sequence of 21 integers can be written as

$$n-10,\ n-9,\ n-8,\ \ldots,\ n-1,\ n,\ n+1,\ \ldots,\ n+8,\ n+9,\ n+10\,.$$

The sum of these 21 expressions is $21n$.

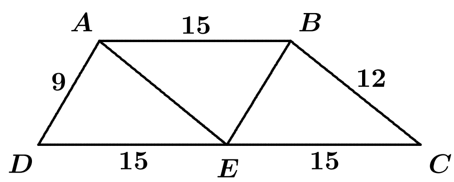
For this sum to be positive, we must have $n > 0$. Therefore, the 21^{st} positive integer that can be written in this form occurs when $n = 21$. Thus, 441 is the 21^{st} number in the list.

Also solved by JACLYN CHANG, student, Western Canada High School, Calgary, AB; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. There was one incorrect solution submitted.

M415. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

In trapezoid $ABCD$, AB and CD are parallel. If $AB = 15$, $CD = 30$, $AD = 9$, and $BC = 12$, determine the area of the trapezoid.

I. *Solution by Jaclyn Chang, student, Western Canada High School, Calgary, AB.*

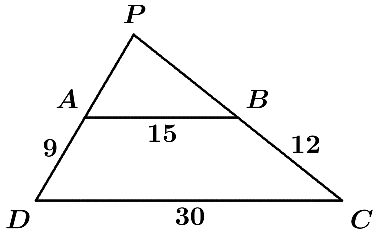


Let E be the midpoint of DC. Then $DE = EC = \frac{1}{2}DC = 15$. Join A and B to E.

Since AB is parallel to each of DE and EC and $AB = DE = EC = 15$, then $ABED$ and $ABCE$ are both parallelograms. Thus, $BE = AD = 9$ and $AE = BC = 12$. Therefore, triangles ADE, EBA and BEC are congruent, each having side lengths 9, 12, and 15. Each of these triangles is right-angled, as each is a 3-4-5 triangle scaled up by a factor of 3. Thus, the area of each of $\frac{1}{2}(9)(12) = 54$, and so the area of the trapezoid is $3(54) = 162$.

II. *Solution by Hugo Luyo Sánchez, Pontificia Universidad Católica del Peru, Lima, Peru.*

Extend DA and CB to meet at point P.



Since AB is parallel to DC, then $\angle PAB = \angle PDC$, and so $\triangle PAB$ is similar to $\triangle PDC$. Since $AB = \frac{1}{2}DC$, then $PA = \frac{1}{2}PD$ and $PB = \frac{1}{2}PC$, or $PA = AD = 9$ and $PB = BC = 12$. Since $\triangle PAB$ has side lengths 9, 12, 15, and $9^2 + 12^2 = 225 = 15^2$, then $\angle APB = 90°$. Therefore, the area of $ABCD$ is the area of $\triangle PDC$ minus the area of $\triangle PAB$, or $\frac{1}{2}(18)(24) - \frac{1}{2}(9)(12) = 216 - 54 = 162$.

Also solved by NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; SZÉP GYUSZI, Dimitrie Leonida Technological High School, Petrosani, Romania; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MITEA MARIANA, No. 2 Secondary School, Cugir, Romania; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; RICARD PEIRÓ, IES "Abastos", Valencia, Spain (two solutions); BRUNO SALGUEIRO FANEGO, Viveiro, Spain; JOCHEM VAN GAALEN, student, Medway High School, Arva, ON; JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA.

M416. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Prove that 9 divides $10^n + 3(4^{n+2}) + 5$ for all nonnegative integers n.

Solution by Jaclyn Chang, student, Western Canada High School, Calgary, AB, modified by the editor.

For each nonnegative integer n, define $P(n)$ to be the proposition "$10^n + 3(4^{n+2}) + 5$ is divisible by 9". We want to prove that $P(n)$ is true for all nonnegative integers n by Mathematical Induction.

When $n = 0$, we have $10^n + 3(4^{n+2}) + 5 = 10^0 + 3(4^2) + 5 = 1 + 3(16) + 5 = 54$, which is divisible by 9, so $P(0)$ is true.

Next, assume that $P(k)$ is true for some nonnegative integer k; that is, assume that $10^k + 3(4^{k+2}) + 5$ is divisible by 9.

Then $P(k+1)$ involves the expression $10^{k+1} + 3(4^{k+3}) + 5$, which we manipulate as follows:

$$\begin{aligned} 10^{k+1} + 3(4^{k+3}) + 5 &= 10(10^k) + 12(4^{k+2}) + 5 \\ &= 9(10^k) + 10^k + 9(4^{k+2}) + 3(4^{k+2}) + 5 \\ &= 9(10^k + 4^{k+2}) + (10^k + 3(4^{k+2}) + 5)\,. \end{aligned}$$

The first term above is divisible by 9. The second term is divisible by 9 by the inductive assumption. Therefore, this expression is divisible by 9, and so $P(k+1)$ is true.

Therefore, by Mathematical Induction, $P(n)$ is true for all nonnegative integers n

Also solved by ARKADY ALT, San Jose, CA, USA (three solutions); SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; SZÉP GYUSZI, Dimitrie Leonida Technological High School, Petrosani, Romania; RICHARD I. HESS, Rancho Palos Verdes, CA, USA;

GEOFFREY A. KANDALL, Hamden, CT, USA (two solutions); CARL LIBIS, Cumberland University, Lebanon, TN, USA; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; MITEA MARIANA, No. 2 Secondary School, Cugir, Romania; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. There was one incomplete solution submitted.

M417. *Proposed by Mihály Bencze, Brasov, Romania.*

Let $M = \{x^2 + 4xy + y^2 : x,\, y \in \mathbb{Z}\}$. Prove that the number 2022 is in M but that the number 11 is not in M.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.

Suppose that $y = 1$. Then $x^2 + 4xy + y^2 = 2022$ is equivalent to $x^2 + 4x = 2021$ or $(x + 2)^2 = 2025$. Since $2025 = 45^2$, then $x = 43$ is a solution. Therefore, $(x, y) = (43, 1)$ is a solution to $x^2 + 4xy + y^2 = 2022$, so $2022 \in M$.

Next, suppose that $11 \in M$; that is, $x^2 + 4xy + y^2 = 11$, or

$$(x + y)^2 + 2xy \;=\; 11$$

for some integers x and y. Since $(x + y)^2 = 11 - 2xy$ and the right side is always odd, then $(x + y)^2$ is odd, so $x + y$ is odd.

Since $x + y$ is odd, then one of x and y is even and the other is odd. Therefore, $2xy$ is a multiple of 4, say $2xy = 4k$. Also, since $x+y$ is odd, then $(x+y)^2$ is 1 more than a multiple of 4, say $(x+y)^2 = 4q+1$. (This is because if $n = 2m + 1$, then $n^2 = (2m + 1)^2 = 4m^2 + 4m + 1 = 4(m^2 + m) + 1$, which is 1 more than a multiple of 4.)

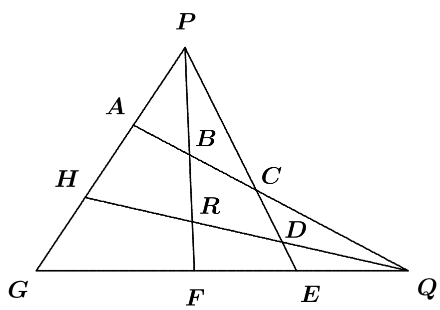
Thus, $(x + y)^2 + 2xy = 11$ gives $4q + 1 + 4k = 11$ or $4(k + q) = 10$, which is a contradiction, since 10 is not a multiple of 4. Thus, $11 \not\in M$.

Also solved by ARKADY ALT, San Jose, CA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; GEOFFREY A. KANDALL, Hamden, CT, USA; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. There were two incomplete solutions and one incorrect solution submitted.

M418. *Proposed by Geoffrey A. Kandall, Hamden, CT, USA.*

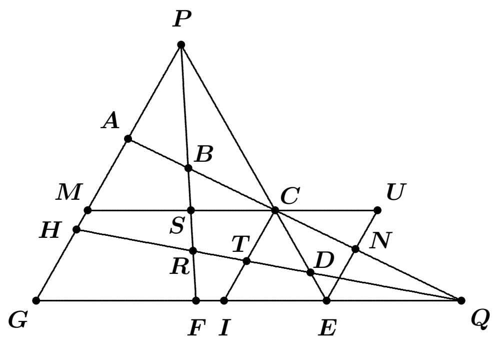
In the diagram, F lies on GE and Q lies on GE extended. Also, A and H are on PG so that QA intersects PF at B, QA intersects PE at C, and QH intersects PE at D. Prove that

$$\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FG} \cdot \frac{GH}{HA} \;=\; 1\,.$$



Solution by Julio Orihuela Bastidas, Universidad Nacional Mayor de San Marcos, Lima, Peru, modified by the editor.

From point C, draw CM parallel to EG with M on GP. Also, from point C, draw CI parallel to GP with I on GQ. Extend MC to point U so that UE is parallel to PG. From these constructions, $GMCI$ and $ICUE$ are parallelograms. Label the intersection of PF and MC as S, the intersection of QH and CI as T, and the intersection of CQ and EU as N.



Since MC is parallel to GE, then $\triangle MPS$ is similar to $\triangle GPF$, which gives $\frac{SM}{FG} = \frac{PS}{PF}$. Also, since MC is parallel to GE, then $\triangle SPC$ is similar to $\triangle FPE$, which gives $\frac{CS}{EF} = \frac{PS}{PF}$. Combining these two equations gives $\frac{SM}{FG} = \frac{CS}{EF}$, or $\frac{EF}{FG} = \frac{CS}{SM}$.

Similarly, since CI is parallel to GA, then $\frac{HA}{GH} = \frac{TC}{IT}$.

Using a version of the Theorem of Menelaus in $\triangle MCA$ with point P gives $\frac{CS}{SM} \cdot \frac{AB}{BC} \cdot \frac{MP}{AP} = 1$, which is equivalent to $\frac{EF}{FG} \cdot \frac{AB}{BC} \cdot \frac{MP}{AP} = 1$, or

$$\frac{AB}{BC} \cdot \frac{EF}{FG} = \frac{AP}{MP}.$$

Using a version of the Theorem of Menelaus in $\triangle ICE$ with point Q gives $\frac{TC}{IT} \cdot \frac{DE}{CD} \cdot \frac{IQ}{EQ} = 1$, which is equivalent to $\frac{HA}{GH} \cdot \frac{DE}{CD} \cdot \frac{IQ}{EQ} = 1$, or

$$\frac{DE}{CD} \cdot \frac{HA}{GH} = \frac{EQ}{IQ}.$$

Since MP is parallel to UE, then $\triangle MPC$ is similar to $\triangle UEC$, which gives, in a similar way to earlier, $\frac{AP}{MP} = \frac{EN}{EU}$. Since $EU = IC$, then $\frac{AP}{MP} = \frac{EN}{IC}$.

Since CI is parallel to NE, then $\triangle ICQ$ is similar to $\triangle ENQ$ and so $\frac{EQ}{IQ} = \frac{EN}{IC}$. Thus, $\frac{AP}{MP} = \frac{EQ}{IQ}$.

Therefore, by combining several equations, we see that $\frac{AB}{BC} \cdot \frac{EF}{FG} = \frac{DE}{CD} \cdot \frac{HA}{GH}$, whence $\frac{AB}{BC} \cdot \frac{EF}{FG} \cdot \frac{CD}{DE} \cdot \frac{GH}{HA} = 1$, as required.

No other solutions were submitted.

M419. *Proposed by Joe Howard, Portales, NM, USA.*

Let a, b, and c be the side lengths of a triangle. Prove that

$$\frac{a(b+c)}{a^2+bc} + \frac{b(c+a)}{b^2+ca} + \frac{c(a+b)}{c^2+ab} \leq 3\,.$$

Solution by Arkady Alt, San Jose, CA, USA.

Due to the symmetry of the inequality, we may take $a \geq b \geq c$. Then

$$\begin{aligned}
3 &- \frac{a(b+c)}{a^2+bc} - \frac{b(c+a)}{b^2+ca} - \frac{c(a+b)}{c^2+ab} \\
&= \left(1 - \frac{a(b+c)}{a^2+bc}\right) + \left(1 - \frac{b(c+a)}{b^2+ca}\right) + \left(1 - \frac{c(a+b)}{c^2+ab}\right) \\
&= \frac{a^2-ab-ac+bc}{a^2+bc} + \frac{b^2-bc-ba+ca}{b^2+ca} + \frac{c^2-ca-cb+ab}{c^2+ab} \\
&= \frac{(a-b)(a-c)}{a^2+bc} + \frac{(b-c)(b-a)}{b^2+ca} + \frac{(c-a)(c-b)}{c^2+ab} \\
&= \frac{(a-b)(a-c)}{a^2+bc} + (b-c)\left(\frac{b-a}{b^2+ca} - \frac{c-a}{c^2+ab}\right) \\
&= \frac{(a-b)(a-c)}{a^2+bc} + (b-c)\left(\frac{a-c}{c^2+ab} - \frac{a-b}{b^2+ca}\right) \\
&= \frac{(a-b)(a-c)}{a^2+bc} + (b-c)\left(\frac{a-c}{c^2+ab} - \frac{a-b}{c^2+ab}\right) \\
&\qquad + (b-c)\left(\frac{a-b}{c^2+ab} - \frac{a-b}{b^2+ca}\right) \\
&= \frac{(a-b)(a-c)}{a^2+bc} + \frac{(b-c)^2}{c^2+ab} + (b-c)(a-b)\frac{b^2+ca-c^2-ab}{(c^2+ab)(b^2+ca)} \\
&= \frac{(a-b)(a-c)}{a^2+bc} + \frac{(b-c)^2}{c^2+ab} + (b-c)(a-b)\frac{(b^2-c^2)-a(b-c)}{(c^2+ab)(b^2+ca)} \\
&= \frac{(a-b)(a-c)}{a^2+bc} + \frac{(b-c)^2}{c^2+ab} + (b-c)^2(a-b)\frac{b+c-a}{(c^2+ab)(b^2+ca)}\,.
\end{aligned}$$

Since a, b, and c are positive, since $a \geq b \geq c$, and since a, b, and c are sides of a triangle (which gives $b+c-a > 0$), then each factor in each term in the final expression is nonnegative, so the initial expression is nonnegative.

Therefore, $\frac{a(b+c)}{a^2+bc} + \frac{b(c+a)}{b^2+ca} + \frac{c(a+b)}{c^2+ab} \leq 3$, as required.

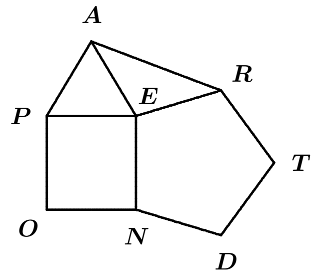
Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA. There was one incomplete solution and one incorrect solution submitted.

Problem of the Month

Ian VanderBurgh

Are you someone who likes to memorize formulas or someone who prefers to remember how to derive them?

Problem 1 (2002 Canadian Open Mathematics Challenge) A regular pentagon is a five-sided figure which has all of its angles equal and all of its side lengths equal. In the diagram, $TREND$ is a regular pentagon, PEA is an equilateral triangle, and $OPEN$ is a square. Determine the size of $\angle EAR$.



When working with polygons, there is some important information that you need to be able to remember or derive. One of the most important pieces of information that you need to know to do this problem is that the sum of the angles in a pentagon is 540°. Some of us are able to remember this sort of thing. How could you figure it out if you didn't know this?

Imagine connecting R to N and D. This divides pentagon $TREND$ into three triangles. What is the sum of the angles in each of these triangles? It is 180°. Therefore, the sum of the angles in the entire pentagon is $3 \times 180^\circ$ or 540°. Can you see why this is true? In a regular pentagon, all angles are equal, so each must be equal to one-fifth of their total. We can now solve Problem 1.

Solution to Problem 1. To determine $\angle EAR$, we first look at the angles around E. We know that $\angle AER + \angle REN + \angle NEP + \angle PEA = 360^\circ$.

Since $\angle PEA$ is an angle in an equilateral triangle, $\angle PEA = 60^\circ$. Since $\angle NEP$ is an angle in a square, $\angle NEP = 90^\circ$. Since $\angle REN$ is an angle in a regular pentagon, $\angle REN = \frac{1}{5}(540^\circ) = 108^\circ$.

Therefore,

$$\begin{aligned} \angle AER &= 360^\circ - \angle REN - \angle NEP - \angle PEA \\ &= 360^\circ - 108^\circ - 90^\circ - 60^\circ \;=\; 102^\circ\,. \end{aligned}$$

Now since PEA is an equilateral triangle, $OPEN$ is a square, and $TREND$ is a regular pentagon, then their side lengths must all be the same, since $OPEN$ and $TREND$ share a side, and since $OPEN$ and PEA share a side. In particular, $AE = ER$.

Therefore, $\triangle ARE$ is isosceles, and so

$$\angle EAR \;=\; \frac{1}{2}(180^\circ - \angle AER) \;=\; \frac{1}{2}(180^\circ - 102^\circ) \;=\; 39^\circ\,. \qquad \blacksquare$$

What about a general polygon with n sides? There are three types of angles that can be important:

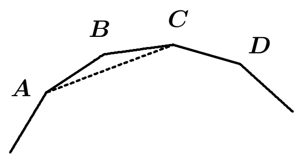
- *Interior Angles*: Some of you may remember that the sum of the interior angles in such a polygon is $(n-2)180^\circ$. Can you see how to derive this formula using a similar method to method above for the pentagon? Pick one of the n vertices and call it A. Vertex A is already connected to the vertex on either side. Connect it to the $n-3$ remaining vertices. (These are all of the n vertices, except for A and its two adjacent vertices.) This forms $n-2$ triangles, each of which has angles that add to 180°. Therefore, the sum of the angles in the polygon is $(n-2)180^\circ$. If the polygon is regular, then all n interior angles are equal and so each must equal $\dfrac{(n-2)180^\circ}{n}$.

- *Exterior Angles*: Sometimes it is much simpler (and more useful) to remember that the sum of the exterior angles in any convex polygon is 360°. The exterior angles are formed by extending each side and looking at the angles formed outside the polygon. You can see a picture in Solution 1 below. You can derive this total using the sum of the interior angles above. Can you come up with another way of deriving this directly by trying to combine all of the exterior angles visually in some way? Try drawing a pentagon, for example, identifying the exterior angles and seeing how you might be able to move all of these angles together.

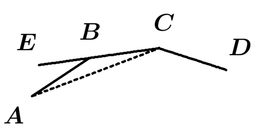
- *Central Angles*: By central angle, we mean the angle formed at the centre of the polygon by the two ends of each side. The total central angle must be 360°, and if the polygon is regular, this is divided into n equal pieces each equal to $\dfrac{360^\circ}{n}$.

Here is a second problem involving a polygon with an unknown number of sides, and two solutions to this problem. Solution 1 uses the exterior angles of the polygon, and Solution 2 uses the central angles.

Problem 2 (2009 Sun Life Financial Canadian Open Mathematics Challenge) A polygon is called *regular* if all of its sides are equal in length and all of its interior angles are equal in size. In the diagram, a portion of a regular polygon is shown. If $\angle ACD = 120^\circ$, how many sides does the polygon have?



Solution 1 to Problem 2. Suppose that the polygon has n sides. Extend CB outside of the polygon to a point E. Since the sum of the exterior angles in a polygon is 360°, then



$\angle ABE = \dfrac{360^\circ}{n}$, because there will be n equal exterior angles. Thus, $\angle ABC = 180^\circ - \dfrac{360^\circ}{n}$ and this will also be the measure of $\angle BCD$, since the polygon is regular.

Since the polygon is regular, then $AB = BC$, so $\triangle ABC$ is isosceles, which means that we have $\angle BAC = \angle BCA$. Therefore,

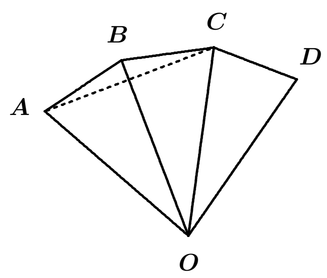
$$\angle BCA = \frac{1}{2}(180^\circ - \angle ABC) = \frac{1}{2}\left(180^\circ - \left(180^\circ - \frac{360^\circ}{n}\right)\right) = \frac{180^\circ}{n}$$

But $\angle BCD = \angle BCA + \angle ACD$, so

$$180^\circ - \frac{360^\circ}{n} = \frac{180^\circ}{n} + 120^\circ \iff 60^\circ = \frac{540^\circ}{n} \iff n = 9\,.$$

Therefore, the polygon has 9 sides. ■

Solution 2 to Problem 2. Suppose that the polygon has n sides. Let O be the centre of the polygon. Join O to each of A, B, C, and D. Since the polygon is regular, then the angle subtended at O by each of the n sides will be equal, and these angles all add to 360°. Since there are n equal central angles, then $\angle AOB = \angle BOC = \angle COD = \dfrac{360^\circ}{n}$. This also tells us that $\angle AOC = \angle AOB + \angle BOC = 2 \cdot \dfrac{360^\circ}{n} = \dfrac{720^\circ}{n}$.



Since the polygon is regular, then we have that $OA = OC = OD$, which tells us that $\triangle AOC$ and $\triangle COD$ are both isosceles. Thus,

$$\angle ACO = \frac{1}{2}(180^\circ - \angle AOC) = \frac{1}{2}\left(180^\circ - \frac{720^\circ}{n}\right) = 90^\circ - \frac{360^\circ}{n}$$

and

$$\angle DCO = \frac{1}{2}(180^\circ - \angle COD) = \frac{1}{2}\left(180^\circ - \frac{360^\circ}{n}\right) = 90^\circ - \frac{180^\circ}{n}\,.$$

Now, $\angle ACD = \angle ACO + \angle DCO$, so

$$120^\circ = 90^\circ - \frac{360^\circ}{n} + 90^\circ - \frac{180^\circ}{n} \iff \frac{540^\circ}{n} = 60^\circ \iff n = 9\,.$$

Therefore, the polygon has 9 sides. ■

Problems involving polygons arise frequently. Try to remember the facts and strategies here, but more importantly, remember how to rederive these facts.

What other solutions to this problem can you find?

THE OLYMPIAD CORNER

No. 288

R.E. Woodrow

We begin this number with a collection of examination sets from Peru that were sent to us by Hugo Luyo, trainer for the Mathematics Competitions in Peru. They were translated with the assistance of Leda Sanchez, Executive Assistant to the Vice-Provost (International) at the University of Calgary. My thanks go to both of them for making these available for the *Corner*.

OLIMPIADA NACIONAL ESCOLAR DE MATEMÁTICA 2009
Level 1
November, 2009

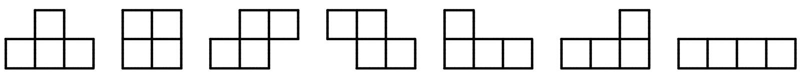
1. If P, E, R, and U are pairwise distinct nonzero digits such that $\overline{PER} + \overline{PRU} + \overline{PUE} + 2009 = \overline{PERU}$, find all the values that $P + E + R + U$ can take.

2. Saladin ("unlucky" one) and Suertudo ("lucky" one) are playing with a die. Each time a player rolls a 6 he gets one point. Suertudo is so lucky that he always gets at least one point in every five consecutive rolls. On the other hand Saladin gets at most one point in every six consecutive rolls. The first one to accumulate four points wins, with Suertudo starting the game and players throwing the die alternately.

(a) Show a sequence of play in which Suertudo wins.

(b) Show a sequence of play in which Saladin wins.

3. Andrés and Bertha play on a 4×4 table with tetrominoes as shown.



Andrés begins the game placing 4 tetrominoes of the same shape on the table without overlaps and leaving no empty space. Then Bertha must write on each square of the table one of the numbers 1, 2, 3 or 4 in such a way that each row and column has no two numbers repeated. Bertha wins if each of the tetrominoes on the table covers 4 distinct numbers.

(a) Show that Bertha can always win the game.

(b) Andrés fills the table with 4 tetrominoes where at least 2 are different. Is it true that in this situation, playing with the same rules, Bertha can always win?

4. Let $k > 1$ be an integer. A positive integer N is a *bimultiple* of k if N is a multiple of k and when the order of the digits of N is reversed, then the resulting number is also a multiple of k. Mario writes a 7-digit number on the board, all digits nonzero. Show that one can erase three of the digits of N so that the remaining 4-digit number is a bimultiple of some $k > 1$.

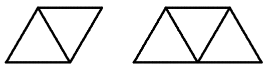
Level 2

1. Let a, b, c, and d be four integers whose sum is 0. Let

$$M = (bc - ad)(ac - bd)(ab - cd) .$$

Show that there is an integer P such that $P^2 = M$.

2. An equilateral triangle of side length 6 is divided into 36 small equilateral triangles of side length 1. The resulting chart is covered by m markers of type A and n markers of type B without doubling or leaving empty spaces. Markers of type A are formed by two equilateral triangles of side length 1 and markers of type B are formed from 3 small triangles, as shown in the figure. Determine all possible values of m.



3. For each positive integer n let d be the largest divisor of n with $d \leq \sqrt{n}$, and define $a_n = \dfrac{n}{d} - d$. Show that in the sequence a_1, a_2, a_3, $\ldots$ each nonnegative integer k appears infinitely often.

4. On a circle $N \geq 5$ points are marked so that the N arcs formed have the same length. A coin is placed on each point, and Ricardo and Tomás play a game with the following rules:

- They play alternately.
- Ricardo starts.
- A player may take a coin only if that coin forms an acute triangle with at least two other coins.

A player loses when he cannot take any coin during his turn.

Does either player have a winning strategy? If so, what is it?

Level 3

1. For each positive integer N let $c(n)$ be the number of decimal digits of N. Let A be a set of positive integers such that if a and b are two distinct elements of A, then $c(a + b) + 2 > c(a) + c(b)$. Find the largest number of elements that A can have.

2. In a quadrilateral $ABCD$, a circle is drawn that is tangent to the sides AB, BC, CD, and DA at the points M, N, P, and Q respectively. Prove that if

$$AM \cdot CP = BN \cdot DQ\,,$$

then $ABCD$ can be inscribed in a circle.

3. (a) There are 8 points placed on a circle. We say that Juliana performs "operation T" if she chooses 3 such points and paints the sides of the triangle they determine in such a way that each painted triangle has at most one vertex in common with a previously painted triangle.

What is the greatest number of operations T that Juliana can make?

(b) If in part (a) you have 7 points instead of 8 points, then what is the greatest number of operations T Juliana can make?

4. Let n be a positive integer. A rectangular $4 \times n$ array is tiled by $2n$ dominoes, and each point lying underneath a corner of a domino is painted red. What is the smallest number of red points that can be obtained?

Next we give the problems of the selection test for the Swiss Olympiad Team for 2006, Sélection OIM 2006. Thanks go to Bill Sands, Canadian Team Leader to the IMO in Vietnam for obtaining them for us.

SÉLECTION OIM 2006
29 et 30 avril 2006, 13 et 14 mai 2006

Premier jour — 4.5 heures

1. Dans le triangle ABC soit D le milieu du côté BC et E la projection de C sur AD. On suppose que $\angle ACE = \angle ABC$. Montrer que le triangle ABC est soit isocèle, soit rectangle.

2. Soit $n \geq 5$ un nombre entier. Déterminer le plus grand entier k tel qu'il existe un n-gone avec exactement k angles intérieurs de 90°. (Le n-gone n'a pas besoin d'être convexe.)

3. Soit n un nombre naturel. Chacun des nombres $\{1, 2, \ldots, n\}$ est coloré soit en blanc, soit en noir. On choisit un nombre et on change sa couleur, tout comme la couleur des nombres avec lesquels il a un diviseur commun. Au départ tous les nombres sont blancs. Pour quels n peut-on arriver à une configuration où tous les nombres sont noirs en un nombre fini de changements ?

Deuxième jour — 4.5 heures

4. Soient $1 = d_1 < d_2 < \cdots < d_k = n$ les diviseurs positifs de n. Déterminer tous les n tels que

$$2n = d_5^2 + d_6^2 - 1 .$$

5. Soit ABC un triangle et D un point à l'intérieur. Soit E un point sur la droite AD différent de D. Soient ω_1 et ω_2 les cercles circonscrits des triangles BDE, respectivement CDE. Soit F et G les intersections intérieures respectives de ω_1 et ω_2 avec le côté BC. Soit X le point d'intersection de DG avec AB et Y le point d'intersection de DF avec AC. Montrer que XY est parallèle à BC.

6. Trouver toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$ telles que pour tout $x, y \in \mathbb{R}$ on ait l'égalité suivante

$$f(f(x) - y^2) = f(x)^2 - 2f(x)y^2 + f(f(y)) .$$

Troisième jour — 4.5 heures

7. Les trois zéros réels du polynôme $P(x) = x^3 - 2x^2 - x + 1$ sont $a > b > c$. Trouver la valeur de l'expression

$$a^2b + b^2c + c^2a .$$

8. On aligne les nombres 1, 2, $\ldots$ 2006 le long d'un cercle dans un ordre quelconque. Un coup consiste à échanger deux nombres voisins. Après un nombre fini de coups tous les nombres se trouvent diamétralement opposés à leur position de départ. Montrer qu'au moins une fois on a échangé deux nombres dont la somme valait 2007.

9. Soit ABC un triangle acutangle avec $AB \neq AC$ et l'orthocentre H. Soit M le milieu du côté BC. Soient D sur AB et E sur AC deux points tels que $AE = AD$ et D, H, E se trouvent sur la même droite. Montrer que HM et l'arc commun des cercles circonscrits des triangles ABC et ADE sont orthogonaux.

Quatrième jour — 4.5 heures

10. Soient a, b, c des nombres réels positifs avec $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$. Démontrer l'inégalité suivante :

$$\sqrt{ab + c} + \sqrt{bc + a} + \sqrt{ca + b} \geq \sqrt{abc} + \sqrt{a} + \sqrt{b} + \sqrt{c} .$$

11. Trouver tous les nombres naturels k tels que $3^k + 5^k$ est la puissance d'un nombre naturel d'exposant ≥ 2.

12. Un aéroport contient 25 terminaux qui sont deux à deux reliés par des tunnels. Il y a exactement 50 tunnels principaux qui peuvent être parcourus dans les deux sens, les autres sont à sens unique. Un groupe de quatre terminaux est appelé *connexe* si de chacun d'entre eux on peut accéder à tous les autres en utilisant uniquement les six tunnels qui les relient. Déterminer le nombre maximal de groupes de terminaux connexes.

Next we present the First Round of the 17th Japanese Mathematical Olympiad, written January 8, 2007. Thanks go to Bill Sands, Canadian Team Leader to the IMO in Vietnam, for collecting them for us.

17th JAPANESE MATHEMATICAL OLYMPIAD
First Round
January 8, 2007

1. Let $ABCD$ be a convex quadrilateral with $AB = 3$, $BC = 4$, $CD = 5$, $DA = 6$, and $\angle ABC = 90^\circ$. Find the area of $ABCD$.

2. Determine the ten's digit of $11^{12^{13}}$ (the 12^{13}th power of 11, not the 13th power of 11^{12}).

3. The segment AB and point P lie in a plane, AB is 7 units long, and P is 3 units away from the line AB. Find the smallest possible value of $AP \cdot BP$.

4. The ten's digit of the 4-digit integer n is nonzero. If we take the first 2 digits and the last 2 digits of n as two 2-digit integers, then their product is a divisor of n. Determine all n with this property.

5. Three rectangles lie in a plane such that any two of them have parallel sides. They divide the plane into several regions. Determine the maximum possible number of such regions. (The area contained in no rectangles is one of the regions, so a single rectangle divides the plane into two regions.)

6. We have 15 cards numbered 1, 2, $\dots$, 15. How many ways are there to choose some (at least 1) cards so that all numbers on these cards are larger than or equal to the number of cards chosen?

7. In how many ways can 100 be written as a sum of nonnegative powers of 3? (Two ways are the same if they differ only in the order of the powers.)

8. How many ways are there to cut a cube S into tetrahedra T_1, T_2, $\dots$, T_k with the following properties?

(i) Every vertex of T_1, T_2, $\dots$, T_k is one of the vertices of S.

(ii) For every $i \neq j$, the intersection of T_i and T_j is a common face, a common edge, a common vertex, or empty.

9. How many pairs of integers (a, b) satisfy $a^2b^2 = 4a^5 + b^3$?

10. A set of cards with positive integers on them is given, and the sum of these integers is 2007. For each integer $k = 1, 2, \ldots, 2006$ there is only one way to choose some of these cards so that the sum of the numbers on them is k. (Cards with the same number are considered identical). How many such sets of cards are there?

11. In a mathematical competition, gold medals are given to $\left\lfloor \frac{n}{a} \right\rfloor$ people, silver medals to $\left\lfloor \frac{n}{b} \right\rfloor$ people, and bronze medals to $\left\lfloor \frac{n}{c} \right\rfloor$ people ($a \geq b \geq c$ are integer constants and n is the number of participants). No one gets two or more medals. Determine all triples (a, b, c) with the following property: For all integers $k \geq 3$, there are exactly two values for n such that the number of people without medals is k. Here $\lfloor x \rfloor$ is the largest integer that does not exceed x.

12. There is a village with a population of 2007. This village has no name. You are God of this village and you want villagers to decide the name of this village. Every villager has one idea of the village's name.

Each villager can send a letter to each villager (including himself), and every villager can send any number of letters every day. Letters are collected in the evening and delivered at once the next morning every day. The villager who sends the letter can decide to whom the letter should be delivered. Each villager can send a letter to tell the idea of the name of the village to God only one time. This idea does not need to be the same as the idea which he and the other villagers had thought at first. And every villager's action is only writing a letter.

Every villager can be classified into an honest person or a liar. You and every villager do not know who is an honest person, and who is a liar. But you know that the number of liars is less than or equal to T, and there is one honest person at least in this village.

You can give instructions to every villager only once at noon of one day. An honest person necessarily follows the instruction, but you do not know if a liar follows the instruction. Find the maximum T for which there exists an instruction which fulfills the conditions below.

(i) At last, every honest person sends a letter to God and every honest person sends the same idea of the village's name.

(ii) If every honest person had thought the same idea of the name of the village at first, every honest person sends this idea to God.

And to complete that set we give the Final Round of the 17^{th} Japanese Mathematical Olympiad, written February 11, 2007. Thanks again to Bill Sands, Canadian Team Leader to the IMO in Vietnam, for obtaining them.

17^{th} JAPANESE MATHEMATICAL OLYMPIAD

Final Round

February 11, 2007 (Time: 4 hours)

1. Let n be a positive integer. Two people P, Q play a game in which they alternately call integers m with $1 \leq m \leq n$. Player P calls the first number, and once a number is called it cannot be called again. The game ends when all n numbers have been called. If the sum of the numbers that P has called is divisible by 3, then P wins, otherwise Q wins. Find all n such that P can win the game no matter what Q does.

2. Find all functions f, defined on the positive real numbers and taking real values, such that

$$f(x) + f(y) \leq \frac{f(x+y)}{2} \quad \text{and} \quad \frac{f(x)}{x} + \frac{f(y)}{y} \geq \frac{f(x+y)}{x+y}$$

for all positive real numbers x and y.

3. Let Γ be the circumcircle of triangle ABC. Let Γ_A be the circle tangent to AB, AC and tangent internally to Γ, and let Γ_B and Γ_C be defined similarly. Let Γ_A, Γ_B, Γ_C be tangent to Γ at A', B', C', respectively. Prove that the lines AA', BB', CC' are concurrent.

4. A band of width d in the plane is a set of points whose distance from a line is at most $\frac{d}{2}$. Any three of the points A, B, C, D in the plane lie in a band of width 1. Prove that all of them lie in a band of width $\sqrt{2}$.

5. Let $\lfloor r \rfloor$ be the largest integer not exceeding the real number r. For real positive numbers x let $A(x) = \{\lfloor nx \rfloor \,:\, n \text{ is a positive integer}\}$. Find all irrational numbers $\alpha > 1$ with the property that whenever a positive real number β satisfies $A(\alpha) \supset A(\beta)$, then $\frac{\beta}{\alpha}$ is an integer.

Next we give our readers' solutions to problems of the 37^{th} Austrian Mathematical Olympiad Regional Competition (Qualifying Round) given at [2009 : 290–291].

1. Let $0 < x < y$ be real numbers and

$$H = \frac{2xy}{x+y}, \quad G = \sqrt{xy}, \quad A = \frac{x+y}{2}, \quad \text{and} \quad Q = \sqrt{\frac{x^2+y^2}{2}}$$

be the harmonic, geometric, arithmetic, and quadratic means of x and y, respectively. It is well known that $H < G < A < Q$ holds. Order the intervals $[H, G]$, $[G, A]$, and $[A, Q]$ by length.

Solved by Mohammed Aassila, Strasbourg, France; Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Cománeşti, Romania. We use the write-up of Curtis.

By homogeneity, we may assume that $y = 1$ and $x = t$, with $0 < t < 1$, and we consider the functions

$$H(t) = \frac{2t}{1+t}, \quad G(t) = \sqrt{t}, \quad A(t) = \frac{t+1}{2}, \quad Q(t) = \sqrt{\frac{t^2+1}{2}}.$$

Set $u(t) = G(t) - H(t)$, $v(t) = A(t) - G(t)$, and $w(t) = Q(t) - A(t)$. For each fixed $t \in (0,1)$ we claim that $u(t) < w(t) < v(t)$, so that

$$\text{length}([H,G]) < \text{length}([A,Q]) < \text{length}([G,A]).$$

Case 1. $u(t) < w(t)$

The inequality is successively equivalent to

$$\begin{aligned}
\sqrt{t} - \frac{2t}{1+t} &< \sqrt{\frac{t^2+1}{2}} - \frac{t+1}{2}, \\
\sqrt{t} + \frac{(t-1)^2}{2(t+1)} &< \sqrt{\frac{t^2+1}{2}}, \\
t + \frac{(t-1)^4}{4(t+1)^2} + 2\sqrt{t}\cdot\frac{(t-1)^2}{2(t+1)} &< \frac{t^2+1}{2}, \\
\sqrt{t} &< \frac{t^2+6t+1}{4(t+1)}, \\
0 &< \frac{(\sqrt{t}-1)^4}{4(t+1)},
\end{aligned}$$

which is clearly true.

Case 2. $w(t) < v(t)$

The inequality is successively equivalent to

$$\begin{aligned}
\sqrt{\frac{t^2+1}{2}} - \frac{t+1}{2} &< \frac{t+1}{2} - \sqrt{t}, \\
\sqrt{\frac{t^2+1}{2}} &< (t+1) - \sqrt{t}, \\
\frac{t^2+1}{2} &< t^2 + 3t + 1 - 2\sqrt{t}(t+1), \\
0 &< t^2 - 4t\sqrt{t} + 6t - 4\sqrt{t} + 1, \\
0 &< (\sqrt{t}-1)^4.
\end{aligned}$$

Since this last inequality holds, the claim is proved.

2. Let $n > 1$ be an integer and a a real number. Determine all real solutions $(x_1, x_2, \ldots, x_n)$ of the following system of equations:

$$\begin{aligned} x_1 + ax_2 &= 0\,, \\ x_2 + a^2x_3 &= 0\,, \\ x_3 + a^3x_4 &= 0\,, \\ &\vdots \\ x_{n-1} + a^{n-1}x_n &= 0\,, \\ x_n + a^nx_1 &= 0\,. \end{aligned}$$

Solved by Mohammed Aassila, Strasbourg, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Aassila.

Clearly $(x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0)$ is a solution.

So we suppose that $(x_1, x_2, \ldots, x_n) \neq (0, 0, \ldots, 0)$, in which case $a \neq 0$ and $x_i \neq 0$ for each i. Then from the system of equations we have

$$\frac{x_1}{x_2} = -a\,, \quad \frac{x_2}{x_3} = -a^2\,, \quad \ldots\,, \quad \frac{x_n}{x_1} = -a^n\,.$$

Multiplying these equalities we get

$$\begin{cases} a^{\frac{n(n+1)}{2}} = +1 & \text{if } n \text{ is even,} \\ a^{\frac{n(n+1)}{2}} = -1 & \text{if } n \text{ is odd.} \end{cases}$$

Case 1. n is even.

Then, if $a = 1$ we obtain the solution

$$\begin{cases} x_i = k & \text{for } i \text{ odd,} \\ x_i = -k & \text{for } i \text{ even,} \end{cases}$$

where k is a free parameter. If $a = -1$ and $4 \mid n$, then we obtain the solution

$$\begin{cases} x_i = k & \text{for } i \equiv 1, 2 \pmod 4\,, \\ x_i = -k & \text{for } i \equiv 0, 3 \pmod 4\,, \end{cases}$$

where again k is a free parameter.

Case 2. n is odd.

Here we must have $a = -1$ and $n \equiv 1 \pmod 4$, so that

$$\begin{cases} x_i = k & \text{for } i \equiv 1, 2 \pmod 4\,, \\ x_i = -k & \text{for } i \equiv 0, 3 \pmod 4\,, \end{cases}$$

is the solution, where k is a free parameter.

4. Let $\{h_n\}_{n=1}^{\infty}$ be a harmonic sequence of positive rational numbers. In other words, each h_n is the harmonic mean of its neighbours:

$$h_n = \frac{2h_{n-1}h_{n+1}}{h_{n-1}+h_{n+1}}.$$

Prove that if some term h_j of the sequence is the square of a rational number, then the sequence contains an infinite number of terms h_k that are each squares of rational numbers.

Solved by Arkady Alt, San Jose, CA, USA; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Comăneşti, Romania. We give the write-up by Curtis.

Solving the recursion given in the problem statement for h_{n+1} gives

$$h_{n+1} = \frac{h_{n-1}h_n}{2h_{n-1}-h_n}. \qquad (1)$$

For each n let $x_n = \frac{1}{h_n}$. Then (1) implies that

$$x_{n+1} = 2x_n - x_{n-1}, \qquad (2)$$

which in turn implies that $x_{n+1}-x_n = x_n-x_{n-1}$, so that the sequence $\{x_n\}$ is arithmetic. Let k be its common difference. Note that h_n is the square of a rational number if and only if x_n is the square of a rational number. By reindexing, we may assume that $x_0 = (\frac{a}{b})^2$, where a and b are positive integers. Also, $k = \frac{c}{d}$ for some positive integers c and d. Thus,

$$x_n = x_0 + nk = \frac{a^2}{b^2} + n\left(\frac{c}{d}\right) = \frac{a^2d+nb^2c}{b^2d}.$$

Let t be a positive integer, and set $m = a + tb^2c$. Set $l = \frac{m^2-a^2}{b^2c}$ and $n = ld$. Then

$$l = \frac{2tab^2c+t^2b^4c^2}{b^2c} = 2ta + t^2b^2c,$$

and

$$x_n = x_{ld} = \frac{a^2d+ldb^2c}{b^2d} = \frac{a^2+lb^2c}{b^2}$$
$$= \frac{a^2+(2ta+t^2b^2c)b^2c}{b^2} = \left(\frac{a+b^2ct}{b}\right)^2,$$

providing an infinite subsequence of $\{x_n\}$ consisting of squares of rational numbers. The corresponding terms of the sequence $\{h_n\}$ are also squares of rational numbers.

Next we turn to problems of the 37th Austrian Mathematical Olympiad, National Competition, Final Round, Part 1, given at [2009 : 291].

2. Prove that the sequence $\left\{\dfrac{(n+1)^n n^{2-n}}{7n^2+1}\right\}_{n=0}^{\infty}$ is strictly increasing.

Solved by Arkady Alt, San Jose, CA, USA; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Comănești, Romania. We give Zvonaru's solution, modified by the editor.

We write

$$\frac{(n+1)^n n^{2-n}}{7n^2+1} = \frac{(n+1)^n}{n^n}\cdot\frac{n^2}{7n^2+1} = \left(1+n^{-1}\right)^n\left(7+n^{-2}\right)^{-1} .$$

The portion $(7+n^{-2})^{-1}$ is increasing with n, since $f(n)=n^{-2}$ is a decreasing function of n.

It is known that the other portion $(1+n^{-1})^n$ is increasing, and here is a proof: Starting with the AM–GM Inequality, we successively deduce that

$$\begin{aligned} 1+\underbrace{\left(1+n^{-1}\right)+\cdots+\left(1+n^{-1}\right)}_{n} &> (n+1)\sqrt[n+1]{(1+n^{-1})^n}\,; \\ 1+n+n\cdot n^{-1} &> (n+1)\sqrt[n+1]{(1+n^{-1})^n}\,; \\ \left(\frac{n+2}{n+1}\right)^{n+1} &> \left(1+n^{-1}\right)^n ; \\ \left(1+(n+1)^{-1}\right)^{n+1} &> \left(1+n^{-1}\right)^n . \end{aligned}$$

Since both portions are positive and strictly increasing, the given sequence is also strictly increasing.

3. The incircle of triangle ABC touches the lines BC and AC at D and E, respectively. Prove that if AD and BE are of the same length, then the triangle is isosceles.

Solved by Arkady Alt, San Jose, CA, USA; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Comănești, Romania. We give the solution of Amengual Covas.

In $\triangle ABC$ let $BC=a$, $CA=b$, $AB=c$, and let $s=\frac{1}{2}(a+b+c)$ denote the semiperimeter. By the Law of Cosines in $\triangle ADC$ with $DC=s-c$ we have

$$AD^2 = b^2+(s-c)^2-2b(s-c)\cos C ,$$

and similarly in $\triangle BCE$ with $CE=s-c$ we have

$$BE^2 = a^2+(s-c)^2-2a(s-c)\cos C .$$

Using the Law of Cosines once more in $\triangle ABC$ and combining this with the preceding results, we have

$$\begin{aligned} AD^2 - BE^2 &= b^2 - a^2 - 2(s-c)(b-a)\cos C \\ &= (b-a)\left[b + a - (a+b-c)\cdot\frac{a^2+b^2-c^2}{2ab}\right] \\ &= \frac{b-a}{2ab}(ab^2 + a^2b + bc^2 + b^2c + ca^2 + c^2a - a^3 - b^3 - c^3) \\ &= \frac{b-a}{2ab}[a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c)]\,. \end{aligned}$$

Now, $b+c-a$, $c+a-b$, and $a+b-c$ are all positive, therefore $a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c)$ is positive. It follows that $AD^2 - BE^2 = 0$ if and only if $b-a=0$, that is, $AD = BE$ if and only if $\triangle ABC$ is isosceles with $a = b$.

Next we look at solutions sent in for the 37th Austrian Mathematical Olympiad National Competition, Final Round, Part 2, at [2009 : 291–292].

1. Find the number of nonnegative integers $n \le N$ with the property that the decimal expansion of some multiple of n contains only the digits 2 and 6 (not necessarily the same number of each).

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Cománeşti, Romania. We give Zvonaru's write-up.

Since none of 22, 26, 62, 66 is divisible by 4, it follows that no integer ending in these digits can be divisible by 4. Hence, if n is a multiple of 4, then there is no multiple of n containing only the digits 2 and 6.

If n is a multiple of 5, then any multiple of n ends in 0 or 5, so no multiple of n can contain only the digits 2 and 6.

If n is coprime to 10, then there is a number $11\ldots1$ with all digits 1 [a *rep-unit*] that is divisible by n. [*Ed.: By the Pigeon Hole Principle, two numbers in the sequence* 1, 11, 111, $\ldots$ *leave the same remainder modulo* n. *Thus,* n *divides their difference, which is of the form* $111\ldots1000\ldots0$, *and hence* n *divides* $111\ldots1$ *since* n *is coprime to* 10.]

Now, n divides a t-digit rep-unit $11\ldots1$, hence n divides the $2t$-digit number $\underbrace{22\ldots2}_{t}\,\underbrace{66\ldots6}_{t}$, and this multiple of n contains only digits 2 and 6.

If $n = 2k$ with k coprime to 10, then k divides some t-digit rep-unit $11\ldots1$, and then n divides the $2t$-digit number $\underbrace{22\ldots2}_{t}\,\underbrace{66\ldots6}_{t}$.

We deduce that the number of nonnegative integers $n \le N$ with the required property is

$$N - \left\lfloor\frac{N}{4}\right\rfloor - \left\lfloor\frac{N}{5}\right\rfloor + \left\lfloor\frac{N}{20}\right\rfloor\,.$$

2. Prove that

$$3(a+b+c) \;\geq\; 8\sqrt[3]{abc} \;+\; \sqrt[3]{\frac{a^3+b^3+c^3}{3}}$$

for all positive real numbers a, b, and c. Determine when equality holds.

Solved by Mohammed Aassila, Strasbourg, France; Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; and Oliver Geupel, Brühl, NRW, Germany. We give the solution of Diaz-Barrero.

Applying mean inequalities, we have

$$8\sqrt[3]{abc} + \sqrt[3]{\frac{a^3+b^3+c^3}{3}} \;\leq\; 9\sqrt[3]{\frac{8abc + \dfrac{a^3+b^3+c^3}{3}}{9}}$$

$$= 9\sqrt[3]{\frac{24abc + a^3+b^3+c^3}{27}} \;=\; 3\sqrt[3]{24abc + a^3+b^3+c^3}\,.$$

It then suffices to prove any of the following inequalities

$$\begin{aligned} \sqrt[3]{24abc + a^3+b^3+c^3} &\;\leq\; a+b+c\,, \\ 24abc + a^3+b^3+c^3 &\;\leq\; (a+b+c)^3\,, \\ a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2 &\;\geq\; 6abc\,, \end{aligned}$$

and the last inequality follows from the AM–GM Inequality. Indeed, we have

$$a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2 \;\geq\; 6\sqrt[6]{a^6b^6c^6} \;\geq\; 6abc\,.$$

Equality holds if and only if $a = b = c$, and we are done.

3. Given triangle ABC, let point R be on the extension of AB beyond B with $BR = BC$, and let point S be on the extension of AC beyond C with $CS = CB$. Let the diagonals of $BRSC$ intersect in the point A', and construct the points B' and C' similarly. Prove that the area of the hexagon $AC'BA'CB'$ is the sum of the areas of triangles ABC and $A'B'C'$.

Solved by Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; and Titu Zvonaru, Comăneşti, Romania. We use Bataille's submission.

Let $[\cdot]$ denote the area of the figure it encloses. We will show that

$$[A'BC] + [B'CA] + [C'AB] \;=\; [ABC]$$

and

$$[A'B'C'] \;=\; [ABC]\,.$$

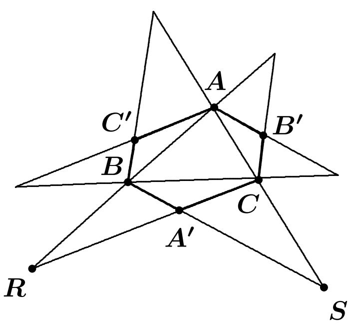
This clearly implies the required result.

Let $BC = a$, $CA = b$, and $AB = c$.

Since $b\overrightarrow{CS} = a\overrightarrow{AC}$ and $c\overrightarrow{BR} = a\overrightarrow{AB}$, we have $bS = -aA + (a+b)C$ and $cR = -aA + (a+c)B$.

It follows that

$$\begin{aligned} & bS + (a+c)B \\ = \ & -aA + (a+c)B + (a+b)C \\ = \ & cR + (a+b)C\,, \end{aligned}$$



and hence

$$\begin{aligned} 2sA' &= bS + (a+c)B \;=\; cR + (a+b)C \\ &= -aA + (a+c)B + (a+b)C\,, \end{aligned} \qquad (1)$$

where $2s = a + b + c$. In particular, we obtain the first relation below and the other two follow similarly:

$$\frac{[A'BC]}{[ABC]} = \frac{a}{2s}\,, \qquad \frac{[B'CA]}{[ABC]} = \frac{b}{2s}\,, \qquad \frac{[C'AB]}{[ABC]} = \frac{c}{2s}\,;$$

and hence

$$[A'BC] + [B'CA] + [C'AB] \;=\; \frac{a+b+c}{2s}[ABC] \;=\; [ABC]\,.$$

In addition, from (1) we also deduce that

$$2s(A + A') \;=\; (b+c)A + (c+a)B + (a+b)C\,.$$

Clearly, the same result $(b+c)A + (c+a)B + (a+b)C$ will be obtained for $2s(B+B')$ and $2s(C+C')$, proving that AA', BB', CC' have the same midpoint. This means that $\triangle ABC$ and $\triangle A'B'C'$ are symmetrical about some point, and hence they have the same area. This completes the proof.

4. Determine all rational numbers x such that $1 + 105 \cdot 2^x$ is the square of a rational number.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania. We give Geupel's solution.

It is readily checked that $-8, -6, -4$, 3, and 4 are solutions and we shall prove there are no others.

Let x be a solution. Then there are integers a, b, c, and d such that b, d are positive, $\gcd(a,b) = \gcd(c,d) = 1$, $x = \frac{a}{b}$, and $1 + 105 \cdot 2^{a/b} = \frac{c^2}{d^2}$. We obtain $2^a = \left(\frac{c^2 - d^2}{105d^2}\right)^b$. The multiplicity of the factor 2 in the right term is a multiple of b. Hence, $b \mid a$, which implies that $b = 1$ and

$$c^2 \;=\; d^2\left(1 + 105 \cdot 2^a\right) \,. \tag{1}$$

Since 106 is not a square, $a \neq 0$, and we consider the cases $a < 0$ and $a > 0$.

First, let $a < 0$ with $m = -a$. From (1), we have

$$2^m c^2 \;=\; d^2\left(2^m + 105\right) \,. \tag{2}$$

If m were odd, then the multiplicity of the factor 2 in (2) would be odd on the left but even on the right, a contradiction. Thus, m is even, say $m = 2n$. We obtain $\left(2^n c\right)^2 = d^2\left(4^n + 105\right)$. Thus, $4^n + 105$ is a square of an integer $q > 0$, and the equation $3 \cdot 5 \cdot 7 = \left(q - 2^n\right)\left(q + 2^n\right)$ yields a factorization of 105 into two positive integers. There is no solution with $q - 2^n = 1$. If $q - 2^n = 3$, then $a = -8$, while if $q - 2^n = 5$ or $q - 2^n = 7$ then $a = -6$ or $a = -4$, respectively. Otherwise we have $q - 2^n \geq 15$, which yields $q + 2^n \leq 7 < q - 2^n$, a contradiction.

Second, let $a > 0$. From (1) we see that $d \mid c$, therefore $d = 1$ and

$$2^a \cdot 105 \;=\; (c-1)(c+1) \,.$$

Checking the cases $1 \leq a \leq 8$, we find the solutions $a = 3$ and $a = 4$. Assume $a \geq 9$. Without loss of generality let $c > 0$. One of the numbers $c - 1$ and $c + 1$ must be divisible by 2^{a-1} and is therefore not less than 256. The other number is not greater than $2 \cdot 105 = 210$, a contradiction.

This completes the proof.

5. Find all monotonic functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\big(-f(x)\big) \;=\; f\big(f(x)\big) \;=\; f(x)^2 \,.$$

(A function f is monotonic if either $f(a) \leq f(b)$ for all $a < b$ or $f(a) \geq f(b)$ for all $a < b$.)

Solved by Michel Bataille, Rouen, France; and Chip Curtis, Missouri Southern State University, Joplin, MO, USA. We give Bataille's solution.

Let f be a monotonic function such that $f(-f(x)) = f(f(x)) = f(x)^2$ for all real numbers x, and let $d(x) = x(f(x) - f(0))$. Since f is monotonic, we have $d(x) \cdot d(y) \geq 0$ for all real x, y.

By a simple calculation, $d(f(0)) \cdot d(-f(0)) = -f(0)^4(f(0) - 1)^2$, so $f(0) = 0$ or $f(0) = 1$.

First, suppose $f(0) = 0$, so that $d(x) = xf(x)$. For any x, the number

$$d(f(x)) \cdot d(-f(x)) \;=\; f(x)f(f(x)) \cdot (-f(x)f(-f(x)) \;=\; -f(x)^6$$

is nonnegative, hence $f(x) = 0$. Thus, f is the constant function $x \mapsto 0$.

Second, suppose $f(0) = 1$, so that $d(x) = x(f(x) - 1)$. Then,

$$d(f(x)) \cdot d(-f(x)) \;=\; -f(x)^2(f(x)^2 - 1)^2\,,$$

hence $f(x) \in \{0,1,-1\}$ for all x. But if for some x_0 we had $f(x_0) = 0$, we would have $0 = f(x_0)^2 = f(f(x_0)) = f(0)$, contradicting $f(0) = 1$. It follows that $f(x) = 1$ or -1 for all x. Since $f(-1) = f(-f(0)) = f(0)^2 = 1$ and similarly $f(1) = 1$, the monotonicity of f implies $f(x) = 1$ for all $x \in [-1,1]$ and $f(x) = 1$ for $x \geq 1$ if f is increasing, $f(x) = 1$ for $x \leq 1$ if f is decreasing.

Now, suppose that f is increasing and different from the constant function $x \mapsto 1$. Let the set $\{x \in \mathbb{R} :\; x < -1 \text{ and } f(x) = -1\}$ have a as its lowest upper bound. We have $a \leq -1$ and if $x < a$, then $f(x') = -1$ for some x' with $x < x' \leq a$, so that $f(x) = -1$. As a result, $f(x) = -1$ for $x < a$ and $f(x) = 1$ for $x > a$ and f is one of the functions ϕ, ϕ_a, ψ_a defined by $\phi(x) = -1$ $(x < -1)$ and $\phi(x) = 1$ $(x \geq -1)$; $\phi_a(x) = -1$ $(x < a)$ and $\phi_a(x) = 1$ $(x \geq a)$; $\psi_a(x) = -1$ $(x \leq a)$ and $\psi_a(x) = 1$ $(x > a)$, where $a < -1$ in each instance.

Similarly, if f is decreasing, then f is one of the functions λ, λ_b, μ_b defined by $\lambda(x) = 1$ $(x \leq 1)$ and $\lambda(x) = -1$ $(x > 1)$; $\lambda_b(x) = 1$ $(x \leq b)$ and $\lambda_b(x) = -1$ $(x > b)$; $\mu_b(x) = 1$ $(x < b)$ and $\mu_b(x) = -1$ $(x \geq b)$, where $b > 1$ in each instance.

Conversely, the constant functions $x \mapsto 0$ and $x \mapsto 1$ and the functions ϕ, ϕ_a, ψ_a, λ, λ_b, μ_b where $a < -1$ and $b > 1$ are monotonic on $\mathbb{R}$ and satisfy $f(-f(x)) = f(f(x)) = f(x)^2$ for all x (the nonconstant ones satisfy $f(x) = \pm 1$ and $f(1) = f(-1) = 1$).

We conclude that these are all of the solutions to the given equation.

6. Let A be a nonzero integer. Find all integer solutions of the following system of equations:

$$\begin{aligned} x + y^2 + z^3 &= A\,, \\ x^{-1} + y^{-2} + z^{-3} &= A^{-1}\,, \\ xy^2z^3 &= A^2\,. \end{aligned}$$

Solved by Mohammed Aassila, Strasbourg, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Edward T.H. Wang and Kaiming Zhao, Wilfrid Laurier University, Waterloo, ON; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give the write-up of Wang and Zhao.

We solve the more general problem where x, y, z, and A are real numbers and $A \neq 0$. We shall show that if $A > 0$, then there are no solutions, while if $A < 0$ then there are exactly four solutions given by $(x,y,z) = (A, \pm\sqrt[4]{-A}, -\sqrt[6]{-A})$ or $(-\sqrt{-A}, \pm\sqrt[4]{-A}, \sqrt[3]{A})$. If we restrict x, y, z, and A to be integers, then there are four solutions if $A = -n^{12}$ for some positive integer n and no solutions otherwise.

We first set $s = y^2$ and $t = z^3$. Then the given system becomes

$$x + s + t = A\,, \qquad (1)$$
$$x^{-1} + s^{-1} + t^{-1} = A^{-1}\,, \qquad (2)$$
$$xst = A^2\,. \qquad (3)$$

From (2) and (3) we obtain

$$xs + st + tx = A\,. \qquad (4)$$

From (1), (3), and (4) we see that x, s, and t are the roots of the cubic equation $f(u) = u^3 - Au^2 + Au - A^2 = 0$.

Since $f(u) = (u - A)(u^2 + A)$ the roots of $f(u)$ are A and $\pm\sqrt{-A}$. Hence, there is only one real solution if $A > 0$, while if $A < 0$ then x, s, t are just A, $\sqrt{-A}$, $-\sqrt{-A}$ in some order.

Since $s = y^2$ is nonnegative, then we must have $y^2 = \sqrt{-A}$, so that $(x, s, t) = (A, \sqrt{-A}, -\sqrt{-A})$ or $(-\sqrt{-A}, \sqrt{-A}, A)$. That is $(x, y, z) = (A, \pm\sqrt[4]{-A}, -\sqrt[6]{-A})$ or $(-\sqrt{-A}, \pm\sqrt[4]{-A}, \sqrt[3]{A})$.

Finally, if A is an integer, then clearly x, y, and z are all integers if and only if $-A = n^{12}$ for some natural number n.

This completes our proof.

Next we open our file of readers' solutions to problems of the Brazilian Mathematical Olympiad 2005, given in the *Corner* at [2009 : 292–293].

1. A positive integer is a *palindrome* if reversing its digits leaves it unchanged (for example, 481184, 131, and 2 are palindromes). Find all pairs (m, n) of positive integers such that $\underbrace{111\ldots1}_{m\text{ ones}} \times \underbrace{111\ldots1}_{n\text{ ones}}$ is a palindrome.

Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

If $m \geq 10$ and $n \geq 10$, the product can be computed as the sum of m rows of n ones each, with each row shifted one digit to the right of the preceding one. Hence, the eighth digit from the right will be the sum of eight ones, and is thus 8. The eighth digit from the left will be the sum of eight ones plus another one from a carry, and is thus 9. In this case, therefore, the product is not a palindrome.

Otherwise, we may assume by symmetry that $m \leq 9$. Again regarding the product as the sum of m rows of n ones each, there are no carries, so the product is $123\cdots mmm\cdots mmm\cdots 321$, a palindrome.

2. Determine the smallest real number C such that

$$C\left(x_1^{2005} + x_2^{2005} + \cdots + x_5^{2005}\right) \geq x_1x_2x_3x_4x_5\left(x_1^{125} + x_2^{125} + \cdots + x_5^{125}\right)^{16}$$

for all positive real numbers x_1, x_2, x_3, x_4, and x_5.

Solved by Mohammed Aassila, Strasbourg, France; Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; and Titu Zvonaru, Comănești, Romania. We give Bataille's generalization.

The smallest suitable C is 5^{15}.

Taking $x_1 = x_2 = x_3 = x_4 = x_5 = 1$, we see that $C \geq 5^{15}$. Thus, it suffices to prove the inequality for all positive x_1, x_2, x_3, x_4, x_5 when $C = 5^{15}$. In fact we prove a generalization: If m, n and x_1, x_2, $\ldots$, x_k are positive real numbers and $p > 1$, then

$$x_1^m \cdots x_k^m (x_1^n + \cdots + x_k^n)^p \leq k^{p-1}(x_1^{km+np} + \cdots + x_k^{km+np}) .$$

Let R denotes the righthand side. We have

$$\begin{aligned} R &= \left(x_1^{\frac{m}{p}} x_2^{\frac{m}{p}} \cdots x_k^{\frac{m}{p}} (x_1^n + x_2^n + \cdots + x_k^n) \right)^p \\ &= \Big((x_1^{m+np} x_2^m \cdots x_k^m)^{\frac{1}{p}} + (x_1^m x_2^{m+np} x_3^m \cdots x_k^m)^{\frac{1}{p}} \\ &\qquad\qquad + \cdots + (x_1^m x_2^m \cdots x_{k-1}^m x_k^{m+np})^{\frac{1}{p}} \Big)^p \\ &\leq k^{p-1} (x_1^{m+np} x_2^m \cdots x_k^m + x_1^m x_2^{m+np} x_3^m \cdots x_k^m \\ &\qquad\qquad + \cdots + x_1^m x_2^m \cdots x_{k-1}^m x_k^{m+np}) \qquad (1) \end{aligned}$$

where (1) follows from the inequality of means:

$$\left(\frac{a_1^{1/p} + \cdots + a_k^{1/p}}{k} \right)^p \leq \frac{a_1 + \cdots + a_k}{k}$$

for positive a_1, a_2, $\ldots$, a_k.

Now, using the weighted AM–GM Inequality we have, for example,

$$\begin{aligned} & x_1^{m+np} x_2^m \cdots x_k^m \\ & \leq \frac{m+np}{km+np} x_1^{km+np} + \frac{m}{km+np} x_2^{km+np} + \cdots + \frac{m}{km+np} x_k^{km+np} . \end{aligned}$$

Treating the other terms of (1) in the same way and adding up gives

$$R \leq k^{p-1} (x_1^{km+np} + x_2^{km+np} + \cdots + x_k^{km+np}) ,$$

as desired.

5. Let ABC be an acute triangle and let F be its Fermat point, that is, the interior point of ABC such that $\angle AFB = \angle BFC = \angle CFA = 120^\circ$. For each of the triangles ABF, BCF, and CAF, draw its Euler line, that is, the line connecting its circumcentre and its centroid.

Prove that these three lines are concurrent.

Comment by Mohammed Aassila, Strasbourg, France.

This is problem A323 from Kömal (September 2003 issue).

6. Let b be an integer and let a and c be positive integers. Prove that there exists a positive integer x such that $a^x + x \equiv b \pmod{c}$, that is, prove there exists a positive integer x such that c divides $a^x + x - b$.

Comment by Mohammed Aassila, Strasbourg, France.

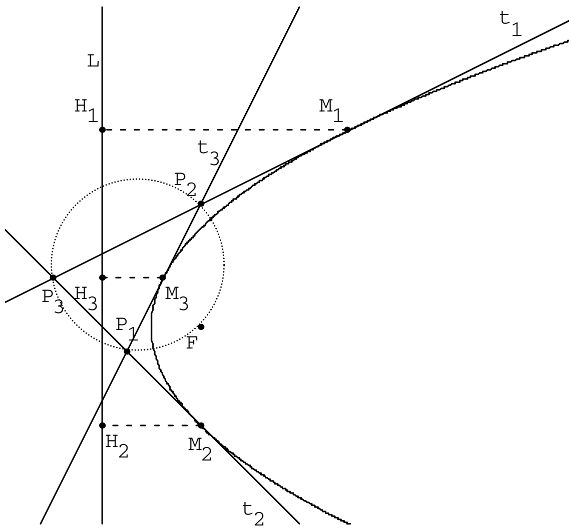
This is a result by Gergely Zábrábi: "An interesting theorem in number theory," Century 2 of Kömal, Volume 2, 1998–2000, pp. 162-163.

Lastly, we give a solution to a problem from the Croatian Mathematical Olympiad 2006, National Competition, 4th Grade, at [2009 : 293–294].

1. Prove that three tangents to a parabola always form the sides of a triangle whose altitudes intersect on the directrix of the parabola.

Solved by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; and Chip Curtis, Missouri Southern State University, Joplin, MO, USA. We give Bataille's presentation.

Let M_1, M_2, M_3 be the points of contact of the three given tangents t_1, t_2, t_3 to the parabola and let H_1, H_2, H_3 be their orthogonal projections onto the directrix L (see figure at right). If F denotes the focus of the parabola, then t_i is the perpendicular bisector of FH_i for $i = 1, 2, 3$ (a well-known result). It follows that the orthogonal projections of F onto t_1, t_2, t_3 are three points on the tangent t to the parabola at its vertex (the image of L under the homothety with centre F and factor $\frac{1}{2}$). As a result, F is on the circumcircle of $\triangle P_1P_2P_3$ and the tangent t is the Simson line of F relative to this triangle. We know this Simson line bisects the segment FH, where H is the orthocentre of $\triangle P_1P_2P_3$. It follows that H is on the image of t under the homothety with centre F and factor 2, that is, H is on L and L is the Steiner line of F relative to $\triangle P_1P_2P_3$.



That's all the material for this number. Send me your nice solutions!

BOOK REVIEWS

Amar Sodhi

Explorations in Geometry
By Bruce Shawyer
World Scientific Publishing Co. Pte. Ltd., 2010
ISBN13: 978-9814295864, Soft cover, 306+xii pages, CAD$35.50
Reviewed by ***J. Chris Fisher****, University of Regina, Regina, SK.*

Finally a book has appeared that is suitable for the Euclidean geometry class that I have taught at my university for many years. Such a class is directed toward education students specializing in secondary-school mathematics, although it also attracts quite a few other students who take the course as an elective. Shawyer's students, like mine, had studied trigonometry and Euclidean geometry in high school (according to the curriculum guides), but by the time they arrived in our classrooms they seem to have long ago forgotten whatever it was they might have been exposed to. For many years there was no adequate source material for these students, hence the author's motivation for turning his class notes into a text book. The only other text I have found that is both well written and aimed at the right level is I. Martin Isaacs' *Geometry for College Students*, which was reviewed in ***CRUX with Mayhem*** [2002 : 505-506]; Isaacs' book, whatever its merits, is overpriced by a factor of four, which makes it far less attractive than this new, reasonably priced paperback. Shawyer's book will certainly also prove useful to students who wish to master the material on their own, perhaps to prepare themselves for a mathematics competition, or just for the enjoyment of solving interesting problems. As the title proclaims, the goal here is to explore geometry. Shawyer's approach is exploratory; his advice for solving a geometry problem: first draw an accurate figure, add helping lines, then revise the figure retaining those parts that seem most relevant to what was given and what was required.

The book comes in three parts. The first 113 pages, arranged into six chapters, briefly describe the tools required for solving geometry problems. It starts with a very quick review of the basic Euclidean theorems; then come chapters on trigonometry, concurrency and collinearity, basic formulae (involving R, r, s, etc.), conic sections (with coordinates), and constructions. An unusual feature, incorporated into the section on the construction of regular polygons, is a discussion of the accuracy of the rule of thumb used by carpenters and draughtsmen for determining the centre of a regular polygon. The second part of the book, covering just 14 pages, is devoted to three fascinating problems that caught the author's fancy. Many of you will have seen his earlier version of the third problem, "Remarkable Bisections", in ***CRUX with Mayhem*** [2006 : 434-435], which by coincidence was closely related to a 2002 Hungarian Olympiad problem [2006 : 150; 2007 : 160-161, 415-417]. The final 176 pages consist of two chapters; the first lists 124 miscellaneous problems; the second presents their solutions together with solutions to many of the other problems that appear throughout the text. The problems have been carefully selected to illustrate the many different tools that had been discussed earlier; the solutions are complete and very well explained. It is these last two chapters that make the text worthwhile.

My enthusiasm for the book, however, is dampened by what I believe to be shortcomings in Chapter 1. The author obviously wishes to keep his summary brief, but he fails to address the issue of what constitutes a proof. Nowhere is the reader told what can be used to justify a claim. The majority of students who take an introductory course in geometry have never seen a valid argument! I would expect Chapter 1 to list the theorems that are to be accepted as known; I believe, moreover, that every result discussed in that chapter should come with a proof that is complete down to the last detail. Instead, the author begins by advising his readers to see Euclid's *Elements* for a definitive account. I object! A sketchy "review" might be fine for the skillful self-learner, but it forces an instructor to devote valuable class time to supplying (boring!) details. The reason for writing the book in the first place was the lack of adequate source material—Euclid is not the source I would recommend to students who lack the required skill, background, and tenacity. The first theorem discussed in the book is that the sum of the angles of a triangle has measure equal to two right angles. His only explanation: "This can be deduced from Fig. 1.1", a figure that shows a line through vertex A that is parallel to the base BC. The figure does not indicate which angles happen to be equal; nowhere in the text is it mentioned what alternate interior angles are, or when they might be equal (although this property is tacitly assumed throughout). On page two the author makes a claim followed by the question "Why?" in parentheses—where are the struggling students going to discover the answer? Is the student to learn from this that he should insert a challenge to the reader (or maybe an "it's obvious") whenever he comes across something he cannot explain? A few sentences earlier come two definitions of similar triangles—one involving corresponding angles, the other the ratios of corresponding sides; the author claims that "it is easy to see that the two definitions are equivalent." Even if that were indeed easy to see, I fail to see how such a claim belongs in a chapter that should be showing how proofs work. It is only in the valuable final chapter (where answers to most of the problems in the text have been produced in full) that the students get to see good proofs. Thus, my recommendation for the text comes with a warning to potential instructors: Be prepared to devote most of the first month of your course to filling in missing details while providing exemplary valid arguments. Until the perfect geometry text comes along, *Explorations in Geometry* can serve as a suitable alternative.

On a Trigonometric Inequality and the Sum of Perimeters of n-gons

Erhard Braune

Abstract

Let $a \geq b > 0$ and $0 < x < \frac{\pi}{2}$. We prove that

$$a^2 \frac{\tan x}{x} + b^2 \left(\frac{\sin x}{x}\right)^2 \; > \; 2ab + b^2 f(x)\,,$$

where $f(x) = x\frac{\tan x}{2}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\left(1 - \frac{\sin x}{x}\right)$, which sharpens and generalizes an inequality of J. Wilker. Taking $a = b = 1$ allows us to sharpen a lower bound for the sum of the perimeters of the inscribed and circumscribed regular n-gons of a circle.

J. Wilker [4] posed the problem to show that

$$\frac{\tan x}{x} + \left(\frac{\sin x}{x}\right)^2 \; > \; 2 \quad \text{for} \quad 0 < x < \frac{\pi}{2}\,; \tag{2}$$

proofs were given by Anglesio, Pinelis and others.

In this note we sharpen and generalize (2): Let $a \geq b > 0$; then

$$a^2 \frac{\tan x}{x} + b^2 \left(\frac{\sin x}{x}\right)^2 \; > \; 2ab + b^2 f(x) \quad \text{for} \quad 0 < x < \frac{\pi}{2}\,, \tag{3}$$

where

$$f(x) \; = \; x\frac{\tan x}{2}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\left(1 - \frac{\sin x}{x}\right).$$

As an application of (3), we obtain the lower bound below for the sum of the perimeters o_n and O_n, where o_n (resp. O_n) denotes the perimeter of the inscribed (resp. circumscribed) regular n-gon of a circle with radius R:

$$o_n + O_n > 4\pi R\left[1 + \frac{2(\pi-2)}{\pi^2 n^4}\left(3n^2 - 4 + \frac{\pi(3n^2+\pi^2)(3n^2-1)^2}{6n^6}\right)\right] \tag{4}$$

for all integers $n \geq 3$, which sharpens the inequality

$$o_n + O_n \; > \; 4\pi R \tag{5}$$

mentioned in Bottema et al., [2], problem 16.22, p. 143.

Proof of inequality (3).

Since $\cos x < \left(\frac{\sin x}{x}\right)^3$ for $0 < x < \frac{\pi}{2}$ (see [1] for a proof) we have

$$1 - (\cos x)^{\frac{1}{3}} > 1 - \frac{\sin x}{x}\,;$$

$$a^2(1 - (\cos x)^{\frac{1}{3}}) > b^2\left(\cos x + 2\sin^2\frac{x}{2}\right)\left(1 - \frac{\sin x}{x}\right)\,;$$

$$a^2\tan x\left(1 - (\cos x)^{\frac{1}{3}}\right) > b^2\sin x\left(1 - \frac{\sin x}{x}\right) + 2b^2\tan x\sin^2\frac{x}{2}\left(1 - \frac{\sin x}{x}\right)\,;$$

$$a^2\tan x + b^2\frac{\sin^2 x}{x} > a^2\tan x(\cos x)^{\frac{1}{3}} + b^2\sin x + 2b^2\tan x\sin^2\frac{x}{2}\left(1 - \frac{\sin x}{x}\right)\,. \qquad (6)$$

The following inequality posed by M.S. Klamkin (see [1] for a proof)

$$a^2\tan x(\cos x)^{\frac{1}{3}} + b^2\sin x \;\geq\; 2abx \qquad (7)$$

holds for a, $b > 0$ and $0 \leq x \leq \frac{\pi}{2}$, so by using (6) we obtain what is required:

$$a^2\tan x + b^2\frac{\sin^2 x}{x} > 2abx + 2b^2\tan x\sin^2\frac{x}{2}\left(1 - \frac{\sin x}{x}\right)\,,$$

$$a^2\frac{\tan x}{x} + b^2\left(\frac{\sin x}{x}\right)^2 > 2ab + b^2\frac{x\tan x}{2}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\left(1 - \frac{\sin x}{x}\right)\,.$$

Proof of inequality (4).

In (3) take $a = b = 1$; then we successively deduce that

$$\frac{\tan x}{x} + \left(\frac{\sin x}{x}\right)^2 > 2 + f(x)\,;$$

$$\tan x > 2x - \frac{\sin^2 x}{x} + xf(x)\,;$$

$$\tan x + \sin x > 2x + \sin x\left(1 - \frac{\sin x}{x}\right) + xf(x) = 2x + \left(1 - \frac{\sin x}{x}\right)\left[\sin x + \frac{x^2\tan x}{2}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\right]\,. \qquad (8)$$

The following inequality is established in [3] for $0 < x \leq \frac{\pi}{2}$,

$$\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) \;\leq\; \frac{\sin x}{x} \;\leq\; \frac{2}{\pi} + \frac{\pi - 2}{\pi^3}(\pi^2 - 4x^2) \qquad (9)$$

with equality if and only if $x = \frac{\pi}{2}$.

For such x, the next three inequalities follow easily from (9) and the last follows from the McLaurin expansion of $\tan x$:

$$1-\frac{\sin x}{x}\geq\frac{4(\pi-2)}{\pi}\left(\frac{x}{\pi}\right)^{2};\qquad \sin x\geq\frac{3x}{\pi}\left(1-\frac{4}{3}\left(\frac{x}{\pi}\right)^{2}\right);$$

$$\frac{\sin\frac{x}{2}}{\frac{x}{2}}\geq\frac{3}{\pi}\left(1-\frac{1}{3}\left(\frac{x}{\pi}\right)^{2}\right);\qquad \tan x>x\left(1+\frac{x^{2}}{3}\right).$$

Hence, using (8), we obtain

$$\begin{aligned}&\tan x+\sin x\\&>2x+\frac{4(\pi-2)}{\pi}\left(\frac{x}{\pi}\right)^{2}\left[\frac{3x}{\pi}\left(1-\frac{4}{3}\left(\frac{x}{\pi}\right)^{2}\right)+\frac{x^{3}}{2}\left(1+\frac{x^{2}}{3}\right)\frac{9}{\pi^{2}}\left(1-\frac{1}{3}\left(\frac{x}{\pi}\right)^{2}\right)^{2}\right]\\&=2x\left[1+\frac{2(\pi-2)}{\pi^{2}}\left(\frac{x}{\pi}\right)^{2}\left(3-4\left(\frac{x}{\pi}\right)^{2}+\frac{9x^{2}}{2\pi}\left(1+\frac{\pi^{2}}{3}\left(\frac{x}{\pi}\right)^{2}\right)\left(1-\frac{1}{3}\left(\frac{x}{\pi}\right)^{2}\right)^{2}\right)\right]\end{aligned}$$

and taking $x=\frac{\pi}{n}$ for $n\geq 3$ finally yields

$$\begin{aligned}o_n+O_n&=2Rn\left(\tan\frac{\pi}{n}+\sin\frac{\pi}{n}\right)\\&>2Rn\cdot\frac{2\pi}{n}\left[1+\frac{2(\pi-2)}{\pi^{2}n^{2}}\left(3-\frac{4}{n^{2}}+\frac{9\pi}{2n^{2}}\left(1+\frac{\pi^{2}}{3n^{2}}\right)\left(1-\frac{1}{3n^{2}}\right)^{2}\right)\right]\\&=4\pi R\left[1+\frac{2(\pi-2)}{\pi^{2}n^{4}}\left(3n^{2}-4+\frac{\pi(\pi^{2}+3n^{2})(3n^{2}-1)^{2}}{6n^{6}}\right)\right].\end{aligned}$$

Acknowledgement. I am grateful to the referee and University Professor Dr. Glen Anderson for some helpful suggestions.

References

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[4] J.B. Wilker, Problem E3306, *American Mathematical Monthly*, **96**(1), 1989, p. 55.

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PROBLEMS

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1er avril 2010**. *Une étoile (⋆) après le numéro indique que le problème a été soumis sans solution.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

3564. *Proposé par Pham Van Thuan, Université de Science de Hanoï, Hanoï, Vietnam.*

Soit a, b, c, d quatre nombres réel positifs. Montrer que

$$a^3+b^3+c^3+d^3+\frac{32abcd}{a+b+c+d} \;\geq\; 3(abc+bcd+cda+dab)\,.$$

3565. *Proposé par Max Diaz, étudiant, Collège Saint Jean Bosco, Huancayo, Junin, Pérou.*

Trouver tous les entiers positifs n tels que $\sigma(\tau(n)) = n$, où $\tau(m)$ et $\sigma(m)$ sont respectivement le nombre de diviseurs positifs de l'entier m et leur somme.

3566. *Proposé par un inconnu.*

Deux points A et C non symétriques étant donnés sur un cercle de centre O, on choisit B sur le plus petit des deux arcs ainsi définis. Soit ℓ la tangente au cercle en B, et soit respectivement P et Q ses points d'intersection avec les bissectrices des angles AOB et BOC. Montrer que si $E = AC \cap OQ$, alors PE est perpendiculaire à OQ.

3567. *Proposé par Albert Stadler, Herrliberg, Suisse.*

Montrer que

$$\int_0^\infty \frac{e^{-x}\left(1-e^{-2x}\right)\left(1-e^{-4x}\right)\left(1-e^{-6x}\right)}{x(1-e^{-14x})}\,dx \;=\; \ln 2\,.$$

3568. *Proposé par Albert Stadler, Herrliberg, Suisse.*

Soit n un entier non négatif et a_k le coefficient de z^k dans le développement de McLaurin de $(z-1)^n \ln(1-z)$. Montrer que

$$a_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \quad \text{et} \quad a_k = \frac{-1}{(n+1)\binom{k}{n+1}}\,,\; k>n\,.$$

3569★. *Proposé par Jian Liu, East China Jiaotang University, Nanchang City, Chine.*

Soit P un point à l'intérieur et Q un point à l'extérieur d'un triangle ABC. On désigne respectivement par w_1, w_2, w_3 les longueurs des bissectrices des angles BPC, CPA et APB. Décider de la validité de l'inégalité

$$PA \cdot QA + PB \cdot QB + PC \cdot QC \ \geq \ 4(w_1w_2 + w_2w_3 + w_3w_1)$$

[Voir `http ://www.emis.de/journals/JIPAM/article1162.html ?sid=1162` où le proposeur montre l'inégalité ci-dessus lorsque Q est à l'intérieur du triangle.]

3570. *Proposé par Arkady Alt, San José, CA, É-U.*

Soit respectivement r, r_a, r_b, r_c, et R les rayons du cercle inscrit, des cercles exinscrits et du cercle circonscrit d'un triangle ABC dont les côtés mesurent a, b et c. Montrer que

$$\frac{r_a^2}{a^2 + r_a^2} + \frac{r_b^2}{b^2 + r_b^2} + \frac{r_c^2}{c^2 + r_c^2} \ \geq \ \frac{4R + r}{4R - r}\,.$$

3571. *Proposé par Arkady Alt, San José, CA, É-U.*

Soit n un entier positif. Parmi toutes les progressions arithmétiques croissantes x_1, x_2, $\ldots$, x_n telles que $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$, trouver la progression avec la plus grande différence commune d.

3572. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit a, b, c trois nombres réels positifs tels que $a + b + c = 1$. Montrer que

$$\left(\sum_{\text{cyclique}} \frac{ab}{c + ab}\right) + \frac{1}{4} \prod_{\text{cyclique}} \left(\frac{a + \sqrt{ab}}{a + b}\right) \ \geq \ 1\,.$$

3573. *Proposé par A.A. Dzhumadil'daeva, Almaty Republic Physics and Mathematics School, Almaty, Kazakhstan.*

On dénote par $(2n + 1)!! = 1 \cdot 3 \cdots (2n + 1)$ le double factoriel. Ainsi, $7!! = 105$. On convient que $0!! = (-1)!! = 1$. Montrer que pour tout entier n non négatif, on a

$$\sum_{\substack{i+j+k=n \\ i,j,k \geq 0}} \binom{n}{i, j, k} (2i - 1)!!(2j - 1)!!(2k - 1)!! \ = \ (2n + 1)!!\,.$$

3574. *Proposé par Michel Bataille, Rouen, France.*

Soit x, y et z trois nombres réels tels que $x+y+z=0$. Montrer que

$$\sum_{\text{cyclique}} \cosh x \ \le \ \sum_{\text{cyclique}} \cosh^2\left(\frac{x-y}{2}\right) \ \le \ 1+2\sum_{\text{cyclique}} \cosh x\,.$$

3575. *Proposé par Michel Bataille, Rouen, France.*

Caractériser les droites passant par le centre du cercle inscrit d'un triangle ABC et coupant respectivement les côtés AB et AC en D et E de sorte que $DE = DB + EC$, et déterminer le nombre de ces droites en fonction des angles en B et C.

. . . .

3564. *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let a, b, c, d be positive real numbers. Prove that

$$a^3+b^3+c^3+d^3+\frac{32abcd}{a+b+c+d} \ \ge \ 3(abc+bcd+cda+dab)\,.$$

3565. *Proposed by Max Diaz, student, San Juan Bosco High School, Huancayo, Junin, Peru.*

Find all positive integers n such that $\sigma(\tau(n)) = n$, where $\tau(m)$ and $\sigma(m)$ are, respectively, the number of positive divisors of the integer m and the sum of all the positive divisors of the integer m.

3566. *Proposed by an unknown proposer.*

Given points A and C on a circle with centre O, choose B on the shorter arc AC. Let ℓ be the line tangent to the circle at B, and let P and Q be the points where ℓ intersects the bisectors of $\angle AOB$ and $\angle BOC$, respectively. Prove that if $E = AC \cap OQ$, then PE is perpendicular to OQ.

3567. *Proposed by Albert Stadler, Herrliberg, Switzerland.*

Prove that

$$\int_0^\infty \frac{e^{-x}\left(1-e^{-2x}\right)\left(1-e^{-4x}\right)\left(1-e^{-6x}\right)}{x(1-e^{-14x})}\,dx \ = \ \ln 2\,.$$

3568. *Proposed by Albert Stadler, Herrliberg, Switzerland.*

Let n be a nonnegative integer and let a_k be the coefficient of z^k in the McLaurin expansion of $(z-1)^n \ln(1-z)$. Prove that

$$a_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \quad \text{and} \quad a_k = \frac{-1}{(n+1)\binom{k}{n+1}}\,, \ k > n\,.$$

3569★. *Proposed by Jian Liu, East China Jiaotang University, Nanchang City, China.*

Let the point P lie inside the triangle ABC and let the point Q lie outside the triangle. Let w_1, w_2, w_3 denote the lengths of the angle bisectors of $\angle BPC$, $\angle CPA$, $\angle APB$, respectively. Does the inequality

$$PA \cdot QA + PB \cdot QB + PC \cdot QC \ \geq\ 4(w_1w_2 + w_2w_3 + w_3w_1)$$

hold? [*At* `http://www.emis.de/journals/JIPAM/article1162.html?sid=1162` *the proposer's inequality is proved when Q lies* inside *the triangle.*]

3570. *Proposed by Arkady Alt, San Jose, CA, USA.*

Let r, r_a, r_b, r_c, and R be, respectively, the inradius, the exradii, and the circumradius of triangle ABC with side lengths a, b, c. Prove that

$$\frac{r_a^2}{a^2+r_a^2} + \frac{r_b^2}{b^2+r_b^2} + \frac{r_c^2}{c^2+r_c^2} \ \geq\ \frac{4R+r}{4R-r}\,.$$

3571. *Proposed by Arkady Alt, San Jose, CA, USA.*

Let $n \geq 1$ be an integer. Among all increasing arithmetic progressions $x_1, x_2, \ldots, x_n$ such that $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$, find the progression with the greatest common difference d.

3572. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let a, b, c be positive real numbers such that $a+b+c = 1$. Prove that

$$\left(\sum_{\text{cyclic}} \frac{ab}{c+ab}\right) + \frac{1}{4}\prod_{\text{cyclic}}\left(\frac{a+\sqrt{ab}}{a+b}\right) \ \geq\ 1\,.$$

3573. *Proposed by A.A. Dzhumadil'daeva, Almaty Republic Physics and Mathematics School, Almaty, Kazakhstan.*

Let $(2n+1)!! = 1\cdot 3\cdots(2n+1)$ be the double factorial, so (for example) $7!! = 105$. Make the convention that $0!! = (-1)!! = 1$. Prove that for any nonnegative integer n,

$$\sum_{\substack{i+j+k=n\\ i,j,k\geq 0}} \binom{n}{i,j,k}(2i-1)!!(2j-1)!!(2k-1)!! \ =\ (2n+1)!!\,.$$

3574. *Proposed by Michel Bataille, Rouen, France.*

Let x, y, and z be real numbers such that $x + y + z = 0$. Prove that

$$\sum_{\text{cyclic}} \cosh x \ \le\ \sum_{\text{cyclic}} \cosh^2\left(\frac{x-y}{2}\right) \ \le\ 1 + 2\sum_{\text{cyclic}} \cosh x\,.$$

3575. *Proposed by Michel Bataille, Rouen, France.*

Let ABC be a triangle with incentre I. Characterize the lines through I intersecting the sides AB and AC at D and E, respectively, such that $DE = DB + EC$ and determine how many such lines there are in terms of $\angle B$ and $\angle C$.

Some of you had asked about open problems in ***CRUX with Mayhem***, that is, problems posed but for which no (correct) solution has yet been received. We have made some progress on tabulating those, and by the end of the year we hope to be able to present a list.

The Editor-in-Chief again asks all solvers who submit solutions to any of the problems appearing in ***CRUX with Mayhem*** to please start each new solution on a fresh page, and to always include the name, affiliation, and contact information (including email address) at the top of the page.

Finally, we thank our readers for the feedback we have received on the quality of the problems that have appeared, and for all the problem proposals that we continue to receive.

Best Wishes,
Václav (Vazz) Linek

SOLUTIONS

Aucun problème n'est immuable. L'éditeur est toujours heureux d'envisager la publication de nouvelles solutions ou de nouvelles perspectives portant sur des problèmes antérieurs.

3439. Replacement. [2009 : 395, 397] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let Γ be a circle with centre O and radius R. Line t is tangent to Γ at the point A, and P is a point on t distinct from A. The line ℓ distinct from t passes through P and intersects Γ at the points B and C. The point K is on the line AC and such that $PK \| AB$, and the point L is on the line AB and such that $PL \| AC$. Prove that $KL \perp OP$.

Solution by John G. Heuver, Grande Prairie, AB.*

Consider the radical axis of the point-circle P and the given circle Γ. Since $\angle LPB = \angle ACB$ (alternate interior angles formed by the transversal PC and the parallel lines PL and CK — if BC has been labeled with B between P and C), while $\angle ACB = \angle LAP$ (angle subtended by chord AB equals the angle between the chord and tangent), it follows that triangles LBP and LPA are similar, whence

$$\frac{LP}{LA} = \frac{LB}{LP}, \quad \text{or} \quad LP^2 = LA \cdot LB\,.$$

In the same way triangles KAP and KPC are similar since $\angle APK = \angle PAB = \angle PCK$, and hence,

$$\frac{KA}{KP} = \frac{KP}{KC}, \quad \text{or} \quad KP^2 = KA \cdot KC\,.$$

Because both K and L have equal powers with respect to both circles P and Γ, we conclude that K and L lie on their radical axis, whence KL is perpendicular to their line of centres PO.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina*; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; OLIVER GEUPEL, Brühl, NRW, Germany*; JOEL SCHLOSBERG, Bayside, NY, USA; MOSCA SEBASTIANO, Pescara, Italy and ERCOLE SUPPA, Teramo, Italy; EDMUND SWYLAN, Riga, Latvia*; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.*

The editors are grateful to our sharp-eyed readers Salem Malikić and Titu Zvonaru for bringing to our attention that the original problem 3439 had appeared three years earlier as problem 3160 [2006 : 305, 308; 2007 : 316-317]. Because at that time we published two solutions, we have decided to publish no further solutions though we have received some rather nice candidates from those solvers of the replacement problem whose names are starred above, from a handful who also sent us solutions a few years back, and from RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; the HUNEDOARA PROBLEM SOLVING GROUP, Hunedoara, Romania; and CRISTINEL MORTICI, Valahia University of Târgovişte, Romania.

3463. [2009 : 395, 397] *Proposed by Michel Bataille, Rouen, France.*

Let Γ be a circle with centre O and radius r, and let P be a point with $OP > r$. Let $\mathcal{L}$ be the set of all lines ℓ such that $P \not\in \ell$ and ℓ intersects Γ at points A, B such that $PA \cdot PB = OP^2 - r^2$. Show that $\mathcal{L}$ is a pencil of concurrent lines.

Solution by Joel Schlosberg, Bayside, NY, USA.

Let Q be the inverse of P with respect to Γ (namely, the point on the ray OP satisfying $OP \cdot OQ = r^2$). Since P is assumed to lie outside Γ, Q must lie inside, and any line passing through Q intersects Γ in two distinct points. The statement of the problem is (obviously) not quite correct: We will prove that $\mathcal{L}$ is the pencil of all lines through Q except the line OP (which is omitted as part of the definition of $\mathcal{L}$). Indeed, we shall see that for any line ℓ that intersects Γ in points A and B, $PA \cdot PB = OP^2 - r^2$ holds if and only if ℓ passes through P or Q. We begin with two lemmas; we provide a proof for the second, while a proof of the first can be found in any reference dealing with the power of a point with respect to a circle.

Lemma 1 If ℓ is a line through P which intersects Γ at points R and S (with $R = S$ when ℓ is tangent), then $PR \cdot PS = OP^2 - r^2$. ∎

Lemma 2 If ℓ is a line passing through Q which intersects Γ at points R and S, then $PR \cdot PS = OP^2 - r^2$.

Proof. When $P \in RS$, then $PR \cdot PS$ is the power of P with respect to Γ, and the result holds by Lemma 1.

Assume then that P does not lie on RS. Since $\angle POR = \angle ROQ$ and $\frac{OP}{OR} = \frac{OR}{OQ}$, $\triangle OPR \sim \triangle ORQ$, and so $\frac{PR}{OR} = \frac{RQ}{OQ}$; by the same reasoning, $\frac{PS}{OS} = \frac{SQ}{OQ}$. Let TU be the diameter of Γ passing through Q; then,

$$RQ \cdot QS = TQ \cdot QU = (r + OQ)(r - OQ) = r^2 - OQ^2\,,$$

whence

$$\begin{aligned} PR \cdot PS &= \frac{OR \cdot OS \cdot RQ \cdot QS}{OQ^2} = \frac{r^2\left(r^2 - OQ^2\right)}{OQ^2} \\ &= \left(\frac{r^2}{OQ}\right)^2 - r^2 = OP^2 - r^2\,, \end{aligned}$$

as desired. ∎

We turn now to the problem. We take ℓ to be an arbitrary line distinct from OP that intersects Γ in points A and B. The second lemma implies that if ℓ passes through Q, then $PA \cdot PB = OP^2 - r^2$ and, therefore, $\ell \in \mathcal{L}$, as desired. Conversely, suppose that $\ell \in \mathcal{L}$ (that is, we assume that $P \not\in \ell$ and $PA \cdot PB = OP^2 - r^2$), and let C and D be the second points of intersection of Γ with PA and QA, respectively (where $C = A$ when PA is tangent). We are to show that $Q \in \ell$; it suffices to prove that $B = D$. Using in turn Lemma 1, Lemma 2, and the definition of A and B, we obtain

$$PA \cdot PC \; = \; OP^2 - r^2 \; = \; PA \cdot PD \; = \; PA \cdot PB\,.$$

It follows that B, C, and D are each in the intersection of Γ and the circle with centre P and radius $\dfrac{OP^2 - r^2}{PA}$, which is distinct from Γ since their centres, O and P, are different. Since these circles can intersect in at most two points, the points B, C, and D cannot be distinct. From the way B and C were defined, the possibility $C = B$ would imply that $P \in AB = \ell$, contradicting our assumption that $P \notin \ell$. Should $C = D$, then AC would be a diameter of Γ, whence the circle with centre P and radius $PC = PD = PB$ would be tangent to Γ at the common point $B = C = D$, again contradicting the assumption $P \notin \ell$. We are left with $B = D$, which completes our solution.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; EDMUND SWYLAN, Riga, Latvia; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3464. Correction. [2009 : 395, 398; 463, 465] *Proposed by Michel Bataille, Rouen, France.*

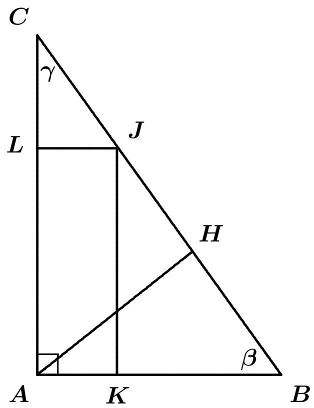
Let ABC be a triangle with $\angle A = 90^\circ$ and H be the foot of the altitude from A. let J be the point on the hypotenuse BC such that $CJ = HB$ and let K, L be the projections of J onto AB, AC, respectively. Prove that

$$\mathcal{M}\left(-\tfrac{2}{3}; AK, AL\right) \;\le\; \tfrac{1}{2}\mathcal{M}\left(-2; AB, AC\right)\,,$$

where $\mathcal{M}\left(\alpha; x, y\right) = \left(\dfrac{x^\alpha + y^\alpha}{2}\right)^{1/\alpha}$.

Solution by John G. Heuver, Grande Prairie, AB.

Let $\beta = \angle ABC$ and $\gamma = \angle ACB$. Then



$$\begin{aligned}
AB &= BC\sin\gamma\,;\\
AC &= BC\sin\beta\,;\\
CJ = HB = AB\sin\gamma &= BC\sin^2\gamma\,;\\
AK = LJ = CJ\cos\beta &= BC\sin^3\gamma\,;\\
CL = CJ\sin\beta &= BC\sin\beta\sin^2\gamma\,;\\
AL = AC - CL &= BC\sin\beta\\
&\quad - BC\sin\beta\sin^2\gamma\\
&= BC\cos^3\gamma\,.
\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{M}\left(-\frac{2}{3}; AK, AL\right) &= \left(\frac{BC^{\frac{-2}{3}}\sin^{-2}(\gamma) + BC^{\frac{-2}{3}}\cos^{-2}(\gamma)}{2}\right)^{-3/2}, \\ &= 2^{\frac{3}{2}} BC\sin^3\gamma\cos^3\gamma \\ \mathcal{M}(-2; AB, AC) &= \left(\frac{BC^{-2}\sin^{-2}(\gamma) + BC^{-2}\sin^{-2}(\beta)}{2}\right)^{-1/2} \\ &= 2^{\frac{1}{2}} BC\sin\gamma\cos\gamma\,.\end{aligned}$$

The desired inequality then becomes

$$2^{\frac{3}{2}} BC\sin^3\gamma\cos^3\gamma \;\leq\; 2^{-\frac{1}{2}} BC\sin\gamma\cos\gamma\,,$$

or equivalently

$$4\sin^2\gamma\cos^2\gamma \;\leq\; 1\,.$$

Finally, the last inequality is just $\sin^2(2\gamma) \leq 1$, which is obvious. Equality holds if and only if $\beta = \gamma = \frac{\pi}{4}$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3465. [2009 : 395, 398] *Proposed by Xavier Ros (student) and José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Prove that $\displaystyle\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij(i+j)} < \frac{\pi^2}{6} + \frac{1}{2} + \frac{3}{4}\log 2$.

Solution by George Apostolopoulos, Messolonghi, Greece.

We have

$$\begin{aligned}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij(i+j)} &= \sum_{i=1}^{\infty}\frac{1}{i}\sum_{j=1}^{\infty}\frac{1}{j(i+j)} = \sum_{i=1}^{\infty}\frac{1}{i}\sum_{j=1}^{\infty}\left(\frac{1}{j} - \frac{1}{i+j}\right) \\ &= \sum_{i=1}^{\infty}\frac{1}{i^2}\sum_{j=1}^{i}\frac{1}{j} = \sum_{j=1}^{\infty}\frac{1}{j}\sum_{i=j}^{\infty}\frac{1}{i^2}\,. \qquad (1)\end{aligned}$$

In equation (1) the term corresponding to $j = 1$ in the outer sum is

$$\sum_{i=1}^{\infty}\frac{1}{i^2} = \frac{\pi^2}{6}\,. \qquad (2)$$

On the other hand, we can estimate the rest of the sum as follows

$$\sum_{j=2}^{\infty}\frac{1}{j}\sum_{i=j}^{\infty}\frac{1}{i^2} \;<\; \sum_{j=2}^{\infty}\frac{1}{j}\sum_{i=j}^{\infty}\frac{1}{i(i-1)} \;=\; \sum_{j=2}^{\infty}\frac{1}{j(j-1)} \;=\; 1\,. \qquad (3)$$

Combining (2) and (3) we have

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij(i+j)} \;<\; 1+\frac{\pi^2}{6}\,,$$

which implies the given inequality since $\frac{\pi^2}{6}+1 < \frac{\pi^2}{6}+\frac{1}{2}+\frac{3}{4}\ln 2$, which holds because $e < 2\sqrt{2}$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece (2nd solution); OVIDIU FURDUI, Campia Turzii, Cluj, Romania; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; TOM LEONG, The University of Scranton, Scranton, PA, USA; CRISTINEL MORTICI, Valahia University of Târgoviște, Romania; ALBERT STADLER, Herrliberg, Switzerland; and the proposer. One incorrect solution was submitted.

Though our featured solution improves the given upper bound by methods similar to those used by the proposers, there is still considerable room for improvement.

Indeed many solvers pointed out that Crux #2984 [2004 :431, 433; 2005 : 478–480] asked to show that $\sum\limits_{i=1}^{\infty}\sum\limits_{j=1}^{\infty}\frac{1}{ij(i+j)} = 2\sum\limits_{n=1}^{\infty}\frac{1}{n^3}$*, which is* $2\zeta(3) = 2.4041\ldots$*, while on the other hand* $\frac{\pi^2}{6}+\frac{1}{2}+\frac{3}{4}\log 2 = 2.6647\ldots$*, an error of roughly* 11%.

Lau pointed out that the expression $\sum\limits_{i=1}^{\infty}\frac{1}{i^2}\sum\limits_{j=1}^{i}\frac{1}{j}$*, which occurs in (1) above, was shown by Euler in 1775 to be equal to* $2\zeta(3)$.

3466. [2009 : 396, 398] *Proposed by Tuan Le, student, Fairmont High School, Anaheim, CA, USA.*

Let x, y, and z be positive real numbers such that $xyz \ge 10+6\sqrt{3}$. Prove that

$$\frac{y}{x+y^3+z^2}+\frac{z}{y+z^3+x^2}+\frac{x}{z+x^3+y^2} \;\le\; \frac{1}{2}\,.$$

Solution by Thanos Magkos, 3rd High School of Kozani, Kozani, Greece.

Clearly, $xyz \neq 0$. Put

$$U = \left(x^{1/2},\, y^{-1/2},\, 1\right)\,, \qquad V = \left(x^{1/2},\, y^{3/2},\, z\right)\,.$$

By the Cauchy–Schwarz Inequality,

$$\left(x+y^{-1}+1\right)(x+y^3+z^2) \;=\; \|U\|^2\|V\|^2 \;\ge\; (U\cdot V)^2 \;=\; (x+y+z)^2\,.$$

Thus,

$$x + y^3 + z^2 \;=\; \frac{x^2}{x} + \frac{y^2}{y^{-1}} + \frac{z^2}{1} \;\geq\; \frac{(x+y+z)^2}{x + y^{-1} + 1} \;=\; \frac{y(x+y+z)^2}{xy + y + 1}$$

Hence,

$$\frac{y}{x + y^3 + z^2} \;\leq\; \frac{xy + y + 1}{(x+y+z)^2}\,,$$

with two similar inequalities for the other two fractions. The left side of the given inequality is then at most equal to

$$\frac{xy+y+1}{(x+y+z)^2} + \frac{yz+z+1}{(x+y+z)^2} + \frac{zx+x+1}{(x+y+z)^2}$$
$$=\; \frac{xy+yz+zx+x+y+z+3}{(x+y+z)^2}\,.$$

It now suffices to prove that

$$(x+y+z)^2 \;\geq\; 2(xy+yz+zx) + 2(x+y+z) + 6\,,$$

or equivalently

$$x^2 + y^2 + z^2 \;\geq\; 2(x+y+z) + 6\,.$$

Since $3(x^2 + y^2 + z^2) \geq (x+y+z)^2$, it suffices to prove that

$$(x+y+z)^2 \;\geq\; 6(x+y+z) + 18\,,$$

which holds whenever $x + y + z \geq 3(1 + \sqrt{3})$. This is true, since by the AM–GM Inequality we have

$$x + y + z \;\geq\; 3\sqrt[3]{xyz} \;\geq\; 3\sqrt[3]{10 + 6\sqrt{3}} \;=\; 3(1 + \sqrt{3})\,.$$

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; KEE-WAI LAU, Hong Kong, China; and the proposer. One incomplete solution was submitted.

3467. [2009 : 396, 398] *Proposed by Tuan Le, student, Fairmont High School, Anaheim, CA, USA.*

Let x, y, and z be positive real numbers. Prove that

$$\sqrt[3]{\frac{x^3 + y^3 + z^3}{xyz}} + \sqrt{\frac{xy + yz + xz}{x^2 + y^2 + z^2}} \;\geq\; \sqrt[3]{3} + 1\,.$$

Solution by Francisco Javier García Capitán, IES Álvarez Cubero, Priego de Córdoba, Spain.

From the Power Mean Inequality it follows that

$$\sqrt[3]{x^3+y^3+z^3} \;\geq\; \sqrt[3]{3}\sqrt{\frac{x^2+y^2+z^2}{3}}\,. \qquad (1)$$

Starting with the AM-GM Inequality, we successively deduce that:

$$\begin{aligned} \frac{xy+yz+zx}{3} &\geq \sqrt[3]{x^2y^2z^2}\,, \\ \sqrt{\frac{xy+yz+zx}{3}} &\geq \sqrt[3]{xyz}\,, \\ \frac{1}{\sqrt[3]{xyz}} &\geq \sqrt{\frac{3}{xy+yz+zx}}\,. \end{aligned} \qquad (2)$$

From (1) and (2) we obtain

$$\begin{aligned} &\sqrt[3]{\frac{x^3+y^3+z^3}{xyz}} + \sqrt{\frac{xy+yz+zx}{x^2+y^2+z^2}} \\ &\geq \sqrt[3]{3}\sqrt{\frac{x^2+y^2+z^2}{xy+yz+zx}} + \sqrt{\frac{xy+yz+zx}{x^2+y^2+z^2}} = \sqrt[3]{3}R + \frac{1}{R}\,, \end{aligned}$$

where $R = \sqrt{\dfrac{x^2+y^2+z^2}{xy+yz+zx}}$.

Since

$$\begin{aligned} &x^2+y^2+z^2-(xy+yz+zx) \\ &= \frac{1}{2}\big((x-y)^2+(y-z)^2+(z-x)^2\big) \geq 0\,, \end{aligned}$$

we have $R \geq 1$, with equality if and only if $x=y=z$.

Hence, $\sqrt[3]{3}R + \dfrac{1}{R} - (\sqrt[3]{3}+1) = (R-1)\left(\sqrt[3]{3} - \dfrac{1}{R}\right) \geq 0$, from which the given inequality follows immediately and equality holds if and only if $x=y=z$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; MARIAN DINCĂ, Bucharest, Romania; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria and GERHARD KIRCHNER, University of Innsbruck, Innsbruck, Austria; CRISTINEL MORTICI, Valahia University of Târgovişte, Romania; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Perfetti pointed out that the same problem by the same proposer also appeared as Problem #143 on page 697 of La Gaceta de la RSME, *Vol. 12, No. 4, 2009. The website is* `http://www.rsme.es/gacetadigital/abrir.php?id=895`.

Janous and Kirchner commented that the following stronger inequality holds:

$$\sqrt[3]{\frac{x^3+y^3+z^3}{3xyz}} + \sqrt{\frac{xy+yz+xz}{x^2+y^2+z^2}} \geq 2\,.$$

Indeed, by multiplying across inequalities (1) and (2) in the featured solution and then dividing by $\sqrt[3]{3}$, one sees that the leftmost expression above is bounded below by R, hence the entire left side is bounded below by $R+1/R \geq 2$.

3468. [2009 : 396, 398] *Proposed by Joseph DeVincentis, Salem, MA, USA; Bernardo Recamán, Instituto Alberto Merani, Bogotá, Colombia; Peter Saltzman, Berkeley, CA, USA; and Stan Wagon, Macalester College, St. Paul, MN, USA.*

Find all positive integers n for which one can match each even number from $\{2, 4, \ldots, 2n\}$ with exactly one odd number from $\{1, 3, \ldots, 2n-1\}$ so that the sums of the resulting n pairs are pairwise relatively prime. For example, if $n=3$ a solution is $2+3=5$, $4+5=9$, $6+1=7$.

Solution by Oliver Geupel, Brühl, NRW, Germany; and Joel Schlosberg, Bayside, NY, USA, modified slightly by the editor.

We shall prove that a solution exists if and only if $1 \leq n \leq 21$, or $n=23$, 29. First, we list the matchings for these values of n in the table below:

$1 \leq n \leq 3$	$1+2=3$	$3+4=7$	$5+6=11$
$4 \leq n \leq 8$	$1+4=5$	$3+8=11$	$5+2=7$
	$7+6=13$	$9+10=19$	$11+12=23$
	$13+14=27$	$15+16=31$	
$n=9$	$1+2=3$	$3+8=11$	$5+12=17$
	$7+16=23$	$9+4=13$	$11+18=29$
	$13+6=19$	$15+10=25$	$17+14=31$
$n=10$	$1+6=7$	$3+10=13$	$5+4=9$
	$7+16=23$	$9+8=17$	$11+20=31$
	$13+12=25$	$15+14=29$	$17+2=19$
	$19+18=37$		
$11 \leq n \leq 12$	$1+6=7$	$3+8=11$	$5+12=17$
	$7+16=23$	$9+4=13$	$11+14=25$
	$13+18=31$	$15+22=37$	$17+2=19$
	$19+10=29$	$21+20=41$	$23+24=47$
$n=13$	$1+6=7$	$3+8=11$	$5+12=17$
	$7+16=23$	$9+4=13$	$11+14=25$
	$13+18=31$	$15+22=37$	$17+24=41$
	$19+10=29$	$21+26=47$	$23+20=43$
	$25+2=27$		

(Continued on next page)

$14 \le n \le 15$	$1+6=7$	$3+8=11$	$5+14=19$
	$7+16=23$	$9+4=13$	$11+18=29$
	$13+12=25$	$15+22=37$	$17+24=41$
	$19+28=47$	$21+10=31$	$23+20=43$
	$25+2=27$	$27+26=53$	$29+30=59$
$16 \le n \le 18$	$1+4=5$	$3+6=9$	$5+8=13$
	$7+10=17$	$9+2=11$	$11+12=23$
	$13+16=29$	$15+22=37$	$17+14=31$
	$19+24=43$	$21+20=41$	$23+26=49$
	$25+28=53$	$27+32=59$	$29+18=47$
	$31+30=61$	$33+34=67$	$35+36=71$
$19 \le n \le 21$	$1+4=5$	$3+8=11$	$5+2=7$
	$7+6=13$	$9+10=19$	$11+12=23$
	$13+16=29$	$15+22=37$	$17+14=31$
	$19+24=43$	$21+20=41$	$23+28=51$
	$25+34=59$	$27+26=53$	$29+18=47$
	$31+30=61$	$33+38=71$	$35+32=67$
	$37+36=73$	$39+40=79$	$41+42=83$
$n \in \{23, 29\}$	$1+4=5$	$3+8=11$	$5+12=17$
	$7+6=13$	$9+10=19$	$11+18=29$
	$13+24=37$	$15+16=31$	$17+26=43$
	$19+22=41$	$21+2=23$	$23+30=53$
	$25+34=59$	$27+20=47$	$29+32=61$
	$31+36=67$	$33+38=71$	$35+14=49$
	$37+42=79$	$39+44=83$	$41+40=81$
	$43+46=89$	$45+28=73$	$47+50=97$
	$49+52=101$	$51+58=109$	$53+54=107$
	$55+48=103$	$57+56=113$	

Next we consider such a matching for $n \ge 22$ and $n \ne 23$, 29. Since each sum is an odd number between 3 and $4n-1$ and the sums are pairwise relatively prime, each sum must have at least one odd prime divisor q with $q \le 4n-1$ such that q does not divide any of the other sums. Since the number of odd primes between 3 and $4n-1$, inclusive, is $\pi(4n-1)-1$ where π is the prime counting function, and there are n sums, we must have $\pi(4n-1)-1 \ge n$. We consider three mutually exclusive cases:

Case 1: $24 \le n \le 25$ or $30 \le n \le 38$.

Let $f(k) = \pi(4k-1)-1$. By using a table of prime numbers we see that $f(k) = k-1$ for $k = 24$, 25, 30; $f(k) = k-2$ for $k = 31$, 32, 33, 35; $f(k) = k-3$ for $k = 34$, 36, 38; and $f(k) = k-4$ for $k = 37$. Hence, the matching does not exist in this case.

Case 2: $n \ge 39$.

It is known that $\pi(n) < \dfrac{1.25506n}{\ln n}$ for all $n > 1$ (see Eric Weisstein, "Prime Counting Function", from *MathWorld—A Wolfram Web Resource*, `http://mathworld.wolfram.com/primecountingfunction.html`), so that

$$\pi(4n-1)-1 \;<\; \pi(4n-1) \;<\; \frac{1.25506(4n-1)}{\ln(4n-1)}\,. \tag{1}$$

Since $\frac{1}{4}(e^{4\times 1.25506}+1) \approx 38.1 < n$, we have

$$\ln(4n-1) \;>\; 4\times 1.25506 \;>\; \frac{1.25506(4n-1)}{n}\,. \tag{2}$$

From (1) and (2) we deduce that $\pi(4n-1)-1 < n$, so the matching cannot exist in this case.

Case 3: $n \in \{22,\, 26,\, 27,\, 28\}$.

In this case, $n = \pi(4n-1)-1$. By the Pigeonhole Principle, in order for the n sums to be pairwise coprime, each sum is either an odd prime or an odd prime power, and all these primes are distinct. Hence, for every odd prime $p \leq 4n-1$, there is an integer $a(p)$ with $1 \leq a(p) \leq \lfloor \log_p(4n-1) \rfloor$ and such that $p^{a(p)}$ is one of the n sums. The n sums must add up to $1+2+\cdots+2n = 2n^2+n$, that is

$$\sum_{3\leq p\leq 4n-1} p^{a(p)} \;=\; 2n^2+n\,.$$

Since $87 \leq 4n-1 \leq 111$, the value of $\lfloor \log_p(4n-1) \rfloor$ is 4, 2, or 1 depending on whether (resp.) $p=3$, $p \in \{5,\,7\}$, or $p \geq 11$. We then have

$$\sum_{3\leq p\leq 4n-1} p^{a(p)} \;=\; 3^{a(3)} + 5^{a(5)} + 7^{a(7)} + \left(\sum_{11\leq p\leq 4n-1} p\right),$$

where $a_3 \in \{1,\,2,\,3,\,4\}$ and $a_5,\, a_7 \in \{1,\,2\}$.

Hence,

$$3^{a(3)} + 5^{a(5)} + 7^{a(7)} \;=\; 2n^2+n-\left(\sum_{11\leq p\leq 4n-1} p\right). \tag{3}$$

However, direct computations show that $3^{a(3)} + 5^{a(5)} + 7^{a(7)} \in \{15,$ $21,\, 35,\, 39,\, 41,\, 57,\, 59,\, 63,\, 77,\, 81,\, 83,\, 93,\, 101,\, 113,\, 135,\, 155\}$, while on the other hand for $n \in \{22,\, 26,\, 27,\, 28\}$ we have

$$2n^2+n-\left(\sum_{11\leq p\leq 4n-1} p\right) \in \{131, 133\}\,,$$

contrary to (3). Therefore, the matching also does not exist in this last case.

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; and the proposers. One partially incorrect solution was submitted.

The proposers did not use nearly so sharp an estimate for $\pi(x)$ as our featured solution. They noted a prime $p \geq 30$ is of the form $p = 30k + r$ for some $r \in \{1, 7, 11, 13, 17, 19, 23, 29\}$ and that the first seven r's are distinct modulo 7, hence (for fixed k) not all can yield a prime. At $n = 54$ they observed a shortage of 8 primes to build a matching, thus they concluded that for $n \geq 54$, the shortage of primes could never be made up.

3469. [2009 : 396, 398] *Proposed by Mihaela Blanariu, Columbia College Chicago, Chicago, IL, USA.*

Let $p \geq 2$ be a real number. Find the limit

$$\lim_{n\to\infty} \frac{1 + \sqrt{2}(2!)^p + \sqrt[3]{3}(3!)^p + \cdots + \sqrt[n]{n}(n!)^p}{(n!)^p}.$$

Solution by Joel Schlosberg, Bayside, NY, USA and Kee-Wai Lau, Hong Kong, China.

For $1 \leq k \leq n-1$, by the well-known fact that $\sqrt[x]{x} \leq \sqrt[e]{e}$ for all $x > 0$, we have

$$0 < \sqrt[k]{k}(k!)^p \leq \sqrt[e]{e}\left[(n-1)!\right]^p,$$

so

$$\begin{aligned}
\sqrt[n]{n} &< \frac{1 + \sqrt{2}(2!)^p + \sqrt[3]{3}(3!)^p + \cdots + \sqrt[n]{n}(n!)^p}{(n!)^p} \\
&\leq \frac{(n-1)\sqrt[e]{e}\,[(n-1)!]^p}{(n!)^p} + \sqrt[n]{n} \\
&= \frac{\sqrt[e]{e}\left(1 - \frac{1}{n}\right)}{n^{p-1}} + \sqrt[n]{n} \\
&\leq \frac{\sqrt[e]{e}\left(1 - \frac{1}{n}\right)}{n} + \sqrt[n]{n}.
\end{aligned}$$

Since

$$\lim_{n\to\infty} \frac{\sqrt[e]{e}\left(1 - \frac{1}{n}\right)}{n} = 0$$

and $\lim_{n\to\infty} \sqrt[n]{n} = 1$, it follows that

$$\lim_{n\to\infty} \frac{1 + \sqrt{2}(2!)^p + \sqrt[3]{3}(3!)^p + \cdots + \sqrt[n]{n}(n!)^p}{(n!)^p} = 1.$$

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, *Rouen, France; KEITH EKBLAW, Walla Walla, WA, USA; OVIDIU FURDUI, Campia Turzii, Cluj, Romania; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; CRISTINEL MORTICI, Valahia University of Târgovişte, Romania; MOUBINOOL OMARJEE, Paris, France; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.*

Bataille, Furdui, Janous, and Mortici showed that the result holds for all $p > 0$. Many solvers used the Stolz–Cesàro Theorem in their solutions; in particular, Janous gave the generalization that if $x_n \to \alpha$, $y_n \to \infty$, and the sequence $\{y_n\}$ is eventually monotonic, then $(x_1y_1 + x_2y_2 + \cdots + x_ny_n)/y_n \to \alpha$.

3470. [2009 : 396, 399] *Proposed by Mihaela Blanariu, Columbia College Chicago, Chicago, IL, USA.*

Let $p \geq 2$ be a real number. Find the limit

$$\lim_{n\to\infty} \frac{1 + \left(\sqrt{2!}\right)^p + \left(\sqrt[3]{3!}\right)^p + \cdots + \left(\sqrt[n]{n!}\right)^p}{n^{p+1}}\,.$$

Solution by Arkady Alt, San Jose, CA, USA.

Since

$$\left(\frac{n+1}{e}\right)^n \;<\; n! \;<\; (n+1)\left(\frac{n+1}{e}\right)^n ,$$

then

$$\frac{n+2}{en} \;<\; \frac{\sqrt[n+1]{(n+1)!}}{n} \;<\; \sqrt[n+1]{n+2}\left(\frac{n+2}{en}\right).$$

Therefore, by the Squeeze Theorem for limits, we have that

$$\lim_{n\to\infty} \frac{\sqrt[n+1]{(n+1)!}}{n} \;=\; \frac{1}{e}\,.$$

Let $S_n = 1+(\sqrt{2!})^p+(\sqrt[3]{3!})^p+\cdots+(\sqrt[n]{n!})^p$ and $a_n = n^{p+1}$. According to the Stolz–Cesàro Theorem, if $\{a_n\}$ is an increasing sequence such that $\lim\limits_{n\to\infty} a_n = \infty$ and $\lim\limits_{n\to\infty} \dfrac{S_{n+1}-S_n}{a_{n+1}-a_n}$ exists, then $\lim\limits_{n\to\infty} \dfrac{S_n}{a_n} = \lim\limits_{n\to\infty} \dfrac{S_{n+1}-S_n}{a_{n+1}-a_n}$. We have

$$\begin{aligned}\frac{S_{n+1}-S_n}{a_{n+1}-a_n} &= \frac{\left(\sqrt[n+1]{(n+1)!}\right)^p}{(n+1)^{p+1}-n^{p+1}} \\ &= \frac{\left(\sqrt[n+1]{(n+1)!}\right)^p}{n^p}\cdot\frac{1}{n\left(\left(1+\frac{1}{n}\right)^{p+1}-1\right)}\,.\end{aligned}$$

Since $\lim_{n\to\infty} n\left(\left(1+\frac{1}{n}\right)^{p+1}-1\right) = p+1$ and $\lim_{n\to\infty} \frac{\sqrt[n+1]{(n+1)!}}{n} = \frac{1}{e}$, then

$$\lim_{n\to\infty} \frac{S_n}{a_n} = \frac{1}{(p+1)e^p}$$

is the desired limit.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; OVIDIU FURDUI, Campia Turzii, Cluj, Romania; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; CRISTINEL MORTICI, Valahia University of Târgovişte, Romania; MOUBINOOL OMARJEE, Paris, France; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. One incorrect solution was submitted.

3471. [2009 : 396, 399] *Proposed by Cătălin Barbu, Bacău, Romania.*

Let ABC be an acute triangle and M, N, P be the midpoints of the minor arcs BC, CA, AB; respectively. If $[XYZ]$ denotes the area of triangle XYZ, prove that $[MBC]+[NCA]+[PAB] \geq s(3r-R)$, where s, r, and R are the semiperimeter, the inradius, and the circumradius of triangle ABC, respectively.

Solution by Edmund Swylan, Riga, Latvia, expanded by the editor.

The circumcentre O of ABC lies inside it, since the triangle is acute. Let a, b, c be the lengths of the sides opposite A, B, C, respectively, and let $[XYZW]$ denote the area of quadrilateral $XYZW$. Then

$$\begin{aligned}
&[ABC]+[MBC]+[NCA]+[PAB] \\
&\quad = [MBOC]+[NAOC]+[PAOB] \\
&\quad = \frac{1}{2}aR+\frac{1}{2}bR+\frac{1}{2}cR = sR\,.
\end{aligned}$$

From $[ABC] = rs$ we get $[MBC]+[NCA]+[PAB] = s(R-r)$, so the inequality to be proved is equivalent to $R \geq 2r$, which is just Euler's inequality.

Also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; MARIAN DINCĂ, Bucharest, Romania; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; CRISTINEL MORTICI, Valahia University of Târgovişte, Romania; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer. One incorrect solution was submitted.

3472. [2009 : 397, 399] *Proposed by Xavier Ros, student, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let x, y, and z be positive real numbers with $x \le y \le z$. Prove that

$$\log x \cdot \log\left(\frac{1+y}{1+z}\right) + \log y \cdot \log\left(\frac{1+z}{1+x}\right) + \log z \cdot \log\left(\frac{1+x}{1+y}\right) \ge 0\,.$$

Solution by Cristinel Mortici, Valahia University of Târgoviște, Romania.

A convex function f satisfies the inequality below whenever $x \le y \le z$:

$$x(f(y) - f(z)) + y(f(z) - f(x)) + z(f(x) - f(y)) \ge 0\,, \qquad (1)$$

since

$$\left(\frac{z-y}{z-x}\right) f(x) + \left(\frac{y-x}{z-x}\right) f(z) \ge f\left(\frac{z-y}{z-x} \cdot x + \frac{y-x}{z-x} \cdot z\right) = f(y)\,;$$

and the converse also holds. The function $f(x) = \ln(1 + e^x)$ is convex, since

$$f''(x) = \frac{e^x}{(1+e^x)^2} > 0\,.$$

Applying (1) to the function f, we obtain

$$x \ln\left(\frac{1+e^y}{1+e^z}\right) + y \ln\left(\frac{1+e^z}{1+e^x}\right) + z \ln\left(\frac{1+e^x}{1+e^y}\right) \ge 0\,.$$

The conclusion follows upon replacing x, y, z by $\ln x$, $\ln y$, $\ln z$, respectively.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; and the proposer. Two incomplete solutions were submitted.

3473. [2009 : 397, 399] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria, in memory of Jim Totten.*

Let c be a fixed real number such that $0 < c \le 1$.

(a) For all positive real numbers μ prove that

$$\frac{2}{\sqrt{c\mu+1}} + \frac{\mu}{\sqrt{c+\mu^2}} \le \frac{3}{\sqrt{c+1}}\,.$$

(b)★. Determine all positive real numbers λ such that

$$\frac{\lambda}{\sqrt{c\mu+1}} + \frac{\mu}{\sqrt{c+\mu^2}} \le \frac{\lambda+1}{\sqrt{c+1}}$$

holds for all positive real numbers μ.

Similar solutions by Michel Bataille, Rouen, France and Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

(a) Let

$$f(x) \;=\; 2(cx+1)^{-\frac{1}{2}} + x(x^2+c)^{-\frac{1}{2}}\,.$$

Then f is differentiable on $(0,\infty)$, with derivative

$$\begin{aligned} f'(x) &= -c(cx+1)^{-\frac{3}{2}} + (x^2+c)^{-\frac{1}{2}} - x^2(x^2+c)^{-\frac{3}{2}} \\ &= -c(cx+1)^{-\frac{3}{2}} + c(x^2+c)^{-\frac{3}{2}}\,. \end{aligned}$$

Furthermore,

$$\begin{aligned} f'(x) \geq 0 \iff c(x^2+c)^{-\frac{3}{2}} \geq c(cx+1)^{-\frac{3}{2}} \iff \\ x^2+c \leq cx+1 \iff (x-1)(x+1-c) \leq 0 \iff x \leq 1\,. \end{aligned}$$

Thus, f is increasing on $(0,1]$ and decreasing on $[1,\infty)$. Hence, $f(x)$ has an absolute maximum at $x=1$, and thus $f(\mu) \leq f(1)$ for all positive real numbers μ.

(b) We prove that this inequality holds for all positive real numbers μ if and only if $\lambda = 2$. We proved in (a) the "if" part.

Now suppose that λ is such that

$$\frac{\lambda}{\sqrt{c\mu+1}} + \frac{\mu}{\sqrt{c+\mu^2}} \;\leq\; \frac{\lambda+1}{\sqrt{c+1}}$$

holds for all positive real numbers μ.

Then the function

$$f_\lambda(x) \;=\; \lambda(cx+1)^{-\frac{1}{2}} + x(x^2+c)^{-\frac{1}{2}}$$

has an absolute maximum at $x=1$, and thus $f'_\lambda(1) = 0$.

Since

$$f'_\lambda(x) \;=\; -\frac{c\lambda}{2}(cx+1)^{-\frac{3}{2}} + c(x^2+c)^{-\frac{3}{2}}\,,$$

we obtain

$$0 \;=\; -\frac{c\lambda}{2}(c+1)^{-\frac{3}{2}} + c(c+1)^{-\frac{3}{2}} \;=\; c(c+1)^{-\frac{3}{2}}\left(1-\frac{\lambda}{2}\right)\,.$$

Thus, $\lambda = 2$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; KEE-WAI LAU, Hong Kong, China; and PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.

Part (a) was also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; JOE HOWARD, Portales, NM, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer. One incomplete solution to part (b) was submitted.

3474★. [2009 : 397, 399] *Proposed by Silouanos Brazitikos and Christos Patilas, Trikala, Greece.*

Let x, y, and z be positive real numbers with $xyz = 1$. Prove or disprove that $2\sqrt{3\left(x^y + y^z + z^x\right)} + x + y + z \geq 9$.

[*Ed.: The proposers indicate that the inequality is supported by computer computations.*]

Editor: No solution to this problem has been received. Therefore, it remains open.