SKOLIAD No. 124

Lily Yen and Mogens Hansen

Please send your solutions to problems in this Skoliad by **1 July, 2010**. A copy of ***CRUX with Mayhem*** will be sent to one pre-university reader who sends in solutions before the deadline. The decision of the editors is final.

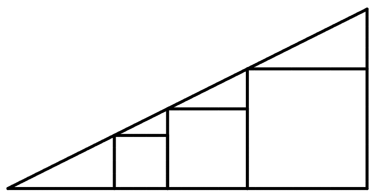
Our contest this month is the selected problems from the 10th Annual Christopher Newport University Regional Mathematics Contest, 2009. Our thanks go to Ron Persky, Christopher Newport University, Newport News, USA, for providing us with this contest and for permission to publish it.

10th Annual Christopher Newport University Regional Mathematics Contest 2009
Selected problems

1. Elves and ogres live in the land of Pixie. The average height of the elves is 80 cm, the average height of the ogres is 200 cm, and the average height of the elves and the ogres together is 140 cm. If 36 elves live in Pixie, how many ogres live there?

2. You are given a two-digit positive integer. If you reverse the digits of your number, the result is a number which is 20% larger than your original number. What is your original number?

3. Three squares are placed side-by-side inside a right-angled triangle as shown in the diagram. The side length of the smallest of the three squares is 16. The side length of the largest of the three squares is 36. What is the side length of the middle square?



4. Friends Maya and Naya ordered finger food in a restaurant, Maya ordering chicken wings and Naya ordering bite-size ribs. Each wing cost the same amount, and each rib cost the same amount, but one wing was more expensive than one rib. Maya received 20% more pieces than Naya did, and Maya paid 50% more in total than Naya did. The price of one wing was what percentage higher than the price of one rib?

5. A 9×12 rectangular piece of paper is folded once so that a pair of diagonally opposite corners coincide. What is the length of the crease?

6. In calm weather, an aircraft can fly from one city to another 200 kilometres north of the first and back in exactly two hours. In a steady north wind, the round trip takes five minutes longer. Find the speed (in kilometres per hour) of the wind.

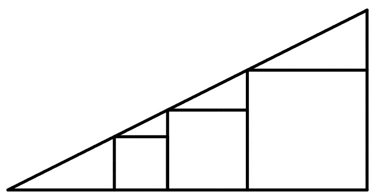
7. A rectangular floor, 24 feet $\times$ 40 feet, is covered by squares of sides 1 foot. A chalk line is drawn from one corner to the diagonally opposite corner. How many tiles have a chalk line segment on them?

10[ième] Concours mathématique annuel de l'Université Christopher Newport 2009
problèmes choisis

1. Un certain nombre de fées et d'ogres demeurent au Pays imaginaire. Les fées sont de taille moyenne 80 cm, comparativement à 200 cm pour les ogres. Enfin la taille moyenne de la totalité des fées et des ogres est de 140 cm. Si 36 fées demeurent au Pays imaginaire, déterminer le nombre d'ogres qui y demeurent.

2. On vous donne un entier à deux chiffres. Si on inverse l'ordre des deux chiffres, l'entier qui en résulte est 20% plus élevé que l'entier original. Déterminer l'entier original.

3. Trois carrés sont placés côte à côte à l'intérieur d'un triangle rectangle, tel qu'illustré. Le plus petit des carrés a un côté de longueur 16, tandis que le plus grand a un côté de longueur 36. Déterminer la longueur du côté du triangle au milieu.



4. Madeleine et Nadine ont décidé de manger sur le pouce, Madeleine ayant commandé des ailes de poulet contrairement à Nadine qui a choisi des côtelettes. Toute aile de poulet a le même prix et toute côtelette a son propre prix, une aile de poulet étant plus coûteuse qu'une côtelette. Madeleine a obtenu un nombre d'ailes 20% plus élevé que le nombre de côtelettes reçues par Nadine ; aussi, Madeleine a déboursé 50% de plus au total que Nadine. Le prix d'une aile de poulet est plus élevé que le prix d'une côtelette par quel pourcentage ?

5. Un morceau de papier rectangulaire de taille 9×12 est plié une seule fois, de façon à ce qu'une paire de coins diagonalement opposés se rejoignent. Déterminer la longueur de la plissure.

6. Dans l'absence de vent, un avion effectue un trajet aller-retour d'une ville à une autre, située à 200 kilomètres au nord, dans exactement deux heures. Avec un vent constant du nord, ce trajet prend cinq minutes de plus. Déterminer la vitesse du vent, en kilomètres à l'heure.

7. Un plancher rectangulaire de taille 24 pieds $\times$ 40 pieds, est pavé de tuiles carrées de côtés 1 pied. Une ligne droite est tracée d'un coin à son coin diagonalement opposé. Déterminer le nombre de tuiles qui incluent un segment de cette ligne.

Next follow solutions to the Cariboo College High School Mathematics Contest, 1990, Junior Final, Part B, which appeared in Skoliad 118 in the Totten Commemorative issue [2009 : 263–265].

1. A boy on a bicycle coasts down from the top of a hill. He covers 4 metres in the first second and in each succeeding second covers 5 metres more than in the previous second. He reaches the bottom of the hill in 11 seconds.

(a) How long is the hill?

(b) What is the boy's average speed in metres per second?

(c) What distance did he cover in the last second?

Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

The distance travelled in each second is easily calculated:

second:	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th	11th
metres:	4	9	14	19	24	29	34	39	44	49	54

Adding the distances yields that the hill is 319 m long. The boy's average speed is therefore $\frac{319}{11}$ m/s $= 29$ m/s. That he covered 54 metres in the last second is in the table above.

Also solved by LENA CHOI, student, École Banting Middle School, Coquitlam, BC; ROWENA HO, student, École Banting Middle School, Coquitlam, BC; MONICA HSIEH, student, Burnaby North Secondary School, Burnaby, BC; JULIA PENG, student, Campbell Collegiate, Regina, SK; SZERA PINTER, student, Moscrop Secondary School, Burnaby, BC; ALISON TAM, student, Burnaby South Secondary School, Burnaby, BC; and KENRICK TSE, student, Quilchena Elementary School, Vancouver, BC.

2. Two golfers, on their way to the course, reached a railway crossing just as a 2.5 km train arrived. Rather than waiting, they decided to go on to the next crossing 1 km along in the direction the train was going. They travelled at 50 km/h while the train travelled at 70 km/h.

(a) How long did they have to wait for the train to clear the crossing?

(b) Rather than travelling at 50 km/h, how fast would they have had to travel to reach the crossing just as the train was clearing the crossing?

Solution by Alison Tam, student, Burnaby South Secondary School, Burnaby, BC.

The train needs to travel 3.5 km to completely pass the second crossing. That takes $\frac{3.5\text{ km}}{70\text{ km/h}} = 0.05$ h, or 3 minutes. The golfers then just travel 1 km, which takes them $\frac{1\,\text{km}}{50\,\text{km/h}} = 0.02\,\text{h}$ or 1.2 minutes. The golfers therefore must wait for $3 - 1.2 = 1.8$ minutes, or 1 minute and 48 seconds.

If the golfers do not want to wait at the second crossing, then they must spend 3 minutes (or $0.05\,\text{h}$) getting there, since that is how long the train takes to pass. Thus, their speed must be $\frac{1\,\text{km}}{0.05\,\text{h}} = 20\,\text{km/h}$.

Also solved by LENA CHOI, student, École Banting Middle School, Coquitlam, BC; JULIA PENG, student, Campbell Collegiate, Regina, SK; SZERA PINTER, student, Moscrop Secondary School, Burnaby, BC; and KENRICK TSE, student, Quilchena Elementary School, Vancouver, BC.

3. A student asks you to choose a number from 1 to 9 and multiply it by 109, then asks you to find the sum of the digits in the product. Knowing the sum of the digits, the student is able to tell you the number with which you began. Explain how this can be done.

Solution by Szera Pinter, student, Moscrop Secondary School, Burnaby, BC.

The table shows the calculation of the desired digit sum in each case. Since all the digit sums are different, the original number can easily be recovered. Indeed, simply subtract 9 from the digit sum.

$1 \times 109 = 109$	$\rightarrow 10$
$2 \times 109 = 218$	$\rightarrow 11$
$3 \times 109 = 327$	$\rightarrow 12$
$4 \times 109 = 436$	$\rightarrow 13$
$5 \times 109 = 545$	$\rightarrow 14$
$6 \times 109 = 654$	$\rightarrow 15$
$7 \times 109 = 763$	$\rightarrow 16$
$8 \times 109 = 872$	$\rightarrow 17$
$9 \times 109 = 981$	$\rightarrow 18$

You can also predict the digit sums without actually calculating them. If the original single-digit number is x, then $9x$ is obviously divisible by 9, so its digit sum must be divisible by 9. Since $9x \leq 81$, the digit sum of $9x$ must therefore equal 9. The digit sum of $100x$ is clearly x, so the digit sum of $109x$ is $x + 9$. Hence, x can be recovered from the digit sum by subtracting 9.

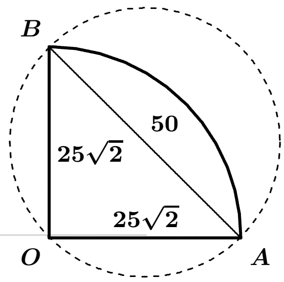
Also solved by CINDY CHEN, student, Burnaby North Secondary School, Burnaby, BC; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; ROWENA HO, student, École Banting Middle School, Coquitlam, BC; JULIA PENG, student, Campbell Collegiate, Regina, SK; and KENRICK TSE, student, Quilchena Elementary School, Vancouver, BC.

4. Suppose you throw 5 darts at a round board with a radius of $25\sqrt{2}\,\text{cm}$. If all 5 darts stick in the board, show that at least two of them must be within $50\,\text{cm}$ of each other.

Solution by Julia Peng, student, Campbell Collegiate, Regina, SK.

Divide the dart board into four parts using perpendicular diameters, like so: $\oplus$. (Let each quarter circle include its boundary.) With five darts, clearly at least one of the four quarter circles must contain at least two darts.

Consider such a quarter circle (see the thick lined part of the figure). By the Pythagorean Theorem,



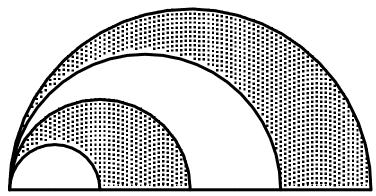
$$\begin{aligned} |AB|^2 &= |OA|^2 + |OB|^2 \\ &= 25^2 \cdot 2 + 25^2 \cdot 2 = 25^2 \cdot 2^2 , \end{aligned}$$

so $|AB| = 25 \cdot 2 = 50$. Then draw the circle through points O, A, and B. Since $\angle AOB = 90$, the centre of that circle is the midpoint of AB. Thus AB is a diameter for the dotted circle in the figure, and the distance between any two darts within the thick quarter circle is at most $|AB|$, as required.

Also solved by KENRICK TSE, student, Quilchena Elementary School, Vancouver, BC.

The idea that if four parts contain five darts then at least one part must contain at least two darts is known as the Pigeonhole Principle, *which is useful in many contest problems.*

5. The diameter, AB, of a circle is divided into 4 equal parts by the points C, D, and E. Semicircles are drawn on AC, AD, AE, and AB as shown. Find the ratio of the area of the shaded parts to the area of the unshaded parts.



Solution by Kenrick Tse, student, Quilchena Elementary School, Vancouver, BC, modified by the editor.

Let r be the radius of the smallest semicircle. Then the other semicircles have radii $2r$, $3r$, and $4r$, respectively. Thus their areas are, respectively, $\frac{1}{2}\pi r^2$, $\frac{1}{2}\pi(2r)^2 = 2\pi r^2$, $\frac{1}{2}\pi(3r)^2 = \frac{9}{2}\pi r^2$, and $\frac{1}{2}\pi(4r)^2 = 8\pi r^2$.

The area of the innermost shaded band is now the difference in area between the two smallest semicircles, thus $2\pi r^2 - \frac{1}{2}\pi r^2 = \frac{3}{2}\pi r^2$. Similarly, the outer shaded band has area $8\pi r^2 - \frac{9}{2}\pi r^2 = \frac{7}{2}\pi r^2$. Thus the shaded region has area $\frac{3}{2}\pi r^2 + \frac{7}{2}\pi r^2 = 5\pi r^2$.

Now we see that the unshaded region has area $8\pi r^2 - 5\pi r^2 = 3\pi r^2$. It follows that the desired ratio is $\frac{5\pi r^2}{3\pi r^2} = \frac{5}{3}$, or $5 : 3$.

Also solved by MONICA HSIEH, student, Burnaby North Secondary School, Burnaby, BC; JULIA PENG, student, Campbell Collegiate, Regina, SK; and SZERA PINTER, student, Moscrop Secondary School, Burnaby, BC.

This issue's prize of one copy of ***CRUX with MAYHEM*** for the best solutions goes to Julia Peng, student, Campbell Collegiate, Regina, SK.

We hope that our readers will continue to share their joy of mathematics by submitting solutions to our problems.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff members are Monika Khbeis (Our Lady of Mt. Carmel Secondary School, Mississauga, ON) and Eric Robert (Leo Hayes High School, Fredericton, NB).

Mayhem Problems

*Please send your solutions to the problems in this edition by **15 August 2010**. Solutions received after this date will only be considered if there is time before publication of the solutions. The Mayhem Staff ask that each solution be submitted on a separate page and that the solver's name and contact information be included with each solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

M432. *Proposed by the Mayhem Staff.*

Determine the value of d with $d > 0$ so that the area of the quadrilateral with vertices $A(0, 2)$, $B(4, 6)$, $C(7, 5)$, and $D(d, 0)$ is 24.

M433. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

In triangle ABC, $AB < BC$, L is the midpoint of AC, and M is the midpoint of AB. Also, P is the point on LM such that $MP = MA$. Prove that $\angle PBA = \angle PBC$.

M434. *Proposed by Heisu Nicolae, Pîrjol Secondary School, Bacău, Romania.*

Determine all eight-digit positive integers $abcdefgh$ which satisfy the relations $a^3 - b^2 = 2$, $c^3 - d^2 = 4$, $2^e - f^2 = 7$, and $g^3 - h^2 = -1$.

M435. *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that

$$\sum_{k=1}^{n} \sqrt{1 + \frac{1}{k^2} + \frac{1}{(k+1)^2}} = \frac{n(n+2)}{n+1}.$$

M436. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Determine the smallest possible value of $x + y$, if x and y are positive integers with $\frac{2008}{2009} < \frac{x}{y} < \frac{2009}{2010}$.

M437. *Proposed by Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain.*

Let $\lfloor x \rfloor$ denote the greatest integer not exceeding x. For example, $\lfloor 3.1 \rfloor = 3$ and $\lfloor -1.4 \rfloor = -2$. Let $\{x\}$ denote the fractional part of the real number x, that is, $\{x\} = x - \lfloor x \rfloor$. For example, $\{3.1\} = 0.1$ and $\{-1.4\} = 0.6$. Determine all rational numbers x such that $x \cdot \{x\} = \lfloor x \rfloor$.

....

M432. *Proposé par l'Équipe de Mayhem.*

Déterminer la valeur de d, $d > 0$, telle que l'aire du quadrilatère de sommets $A(0, 2)$, $B(4, 6)$, $C(7, 5)$ et $D(d, 0)$ soit de 24.

M433. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Dans un triangle ABC avec $AB < BC$, soit L et M les points milieux respectifs de AC et AB. De plus, soit P le point sur LM tel que l'on ait $MP = MA$. Montrer que $\angle PBA = \angle PBC$.

M434. *Proposé par Heisu Nicolae, École secondaire Pîrjol, Bacău, Roumanie.*

Trouver tous les entiers positifs à huit chiffres $abcdefgh$ satisfaisant les relations $a^3 - b^2 = 2$, $c^3 - d^2 = 4$, $2^e - f^2 = 7$ et $g^3 - h^2 = -1$.

M435. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Montrer que

$$\sum_{k=1}^{n} \sqrt{1 + \frac{1}{k^2} + \frac{1}{(k+1)^2}} \;=\; \frac{n(n+2)}{n+1}\,.$$

M436. *Proposé par Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie.*

Trouver la plus petite valeur possible de $x+y$, si x et y sont des entiers positifs tels que $\frac{2008}{2009} < \frac{x}{y} < \frac{2009}{2010}$.

M437. *Proposé par Samuel Gómez Moreno, Université de Jaén, Jaén, Espagne.*

On note $\lfloor x \rfloor$ le plus grand entier n'excédant pas x. Par exemple, on a $\lfloor 3.1 \rfloor = 3$ et $\lfloor -1.4 \rfloor = -2$. On note $\{x\}$ la partie fractionnaire du nombre réel x, c'est-à-dire $\{x\} = x - \lfloor x \rfloor$. Par exemple, on a $\{3.1\} = 0.1$ et $\{-1.4\} = 0.6$. Trouver tous les nombres rationnels x tels que $x \cdot \{x\} = \lfloor x \rfloor$.

Mayhem Solutions

Totten–M1. *Proposed by Shawn Godin, Cairine Wilson Secondary School, Orleans, ON.*

Ancient Egyptians wrote all fractions in terms of distinct unit fractions (that is, in terms of distinct fractions with numerators of 1). For example, instead of writing $\frac{11}{12}$, they would write $\frac{1}{2} + \frac{1}{3} + \frac{1}{12}$. The unit fraction $\frac{1}{2}$ can be written in terms of other unit fractions as $\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$. Find an infinite family of unit fractions each of which can be written as the sum of two unit fractions.

Solution by Katherine Janell Eyre, student, Angelo State University, San Angelo, TX, USA.

We show that any unit fraction $\frac{1}{n}$ with n a positive integer and $n \geq 2$ can be written as the sum of two distinct unit fractions. To do this, we need to find positive integers x and y such that $\frac{1}{n} = \frac{1}{x} + \frac{1}{y}$. By multiplying through by xyn and manipulating, we obtain the equivalent equations

$$\begin{aligned} xy &= yn + xn\,; \\ xy - yn - xn + n^2 &= n^2\,; \\ (x-n)(y-n) &= n^2\,. \end{aligned}$$

Since a possible factorization of n^2 is $1 \cdot n^2$, then we can let $x - n = 1$ (and so $x = n + 1$) and $y - n = n^2$ (and so $y = n + n^2 = n(n+1)$).

Thus,

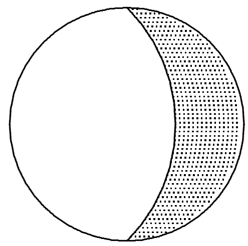
$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}$$

for $n \geq 2$. This represents an infinite family of unit fractions each of which can be written as the sum of two distinct unit fractions.

Also solved by SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; and D.J. SMEENK, Zaltbommel, the Netherlands. There was one incorrect solution submitted.

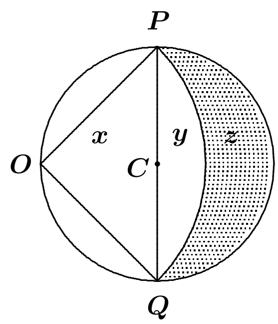
Totten–M2. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

The boundary of the shadow on the moon is always a circular arc. On a certain day, the moon is seen with the shadow passing through diametrically opposite points. If the centre of the circular arc forming the shadow is on the circumference of the moon, determine the exact proportion of the moon that is not in shadow.



Solution by Geoffrey A. Kandall, Hamden, CT, USA.

Let r denote the radius of the moon and C its centre, and R denote the radius of the circular arc forming the shadow and O the centre of this arc. Let P and Q be the points where the shadow intersects the circumference of the moon. Let x denote the area of $\triangle POQ$, y the area of the region between PQ and the arc through P and Q centred at O, and z the area of the region between the two arcs through P and Q.



Since $PC = CQ = r$, $OP = OQ = R$, and $\angle POQ = 90^\circ$ (because PQ is a diameter), then $\sqrt{2}R = 2r$, and so $R = \sqrt{2}r$.

Since $\angle POQ = 90^\circ$, then the sum of areas x and y is one-quarter of the area of the circle centred at O, or $x + y = \frac{1}{4}\pi R^2 = \frac{1}{4}\pi(\sqrt{2}r)^2 = \frac{1}{2}\pi r^2$.

Also, the sum of the areas y and z is one-half of the area of the circle centred at C, or $y + z = \frac{1}{2}\pi r^2$.

Thus, $x + y = y + z$ and so $x = z$. Since x equals the area of $\triangle POQ$ which is right-angled at O, then $z = x = \frac{1}{2}(\sqrt{2}r)^2 = r^2$.

Thus, the area of the region that is not in the shadow equals the area of the entire circle centred at C minus z, or $\pi r^2 - r^2 = r^2(\pi - 1)$ and the exact proportion of the moon that is not in shadow is $\dfrac{r^2(\pi - 1)}{\pi r^2} = \dfrac{\pi - 1}{\pi}$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and RICARD PEIRÓ, IES "Abastos", Valencia, Spain.

Totten–M3. *Proposed by John Ciriani, Kamloops, BC.*

Prove that the quadratic equation $ax^2 + bx + c = 0$ does not have a rational root if a, b, and c are odd integers.

Solution by Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain.

Suppose that the rational number $\frac{p}{q}$, where p and q are relatively prime integers, is a root of the quadratic equation $ax^2 + bx + c = 0$. Then we have that $a\left(\frac{p}{q}\right)^2 + b\left(\frac{p}{q}\right) + c = 0$, which yields $ap^2 + bpq + cq^2 = 0$.

We are given that a, b, and c are odd integers. Since p and q are relatively prime, they cannot both be even. There are three cases to consider:

- If p is even and q is odd, then ap^2 and bpq are even and cq^2 is odd, so $ap^2 + bpq + cq^2$ is odd, and thus cannot equal 0.
- Similarly, if p is odd and q is even, then $ap^2 + bpq + cq^2$ is odd, and thus cannot equal 0.
- If p is odd and q is odd, then ap^2, bpq, and cq^2 are all odd, so again we see that $ap^2 + bpq + cq^2$ is odd, and thus cannot equal 0.

Therefore, the quadratic equation $ax^2 + bx + c = 0$ with a, b, and c all odd integers cannot have a rational root.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; KATHERINE JANELL EYRE, student, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MATT HUBBS, student, Missouri State University, Springfield, MO, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; KONSTANTINOS AL. NAKOS, Agrinio, Greece; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA.

Miguel Amengual Covas pointed out that this problem appeared as problem no. 178 in ***Eureka****, Vol. 2, No. 8 (October), 1976, p. 171.* [Ed.: **Eureka** became **Crux Mathematicorum**, which eventually became **Crux Mathematicorum with Mathematical Mayhem**.]

A similar argument to the featured one shows that an even degree polynomial with odd integer coefficients cannot have a rational root.

Totten–M4. *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

In a survey, some students were asked whether they liked the colour orange. Exactly 2% of the boys in the survey liked orange, while exactly 59% of the girls in the survey liked orange. Altogether, exactly 17% of the students in the survey liked orange. Find the smallest possible number of students in the survey.

Solution by Geoffrey A. Kandall, Hamden, CT, USA.

Let B be the number of boys in the survey and G the number of girls. From the given information, $0.02B$ is the number of boys that liked orange, $0.59G$ is the number of girls that liked orange, and $0.17(B+G)$ is the total number of students that liked orange. Each of these three quantities must be an integer.

Note that $0.02B + 0.59G = 0.17(B+G)$ which gives $2B + 59G = 17B + 17G$, or $42G = 15B$, or $5B = 14G$, or $G : B = 5 : 14$. Thus, there is an integer t such that $G = 5t$ and $B = 14t$. Since we seek the minimum value of $B + G = 19t$, we want to find the minimum value of t.

But $0.17(B+G) = \frac{17}{100}(5t + 14t) = \frac{17 \cdot 19}{100}t$ is also an integer; that is, $100 \mid 17 \cdot 19 \cdot t$. Since $\gcd(17 \cdot 19, 100) = 1$, then $100 \mid t$. The smallest such positive value of t is therefore $t = 100$.

This value of t gives $G = 5t = 500$ and $B = 14t = 1400$, which are admissible values of G and B since 2% of B is 10 and 59% of G is 295, and these are integers. Above we saw that 17% of $B + G$ is an integer since $100 \mid t$.

Therefore, the smallest possible number of students surveyed is 1900.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; and RICARD PEIRÓ, IES "Abastos", Valencia, Spain.

Totten–M5. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $a \neq 1$ be a positive real number. Determine all pairs of positive integers (x, y) such that $\log_a x - \log_a y = \log_a(x - y)$.

Solution by Geoffrey A. Kandall, Hamden, CT, USA.

Suppose that x and y satisfy $\log_a x - \log_a y = \log_a(x - y)$. Since $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$, then $\log_a(x - y) = \log_a \left(\frac{x}{y}\right)$ and so by the one-to-one property of $f(u) = \log_a u$, it follows that $\frac{x}{y} = x - y$.

From this we obtain $x = xy - y^2$ or $y^2 = x(y - 1)$ which gives

$$\begin{aligned} x &= \frac{y^2}{y-1} = \frac{y^2 - 1 + 1}{y-1} \\ &= \frac{(y-1)(y+1)+1}{y-1} = y + 1 + \frac{1}{y-1}\,. \end{aligned}$$

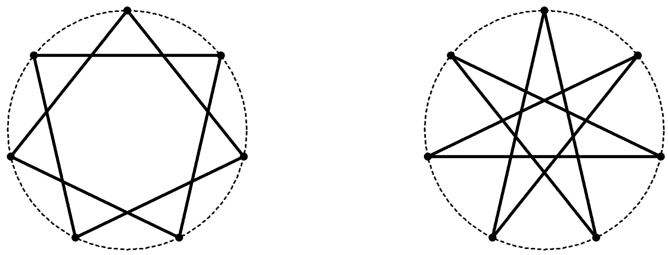
Since we are seeking pairs of positive integers (x, y), then $\frac{1}{y-1}$ is an integer. Thus, $y - 1 = 1$, or $y = 2$, and so $x = \frac{y^2}{y-1} = 4$.

Therefore, the only pair (x, y) satisfying the relation is $(4, 2)$.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; LUIS J. BLANCO (student) and ANGEL PLAZA, University of Las Palmas de Gran Canaria, Spain; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; KONSTANTINOS AL. NAKOS, Agrinio, Greece; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA.

Totten–M6. *Proposed by Suzanne Feldberg, Thompson Rivers University, Kamloops, BC.*

It is widely known how to draw a 5-pointed star quickly. To make it symmetric, one places 5 vertices at $72°$ intervals about a circle and connects the vertices with line segments of equal length without lifting one's pen. By starting from a fixed point and using the same method, one can draw two different (and symmetric) 7-pointed stars without lifting one's pen.



How many different 6-pointed, 8-pointed, or 9-pointed stars can one draw this way? How many different n-pointed stars can one draw this way?

Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA, modified by the editor.

There are no such 6-pointed stars, one such 8-pointed star, and two such 9-pointed stars.

Suppose that $n \geq 5$. We can define such an n-pointed star more precisely as joining n regularly spaced points on a circle by starting at a fixed point and drawing n congruent lines connecting points which are successively k spaces apart, with $1 < k < \frac{1}{2}n$. We exclude $k = 1$ since this would create an n-gon. We exclude $k = \frac{1}{2}n$ (in the case where n is even), because only two lines would be drawn. We exclude $\frac{1}{2}n < k < n$ because these would create congruent stars to the stars counted above by drawing them in the opposite direction.

We note also that k cannot be a divisor of n, since after $\frac{n}{k}$ lines we would be back to the original starting point, and hence we would not have drawn n lines.

Similarly, if k and n are not coprime, they share a common divisor and the only points which would be connected when drawing the star connecting points which are k spaces apart would be the same points as for the star connecting points which are $\gcd(k, n)$ spaces apart. Since $\gcd(k, n)$ is a divisor of n, not all of the points would be used.

Thus, in order to complete an n-pointed star, n and k must be coprime and $1 < k < \frac{1}{2}n$. The number of ways in which an n-pointed star can be drawn equals the number of such k. There are $\phi(n)$ integers k coprime to n satisfying $1 \leq k \leq n$, and $\frac{1}{2}\phi(n)$ integers k coprime to n satisfying $1 \leq k \leq \frac{1}{2}n$ (noting that $\frac{1}{2}n$ is either not an integer or is not coprime to n). Since we omit $k = 1$ which is coprime to n, then there are

$$\frac{1}{2}\phi(n) - 1$$

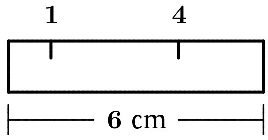
stars that can be drawn. This is well-defined, since $\phi(n)$ is even for $n > 4$.

This agrees with the stated results for $n = 6$, $n = 8$, and $n = 9$.

A partially complete solution was also submitted.

Totten–M7. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

An unmarked ruler is known to be exactly 6 cm in length. It is possible to exactly measure all integer lengths from 1 cm to 6 cm using only 2 marks, as shown in the diagram, at 1 cm and 4 cm, since $2 = 6 - 4$, $3 = 4 - 1$, and $5 = 6 - 1$.



Suppose that an unmarked ruler is known to be exactly 30 cm in length.

(a) Find a way of placing 9 or fewer marks on the ruler to be able to exactly measure all integer lengths from 1 cm to 30 cm.

(b) Prove that at least 7 marks are needed to be able to exactly measure all integer lengths from 1 cm to 30 cm.

(c)★ Determine the smallest number of marks required on the ruler to be able to exactly measure all integer lengths from 1 cm to 30 cm.

Solution to (a) *by the Mayhem Staff; solution to* (b) *by Richard I. Hess, Rancho Palos Verdes, CA, USA, modified by the editor.*

(a) If marks are placed 1, 2, 3, 4, 5, 10, 15, 20, and 25 cm from the left end, then these distances can be measured from that end of the ruler. Distances from 6 cm to 9 cm inclusive can be measured from the 10 cm mark to the appropriate mark at 1, 2, 3, or 4 cm. Distances from 11 cm to 14 cm inclusive can be measured in a similar way using the 15 cm mark. Similarly, we can measure distances from 16 cm to 19 cm, 21 cm to 24 cm, and 25 cm to 29 cm. Therefore, all distances from 1 cm to 30 cm can be measured.

(b) Suppose there are k marks. Each mark can potentially measure a different distance to each end of the ruler. Thus, the marks can measure up to $2k$ distinct distances using the ends of the ruler. Also, each pair of marks can potentially measure a different distance. There are $\binom{k}{2}$ such pairs. Therefore, at most $2k + \binom{k}{2} = \frac{1}{2}k(k+3)$ distances can be measured using k marks. If $k = 6$, this total is 27. Therefore, at least 7 marks are needed to measure the 30 distances.

No complete solution to part (c) *was received, so this part remains open.*

Using a computer, both Hess, and Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain verified that at least 8 *marks are needed in part* (c). *In addition, Gómez Moreno provided some web links to information about "perfect rulers" and "optimal rulers".*

Totten–M8. *Proposed by Edward J. Barbeau, University of Toronto, Toronto, ON.*

Let T be the set of all ordered triples (a, b, c) of positive integers such that $a < b < c$. We say that two triples (a, b, c) and (u, v, w) are equivalent if $a : b : c = u : v : w$. We use this relation to partition T into equivalence classes. The triple (a, b, c) is *geometric* if $ac = b^2$ (that is, its terms form a geometric sequence) and *harmonic* if $\frac{1}{a} + \frac{1}{c} = \frac{2}{b}$ (that is, the reciprocals of its terms form an arithmetic sequence).

(a) Verify that if (a,b,c) is geometric, then all triples equivalent to it are also geometric.

(b) Verify that if (a,b,c) is harmonic, then all triples equivalent to it are also harmonic.

(c) Let G be the set of equivalence classes of geometric triples and H be the set of equivalence classes of harmonic triples. Determine a one-to-one correspondence between G and H.

Solution to (a) *and* (b) *by Jaclyn Chang, student, Western Canada High School, Calgary, AB; solution to* (c) *by the proposer, each modified by the editor.*

Throughout the solution, we use the notation $(a,b,c) \equiv (u,v,w)$ to mean that the triples (a,b,c) and (u,v,w) are equivalent members of T. If $(a,b,c) \equiv (u,v,w)$, then $a : b : c = u : v : w$ by definition, so for some positive real number k we have $(u,v,w) = (ka,kb,kc)$.

(a) Suppose that (a,b,c) is geometric. Then by definition, $ac = b^2$. If $(u,v,w) \equiv (a,b,c)$, then $(u,v,w) = (ka,kb,kc)$ for some positive real number k. Since $ac = b^2$, then

$$uw \;=\; (ka)(kc) \;=\; k^2ac \;=\; k^2b^2 \;=\; (kb)^2 \;=\; v^2\,,$$

and so (u,v,w) is geometric.

(b) Suppose that (a,b,c) is harmonic. By definition, $\frac{1}{a} + \frac{1}{c} = \frac{2}{b}$. If $(u,v,w) \equiv (a,b,c)$, then $(u,v,w) = (ka,kb,kc)$ for some positive real number k. Since $\frac{1}{a} + \frac{1}{c} = \frac{2}{b}$, then

$$\frac{1}{u} + \frac{1}{w} \;=\; \frac{1}{ka} + \frac{1}{kc} \;=\; \frac{1}{k}\left(\frac{1}{a} + \frac{1}{c}\right) \;=\; \frac{1}{k}\cdot\frac{2}{b} \;=\; \frac{2}{kb} \;=\; \frac{2}{v}\,,$$

so (u,v,w) is harmonic.

(c) Suppose that (a,b,c) is a geometric triple of positive integers with $a < b < c$. Then we can write $(a,b,c) = (a,ar,ar^2)$ for some positive real number r. Define a function f by

$$f(a,b,c) \;=\; f(a,ar,ar^2) \;=\; (a(r{+}1), 2ar, ar(r{+}1)) \;=\; (a{+}b, 2b, b{+}c)\,.$$

Note that $a+b < b+b < b+c$, since $a < b < c$, so $(a+b, 2b, b+c)$ is a triple in T. Furthermore,

$$\frac{1}{a(r+1)} + \frac{1}{ar(r+1)} \;=\; \frac{r+1}{ar(r+1)} \;=\; \frac{1}{ar} \;=\; \frac{2}{2ar}\,,$$

so $f(a,b,c)$ is a harmonic triple. Note also that if $(u,v,w) \equiv (a,b,c)$ with $(u,v,w) = (ka,kb,kc)$, then

$$\begin{aligned} f(u,v,w) \;&=\; f(ka,kb,kc) \;=\; (ka+kb, 2kb, kb+kc) \\ &=\; k(a+b, 2b, b+c) \;=\; k\cdot f(a,b,c)\,. \end{aligned}$$

This tells us that f maps equivalent triples to equivalent triples, which means that f is a well-defined map from G to H.

Suppose now that (A, B, C) is a harmonic triple of positive integers with $A < B < C$. Define $g(A, B, C) = (A^2, AC, C^2)$. Then the components of $g(A, B, C)$ are integers with $A^2 < AC < C^2$ so $g(A, B, C)$ is in T. Note that $A^2C^2 = (AC)^2$, so $g(A, B, C)$ is geometric. By directly calculating as we did above, we can show that if $(U, V, W) \equiv (A, B, C)$, then $g(U, V, W) \equiv g(A, B, C)$. This tells us that g maps equivalent triples to equivalent triples, so g is a well-defined map from H to G.

Finally, we need to show that $g(f(a, b, c)) \equiv (a, b, c)$ and also that $f(g(A, B, C)) \equiv (A, B, C)$. This will show that f and g are inverses of each other.

If (a, b, c) is geometric, then $(a, b, c) = (a, ar, ar^2)$ for some real number r, so

$$\begin{aligned} g(f(a,b,c)) &= g(a(r+1), 2ar, ar(r+1)) \\ &= (a^2(r+1)^2, a^2r(r+1)^2, a^2r^2(r+1)^2) \\ &= a(r+1)^2 \cdot (a, ar, ar^2) \\ &\equiv (a, b, c)\,. \end{aligned}$$

If (A, B, C) is harmonic, then $\frac{1}{A} + \frac{1}{C} = \frac{2}{B}$ and so $\frac{A+C}{AC} = \frac{2}{B}$, or $B = \frac{2AC}{A+C}$. Thus,

$$\begin{aligned} f(g(A,B,C)) &= f(A^2, AC, C^2) \\ &= (A^2 + AC, 2AC, AC + C^2) \\ &= (A+C) \cdot \left(A, \frac{2AC}{A+C}, C\right) \\ &= (A+C) \cdot (A, B, C) \\ &\equiv (A, B, C)\,. \end{aligned}$$

Therefore, f and g are inverses of each other, so they each define a one-to-one correspondence between G and H.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA.

Totten–M9. *Proposed by Kirk Evenrude, Kamloops, BC.*

A train 900 m long, travelling at 90 km/h, approaches a 100 m long bridge.

(a) How many seconds does it take the train to clear the bridge?

(b) Suppose that, just as the train reaches the bridge, it begins to slow down at the rate of $0.2\,\text{m/s}^2$. Now how long does it take to clear the bridge?

Solution by Geoffrey A. Kandall, Hamden, CT, USA.

(a) The train's given speed of $90\,\text{km/h}$ is equivalent to $90\,000\,\text{m/h}$. Since there are 60 seconds in 1 minute and 60 minutes in 1 hour, then there are $60 \cdot 60 = 3600$ seconds in one hour. Thus, the train's speed is equivalent to $90\,000 \div 3600 = 25\,\text{m/s}$.

To clear the bridge, the train must travel $1000\,\text{m}$; that is, the front of the train travels $900\,\text{m}$ from the start of the bridge to the end of the bridge, then an additional $100\,\text{m}$ as the rear of the train finishes crossing the bridge.

To travel $1000\,\text{m}$ at $25\,\text{m/s}$, it takes the train $1000 \div 25 = 40$ seconds.

(b) The velocity in metres per second of the train t seconds after the train reaches the bridge is $v(t) = 25 - 0.2t$, since the train decelerates at $0.2\,\text{m/s}^2$. Note that if $t < 125$, then $v(t) > 0$, so the train is moving forwards when $0 \le t < 125$. (Also if the train continues "slowing down" for $t > 125$, then $v(t) < 0$, so the train moves backwards in that case.)

The distance in metres that the front of the train travels t seconds after the train reaches the bridge is $s(t) = 0 + 25t - \frac{1}{2}(0.2)t^2 = 25t - 0.1t^2$. (This formula is a standard formula in physics involving the initial displacement, initial velocity, and constant acceleration.)

To determine the time required to clear the bridge, we solve the equation $s(t) = 1000$, or equivalently

$$\begin{aligned} 25t - 0.1t^2 &= 1000\,; \\ 0 &= t^2 - 250t + 10000\,; \\ 0 &= (t-50)(t-200)\,. \end{aligned}$$

Therefore, $t = 50$ or $t = 200$. We want the smallest solution, so the time to clear the bridge is $50\,\text{s}$. (The second solution comes from the case where the train clears the bridge, continues to slow down, comes to a stop, and then continues backwards until reaching the bridge again.)

Also solved by PAUL BRACKEN, University of Texas, Edinburg, TX, USA; JACLYN CHANG, student, Western Canada High School, Calgary, AB; EMILY HENDRYX, student, Angelo State University, San Angelo, TX, USA; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and RICARD PEIRÓ, IES "Abastos", Valencia, Spain.

Totten–M10. *Proposed by Nicholas Buck, College of New Caledonia, Prince George, BC.*

Show that if p is a prime number, and A and B are positive integers such that p divides A, p^2 does not divide A, and p does not divide B, then the Diophantine equation $Ax^2 + By^2 = p^{2008}$ does not have any solutions in positive integers x and y.

Solution by Geoffrey A. Kandall, Hamden, CT, USA.

For each nonnegative integer n, let $P(n)$ be the statement "The equation $Ax^2 + By^2 = p^n$ has a positive integer solution (x, y)". We will show that $P(n)$ is false for all such integers n.

Note first that $P(0)$ and $P(1)$ are false, since the fact that A, B, x, and y are positive integers and $A \geq p$ (because A is divisible by p) implies that $Ax^2 + By^2 \geq Ax^2 + 1 \geq p + 1$, so $Ax^2 + By^2 \neq 1$ and $Ax^2 + By^2 \neq p$ for all positive integers x and y.

Assume that $P(n)$ is true for some integers n, and that $k \geq 2$ is the smallest positive integer for which $P(k)$ is true, say with $Ar^2 + Bs^2 = p^k$.

Since $p \mid A$ and $p^2 \nmid A$, then $A = pC$ for some positive integer C with $p \nmid C$. Thus, $pCr^2 + Bs^2 = p^k$, and so $Bs^2 = p^k - pCr^2$. Since the right side is divisible by p, then $p \mid Bs^2$. Since $p \nmid B$, then $p \mid s^2$, which means that $p \mid s$; say $s = pt$ for some positive integer t.

Then $pCr^2 + Bp^2t^2 = p^k$, which yields $Cr^2 + Bpt^2 = p^{k-1}$, or equivalently $Cr^2 = p^{k-1} - Bpt^2$. By an argument similar to the above, $p \mid r$, and so $r = pu$ for some positive integer u.

Therefore, $Cp^2u^2 + Bpt^2 = p^{k-1}$, or $Cpu^2 + Bt^2 = p^{k-2}$, or finally $Au^2 + Bt^2 = p^{k-2}$. But this means that $(x, y) = (u, t)$ is a positive integer solution to $Ax^2 + By^2 = p^{k-2}$, which implies that $P(k - 2)$ is true, which contradicts the minimality of k.

Therefore, $P(n)$ is false for each n; in particular, $Ax^2 + By^2 = p^{2008}$ has no positive integer solutions.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; and RICHARD I. HESS, Rancho Palos Verdes, CA, USA.

Problem of the Month

Ian VanderBurgh

Any commuters in the crowd?

Problem 1 (2009 Sun Life Financial Canadian Open Mathematics Challenge) Suppose that f and g are functions. The real number x is a *real fixed point* of f if $f(c) = c$. We say that f and g commute if $f(g(x)) = g(f(x))$ for all real numbers x.

(a) If $f(x) = x^2 - 2$, determine all real fixed points of f.

(b) If $f(x) = x^2 - 2$, determine all cubic polynomials g that commute with f.

To solve the first part we must understand and apply the definition.

Solution to part (a). We want to find all real numbers c for which $f(c) = c$. Since $f(x) = x^2 - 2$, then we want to solve $c^2 - 2 = c$, or $c^2 - c - 2 = 0$. Factoring, we obtain $(c - 2)(c + 1) = 0$, and so $c = 2$ or $c = -1$. We can check that $f(2) = 2$ and $f(-1) = -1$, so the real fixed points of f are $x = 2$ and $x = -1$. ∎

We could solve the second part using a lot of algebra.

Solution 1 to part (b). Here $g(x)$ is a cubic polynomial, so suppose that $g(x) = ax^3 + bx^2 + dx + e$ for some real numbers a, b, d, e with $a \neq 0$.

For $f(g(x)) = g(f(x))$, we need

$$\begin{aligned} f(ax^3 + bx^2 + dx + e) &= g(x^2 - 2)\,; \\ (ax^3 + bx^2 + dx + e)^2 - 2 &= a(x^2 - 2)^3 + b(x^2 - 2)^2 + d(x^2 - 2) + e\,. \end{aligned}$$

After some painful algebra, we find that

$$\begin{aligned} &(ax^3 + bx^2 + dx + e)^2 - 2 \\ &\quad = a^2x^6 + (2ab)x^5 + (2ad + b^2)x^4 + (2ae + 2bd)x^3 \\ &\qquad\qquad + (2be + d^2)x^2 + (2de)x + (e^2 - 2) \end{aligned}$$

and

$$\begin{aligned} &a(x^2 - 2)^3 + b(x^2 - 2)^2 + d(x^2 - 2) + e \\ &\quad = ax^6 + (-6a + b)x^4 + (12a - 4b + d)x^2 + (-8a + 4b - 2d + e)\,. \end{aligned}$$

Since these two expressions are equal for all values of x, then we can equate coefficients to obtain the following seven (!) equations:

$$\begin{aligned} a^2 &= a\,, \\ 2ab &= 0\,, \\ 2ad + b^2 &= -6a + b\,, \\ 2ae + 2bd &= 0\,, \\ 2be + d^2 &= 12a - 4b + d\,, \\ 2de &= 0\,, \\ e^2 - 2 &= -8a + 4b - 2d + e\,. \end{aligned}$$

Take a deep breath. While there are seven equations, there are only four variables. In what order should we work through these equations?

Since $a \neq 0$, the first equation gives $a = 1$. The second equation becomes $2b = 0$, which yields $b = 0$. The third equation becomes $2d = -6$, which yields $d = -3$. The fourth equation becomes $2e = 0$, which yields $e = 0$. We can check that these values of a, b, d, e satisfy the remaining three equations.

Therefore, $g(x) = x^3 - 3x$ is the only cubic polynomial that can commute with $f(x)$. The algebra above confirms that it does indeed commute with $f(x)$. ■

That was pretty gruesome. I was pretty surprised when we found that $b = e = 0$. Does this tell us anything special about $g(x)$? It tells us that the function $g(x)$ is an odd function, since

$$\begin{aligned} g(-x) &= (-x)^3 - 3(-x) \\ &= -x^3 + 3x \\ &= -(x^3 - 3x) \\ &= -g(x)\,. \end{aligned}$$

Is there a way of showing that $g(x)$ is odd in advance, before doing any of this expansion?

The solution below gives us a way of showing that $b = e = 0$ and that the function $g(x)$ is odd. Is this way easier? Likely not. But, it may be more portable, because of one key fact that it uses:

> If $p(x)$ is a polynomial of degree n, then the equation $p(x) = 0$ has at most n solutions.

We'll actually use this fact in a slightly different form:

> If $p(x)$ is a polynomial of degree at most n and the equation $p(x) = 0$ has more than n solutions, then $p(x)$ must be the zero polynomial.

Solution 2 to part (b). Suppose that $g(x)$ is a cubic polynomial for which $f(g(x)) = g(f(x))$. To show that $g(x)$ is an odd function, we want to show that $g(-x) = -g(x)$ for all x.

We note first that f is an even function, since

$$f(-x) \;=\; (-x)^2 - 2 \;=\; x^2 - 2 \;=\; f(x)$$

for all x.

We know that $f(g(x)) = g(f(x))$ and $f(g(-x)) = g(f(-x))$. Since f is even, this second equation becomes $f(g(-x)) = g(f(x))$. Thus, we have that $f(g(-x)) = f(g(x))$ for all x.

Since $f(x) = x^2 - 2$, then for every real number x we have

$$\begin{aligned}
[g(-x)]^2 - 2 &= [g(x)]^2 - 2\,;\\
[g(-x)]^2 &= [g(x)]^2\,;\\
[g(-x)]^2 - [g(x)]^2 &= 0\,;\\
[g(-x) - g(x)][g(-x) + g(x)] &= 0\,.
\end{aligned}$$

Therefore, for all x, we have $g(-x) + g(x) = 0$ or $g(-x) - g(x) = 0$. Using the same notation from Solution 1, we note that

$$\begin{aligned}
g(-x) &= a(-x)^3 + b(-x)^2 + d(-x) + e\\
&= -ax^3 + bx^2 - dx + e\,.
\end{aligned}$$

Therefore, $g(-x) + g(x) = 0$ is equivalent to

$$(-ax^3 + bx^2 - dx + e) + (ax^3 + bx^2 + dx + e) \;=\; 0\,,$$

or $bx^2 + e = 0$. Also, $g(-x) - g(x) = 0$ is equivalent to

$$(-ax^3 + bx^2 - dx + e) - (ax^3 + bx^2 + dx + e) \;=\; 0\,,$$

or $ax^3 + dx = 0$.

Therefore, for every x, at least one of $bx^2 + e = 0$ and $ax^3 + dx = 0$ is true. Since there are infinitely many possible x, then one of these equations is true for infinitely many x. Our key fact tells that this means that one of the left sides must be the zero polynomial. Therefore, either $b = e = 0$ or $a = d = 0$. Since $g(x)$ is a cubic polynomial, then $a \neq 0$. Thus, $b = e = 0$ and so $g(x) = ax^3 + dx$.

Finally, we need to determine the values of a and d, and so we must find a and d so that

$$\begin{aligned} (ax^3 + dx)^2 - 2 &= a(x^2 - 2)^3 + d(x^2 - 2)\,; \\ a^2x^6 + 2adx^4 + d^2x^2 - 2 &= ax^6 - 6ax^4 + (12a + d)x^2 + (-8a - 2d)\,. \end{aligned}$$

Equating the first two coefficients, we obtain $a^2 = a$ and $2ad = -6a$. Since $a \neq 0$, the first gives $a = 1$ from which the second gives $2d = -6$ or $d = -3$. Therefore, $g(x) = x^3 - 3x$ as in Solution 1. ■

A really good question to ask at this point is whether anything was gained in Solution 2. We did less algebraic work, but we had to do a fair bit more deep thinking. I would argue that, in fact, Solution 2 is worth it, because it gives us another tool in our toolbox. What if we wanted to find a seventh degree polynomial that commuted with $f(x)$? Solution 1 would become a fair bit uglier, whereas the method of Solution 2 would still work; the initial set-up would be no more complicated while the algebraic work at the end would be significantly easier than in the Solution 1 approach.

As a postscript, I will leave you with part (c) from this problem as it appeared:

(c) Suppose that p and q are real-valued functions that commute. If $2[q(p(x))]^4 + 2 = [p(x)]^4 + [p(x)]^3$ for all real numbers x, prove that q has no real fixed points.

Happy commuting!

THE OLYMPIAD CORNER

No. 285

R.E. Woodrow

We begin this number with the problems of the Estonian IMO team selection test 2007. Thanks go to Bill Sands, Canadian Team Leader to the IMO in Vietnam, for collecting them for our use.

ESTONIAN IMO TEAM SELECTION CONTEST 2007

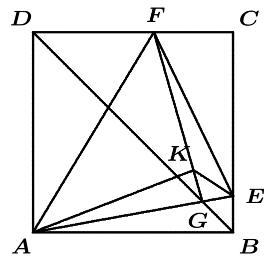
1. A switchboard has one row of n switches, where each switch can be either up or down. If a switch flips down from the up position, then its right neighbour (if present) automatically flips. At the start all switches are down. The operator of the board first flips the leftmost switch, then flips the second leftmost switch twice, and so forth until eventually he flips the rightmost switch n times. After all these flips, how many switches are up?

2. Let D be the foot of the altitude of triangle ABC drawn from vertex A. Let E, F be the points symmetric to D with respect to the lines AB, AC, respectively. Let triangles BDE, CDF have inradii r_1, r_2 and circumradii R_1, R_2, respectively. If S_K denotes the area of figure K, prove that

$$|S_{ABD} - S_{ACD}| \;\geq\; |r_1R_1 - r_2R_2|\,.$$

3. Let n be a natural number, $n \geq 2$. Prove that if $\dfrac{b^n - 1}{b - 1}$ is a prime power for some positive integer b, then n is prime.

4. In square $ABCD$ the points E and F are chosen in the interior of sides BC and CD, respectively. The line drawn from F perpendicular to AE passes through the intersection point G of AE and BD. A point K is chosen on FG such that $|AK| = |EF|$. Find $\angle EKF$.



5. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that for all x, $y \in \mathbb{R}$,

$$f(x + f(y)) \;=\; y + f(x + 1)\,.$$

6. Consider a 10×10 grid. A move consists of colouring 4 unit squares that lie at the intersection of some two rows and two columns. A move is permitted if at least one of the 4 squares is previously uncoloured. What is the *largest* possible number of moves that can be taken to colour the whole grid?

Next are the 10th Grade problems of the XXXIII Russian Mathematical Olympiad, Final Round 2006–2007. Thanks again go to Bill Sands, Canadian Team Leader to the IMO in Vietnam, for collecting them for our use.

XXXIII RUSSIAN MATHEMATICAL OLYMPIAD 2006–2007

Final Round

10th Grade

1. (*A. Polyansky*) The surface of a $9 \times 9 \times 9$ cube is divided into unit squares, and is completely covered by 2×1 non-overlapping paper rectangles (each rectangle covers two adjacent squares). Prove that the number of rectangles which are bent over an edge is odd.

2. (*A. Khrabrov*) Given a polynomial $P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$, let $m = \min\{a_0,\, a_0 + a_1,\, \ldots,\, a_0 + a_1 + \cdots + a_n\}$. Prove that $P(x) \geq mx^n$ for all $x \geq 1$.

3. (*V. Astakhov*) In an acute triangle ABC, BB_1 is a bisector. Point K is chosen on the smaller arc BC of the circumcircle, such that B_1K and AC are perpendicular. Point L is chosen on line AC such that BL and AK are also perpendicular. Line BB_1 meets the smaller arc AC at point T. Prove that points K, L, T are collinear.

4. (*K. Knop, O. Leontyeva*) A magician and his assistant want to develop the following trick: A spectator writes an arbitrary sequence of N decimal digits on a blackboard. The assistant then covers two adjacent digits. Then the magician enters the room, looks at the blackboard, and announces which digits are covered. Find the smallest value of N for which they can ensure that they can always perform this trick successfully.

5. (*N. Agakhanov*) A set of $n > 2$ vectors is given in the plane. A vector in the set is *long* if its length is not less than the length of the sum of all the other vectors in the set. Prove that if each vector in the set is long, then the sum of all the vectors in the set is the zero vector.

6. (*S. Berlov*) Two circles ω_1 and ω_2 intersect at points A and B. Let PQ and RS be the segments of common tangents to these circles (points P and R lie on ω_1, while points Q and S lie on ω_2). Ray RB intersects ω_2 again at point W. If $RB \| PQ$, find the ratio RB/BW.

7. (*D. Karpov*) A convex polyhedron F has a vertex A of degree 5, while all other vertices are of degree 3. A colouring of the edges of F with three colours is *good*, if all three colours are present at each vertex $B \neq A$. Prove that if the number of good colourings is not divisible by 5, then there exists a good colouring such that three edges meeting at A have the same colour.

8. (*A. Golovanov*) Dima computed the decimal expansions of the numbers $\frac{1}{80!}, \frac{1}{81!}, \ldots, \frac{1}{99!}$, which he wrote on 20 infinite strips of paper (for example, the last strip contains the number $\frac{1}{99!} = 0.\underbrace{00\ldots00}_{155 \text{ zeros}}10715\ldots$). From one of these strips, Sasha cuts a fragment consisting of N consecutive digits and no decimal point. Find the largest N for which Sasha can cut such a fragment without Dima being able to deduce which one of 20 strips was cut.

We continue with the problems of the 11th Grade, Final Round of the XXXIII Russian Mathematical Olympiad 2006-2007. Thanks again go to Bill Sands, Canadian Team Leader to the IMO in Vietnam, for collecting these problems for our use.

XXXIII RUSSIAN MATHEMATICAL OLYMPIAD 2006-2007

Final Round

11th Grade

1. *(N. Agakhanov)* The product $f(x) = \cos x \cos 2x \cos 3x \cdots \cos 2^k x$ is written on the blackboard, where $k \geq 10$. Prove that it is possible to replace one "$\cos$" by a "$\sin$" such that the new product, $f_1(x)$, satisfies the inequality $|f_1(x)| \leq 3 \cdot 2^{-1-k}$ for all real x.

2. *(A. Polyansky)* The incircle of a triangle ABC touches sides BC, AC, and AB at points A_1, B_1, and C_1, respectively. Segment AA_1 intersects the incircle again at point Q. Line ℓ is parallel to BC and passes through A. Lines A_1C_1 and A_1B_1 intersect ℓ at points P and R, respectively. Prove that $\angle PQR = \angle B_1QC_1$.

3. [*Ed.: This is the same as problem 4 of the 10th Grade contest*].

4. *(A. Golovanov)* An infinite sequence $\{x_n\}_{n=1}^{\infty}$ is defined as follows. The first term x_1 is a rational number greater than 1, and $x_{n+1} = x_n + \frac{1}{\lfloor x_n \rfloor}$ for all positive integers n, where $\lfloor x_n \rfloor$ is the greatest integer not exceeding x. Prove that this sequence contains an integer.

5. *(F. Petrov)* At each vertex of a convex 100-gon, two distinct numbers are written. Prove that one can select one number at each vertex, such that the selected numbers at two adjacent vertices are always distinct.

6. *(N. Agakhanov, I. Bogdanov)* Determine if there exist three nonzero real numbers a, b, c such that for every $n > 3$ there exists a polynomial $P_n(x)$ of the form $P_n(x) = x^n + \cdots + x^3 + ax^2 + bx + c$ which has only integer roots.

7. *(A. Zaslavsky)* Given a tetrahedron T, a pair of its skew edges a and b is *large* if T is covered by the spheres with diameters a and b. Does a large pair necessarily exist?

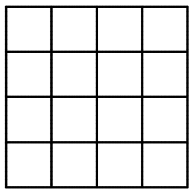
8. *(I. Bogdanov, G. Chelnokov)* Among N cities some pairs of them are connected by nonstop air shuttles. For each k with $2 \leq k \leq N$ and for any set of k cities, there are at most $2k - 2$ airlines connecting these cities. Prove that all flights can be distributed between two companies such that every circular trip uses the airlines of both companies.

Next we give the problems of the 20th Korean Mathematical Olympiad, Final Round, March 24–25, 2007. Thanks again go to Bill Sands, Canadian Team Leader to the IMO in Vietnam, for collecting them for us.

20th KOREAN MATHEMATICAL OLYMPIAD
Final Round
March 24, 2007 — Time 4.5 hours

1. Triangle ABC is acute with circumcircle Γ and circumcentre O. The circle Γ' has centre O', is tangent to O at A and to the side BC at D, and intersects the lines AB and AC again at E and F, respectively. The lines OO' and EO' intersect Γ' again at A' and G, respectively. The lines BO and $A'G$ intersect at H. Prove that $DF^2 = AF \cdot GH$.

2. Consider the sixteen tiles fixed on a wall as shown below. How many ways are there to write either 0 or 1 on each tile so that the product of the two numbers written on every neighbouring pair of tiles (sharing a common side) is always 0?



3. Find all triples (x, y, z) of positive integers satisfying $1 + 4^x + 4^y = z^2$.

March 25, 2007 — Time 4.5 hours

4. Find all pairs (p, q) of primes such that $p^p + q^q + 1$ is divisible by pq.

5. For the vertex A of $\triangle ABC$, let A' be the point of intersection of the angle bisector at A with side BC, and let ℓ_A be the distance between the feet of the perpendiculars from A' to the lines AB and AC, respectively. Define ℓ_B and ℓ_C similarly, and let ℓ be the perimeter of $\triangle ABC$. Prove that

$$\frac{\ell_A \ell_B \ell_C}{\ell^3} \leq \frac{1}{64}.$$

6. Let $\mathbb{N}$ be the set of positive integers, and let $f : \mathbb{N} \to \mathbb{N}$ satisfy

$$kf(n) \ \leq \ f(kn) \ \leq \ kf(n) + k - 1$$

for all k, $n \in \mathbb{N}$.

(a) Prove that $f(a) + f(b) \leq f(a+b) \leq f(a) + f(b) + 1$ for all a, $b \in \mathbb{N}$.

(b) Show that if f satisfies $f(2007n) \leq 2007f(n) + 2005$ for all $n \in \mathbb{N}$, then $f(2007c) = 2007f(c)$ for some $c \in \mathbb{N}$.

As final sets of problems for this number, we first give the 2006/7 British Mathematical Olympiad, Round 1. Thanks are extended to Bill Sands, Canadian Team Leader to the IMO in Vietnam, for collecting these problems for us.

2006/7 BRITISH MATHEMATICAL OLYMPIAD
Round 1

1. Find four prime numbers less than 100 which are factors of $3^{32} - 2^{32}$.

2. In the convex quadrilateral $ABCD$, points M, N lie on the side AB such that $AM = MN = NB$, and points P, Q lie on the side CD such that $CP = PQ = QD$. Prove that

$$\text{Area of } AMCP \ = \ \text{Area of } MNPQ \ = \ \frac{1}{3}(\text{ Area of } ABCD)\,.$$

3. The number 916238457 is an example of a nine-digit number which contains each of the digits 1 to 9 exactly once. It also has the property that the digits 1 to 5 occur in their natural order, while the digits 1 to 6 do not. How many such numbers are there?

4. Two touching circles S and T share a common tangent which meets S at A and T at B. Let AP be a diameter of S and let the tangent from P to T touch it at Q. Show that $AP = PQ$.

5. For positive real numbers a, b, c prove that

$$(a^2 + b^2)^2 \ \geq \ (a+b+c)(a+b-c)(b+c-a)(c+a-b)\,.$$

6. Let n be an integer. Show that, if $2 + 2\sqrt{1 + 12n^2}$ is an integer, then it is a perfect square.

Lastly, we have the problems of Round 2 of the 2006/7 British Mathematical Olympiad, courtesy of Bill Sands.

2006/7 BRITISH MATHEMATICAL OLYMPIAD
Round 2

1. Triangle ABC has integer-length sides, and $AC = 2007$. The internal bisector of $\angle BAC$ meets BC at D. Given that $AB = CD$, determine AB and BC.

2. Show that there are infinitely many pairs of positive integers (m, n) such that

$$\frac{m+1}{n} + \frac{n+1}{m}$$

is a positive integer.

3. Let ABC be an acute-angled triangle with $AB > AC$ and $\angle BAC = 60^\circ$. Denote the circumcentre by O and the orthocentre by H and let OH meet AB at P and AC at Q. Prove that $PO = HQ$.
Note: The circumcentre of triangle ABC is the centre of the circle which passes through the vertices A, B and C. The orthocentre is the point of intersection of the perpendiculars from each vertex to the opposite side.

4. In the land of Hexagonia, the six cities are connected by a rail network such that there is a direct rail line connecting each pair of cities. On Sundays, some lines may be closed for repair. The passengers' rail charter stipulates that any city must be accessible by rail from any other (not necessarily directly) at all times. In how many different ways can some of the lines be closed subject to this condition?

Now we return to solutions from our readers to problems given in the February 2009 number of the *Corner* and solutions to the Thai Mathematical Olympiad Examinations 2005, given at [2009 : 22].

1. Let $P(x)$, $Q(x)$, and $R(x)$ be polynomials satisfying

$$2xP\left(x^3\right) + Q\left(-x-x^2\right) = \left(1+x+x^2\right)R(x)\,.$$

Show that $x - 1$ is a factor of $P(x) - Q(x)$.

Solved by Matthew Babbitt, home-schooled student, Fort Edward, NY, USA; Michel Bataille, Rouen, France; and Titu Zvonaru, Comănești, Romania. We give Babbitt's write-up.

We need to prove that $x-1$ is a factor of $P(x)-Q(x)$. In other words, we need to show that 1 is a root of $P(x) - Q(x)$, or $P(1) - Q(1) = 0$.

Since $x^3 - 1 = (x - 1)(x^2 + x + 1)$, the roots of $x^2 + x + 1$ are third roots of unity, namely $w = \dfrac{-1 + i\sqrt{3}}{2}$ and $w^2 = \dfrac{-1 - i\sqrt{3}}{2}$. Note that $-x - x^2 = (-1 - x - x^2) + 1 = -0 + 1 = 1$ and $x^3 = 1$. Thus, we have

$$2wP(1) + Q(1) \;=\; 2w^2P(1) + Q(1) \;=\; 0\,.$$

Hence

$$2(w^2 - w)P(1) \;=\; 0 \implies P(1) \;=\; 0\,.$$

We now have $Q(1) = 0$ and $P(1) - Q(1) = 0$, hence $(x - 1) | (P(x) - Q(x))$.

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\big(x + y + f(xy)\big) \;=\; f\big(f(x + y)\big) \;+\; xy$$

for all x, $y \in \mathbb{R}$.

Solution by Michel Bataille, Rouen, France.

For $a \in \mathbb{R}$, let $f_a : \mathbb{R} \to \mathbb{R}$ be defined by $f_a(x) = x + a$. It is readily checked that f_a is a solution for each real number a. We show that there are no other solutions. To this aim, let f be an arbitrary solution and let u, v be any real numbers. Choose $s \in \mathbb{R}$ such that $s^2 > 4u$ and $s^2 > 4v$. Then the quadratic $X^2 - sX + u = 0$ has two solutions x_1, y_1 satisfying $x_1 + y_1 = s$ and $x_1y_1 = u$. Taking $x = x_1$, $y = y_1$ in the given equation yields $f(s + f(u)) = f(f(s)) + u$. Similarly, we obtain $f(s + f(v)) = f(f(s)) + v$. It follows that if $f(u) = f(v)$, then $u = v$, that is, f is injective.

Now, let $a = f(0)$. Setting $y = 0$ in the functional equation, we obtain $f(x + a) = f(f(x))$ for each x, hence $f(x) = x + a$ and $f = f_a$, as desired.

3. Let a, b, and c be positive real numbers. Prove that

$$1 \;+\; \frac{3}{ab + bc + ca} \;\geq\; \frac{6}{a + b + c}\,.$$

Solved by George Apostolopoulos, Messolonghi, Greece; Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; Joe Howard, Portales, NM, USA; and Titu Zvonaru, Comăneşti, Romania. We give the write-up by Howard.

By the well-known inequality $(a + b + c)^2 \geq 3(ab + bc + ca)$, we have

$$\frac{1}{\sqrt{3(ab + bc + ca)}} \;\geq\; \frac{1}{a + b + c}\,.$$

By the AM-GM Inequality and the above, we have

$$1 + \frac{3}{ab+bc+ca} \geq 2\sqrt{\frac{3}{ab+bc+ca}} = \frac{6}{\sqrt{3(ab+bc+ca)}} \geq \frac{6}{a+b+c},$$

and the result is proved.

4. Let n be a positive integer. Prove that $n(n+1)(n+2)$ is not a perfect square.

Solved by Arkady Alt, San Jose, CA, USA; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Matthew Babbitt, home-schooled student, Fort Edward, NY, USA; Edward T.H. Wang and Dexter Wei, Wilfrid Laurier University, Waterloo, ON; and by Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the solution of Amengual Covas.

Assume on the contrary that $n(n+1)(n+2)$ is a perfect square for some positive integer n.

Since $\gcd(n, n+1) = 1$ and $\gcd(n+1, n+2) = 1$, we have that $\gcd(n+1, n(n+2)) = 1$, and therefore each of the numbers $n+1$ and $n(n+2)$ is a perfect square. Writing $n+1 = u^2$ and $n(n+2) = v^2$, where u, v are integers, yields $u^4 - 1 = v^2$. This equation can be written in the form $(u^2 - v)(u^2 + v) = 1$, whence $u^2 - v = 1$ and $u^2 + v = 1$. Solving, we find that $u^2 = 1$, $v = 0$. This implies that $n = 0$, which contradicts that n is a positive integer.

Thus our assumption was wrong, and consequently $n(n+1)(n+2)$, where n is a positive integer, is not a perfect square.

5. Find the least positive integer n such that $2549 \mid \left(n^{2545} - 2541\right)$.

Solution by Matthew Babbitt, home-schooled student, Fort Edward, NY, USA.

The given statement is equivalent to $n^{2545} \equiv -8 \pmod{2549}$. Note that 2549 is prime. Since n^{2545} is not a multiple of 2549, n is not a multiple of 2549. By Fermat's Little Theorem, $n^{2548} \equiv 1 \pmod{2549}$. Hence, $-8n^3 \equiv 1 \pmod{2549}$, or $(-2n)^3 \equiv 1 \equiv n^{2548} \pmod{2549}$. Since 3 is not a divisor of 2548, it follows that $-2n \equiv 1 \pmod{2549}$, which means that $2n \equiv 2548 \pmod{2549}$. The least positive integer that satisfies this last congruence is $n = \frac{2548}{2} = 1274$.

6. Do there exist positive integers x, y, and z such that

$$2548^x + (-2005)^y = (-543)^z\,?$$

Solved by Matthew Babbitt, home-schooled student, Fort Edward, NY, USA; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give Babbitt's write-up.

The answer is no.

Assume for the sake of contradiction that there are positive integer solutions to the equation. Note that $2548 \equiv 1 \pmod 3$, $-2005 \equiv 2 \pmod 3$, and $-543 \equiv 0 \pmod 3$. Hence the first term is $1 \pmod 3$ and the third term is $0 \pmod 3$. Therefore, the second term must be $0 - 1 \equiv 2 \pmod 3$. Hence, y is odd, and so the second term is negative. Let $y = 2y_1 - 1$. We are now looking for positive integer solutions to $2548^x - 2005^{2y_1-1} = (-543)^z$.

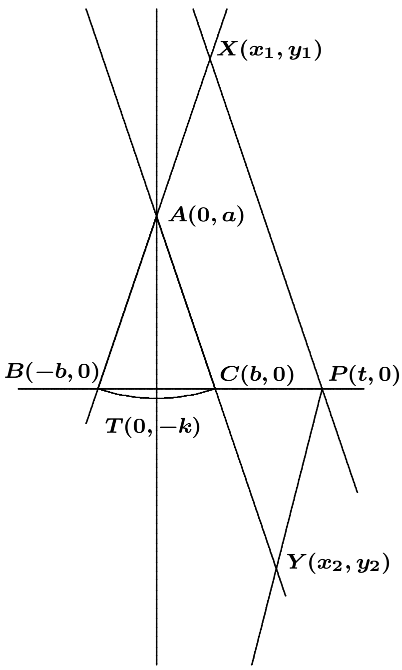
Note that $2548 \equiv 0 \pmod 4$ and $2005 \equiv -543 \equiv 1 \pmod 4$. Therefore, the first term is $0 \pmod 4$ and the second and third terms are $1 \pmod 4$. Thus, $0 - 1 \equiv 3 \equiv 1 \pmod 4$, which is a contradiction.

Therefore, there are no positive integer solutions to the equation.

10. Assume ABC is an isosceles triangle with $AB = AC$. Suppose that P is a point on the extension of side BC. X and Y are points on lines AB and AC such that $PX \| AC$ and $PY \| AB$. Let T be the midpoint of arc BC. Prove that $PT \perp XY$. (Iran 2004)

Solved by Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA.

We introduce coordinates. Let a and b be positive reals. Let the vertices of triangle ABC be $A(a, 0)$, $B(-b, 0)$, and $C(b, 0)$. The point $P(t, 0)$ is on the x-axis to the right of $C(b, 0)$, so that $t > b$. We also introduce the points $X(x_1, y_1)$ and $Y(x_2, y_2)$. The arc in the figure is part of the circumcircle of triangle ABC.



The origin $(0, 0)$ is the midpoint of side AC, and since the triangle is isosceles with $AB = AC$, it follows that the centre of the circumscribed circle is the midpoint of the line segment AT and AT is a diameter. The point T has coordinates $T(0, -k)$, k a positive real number.

Let $\angle A = 2w$; we have

$$|AC| = \sqrt{a^2 + b^2}\,,$$
$$\sin w = \frac{b}{\sqrt{a^2 + b^2}}\,,$$
$$\cos w = \frac{a}{\sqrt{a^2 + b^2}}\,.$$

From the Law of Sines in the triangle ABC, we also have

$$2R \sin A = 2b\,, \qquad R = \frac{b}{\sin A}\,,$$

where R is the radius of the circumscribed circle. Furthermore, we have that $\sin A = \sin 2w = 2\sin w\cos w$, hence

$$\sin A \;=\; \sin 2w \;=\; \frac{2ab}{a^2+b^2}\,, \quad \text{and} \quad R \;=\; \frac{b}{\sin A} \;=\; \frac{a^2+b^2}{2a}\,.$$

From the figure, $|AT| = 2R = a + k$, hence

$$k \;=\; 2R - a \;=\; \left(\frac{a^2+b^2}{a}\right) - a \;=\; \frac{b^2}{a}\,.$$

and $T(0,-k) = T\left(0, -\frac{b^2}{a}\right)$. Next, we express x_1, x_2, y_1, y_2 in terms of a and b. First, the slope of line PX is $m_{PX} = -\frac{a}{b}$; an equation for the line PX is thus

$$y \;=\; -\frac{a}{b}(x-t)\,. \tag{1}$$

The slope of line PY is $m_{PY} = \frac{a}{b}$; an equation for line PY is thus

$$y \;=\; \frac{a}{b}(x-t)\,. \tag{2}$$

An equation for line AB is

$$y \;=\; \frac{a}{b}x + a\,. \tag{3}$$

An equation for line AC is

$$y \;=\; -\frac{a}{b}x + a\,. \tag{4}$$

The point $X(x_1, y_1)$ lies at the intersection of the lines AB and PX. By solving the system of equations (1) and (3), we find that

$$x_1 \;=\; \frac{t-b}{2}\,, \qquad y_2 \;=\; \frac{a(t+b)}{2b}\,. \tag{5}$$

Similarly, $Y(x_2, y_2)$ is the point of intersection of the lines AC and PY. By solving the system of two equations (2) and (4), we find that

$$x_2 \;=\; \frac{b+t}{2}\,, \qquad y_2 \;=\; \frac{a(b-t)}{2b}\,. \tag{6}$$

Therefore, the slope of XY is

$$m_{XY} \;=\; \frac{y_2-y_1}{x_2-x_1} \;=\; -\frac{at}{b^2}\,. \tag{7}$$

The slope of line PT is

$$m_{PT} \;=\; \frac{0-(-k)}{t-0} \;=\; \frac{k}{t} \;=\; \frac{b^2}{at}\,. \tag{8}$$

By (7) and (8), we have $m_{XY} \cdot m_{PT} = -1$, which proves that $XY \perp PT$.

Now we turn to solutions to the 46th Ukrainian Mathematical Olympiad 2006, Final Round, given at [2009 : 23–24].

1. (V.V. Plakhotnyk) Prove that for any rational numbers a and b the graph of the function

$$f(x) \;=\; x^3 - 6abx - 2a^3 - 4b^3\,, \quad x \in \mathbb{R}$$

has exactly one point in common with the x-axis.

Solved by Matthew Babbitt, home-schooled student, Fort Edward, NY, USA; Michel Bataille, Rouen, France; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give Bataille's version.

We will make use of the following result proved at the end: $x^3-3px+2q$ vanishes exactly once for $x \in \mathbb{R}$ if and only if $q^2 > p^3$ or $p = q = 0$.

Here, $p = 2ab$ and $q = -(a^3 + 2b^3)$, hence $q^2 > p^3$ can be rewritten as $(a^3 + 2b^3)^2 > 8a^3b^3$, that is, $(a^3 - 2b^3)^2 > 0$. This is certainly true if $a^3 \neq 2b^3$. However, $a^3 = 2b^3$ cannot occur if a, $b \neq 0$, since otherwise the number 2 would be the cube of a nonzero rational number, which is impossible (if $m^3 = 2n^3$ for positive integers m and n, then a contradiction arises: the exponent of 2 in the standard factorization is a multiple of 3 on the left but not on the right).

Since $p = q = 0$ when $a = b = 0$, the condition $q^2 > p^3$ or $p = q = 0$ is satisfied for all rational numbers a and b, and the result follows.

We now prove the result used above. Let $P(x) = x^3 - 3px + 2q$. If $p \leq 0$, then for a, $b \in \mathbb{R}$ with $a \neq b$,

$$\frac{P(a) - P(b)}{a - b} \;=\; a^2 + ab + b^2 - 3p \;>\; 0\,,$$

hence P is increasing on $\mathbb{R}$ and vanishes only once.

If $p > 0$, then P is increasing on $(-\infty, -\sqrt{p})$ and $(\sqrt{p}, \infty)$ and decreasing on $(-\sqrt{p}, \sqrt{p})$. An easy calculation gives $P(-\sqrt{p}) = 2(q + p\sqrt{p})$ and $P(\sqrt{p}) = 2(q - p\sqrt{p})$, so that $P(-\sqrt{p}) > P(\sqrt{p})$ and we have that $P(-\sqrt{p}) \cdot P(\sqrt{p}) = 4(q^2 - p^3)$. It follows that if $q^2 < p^3$, then P vanishes once in $(-\sqrt{p}, \sqrt{p})$ as well as in $(-\infty, -\sqrt{p})$ and in $(\sqrt{p}, \infty)$.

If $0 < p^3 < q^2$, then $P(-\sqrt{p})$ and $P(\sqrt{p})$ have the same sign and P vanishes only once. Lastly if $p^3 = q^2 \neq 0$, then

$$P(x) \;=\; \left(x - \frac{q}{p}\right)^2 \left(x + \frac{2q}{p}\right)\,,$$

and P vanishes twice. The required result follows from these observations.

5. (O.O. Kurchenko) Prove that for any real numbers x and y

$$|\cos x| \;+\; |\cos y| \;+\; |\cos(x+y)| \;\geq\; 1\,.$$

Solved by Arkady Alt, San Jose, CA, USA; and Michel Bataille, Rouen, France. We give the argument of Bataille.

Let $v = e^{-2ix}$ and $w = e^{2iy}$. Then,

$$|1+v| = \left|e^{-ix}(e^{ix}+e^{-ix})\right| = \left|e^{-ix}(2\cos x)\right| = 2|\cos x|$$

and similarly,

$$|1+w| = 2|\cos y|$$

and

$$|v+w| = \left|e^{i(y-x)}(e^{i(x+y)}+e^{-i(x+y)})\right| = 2|\cos(x+y)|\,.$$

Now, using the Triangle Inequality, we obtain

$$\begin{aligned} 2 &= |(1+v)+(1+w)-(v+w)| \\ &\le |1+v|+|1+w|+|v+w| \\ &= 2\big(|\cos x|+|\cos y|+|\cos(x+y)|\big)\,, \end{aligned}$$

and the result follows.

6. (T.M. Mitelman) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\left(x^3+y^3\right) = x^2 f(x) + y f(y^2)$$

for all real numbers x and y.

Solution by Michel Bataille, Rouen, France.

The solutions are the functions $f_m(x) = mx$, where m a real number.

It is readily checked that these functions satisfy the identity. We now show that there are no other solutions. To this aim, let f be any solution. Taking $x = y = 0$ in the identity yields $f(0) = 0$; also, with only $y = 0$, we obtain $f(x^3) = x^2 f(x)$, and with only $x = 0$, we obtain $f(y^3) = y f(y^2)$. Thus, for all real numbers x,

$$f(x^3) = x^2 f(x) = x f(x^2)\,.$$

From the identity we now obtain $f(x^3+y^3) = f(x^3)+f(y^3)$. Since any real number is the cube of a real number, it follows that

$$f(a+b) = f(a)+f(b)$$

for all real numbers a and b.

Consequently, f is odd (take $b = -a$), and $f(na) = nf(a)$ if $n \in \mathbb{Z}$ and $a \in \mathbb{R}$. Substituting $x+1$ and $x-1$ for x and y in the identity, we obtain on the one hand

$$f((x+1)^3+(x-1)^3) = f(2x^3+6x) = 2f(x^3)+6f(x) = 2x^2 f(x)+6f(x)\,,$$

while on the other hand, using the identity and the facts established so far,

$$\begin{aligned} & f\big((x+1)^3 + (x-1)^3\big) \\ = \; & (x+1)^2\big(f(x) + f(1)\big) + (x-1)^2\big(f(x) - f(1)\big) \\ = \; & 2x^2 f(x) + 2f(x) + 4xf(1) \,. \end{aligned}$$

By comparison, we see that $f(x) = xf(1)$ for all x, so $f(x) = f_m(x)$ with $m = f(1)$ and we are done.

Next we turn to solutions from our readers to problems of the Czech-Polish-Slovak Mathematical Competition 2006 given at [2009 : 24–25].

1. Five distinct points A, B, C, D, and E lie in this order on a circle of radius r and satisfy $AC = BD = CE = r$. Prove that the orthocentres of the triangles ACD, BCD, and BCE are the vertices of a right-angled triangle.

Solved by Michel Bataille, Rouen, France; and Titu Zvonaru, Comăneşti, Romania. We give Bataille's solution.

Let O be the centre of the circle containing A, B, C, D, E. Let H_1, H_2, H_3 and G_1, G_2, G_3 be the orthocentres and centroids of the triangles ACD, BCD, BCE, respectively. Since these triangles have O as a common circumcentre, we have $\overrightarrow{OH_k} = 3\overrightarrow{OG_k}$ for each k. Thus, triangles $H_1H_2H_3$ and $G_1G_2G_3$ are homothetic, and the problem is equivalent to showing that the triangle $G_1G_2G_3$ is right-angled.

Without loss of generality we take $r = 1$ and we shall use complex affixes. Observing that OAC, OCE, OBD are equilateral triangles with the same orientation, we may suppose that the affixes of O, A, C, E, B, D are 0, 1, $-\omega^2$, ω, u, $-u\omega^2$, respectively, where $\omega = e^{2\pi i/3}$ and $u = e^{\alpha i}$ for some $\alpha \in \left(0, \frac{\pi}{3}\right)$. The affixes g_1, g_2, g_3 of G_1, G_2, G_3 are then given by

$$3g_1 = 1 - \omega^2 - u\omega^2 \,, \qquad 3g_2 = u - \omega^2 - u\omega^2 \,, \qquad 3g_3 = u - \omega^2 + \omega \,.$$

Now, the vectors $3\overrightarrow{G_2G_1}$ and $3\overrightarrow{G_2G_3}$ have affixes

$$3(g_1 - g_2) \;=\; 1 - u \,, \qquad 3(g_3 - g_2) \;=\; \omega + u\omega^2 \,.$$

From $1 - u = 1 - e^{\alpha i} = e^{\alpha i/2}(e^{\alpha(-i)/2} - e^{\alpha i/2}) = -2ie^{\alpha i/2}\sin(\alpha/2)$ and

$$\omega + u\omega^2 \;=\; e^{2\pi i/3} + e^{(\alpha - \frac{2\pi}{3})i} \;=\; 2e^{\alpha i/2}\cos\left(\frac{2\pi}{3} - \frac{\alpha}{2}\right)$$

we deduce that $\dfrac{g_3 - g_2}{g_1 - g_2} = ki$ for some real number k. This means that $\overrightarrow{G_2G_1}$ and $\overrightarrow{G_2G_3}$ are orthogonal; that is, triangle $G_1G_2G_3$ is right-angled.

2. There are n children sitting at a round table. Erika is the oldest among them and she has n candies. No other child has any candy. Erika distributes the candies as follows. In every round, all the children with at least two candies show their hands. Erika chooses one of them and he/she gives one candy to each of the children sitting next to him/her. (So in the first round Erika must choose herself to begin the distribution.) For which $n \geq 3$ is it possible to redistribute the candies so that each child has exactly one candy?

Solution by Oliver Geupel, Brühl, NRW, Germany.

The redistribution is possible if and only if n is odd.

To prove this, let $\mathbb{N}$ be the set of positive integers, and let $\mapsto$ denote the binary relation on $\mathbb{N}^n$ such that $(a_1, a_2, \ldots, a_n) \mapsto (a'_1, a'_2, \ldots a'_n)$ if and only if there is an index k such that $a'_{k-1} = a_{k-1} + 1$, $a'_k = a_k - 2$, and $a'_{k+1} = a_{k+1} + 1$, where indices are taken modulo n, that is, $a_{k+n} = a_k$.

First, assume that

$$\begin{aligned}(n, 0, 0, \ldots, 0) &= (a_1^{(0)}, a_2^{(0)}, \ldots, a_n^{(0)}) \mapsto (a_1^{(1)}, a_2^{(1)}, \ldots, a_n^{(1)}) \\ &\mapsto \cdots \mapsto (a_1^{(m)}, a_2^{(m)}, \ldots, a_n^{(m)}) = (1, 1, \ldots, 1)\,.\end{aligned}$$

For $1 \leq k \leq n$, let b_k denote the number of indices $j \in \{1, 2, \ldots\ m\}$ such that $a_k^{(j)} = a_k^{(j-1)} - 2$. Then, for $2 \leq k \leq n$, the number of indices $j \in \{1, \ldots, m\}$ such that $a_k^{(j)} = a_k^{(j-1)} + 1$ is $a_k^{(m)} - a_k^{(0)} + 2b_k = 2b_k + 1$, and on the other hand is $b_{k-1} + b_{k+1}$, since an increase in the k^{th} component results from a decrease of a neighbour. Hence, $b_{k+1} = 2b_k - b_{k-1} + 1$ for $2 \leq k \leq n$. By induction one has $b_{k+1} = kb_2 - (k-1)b_1 + k(k-1)/2$. Hence, $b_1 = b_{n+1} = nb_2 - (n-1)b_1 + n(n-1)/2$, and thus $b_1 = b_2 + (n-1)/2$. Consequently, n is odd.

For convenience, let a^k denote the sequence a, a, $\ldots$, a consisting of k occurrences of the symbol a, and let $\mapsto^+$ denote the transitive closure of the relation $\mapsto$.

We will show by mathematical induction that for each $k > 0$ there is a path $(0^k, 2k+1, 0^k) \mapsto^+ (1^{2k+1})$ such that both the first and the last component of our n-tuple have only one increasing step each.

The statement is obvious for $k = 1$. Assume it holds for k. We obtain

$$\begin{aligned}(0^{k+1}, 2k+3, 0^{k+1}) &\mapsto^+ (0, 1^k, 3, 1^k, 0) && \text{(by hypothesis)} \\ &\mapsto^+ (1, 0, 1^{k-1}, 3, 1^{k-1}, 0, 1) && \text{(shift from the centre outwards)} \\ &\mapsto^+ (1^2, 0, 1^{k-2}, 3, 1^{k-2}, 0, 1^2) && \text{(shift from the centre outwards)} \\ &\mapsto^+ \cdots \mapsto^+ (1^{k+3}) && \text{(iterate shifting from the centre outwards)}\,.\end{aligned}$$

This completes the proof.

3. The sum of four real numbers is 9 and the sum of their squares is 21. Prove that these four numbers can be labelled as a, b, c, and d so that the inequality $ab - cd \geq 2$ holds.

Solution by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.

Without loss of generality we can assume that $a \geq b \geq c \geq d$. We will consider two cases: $a + b \geq 5$ and $a + b < 5$.

Case 1. If $a + b \geq 5$, then taking into account that $c^2 + d^2 \geq 2cd$, we have

$$\begin{aligned} a^2 + b^2 + 2ab &= (a+b)^2 \geq 25 \\ &= 4 + a^2 + b^2 + c^2 + d^2 \geq 4 + a^2 + b^2 + 2cd\,, \end{aligned}$$

from which follows $ab - cd \geq 2$.

Case 2. Now we will see that $a + b < 5$ is not possible. In fact, if $a + b < 5$, then $c+d > 4$ and therefore $4 < c+d \leq a+b < 5$. From $(a-d)(b-c) \geq 0$ and $(a-b)(c-d) \geq 0$ it follows that $ab + cd \geq ac + bd \geq ad + bc$. Since

$$\begin{aligned} &(ab + cd) + (ac + bd) + (ad + bc) \\ &= \frac{(a+b+c+d)^2 - (a^2+b^2+c^2+d^2)}{2} = 30\,, \end{aligned}$$

we have $ab+cd \geq 10$. From $4 < c+d \leq a+b < 5$ and $(a+b)+(c+d) = 9$, it follows that $(a+b)(c+d) \geq 20$, and hence from

$$(a+b)^2 + (c+d)^2 + 2(a+b)(c+d) = 81$$

we obtain $(a+b)^2 + (c+d)^2 < 41$. We now have

$$\begin{aligned} 41 &= 21 + 2 \cdot 10 \\ &\leq (a^2 + b^2 + c^2 + d^2) + 2(ab + cd) \\ &= (a+b)^2 + (c+d)^2 \\ &< 41\,, \end{aligned}$$

a contradiction, and we are done.

Next we look at readers' solutions to problems of the XXI Olimpiadi Italiano della Matematica, Cesenatico 2006 given at [2009 : 25–26].

1. Rose and Savino play a game with a deck of traditional Neapolitan playing cards which consists of 40 cards of four different suits, numbered 1 to 10. At the start each player has 20 cards. Taking turns, one shows a card on the table. Whenever some cards on the table add to exactly 15, these are then removed from the game (if the sum 15 can be obtained in more than one way, the player who last moved decides which cards adding to 15 to remove). At the end of the game only one card, a 9, is left on the table. Savino holds two cards numbered 3 and 5, and Rose holds one card. What is the number of Rose's card?

Solved by Matthew Babbitt, home-schooled student, Fort Edward, NY, USA; John Grant McLoughlin, University of New Brunswick, Fredericton, NB and Titu Zvonaru, Comănești, Romania. We give the write-up of Babbitt.

We shall prove that Rose is holding an 8.

Let Rose's card have a value of R. Let the number of sweeps occurring in the game be d. The sum of all the cards is $4(1+2+\cdots+10) = 4{\cdot}55 = 220$. Therefore, $220 = 3+5+9+R+15d$, which implies $203 = R+15d$. Since R is less than 15, R is the remainder when 203 is divided by 15. Now, $\frac{203}{15} = 13 + \frac{8}{15}$, hence $R = 8$.

2. Find all values of m, n, and p such that

$$p^n + 144 = m^2 ,$$

where m and n are positive integers and p is a prime number.

Solved by Matthew Babbitt, home-schooled student, Fort Edward, NY, USA; Michel Bataille, Rouen, France; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comănești, Romania. We give Babbitt's solution.

The only such triples are $(m,n,p) \in \{(13,2,5), (20,8,2), (15,4,3)\}$.

Write $p^n = m^2 - 144 = (m-12)(m+12)$. Since p is a prime, $m-12$ and $m+12$ must be integral powers of p. Either p divides each factor $m \pm 12$, or $m - 12 = 1$ and $m = 13$. The second case yields $(m,n,p) = (13,2,5)$. The first case yields $p \mid [(m+12)-(m-12)] = 24$, or $p \in \{2, 3\}$.

If $p = 2$, then we need to find two powers of 2 whose difference is 24. Note that $m - 12 < 32$, since $2^{n+1} - 2^n > 24$ for $n \geq 5$. So $m - 12$ is one of 2^1, 2^2, 2^3, 2^4. Checking each case yields only $m - 12 = 8 = 2^3$, or $m = 20$, and thus $(m,n,p) = (20,8,2)$.

If $p = 3$, then we need to find two powers of 3 whose difference is 24. Now $m - 12 < 27$, since $3^{n+1} - 3^n > 24$ for $n \geq 3$. So $m - 12$ is either 3 or 9. Only $m - 12 = 3$ works, yielding $(m,n,p) = (15,4,3)$.

Therefore, these are the only triples.

3. Let A and B be two points on a circle Γ such that AB is not a diameter. Let P be a point on Γ different from A and B, and let H be the orthocentre of the triangle ABP. Find the locus of H as P varies over all points of Γ different from A and B.

Solved by Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Comănești, Romania. We give Bataille's write-up.

Let O be the centre of Γ and M be the midpoint of AB. When P traverses $\Gamma - \{A, B\}$, the fixed point O remains the circumcentre of $\triangle PAB$.

It is well known that $\overrightarrow{PH} = 2\overrightarrow{OM}$. It follows that H is the image of P under $\mathcal{T}$, where $\mathcal{T}$ is the translation by the vector $2\overrightarrow{OM}$. Let $\Gamma' = \mathcal{T}(\Gamma)$. The circle Γ' has the same radius as Γ and its centre is the symmetric point of O about M. The locus of H is clearly $\Gamma' - \{A', B'\}$, where $A' = \mathcal{T}(A)$ and $B' = \mathcal{T}(B)$.

5. Consider the inequality

$$(x_1 + \cdots + x_n)^2 \ \geq \ 4(x_1x_2 + x_2x_3 + \cdots + x_nx_1)\,.$$

(a) Determine for which $n \geq 3$ the inequality holds true for all possible choices of positive real numbers x_1, x_2, ..., x_n.

(b) Determine for which $n \geq 3$ the inequality holds true for all possible choices of *any* real numbers x_1, x_2, ..., x_n.

Solved by Edward T.H. Wang and Dexter Wei, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Cománeşti, Romania. We give the write-up of Wang and Wei.

For part (a) we will show that the inequality holds for all $n \geq 4$ but fails for $n = 3$. When $n = 3$, $(x_1, x_2, x_3) = (2, 1, 1)$ provides a counterexample. Next, we consider the case $n = 4$. We have

$$\begin{aligned}
& (x_1 + x_2 + x_3 + x_4)^2 - 4(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1) \\
= \ & x_1^2 + x_2^2 + x_3^2 + x_4^2 + 2(x_1x_3 + x_2x_4) \\
& \qquad\qquad\qquad - 2(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1) \\
= \ & (x_1 - x_2 + x_3 - x_4)^2 \ \geq \ 0\,,
\end{aligned}$$

so the inequality in fact holds for all real numbers x_1, x_2, x_3, and x_4.

Now we use induction to show that the inequality holds for $n \geq 5$ and for all *nonnegative* real numbers x_1, x_2, ..., x_n. Suppose that $n \geq 4$ and $(x_1 + x_2 + \cdots + x_n)^2 \geq 4(x_1x_2 + x_2x_3 + \cdots + x_nx_1)$, where $x_i \geq 0$ for each i. We are to show that

$$(x_1 + x_2 + \cdots + x_{n+1})^2 \ \geq \ 4(x_1x_2 + x_2x_3 + \cdots + x_{n+1}x_1)\,. \qquad (1)$$

Since both sides of (1) are invariant under cyclic permutations of the indices, we may assume, without loss of generality, that $x_1 \geq x_{n+1}$.

Then, by the induction hypothesis, we have

$$\begin{aligned}
& (x_1 + x_2 + \cdots + x_{n+1})^2 \ = \ (x_1 + x_2 + \cdots + (x_n + x_{n+1}))^2 \\
\geq \ & 4(x_1x_2 + x_2x_3 + \cdots + x_{n-1}(x_n + x_{n+1}) + (x_n + x_{n+1})x_1) \\
= \ & 4(x_1x_2 + x_2x_3 + \cdots + x_nx_{n+1} + x_{n+1}x_1) \\
& \qquad\qquad\qquad + 4x_{n-1}x_{n+1} + 4x_n(x_1 - x_{n+1}) \\
\geq \ & 4(x_1x_2 + x_2x_3 + \cdots + x_{n+1}x_1)\,.
\end{aligned}$$

This completes the induction and hence, the proof of our claim.

For part (b) we claim that the inequality holds only for $n = 4$.

That it fails for $n = 3$ and holds for $n = 4$ was shown in part (a). Hence, it remains to show that it fails for all $n \geq 5$. For convenience, we let A and B denote the left side and right side of the inequality, respectively.

If $n = 2k$ where $k \geq 2$, then we let $x_1 = 1$ for $i = 1, 2, \ldots, k$, and $x_i = -1$ for $i = k+1, k+2, \ldots, 2k = n$. Then $A = 0$, while on the other hand $B = 4[(n-2)-2] = 4(n-4) > 0$.

If $n = 2k+1$ where $k \geq 2$, then we let $x_1 = 1$ for $i = 1, 2, \ldots, k+1$, and $x_i = -1$ for $i = k+2, k+3, \ldots, 2k+1 = n$. Then $A = 1$, while on the other hand $B = 4[(2k-1)-2] = 4(2k-3) = 4(n-4) \geq 4$.

This completes the proof.

Next we open our file of solutions to problems posed in the March 2009 number of the *Corner*, beginning with a solution to a problem of the 19th Korean Mathematical Olympiad 2006 given at [2009 : 80–81].

4. Given three distinct real numbers a_1, a_2, and a_3, define three real numbers b_1, b_2, and b_3 as follows

$$b_j = \left(1 + \frac{a_j a_i}{a_j - a_i}\right)\left(1 + \frac{a_j a_k}{a_j - a_k}\right), \qquad \text{where} \qquad \{i,j,k\} = \{1,2,3\}.$$

Prove that

$$1 + |a_1 b_1 + a_2 b_2 + a_3 b_3| \leq \big(1 + |a_1|\big)\big(1 + |a_2|\big)\big(1 + |a_3|\big).$$

When does equality hold?

Solved by Michel Bataille, Rouen, France; and Oliver Geupel, Brühl, NRW, Germany. We give Bataille's write-up.

For convenience, let

$$c_1 = \frac{a_2 a_3}{a_2 - a_3}, \qquad c_2 = \frac{a_3 a_1}{a_3 - a_1}, \qquad c_3 = \frac{a_1 a_2}{a_1 - a_2};$$

so $b_1 = (1+c_3)(1-c_2)$, $b_2 = (1-c_3)(1+c_1)$, and $b_3 = (1+c_2)(1-c_1)$. Simple calculations then yield

$$\begin{aligned} &a_1 b_1 + a_2 b_2 + a_3 b_3 \\ &\quad = a_1 + a_2 + a_3 + a_1 a_2 + a_2 a_3 + a_3 a_1 - a_1 c_2 c_3 - a_2 c_3 c_1 - a_3 c_1 c_2 \end{aligned}$$

and

$$\begin{aligned} &-a_1 c_2 c_3 - a_2 c_3 c_1 - a_3 c_1 c_2 \\ &\quad = \frac{a_1 a_2 a_3 \cdot [a_1^2(a_3 - a_2) + a_2^2(a_1 - a_3) + a_3^2(a_2 - a_1)]}{(a_2 - a_3)(a_3 - a_1)(a_1 - a_2)} \\ &\quad = a_1 a_2 a_3 . \end{aligned}$$

Thus, using the Triangle Inequality,

$$\begin{aligned}
&1 + |a_1b_1 + a_2b_2 + a_3b_3| \\
&\quad = \; 1 + |a_1 + a_2 + a_3 + a_1a_2 + a_2a_3 + a_3a_1 + a_1a_2a_3| \\
&\quad \le \; 1 + |a_1| + |a_2| + |a_3| + |a_1a_2| + |a_2a_3| + |a_3a_1| + |a_1a_2a_3| \\
&\quad = \; 1 + |a_1| + |a_2| + |a_3| + |a_1||a_2| + |a_2||a_3| + |a_3||a_1| + |a_1||a_2||a_3| \\
&\quad = \; (1 + |a_1|)(1 + |a_2|)(1 + |a_3|)\,.
\end{aligned}$$

and the desired inequality follows.

Equality holds if and only if it holds where we applied the Triangle Inequality, that is, if and only if a_1, a_2, a_3, a_1a_2, a_2a_3, a_3a_1, $a_1a_2a_3$ all have the same sign. Due to the presence of the products a_1a_2, a_2a_3, a_3a_1, this sign is the positive one and equality occurs precisely when a_1, a_2, a_3 are nonnegative.

Now we turn to solutions from readers to problems of the Olympiade Suisse de mathématiques 2005, tour final, given at [2009 : 82–83].

3. Pour tout $a_1, \ldots, a_n > 0$, prouver l'inégalité suivante et déterminer tous les cas d'égalité

$$\sum_{k=1}^{n} ka_k \;\le\; \binom{n}{2} \;+\; \sum_{k=1}^{n} a_k^k\,.$$

Solutions and comments by Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comăneşti, Romania. We give Bataille's comment.

Ce probléme a déjà été proposé lors de la 52ième olympiade polonaise en 2001. Son énoncé et sa solution se trouvent dans ***CRUX with MAYHEM***: [2004 : 19; 2005 : 445].

6. Soient a, b, c des nombres réels positifs avec $abc = 1$. Déterminer toutes les valeurs que peut prendre la somme

$$\frac{1+a}{1+a+ab} \;+\; \frac{1+b}{1+b+bc} \;+\; \frac{1+c}{1+c+ca}\,.$$

Solved by George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; and Titu Zvonaru, Comăneşti, Romania. We give the write-up of Apostolopoulos.

Since $abc = 1$, we have

$$\frac{1+b}{1+b+bc} \;=\; \frac{1+b}{1+b+\frac{1}{a}} \;=\; \frac{a(1+b)}{1+a+ab}\,,$$

so

$$\frac{1+a}{1+a+ab} + \frac{1+b}{1+b+bc} = \frac{1+a+a(1+b)}{1+a+ab} = 1 + \frac{a}{1+a+ab}$$

and

$$\frac{1+c}{1+c+ca} = \frac{1+\frac{1}{ab}}{1+\frac{1}{ab}+\frac{1}{b}} = \frac{ab+1}{1+a+ab}\,.$$

Finally,

$$\begin{aligned} & \frac{1+a}{1+a+ab} + \frac{1+b}{1+b+bc} + \frac{1+c}{1+c+ca} \\ &= \left(1+\frac{a}{1+a+b}\right) + \frac{ab+1}{1+a+ab} \\ &= 1+\left(\frac{1+a+ab}{1+a+ab}\right) = 1+1 = 2\,. \end{aligned}$$

7. Soit $n \geq 1$ un nombre naturel. Déterminer toutes les solutions entières positives de l'équation

$$7 \cdot 4^n = a^2+b^2+c^2+d^2\,.$$

Solution by Michel Bataille, Rouen, France.

The following quadruples and their permutations yield all solutions:

$$\begin{aligned} &(2^n, 2^n, 2^n, 2^{n+1})\,, \\ &(2^{n-1}, 2^{n-1}, 2^{n-1}, 5 \cdot 2^{n-1})\,, \\ &(2^{n-1}, 3 \cdot 2^{n-1}, 3 \cdot 2^{n-1}, 3 \cdot 2^{n-1})\,. \end{aligned}$$

It is readily checked that each quadruple yields a solution. Conversely, let (a,b,c,d) be any solution. Then, $a^2+b^2+c^2+d^2 \equiv 0 \pmod 4$ and, since $x^2 \equiv 0$ or $1 \pmod 4$ according as the integer x is even or odd, we must have $a^2 \equiv b^2 \equiv c^2 \equiv d^2 \equiv 0 \pmod 4$ or $a^2 \equiv b^2 \equiv c^2 \equiv d^2 \equiv 1 \pmod 4$. In the latter case, a, b, c, d are odd, hence $a^2 \equiv b^2 \equiv c^2 \equiv d^2 \equiv 1 \pmod 8$, and we can set

$$a^2 = 1+8a_1\,, \quad b^2 = 1+8b_1\,, \quad c^2 = 1+8c_1\,, \quad d^2 = 1+8d_1\,,$$

where a_1, b_1, c_1, d_1 are nonnegative integers. From the equation

$$1+2(a_1+b_1+c_1+d_1) = 7 \cdot 4^{n-1}$$

we see that $n = 1$ and $a_1+b_1+c_1+d_1 = 3$. Up to permutations, (a_1,b_1,c_1,d_1) is one of the quadruples $(0,0,0,3)$, $(0,0,1,2)$, $(0,1,1,1)$, whence $(a,b,c,d) = (1,1,1,5)$ or $(1,3,3,3)$ (up to permutations).

Now, suppose that a, b, c, d are all even positive integers. We write them as

$$a = 2^{\alpha}t\,, \quad b = 2^{\beta}u\,, \quad c = 2^{\gamma}v\,, \quad d = 2^{\delta}w\,,$$

for some odd positive integers t, u, v, w and positive integers α, β, γ, δ for which we assume $\alpha \le \beta \le \gamma \le \delta$ without loss of generality. Then,

$$2^{2\alpha}(t^2 + 2^{2(\beta-\alpha)}u^2 + 2^{2(\gamma-\alpha)}v^2 + 2^{2(\delta-\alpha)}w^2) \;=\; 7 \cdot 2^{2n}\,.$$

If $\beta > \alpha$, then $\alpha = n$ and $t^2 + 2^{2(\beta-\alpha)}u^2 + 2^{2(\gamma-\alpha)}v^2 + 2^{2(\delta-\alpha)}w^2 = 7$, which is clearly impossible.

Thus, $\alpha = \beta$, and

$$2^{2\alpha}(t^2 + u^2 + 2^{2(\gamma-\alpha)}v^2 + 2^{2(\delta-\alpha)}w^2) \;=\; 7 \cdot 2^{2n}\,.$$

Since t^2+u^2 is twice an odd integer, we must have $\gamma = \alpha$ (otherwise the total exponent of 2 would be odd on the left and even on the right) and finally,

$$2^{2\alpha}(t^2 + u^2 + v^2 + 2^{2(\delta-\alpha)}w^2) \;=\; 7 \cdot 2^{2n}\,.$$

If $\delta = \alpha$, then $2^{2\alpha}(t^2 + u^2 + v^2 + w^2) = 7 \cdot 2^{2n}$, and hence $\alpha < n$ and $t^2 + u^2 + v^2 + w^2 = 7 \cdot 2^{2(n-\alpha)}$. By the previous results, $\alpha = n - 1$ and $(t,u,v,w) = (1,1,1,5)$ or $(1,3,3,3)$ (up to permutations), and thus $(a,b,c,d) = (2^{n-1}, 2^{n-1}, 2^{n-1}, 5\cdot 2^{n-1})$ or $(2^{n-1}, 3\cdot 2^{n-1}, 3\cdot 2^{n-1}, 3\cdot 2^{n-1})$ (up to permutations).

If $\delta > \alpha$, then $t^2 + u^2 + v^2 + 2^{2(\delta-\alpha)}w^2$ is odd, hence

$$2^{2\alpha}(t^2 + u^2 + v^2 + 2^{2(\delta-\alpha)}w^2) \;=\; 7 \cdot 2^{2n}\,.$$

In this last case, the only possibility is $t = u = v = 1$ and $2^{2(\delta-\alpha)}w^2 = 4$, so that $\delta = \alpha + 1$ and $w = 1$. It follows that $(a,b,c,d) = (2^n, 2^n, 2^n, 2^{n+1})$.

9. Trouver toutes les fonctions $f : \mathbb{R}^+ \to \mathbb{R}^+$, vérifiant la condition suivante pour tout x,$y > 0$:

$$f\big(yf(x)\big)(x+y) \;=\; x^2\big(f(x) + f(y)\big)\,.$$

Solution by Michel Bataille, Rouen, France.

The function $f(x) = \dfrac{1}{x}$ is the unique solution. To this aim, let f be a solution and let (E) denote the identity. Taking $y = x = 1$ in (E) yields $f(a) = a$ where $a = f(1)$. Taking $x = a$, $y = 1$ in (E) yields $2a^2 - a - 1 = 0$, and since $a > 0$ we have $a = 1$ and $f(1) = 1$. Finally, setting $x = 1$ in (E) with arbitrary y, we obtain $f(y)(1+y) = 1 + f(y)$, whence $f(y) = \dfrac{1}{y}$.

That completes the *Corner* for this issue. Send me your nice solutions and generalizations.

BOOK REVIEWS

Amar Sodhi

Mythematics: Solving the Twelve Labors of Hercules
By Michael Huber, Princeton University Press, Princeton and Oxford, 2009
ISBN 978-0-691-13575-5, hardcover, 183+xix pages, US$24.95
Reviewed by ***Edward Barbeau****, University of Toronto*

The twelve labours of the classical Greek hero, Hercules, were performed at the behest of Eurystheus in atonement for Hercules' murder of his children and those of his brother in a fit of madness. As they stand, these labours do not appear to offer much potential for mathematical elaboration, but the author uses them as a pretext to create a number of problems that can be given to mathematical undergraduates of the first two years.

Each of the twelve chapters opens with the description by the Greek author Apollodorus of one of the labours, followed by the statement and finally the solutions of two or three problems. Appendices relate problems and their mathematical areas, fill in the tale of Hercules up to the time of the labours, discuss various versions of the Hercules legend, and provide a brief primer on the Laplace transform.

For example, the twelfth labour is to bring to Eurystheus Cerberus, the three-headed dog that guards Hades. In the first of the two problems, Hercules is located at the point $(2, 1, -7)$ on the surface $z = -2x + y^2 - 2xy$ and has to make his descent to Hades. The reader is asked what the rate of descent is above the line $y = x$ and also for the direction of steepest descent; the solution of this uses a vector flow diagram. To capture Cerberus, Hercules must render the beast unconscious by sufficiently impeding the blood flow to all of the three heads by choking each of the three necks in turn. This is a problem of exponential decay in which the blood flow has to be reduced from 6 to below 2.5 ml per second; he has to contend with the slow restoration of the flow when each neck is released. The solution uses a spreadsheet. As a bonus, the reader is challenged by one of the three Sudoku puzzles in the book; this one has the novel feature of requiring that each row, column and block is filled by one each of the first six digits and three sevens.

Many of the problems in the book are familiar ones provided with a new and sometimes artificial setting: the shooting towards a target of an arrow subject to gravity, the Monty Hall (car and goats) problem, minimizing the time to cross a river to a point along the opposite bank, maximizing the angle subtended at the eye by a picture hanging on a wall, resonance of an undamped harmonic oscillator with a forcing function, logistic growth. Even though the book does not have a real spark, the treatment of the problems is solid and the sourcing of the myth is meticulous.

PROBLEMS

Solutions to problems in this issue should arrive no later than ***1 October 2010****. An asterisk* ($\star$) *after a number indicates that a problem was proposed without a solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

3527. *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let a, b, and c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$\sum_{\text{cyclic}}\left(a^2b+\frac{3}{2}\right)\left(b^2c+\frac{3}{2}\right) \;\le\; \frac{75}{4}\,.$$

3528. *Proposed by Hiroshi Kinoshita and Katsuhiro Yokota, Tokyo, Japan.*

The incircle of triangle ABC touches the sides BC, AC, AB at the points A', B', C', respectively. Let ρ, r_a, r_b, r_c denote the inradii of the circles $A'B'C'$, $AB'C'$, $BC'A'$, $CA'B'$, respectively, and let r be the inradius of the triangle ABC. Prove that

$$r \;=\; \frac{1}{2}(\rho+r_a+r_b+r_c)\,.$$

3529. *Proposed by Michel Bataille, Rouen, France.*

Let A be a point on a circle Γ with centre O and t be the tangent to Γ at A. Triangle POQ is such that P is on Γ, Q is on t, and $\angle POQ=90^\circ$. Find the envelope of the perpendicular to AP through Q as $\triangle POQ$ varies.

3530. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $f:[0,1]\to\mathbb{R}$ be an integrable function which is continuous at 1. Let k be a fixed positive integer, and let

$$a_n \;=\; \int_0^1 \frac{f(x)}{(1+x^n)(1+x^{n+k})}\,dx\,.$$

Find $L=\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} n(L-a_n)$.

3531. *Proposed by K.S Bhanu, Institute of Science, Nagpur, India, and M.N. Deshpande, Nagpur, India.*

Let a, b be positive integers. On the real line, A stands at $-a$ and B stands at b. A fair coin is tossed, and if it shows heads then A moves one unit to the right, while if it shows tails then B moves one unit to the left. The process stops when A or B reaches the origin.

Let $P_A(a,b)$ be the probability that A reaches the origin before B, and define $P_B(a,b)$ similarly. Prove that

$$E(a,b) \;=\; 2aP_A(a+1,b) + 2bP_B(a,b+1)\,,$$

where $E(a,b)$ is the expected number of tosses before the process terminates.

3532. *Proposed by Michel Bataille, Rouen, France.*

Let triangle ABC have circumradius R, inradius r, and let δ_a, δ_b, δ_c be the distances from the centroid to the sides BC, CA, AB, respectively. Prove that

$$r \;\le\; \frac{\sqrt{\delta_a}+\sqrt{\delta_b}+\sqrt{\delta_c}}{3} \;\le\; \sqrt{\frac{R}{2}}\,.$$

3533. *Proposed by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Let a, b, c be positive real numbers such that $a+b+c=1$. Let m and n be positive real numbers satisfying $6m \le 5n$. Prove that

$$\frac{ma+nbc}{b+c} + \frac{mb+nca}{c+a} + \frac{mc+nab}{a+b} \;\le\; \frac{3m+n}{2}\,.$$

3534. *Proposed by Mihály Bencze, Brasov, Romania.*

Let x_1, x_2, $\dots$, x_n be positive real numbers, where $n \ge 2$, and let $\alpha \ge 1$. Prove that

$$\begin{aligned}(n-1)^{\alpha-1}&\left(\sum_{k=1}^{n} x_k^{\alpha}\right)\left(\sum_{k=1}^{n} x_k\right)^{\alpha}\\ \ge\;\; 2^{\alpha}&\left(\sum_{1\le i<j\le n} x_i x_k\right)^{\alpha} + (n-1)^{\alpha-1}\left(\sum_{k=1}^{n} x_k^{\alpha+1}\right)\left(\sum_{k=1}^{n} x_k\right)^{\alpha-1}.\end{aligned}$$

3535. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let a, b, and c be positive real numbers and let $\alpha \ge 0$. Prove that

$$\left(\frac{a^2+bc}{b+c}\right)^{\alpha} + \left(\frac{b^2+ca}{c+a}\right)^{\alpha} + \left(\frac{c^2+ab}{a+b}\right)^{\alpha} \;\ge\; 3^{1-\alpha}(a+b+c)^{\alpha}\,.$$

3536. *Proposed by Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain.*

Find all positive integers n and k such that the equation $\{x^{2n}\} = \{x\}$ has 2010 roots inside the interval $[k, k+1)$, where $\lfloor x \rfloor$ is the greatest integer not exceeding x and $\{x\} = x - \lfloor x \rfloor$.

3537. *Proposed by Marian Marinescu, Monbonnot, France.*

Let $f : [0,1] \to \mathbb{R}$ be continuous, and let $g : [0,1] \to \mathbb{R}$ be monotonic and differentiable with $g(0) = 0$. Prove that there is a number $0 < a < 1$ such that

$$\int_0^a f(x)g(x)\,dx \;=\; \left(\int_0^1 f(x)\,dx\right)\left(\int_0^a g(x)\,dx\right)\,.$$

3538. *Proposed by Victor Oxman, Western Galilee College, Israel.*

In the plane you are given a triangle ABC with its internal angle bisector BD, a point E on the side BC such that ED is the bisector of angle AEC, and the circumcircle of the triangle ABC (but not its centre). Construct the centre of that circle using only a straightedge.

[*Ed.: The Poncelet–Steiner Theorem says that given a circle with its centre, we can carry out all the ruler-compass constructions in the plane of that circle by straightedge only. See Crux problems 2694, 2695, and 2696* [*2002 : 553-557*].]

. . . .

3527. *Proposé par Pham Kim Hung, étudiant, Université de Stanford, Palo Alto, CA, É-U.*

Soit a, b, et c trois nombres réel non-negatifs tel que $a+b+c=3$. Montrer que

$$\sum_{\text{cyclic}} \left(a^2b + \frac{3}{2}\right)\left(b^2c + \frac{3}{2}\right) \;\leq\; \frac{75}{4}\,.$$

3528. *Proposé par Hiroshi Kinoshita et Katsuhiro Yokota, Tokyo, Japon.*

Le cercle inscrit d'un triangle ABC touche respectivement les côtés BC, AC et AB aux points A', B' et C'. On dénote respectivement par ρ, r_a, r_b et r_c les rayons des cercles inscrits des triangles $A'B'C'$, $AB'C'$, $BC'A'$ et $CA'B'$, et soit r le rayon du cercle inscrit du triangle ABC. Montrer que

$$r \;=\; \frac{1}{2}(\rho + r_a + r_b + r_c)\,.$$

3529. *Proposé par Michel Bataille, Rouen, France.*

Soit A un point sur un cercle Γ de centre O et t la tangente à Γ en A. Soit POQ le triangle tel que P soit sur Γ, Q sur t et $\angle POQ = 90°$. Trouver l'enveloppe des perpendiculaires à AP passant par Q lorsque le triangle POQ varie.

3530. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Soit $f : [0,1] \to \mathbb{R}$ une fonction integrable, continue en 1. Soit k un entier positif fixé et soit

$$a_n = \int_0^1 \frac{f(x)}{(1+x^n)(1+x^{n+k})}\,dx\,.$$

Trouver $L = \lim\limits_{n\to\infty} a_n$ et $\lim\limits_{n\to\infty} n(L - a_n)$.

3531. *Proposé par K.S. Bhanu, Institut des Sciences, Nagpur, Inde, et M.N. Deshpande, Nagpur, Inde.*

Soit a et b deux entiers positifs. On place deux points sur la droite réelle, A en $-a$ et B en b. On lance une pièce de monnaie non pipée ; si elle tombe sur pile, on déplace B d'une unité vers la gauche, si elle tombe sur face, on déplace A d'une unité vers la droite. Le jeu s'arrête dès que l'un des points atteint l'origine.

Soit $P_A(a,b)$ la probabilité que A atteigne l'origine avant B et $P_B(a,b)$ celle du cas contraire. Montrer que

$$E(a,b) = 2aP_A(a+1,b) + 2bP_B(a,b+1)\,,$$

où $E(a,b)$ est le nombre de lancers attendu avant la fin du jeu.

3532. *Proposé par Michel Bataille, Rouen, France.*

Dans un triangle ABC, on note R le rayon du cercle circonscrit, r celui du cercle inscrit, et soit δ_a, δ_b et δ_c les distances respectives de son centre de gravité aux côtés BC, CA et AB. Montrer que

$$r \le \frac{\sqrt{\delta_a} + \sqrt{\delta_b} + \sqrt{\delta_c}}{3} \le \sqrt{\frac{R}{2}}\,.$$

3533. *Proposé par Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Soit a, b et c trois nombres réels positifs tels que $a+b+c=1$. Soit m et n deux nombres réels positifs tels que $6m \le 5n$. Montrer que

$$\frac{ma+nbc}{b+c} + \frac{mb+nca}{c+a} + \frac{mc+nab}{a+b} \le \frac{3m+n}{2}\,.$$

3534. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit $x_1, x_2, \ldots, x_n$, $n \geq 2$, des nombres réels positifs et $\alpha \geq 1$. Montrer que

$$(n-1)^{\alpha-1}\left(\sum_{k=1}^{n} x_k^{\alpha}\right)\left(\sum_{k=1}^{n} x_k\right)^{\alpha}$$
$$\geq\; 2^{\alpha}\left(\sum_{1\leq i<j\leq n} x_i x_k\right)^{\alpha} + (n-1)^{\alpha-1}\left(\sum_{k=1}^{n} x_k^{\alpha+1}\right)\left(\sum_{k=1}^{n} x_k\right)^{\alpha-1}.$$

3535. *Proposé par Walther Janous, Ursulinengymnasium, Innsbruck, Autriche.*

Soit a, b et c trois nombres réels positifs et $\alpha \geq 0$. Montrer que

$$\left(\frac{a^2+bc}{b+c}\right)^{\alpha} + \left(\frac{b^2+ca}{c+a}\right)^{\alpha} + \left(\frac{c^2+ab}{a+b}\right)^{\alpha} \;\geq\; 3^{1-\alpha}(a+b+c)^{\alpha}.$$

3536. *Proposé par Samuel Gómez Moreno, Université de Jaén, Jaén, Espagne.*

Trouver tous les entiers positifs n et k tels que l'équation $\{x^{2n}\} = \{x\}$ ait 2010 racines à l'intérieur de l'intervalle $[k, k+1)$, où $\lfloor x \rfloor$ est le plus grand entier n'excédant pas x et $\{x\} = x - \lfloor x \rfloor$.

3537. *Proposé par Marian Marinescu, Monbonnot, France.*

Soit $f : [0,1] \to \mathbb{R}$ continue, et $g : [0,1] \to \mathbb{R}$ monotone et différentiable avec $g(0) = 0$. Montrer qu'il existe un nombre $0 < a < 1$ tel que

$$\int_0^a f(x)g(x)\,dx \;=\; \left(\int_0^1 f(x)\,dx\right)\left(\int_0^a g(x)\,dx\right).$$

3538. *Proposé par Victor Oxman, Western Galilee College, Israël.*

Dans le plan, on considère un triangle ABC dont on donne la bissectrice intérieure BD, un point E sur le côté BC tel que ED soit la bissectrice de l'angle AEC, et enfin le cercle circonscrit de ABC, mais pas son centre. Avec la règle seulement, trouver le centre de ce cercle.

[*Ed. : Le théorème de Poncelet-Steiner dit qu'étant donné un cercle avec son centre, on peut exécuter toutes les constructions avec la règle et le compas dans le plan de ce cercle à l'aide de la règle seulement. Voir les problèmes de Crux 2694, 2695 et 2696* [*2002 : 553-557*].]

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3425. Corrigenda. *The following corrections should replace the corresponding text in* ***CRUX with MAYHEM*** *Vol. 36, No. 2 at the indicated place. Our apologies for these errors.*

p. 125, line 26: $\frac{1}{u} \leq \frac{e^u}{u} = \frac{e^v}{v} \leq \frac{e^{0.5}}{u}$

p. 125, line 31: $-(v-1)e^{-v} + \frac{1}{v} + (v-2)e^{\frac{1}{4}-\frac{v}{2}}$

p. 126, line 10: $1 - t - 2\sqrt{1-t} \geq \frac{-1-\ln(t^2+0.5t+1)}{t^2+0.5t+1}$

p. 126, line 14: $-(12t^3+2t^2+12t-11)$ in the numerator of $\frac{\Psi'(t)}{t}$, with corresponding changes on lines 18, 19, and 23.

p. 126, line 16: $11 - 12t^3 - 2t^2 - 12t \geq 0$

p. 126, line 20: $\frac{1}{3}(18t^2 - 13t + 3) > 0$

p. 126, line 24: $5(2t-1)(2t^2+t+2)-$

p. 126, line 25: $\begin{cases} 3.32 - 2.496 > 0\,, & \text{if } 0.6 \leq t \leq 0.7\,, \\ 7.36 - 6.024 > 0\,, & \text{if } 0.7 \leq t \leq 0.8\,. \end{cases}$

p. 127, line 6: $\varphi(0) = 0$

3426. [2009 : 172, 174] *Proposed by Salvatore Tringali, student, Mediterranea University, Reggio Calabria, Italy.*

Find all prime numbers p, q, and r such that $p+q=(p-q)^r$.

Solution by Harry Sedinger, St. Bonaventure University, St. Bonaventure, New York, USA.

Clearly, $p \neq q$. Assume $p > q$ and let $p - q = n$. The equation then becomes $n + 2q = n^r$, or

$$2q \;=\; n(n^{r-1}-1)\,.$$

If $r = 2$, then $2q = n(n-1)$, which has a solution with q prime only if $n = 3$; and then $q = 3$.

If $r = 3$, then $2q = n(n^2 - 1) = (n - 1)n(n + 1)$, so that $n = 2$, $q = 3$, $p = 5$ and we have a solution, $(p, q, r) = (5, 3, 3)$.

If $r > 3$, then

$$2q \;=\; n(n^{r-1} - 1) \;=\; n(n - 1)(n^{r-2} + \cdots + 1)\,,$$

which implies $n = 2$, and then $q = 2^{r-2} + \cdots + 1 = 2^{r-1} - 1 \geq 7$. Now, $q = 2^{r-1} - 1$, 2^{r-1}, and $p = q + 2 = 2^{r-1} + 1$ are three consecutive integers, so that one of them must be divisible by 3. Since 2^{r-1} is not divisible by 3, then either q or p is divisible by 3, and therefore not a prime. Thus there are no solutions with $r > 3$.

We have shown that, under the assumption $p > q$, the given equation has a solution only if $r = 3$. Now, if we assume $p < q$, then r must be even and we can rewrite the given equation as $q + p = (q - p)^r$. But we have already shown that this equation has a solution with $q > p$ only if $r = 3$. This shows that the solution found above is unique.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; CRISTINEL MORTICI, Valahia University of Târgoviște, Romania; TROY MULHOLLAND, student, St. Bonaventure University, New York, USA (2 solutions); MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; JOEL SCHLOSBERG, Bayside, NY, USA; DIGBY SMITH, Mount Royal College, Calgary, AB; ALBERT STADLER, Herrliberg, Switzerland; EDMUND SWYLAN, Riga, Latvia; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Note that the above argument does not rely on r being a prime. Barbara, Mortici, and the Missouri State University Problem Solving Group also submitted solutions that relaxed the restriction on r.

3427. [2009 : 172, 174] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

The numbers a, b, c, and d all lie in the interval $(1, \infty)$ and are such that $a + b + c + d = 16$. Prove that

$$\sum_{\text{cyclic}} \log_a\left(\sqrt[4]{bcd} + a\right) \;\geq\; \frac{11}{2}\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

Repeated application of Arithmetic Mean – Geometric Mean Inequality yields the following chain of inequalities:

$$\sum_{\text{cyclic}} \log_a\left(\sqrt[4]{bcd} + a\right) \geq \sum_{\text{cyclic}} \log_a\left(2(bcd)^{\frac{1}{8}} a^{\frac{1}{2}}\right)$$

$$= \sum_{\text{cyclic}} \left[\frac{\ln 2}{\ln a} + \frac{1}{8}\left(\frac{\ln b}{\ln a} + \frac{\ln c}{\ln a} + \frac{\ln d}{\ln a}\right) + \frac{1}{2}\right]$$

$$= \frac{1}{8}\left[\left(\frac{\ln a}{\ln b} + \frac{\ln b}{\ln a}\right) + \left(\frac{\ln a}{\ln c} + \frac{\ln c}{\ln a}\right) + \left(\frac{\ln a}{\ln d} + \frac{\ln d}{\ln a}\right) + \left(\frac{\ln b}{\ln c} + \frac{\ln c}{\ln b}\right) + \left(\frac{\ln b}{\ln d} + \frac{\ln d}{\ln b}\right) + \left(\frac{\ln c}{\ln d} + \frac{\ln d}{\ln c}\right)\right] + 2 + \ln 2 \sum_{\text{cyclic}} \frac{1}{\ln a}$$

$$\geq \frac{1}{8}(2+2+2+2+2+2) + 2 + \ln 2 \sum_{\text{cyclic}} \frac{1}{\ln a}$$

$$= \frac{7}{2} + \ln 2 \sum_{\text{cyclic}} \frac{1}{\ln a}\,.$$

By the Arithmetic Mean – Harmonic Mean Inequality we have

$$\sum_{\text{cyclic}} \frac{1}{\ln a} \geq \frac{16}{\sum_{\text{cyclic}} \ln a} = \frac{16}{\ln(abcd)} \geq \frac{16}{\ln\left(\frac{a+b+c+d}{4}\right)^4} = \frac{16}{4\ln 4} = \frac{2}{\ln 2}\,.$$

Thus,

$$\sum_{\text{cyclic}} \log_a\left(\sqrt[4]{bcd} + a\right) \geq \frac{7}{2} + \ln 2\left(\frac{2}{\ln 2}\right) = \frac{11}{2}\,.$$

By the condition for equality in the AM–GM–HM Inequality, equality holds if and only if $a = b = c = d = 4$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

Apostolopoulos and Barbara gave similar solutions, but using Jensen's inequality for $1/\ln(x)$ *instead of the AM–HM Inequality.*

3428★. [2009 : 172, 174] *Proposed by J. Walter Lynch, Athens, GA, USA.*

Fix an integer $n > 2$ and let I be the interval of all positive ratios r such that there exists an n-gon whose sides consist of n terms of a geometric sequence with common ratio r. Prove that the endpoints of I are reciprocals of each other.

[*Ed.*: The proposer refers to ***Crux*** M67 [2003 : 430-431] and 3082 [2006 : 477] for the special cases $n = 3$ and $n = 4$.]

Composite of solutions by Roy Barbara, Lebanese University, Fanar, Lebanon and Oliver Geupel, Brühl, NRW, Germany.

We will prove that there exists a real number ρ between $2 - \frac{2}{n}$ and 2 such that the range I of possible ratios r consists of those numbers in the open interval $\left(\frac{1}{\rho}, \rho\right)$.

By considering a regular n-gon, we see that $1 \in I$. Next, let $r \in I$; that is, we assume that there is an n-gon whose sides are a, ar, $\dots$, ar^{n-1} in some order. That same polygon has sides b, $\frac{b}{r}$, $\dots$, $\frac{b}{r^{n-1}}$ for $b = ar^{n-1}$. This observation shows that $r \in I$ if and only if $\frac{1}{r} \in I$. It remains to show that for $r > 1$, the values of r in I form a finite open interval. A necessary and sufficient condition that a, ar, $\dots$, ar^{n-1} be the sides of an n-gon (when $a > 0$ and $r > 1$) is that

$$ar^{n-1} \; < \; a + ar + \cdots + ar^{n-2} \,.$$

This is equivalent to

$$r^{n-1} \; < \; \frac{r^{n-1} - 1}{r - 1} \,,$$

and, therefore, to $f(r) = r^n - 2r^{n-1} + 1 < 0$. Because the derivative of $f(r)$ is zero when $r = 2 - \frac{2}{n}$, negative to the left, and positive to the right, $r^n - 2r^{n-1} + 1$ strictly decreases on the interval $\left[1, 2 - \frac{2}{n}\right]$, and strictly increases when $r > 2 - \frac{2}{n}$. Because $f(1) = 0$, we deduce that $f(2 - \frac{2}{n}) < 0$. Because $f(2) = 1 > 0$, there must be a unique value of r, call it ρ, between $2 - \frac{2}{n}$ and 2 where $f(\rho) = 0$. It follows immediately that I must be the open interval $\left(\frac{1}{\rho}, \rho\right)$, whose endpoints are reciprocals of one another, as claimed.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; and ALBERT STADLER, Herrliberg, Switzerland.

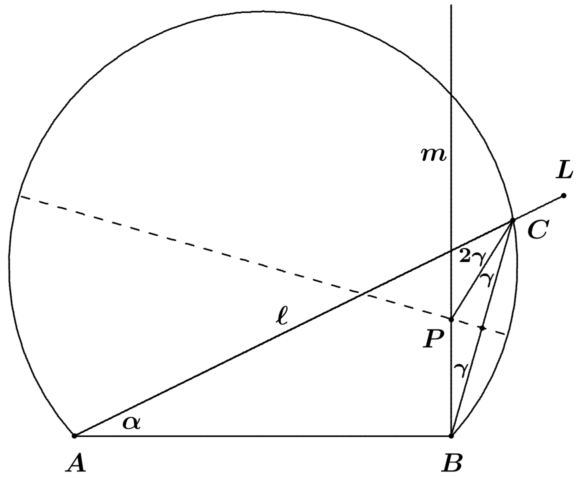
3429. [2009 : 172, 175] *Proposed by Václav Konečný, Big Rapids, MI, USA.*

The line ℓ passes through the point A and makes an acute angle with the segment AB. The line m passes through B and is perpendicular to AB. Construct a point C on the line ℓ and a point P on the line m such that the triangle BPC is isosceles with $BP = PC$ and

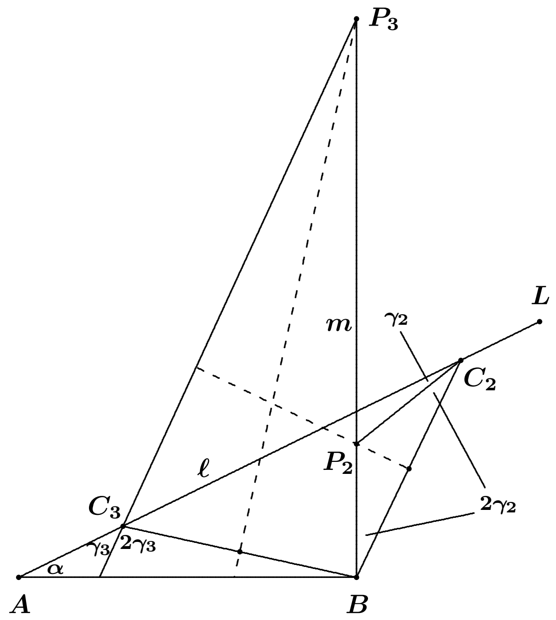
(a) the line CP trisects $\angle BCA$,

(b) the line CP bisects $\angle BCA$.

Composite of solutions by Michel Bataille, Rouen, France and by Oliver Geupel, Brühl, NRW, Germany.



(a) Fix a point L on ℓ such that $\alpha = \angle BAL$ is acute; for each position of C on AL define P to be the intersection of m with the perpendicular bisector of BC (to satisfy the requirement that P lies on m and $BP = PC$), and set $\gamma = \frac{1}{3}\angle BCA$. We shall first establish that when $\alpha < 45^\circ$, there are three positions of C on AL for which CP trisects $\angle BCA$. Clearly A cannot lie between L and such a point C; nor can $\angle CBA$ be too large (that is, it cannot exceed $135^\circ - \frac{\alpha}{2}$). In both these cases the entire line CP, except for the point C, would lie outside $\triangle ABC$, hence could not trisect an interior angle. Consider the case when A and C are on opposite sides of m, and C satisfies $\angle CBA = 90^\circ + \gamma$; that is, $\angle PCB = \gamma$. Using the angles of $\triangle ABC$, namely α, $90^\circ + \gamma$, and 3γ, we determine that $\gamma = 22.5^\circ - \frac{\alpha}{4}$. The second position of C, also separated from A by m, satisfies $\angle CBA = 90^\circ + 2\gamma$; that is, $\angle PCB = 2\gamma$, and $\gamma = 18^\circ - \frac{\alpha}{5}$. Finally, C can be on the same side of m as A: here $\angle CBA = 2\gamma - 90^\circ$, and $\gamma = 54^\circ - \frac{\alpha}{5}$; it follows that we must have $\alpha < 45^\circ$ in order that $\angle CBA = 2\gamma - 90^\circ = 18^\circ - \frac{2\alpha}{5} > 0$.



We now turn to the constructions. For the first position of C we construct on the same side of AB as L the circular arc that consists of all points X for which $\angle AXB = \frac{3}{4}(90^\circ - \alpha)$. This arc meets the ray AL at the desired point C. To see why, note that in triangle BCP, $BP = PC$ as required; moreover, the angles at B and C are equal. By construction, $\angle ACB = \frac{3}{4}(90^\circ - \alpha)$, whence $\angle CBA = \frac{5}{4}90^\circ - \frac{\alpha}{4} > 90^\circ$, and the acute angle between BC and m, namely $\angle CBP$, satisfies

$$\angle CBP = \frac{5}{4}90^\circ - \frac{\alpha}{4} - 90^\circ = 22.5^\circ - \frac{\alpha}{4} = \frac{1}{3}\angle BCA\,.$$

Since $\angle CBP = \angle BCP$, it follows that CP trisects $\angle BCA$ as claimed.

A similar construction works for the other two positions of C in the sense that we draw the arc of an appropriate circle subtended by the chord AB — in the second case AB subtends an angle of $54^\circ - \frac{3\alpha}{5}$, while in the third case the angle subtended is $162^\circ - \frac{3\alpha}{5}$ — and we define C to be the point where the circle meets AL. Note, however, that the construction for the first position of C requires dividing the angle $90^\circ - \alpha$ into four equal pieces, which can easily be achieved by ruler and compass. By contrast, the construction for the remaining two positions of C requires dividing $90^\circ - \alpha$ and $270^\circ - \alpha$ into five equal pieces, hence in general it cannot be achieved using the traditional Euclidean tools.

(b) There is exactly one position of C on AL for which there exists a point P on m such that $BP = PC$ and CP bisects $\angle BCA$; the point C must satisfy $\angle CBA = 120^\circ - \frac{\alpha}{3}$. For the construction, on the same side of AB as L draw the circular arc that consists of all points X for which $\angle AXB = \frac{2}{3}(90^\circ - \alpha)$. This arc meets the ray AL at the desired point C, and again we define P to be the point where the perpendicular bisector of the chord BC meets m so that $BP = PC$ and the angles at B and C are equal. Moreover, since $\angle CBA = 120^\circ - \frac{\alpha}{3}$, we have

$$\angle PCB = \angle CBP = 120^\circ - \frac{\alpha}{3} - 90^\circ = 30^\circ - \frac{\alpha}{3} = \frac{1}{2}\angle AXB = \frac{1}{2}\angle ACB\,.$$

In other words, CP bisects $\angle BCA$, as desired. Here again, the construction cannot generally be accomplished with only ruler and compass (because of the necessity of trisecting the angle $90^\circ - \alpha$). It is amusing to observe that when the tools are restricted to ruler and compass, the bisection (in part (b)) is not possible while a trisection (in part (a)) is possible.

Also solved by RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; EDMUND SWYLAN, Riga, Latvia; and the proposer.

3430. [2009 : 173, 175] *Proposed by Michel Bataille, Rouen, France.*

Let n be a positive integer. Determine the coefficients of the unique polynomial $P_n(x)$ for which the relation

$$\cos^{2n}\theta + \sin^{2n}\theta \;=\; P_n\left(\sin^2(2\theta)\right)$$

holds for all real numbers θ.

Similar solutions by Joel Schlosberg, Bayside, NY, USA and Paul Deiermann, Southeast Missouri State University, Cape Girardeau, MO, USA.

We shall show that

$$P_n(x) \;=\; \sum_{l=0}^{\lfloor n/2\rfloor} 2^{-2l}\binom{n-l}{l}\left(\frac{n}{n-l}\right)(-1)^l x^l\,.$$

Let $x = \sin^2 2\theta$, so that $1 - x = \cos^2 2\theta$ and $0 \le x \le 1$ for all θ. The double-angle formulae along with $\cos 2\theta = \pm\sqrt{1-x}$ yield

$$\cos^2\theta = \frac{1 \pm \sqrt{1-x}}{2} \qquad \text{and} \qquad \sin^2\theta = \frac{1 \mp \sqrt{1-x}}{2}\,.$$

Using the Binomial Theorem, we see that

$$\begin{aligned}
P_n(x(\theta)) &= \cos^{2n}\theta + \sin^{2n}\theta \\
&= 2^{-n}\left([1 \pm \sqrt{1-x}]^n + [1 \mp \sqrt{1-x}]^n\right) \\
&= 2^{-n}\sum_{k=0}^{n}\binom{n}{k}(1-x)^{k/2}[(\pm 1)^k + (\mp 1)^k] \\
&= 2^{-(n-1)}\sum_{\substack{k=0 \\ k \text{ even}}}^{2\lfloor n/2\rfloor}\binom{n}{k}(1-x)^{k/2} \\
&= 2^{-(n-1)}\sum_{q=0}^{\lfloor n/2\rfloor}\binom{n}{2q}(1-x)^q \\
&= 2^{-(n-1)}\sum_{q=0}^{\lfloor n/2\rfloor}\binom{n}{2q}\sum_{l=0}^{q}\binom{q}{l}(-1)^l x^l \\
&= 2^{-(n-1)}\sum_{l=0}^{\lfloor n/2\rfloor}\sum_{q=l}^{\lfloor n/2\rfloor}\binom{n}{2q}\binom{q}{l}(-1)^l x^l\,.
\end{aligned}$$

Next, we use the known "Moriarty" formula (see formula (1.2) of [1])

$$\sum_{k=j}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{k}{j} \;=\; 2^{n-2j-1}\binom{n-j}{j}\left(\frac{n}{n-j}\right)$$

to complete the proof:

$$P_n(x) = 2^{-(n-1)} \sum_{l=0}^{\lfloor n/2\rfloor} \sum_{q=l}^{\lfloor n/2\rfloor} \binom{n}{2q}\binom{q}{l}(-1)^l x^l$$
$$= \sum_{l=0}^{\lfloor n/2\rfloor} 2^{-2l}\binom{n-l}{l}\left(\frac{n}{n-l}\right)(-1)^l x^l .$$

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ALBERT STADLER, Herrliberg, Switzerland; and the proposer

Deiermann also gave the following recursion for the polynomial in the problem:

$$P_n(x) = 1 - \sum_{k=1}^{\lfloor n/2\rfloor} \binom{n}{k} 2^{-2k} x^k P_{n-2k}(x) .$$

References

[1] M. Shattuck, Combinatorial Proofs of Some Moriarty-Type Binomial Coefficient Identities, *Integers: Electronic Journal of Combinatorial Number Theory* **6** (2006), No. A35, p. 1 http://www.emis.de/journals/INTEGERS/papers/g35/g35.pdf

3431. *Proposed by Michel Bataille, Rouen, France.*

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function that satisfies

$$f(x+y) = f\bigl(f(x)\cdot f(y)\bigr)$$

for all real numbers x and y. Prove that f is constant.

Solution by Charles R. Diminnie, Angelo State University, San Angelo, TX, USA.

If $f(0) = 0$, then for all real x we have

$$f(x) = f(x+0) = f(f(x)\cdot f(0)) = f(0) = 0 .$$

and f is constant.

If $f(0) \neq 0$, then $a = f(0)^2 > 0$, and

$$f(a) = f(f(0)\cdot f(0)) = f(0+0) = f(0) .$$

Since f is continuous on $[0, a]$, f attains a maximum and a minimum on $[0, a]$. Furthermore, $f(0) = f(a)$ implies that at least one of these values is attained at a point $b \in (0, a)$. Without loss of generality, we can assume that f has a maximum at b.

Let $c = \min\{a - b,\, b\}$.

Let $x \in [0, c]$. Define $g : [b - x, b] \to \mathbb{R}$ by

$$g(t) = f(t+x) - f(t) .$$

By the definition of c we have $0 \leq b - x \leq b + x \leq a$, thus

$$g(b-x) \;=\; f(b) - f(b-x) \;\geq\; 0\,,$$

and

$$g(b) \;=\; f(b+x) - f(b) \;\leq\; 0\,.$$

Since g is continuous on $[b-x, b]$, by the Intermediate Value Theorem there exists some $y \in [b-x, b]$ such that $g(y) = 0$, or $f(y) = f(x+y)$.

It follows that

$$\begin{aligned} f(x) &= f((x+y)-y) \;=\; f(f(x+y)\cdot f(-y)) \\ &= f(f(y)\cdot f(-y)) \;=\; f(y-y) \;=\; f(0)\,, \end{aligned}$$

hence, $f(x) = f(0)$ for all $x \in [0, c]$.

Finally, for all real numbers x we have

$$\begin{aligned} f(x) &= f(x+0) \;=\; f(f(x)\cdot f(0)) \\ &= f(f(x)\cdot f(c)) \;=\; f(x+c)\,. \end{aligned}$$

Thus f is periodic with period c. Since f is constant on $[0, c]$, and periodic with period c, it follows that f is constant on all of $\mathbb{R}$.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA (second solution); OLIVER GEUPEL, Brühl, NRW, Germany; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer. One incorrect solution was submitted.

The continuity of f is needed, as the following example by Nicolae Strungaru shows. Let $\{1, \sqrt{2}\} \cup B$ be a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$. Each $x \in \mathbb{R}$ then has a unique expression $x = r_x + s_x\sqrt{2} + \sum_{b\in B} x_b b$, for suitable r_x, s_x, x_b in $\mathbb{Q}$. If $f(x) = 1 + s_x\sqrt{2}$, then f satisfies the identity of our problem but f is not constant.

3432. *Proposed by Michel Bataille, Rouen, France.*

Let a, b, and c be real numbers satisfying $a < 2(b+c)$, $b < 2(c+a)$, and $c < 2(a+b)$. Prove that

$$3 \;\leq\; \frac{4\left(a^3+b^3+c^3\right)+15abc}{(a+b+c)(ab+bc+ca)} \;<\; 6\,.$$

Solution by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.

First we prove that $a+b+c$ and $ab+bc+ca$ are positive quantities.

Adding the three constraint inequalities yields $a+b+c < 4(a+b+c)$, hence $0 < 3(a+b+c)$ and $a+b+c$ is positive.

To prove that $ab + bc + ca > 0$, we may assume $a = \max\{a, b, c\}$. Since $a + b + c > 0$, we have $a > 0$. Now we obtain

$$\begin{aligned} ab + bc + ca &= a(b + c) + bc \\ &> a(a - b - c) + bc \\ &= (a - b)(a - c) \ \geq \ 0\,, \end{aligned}$$

and hence $ab + bc + ca$ is positive.

By multiplying by $(a + b + c)(ab + bc + ca)$ and collecting terms, we see that the given inequality is equivalent to the following two inequalities:

$$-4\left(a^3 + b^3 + c^3\right) + 6(a + b + c)(ab + bc + ca) - 15abc \ > \ 0\,, \quad (1)$$
$$4\left(a^3 + b^3 + c^3\right) - 3(a + b + c)(ab + bc + ca) + 15abc \ \geq \ 0\,. \quad (2)$$

After some algebraic work, the left side of (1) can be rewritten to obtain

$$(2b + 2c - a)(2c + 2a - b)(2a + 2b - c) \ > \ 0\,,$$

which is clearly true due to the hypotheses.

Further algebraic work shows that the left side of (2) can be rewritten to obtain

$$(a - b)^2(2a + 2b - c) + (b - c)^2(2b + 2c - a) + (c - a)^2(2c + 2a - b) \geq 0\,,$$

which is also true because of the hypotheses.

The inequality is proved.

Equality holds if and only if $a = b = c$, with a positive common value.

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; and the proposer. There were two incomplete solutions submitted.

The proposer remarked that some variables can be negative, for instance $a = -1$, $b = 4$, and $c = 4$ satisfy the constraints.

3433. [2009 : 173, 175] *Proposed by an unknown proposer.*

For each positive integer n prove that

$$\sum_{k=0}^{n} \frac{1}{2k+1}\binom{2k}{k}\binom{2n-2k}{n-k} \ = \ \frac{2^{4n}}{2n+1}\binom{2n}{n}^{-1}.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

Our solution employs generating functions.

The Maclaurin series below are well known (see for example 1.641, #2 and 1.515, #4 in *Table of Integrals, Series, and Products* by I.S. Gradshteyn and I.M. Ryzhik, Academic Press, 5th edition, 1996):

$$\sinh^{-1} x = \ln\left(x + \sqrt{1+x^2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n(2n+1)}\binom{2n}{n} x^{2n+1};$$

$$\frac{\sinh^{-1} x}{\sqrt{1+x^2}} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{2n+1}\binom{2n}{n}^{-1} x^{2n+1};$$

where both series expansions are valid for $x^2 \le 1$.

Let $f(x) = \sinh^{-1} x$ for $x^2 \le 1$. Then we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}\left(\sum_{k=0}^{n} \frac{1}{2k+1}\binom{2k}{k}\binom{2n-2k}{n-k}\right) x^{2n+1}$$

$$= \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n(2n+1)}\binom{2n}{n} x^{2n+1}\right)\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}\binom{2n}{n} x^{2n}\right)$$

$$= f(x)f'(x) = \frac{\sinh^{-1} x}{\sqrt{1+x^2}} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{2n+1}\binom{2n}{n}^{-1} x^{2n+1}.$$

Comparing the coefficients of the terms x^{2n+1} yields the identity. The calculations are justified by the convergence of the power series for $x^2 \le 1$.

Also solved by MICHEL BATAILLE, Rouen, France (two solutions); PAUL BRACKEN, University of Texas, Edinburg, TX, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

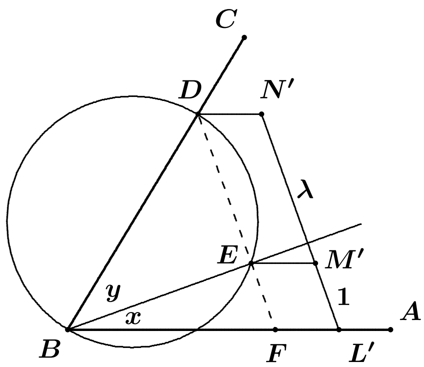
Bataille pointed out that this problem is equivalent to problem 11356 of the American Mathematical Monthly (*Vol. 115, No. 4, April, 2008, solution to appear*). [*Ed: If we multiply the identity in problem 11356 by* $\binom{2n}{n}$*, then we obtain our problem. It is not clear whether these two problems were submitted by the same proposer.*] *Janous located the identity as item 4.2.6.17 on p. 623 of* Integrals and Series (Elementary Functions)*, Nauka, Moscow, 1981 (in Russian).*

Stan Wagon remarked that by applying `Mathematica` *and entering the summation in the proposed identity, the answer on the right side comes out immediately.*

3434. [2009 : 173, 175] *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Given the line segment LMN with $LM : MN = 1 : \lambda$ and $\lambda > 0$, and given the triangle ABC with $\angle ABC = x + y$ and $\dfrac{\tan x}{\tan y} = \dfrac{1}{\lambda}$, construct the angle x using only a straight edge and compass.

I. *Solution by Václav Konečný, Big Rapids, MI, USA.*

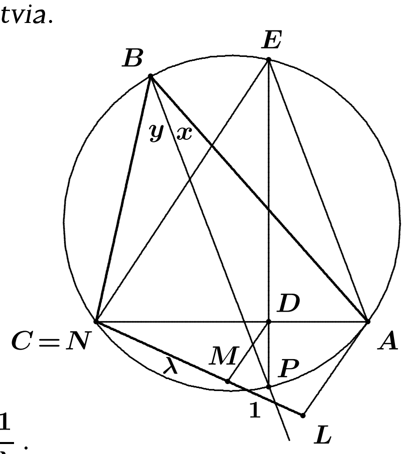


Let L' be a convenient point on the halfline BA that defines $\angle ABC$, and let N' be some point within that angle for which $L'N' = LN$, and M' be the point on the segment $L'N'$ for which $L'M' = LM$. Let the parallel to AB through N' meet the halfline BC at D. Construct the circle with diameter BD and call it Γ. Let the parallel to BA through M' meet Γ for the first time at the point E. Claim: $\angle ABE$ is the desired angle x. To see why, let F be at the intersection of the lines DE and BA. Because $L'F$, $M'E$, and $N'D$ are parallel segments, we have $FE : ED = L'M' : M'N' = 1 : \lambda$. Define $x = \angle ABE$ and $y = \angle EBC$. Because BD is a diameter of Γ, BE is perpendicular to DE. In the right triangles BEF and BED we have

$$\frac{\tan x}{\tan y} = \frac{FE}{BE}\Big/\frac{ED}{BE} = \frac{FE}{ED} = \frac{1}{\lambda},$$

as desired. By definition, $\angle ABC = x + y$. Note that the triangle was not needed for the construction—the vertices A and C were used only to name the given angle at B.

II. *Solution by Edmund Swylan, Riga, Latvia.*



Place the segment LN so that N coincides with C and L lies within $\angle ABC$ but not on AC. Denote by D the point where the parallel to AL through M meets AC. Then we have $AD : DC = 1 : \lambda$. Draw the circumcircle of $\triangle ABC$, and draw the chord EP through D perpendicular to AC with E on the same side of AC as B. In the right triangles DEA and CED we have

$$\frac{\tan\angle DEA}{\tan\angle CED} = \frac{AD}{DE}\Big/\frac{DC}{DE} = \frac{AD}{DC} = \frac{1}{\lambda}.$$

But, $\angle PBA = \angle PEA = \angle DEA$, while $\angle CBP = \angle CED$, whence BP is the desired halfline that partitions $\angle ABC$ into the angles $x = \angle ABP$ and $y = \angle PBC$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

3435. [2009 : 173, 176] *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Let a, b, c, and d be positive integers. Prove that

$$\frac{1}{a+b+c+d+2} - \frac{1}{(a+1)(b+1)(c+1)(d+1)} \le \frac{5}{48}\,.$$

Solution by Albert Stadler, Herrliberg, Switzerland.

Let $s = a+b+c+d$. For $s > 7$ the inequality follows from

$$\frac{1}{a+b+c+d+2} - \frac{1}{(a+1)(b+1)(c+1)(d+1)} \le \frac{1}{10} < \frac{5}{48}\,.$$

By assumption each of a, b, c, d is at least 1, and we now consider the case where $s \le 7$. By the AM–GM Inequality,

$$\begin{aligned}(a+1)(b+1)(c+1)(d+1) &\le \left(\frac{a+1+b+1+c+1+d+1}{4}\right)^4 \\ &= \left(1+\frac{a+b+c+d}{4}\right)^4 \\ &= \left(1+\frac{s}{4}\right)^4\,.\end{aligned}$$

Then for $s = 5$, 6, 7 we have

$$\begin{aligned}\frac{1}{a+b+c+d+2} &- \frac{1}{(a+1)(b+1)(c+1)(d+1)} \\ &\le \frac{1}{s+2} - \frac{1}{\left(1+\frac{s}{4}\right)^4} \\ &< \frac{5}{48}\,.\end{aligned}$$

If $a = b = c = d = 1$, then $s = 4$ and equality holds.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; CRISTINEL MORTICI, Valahia University of Târgovişte, Romania; JOEL SCHLOSBERG, Bayside, NY, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

Both Geupel and Zvonaru proved that the inequality holds for nonnegative integers. Hess observed that the inequality fails for $a = b = c = d = 1.1$, hence the condition that all variables be integers cannot be relaxed.

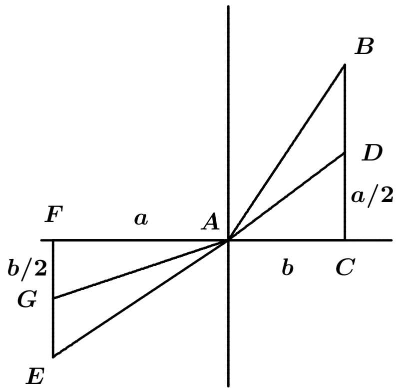
3436. [2009 : 174, 176] *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Let ABC be a right-angled triangle with hypotenuse AB. Let m_a and m_b be the lengths of the medians to the sides BC and AC, respectively. Prove that

$$\frac{\sqrt{5}}{2} \le \frac{m_a + m_b}{a+b} < \frac{3}{2}\,.$$

Solution by Missouri State University Problem Solving Group, Springfield, MO, USA.

Introduce a coordinate system with vertex A at the origin and AC along the x-axis. The vertices of the triangle are then $A(0,0)$, $B(b,a)$, and $C(b,0)$; with $D\left(b,\frac{a}{2}\right)$ the midpoint of side BC. Introduce the points $E(-a,-b)$ and $F(-a,0)$; then $\triangle EAF$ is congruent to $\triangle ABC$. Furthermore, $G\left(-a,-\frac{b}{2}\right)$ is the midpoint of EF, so $m_a = AD$ and $m_b = AG$. By the triangle inequality in $\triangle ACD$, we have $AD < AC + CD = b + \frac{a}{2}$, and likewise the triangle inequality in $\triangle AFG$ yields $AG < AF + FG = a + \frac{b}{2}$. Hence,



$$m_a + m_b < \frac{3}{2}(a+b)\,.$$

Similarly, the triangle inequality applied to $\triangle ADG$ yields

$$AD + AG \ge DG = \sqrt{(a+b)^2 + \left(\frac{a+b}{2}\right)^2}\,,$$

and therefore,

$$m_a + m_b \ge \frac{\sqrt{5}}{2}(a+b)\,,$$

which completes the proof.

Also solved by ARKADY ALT, San Jose, CA, USA; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; MICHEL BATAILLE, Rouen, France; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Big Rapids, MI, USA (2 solutions); KEE-WAI LAU, Hong Kong, China; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; SALEM

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3437. [2009 : 174, 176] *Proposed by Pham Huu Duc, Ballajura, Australia and Vo Quoc Ba Can, Can Tho University of Medicine and Pharmacy, Can Tho, Vietnam.*

Let a, b, and c be positive real numbers. Prove that

$$\sqrt{\frac{a^2+bc}{b+c}} + \sqrt{\frac{b^2+ca}{c+a}} + \sqrt{\frac{c^2+ab}{a+b}} \geq \sqrt{3(a+b+c)}\,.$$

Solution by Titu Zvonaru, Comăneşti, Romania.

[*Ed: Throughout we shall use* $\sum$ *for* $\sum\limits_{\text{cyclic}}$].

By the Generalized Hölder Inequality, we easily deduce that

$$\left(\sum\sqrt{\frac{a^2+bc}{b+c}}\right)\left(\sum\sqrt{\frac{a^2+bc}{b+c}}\right)\left(\sum(a^2+bc)^2(b+c)\right) \geq \left(\sum(a^2+bc)\right)^3. \qquad (1)$$

Hence, it suffices to prove that

$$\left(\sum(a^2+bc)\right)^3 \geq 3(a+b+c)\left(\sum(a^2+bc)^2(b+c)\right). \qquad (2)$$

By tedious but straightforward computations, we find that (2) is equivalent, in succession, to

$$\left(\sum(a^2+bc)\right)^3 \geq 3\sum(a^2+bc)^2(b^2+c^2+ab+ac+2bc)\,;$$

$$\begin{aligned}\sum(a^2+bc)^3 &+ 3\sum(a^2+bc)^2(b^2+ca)\\ &+3\sum(a^2+bc)(b^2+ca)^2 + 6(a^2+bc)(b^2+ca)(c^2+ab)\\ &\geq 3\sum(a^2+bc)^2(b^2+ca+c^2+ab) + 6\sum(a^2+bc)^2bc\,;\end{aligned}$$

$$\begin{aligned}\sum[(a^2+bc)^3-(a^2+bc)^2bc] &+ 6(a^2+bc)(b^2+ca)(c^2+ab)\\ &\geq 5\sum(a^2+bc)^2bc\,;\end{aligned}$$

$$\frac{\sum a^2(a^2+bc)^2 + 12a^2b^2c^2 + 6\sum a^3b^3 + 6\sum a^4bc}{}$$

$$\geq 5\sum(a^2+bc)^2bc\,;$$

$$15a^2b^2c^2 + \sum a^6 + 8\sum a^4bc + 6\sum a^3b^3$$

$$\geq 5\sum a^4bc + 10\sum a^2b^2c^2 + 5\sum b^3c^3\,;$$

$$\sum a^6 + 3\sum a^4bc + \sum a^3b^3 \geq 15a^2b^2c^2\,.$$

The last inequality above is true, since by the AM–GM Inequality we have $\sum_{\text{cyclic}} a^6 \geq 3a^2b^2c^2$, $\sum_{\text{cyclic}} a^4bc \geq 3a^2b^2c^2$, and $\sum_{\text{cyclic}} a^3b^3 \geq 3a^2b^2c^2$.

The required inequality now follows from (1) and (2).

Note that equality holds if and only if $a = b = c$.

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

3438★. *Proposed by Vo Quoc Ba Can, Can Tho University of Medicine and Pharmacy, Can Tho, Vietnam.*

Let a, b, and c be nonnegative real numbers. Prove the inequality below for all $\kappa \geq 0$, or give a counterexample:

$$\sum_{\text{cyclic}} \sqrt{\frac{a^2+\kappa bc}{b^2+c^2}} \;\geq\; 2 + \sqrt{\frac{\kappa}{2}}\,.$$

Solution by Albert Stadler, Herrliberg, Switzerland.

We prove that the statement is false for $\kappa > 32$.

Let $a = 0$ and $x = \dfrac{b}{c}$, where $b \neq c$. Then the inequality reads

$$\sqrt{\frac{\kappa x}{x^2+1}} + x + \frac{1}{x} \;\geq\; 2 + \sqrt{\frac{\kappa}{2}}\,,$$

which is equivalent to

$$\kappa \;\leq\; \left(\frac{x + \dfrac{1}{x} - 2}{\sqrt{\dfrac{1}{2}} - \sqrt{\dfrac{x}{x^2+1}}}\right)^2,$$

which in turn is equivalent to

$$\kappa \;\leq\; \frac{4(x^2+1)^2}{x^2}\left(\sqrt{\frac{1}{2}} + \sqrt{\frac{x}{x^2+1}}\right)^2 \;=\; \frac{2(x^2+1)}{x}\left(\sqrt{\frac{x^2+1}{x}} + \sqrt{2}\right)^2.$$

The function $f(t) = \frac{2(t^2+1)}{t}\left(\sqrt{\frac{t^2+1}{t}} + \sqrt{2}\right)^2$ attains its minimum at $t=1$, and that minimum value is 32.

So, if $\kappa > 32$, then a triple (a, b, c) exists (with x sufficiently close to 1) that violates the inequality.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; and RICHARD I. HESS, Rancho Palos Verdes, CA, USA.

Stadler also proved that the inequality holds for $\kappa = 0$. He commented that numerical evidence suggests that the inequality holds true for $0 \le \kappa \le 32$.