Contributor Profiles: Christopher J. Bradley

As a child, Bradley displayed signs of being mathematically precocious, disconcerting some visitors at a very early age by being able to read a clock face and telling them when they should leave in order to catch the bus home, and boring everyone in sight with mathematical amusements. Playing trains was always accompanied by reference to the timetables in Bradshaw. His trains, unlike those on the national networks, were always punctual. He could also play an adventurous and unorthodox hand of bridge at the age of five.

Bradley gained entry to Clare College, Cambridge at age 16, but on good advice delayed going for a year. In his final year, Bradley won the Mayhew University Prize. He migrated to Oxford for his D.Phil., and in 1963 was elected to a Fellowship at Jesus College. His early research was on the representation theory of groups.

It was a rich and secure life in Oxford in those days, but he preferred teaching to research, and in 1977 Bradley took the unusual step of leaving university life to become a schoolteacher. He feels he has been fortunate to have had two careers, both of which he enjoyed enormously.

Eventually, he became Head of Mathematics, and then Deputy Head (Curriculum) at Clifton College, Bristol, and during this period he was a Chief Examiner for International Baccalaureate. He then became involved with the Olympiad world, specializing in geometry and to a lesser extent in number theory. He had the good fortune (so he describes it) in the early 1990s of being introduced to ***Crux Mathematicorum***, and contributed to its pages for many years. The discipline of regular problem solving helped him to extend and broaden his knowledge. He was Deputy Leader of the UK IMO team for a few years, and he is still involved in team training.

If his name no longer appears in the pages of ***Crux Mathematicorum***, it is because (in retirement) he has embarked on a third career, research and writing, which has left him little time to contribute. He has written three books, one on crystallographic space groups, one on the appearance of integers in geometry, and one on plane euclidean geometry, co-authored with Tony Gardiner. Other booklets are under preparation.

SKOLIAD No. 95

Robert Bilinski

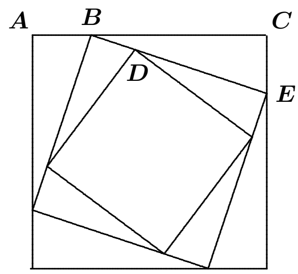
Please send your solutions to the problems in this edition by ***1 March, 2007***. A copy of **MATHEMATICAL MAYHEM Vol. 5** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

Our items for this issue come from the 5th annual CNU Regional Mathematics Contest. Only a selection of the problems has been included. Thanks go to R. Porsky, Christopher Newport University (CNU), Newport News, VA. [Some of these questions also appear on the 2004 Calgary Junior Mathematics Contest.]

5th Annual CNU Regional High School Mathematics Contest
Saturday November 13, 2004

1. Mr. Smith pours a full cup of coffee and drinks $\frac{1}{2}$ of it, deciding it is too strong and needs some milk. So he fills the cup with milk, stirs it, and tastes again, drinking another $\frac{1}{4}$ cup. Once again he fills the cup with milk, stirs it, and finds that this is just as he likes it. What ratio $\frac{\text{amount of coffee}}{\text{amount of milk}}$ does Mr. Smith like?

2. You have three inscribed squares, with the corners of each inner square at the $\frac{1}{4}$ point along the sides of its outer square. (Thus, for example, $AB = \frac{1}{4}AC$ and $BD = \frac{1}{4}BE$.) The area of the largest square is 64 cm^2. What is the area of the smallest square?



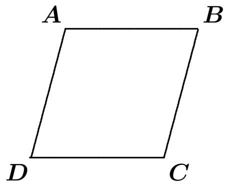
3. Solve the equation $\cos 2x = \cos x$ for $0 \le x < 2\pi$.

4. The centre of a circle of radius 1 cm is on the circumference of a circle of radius 3 cm. How far (in cm) from the centre of the big circle do the common tangents of the two circles meet?

5. One root of $2hx^2 + (3h - 6)x - 9 = 0$ is the negative of the other. Find the value of h.

6. Solve the equation $\sqrt{16x+1} - 2\sqrt[4]{16x+1} = 3$.

7. In the figure $ABCD$, all four sides have length 10 and the area is 60. What is the length of the shorter diagonal AC?



8. A man has 1000 equilateral triangular pieces of mosaic, all of side length 1 cm. He constructs the largest possible mosaic in the form of an equilateral triangle.

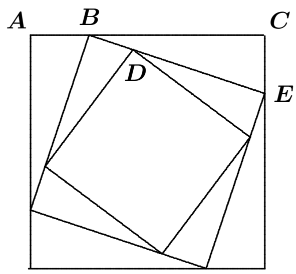
(a) What is the side length of the mosaic?

(b) How many pieces will he have left over?

5ᵉ Concours Annuel CNU Régional de Mathématique du Secondaire
Samedi, le 13 Novembre 2004

1. M. Smith se verse une tasse pleine de café et en boit la moitié. Décidant que le café est trop fort, il remplit la tasse de lait, mélange le contenu et goûte de nouveau, buvant de nouveau $\frac{1}{4}$ de tasse. Il remplit de nouveau la tasse de lait, mélange le contenu et trouve le café alors à son goût. Quel ratio $\frac{\text{quantité de café}}{\text{quantité de lait}}$ est au goût de M. Smith?

2. On a 3 carrés inscrits, de telle sorte que les sommets d'un carré intérieur sont au $\frac{1}{4}$ des côtés de son carré extérieur (par exemple, $AB = \frac{1}{4}AC$ et $BD = \frac{1}{4}BE$). L'aire du carré le plus grand est 64 cm^2. Quelle est l'aire du plus petit?



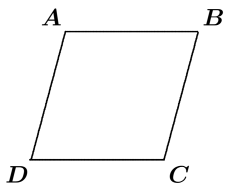
3. Résoudre l'équation $\cos 2x = \cos x$ pour $0 \leq x < 2\pi$.

4. Le centre d'un cercle de rayon 1 cm est sur la circonférence d'un cercle de rayon 3 cm. À quelle distance du centre du gros cercle se trouve le point d'intersection des tangentes communes des 2 cercles?

5. Une racine de $2hx^2 + (3h - 6)x - 9 = 0$ est l'opposé de l'autre. Trouver la valeur de h.

6. Résoudre l'équation $\sqrt{16x+1} - 2\sqrt[4]{16x+1} = 3$.

7. Les quatre côtés de la figure $ABCD$ mesurent 10 et son aire vaut 60. Quelle est la longueur de la diagonale la plus courte AC?



8. Un homme a 1000 triangles équilatéraux de côté 1 cm. Il décide de construire avec eux une mosaïque ayant pour forme le plus grand triangle équilatéral possible.

(a) Quel est le côté de cette mosaïque?

(b) Combien de morceaux lui reste-t-il?

Next we give solutions to the 2005 Maritime Mathematics Competition [2005 : 482–483].

2005 Maritime Mathematics Competition
March 3, 2005

1. A gardener owns a riding lawn mower and a push mower. It takes her 3 hours to cut the entire lawn with the push mower but only 75 minutes with the riding mower. One particular day, she cuts a portion of the lawn with the push mower and the rest with the riding lawn mower. If the total time to mow the lawn was 96 minutes, what fraction of the lawn was cut with the riding mower?

Official solution.

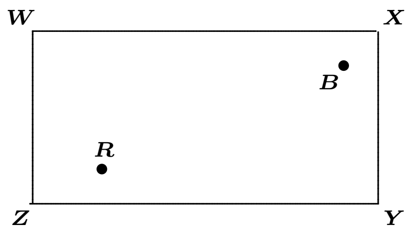
Suppose that the gardener used the riding mower for x minutes and the push mower for $96 - x$ minutes. The fraction of the lawn cut by the riding mower is then $x/75$, and the fraction cut by the push mower is $(96-x)/180$. Now we have

$$\frac{x}{75} + \frac{96-x}{180} = 1\,,$$

which gives $x = 60$. The fraction of the lawn cut by the riding mower is then $60/75 = 4/5$.

One incorrect solution was received.

2. Les bandes (côtés intérieurs) d'une table de billard forment le rectangle $WXYZ$ comme dans le diagramme ci-dessous. La bande WZ a cinq pieds de long. La bande WX a dix pieds de long. Une boule rouge (R) est située à un pied de YZ et à deux pieds de WZ. Une boule bleue (B) est située à un pied de WX et à un pied de XY. Nous voulons frapper la boule bleue pour qu'elle frappe la bande YZ, avec l'angle d'incidence égal à l'angle de réflexion, pour ensuite frapper la boule rouge. Quel point sur la bande YZ devons-nous viser?



Solution par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC.

Modélisons la situation par une fonction valeur absolue. Soit y_1 la partie décroissante de la fonction et y_2 sa partie croissante. Par symétrie, on a $y_1 = -mx + b$ et $y_2 = mx - b$.

En R, on a : $y_1 = -mx + b$ qui donne $1 = -2m + b$.

En B, on a : $y_2 = mx - b$ qui donne $4 = 9m - b$.

On résoud ce système pour obtenir $m = \frac{5}{7}$ et $b = \frac{17}{7}$. L'abscisse à l'origine est le point que l'on recherche et il vaut $x = \frac{17}{5} = 3,4$.

3. Trois étudiants jouent un jeu où le perdant de chaque partie doit doubler l'argent de chacun des deux autres joueurs. Après trois parties, chaque joueur a perdu une fois et possède $24. Combien d'argent possédait chaque étudiant au début du jeu ?

Solution par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC.

Soit X, Y et Z les noms des trois joueurs. Alors, on utilisera x_0, x_1 et x_2 pour désigner les bourses au début de chaque partie pour X. Il en sera de même pour y_0, y_1 et y_2 et z_0, z_1 et z_2.

Soit Z ayant perdu la dernière partie, alors avant cette partie les montants valaient :

$$x_2 = \frac{24}{2} = 12\,, \qquad y_2 = \frac{24}{2} = 12\,, \qquad z_2 = 24 + x_2 + y_2 = 48\,.$$

Soit Y ayant perdue la deuxième partie, alors avant cette partie, les montants valaient :

$$x_1 = \frac{12}{2} = 6\,, \qquad z_1 = \frac{48}{2} = 24\,, \qquad y_1 = 12 + x_1 + z_1 = 42\,.$$

Soit X ayant perdue la première partie, alors avant cette partie, les montants valaient :

$$y_0 = \frac{42}{2} = 21\,, \qquad z_0 = \frac{24}{2} = 12\,, \qquad x_0 = 6 + y_0 + z_0 = 39\,.$$

Les montants initiaux sont donc $39, $21 et $12.

4. Find all integers a for which the equation $x^3 - 13x + a = 0$ has three integer roots.

Official solution.

Let r_1, r_2, and r_3 be the roots of the equation. We may then write

$$\begin{aligned} x^3 - 13x + a &= (x - r_1)(x - r_2)(x - r_3) \\ &= x^3 - (r_1 + r_2 + r_3)x^2 \\ &\qquad +(r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3\,; \end{aligned}$$

equating coefficients on like powers of x gives

$$\begin{aligned} r_1 + r_2 + r_3 &= 0\,, \\ r_1r_2 + r_1r_3 + r_2r_3 &= -13\,, \\ -r_1r_2r_3 &= a\,. \end{aligned}$$

Now

$$r_1^2+r_2^2+r_3^2 \;=\; (r_1+r_2+r_3)^2-2(r_1r_2+r_1r_3+r_2r_3) \;=\; 0^2-2(-13) \;=\; 26\,.$$

Since r_1, r_2, and r_3 are integers, their squares are also integers. We seek, therefore, three squares of integers that sum to 26. The only possibilities are $\{0,\,1,\,25\}$ and $\{1,\,9,\,16\}$. If $\{r_1^2,\,r_2^2,\,r_3^2\} = \{0,\,1,\,25\}$, then we have $\{r_1,\,r_2,\,r_3\} = \{0,\,\pm 1,\,\pm 5\}$, which is impossible since $r_1 + r_2 + r_3 = 0$. Therefore, we must have $\{r_1^2,\,r_2^2,\,r_3^2\} = \{1,\,9,\,16\}$. Hence, there are two possibilities for the three roots, namely $\{1,\,3,\,-4\}$ and $\{-1,\,-3,\,4\}$. Finally, there are two possible values for $a = -r_1r_2r_3$, namely 12 and -12.

Also solved by Jean-François Désilets, student, Collège Montmorency, Laval, QC.

5. Triangle ABC is right-angled at A. Let x and y denote the lengths of the sides AB and AC, respectively. Suppose that the point D on BC is such that $\angle DAC = 30^\circ$. Determine the length of AD in terms of x and y.

Official solution, modified by the editors.

Place a coordinate system on the triangle so that A is at the origin, B is at $(x,0)$, and C is at $(0,y)$. The equation of the line through B and C is then $Y = (-y/x)X + y$. Since $\angle DAC = 30^\circ$, the line through A and D has slope $\cot 30^\circ = \sqrt{3}$, and its equation is $Y = \sqrt{3}\,X$. We now find the coordinates of D, the point of intersection of the two lines. We have $\sqrt{3}\,X = (-y/x)X + y$, which gives $X = \dfrac{xy}{y + x\sqrt{3}}$. Then $Y = \dfrac{xy\sqrt{3}}{y + x\sqrt{3}}$.

Now the length of AD is

$$\sqrt{\left(\frac{xy}{y + x\sqrt{3}}\right)^2 + \left(\frac{xy\sqrt{3}}{y + x\sqrt{3}}\right)^2} \;=\; \frac{2xy}{y + x\sqrt{3}}\,.$$

6. Evaluate the following sum.

$$\frac{1}{1^4+1^2+1}+\frac{2}{2^4+2^2+1}+\cdots+\frac{2005}{2005^4+2005^2+1}\,.$$

Official solution.

The general term in the series has the form

$$\begin{aligned}\frac{n}{n^4+n^2+1} &= \frac{n}{(n^2+1)^2-n^2} = \frac{n}{(n^2-n+1)(n^2+n+1)}\\ &= \frac{1}{2}\left(\frac{1}{n^2-n+1}-\frac{1}{n^2+n+1}\right)\,.\end{aligned}$$

Now $n^2+n+1=(n+1)^2-(n+1)+1$, which implies that the given sum may be written as a telescoping series as follows:

$$\begin{aligned}&\frac{1}{2}\left(\frac{1}{1}-\frac{1}{3}\right)+\frac{1}{2}\left(\frac{1}{3}-\frac{1}{7}\right)+\frac{1}{2}\left(\frac{1}{7}-\frac{1}{13}\right)+\cdots\\ &\qquad\qquad+\frac{1}{2}\left(\frac{1}{2005^2-2005+1}-\frac{1}{2005^2+2005+1}\right)\\ &= \frac{1}{2}\left(1-\frac{1}{2005^2+2005+1}\right) = \frac{1}{2}\left(\frac{2005^2+2005}{2005^2+2005+1}\right)\\ &= \frac{2011015}{4022031}\,.\end{aligned}$$

Also solved by Jean-François Désilets, student, Collège Montmorency, Laval, QC.

That brings us to the end of another issue. This month's winner of a past Volume of Mayhem is Jean-François Désilets. Congratulations Jean-François! Continue sending in your contests and solutions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

Mayhem Problems

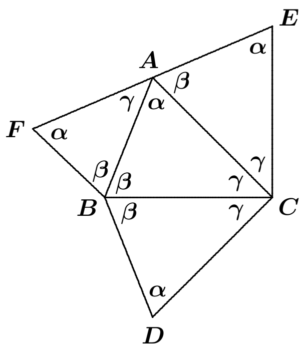
Please send your solutions to the problems in this edition by **1 January 2007**. *Solutions received after this date will only be considered if there is time before publication of the solutions.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

M251. *Proposed by K.R.S. Sastry, Bangalore, India.*

Let α, β, γ be the angle measures at angles A, B, C, respectively, in $\triangle ABC$. On the sides of $\triangle ABC$, externally, are triangles DBC, EAC, and FBA as in the diagram.



Prove that $AD = EF$ if and only if $\alpha = \pi/2$.

M252. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let x, y, z be positive real numbers. Prove that

$$\left(\frac{x}{y}+\frac{z}{\sqrt[3]{xyz}}\right)^2+\left(\frac{y}{z}+\frac{x}{\sqrt[3]{xyz}}\right)^2+\left(\frac{z}{x}+\frac{y}{\sqrt[3]{xyz}}\right)^2 \;\geq\; 12\,.$$

M253. *Proposed by Fabio Zucca, Politecnico di Milano, Milano, Italy.*

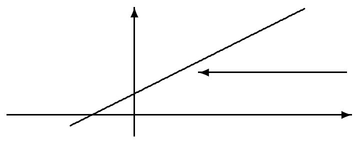
Consider the set of lattice points $\{(x,y)\}$ where x and y are integers such that $0 \le x \le 7$ and $0 \le y \le 7$. Two points are selected at random from this set. All points have the same probability of being selected and the points need not be distinct. Find the probability that the area of the triangle (possibly degenerate) formed by these two points and the point $(0,0)$ is an integer (possibly 0).

M254. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Evaluate the summation $S_{2006} = \sum\limits_{k=1}^{2006} (-1)^k \frac{k^2-3}{(k+1)!}$. [Recall that $n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$; for example, $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$.]

M255. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

The line with slope $\lambda > 0$ acts like a mirror to a ray of light coming along a line parallel to the x–axis. Determine the slope of the reflected ray.



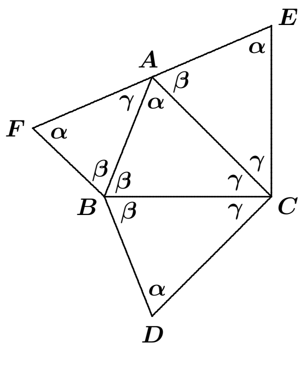
M256. *Proposed by the Mayhem Staff.*

Find a quadratic polynomial $f(x)$ such that, if n is a positive integer consisting of the digit 5 repeated k times, then $f(n)$ consists of the digit 5 repeated $2k$ times. (For example, $f(555) = 555555$.)

M251. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

Soit α, β et γ les mesures respectives des angles A, B et C dans le triangle ABC. Sur les côtés du triangle ABC, on construit extérieurement les triangles DBC, EAC et FBA, comme indiqué dans la figure.

Montrer que $AD = EF$ si et seulement si $\alpha = \pi/2$.



M252. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit x, y et z trois nombres réels positifs. Montrer que

$$\left(\frac{x}{y} + \frac{z}{\sqrt[3]{xyz}}\right)^2 + \left(\frac{y}{z} + \frac{x}{\sqrt[3]{xyz}}\right)^2 + \left(\frac{z}{x} + \frac{y}{\sqrt[3]{xyz}}\right)^2 \ge 12\,.$$

M253. *Proposé par Fabio Zucca, Politecnico di Milano, Milano, L'italie.*

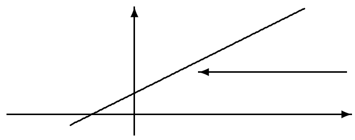
On considère l'ensemble des points $\{(x, y)\}$ d'un réseau où x et y sont des entiers tels que $0 \leq x \leq 7$ et $0 \leq y \leq 7$. On choisit deux points de cet ensemble au hasard. Tous les points ont la même probabilité d'être choisis et les points peuvent ne pas être distincts. Trouver la probabilité pour que l'aire du triangle (peut-être dégénéré) formé par ces points et le point $(0, 0)$ soit un entier (peut-être 0).

M254. *Proposé par Edward T.H. Wang, Université Wilfrid Laurier, Waterloo, ON.*

Evaluer la somme $S_{2006} = \sum_{k=1}^{2006} (-1)^k \frac{k^2 - 3}{(k+1)!}$. [On rappelle que $n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$; par exemple, $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$.]

M255. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Une droite de pente $\lambda > 0$ agit comme un miroir sur un rayon lumineux suivant une droite parallèle à l'axes des x. Déterminer la pente du rayon réfléchi.



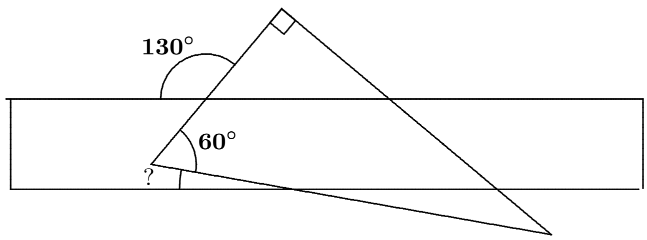
M256. *Proposé par l'Équipe de Mayhem.*

Trouver un polynôme quadratique $f(x)$ tel que, si n est un entier positif formé du chiffre 5 répété k fois, alors $f(n)$ est formé du chiffre 5 répété $2k$ fois. (Par exemple, $f(555) = 555555$.)

Mayhem Solutions

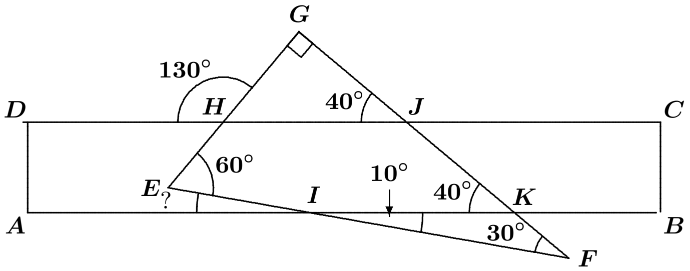
M201. *Proposed by Robert Bilinski, Collège Montmorency, Laval, QC.*

A student drops his 30°–60°–90° triangle on his ruler so that a 130° angle appears as in the diagram below. What is the measure of the other marked angle?



Solution by Glenier L. Bello-Burguet, student, 3[ro] de ESO Instituto Hermanos D'Elhuyar, Logroño, Spain.

We label the diagram as shown. The angle we seek is $\angle AIE = 10°$.



We are given $\angle GHD = 130°$ and also the angles of triangle EFG, namely $\angle GEF = 60°$, $\angle FGE = 90°$, $\angle EFG = 30°$. Thus, $\angle HJG = 40°$ (since $\angle HJG + \angle HGJ = \angle GHD$), and then the corresponding $\angle JKI$ also equals $40°$. Hence, $\angle KIF = 10°$ (because $\angle KIF + \angle EFG = \angle JKI$) and, therefore, also $\angle AIE = 10°$.

Also solved by James T. Bruening, Southeast Missouri State University, Cape Girardeau, MO, USA.

M202. *Proposed by Edward J. Barbeau, University of Toronto, Toronto, ON.*

(a) By the end of the season, Samuel had made more than 50% of his foul shots, even though at the start of the season his average was below 50%. Show that there was a time during the season when his average was exactly 50%.

(b) For what other percentages p can you be certain that the average was exactly p at some time when you know only that the average was below p and then above p at a later time?

Solution by the Mayhem Staff.

(a) For the purpose of contradiction, assume that there was no time during the season when Samuel's average was exactly 50%. Then there must have been a time when, by sinking a foul shot, he increased his average from a fraction $a/b < 1/2$ to $(a+1)/(b+1) > 1/2$, where a and b are integers with $a \geq 0$ and $b > 0$. Then $2a < b$ and $2a + 2 > b + 1$. Together, these inequalities imply that $2a + 1 < b + 1 < 2a + 2$, which is impossible, since the integer $b + 1$ cannot lie strictly between two consecutive integers. This contradiction establishes the existence of a point in the season when Samuel had an average of 50%.

(b) By generalizing the argument in part (a) (using a fraction c/d in place of $1/2$), we can show that the percentages we seek are those of the form $c/(c+1)$, where c is a positive integer.

[*Ed:* This problem was explored in the Problem of the Month feature in the April 2006 issue of ***Crux Mathematicorum*** [2], and there is even a Proof Without Words treatment which has recently appeared [1].]

References

[1] Robert J. MacG. Dawson, Putnam Proof Without Words, *Mathematics Magazine*, 79 (2006), p. 149.

[2] Ian VanderBurgh, Problem of the Month, *Crux Mathematicorum with Mathematical Mayhem*, 32 (2006), pp. 147–148.

M203. *Proposed by Richard K. Guy, University of Calgary, Calgary, AB.*

Chuck goes into the local 7-11 store and buys four items. The bill totals \$7.11. He notices that the product of the four prices is exactly 7.11. What are the prices of the four items?

[*Ed:* The proposer has indicated that the problem does not originate with him, nor does he know its origin.]

Solution by the University of Regina Problem Group.

The items cost \$1.20, \$1.25, \$1.50, and \$3.16. We found the problem in *How To Solve It: Modern Heuristics*, by Zbigniew Michalewicz and David B. Fogel (Springer, 1998), Puzzle III, pages 49–53. The problem seems not to have originated with these authors, because they refer to it as the "so-called '7–11' problem", but they do not provide a reference. Our approach was essentially the same as theirs: convert the prices into cents (so that we can use integer arithmetic), systematically list all possible combinations of feasible prices, then check which prices satisfy the problem's conditions.

When we multiply each of the four prices by 100, the problem becomes finding four integers w, x, y, z for which

$$wxyz = 711000000 \qquad \text{and} \qquad w+x+y+z = 711\,.$$

Our task is to factor $711000000 = 2^6 \cdot 3^2 \cdot 5^6 \cdot 79$ into four numbers that sum to 711. Because the fourth root of the product is approximately 163, and $4 \cdot 163 = 652$, we seek factors somewhere around 163 while being aware that there is considerable leeway.

Since 79 is the largest prime factor, it makes sense to begin with that: we try letting w be a small multiple of 79. To our surprise there turns out to be no solution with $w = 2 \cdot 79 = 158$ or $w = 3 \cdot 79 = 237$. We next try $w = 4 \cdot 79 = 316$, and look for values of x, y, z that satisfy

$$xyz = 2^4 \cdot 3^2 \cdot 5^6 \qquad \text{and} \qquad x+y+z = 395\,.$$

For the sum to be divisible by 5 (and since setting $x = 5^6$ would make the sum too large), each of the three prices must be a multiple of 5. We therefore set $x = 5x'$, $y = 5y'$, and $z = 5z'$, and thereby reduce the problem to finding x', y', z' that satisfy

$$x'y'z' \;=\; 2^4 \cdot 3^2 \cdot 5^3 \qquad \text{and} \qquad x' + y' + z' \;=\; 79\,.$$

We can simplify matters further by noting that for a sum of 79, not all three unknowns can be multiples of 5; consequently, at least one of them, say x', has to be a multiple of 25. Setting $x' = 25t$, we now must solve

$$ty'z' \;=\; 2^4 \cdot 3^2 \cdot 5 \qquad \text{and} \qquad 25t + y' + z' \;=\; 79\,.$$

It is easily seen that $t \neq 2$ (since $y'z' = 2^3 \cdot 3^2 \cdot 5 = 360$ and $y' + z' = 29$ have no common solution), and also $t \neq 3$ (since $y'z' = 2^4 \cdot 3 \cdot 5 = 240$ and $y' + z' = 4$ have no common solution). We are left with $t = 1$, which gives $y'z' = 720$, and $y' + z' = 54$. This is satisfied by $y' = 24$ and $z' = 30$. Therefore, one solution is $x = 5 \cdot 1 \cdot 25 = 125$, $y = 5 \cdot 24 = 120$, and $z = 5 \cdot 30 = 150$, as claimed.

To convince oneself that the solution is unique, it seems unavoidable to investigate all multiples of 79, the approach that is outlined in the cited text. The problem is small enough that a computer can quickly check all sets of four numbers that sum to 7.11 to see if their product is 7.11. Two near misses occur: the prices \$1.25, \$1.25, \$1.44, and \$3.16 sum to \$7.10, while their product (as numbers) is 7.11; the prices \$.75, \$2.00, \$2.00, and \$2.37 sum to \$7.12, also with a product of 7.11.

M204. *Proposed by Geneviève Lalonde, Massey, ON.*

Suppose that there is a line of 2005 buttons numbered 1 through 2005. Above each button is a counter initially set to 0. Each time a button is pushed, the corresponding counter advances by 1. A set of 2005 people now proceed down the line of buttons. The first person pushes every button, the second person pushes every second button starting at button #2, the third person pushes every third button starting at button #3, and so on, so that the 2005th person pushes only button #2005. When everyone has gone, which buttons' counters will read 4?

Please provide a description of the set of buttons, rather than the actual list.

Solution by the proposer.

The first button whose counter will read 4 is button #6. It is pushed by persons numbered 1, 2, 3, and 6. The numbers 1, 2, 3, and 6 are the positive divisors of 6, including 1 and the number itself. A button's counter will obviously read 4 if the number n on the button has exactly four positive divisors, including 1 and the number itself.

Let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ be the prime factorization of n, where the p_is are distinct primes and the a_is are positive integers. It is well known that the number of distinct divisors of n is $(a_1+1)(a_2+1)\cdots(a_k+1)$, which includes 1 and the number itself. [The reason for this is as follows: each divisor of n has the form $p_1^{c_1} p_2^{c_2} \cdots p_k^{c_k}$, with $0 \leq c_i \leq a_i$ for $1 \leq i \leq k$. Since there are $a_i + 1$ possible values for each c_i, the formula follows. Note that if $c_i = 0$ for all $i = 1, 2, \ldots, k$, the corresponding divisor would be 1.] Hence, n has 4 divisors if and only if $(a_1+1)(a_2+1)\cdots(a_k+1) = 4$, which is true if and only if either $k = 1$ and $a_1 = 3$ or $k = 2$ and $a_1 = a_2 = 1$.

Thus, the counter of button #n will read 4 if and only if $n = p^3$, where p is a prime, or $n = pq$, where p and q are distinct primes such that $n \leq 2005$.

M205. *Proposed by John Katic, Ottawa, ON.*

Show that for every triangle ABC,

$$1 \;\leq\; \cos A + \cos B + \cos C \;\leq\; \frac{3}{2}\,.$$

Solution by Alper Cay, Uzman Private School, Kayseri, Turkey, with details added by the editor.

Let a, b, c be the lengths of the sides opposite the angles A, B, C, respectively, and let $s = \frac{1}{2}(a+b+c)$ be the semiperimeter of $\triangle ABC$. It is known that

$$\cos A + \cos B + \cos C \;=\; 1 + 4\,\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\,.$$

(See, for example, problem 2760 [2003 : 342–343].) Furthermore, we have $\sin\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{bc}}$, with similar expressions for $\sin\left(\frac{B}{2}\right)$ and $\sin\left(\frac{C}{2}\right)$. Therefore,

$$\cos A + \cos B + \cos C \;=\; 1 + 4\,\frac{(s-a)(s-b)(s-c)}{abc}\,.$$

Thus, the inequalities that we want to prove are equivalent to

$$0 \;\leq\; 4\,\frac{(s-a)(s-b)(s-c)}{abc} \;\leq\; \frac{1}{2}\,;$$

that is,

$$0 \;\leq\; (-a+b+c)(a-b+c)(a+b-c) \;\leq\; abc\,. \qquad (1)$$

By the Triangle Inequality, each of the factors $-a+b+c$, $a-b+c$, and $a+b-c$ is positive. Therefore, the left inequality in (1) is true. To obtain the right inequality, we apply the AM–GM Inequality as follows:

$$\begin{aligned}
\sqrt{(-a+b+c)(a-b+c)} &\;\leq\; \frac{(-a+b+c)+(a-b+c)}{2} \;=\; c\,,\\
\sqrt{(a-b+c)(a+b-c)} &\;\leq\; \frac{(a-b+c)+(a+b-c)}{2} \;=\; a\,,\\
\sqrt{(a+b-c)(-a+b+c)} &\;\leq\; \frac{(a+b-c)+(-a+b+c)}{2} \;=\; b\,.
\end{aligned}$$

Multiplying these three inequalities gives the desired result.

Also solved by JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; and MIHÁLY BENCZE, Brasov, Romania.

M206. *Proposed by Bill Arden, Rideau High School, Ottawa, ON.*

Let n be a composite number such that $a^{n-1} - 1$ is divisible by n for every number a that does not have a factor in common with n. Prove that at least 3 distinct primes divide n.

Editorial Comments: A full understanding of the solution to this innocent-looking problem is actually beyond the scope of this Column. (The editors goofed in letting this problem be printed in Mayhem!) We will instead give a brief history of the origin of this problem and the known results.

As is perhaps known to some high school students, Fermat's Little Theorem states that, if n is a prime, and a is a positive integer such that a and n are relatively prime, then $a^{n-1} \equiv 1 \pmod{n}$. This raises the natural question whether the congruence can hold if n is composite. The answer turns out to be yes, and it leads to the concept of *pseudoprimes*. Specifically, given an integer $a \geq 2$, a composite number n is called a pseudoprime to the base a if a and n are relatively prime and if $a^{n-1} \equiv 1 \pmod{n}$. For example, $n = 341 = 11 \times 31$ is a pseudoprime to the base 2, since it can be verified that $2^{340} \equiv 1 \pmod{341}$. Similarly, we can show that 91 is a pseudoprime to the base 3.

One could now ask if there is a composite n which is a pseudoprime to the base a for **all** positive integers a that are relatively prime to n. The answer, surprisingly, is still yes. Such an integer is called a *Carmichael number*, named after Robert Daniel Carmichael (1879–1967). The smallest such number is $561 = 3 \times 11 \times 17$. Another is $2821 = 7 \times 13 \times 31$.

Many interesting properties of Carmichael numbers are known. For example, a Carmichael number is square-free (a product of distinct primes) and has at least three distinct odd prime divisors (this is exactly the problem M206). In fact, a complete characterization of a Carmichael number is known: namely, n is a Carmichael number if and only if $n = p_1 p_2 \cdots p_k$ where the p_is are distinct primes such that $(p_i - 1) \mid (n - 1)$ for all $i = 1, 2, \ldots, k$.

In 1912 Carmichael conjectured that there exist infinitely many such numbers. This was a very difficult conjecture, and it took 80 years before it was finally proved in the affirmative by the following three mathematicians: Alford, Granville, and Pomerance.

Readers interested in this topic are encouraged to consult the references listed below. Most of the information given above can be found in [1].

A complete solution to the current problem was submitted by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina. His proof, however, is quite technical and used, among other things, known results from number theory about primitive roots.

References

[1] Kenneth H. Rosen, *Elementary Number Theory and its Applications*, 5th ed., Addison Wesley and Longman, 2005.

[2] W. Sierpinski, *Elementary Theory of Numbers*, 2nd ed., North-Holland, Arsterdam, 1987.

Problem of the Month

Ian VanderBurgh

Next in the sequence of Problems of the Month comes a problem (actually, two problems) about sequences.

Problem 1 (2006 Pascal Contest)

John writes a number with 2187 digits on the blackboard, each digit being a 1 or a 2. Judith creates a new number from John's number by reading his number from left to right and wherever she sees a 1, writing 112, and wherever she sees a 2, writing 111. (For example, if John's number begins 2112, then Judith's number would begin 111112112111.) After Judith finishes writing her number, she notices that the left-most 2187 digits in her number and in John's number are the same. How many times do five 1s occur consecutively in John's number?

This is quite something to try to wrap your head around! Let's start off with a slightly different problem, which is a bit easier to sort out:

Problem 2. John writes a number with 3000 digits on the blackboard, each digit being a 1 or a 2. Judith creates a new number from John's number by reading his number from left to right and wherever she sees a 1, writing 122, and wherever she sees a 2, writing 111. (For example, if John's number begins 2112, then Judith's number would begin 111122122111.) After Judith finishes writing her number, she notices that the left-most 2500 digits in her number and in John's number are the same. What are the digits in positions 2006 to 2012? (Problem 2 appeared on one of the earlier drafts of the 2006 Pascal Contest, but was removed in favour of Problem 1.)

It is tough to know where to start this problem; as the song says, though, let's start at the very beginning (it's a very good place to start).

Solution to Problem 2: If John's number begins with a 1, then Judith's will begin 122; if John's number begins with a 2, Judith's will begin 111. Since Judith's number begins with a 1 regardless of what John's number begins with, and since their numbers are the same for the first 2500 digits, John's number must begin with a 1. Thus, Judith's begins 122, and therefore John's begins 122. (We call these three digits Stage 1.)

We can then use this beginning as the "seed" with which to create the number. Since John's number begins 122, then Judith's begins 122111111 (since each 1 becomes 122 and each 2 becomes 111). Thus, John's number begins 122111111 (Stage 2).

Continuing in a similar way, each digit at each Stage generates three digits at the next Stage, which means that each Stage is three times as long as the previous one. Thus, successive stages consist of 3, 9, 27, 81, 243, 729, and 2187 digits. In general, Stage n consists of 3^n digits.

Consider Stage 7, which has length $3^7 = 2187$. This includes the digits in positions 2006 to 2012, the positions that interest us. How can we figure out what these digits are? One strategy is to trace backwards through the Stages the positions of the digits which generate these digits, until we come to one of the early Stages, where we can easily determine the digits. We then use these known digits to move forward again through the Stages to determine the desired digits at Stage 7.

In general, the first k digits from one Stage generate the first $3k$ digits at the next stage. This tells us that digit k from one Stage generates digits $3k-2$, $3k-1$, and $3k$ at the next stage. We would like digits 2006 to 2012 at Stage 7. Digit 669 from Stage 6 generates digits 2005 to 2007 at Stage 7, digit 670 generates digits 2008 to 2010 at Stage 7, and digit 671 generates digits 2011 to 2013 at Stage 7. If we can determine digits 669 to 671 at Stage 6, then we can get digits 2006 to 2012 at Stage 7. (These digits at Stage 6 actually will give us digits 2005 to 2013 at Stage 7, which is a bit more than we need.)

Perhaps a chart might be in order:

Stage	Digit(s)		Stage	Digit(s)		Digit(s)
7	2006–2012	from	6	669–671	giving	2005–2013
6	669–671	from	5	223–224	giving	667–672
5	223–224	from	4	75	giving	223–225
4	75	from	3	25	giving	73–75
3	25	from	2	9	giving	25–27

Aha! We know that digit number 9 at Stage 2 is a 1. Hence, remembering that a 1 at one stage gives 122 at the next stage and a 2 at one stage gives 111 at the next stage, we can then trace these digits forward through the Stages. Probably making another table will help:

Stage	Digit(s)	String		Stage	Digits	String
2	9	1	gives	3	25–27	122
3	25	1	gives	4	73–75	122
4	75	2	gives	5	223–225	111
5	223–224	11	gives	6	667–672	122122
6	669–671	212	gives	7	2005–2013	111122111

Thus, the digits in positions 2006 to 2012 are 1112211.

Making a table seemed to be pretty helpful here. Often, it is a really good way to consolidate or organize a bunch of information. Now Problem 1 certainly looks similar (be careful, though—the digit replacement scheme is slightly different).

Solution to Problem 1: Conveniently, here we are looking at 2187 digits, which exactly matches the length of Stage 7 in our analysis above. (This seems unlikely to be a coincidence!) Here, the digit replacement scheme is that 1 becomes 112 and 2 becomes 111. Again, John's number must start with a 1, and thus must start 112 (Stage 1).

How can 11111 (that is, five consecutive 1s) appear at Stage 7? They must come from 21 at Stage 6. They cannot come from 22 because there cannot be two consecutive 2s at any stage. (Think about why this is true.) In fact, there cannot be more than five consecutive 1s at any stage. (Think about this as well.)

Thus, counting the number of occurrences of 11111 at Stage 7 is the same as counting the number of occurrences of 21 at Stage 6. But each 2 is always followed by a 1; whence, this is the same as the number of 2s at Stage 6. (Wait! If the 2 occurs at the very end of Stage 6, then it is not followed by a 1. It turns out that Stage 6 ends in a 1—a third thing to think about!)

How can we figure out the number of 2s at Stage 6? The only way to get a 2 at Stage 6 is from a 1 at Stage 5, which gives 112. (And each 1 at Stage 5 gives exactly one 2 at Stage 6).

At this point, trying to write all of this out in words is becoming painful. Thus, some notation would be helpful. Let's use a_n to represent the number of 1s at Stage n and b_n to represent the number of 2s at Stage n. Notice that $a_1 = 2$ and $b_1 = 1$. Also, $b_{n+1} = a_n$ (each 1 at Stage n generates exactly one 2 at Stage $n + 1$) and $a_{n+1} = 2a_n + 3b_n$ (each 1 at Stage n generates two 1s and each 2 generates three 1s). Again, a table:

n	a_n	b_n
1	2	1
2	7	2
3	20	7
4	61	20
5	182	61

Therefore, the number of 1s at Stage 5 is 182, which implies that there are 182 occurrences of 11111 at Stage 7 (that is, in the first 2187 digits of the full number).

Yes, this problem was pretty difficult to be on a Grade 9 Contest(!). But it required very little specialized knowledge, and lots could be discovered through some fiddling, which was why we felt comfortable including it. And there is a lot more that could be asked, and lots of interesting investigation to be done (including lots that could be investigated with a computer). Happy hunting! It is also interesting to look at what happens by modifying the digit replacement schemes, even allowing the 1 and 2 to be replaced by strings of different lengths.

Pólya's Paragon

Playing Games with Mathematics (Part I)

John Grant McLoughlin

Recreational mathematics provides a wonderful vehicle for developing mathematical thinking. Game-playing arouses curiosity about the principles underlying the structures and strategies associated with the games. This is particularly evident when the players are students at a mathematical event such as the University of New Brunswick Math Camp. This column and its successor are based on a recent contribution to that event, in May 2006. Four of the games and challenges that were posed at the camp are shared here, followed by an additional problem involving polynomials.

The purpose of the challenges is threefold: to broaden one's knowledge of recreational mathematics; to engage in playful mathematical activity; and to develop mathematical arguments and proofs in response to the challenges. This latter point will be the focus of Part II, in which these challenges are to be discussed. It should be noted that this two-part model was used at the camp also. The challenges were shared one afternoon in an introductory session, and the discussion of the mathematical principles, including insights gained from participating students, took place the next evening.

1. Sim

Six dots are drawn on a piece of paper to form the vertices of a hexagon. Two players are each assigned a colour. The players take turns joining any two of the dots with a line segment, using their assigned colours. The loser is the player who completes a triangle with three of the original six dots as its vertices and with all three edges the same colour.

Challenge: Prove that there must always be a loser (and a winner).

2. 31

This mental math game involves a running total which starts at zero. Each player has the choice to add 1, 2, 3, 4, 5, or 6 to the total. Players alternate turns. The winner is the player who is able to bring the total to 31.

For example, player B is the winner in the following game:

Player	A	B	A	B	A	B	A	B
Chosen number	6	4	2	4	4	1	5	5
Total	6	10	12	16	20	21	26	31

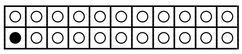
Challenge: Determine a winning strategy. You may choose to play first or second.

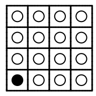
3. Chomp

Counters are placed in a rectangular grid such that one counter appears in each small rectangle. The counter in the bottom left-hand corner is a different colour than the others. Players take turns selecting one counter. If the counter selected occupies the bottom left-hand corner of a rectangle on the grid, all the counters in that rectangle are removed. The object is to force your opponent to select the differently coloured counter (the one in the bottom left-hand corner).

Try playing this game with rectangular boards of different sizes.

Challenge: Suppose that you play two games of Chomp in which the boards are $2 \times n$ and $k \times k$, examples of which are shown. Determine a winning strategy in each case. You may choose to play first or second.





4. Fifteen Finesse

The numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 are available for use in this game. Each number can be used only once. Two players alternate turns selecting one of the available numbers. To win the game, a player must obtain exactly three numbers that sum to 15. (Neither a pair of numbers, such as 7 and 8, nor a set of four numbers, such as 1, 3, 5, and 6, constitutes a winning combination.) The game ends in a draw if no player is able to acquire three numbers that sum to 15.

Challenge: Explain the underlying structure of the game, and suggest strategies that may help to win a game.

5. A Polynomial in Transition

Consider the polynomial $x^2 + 10x + 20$. Under the conditions below, is it possible to convert this polynomial to $x^2 + 20x + 10$? Justify your answer.

Conditions:

(i) On each step you may only change the constant term or the coefficient of x (but not both).

(ii) The change must be an increase of 1 or a decrease of 1.

(iii) The change must NOT produce a polynomial that can be factored into the form $(x+m)(x+n)$ where m and n are integers. For example, you could not begin by reducing 10 to 9, since $x^2+9x+20 = (x+5)(x+4)$.

Various sources have contributed ideas/games for this presentation, including books by Brian Bolt, Ian Stewart, Martin Gardner, and others. The polynomial problem is due to Ed Barbeau. A bibliography will be included in Part II. As usual, comments on any aspect of Pólya's Paragon are welcomed.

THE OLYMPIAD CORNER

No. 255

R.E. Woodrow

Welcome back from the Summer break (at least in Canada!). To start your problem-solving engines we give the 2003 Belarus Mathematical Olympiad. Thanks go to Andy Liu, Canadian Team Leader to the 2003 IMO in Japan, for supplying us with these problems.

BELARUS MATHEMATICAL OLYMPIAD 2003
June 21-22, 2003

1. Let T be the set of all ordered triples of non-negative integers. Find all functions f from T to the real numbers such that $f(x,y,z) = 0$ when $xyz = 0$, and, when $xyz \neq 0$,

$$\begin{aligned} &f(x,y,z)\\ &\quad = 1 + \tfrac{1}{6}\big[f(x+1,y-1,z) + f(x-1,y+1,z) + f(x+1,y,z-1)\\ &\qquad\quad + f(x-1,y,z+1) + f(x,y+1,z-1) + f(x,y-1,z+1)\big]\,. \end{aligned}$$

2. Define a *k–clique* to be a set of k people each of whom is acquainted with all of the others. At a certain party, every pair of 3–cliques has at least one person in common, and there are no 5–cliques. Prove that there are two or fewer people at the party whose departure leaves no 3–cliques.

3. Find all functions f from the real numbers to the real numbers such that, for any real numbers x and y,

$$f(xy)\big(f(x) - f(y)\big) = (x-y)f(x)f(y)\,.$$

4. From a point P outside triangle ABC, the feet of the perpendiculars to BC, CA, and AB are D, E, and F, respectively. If triangles PAF, PBD, and PCE all have equal area, prove that ABC also has the same area.

5. Twenty-one girls and twenty-one boys took part in a mathematics competition. Each contestant solved at most six problems. For any pair consisting of a girl and a boy, there was at least one problem solved by both contestants. Prove that there was a problem solved by at least three girls and at least three boys.

6. The sequence $\{a_n\}$ is defined by $a_1 = 11^{11}$, $a_2 = 12^{12}$, $a_3 = 13^{13}$ and $a_{n+3} = |a_{n+2} - a_{n+1}| + |a_{n+1} - a_n|$ for all non-negative integers n. Determine a_n when $n = 14^{14}$.

As a second set of problems we give the Selection Problems for the Indian Team for IMO 2003. Thanks again go to Andy Liu, Canadian Team Leader, for collecting them for our use.

PROBLEMS TO SELECT INDIAN IMO TEAM 2003

1. Let A', B', C' be the mid-points of the sides BC, CA, AB, respectively, of an acute non-isoceles triangle ABC, and let D, E, F be the feet of the altitudes through the vertices A, B, C on these sides, respectively. Consider the arc DA' of the nine-point circle of triangle ABC lying outside the triangle. Let the point of trisection of this arc closer to A' be A''. Define analogously the points B'' (on arc EB') and C'' (on arc FC'). Show that triangle $A''B''C''$ is equilateral.

2. Find all triples (a, b, c) of positive integers such that

(i) $a \leq b \leq c$;

(ii) $\gcd(a, b, c) = 1$; and

(iii) $a^3 + b^3 + c^3$ is divisible by each of the numbers a^2b, b^2c, c^2a.

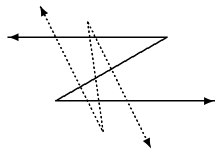
3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all x, y in $\mathbb{R}$, we have

$$f(x+y) + f(x)f(y) \;=\; f(x) + f(y) + f(xy)\,.$$

4. There are four lines in the plane, no three concurrent, no two parallel, and no three forming an equilateral triangle. If one of them is parallel to the *Euler line* of the triangle formed by the other three lines, prove that a similar statement holds for each of the other lines.

5. On the real number line, paint red all points that correspond to integers of the form $81x + 100y$, where x and y are positive integers. Paint the remaining integer points blue. Find a point P on the line such that, for every integer point T, the reflection of T with respect to P is an integer point of a different colour than T.

6. A zig-zag in the plane consists of two parallel half-lines connected by a line segment. Find z_n, the maximum number of regions into which n zig-zags can divide the plane. For example, $z_1 = 2$ and $z_2 = 12$ (see the diagram). Of these z_n regions, how many are bounded? [The zig-zags can be as narrow as you please.] Express your answers as polynomials in n of degree not exceeding 2.



7. Let $P(x)$ be a polynomial with integer coefficients such that $P(n) > n$ for all positive integers n. Suppose that for each positive integer m, there is a term in the sequence $P(1)$, $P(P(1))$, $P(P(P(1)))$, $\ldots$ which is divisible by m. Show that $P(x) = x + 1$.

8. Let ABC be a triangle, and let r, r_1 ,r_2, r_3 denote its inradius and the exradii opposite the vertices A, B, C, respectively. Suppose $a > r_1$, $b > r_2$, $c > r_3$. Prove that

(a) triangle ABC is acute, (b) $a + b + c > r + r_1 + r_2 + r_3$.

9. Let n be a positive integer and $\{A, B, C\}$ a partition of $\{1, 2, \ldots, 3n\}$ such that $|A| = |B| = |C| = n$. Prove that there exist $x \in A$, $y \in B$, $z \in C$ such that one of x, y, z is the sum of the other two.

10. Let n be a positive integer greater than 1, and let p be a prime such that n divides $p - 1$ and p divides $n^3 - 1$. Prove that $4p - 3$ is a square.

As a final set of problems for this issue we give the 2003 German Mathematical Olympiad, Final Round, Grades 12-13.

GERMAN MATHEMATICAL OLYMPIAD 2003

Final Round, Grades 12-13

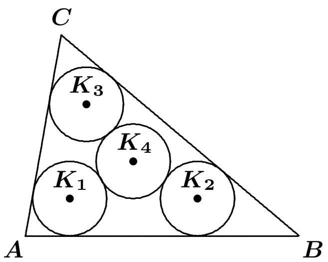
June 22-25, 2003

1. Determine all pairs (x, y) of real numbers x, y which satisfy

$$\begin{aligned} x^3 + y^3 &= 7\,, \\ xy(x+y) &= -2\,. \end{aligned}$$

2. In the interior of a triangle ABC, circles K_1, K_2, K_3, and K_4 of the same radii are defined such that K_1, K_2, and K_3 touch two sides of the triangle and K_4 touches K_1, K_2, and K_3, as shown in the figure.

Prove that the centre of K_4 is located on the line through the incentre and the circumcentre.

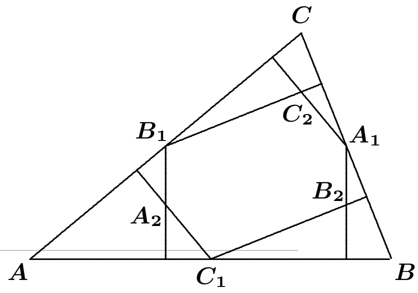


3. The caterpillar *Nummersatt* is sitting in the middle square of an $N \times N$ board, where N is an odd integer with $N \geq 3$. The other squares of the board each contain a positive integer, and all of these integers are different. Nummersatt wants to find a way off the board. The caterpillar can move only between adjacent squares (squares having a common side), or off the board from one of the outermost squares, having once reached such a square. On reaching a new square, Nummersatt has to eat the number on that square. The number n weighs $\frac{1}{n}$ kg, and Nummersatt cannot eat more than 2 kg.

Decide whether numbers can be distributed on the board so that there is no way off the board for Nummersatt

(a) for $N = 2003$, (b) for all odd integers $N \geq 3$.

4. Let A_1, B_1, and C_1 be the midpoints of the sides of the acute-angled triangle ABC. The 6 lines through these points perpendicular to the other sides meet in the points A_2, B_2, and C_2, as shown in the figure. Prove that the area of the hexagon $A_1C_2B_1A_2C_1B_2$ equals half of the area of $\triangle ABC$.



5. If n is a positive integer, let $a(n)$ be the smallest positive number for which $(a(n))!$ is divisible by n. Determine all positive integers n satisfying

$$\frac{a(n)}{n} = \frac{2}{3}\,.$$

6. Prove that there are infinitely many pairs (a, b) of positive integers with $a > b$ having the following properties:

(i) the greatest common divisor of a and b equals 1;

(ii) a is a divisor of $b^2 - 5$.

(iii) b is a divisor of $a^2 - 5$.

Now we turn to solutions from our readers to some of the problems from the February 2005 *Corner*. The first group of solutions are to problems of the Icelandic Mathematical Contest 2000-2001 given in [2005 : 26–27].

1. Let x and y be positive real numbers such that $xy = 1$. Prove that

$$\frac{x}{y} + \frac{y}{x} \geq 2\,.$$

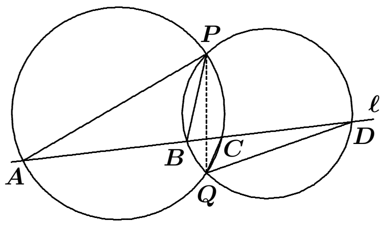
Solved by Robert Bilinski, Collège Montmorency, Laval, QC; Pierre Bornsztein, Maisons-Laffitte, France; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; R. Laumen, Deurne, Belgium; Geoffrey A. Kandall, Hamden, CT, USA; and Vedula N. Murty, Dover, PA, USA. We give Kandall's write-up.

The condition $xy = 1$ is unnecessary. For any positive real numbers x and y,

$$\frac{x}{y} + \frac{y}{x} = \left(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}}\right)^2 + 2 \geq 2\,.$$

2. Two circles intersect at points P and Q. A line ℓ that intersects the line segment PQ intersects the two circles at the points A, B, C, and D (in that order along the line ℓ). Prove that $\angle APB = \angle CQD$.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Pierre Bornsztein, Maisons-Laffitte, France; Paolo Custodi, Fara Novarese, Italy; R. Laumen, Deurne, Belgium; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Laumen's write-up.



We have $\angle PBD = \angle PQD$ (since these angles are subtended by the same arc) and $\angle PQD = \angle CQD + \angle CQP$. Hence, $\angle PBD = \angle CQD + \angle CQP$. Also, since $\angle PBD$ is an exterior angle to $\triangle APB$, we have $\angle PBD = \angle APB + \angle PAB$. Thus,

$$\angle APB + \angle PAB = \angle CQD + \angle CQP .$$

But $\angle PAB = \angle PAC = \angle CQP$ (since these angles are subtended by the same arc). Therefore, $\angle APB = \angle CQD$.

3. Richard is walking up a stair that has 10 steps. With each stride he goes up either one step or two steps. In how many different ways can Richard go up the stairs?

Solved by Robert Bilinski, Collège Montmorency, Laval, QC; Pierre Bornsztein, Maisons-Laffitte, France; Paolo Custodi, Fara Novarese, Italy; and Geoffrey A. Kandall, Hamden, CT, USA. We give Custodi's write-up, expanded by the editor.

To count the number of ways in which Richard climbs 2–step blocks k times, we are really looking at the number of ways of arranging a sequence of k 2–step blocks and $10 - 2k$ 1–step blocks, which is $\binom{10-k}{k}$. Thus, the number of different ways Richard can go up the stairs is

$$\binom{10}{0} + \binom{9}{1} + \binom{8}{2} + \binom{7}{3} + \binom{6}{4} + \binom{5}{5} = 89 .$$

Generalizing, if there are $2n$ steps, the number of ways is $\sum_{k=0}^{n} \binom{2n-k}{k}$.

4. In Flora's number-set there are the numbers

$$2^n - 1 ,\quad 3^{2n} - 1 ,\quad 4^{3n} - 1 ,\quad 5^{4n} - 1 ,\quad 6^{5n} - 1 ,\quad 7^{6n} - 1 ,\quad 8^{7n} - 1 ,\quad 9^{8n} - 1 ,$$

for each natural number n, and there are no other numbers in the set. How many square numbers does the set contain?

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Paolo Custodi, Fara Novarese, Italy; Geoffrey A. Kandall, Hamden, CT, USA; and R. Laumen, Deurne, Belgium. We give Kandall's write-up.

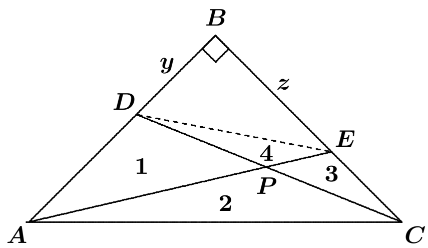
The numbers $3^{2n} - 1$, $5^{4n} - 1$, $7^{6n} - 1$, and $9^{8n} - 1$ can be eliminated, since tbey are 1 less than a square. The numbers $2^n - 1$ for $n \geq 2$, $4^{3n} - 1$,

$6^{5n} - 1$, and $8^{7n} - 1$ can be eliminated, since each is congruent to $-1 \pmod 4$, whereas a square must be congruent to 0 or $1 \pmod 4$.

This leaves only the one square $2^1 - 1 = 1$.

5. Triangle ABC is isosceles with a right angle at B and $AB = BC = x$. Point D on the side AB and point E on the side BC are chosen such that $BD = BE = y$. The line segments AE and CD intersect at the point P. What is the area of the triangle APC, expressed in terms of x and y?

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; Robert Bilinski, Collège Montmorency, Laval, QC; Pierre Bornsztein, Maisons-Laffitte, France; Paolo Custodi, Fara Novarese, Italy; Geoffrey A. Kandall, Hamden, CT, USA; and Edward T.H. Wang and Kaiming Zhao, Wilfrid Laurier University, Waterloo, ON. We give the generalization by Wang and Zhao.



We consider the more general problem of determining the area of $\triangle APC$ if $BD = y$ and $BE = z$. The given problem is the special case where $y = z$.

We connect DE and let $\triangle_1$, $\triangle_2$, $\triangle_3$, $\triangle_4$ denote the areas of the triangles labelled by 1, 2, 3, 4, respectively. We are to determine $\triangle_2$ in terms of x, y, and z.

From the diagram, it is easily seen that

$$\triangle_1 + \triangle_2 = \tfrac{1}{2}x(x-y)\,, \qquad (1)$$

$$\triangle_1 + \triangle_4 = \tfrac{1}{2}z(x-y)\,, \qquad (2)$$

$$\text{and} \quad \triangle_2 + \triangle_3 = \tfrac{1}{2}x(x-z)\,. \qquad (3)$$

Since $\dfrac{\triangle_1}{\triangle_2} = \dfrac{DP}{CP} = \dfrac{\triangle_4}{\triangle_3}$, we have, using (2) and (3),

$$\frac{\triangle_1}{\triangle_2} = \frac{\triangle_1 + \triangle_4}{\triangle_2 + \triangle_3} = \frac{\frac{1}{2}z(x-y)}{\frac{1}{2}x(x-z)}\,;$$

that is, $\triangle_1 = \dfrac{z(x-y)}{x(x-z)}\triangle_2$. Substituting into (1), we get

$$\frac{z(x-y)}{x(x-z)}\triangle_2 + \triangle_2 = \tfrac{1}{2}x(x-y)\,,$$

$$\text{or} \quad \big(z(x-y) + x(x-z)\big)\triangle_2 = \tfrac{1}{2}x^2(x-y)(x-z)\,.$$

Hence, $\triangle_2 = \dfrac{x^2(x-y)(x-z)}{2(x^2-yz)}$.

For the original problem, where $y = z$, we have $\triangle_2 = \dfrac{x^2}{2}\left(\dfrac{x-y}{x+y}\right)$.

Remark. Even more generally, we can remove the assumption that $\angle B$ is a right angle and show, using arguments similar to those given above, that $\triangle_2 = \frac{(x-y)(x-z)}{x^2-yz}\triangle$, where $\triangle$ denotes the area of triangle ABC.

6. How many natural numbers are divisible by 2001 and have exactly 2001 natural divisors?

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Paolo Custodi, Fara Novarese, Italy; Geoffrey A. Kandall, Hamden, CT, USA; R. Laumen, Deurne, Belgium; Vedula N. Murty, Dover, PA, USA; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Custodi's write-up.

Let n be a natural number which is divisible by 2001 and also has 2001 divisors. Decompose n into prime factors as $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}$, where $p_1, p_2, \ldots, p_k$ are distinct primes and $a_1, a_2, \ldots, a_k$ are positive integers. Then the number of divisors of n is

$$\prod_{i=1}^{k}(a_i+1) \;=\; 2001 \;=\; 3\cdot 23\cdot 29\,.$$

Since n is divisible by 2001, each prime factor of 2001 is a factor of n. It follows that $k=3$ and $\{a_1, a_2, a_3\} = \{2, 22, 28\}$. Therefore, we must have $\{p_1, p_2, p_3\} = \{3, 23, 29\}$. Now there are six possibilities for n:

$$\begin{array}{lll} 3^2\cdot 23^{22}\cdot 29^{28}\,, & 3^2\cdot 23^{28}\cdot 29^{22}\,, & 3^{22}\cdot 23^2\cdot 29^{28}\,, \\ 3^{22}\cdot 23^{28}\cdot 29^2\,, & 3^{28}\cdot 23^2\cdot 29^{22}\,, & 3^{28}\cdot 23^{22}\cdot 29^2\,. \end{array}$$

Now we have readers' solutions to problems of the Greek Mathematical Competitions Selection Examination for the IMO 2002 given in [2005 : 27].

2. Let x, y, a be real numbers such that

$$x+y \;=\; x^3+y^3 \;=\; x^5+y^5 \;=\; a\,.$$

Determine all the possible values of a.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Geoffrey A. Kandall, Hamden, CT, USA; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Bataille's solution.

The possible values of a are 0, 1, -1, 2, and -2.

Observe that $a=0$ when $y=-x$, $a=1$ when $x=0$ and $y=1$, $a=-1$ when $x=0$ and $y=-1$, $a=2$ when $x=y=1$, and $a=-2$ when $x=y=-1$.

Conversely, suppose that $x+y = x^3+y^3 = x^5+y^5 = a \neq 0$ for some real numbers x and y. We show that a must be 1, -1, 2, or -2.

Let $p = xy$. Then $a = x^3 + y^3 = (x+y)^3 - 3xy(x+y) = a^3 - 3ap$, which yields $a^2 = 1 + 3p$. Also,

$$a = x^5 + y^5 = (x+y)^5 - 5xy(x^3+y^3) - 10xy(x+y) = a^5 - 15ap\,,$$

and hence, $a^4 = 1 + 15p$. Thus, $1 + 15p = (1+3p)^2$, which is easily solved to get $p = 0$ or $p = 1$.

If $p = 0$, then the equation $a^2 = 1 + 3p$ gives $a = 1$ or $a = -1$. If $p = 1$, then the same equation gives $a = 2$ or $a = -2$.

4. Prove that the following inequality holds for every triple (a, b, c) of non-negative real numbers with $a^2 + b^2 + c^2 = 1$:

$$\frac{a}{b^2+1} + \frac{b}{c^2+1} + \frac{c}{a^2+1} \;\geq\; \frac{3}{4}\left(a\sqrt{a} + b\sqrt{b} + c\sqrt{c}\right)^2 .$$

When does equality hold?

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Nick Skombris and Babis Stergiou, Chalkida, Greece. We give the solution by Skombris and Stergiou, modified by the editor.

First we note that if x_1, x_2, x_3 are any real numbers and w_1, w_2, w_3 are any positive real numbers, then, by the Cauchy-Schwarz Inequality,

$$\left(\sum_{i=1}^{3} x_i\right)^2 = \left(\sum_{i=1}^{3} \sqrt{w_i}\frac{x_i}{\sqrt{w_i}}\right)^2 \leq \left(\sum_{i=1}^{3} w_i\right)\left(\sum_{i=1}^{3} \frac{x_i^2}{w_i}\right) . \qquad (1)$$

Now we let $x_1 = a\sqrt{a}$, $x_2 = b\sqrt{b}$, $x_3 = c\sqrt{c}$, $w_1 = a^2(b^2+1)$, $w_2 = b^2(c^2+1)$, and $w_3 = c^2(a^2+1)$. Then

$$\begin{aligned}\sum_{i=1}^{3} \frac{x_i^2}{w_i} &= \frac{\left(a\sqrt{a}\right)^2}{a^2(b^2+1)} + \frac{\left(b\sqrt{b}\right)^2}{b^2(c^2+1)} + \frac{\left(c\sqrt{c}\right)^2}{c^2(a^2+1)} \\ &= \frac{a}{b^2+1} + \frac{b}{c^2+1} + \frac{c}{a^2+1}\,, \qquad (2)\end{aligned}$$

and

$$\sum_{i=1}^{3} w_i = a^2b^2 + b^2c^2 + c^2a^2 + a^2 + b^2 + c^2 = a^2b^2 + b^2c^2 + c^2a^2 + 1\,.$$

Since

$$a^2b^2+b^2c^2+c^2a^2 \leq a^4+b^4+c^4 = (a^2+b^2+c^2)^2 - 2\left(a^2b^2+b^2c^2+c^2a^2\right),$$

we have

$$a^2b^2 + b^2c^2 + c^2a^2 \leq \tfrac{1}{3}\left(a^2+b^2+c^2\right)^2 = \tfrac{1}{3}\,.$$

Thus, $\sum\limits_{i=1}^{3} w_i \leq \frac{1}{3} + 1 = \frac{4}{3}$. Using this result along with (2) in (1), we get

$$\left(a\sqrt{a} + b\sqrt{b} + c\sqrt{c}\right)^2 \leq \frac{4}{3}\left(\frac{a}{b^2+1} + \frac{b}{c^2+1} + \frac{c}{a^2+1}\right),$$

which immediately yields the desired result.

We have equality if and only if $a = b = c = \frac{1}{3}\sqrt{3}$.

Next we look at solutions to problems of the 16^{th} China Mathematical Olympiad, Selected Problems, given in [2005 : 28].

2. Let $X = \{1, 2, \ldots, 2001\}$. Find the minimum positive integer m such that, for each m–element subset W of X, there exist u, $v \in W$ (u and v may be the same) with $u + v = 2^k$ for some positive integer k.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

We prove that $m = 999$.

Let W be a subset of X such that there do not exist u, $v \in W$ with $u + v$ a power of 2. Consider the pairs of the form:

(a) $(47 + k, 2001 - k)$ for $k = 0$ to 976 (these pairs contain all the integers from 47 to 2001, except 1024, and each pair sums to $2048 = 2^{11}$).

(b) $(18 + k, 46 - k)$ for $k = 0$ to 13 (these pairs contain all the integers from 18 to 46, except 32, and each pair sums to $64 = 2^6$).

(c) $(2 + k, 14 - k)$ for $k = 0$ to 5 (these pairs contain all the integers from 2 to 14, except 8, and each pair sums to $16 = 2^4$).

(d) $(15, 17)$ (this pair sums to $32 = 2^5$).

This leaves 1, 8, 16, 32, 1024 as the only the unpaired integers.

Clearly, W does not contain any of the 5 unpaired integers and contains at most one integer from each pair. Thus,

$$|W| \;\leq\; 2001 - 977 - 14 - 6 - 1 - 5 \;=\; 998\,,$$

which leads to $m \leq 999$.

On the other hand, direct checking shows that if

$$W \;=\; \{9, 10, \ldots, 15, 33, 34, \ldots, 46, 1025, 1026, \ldots, 2001\},$$

then W does not contain two elements which add up to a power of 2, and $|W| = 998$. Therefore $m \geq 999$, and we are done.

3. At each vertex of a regular n-sided polygon, there was a magpie. When scared, all the magpies flew away. After a while they all returned, one to each vertex, but not all to their former positions. Find all positive integers n for which there must be 3 magpies such that the triangle formed by the vertices at which they first stood and the triangle formed by the vertices at which they now stand are both acute triangles, both right triangles, or both obtuse triangles.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

We prove that the desired values of n are those such that $n \geq 3$ and $n \neq 5$.

Clearly $n \geq 3$. The case $n = 3$ is trivial since there is only one triangle.

Case 1. n is even.

Let us consider any two diametrically opposite vertices, say A and B, of the polygon. Then, for each other vertex C, the triangle ABC is right-angled. Assume that the magpies which initially stood at A and B now stand at A' and B', respectively.

If A' and B' are diametrically opposite vertices, then, for any third bird, which goes from a vertex C to a vertex C', the triangles ABC and $A'B'C'$ are right-angled. Otherwise, let C' be the vertex diametrically opposite from A'. Then $C' \neq B'$, and the bird which is now standing at C' comes from a vertex $C \neq B$. The triangles ABC and $A'B'C'$ are right-angled again.

This proves that all even values of n (except 2) are solutions.

Case 2. n is odd and $n \geq 5$.

We will prove that, if $n \geq 7$, there are more obtuse than acute triangles among the triangles whose vertices are vertices of the polygon (note that there is no right triangle). Assume we have proved this assertion. Then it is impossible to transform each obtuse triangle into an acute one, so that there necessarily exists an obtuse triangle which is transformed into another obtuse triangle. In that case, n is a solution of the problem.

Now let us prove the assertion. Let $n = 2k+1$, and let $A_1 A_2 \ldots A_{2k+1}$ be the polygon. Then the number of triangles having vertices among $\{A_1, A_2, \ldots, A_{2k+1}\}$ is

$$T_n \;=\; \frac{n(n-1)(n-2)}{6} \;=\; \frac{2k(2k+1)(2k-1)}{6}\,.$$

The triangle $A_i A_1 A_j$ (with $i < j$) has an obtuse angle at A_1 if and only if $i \in \{2, \ldots, k+1\}$ and $j \in \{k+1+i, \ldots, 2k+1\}$. Thus, for a given $i \in \{2, \ldots, k+1\}$, there are $k+1-i$ admissible values for j. It follows that there are $k(k-1)/2$ obtuse triangles with obtuse angle at A_1. The same reasoning applies to the other vertices. Since each obtuse triangle has only one obtuse angle, the total number of obtuse triangles is

$$O_n \;=\; n \cdot k(k-1)/2 \;=\; (2k+1)k(k-1)/2\,.$$

Now, it is straightforward to verify that $O_n > \frac{1}{2} T_n$ for $k \geq 3$, which is $n \geq 7$. This proves the claim.

Case 3. $n = 5$.

In this case, the existence is not assured, as the following example shows: Assume that the magpies are initially at positions $M_1 M_2 M_3 M_4 M_5$, and that after being scared they come back at positions $M_1 M_4 M_2 M_5 M_3$.

Thus, $n = 5$ is not a solution.

4. Let a, b, c, $a+b-c$, $a+c-b$, $b+c-a$, $a+b+c$ be 7 distinct prime numbers such that the sum of two of a, b, c is 800. Let d be the difference between the largest and smallest numbers among the 7 primes. Find the largest possible value of d.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

The largest possible value of d is 1594.

First note that if one of a, b, c is 2, say $a = 2$, then b and c are odd, so that $a+b+c$ is even and greater than 2, which contradicts the hypothesis that it is a prime. Thus a, b, $c \geq 3$ and all the seven primes are odd.

Without loss of generality, we may assume that $a+b = 800$ and $a < b$. Since $a+b-c \geq 3$ (since it is an odd prime number), we deduce that $c \leq 797$. Clearly, the greatest of the seven primes is $a+b+c$. Therefore,

$$d \;\leq\; (a+b+c)-3 \;\leq\; 800+797-3 \;=\; 1594\,.$$

Conversely, note that $800+797 = 1597$ is a prime. And 797 is a prime too. Thus, if $a = 13$, $b = 787$ and $c = 797$ it follows that $a+b+c = 1597$, $a+b-c = 23$, $a+c-b = 3$ and $b+c-a = 1571$ are all primes. And in that case $d = 1594$.

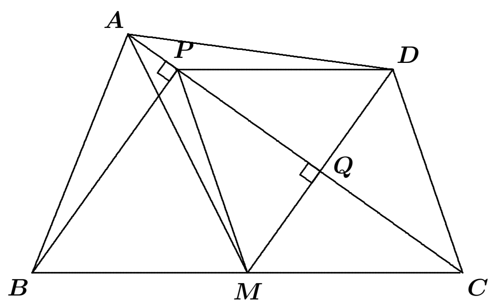
Now we return to the April 2005 number and solutions to problems of the XX Colombian Mathematical Olympiad, Higher Level, which appeared in [2005 : 150–151].

1. [7 points] Let ABC be an isosceles triangle with $AB = AC$. Let M be the mid-point of side BC. The circle with diameter AB cuts side AC at point P. The parallelogram $MPDC$ is constructed so that $PD = MC$ and $PD \parallel MC$. Prove that triangles APD and APM are congruent.

Solution by Geoffrey A. Kandall, Hamden, CT, USA.

It is not necessary to assume that $\triangle ABC$ is isosceles.

Let Q be the point of intersection of AC and MD. Since $MPDC$ is a parallelogram, we have $MQ = QD$. Also, $BPDM$ is a parallelogram; in particular, $BP \parallel MD$. Since $\angle APB$ is a right angle, so is $\angle AQM$. Thus, AC is the perpendicular bisector of MD. Therefore, $AM = AD$ and $PM = PD$, and it follows that $\triangle APD \cong \triangle APM$ by SSS.



2. [7 points] Find all positive integers z for which the equation

$$x(x+z) \;=\; y^2$$

has no solutions x, y that are positive integers.

Solved by Houda Anoun, Bordeaux, France; Pierre Bornsztein, Maisons-Laffitte, France; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Bornsztein's solution.

We will say that a positive integer z is *bad* if the equation

$$x(x+z) \;=\; y^2 \tag{1}$$

has no solution in positive integers (and z is *good* otherwise). We will prove that the bad integers, for which the problem is asking, are 1, 2 and 4.

First note that if (x, y, z) is a solution of (1) in positive integers, then $(2x, 2y, 2z)$ is also a solution. Therefore, if z is good, then $2z$ is also good.

If $z = 2n + 1$, with $n \geq 1$, then z is good because it suffices to choose $x = n^2$ and $y = n(n+1)$ to obtain a solution of (1) in positive integers. It follows that each positive integer which is not a power of 2 is good.

For each positive integer x, we have $x^2 < x(x+4) < (x+2)^2$. If $x(x+4) = y^2$, then we must have $x(x+4) = (x+1)^2$. But this equation is equivalent to $2x = 1$, which has no integer solutions. Thus, $z = 4$ is bad, from which we deduce that 1, 2, and 4 are bad.

Clearly, $(1, 3, 8)$ is a solution of (1), which means that $z = 8$ is good. Then $z = 2^n$ is good for all integers $n \geq 3$, and the proof is complete.

3. [7 points] Let $n \geq 4$ be a fixed integer. Let $S = \{P_1, P_2, \ldots, P_n\}$ be a set of n points in the plane, no three of which are collinear and no four concyclic. Let a_t, $1 \leq t \leq n$, be the number of circles $P_iP_jP_k$ that contain P_t in the interior, and let

$$m(S) \;=\; a_1 + a_2 + \cdots + a_n\,.$$

Prove that there exists a positive integer $f(n)$, depending only on n, such that the points of S are the vertices of a convex polygon if and only if $m(S) = f(n)$.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

We will prove that $f(n) = 2\binom{n}{4} = \frac{1}{12}n(n-1)(n-2)(n-3)$.

Lemma. If $A_1, \ldots, A_n$ are $n \geq 4$ points of the plane, no three of which are collinear, and such that any four of them are always the vertices of a convex quadrilateral, then they are the vertices (in some order) of a convex n–gon.

Proof of the lemma. Suppose, for the purpose of contradiction, that the convex hull $\mathcal{C}$ of the n points is not an n–gon. Then at least one of the points, say A_1, is an interior point of $\mathcal{C}$. Without loss of generality, we may assume that A_2 is a vertex of $\mathcal{C}$. We triangulate $\mathcal{C}$ by using the diagonals from A_2. Since no three points are collinear, we deduce that A_1 is an interior point of one of the triangles, say $A_2A_3A_4$. But this contradicts the hypothesis that A_1, A_2, A_3, A_4 are the vertices of a convex quadrilateral. ∎

Returning to the problem, let A, B, C, D be any four distinct points from S. Let $n_A = 1$ if the circumcircle of the triangle BCD contains A, and $n_A = 0$ otherwise. The numbers n_B, n_C, n_D are defined similarly. Since no three are collinear, the convex hull of the four points is either a triangle or a quadrilateral.

Case 1. The convex hull is a triangle.

Without loss of generality, we may assume that D is an interior point of ABC. Therefore, among the four circles which go through three of the points, only the circumcircle of ABC contains the fourth point. That is, $n_A = n_B = n_C = 0$ and $n_D = 1$.

Thus, in that case, we have $n_A + n_B + n_C + n_D = 1$.

Case 2. The convex hull is a quadrilateral, say $ABCD$.

Let $\alpha = \angle DAB$, $\beta = \angle ABC$, $\gamma = \angle BCD$, and $\delta = \angle CDA$. Then $\alpha + \beta + \gamma + \delta = 2\pi$. Since the four points are not concylic, it follows that either $\alpha + \gamma > \pi$ or $\beta + \delta > \pi$, but not both.

Without loss of generality, we may assume that $\alpha + \gamma > \pi$. Thus, $\angle BCD = \gamma > \pi - \alpha = \angle BAD$, which means that C is an interior point of the circumcircle of ABD. Hence, we have $n_C = 1$.

Similarly, we have $n_A = 1$. Since $\beta + \delta < \pi$, similar reasoning shows that $n_B = n_D = 0$. Thus, in that case, we have $n_A + n_B + n_C + n_D = 2$.

From above, we deduce that for each group of four points in S, say A, B, C, D, we have

$$n_A + n_B + n_C + n_D \ \leq\ 2\,, \tag{1}$$

with equality if and only if A, B, C, D are the vertices of a convex quadrilateral.

There are $\binom{n}{4}$ groups of four points in S. Summing over these groups the inequalities of the form (1), it follows that $m(S) \leq 2\binom{n}{4}$, with equality if and only if each group of four points in S is the set of vertices of a convex quadrilateral. From the lemma, it follows that $m(S) \leq 2\binom{n}{4}$, with equality if and only if the points of S are the vertices of a convex polygon. Thus, letting $f(n) = 2\binom{n}{4}$ proves the claim.

4. [7 points] Let x and y be any two real numbers. Prove that

$$3(x+y+1)^2 + 1 \ \geq\ 3xy\,.$$

Under what conditions does equality hold?

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Houda Anoun, Bordeaux, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give the solution of Amengual Covas.

For any real numbers X and Y,

$$X^2 + Y^2 + XY \;=\; \left(X + \tfrac{1}{2}Y\right)^2 + \tfrac{3}{4}Y^2 \;\geq\; 0\,,$$

and equality holds if and only if $X = Y = 0$.

Let x and y be any two real numbers. Letting $X = x+\frac{2}{3}$ and $Y = y+\frac{2}{3}$ above, we obtain

$$\left(x + \tfrac{2}{3}\right)^2 + \left(y + \tfrac{2}{3}\right)^2 + \left(x + \tfrac{2}{3}\right)\left(y + \tfrac{2}{3}\right) \;\geq\; 0\,.$$

Expanding and multiplying by 3 gives

$$3x^2 + 3y^2 + 3xy + 6x + 6y + 4 \;\geq\; 0\,.$$

This may be written as

$$3(x + y + 1)^2 - 3xy + 1 \;\geq\; 0\,,$$

from which we arrive at the desired inequality.

Equality holds if and only if $x + \frac{2}{3} = y + \frac{2}{3} = 0$; that is, if and only if $x = y = -\frac{2}{3}$.

6. Mr. Leonardo invited a group of children to go for a ride around a lake on his boat, in several turns. He later realized that the following things had happened:

- In each turn, there had been exactly three children on the boat.
- Each pair of children had been together on the boat in exactly one turn.

(a) [2 points] Prove that if Mr. Leonardo invited n children, then n must be a number of the form $6t + 1$ or $6t + 3$, where t is a non-negative integer.

(b) [5 points] Prove that, for any non-negative integer t, Mr. Leonardo can invite $6t + 3$ children under the above conditions.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

(a) There are $\frac{1}{2}n(n-1)$ pairs of children. Since each turn concerns exactly three children, these pairs divide into k groups of three, where k is the number of turns. Thus, $\frac{1}{2}n(n-1) = 3k$, or $n(n-1) = 6k \equiv 0 \pmod 6$.

Let A be one of the children. Since the remaining $n-1$ children divide into disjoint pairs to take a turn with A, it follows that $n \equiv 1 \pmod 6$, $n \equiv 3 \pmod 6$, or $n \equiv 5 \pmod 6$. However, if $n \equiv 5 \pmod 6$, then $n(n-1) \equiv 2 \not\equiv 0 \pmod 6$. Hence, $n \equiv 1 \pmod 6$ or $n \equiv 3 \pmod 6$.

(b) Let $n = 6t + 3$, where t is a non-negative integer. If $t = 0$, only one turn suffices. Thus, we assume that $t \geq 1$.

Note that $n = 3(2t + 1)$, where $2t + 1 \geq 3$ is odd. Thus, we may identify the children with couples (x, r), where $x \in \{1, 2, \ldots, 2t + 1\}$ and $r \in \{0, 1, 2\}$. In the following, the computations are assumed to be modulo $2t + 1$ for the first component and modulo 3 for the second one.

Since $2t+1 \geq 3$ is odd, the number 2 has an inverse modulo $2t+1$; let $\frac{1}{2}$ denote that inverse. Then, the turns are formed by the following triples:

- all the triples of the form $\{(x, 0), (x, 1), (x, 2)\}$;
- all the triples of the form $\left\{(x, r), (y, r), \left(\frac{1}{2}(x + y), r + 1\right)\right\}$, where $x \not\equiv y \pmod{2t + 1}$.

Remark. Triples satisfying the statement of the problem are said to form a *Steiner Triple System* (STS). From (a), a set with cardinality n has a STS only if $n = 1 \pmod 6$ or $n = 3 \pmod 6$. And from (b), the converse is true for $n = 3 \pmod 6$. In fact, the converse also holds for $n = 1 \pmod 6$, as first proved by Kirkman (1847).

References:

[1] T.P. Kirkman, On a problem in combinations, *Cambridge and Dublin Math. J.*, 2 (1847), pp. 191–204.

[2] J.H. van Lint, R.M. Wilson, *A course in Combinatorics*, second edition, Cambridge University Press, p. 236.

Lastly, we look at solutions from our readers to problems of the 53rd Polish Mathematical Olympiad 2001–2002, Final Round [2005 : 151–152].

1. Determine all triples of positive integers a, b, c such that a^2+1 and b^2+1 are prime numbers satisfying $(a^2 + 1)(b^2 + 1) = c^2 + 1$.

Solution by Edward T.H. Wang and Kaiming Zhao, Wilfrid Laurier University, Waterloo, ON.

The only solution is $(a, b, c) = (1, 2, 3)$.

Suppose that a, b, c satisfy

$$(a^2 + 1)(b^2 + 1) \;=\; c^2 + 1\,, \tag{1}$$

where $a^2 + 1 = p$ and $b^2 + 1 = q$ are both prime. If $p = q$, then from (1) we obtain $p^2 = c^2 + 1$, or $(p - c)(p + c) = 1$, which is clearly impossible. Thus, $p \neq q$. Without loss of generality, we may assume $p < q$. Then $a < b < c$.

Rewriting (1) as $a^2b^2 + a^2 + b^2 = c^2$, we have

$$(c - a)(c + a) \;=\; c^2 - a^2 \;=\; b^2(a^2 + 1) \;=\; b^2p \tag{2}$$

$$\text{and}\quad (c - b)(c + b) \;=\; c^2 - b^2 \;=\; a^2(b^2 + 1) \;=\; a^2q\,. \tag{3}$$

Since p and q are primes, we deduce from (2) and (3) that $p \mid (c-a)$ or $p \mid (c+a)$, and $q \mid (c-b)$ or $q \mid (c+b)$. Thus, we have four possible cases:

Case 1. $p \mid (c-a)$ and $q \mid (c-b)$.

Then $pq \mid (c-a)(c-b)$. Hence, $(c^2+1) \mid (c-a)(c-b)$, which is impossible, since $0 < (c-a)(c-b) < c^2 < c^2+1$.

Case 2. $p \mid (c+a)$ and $q \mid (c-b)$.

Then $pq \mid (c+a)(c-b)$. Hence, $(c^2+1) \mid (c+a)(c-b)$, which is impossible, since

$$c^2+1-(c+a)(c-b) \;=\; bc-ac+ab+1 \;=\; c(b-a)+ab+1 \;>\; 0\,.$$

Case 3. $p \mid (c-a)$ and $q \mid (c+b)$.

Then $(c^2+1) \mid (c-a)(c+b)$. Hence, $(c-a)(c+b) = k(c^2+1)$ for some $k \in \mathbb{N}$. However, $(c-a)(c+b) < c(2c) = 2c^2 < 2(c^2+1)$; whence, $k=1$. Therefore, $(c-a)(c+b) = c^2+1$, which can be rewritten as $(b-a)(c-a) = a^2+1 = p$.

Since $b-a < c-a$, we must have $b-a=1$ and $c-a=p$. Since $b=a+1$, we see that a and b have opposite parity; thus, $p=a^2+1$ and $q=b^2+1$ must also have opposite parity. Since $p<q$, we conclude that $p=2$. It follows that $a=1$ and $b=2$, and we obtain the solution $(a,b,c)=(1,2,3)$.

Case 4. $p \mid (c+a)$ and $q \mid (c+b)$.

Then $(c^2+1) \mid (c+a)(c+b)$. Hence, $(c+a)(c+b) = m(c^2+1)$ for some $m \in \mathbb{N}$. However,

$$\begin{aligned}(c+a)(c+b) &< (c+a)(c+b)+(c-a)(c-b)+(a-b)^2+a^2b^2 \\ &= 2c^2+a^2+b^2+a^2b^2 \\ &= 3c^2 < 3(c^2+1)\,,\end{aligned}$$

where we have used (1) in the second-last step. Thus, we see that $m=2$. Then, from $(c+a)(c+b) = 2(c^2+1)$, we obtain

$$(a+b)c+ab \;=\; c^2+2\,. \tag{4}$$

If $p \neq 2$, then both p and q are odd, which implies that a and b are both even. Hence, c is also even. Let $a=2a_1$, $b=2b_1$, and $c=2c_1$. Then (4) becomes $4(a_1+b_1)c_1+4a_1b_1 = 4c_1^2+2$ or $2(a_1+b_1)c_1+2a_1b_1 = 2c_1^2+1$, which is clearly impossible. Hence, $p=2$ and $a=1$. Substituting into (3), we then have $(c-b)(c+b)=q$. Hence, $c-b=1$ and $c+b=q$. Solving, we have $b=(q-1)/2=b^2/2$, from which it follows that $b=2$, and again we are led to the solution $(a,b,c)=(1,2,3)$.

This completes the proof.

2. On sides AC and BC of an acute-angled triangle ABC, rectangles $ACPQ$ and $BKCL$ are erected outwardly. Assuming that these rectangles have equal areas, show that the vertex C, the circumcentre of triangle ABC, and the mid-point of segment PL are collinear.

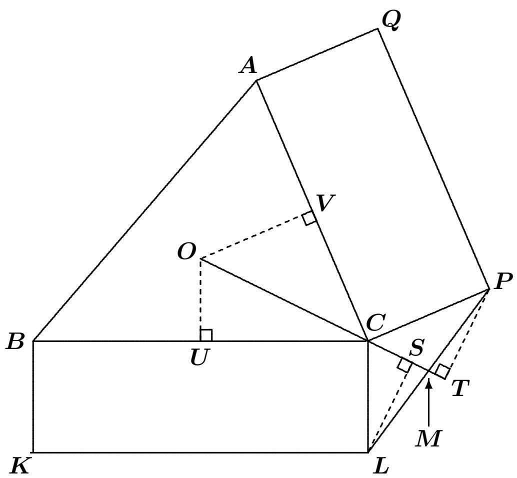
Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; and Geoffrey A. Kandall, Hamden, CT, USA. We first give Amengual Covas' write-up.

Let O be the circumcentre of $\triangle ABC$. Let OC (produced) intersect PL at M. We will prove that M is the mid-point of PL. The desired conclusion follows from this.

Let S and T be the feet of the perpendiculars to OC from L and P, respectively, and let U and V be the feet of the perpendiculars from O to BC and CA, respectively. Since $\angle OCU$ is a complementary angle for both $\angle LCS$ and $\angle COU$, we have $\angle LCS = \angle COU$. It follows that the right triangles OUC and CSL are similar. Hence, $LS/UC = CL/OC$. Similarly, $PT/VC = CP/OC$. Therefore,

$$LS = \frac{UC \cdot CL}{OC} = \frac{\frac{1}{2}[BKLC]}{OC} = \frac{\frac{1}{2}[ACPQ]}{OC} = \frac{VC \cdot CP}{OC} = PT\,.$$

This implies that $LM = MP$; that is M is the mid-point of PL.



Next we give the write-up by Bataille using complex numbers.

We consider the figure to be drawn in the complex plane with origin at the circumcentre O of $\triangle ABC$. We denote by M the mid-point of PL and, generally, by x the complex representation of the point X (so that $m = \frac{1}{2}(p + \ell)$, for example).

Now $CL \cdot CB = CP \cdot CA$, since rectangles $BCLK$ and $ACPQ$ have the same area. Therefore, we may define

$$\varrho = \frac{|\ell - c|}{|a - c|} = \frac{|p - c|}{|b - c|}\,.$$

Without loss of generality, we may suppose that triangle ABC is positively oriented. Letting $\gamma = \angle\overrightarrow{ACB}$, we have $\angle\overrightarrow{ACL} = \gamma + \frac{\pi}{2}$ and

$\angle\overrightarrow{BCP} = -(\gamma + \frac{\pi}{2})$. It follows that

$$\frac{\ell - c}{a - c} = \varrho e^{i(\gamma+\pi/2)} = i\varrho e^{i\gamma}$$

and
$$\frac{p - c}{b - c} = \varrho e^{-i(\gamma+\pi/2)} = -i\varrho e^{-i\gamma}\,.$$

Now,

$$\begin{aligned} m - c &= \tfrac{1}{2}(p + \ell) - c \;=\; \tfrac{1}{2}\big[(\ell - c) + (p - c)\big] \\ &= \tfrac{1}{2}\left[(a - c)\varrho i e^{i\gamma} - (b - c)\varrho i e^{-i\gamma}\right] \\ &= \tfrac{1}{2}\left[\varrho i(ae^{i\gamma} - be^{-i\gamma}) + 2c\varrho\sin\gamma\right]\,. \end{aligned}$$

Since $\angle\overrightarrow{AOB} = 2\gamma$ and $OA = OB$, we get $b = ae^{2i\gamma}$, or $be^{-i\gamma} = ae^{i\gamma}$. Thus, $m - c = (\varrho\sin\gamma)c$, which means that $\overrightarrow{CM} = (\varrho\sin\gamma)\overrightarrow{OC}$. Thus, O, C, and M are collinear.

4. Prove that, for every integer $n \geq 3$ and every sequence of positive numbers $x_1, x_2, \ldots, x_n$, at least one of the following two inequalities is satisfied:

$$\sum_{i=1}^{n} \frac{x_i}{x_{i+1} + x_{i+2}} \geq \frac{n}{2}\,, \qquad \sum_{i=1}^{n} \frac{x_i}{x_{i-1} + x_{i-2}} \geq \frac{n}{2}\,.$$

(Note: Here $x_{n+1} = x_1$, $x_{n+2} = x_2$, $x_0 = x_n$, and $x_{-1} = x_{n-1}$.)

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain. We give the solution of Díaz-Barrero, modified by the editor.

Denote the sums in the inequalities above by S_1 and S_2, respectively. If $S_1 < n/2$ and $S_2 < n/2$, then $S_1 + S_2 < n$. Therefore, it will be sufficient to prove that $S_1 + S_2 \geq n$.

We start with the inequality $\alpha + \dfrac{1}{\alpha} \geq 2$, which is true for all positive real numbers α. For each $i \in \{1, 2, \ldots, n\}$, let $\alpha_i = \dfrac{x_{i-1} + x_i}{x_i + x_{i+1}}$. Then $\alpha_i + \dfrac{1}{\alpha_i} \geq 2$ for each i, and hence,

$$2n \leq \sum_{i=1}^{n}\left(\alpha_i + \frac{1}{\alpha_i}\right) = \sum_{i=1}^{n}\left(\alpha_i + \frac{1}{\alpha_{i+1}}\right),$$

where $\sum\limits_{i=1}^{n} \dfrac{1}{\alpha_i} = \sum\limits_{i=1}^{n} \dfrac{1}{\alpha_{i+1}}$ because the sum is cyclic. Thus,

$$\begin{aligned} 2n &\leq \sum_{i=1}^{n}\left(\frac{x_{i-1} + x_i}{x_i + x_{i+1}} + \frac{x_{i+1} + x_{i+2}}{x_i + x_{i+1}}\right) \\ &= \sum_{i=1}^{n}\left(\frac{x_i + x_{i+1}}{x_i + x_{i+1}} + \frac{x_{i-1} + x_{i+2}}{x_i + x_{i+1}}\right) = n + \sum_{i=1}^{n} \frac{x_{i-1} + x_{i+2}}{x_i + x_{i+1}}\,. \end{aligned}$$

Then

$$\begin{aligned} n &\le \sum_{i=1}^{n} \frac{x_{i-1}}{x_i + x_{i+1}} + \sum_{i=1}^{n} \frac{x_{i+2}}{x_i + x_{i+1}} \\ &= \sum_{i=1}^{n} \frac{x_i}{x_{i+1} + x_{i+2}} + \sum_{i=1}^{n} \frac{x_i}{x_{i-2} + x_{i-1}} \;=\; S_1 + S_2\,, \end{aligned}$$

where we have again made use of the cyclic nature of the sums to shift the index.

6. Let k be a fixed positive integer. The infinite sequence $\{a_n\}$ is defined by the formulae $a_1 = k + 1$ and $a_{n+1} = a_n^2 - ka_n + k$ for $n \geq 1$. Show that if $m \neq n$, then the numbers a_m and a_n are relatively prime.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Let $P(x) = x^2 - kx + k$. Then $P(a_n) = a_{n+1}$ for all integers $n \geq 1$. If $k \neq 4$, then the quadratic polynomial $P(x)$ has no integer root and it follows that $a_n \neq 0$ for all n. If $k = 4$, then $P(x) = (x-2)^2$. Since $a_1 = 5$, it follows by induction that $a_n \geq 5$ for all n, and again we have $a_n \neq 0$ for all n.

Now, let $n \geq 1$ be given. Clearly, we have $a_{n+1} \equiv k \pmod{a_n}$. If $a_{n+t} \equiv k \pmod{a_n}$ for some $t > 0$, then

$$a_{n+t+1} \;\equiv\; k^2 - k^2 + k \;\equiv\; k \pmod{a_n}\ .$$

It follows by induction that $a_p \equiv k \pmod{a_q}$ for all integers p and q such that $p > q$.

Suppose there exist integers p and q such that $p > q \geq 1$ and $\gcd(a_p, a_q) > 1$. Let q be minimal among all such pairs. Let r be a prime which divides both a_p and a_q. Then r divides k, because $a_p \equiv k \pmod{a_q}$. We must have $q > 1$, because r cannot divide both k and $a_1 = k + 1$. Then r divides a_{q-1}^2, from the inductive construction of the sequence, so that a_q and a_{q-1} are not relatively prime. This contradicts the minimality of q, and we are done.

That completes this number of the *Corner*. The back-log is now cleared up and we are publishing reader solutions about one year after the problems appear; so please send me your nice solutions and generalizations, along with Olympiad Contests!

BOOK REVIEWS

John Grant McLoughlin

New Mexico Mathematics Contest Problem Book
By Liong-Shin Hahn, published by the University of New Mexico Press, 2005.
ISBN 0-8263-3534-9, paperbound, 216 pages, US$29.95.
Reviewed by **Catherine Haines**, *Western Technical & Commercial School, Toronto, ON.*

The *New Mexico Mathematics Contest Problem Book* by Liong-Shin Hahn, a retired professor from the University of New Mexico, comprises 166 problems. Many, but not all, of them originally appeared in the annual New Mexico Mathematics Contest. The main body of the book is organized according to the type of mathematics involved, while an appendix gives actual examination papers from 1990 to 1999, when Professor Hahn was in charge of setting them. The first round of the contest is for all secondary students, and the second round is for the top 15% from the first round. The questions in the second round may be extensions of those in the first round.

The first chapter is divided into two sections: the first containing problems in number theory and algebra, and the second containing problems in geometry and combinatorics. Detailed solutions for all problems follow in the second chapter, which is similarly divided into two sections. Analogous problems are grouped together to help the reader to explore more than one aspect of a topic. The solutions are presented clearly, with flashes of humour, and include generous commentary. The author often provides multiple solutions, inviting the reader to learn new strategies and to compare the efficiency and directions of different approaches, and he also encourages the reader to see if it is possible to generalize the problem.

A typical example is a problem showing that no three lattice points can form an equilateral triangle. There are four given solutions: the "brute force solution" (using the distance between points and understandable by any secondary student), a trigonometric solution, a solution using a determinant, as well as "the slick solution", a reductio proof that does not involve any higher methods but does require insight. This is one of a series of related problems concerning lattice points.

Students will find this a good book to work through. Alternatively, one can easily pick out a group of questions on a certain topic, such as magic squares. Since the solutions are separated from the problems, there is no danger of accidentally seeing the solution to a problem. The problems are arranged progressively within each topic so that the earlier problems are more approachable. While working through the problems, the student can expect to develop problem-solving skills.

The problems in this book can be accessed by any secondary student. There is no need to know calculus. Although some solutions use trigonometry (not necessarily known at this level), other methods of solution are given.

A further pair of appendices comprise four calculus competitions and their solutions. This book would make excellent contest practice, say for the Canadian Open, and could be used by teachers as a source of interesting problems for enrichment.

The Contest Problem Book VII:
American Mathematics Competitions 1995–2000 Contests
Compiled and augmented by Harold Reiter, published by the Mathematical Association of America, 2006.
ISBN 0-88385-821-5, paperbound, 200 pages, US$43.95.
Reviewed by ***John T. Siggers****, retired, Thompson Rivers University, Kamloops, BC.*

From the MAA literature, "The American Mathematics Competitions are intended for everyone from the average student at a typical school who enjoys mathematics to the very best student at the most special school". This latest publication from the MAA bears out this philosophy. It contains a brief history of the American Mathematics Competitions since 1950, the contests from 1995 to 2000, solutions to each of these contests, a selection of 23 other challenging problems with solutions, and a classification by topic of all the contest problems in the book.

The brief history discusses the reasons for the various changes made during the last 50 years. This is invaluable for those who design local area school contests. It addresses the problems of student interest and participation, along with contest difficulty level. It also considers the time constraints when writing the contest in an individual school setting. The recent change to AMC 10 and AMC 12 has allowed more flexibility with respect to the topics covered and the difficulty levels of questions.

The 23 additional problems were not deemed to be appropriate for short multiple choice solutions, but are an excellent source of problems for mathematics clubs or a "Problem of the Month". The classification section is excellent for those wishing multiple choice resources or ideas on a specific topic.

This book is ideal as a resource for the classroom teacher, an academic prize, a mathematics contest prize, material for a mathematics club, and/or a source of ideas for mathematics contest problems. With its dimensions of 23 cm $\times$ 15 cm $\times$ 1 cm, it is easily carried and ideal for those tediously long train or airline trips.

It is a "must-have" publication for those involved with high school mathematics contests.

Non-Transitivity in Tournaments

Jerry Lo and David Rhee

From August 1 to 9, 2005, we attended the International Mathematics Tournament of the Towns Summer Seminar, held in Mir Town, Belarus. Sixty students worked on six research projects. One of them, "The Mathematics of Tournaments", was presented by A. Zaslavsky and B. Frenkin, with contributions from A. Tolpygo and G. Tokarev. Our main results were the proofs of Theorems 2 and 4 below, as well as the determination of the parameters α and β used in the definition of the non-transitivity index. We later learned that our work generalizes that of others, such as [1].

A round-robin tournament, or tournament for short, is a competition in which each pair of players plays each other exactly once. For the moment, we do not allow draws. Thus, each game is a win for one of the players. We can record the results in a complete directed graph as follows. Let the participants be represented by vertices. An arrow is drawn from vertex X to vertex Y if player X beats player Y. This graph is also called a tournament.

Three vertices X, Y, and Z are said to form a transitive triple if, whenever X beats Y and Y beats Z, then X must beat Z. A tournament is said to be transitive if every triple is transitive.

Suppose X_i beats X_j in a transitive tournament with n players. By transitivity, X_i beats every player beaten by X_j. Thus, the players may be labelled X_1, X_2, …, X_n so that X_i beats X_j if and only if $i < j$. Hence, there is essentially only one transitive tournament for each fixed value of n.

In a non-transitive tournament, there must be some triple of vertices which is non-transitive. Such vertices form a directed 3–cycle. Hence, the number of such 3–cycles is a measure of how non-transitive a tournament is. This number is defined as the non-transitivity index λ of the tournament.

Theorem 1 If a tournament has an m–cycle, then $\lambda \geq m - 2$.

Proof: We use induction on m. This is trivial for $m = 3$. Suppose the result holds for some $m \geq 3$. Consider a tournament with an $(m+1)$–cycle $(1, 2, \ldots, m+1)$. For $1 \leq i \leq m+1$, consider the arrow between i and $i-2$ (identifying -1 with m and 0 with $m+1$). If, for each $i = 1, \ldots, m+1$, the arrow between i and $i-2$ goes from i to $i-2$, then the number of 3–cycles $(i-2, i-1, i)$ is $m+1$, since we have arrows going from $i-2$ to $i-1$ and from $i-1$ to i along the $(m+1)$–cycle. In this case we are done. Suppose instead that, for some i, the arrow between i and $i-2$ goes from $i-2$ to i. By cyclic symmetry, we may assume that this arrow goes from m to 1. Then we have an m–cycle $(1, 2, \ldots, m)$. By the induction hypothesis, the number of 3–cycles without the vertex $m+1$ is at least $m-2$. Let k be the highest value such that there is an arrow going from $m+1$ to k. Then $1 \leq k$ since we have an arrow going from $m+1$ to 1, and $k \leq m-1$ since the arrow between m and $m+1$ goes from m to $m+1$. Now $(k, k+1, m+1)$ is another 3–cycle, bringing the total number of 3–cycles to at least $m-1$. ∎

An *elbow* in a directed graph is defined as the union of two arrows with a common vertex, which is called the *pivot* of the elbow. There are three kinds of elbows. If both arrows go out from the pivot, it is called an *arrow-from-arrow*; if both arrows go into the pivot, it is called an *arrow-to-arrow*; otherwise, it is called a *broken arrow*. The elbows, especially the broken arrows, will play a central role in our study.

Theorem 2 In a tournament with n players, let the number of wins of the i^{th} player be w_i. Then

$$\lambda = \frac{n(n-1)(2n-1)}{12} - \frac{1}{2}\sum_{i=1}^{n} w_i^2 .$$

Proof: The vertex X_i has w_i outgoing arrows and $n-1-w_i$ incoming arrows, and is the pivot of exactly $w_i(n-1-w_i)$ broken arrows. Hence, the total number of broken arrows is

$$\sum_{i-1}^{n} w_i(n-1-w_i) = (n-1)\binom{n}{2} - \sum_{i=1}^{n} w_i^2 .$$

The number of 3–cycles containing 3 broken arrows is λ, while each transitive triple contains only 1. It follows that the total number of broken arrows is also given by $3\lambda + \binom{n}{3} - \lambda$. Equating the two expression above and simplifying, we have the desired result. ∎

Theorem 3 In a tournament with n players, we have $\lambda \leq \frac{n^3-n}{24}$ if n is odd and $\lambda \leq \frac{n^3-4n}{24}$ if n is even.

Proof: By Theorem 2, $\lambda = \frac{n(n-1)(2n-1)}{12} - \frac{1}{2}\sum_{i=1}^{n} w_i^2$. This remains unchanged if the result of every game is reversed. Therefore, we also have $\lambda = \frac{n(n-1)(2n-1)}{12} - \frac{1}{2}\sum_{i=1}^{n}(n-1-w_i)^2$. Together, these yield

$$\begin{aligned}
\lambda &= \frac{n(n-1)(2n-1)}{12} - \frac{1}{4}\sum_{i=1}^{n}\left(w_i^2 + (n-1-w_i)^2\right) \\
&= \frac{n(n-1)(2n-1)}{12} - \frac{1}{8}\sum_{i=1}^{n}\left((n-1)^2 + (2w_i - n + 1)^2\right) \\
&= \frac{n^3-n}{24} - \frac{1}{8}\sum_{i=1}^{n}(2w_i - n + 1)^2 .
\end{aligned}$$

If n is odd, the maximum occurs when $w_i = (n-1)/2$ for all i, yielding $\lambda \leq (n^3-n)/24$. If n is even, the maximum occurs when $w_i = n/2$ for $n/2$ players and $w_i = n/2 - 1$ for $n/2$ players, yielding $\lambda \leq (n^3-4n)/24$. ∎

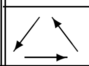
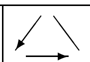
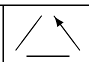
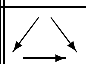
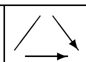
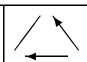
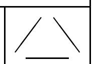
We now allow draws in our tournaments. If the game between players X and Y is a draw, we join vertices X and Y by an ordinary edge as opposed to an arrow. The resulting graph, still called a tournament, is neither an ordinary graph nor a directed graph, but a mixed graph with both edges and arrows.

There are now three additional kinds of transitive triples. In one, X draws with Y, Y draws with Z, and X draws with Z. In the other two kinds, X draws with Y while both beat Z or both are beaten by Z. These triples are vacuously transitive since the question of transitivity never arises. As before, a tournament is said to be transitive if every triple is transitive. However, there are now many transitive tournaments with the same number of players. In addition to the unique transitive tournament without draws, there is at least the transitive tournament in which all games are draws.

The non-transitivity index λ is no longer equal to the number of 3–cycles, because we have two additional kinds of non-transitive triples. In one, X beats Y, Y beats Z, but X draws with Z. In the other kind, X draws with Y, Y draws with Z, but X beats Z. Let their numbers be y and z respectively, and let x be the number of 3–cycles. It would appear that $\lambda = x + y + z$ is a natural definition. However, the degree of non-transitivity represented by each kind of non-transitive triple is not quite the same. Thus, we define $\lambda = x + \alpha y + \beta z$ for some suitably chosen parameters α and β.

How should α and β be chosen? Intuitively, one would think that the value of α should be $\frac{1}{2}$ since replacing an arrow in the wrong direction by an edge makes the non-transitivity only half as bad. As it turned out, our intuition does not lead us astray. However, it would not be so easy to divine a suitable value for β.

We turn once again to the elbows, of which there are three additional kinds, because either of the arrows may be replaced by an ordinary edge. An *arrow-from-edge* consists of an edge and an arrow going out from the pivot. An *arrow-to-edge* consists of an edge and an arrow going into the pivot. A *broken edge* consists of two edges hinged by a pivot. As in the proof of Theorem 2, we count the number of each kind of elbows in each kind of triples. The result is shown in the chart below.

Elbows	Non-transitive Triples			Transitive Triples			
							
Broken Arrow → →	3	1	0	1	0	0	0
Arrow-from-Arrow ← →	0	0	0	1	0	1	0
Arrow-to-Arrow → ←	0	0	0	1	1	0	0
Arrow-from-Edge — →	0	1	1	0	2	0	0
Arrow-to-Edge → —	0	1	1	0	0	2	0
Broken Edge — —	0	0	1	0	0	0	3

Let a, b, c, and d be the respective numbers of the four kinds of transitive triples. The total numbers of the six kinds of elbows are given by $3x+y+a$, $a+c$, $a+b$, $y+z+2b$, $y+z+2c$, and $z+3d$, respectively. We seek an expression involving only x, y, and z. Multiplying these six expressions by 4, -2, -2, 1, 1, and 0, respectively, and then adding, we have $12x+6y+2z$. Rewriting this as $12(x+\frac{1}{2}y+\frac{1}{6}z)$, we see that sensible choices are $\alpha=\frac{1}{2}$ and $\beta=\frac{1}{6}$.

Theorem 4 In a tournament with n players, let the numbers of wins, draws, and losses of the i^{th} player be w_i, d_i, and ℓ_i, respectively. Then

$$\lambda = \frac{n(n-1)(n+1)}{24} - \frac{1}{24}\sum_{i=1}^{n} d_i(d_i+2) - \frac{1}{8}\sum_{i=1}^{n}(w_i-\ell_i)^2 .$$

Proof: From the chart,

$$\begin{aligned} 3x+y+a &= \sum_{i-1}^{n} w_i\ell_i , & a+c &= \sum_{i=1}^{n}\binom{w_i}{2} , \\ a+b &= \sum_{i=1}^{n}\binom{\ell_i}{2} , & y+z+2b &= \sum_{i=1}^{n} w_i d_i , \\ y+z+2c &= \sum_{i=1}^{n} d_i\ell_i , & z+3d &= \sum_{i=1}^{n}\binom{d_i}{2} . \end{aligned}$$

Multiplying these by 8, -4, -4, 2, 2, and 0, respectively, we have

$$\begin{aligned} 24\lambda &= 24x+12y+4z \\ &= 8\sum_{i=1}^{n} w_i\ell_i - 4\sum_{i=1}^{n}\left(\binom{w_i}{2}+\binom{\ell_i}{2}\right) + 2\sum_{i=1}^{n} d_i(w_i+\ell_i) \\ &= \sum_{i=1}^{n}(w_i+d_i+\ell_i)^2 + 2\sum_{i=1}^{n}(w_i+d_i+\ell_i) - \sum_{i=1}^{n} d_i^2 - 2\sum_{i=1}^{n} d_i \\ &\qquad + 6\sum_{i=1}^{n} w_i\ell_i - 3\sum_{i=1}^{n} w_i^2 - 3\sum_{i=1}^{n}\ell_i^2 \\ &= n(n-1)^2 + 2n(n-1) - \sum_{i=1}^{2} d_i(d_i+2) - 3\sum_{i=1}^{n}(w_i-\ell_i)^2 . \end{aligned}$$

The desired result follows immediately. ∎

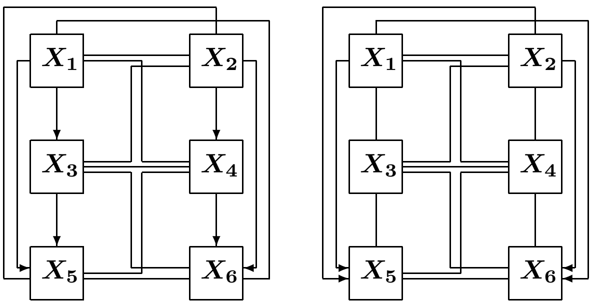
It follows from Theorem 4 that the result in Theorem 3 also holds for tournaments with draws.

A tournament is said to be *semi-transitive* if whenever X beats Y and Y beats Z, then X must beat Z. In other words, the third kind of non-transitive triple is acceptable. In a tournament with draws, the score s of a player with w wins, d draws, and ℓ losses is given by $s=w+\dfrac{d}{2}$.

Theorem 5 For any tournament without draws, there exists a semi-transitive tournament with draws involving the same players such that each has the same score in both tournaments.

Proof: Among all tournaments involving the same players in which each has the same score, consider the one with the highest number of draws. We claim that it is semi-transitive. If not, then there would exist a triple where X beats Y, Y beats Z, and either Z beats X or Z draws with X. In the former case, we replace all three games with draws. In the latter case, let X draw with Y, Y draw with Z and X beat Z instead. These changes do not affect the individual scores but increase the number of draws, contradicting our maximality assumption. ∎

We close this paper with two questions. Consider two semi-transitive tournaments involving the same players in which each has the same score. Is it necessarily true that both tournaments have (i) the same number of draws, or (ii) the same number of non-transitive triples?



Answers. In each of the two tournaments above, X_1 and X_2 have scores of $2\frac{1}{2}$, X_3 and X_4 have scores of 2 while X_5 and X_6 have scores of $1\frac{1}{2}$. Yet there are 9 draws and 18 non-transitive triples in the first but 11 draws and 8 non-transitive triples in the second. Thus, the answer to both questions is negative.

Reference

[1] M.G. Kendall and B. Babbington Smith, On the Method of Paired Comparisons, *Biometrika* **31** (1940), 324–345.

Jerry Lo
7F.-3, No.278, Jinjhou St.
Jhongshan District
Taipei City 104, Taiwan (R.O.C.)
jerry771021@yahoo.com.tw

David Rhee
10410 – 81st Street NW
Edmonton, Alberta, T6L 3A1
Canada
etotheipi1@gmail.com

PROBLEMS

Solutions to problems in this issue should arrive no later than **1 March 2007**. *An asterisk* ($\star$) *after a number indicates that a problem was proposed without a solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

Problem 3149 [2006 : 240, 242] was the same problem as 3073 [2005 : 399, 401]. We are replacing 3149 in this issue. Any solutions for the original 3149 will be treated as solutions to 3073.

3126. Correction. *Proposed by Hidetoshi Fukugawa, Kani, Gifu, Japan.*

Let D be any point on the side BC of triangle ABC. Let Γ_1 and Γ_2 be the incircles of $\triangle ABD$ and $\triangle ACD$, respectively. Let ℓ be the common external tangent to Γ_1 and Γ_2 which is different from BC. If P is the point of intersection of AD and ℓ, show that $2AP = AB + AC - BC$.

3135. Correction. *Proposed by Marian Marinescu, Monbonnot, France.*

Let $\mathbb{R}^+$ be the set of non-negative real numbers. For all a, b, $c \in \mathbb{R}^+$, let $H(a,b,c)$ be the set of all functions $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$h(x) \;=\; h\big(h(ax)\big) + h(bx) + cx$$

for all $x \in \mathbb{R}^+$. Prove that $H(a,b,c)$ is non-empty if and only if $b \le 1$ and $4ac \le (1-b)^2$.

3149. Replacement. *Proposed by David Martinez Ramirez, student, Universidad Nacional Autonoma de Mexico, Mexico.*

Let $P(z)$ be any non-constant complex monic polynomial. Show that there is a complex number w such that $|w| \le 1$ and $|P(w)| \ge 1$.

3150. Correction. *Proposed by Zhang Yun, High School attached to Xi′ An Jiao Tong University, Xi′ An City, Shan Xi, China.*

Let a, b, c be the three sides of a triangle, and let h_a, h_b, h_c be the altitudes to the sides a, b, c, respectively. Prove that

$$\frac{h_a^2}{b^2+c^2} \cdot \frac{h_b^2}{c^2+a^2} \cdot \frac{h_c^2}{a^2+b^2} \;\le\; \left(\frac{3}{8}\right)^3 .$$

3151. *Proposed by Mª Jesús Villar Rubio, Santander, Spain.*

(a) Let $r_1 < 0 < r_2 < r_3$ be the real roots of $8x^3 - 6x + \sqrt{3} = 0$. Prove that
$$r_3^2 = 4r_2^2 - 4r_2^4 \quad \text{and} \quad r_1^2 = 4r_3^2 - 4r_3^4 .$$

(b) Let $s_1 < 0 < s_2 < s_3$ be the real roots of $8x^3 - 6x + 1 = 0$. Prove that
$$r_1^2 + s_2^2 = 1 , \quad s_1^2 + r_2^2 = 1 , \quad \text{and} \quad r_3^2 + s_3^2 = 1 .$$

3152. *Proposed by Michel Bataille, Rouen, France.*

Let $x_1, x_2, \ldots, x_n$ $(n \geq 2)$ be real numbers such that $\sum_{i=1}^{n} x_i = 0$ and $\sum_{i=1}^{n} x_i^2 = 1$. Find the minimum and maximum of $\sum_{i=1}^{n} |x_i|$.

3153. *Proposed by Michel Bataille, Rouen, France.*

For which integers n does the equation
$$\frac{3xy - 1}{x + y} = n$$
have a solution in integers x and y?

3154. *Proposed by Challa K.S.N.M. Sankar, Andhrapradesh, India.*

(a) If $\beta > 1$ is a real constant, determine the number of possible real solutions of the equation
$$x - \beta \log_2 x = \beta - \beta \ln \beta .$$

(b) If $\alpha_1 < \alpha_2$ are two positive real solutions of the equation in (a), and if x_1 and x_2 are any two real numbers satisfying $\alpha_1 \leq x_1 < x_2 \leq \alpha_2$, prove that, for all λ such that $0 < \lambda < 1$,
$$\lambda \log_2 x_1 + (1 - \lambda) \log_2 x_2 \geq \ln\bigl(\lambda x_1 + (1 - \lambda) x_2\bigr) .$$

Determine when equality occurs.

3155. *Proposed by Virgil Nicula, Bucharest, Romania.*

In $\triangle ABC$, let D, E, F be the intersections of the altitudes from A, B, C to the sides BC, CA, AB, respectively. Let H be the orthocentre of $\triangle ABC$, let L be the intersection of AT and the line through B perpendicular to BC, and let T be the intersection of BE and DF.

Show that $BL = BC$ if and only if $\angle ACB = 45^\circ$.

3156. *Proposed by Virgil Nicula, Bucharest, Romania.*

Let Γ be the circumcircle of $\triangle ABC$. Let M be an interior point on the side AB, and let N be an interior point on the side AC. Let D be an intersection point of MN with Γ. Prove that

$$\left|\frac{MB}{MA}\cdot\frac{AC}{DB}-\frac{NC}{NA}\cdot\frac{AB}{DC}\right| \;=\; \frac{BC}{DA}\,.$$

3157. *Proposed by Mihály Bencze, Brasov, Romania.*

Let p be a fixed odd prime number. Let $\alpha(n)$ denote the largest integer k such that p^k is an integral divisor of $1^1\cdot 2^2\cdot 3^3\cdots n^n$. Prove that

$$\lim_{n\to\infty}\frac{\alpha(n)}{n^2} \;=\; \frac{1}{2(p-1)}\,.$$

3158. *Proposed by Mihály Bencze, Brasov, Romania.*

Let $E=\{(x,y)\in\mathbb{N}\times\mathbb{N}\mid x+y \text{ is a perfect square}\}$, and let $N(n)$ be the size of the set $\{(x,y)\in E\mid x\le n \text{ and } y\le n\}$, for $n\in\mathbb{N}$. Prove that

$$\lim_{n\to\infty}\frac{N(n)}{n\sqrt{n}} \;=\; \frac{4}{3}(\sqrt{2}-1)\,.$$

3159. *Proposed by Mihály Bencze, Brasov, Romania.*

Let n be a positive integer, and let γ be Euler's constant. Prove that

$$\gamma-\frac{1}{48n^3} \;<\; 1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln\left(n+\frac{1}{2}+\frac{1}{24n}\right) \;<\; \gamma-\frac{1}{48(n+1)^3}\,.$$

3160. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let $\triangle ABC$ have altitude AD and orthocentre H. Let E be the mid-point of AD and M the mid-point of BC.

(a) If $AD=BC$, prove that $HM=HE$.

(b) Is the converse of (a) true?

3161. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let D be a point on the side BC of $\triangle ABC$, and let P be an arbitrary point on the segment AD. Let BP meet AC at E and CP meet AB at F.

(a) If $AD\perp BC$, prove that $\angle BDF=\angle CDE$.

(b) Is the converse of (a) true?

3162. *Proposed by Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.*

Determine all integer solutions (x, y) of the equation

$$x^5 + y^7 = 2004^{1007}.$$

. . . .

3126. Correction. *Proposé par Hidetoshi Fukugawa, Kani, Gifu, Japon.*

Soit D un point sur le côté BC du triangle ABC. Soit respectivement Γ_1 et Γ_2 les cercles inscrits des triangles ABD et ACD. Soit ℓ la tangente extérieure commune à Γ_1 et Γ_2 et distincte de BC. Si P est le point d'intersection de AD et ℓ, montrer que $2AP = AB + AC - BC$.

3135. Correction. *Proposé par Marian Marinescu, Monbonnot, France.*

Soit $\mathbb{R}^+$ l'ensemble des nombres réels non négatifs. Pour tout a, b et $c \in \mathbb{R}^+$, soit $H(a,b,c)$ l'ensemble de toutes les fonctions $h : \mathbb{R}^+ \to \mathbb{R}^+$ telles que

$$h(x) = h\big(h(ax)\big) + h(bx) + cx$$

pour tout $x \in \mathbb{R}^+$. Montrer que $H(a,b,c)$ est non vide si et seulement si $b \leq 1$ et $4ac \leq (1-b)^2$.

3149. Remplacement. *Proposé par David Martinez Ramirez, étudiant, Universidad Nacional Autonoma de Mexico, Mexique.*

Soit $P(z)$ un polynôme complexe non constant et unitaire. Montrer qu'il existe un nombre complexe w tel que $|w| \leq 1$ et $|P(w)| \geq 1$.

3150. Correction. *Proposé par Zhang Yun, High School attached to Xi*′ *An Jiao Tong University, Xi*′ *An City, Shan Xi, Chine.*

Soit a, b et c les trois côtés d'un triangle, et soit h_a, h_b et h_c les hauteurs abaissées sur les côtés respectifs a, b et c. Montrer que

$$\frac{h_a^2}{b^2+c^2} \cdot \frac{h_b^2}{c^2+a^2} \cdot \frac{h_c^2}{a^2+b^2} \leq \left(\frac{3}{8}\right)^3.$$

3151. *Proposé par Mª Jesús Villar Rubio, Santander, Espagne.*

(a) Soit $r_1 < 0 < r_2 < r_3$ les racines réelles de $8x^3 - 6x + \sqrt{3} = 0$. Montrer que

$$r_3^2 = 4r_2^2 - 4r_2^4 \quad \text{et} \quad r_1^2 = 4r_3^2 - 4r_3^4.$$

(b) Soit $s_1 < 0 < s_2 < s_3$ les racines réelles de $8x^3 - 6x + 1 = 0$. Montrer que

$$r_1^2 + s_2^2 = 1, \quad s_1^2 + r_2^2 = 1, \quad \text{et} \quad r_3^2 + s_3^2 = 1.$$

3152. *Proposé par Michel Bataille, Rouen, France.*

Soit x_1, x_2, …, x_n ($n \geq 2$) des nombres réels tels que $\sum\limits_{i=1}^{n} x_i = 0$ et $\sum\limits_{i=1}^{n} x_i^2 = 1$. Trouver le minimum et le maximum de $\sum\limits_{i=1}^{n} |x_i|$.

3153. *Proposé par Michel Bataille, Rouen, France.*

Pour quels entiers n l'équation

$$\frac{3xy-1}{x+y} \;=\; n$$

a-t-elle des solutions entières x et y ?

3154. *Proposé par Challa K.S.N.M. Sankar, Andhrapradesh, Inde.*

(a) Si $\beta > 1$ est une constante réelle, déterminer le nombre possible de solutions réelles de l'équation

$$x - \beta \log_2 x \;=\; \beta - \beta \ln \beta\,.$$

(b) Si $\alpha_1 < \alpha_2$ sont deux solutions réelles de l'équation en (a), et si x_1 et x_2 sont deux nombres réels quelconques satisfaisant $\alpha_1 \leq x_1 < x_2 \leq \alpha_2$, montrer que, pour tout λ tel que $0 < \lambda < 1$,

$$\lambda \log_2 x_1 + (1-\lambda)\log_2 x_2 \;\geq\; \ln\bigl(\lambda x_1 + (1-\lambda)x_2\bigr)\,.$$

Déterminer quand l'égalité a lieu.

3155. *Proposé par Virgil Nicula, Bucarest, Roumanie.*

Dans le triangle ABC, soit D, E et F les intersections respectives des hauteurs A, B et C avec les côtés BC, CA et AB. Soit H l'orthocentre de ABC, soit T l'intersection de BE et DF, et soit L l'intersection de AT avec la droite passant par B et perpendiculaire à BC.

Montrer que $BL = BC$ si et seulement l'angle $ACB = 45°$.

3156. *Proposé par Virgil Nicula, Bucarest, Roumanie.*

Soit Γ le cercle circonscrit au triangle ABC, M un point intérieur du côté AB et N un point intérieur du côté AC. Si D désigne une intersection de MN avec Γ, montrer que

$$\left|\frac{MB}{MA}\cdot\frac{AC}{DB} - \frac{NC}{NA}\cdot\frac{AB}{DC}\right| \;=\; \frac{BC}{DA}\,.$$

3157. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit p un nombre premier impair donné. On désigne par $\alpha(n)$ le plus grand entier k tel que p^k est un diviseur entier de $1^1 \cdot 2^2 \cdot 3^3 \cdots n^n$. Montrer que

$$\lim_{n\to\infty} \frac{\alpha(n)}{n^2} = \frac{1}{2(p-1)}\,.$$

3158. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit $E = \{(x,y) \in \mathbb{N} \times \mathbb{N} \mid x+y$ est un carré parfait$\}$, et soit $N(n)$ la taille de l'ensemble $\{(x,y) \in E \mid x \le n$ et $y \le n\}$, pour tout $n \in \mathbb{N}$. Montrer que

$$\lim_{n\to\infty} \frac{N(n)}{n\sqrt{n}} = \frac{4}{3}(\sqrt{2}-1)\,.$$

3159. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit n un entier positif et γ la constante d'Euler. Montrer que

$$\gamma - \frac{1}{48n^3} < 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln\left(n + \frac{1}{2} + \frac{1}{24n}\right) < \gamma - \frac{1}{48(n+1)^3}\,.$$

3160. *Proposé par D.J. Smeenk, Zaltbommel, Pays-Bas.*

Dans un triangle ABC d'orthocentre H, soit AD la hauteur issue du sommet A. Soit E le point milieu de AD et M celui de BC.

(a) Si $AD = BC$, montrer que $HM = HE$.

(b) Est-ce que la réciproque de (a) est vraie ?

3161. *Proposé par D.J. Smeenk, Zaltbommel, Pays-Bas.*

Soit D un point du côté BC du triangle ABC, et soit P un point arbitraire du segment AD. Soit E et F les intersections respectives de BP avec AC et de CP avec AB.

(a) Si AD est perpendiculaire à BC, montrer que les angles BDF et CDE sont égaux.

(b) Est-ce que la réciproque de (a) est vraie ?

3162. *Proposé par Eckard Specht, Université Otto-von-Guericke, Magdeburg, Allemagne.*

Déterminer toutes les solutions entières (x,y) de l'équation

$$x^5 + y^7 = 2004^{1007}\,.$$

KLAMKIN SOLUTIONS

These are the solutions to the special section of problems appearing in the September 2005 issue and dedicated to the memory of Murray S. Klamkin.

KLAMKIN–01. [2005 : 327, 330] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

(a) Let x and y be positive real numbers from the interval $\left[0, \frac{1}{2}\right]$. Prove that

$$2 \;\le\; \left(\frac{1-x}{1-y}\right)^{\frac{1}{4}} + \left(\frac{1-y}{1-x}\right)^{\frac{1}{4}} \;\le\; \frac{2}{\left(\sqrt{x}\sqrt{y}+\sqrt{1-x}\sqrt{1-y}\right)^{\frac{1}{2}}}\,.$$

(b)★ Is there a generalization of the above inequality to three or more numbers?

I. *Composite of solutions to* (a) *by Manuel Benito, Óscar Ciaurri, and Emilio Fernández, Logroño, Spain; and Chip Curtis, Missouri Southern State University, Joplin, MO, USA.*

The left inequality follows from the well-known fact that $t + \frac{1}{t} \ge 2$ for $t > 0$, by taking $t = \left(\frac{1-x}{1-y}\right)^{1/4}$.

To prove the right inequality, we make the substitutions $x = \sin^2\alpha$ and $y = \sin^2\beta$, where $\alpha, \beta \in \left[0, \frac{\pi}{4}\right]$. Then the inequality becomes

$$\sqrt{\frac{\cos\alpha}{\cos\beta}} + \sqrt{\frac{\cos\beta}{\cos\alpha}} \;\le\; \frac{2}{\sqrt{\sin\alpha\sin\beta + \cos\alpha\cos\beta}}\,,$$

which is equivalent to $\dfrac{\cos\alpha+\cos\beta}{\sqrt{\cos\alpha\cos\beta}} \le \dfrac{2}{\sqrt{\cos(\alpha-\beta)}}$, or

$$(\cos\alpha+\cos\beta)^2\cos(\alpha-\beta) \;\le\; 4\cos\alpha\cos\beta \tag{1}$$

For notational convenience, we set $u = \frac{1}{2}(\alpha+\beta)$ and $v = \frac{1}{2}(\alpha-\beta)$. Then

$$\begin{aligned}(\cos\alpha+\cos\beta)^2 &= \big(\cos(u+v)+\cos(u-v)\big)^2 \\ &= (2\cos u\cos v)^2 \;=\; 4(1-\sin^2 u)(1-\sin^2 v) \qquad (2)\\ \text{and}\quad \cos(\alpha-\beta) &= \cos(2v) \;=\; 1-2\sin^2 v\,. \qquad (3)\end{aligned}$$

Also,

$$\begin{aligned}\cos\alpha\cos\beta &= \tfrac{1}{2}\big[\cos(\alpha+\beta)+\cos(\alpha-\beta)\big] \\ &= \tfrac{1}{2}\big(\cos(2u)+\cos(2v)\big) \;=\; 1-\sin^2 u-\sin^2 v\,. \qquad (4)\end{aligned}$$

Substituting (2), (3), and (4) into (1), gives

$$(1-\sin^2 u)(1-\sin^2 v)(1-2\sin^2 v) \;\le\; 1-\sin^2 u-\sin^2 v\,,$$

which simplifies to

$$\sin^2 u \sin^2 v \;\le\; 2(1-\sin^2 u)(1-\sin^2 v)\sin^2 v\,. \qquad (5)$$

Now we will prove (5). Since α, $\beta \in \left[0, \frac{\pi}{4}\right]$, we have $u \in \left[0, \frac{\pi}{4}\right]$ and $v \in \left[-\frac{\pi}{8}, \frac{\pi}{8}\right]$. Then $\sin^2 u < \frac{1}{2}$ and $\sin^2 v \le \sin^2 \frac{\pi}{8} < \frac{1}{2}$; hence,

$$2(1-\sin^2 u)(1-\sin^2 v) \;>\; 2\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) \;=\; \tfrac{1}{2} \;\ge\; \sin^2 u\,,$$

from which (5) follows.

II. *Solution to* (a) *and* (b) *by Li Zhou, Polk Community College, Winter Haven, FL, USA, expanded slightly by the editor.*

For any integer $n \ge 2$, any real number r, and any real numbers $x_1, x_2, \ldots, x_n \in [0,1)$, let

$$F_r(x_1,x_2,\ldots,x_n) \;=\; \left(\prod_{i=1}^{n}(1-x_i)^r\right)^{-\frac{1}{n}} \cdot \sum_{i=1}^{n}(1-x_i)^r\,,$$

and
$$G_r(x_1,x_2,\ldots,x_n) \;=\; n\left(\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} + \left(\prod_{i=1}^{n}(1-x_i)\right)^{\frac{1}{n}}\right)^{-r}.$$

We prove that

(i) $F_r(x_1,x_2,\ldots,x_n) \ge n$;

(ii) $F_r(x_1,x_2,\ldots,x_n) \le G_r(x_1,x_2,\ldots,x_n)$ if $r > 0$ and $0 \le x_i \le \dfrac{1}{1+r}$ for all i;

(iii) $F_r(x_1,x_2,\ldots,x_n) \;\ge\; G_r(x_1,x_2,\ldots,x_n)$ if $r \le 0$ or if $r > 0$ and $\dfrac{1}{1+r} \le x_i \le 1$ for all i.

Note that (a) is the special case of (i) and (ii) when $n = 2$ and $r = \frac{1}{2}$.

Proof: Part (i) follows immediately from the AM–GM Inequality. For the other two parts, let $f_r(t) = (1+e^t)^{-r}$. Then

$$\begin{aligned} f_r'(t) &= -r(1+e^t)^{-r-1}e^t \\ \text{and}\quad f_r''(t) &= r(r+1)(1+e^t)^{-r-2}e^{2t} - r(1+e^t)^{-r-1}e^t \\ &= r(1+e^t)^{-r-2}e^t(re^t-1)\,. \end{aligned}$$

Hence, f is convex if $r \le 0$ or if $r > 0$ and $t \ge -\ln r$; and concave if $r > 0$ and $t \le -\ln r$. Now let $t_i = \ln\left(\dfrac{x_i}{1-x_i}\right)$ for $0 < x_i < 1$, $i = 1, 2, \ldots, n$. Then

$$\begin{aligned} \sum_{i=1}^{n} f_r(t_i) &= \sum_{i=1}^{n}\left(1+\frac{x_i}{1-x_i}\right)^{-r} = \sum_{i=1}^{n}(1-x_i)^r \\ &= \left(\prod_{i=1}^{n}(1-x_i)^r\right)^{\frac{1}{n}} \cdot F_r(x_1,x_2,\ldots,x_n) \qquad (6) \end{aligned}$$

and

$$\begin{aligned} nf_r\left(\frac{1}{n}\sum_{i=1}^{n} t_i\right) &= n\left(1+\exp\left(\frac{1}{n}\sum_{i=n}^{n}\ln\left(\frac{x_i}{1-x_i}\right)\right)\right)^{-r} \\ &= n\left(1+\prod_{i=1}^{n}\left(\frac{x_i}{1-x_i}\right)^{\frac{1}{n}}\right)^{-r} \\ &= n\left(\frac{\left(\prod\limits_{i=1}^{n}(1-x_i)\right)^{\frac{1}{n}}}{\left(\prod\limits_{i=1}^{n}x_i\right)^{\frac{1}{n}}+\left(\prod\limits_{i=1}^{n}(1-x_i)\right)^{\frac{1}{n}}}\right)^{r} \\ &= \left(\prod_{i=1}^{n}(1-x_i)^r\right)^{\frac{1}{n}}\cdot G_r(x_1,x_2,\ldots,x_n)\,. \qquad (7) \end{aligned}$$

Furthermore, note that $t_i \geq -\ln r$ is equivalent to $\dfrac{x_i}{1-x_i} \geq \dfrac{1}{r}$, or $\dfrac{1-x_i}{x_i} \leq r$, which in turn is equivalent to $\dfrac{1}{x_i} \leq r+1$, or $\dfrac{1}{x_i} \geq r+1$. Hence, f is convex if $r \leq 0$ or if $r > 0$ and $\frac{1}{r+1} \leq x_i < 1$ for all i; and concave if $r > 0$ and $0 \leq x_i \leq \dfrac{1}{r+1}$.

Using (6) and (7) together with Jensen's Inequality applied to $f_r(t)$ yields (ii) and (iii) for $x_i > 0$. The validity of these inequalities when $x_i = 0$ for any i follows from the continuity of $F_r - G_r$ at 0 in each of its variables.

Part (a) *also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; and the proposer. There was one incomplete solution.*

KLAMKIN–02. [2005 : 327, 330] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

(a) Let x, y, z be positive real numbers such that $x+y+z=1$. Prove that

$$xyz\left(1+\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right) \;\geq\; \frac{28}{27}\,.$$

(b)★ Prove or disprove the following generalization involving n positive real numbers x_1, x_2, …, x_n which sum to 1:

$$\left(\prod_{i=1}^{n} x_i\right)\left(1+\sum_{i=1}^{n}\frac{1}{x_i^2}\right) \;\geq\; \frac{n^3+1}{n^n}\,.$$

I. *Solution to* (a) *by Chip Curtis, Missouri Southern State University, Joplin, MO, USA, modified by the editor.*

Using the condition that $x+y+z=1$, we obtain

$$\begin{aligned} xyz\left(1+\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right) &= xyz+\frac{yz}{x}+\frac{zx}{y}+\frac{xy}{z} \\ &= \frac{(xyz)^2+(yz)^2+(zx)^2+(xy)^2}{xyz} \\ &= \frac{(xyz)^2+(x+y+z)^2\left[(yz)^2+(zx)^2+(xy)^2\right]}{xyz(x+y+z)^3}. \end{aligned}$$

Thus, the given inequality is equivalent to

$$27(xyz)^2+27(x+y+z)^2\left[(yz)^2+(zx)^2+(xy)^2\right] \;\geq\; 28xyz(x+y+z)^3\,,$$

or

$$\begin{aligned} 27s_1-30xyzs_2+54(x^3y^3+y^3z^3+z^3x^3)& \\ -28xyz(x^3+y^3+z^3)-60x^2y^2z^2 &\;\geq\; 0\,, \end{aligned} \tag{1}$$

where

$$\begin{aligned} s_1 &= x^4y^2+x^2y^4+y^4z^2+y^2z^4+z^4x^2+z^2x^4\,, \\ s_2 &= x^2y+xy^2+y^2z+yz^2+z^2x+zx^2\,. \end{aligned}$$

Let S denote the left side of (1), and define T, U, V, and W as follows:

$$\begin{aligned} T &= s_1-2xyz(x^3+y^3+z^3)\,, \\ U &= 2(x^3y^3+y^3z^3+z^3x^3)-xyzs_2\,, \\ V &= s_1-xyzs_2\,, \\ \text{and}\quad W &= s_1-6x^2y^2z^2\,. \end{aligned}$$

Then, by tedious (but straightforward) computations, we determine that $S=14T+27U+3V+10W$; whence, (1) is equivalent to

$$14T+27U+3V+10W \;\geq\; 0\,. \tag{2}$$

To establish (2), it suffices to show that T, U, V, $W\geq 0$.

Consider a majorization relation among the vectors $(4,1,1)$, $(4,2,0)$, $(3,2,1)$, and $(3,3,0)$ in $\mathbb{R}^3$. Since $(4,1,1)\prec(4,2,0)$, $(3,2,1)\prec(3,3,0)$, and $(3,2,1)\prec(4,2,0)$, we have, by Muirhead's Inequality,

$$\begin{aligned} 2xyz(x^3+y^3+z^3) &\leq s_1\,, \\ xyzs_2 &\leq 2(x^3y^3+y^3z^3+z^3x^3)\,, \\ \text{and}\quad xyzs_2 &\leq s_1\,; \end{aligned}$$

that is, T, U, $V\;\geq\;0$. Also, by the AM–GM Inequality, we see that $s_1\geq 6x^2y^2z^2$; that is, $W\geq 0$.

Hence, (2) is proven, and our proof is complete.

II. *Solution to* (b) *by Manuel Benito, Óscar Ciaurri, and Emilio Fernández, Logroño, Spain.*

The claim is false for $n \geq 4$.

Let $f(x_1, x_2, \ldots, x_n)$ denote the left side of the given inequality. If we set $x_1 = x_2 = \cdots = x_{n-1} = k$ and $x_n = 1 - (n-1)k$, where $0 < k < \frac{1}{n}$, then $\sum_{i=1}^{n} x_i = 1$. Since

$$\begin{aligned} & f\big(k, k, \ldots, k, 1-(n-1)k\big) \\ &\quad= k^{n-1}\big(1-(n-1)k\big)\left(1+\frac{n-1}{k^2}+\frac{1}{\big(1-(n-1)k\big)^2}\right) \\ &\quad= k^{n-3}\big(1-(n-1)k\big)\left(k^2+n-1+\frac{k^2}{\big(1-(n-1)k\big)^2}\right), \end{aligned}$$

we have $\lim_{k \to 0+} f\big(k, k, \ldots, k, 1-(n-1)k\big) = 0$.

Also solved by ROBERT B. ISRAEL, University of British Columbia, Vancouver, BC. Part (a) was solved by ARKADY ALT, San Jose, CA, USA; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; VEDULA N. MURTY, Dover, PA, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer. There was one incomplete solution.

KLAMKIN–03. [2005 : 327, 330] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

If a, b, c are positive real numbers, prove that

$$\frac{(a+b+c)^2}{a^2+b^2+c^2} + \frac{1}{2}\left(\frac{a^3+b^3+c^3}{abc} - \frac{a^2+b^2+c^2}{ab+bc+ca}\right) \geq 4\,.$$

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

Since $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$, the given inequality is equivalent to

$$\frac{2(ab+bc+ca)}{a^2+b^2+c^2} + \frac{ab^4+ac^4+bc^4+ba^4+ca^4+cb^4}{2abc(ab+bc+ca)} \geq 3\,. \qquad (1)$$

By the Cauchy–Schwarz Inequality, we have

$$\begin{aligned} \left(\frac{1}{a}+\frac{1}{a}+\frac{1}{b}+\frac{1}{b}+\frac{1}{c}+\frac{1}{c}\right)&\left(ab^4+ac^4+bc^4+ba^4+ca^4+cb^4\right) \\ &\geq \left(b^2+c^2+c^2+a^2+a^2+b^2\right)^2 \\ &= 4\left(a^2+b^2+c^2\right)^2, \end{aligned}$$

from which we obtain

$$\left(\frac{ab+bc+ca}{abc}\right)\left(ab^4+ac^4+bc^4+ba^4+ca^4+cb^4\right) \geq 2\left(a^2+b^2+c^2\right)^2 . \qquad (2)$$

Using the AM–GM Inequality and (2), we then have

$$\frac{2(ab+bc+ca)}{a^2+b^2+c^2}+\frac{ab^4+ac^4+bc^4+ba^4+ca^4+cb^4}{2abc(ab+bc+ca)}$$
$$\geq 3\left(\frac{(ab+bc+ca)\left(ab^4+ac^4+bc^4+ba^4+ca^4+cb^4\right)}{2abc\left(a^2+b^2+c^2\right)^2}\right)^{\frac{1}{3}} \geq 3 ,$$

and (1) is established.

Also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VEDULA N. MURTY, Dover, PA, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

KLAMKIN–04. [2005 : 327, 330] *Proposed by Mihály Bencze, Brasov, Romania.*

Let f_n denote the Fibonacci sequence (that is, $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$). For all integers $k \geq 1$, determine the remainder when f_{kn-r} is divided by f_n^2 for the following cases:

(a) $r = 1$; (b) $r = 2$; (c)★ $r \in \{3, 4, \ldots, k-1\}$.

Solution by Robert B. Israel, University of British Columbia, Vancouver, BC.

We will designate the Fibonacci numbers to start at 0; that is, $F_0 = 0$ and $f_j = F_{j+1}$, for $j = 0, 1, \ldots$. One convenient way to obtain them is using the powers of the matrix $M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. For each positive integer n,

$$M^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix} = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_n + F_{n-1} \end{pmatrix} .$$

It is easy to prove by induction on k that

$$M^{kn} \equiv \begin{pmatrix} F_{n-1}^k & kF_{n-1}^{k-1}F_n \\ kF_{n-1}^{k-1}F_n & F_{n-1}^k + kF_{n-1}^{k-1}F_n \end{pmatrix} \pmod{F_n^2}$$

for all integers k and all integers n. Moreover, using $F_{-j} = (-1)^{j+1}F_j$ and $M^{kn-r} = M^{-r}M^{kn}$, we get

$$F_{kn-1} \equiv (-1)^r\left(kF_{r-1}F_{n-1}^{k-1}F_n - F_rF_{n-1}^k\right) \pmod{F_n^2} .$$

In the other notation, after replacing n by $n+1$ and r by $r+k-1$, we obtain

$$f_{kn-r} \equiv (-1)^{r+k}\left(f_{r+k-2}f_{n-1}^{k} - kf_{r+k-3}f_{n-1}^{k-1}f_n\right) \pmod{f_n^2} .$$

Parts (a) *and* (b) *also solved by the proposer.*

Note that the matrix representation of the Fibonacci numbers also occurs in this issue in the solution of 3044 on page 331.

KLAMKIN–05. [2005 : 328, 330] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let k and n be positive integers with $k < n$, and let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1 \le a_2 \le \cdots \le a_n$. Prove that

$$(a_1 + a_2 + \cdots + a_n)^2 \ \ge\ n(a_1a_{k+1} + a_2a_{k+2} + \cdots + a_na_{n+k})$$

(where the subscripts are taken modulo n) in the following cases:

(a) $n = 2k$; (b) $n = 4k$; (c)★ $2 < \frac{n}{k} < 4$.

Solution by the proposer.

(a) We have to prove that

$$(a_1 + a_2 + \cdots + a_{2k})^2 \ \ge\ 4k(a_1a_{k+1} + a_2a_{k+2} + \cdots + a_ka_{2k}) .$$

Let x be a real number such that $a_k \le x \le a_{k+1}$. Then, obviously,

$$(x - a_1)(a_{k+1} - x) + (x - a_2)(a_{k+2} - x) + \cdots + (x - a_k)(a_{2k} - x) \ \ge\ 0 .$$

Expanding, rearranging, and multiplying by $4k$, we obtain

$$4kx(a_1+a_2+\cdots+a_{2k}) \ \ge\ 4k^2x^2+4k(a_1a_{k+1}+a_2a_{k+2}+\cdots+a_ka_{2k}) . \quad (1)$$

On the other hand, we have the obvious inequality

$$(a_1 + a_2 + \cdots + a_{2k} - 2kx)^2 \ \ge\ 0 ,$$

which can be written as

$$(a_1 + a_2 + \cdots + a_{2k})^2 + 4k^2x^2 \ \ge\ 4kx(a_1 + a_2 + \cdots + a_{2k}) . \quad (2)$$

Adding inequalities (1) and (2), we obtain the desired inequality.

(b) Let $b_i = a_i + a_{2k+i}$ for each integer i, $1 \le i \le 2k$. Clearly, $b_1 \le b_2 \le \cdots \le b_{2k}$. Applying the inequality from part (a), we obtain

$$(b_1 + b_2 + \cdots + b_{2k})^2 \ \ge\ 4k(b_1b_{k+1} + b_2b_{k+2} + \cdots + b_kb_{2k}) ,$$

which is the desired inequality.

There were no other solutions submitted.

KLAMKIN–06. [2005 : 328, 331] *Proposed by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Let Γ be the circumcircle of $\triangle ABC$.

(a) Suppose that the median and the interior angle bisector from A intersect BC at M and N, respectively. Extend AM and AN to intersect Γ at M' and N', respectively. Prove that $MM' \geq NN'$.

(b)★ Suppose that P is a point in the interior of side BC and AP intersects Γ at P'. Find the location of P where PP' is maximal. Is this maximal P constructible by straightedge and compass?

Solution by Manuel Benito, Óscar Ciaurri, and Emilio Fernández, Logroño, Spain.

(a) First we note that the correct inequality should be $MM' \leq NN'$. We use the following well-known (or easy to prove) relations:

$$CN = \frac{ab}{b+c}, \qquad NB = \frac{ac}{b+c}, \qquad AN = \frac{\sqrt{bc}}{b+c}\sqrt{(b+c)^2-a^2}$$

and

$$AM = \frac{\sqrt{2b^2+2c^2-a^2}}{2}.$$

By the Intersecting Chords Theorem,

$$MM' = \frac{CM\cdot MB}{AM} = \frac{a^2}{2\sqrt{2b^2+2c^2-a^2}}$$

and

$$NN' = \frac{CN\cdot NB}{AN} = \frac{a^2\sqrt{bc}}{(b+c)\sqrt{(b+c)^2-a^2}}.$$

Thus, the inequality $MM' \leq NN'$ is equivalent to

$$\frac{1}{2\sqrt{2b^2+2c^2-a^2}} \leq \frac{\sqrt{bc}}{(b+c)\sqrt{(b+c)^2-a^2}},$$

$$\text{or}\quad (b+c)^2 \leq \frac{4bc(2b^2+2c^2-a^2)}{(b+c)^2-a^2}. \qquad (1)$$

To prove (1), we calculate

$$\begin{aligned}
&\frac{4bc(2b^2+2c^2-a^2)}{(b+c)^2-a^2} - (b+c)^2\\
&= \frac{4bc(2b^2+2c^2-a^2)}{(b+c)^2-a^2} - 4bc - (b-c)^2\\
&= \frac{4bc(b-c)^2}{(b+c)^2-a^2} - (b-c)^2 = (b-c)^2\left[\frac{4bc-(b+c)^2+a^2}{(b+c)^2-a^2}\right]\\
&= (b-c)^2\left[\frac{a^2-(b-c)^2}{(b+c)^2-a^2}\right] = (b-c)^2\frac{(a-b+c)(a+b-c)}{(-a+b+c)(a+b+c)}.
\end{aligned}$$

Using the Triangle Inequalities $b+c>a$, $c+a>b$, and $a+b>c$, we see that the last expression above is non-negative, which proves (1). Equality holds if and only if $b=c$.

(b) Let $PB=x$. Clearly, $x\in(0,a)$. Using the Intersecting Chords Theorem and the Cosine Law for $\triangle APB$, we obtain

$$PP' \;=\; \frac{CP\cdot PB}{AP} \;=\; \frac{(a-x)\cdot x}{\sqrt{x^2+c^2-2cx\cos B}}\,.$$

Let

$$L(x) \;=\; \frac{x(a-x)}{\sqrt{x^2+c^2-2cx\cos B}}\,.$$

Consider the function $L(x)$ on the interval $[0,a]$. It is positive on the interval $(0,a)$ and continuous on the interval $[0,a]$. Since $L(0)=L(a)=0$, the function $L(x)$ must have a maximum on the interval $(0,a)$. Let x be a critical point for $L(x)$ on $(0,a)$ such that $L'(x)=0$. Then

$$\begin{aligned} 0 &= \frac{L'(x)}{L(x)} \;=\; [\ln L(x)]' \\ &= \left[\ln x+\ln(a-x)-\tfrac{1}{2}\ln(x^2+c^2-2cx\cos B)\right]' \\ &= \frac{1}{x}-\frac{1}{a-x}-\frac{x-c\cos B}{x^2+c^2-2cx\cos B}\,, \end{aligned}$$

which gives the following equation for the critical point x:

$$x^3-(3c\cos B)x^2+(2c^2+ac\cos B)x-ac^2 \;=\; 0\,.$$

Applying the Cosine Law to $\triangle ABC$, we get $2ac\cos B=a^2+c^2-b^2$, and, simplifying, we can rewrite this cubic equation as

$$2ax^3-3(a^2+c^2-b^2)x^2+a(5c^2+a^2-b^2)x-2a^2c^2 \;=\; 0\,. \qquad (2)$$

Let $Q(x)$ denote the cubic on the left side of the above equation. Note that $Q(0)=-2a^2c^2<0$ and $Q(a)=2a^2b^2>0$, which implies that $Q(x)$ has a root in the interval $(0,a)$. Consider a triangle with sides $a=5$, $b=3$, and $c=4$. Equation (2) becomes

$$5x^3-48x^2+240x-400 \;=\; 0\,.$$

The polynomial $5x^3-48x^2+240x-400$ is irreducible over the field of rationals, implying that its only real root, $x=\frac{1}{5}\big(16-4\sqrt[3]{36}+6\sqrt[3]{6}\big)$, is algebraic of degree 3 over the rationals. Consequently, x is not constructible by straightedge and compass. Hence, our conclusion is: Generally, the point P is not constructible by straightedge and compass.

Part (a) *also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria. There was one incorrect solution.*

KLAMKIN–07. [2005 : 328, 331] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let a, b, c, d be real numbers such that $a > b \geq c > d > 0$. If $ad - bc > 0$, prove that

$$\prod_{k=1}^{n} \left(\frac{a^{\binom{n}{k}} - b^{\binom{n}{k}}}{c^{\binom{n}{k}} - d^{\binom{n}{k}}} \right)^k \;\geq\; \left(\frac{a^{\frac{2^n}{n+1}} - b^{\frac{2^n}{n+1}}}{c^{\frac{2^n}{n+1}} - d^{\frac{2^n}{n+1}}} \right)^{\binom{n+1}{2}} .$$

Combination of the solutions by Walther Janous, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

We first set

$$f(x) \;=\; \ln\left(\frac{a^x - b^x}{c^x - d^x}\right) ,$$

for $x > 0$. Taking logarithms, we see that the given inequality is equivalent to

$$\sum_{k=1}^{n} k \cdot f\left[\binom{n}{k}\right] \;\geq\; \binom{n+1}{2} \cdot f\left(\frac{2^n}{n+1}\right) ;$$

that is,

$$\sum_{k=1}^{n} \frac{2k}{n(n+1)} \cdot f\left[\binom{n}{k}\right] \;\geq\; f\left(\frac{2^n}{n+1}\right) ,$$

where we have

$$\sum_{k=1}^{n} \frac{2k}{n(n+1)} \;=\; 1 .$$

We claim that $f(x)$ is a convex function, but defer the proof until later. Jensen's Inequality then implies that

$$\sum_{k=1}^{n} \frac{2k}{n(n+1)} \cdot f\left[\binom{n}{k}\right] \;\geq\; \sum_{k=1}^{n} f\left(\frac{2k}{n(n+1)} \cdot \binom{n}{k}\right) . \tag{1}$$

Now, the identity $\sum\limits_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$ shows that

$$\sum_{k=1}^{n} \frac{2k}{n(n+1)} \cdot \binom{n}{k} \;=\; \frac{2^n}{n+1} .$$

The stated inequality follows from this equation and (1).

It remains to prove that $f(x)$ is (strictly) convex on $(0, +\infty)$. Since

$$f(x) \;=\; \ln\left(\frac{\left(\frac{a}{b}\right)^x - 1}{\left(\frac{c}{d}\right)^x - 1}\right) + x\ln\left(\frac{b}{d}\right) ,$$

it is sufficient to show that, if $\alpha > \beta > 1$, then $\ln\left(\frac{\alpha^x - 1}{\beta^x - 1}\right)$ is convex. But

$$\ln\left(\frac{\alpha^x - 1}{\beta^x - 1}\right) = \ln\left(\frac{e^{x\ln\alpha} - 1}{e^{x\ln\beta} - 1}\right),$$

and it is then sufficient to show that, if $\gamma > \delta > 0$, then $g(x) = \ln\left(\frac{e^{\gamma x} - 1}{e^{\delta x} - 1}\right)$ is convex. Taking derivatives, we get

$$g''(x) = \frac{\delta^2 e^{\delta x}}{(e^{\delta x} - 1)^2} - \frac{\gamma^2 e^{\gamma x}}{(e^{\gamma x} - 1)^2}.$$

Then $g''(x) > 0$ if and only if $\frac{e^{\gamma x} - 1}{\gamma e^{\frac{1}{2}\gamma x}} > \frac{e^{\delta x} - 1}{\delta e^{\frac{1}{2}\delta x}}$. We can rewrite this inequality as

$$\frac{\sinh\left(\frac{1}{2}\gamma x\right)}{\frac{1}{2}\gamma x} > \frac{\sinh\left(\frac{1}{2}\delta x\right)}{\frac{1}{2}\delta x}.$$

Since $\frac{\sinh x}{x}$ is strictly increasing on $(0, +\infty)$, the preceding inequality holds, and we are done.

There were no other solutions submitted.

KLAMKIN-08. [2005 : 328, 331] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let m and n be positive integers, and let x_1, x_2, $\ldots$, x_m be positive real numbers. If λ is a real number, $\lambda \geq 1$, prove that

$$\left(\prod_{i=1}^{m} x_i\right)^{\frac{1}{m}} \leq \left(\frac{\lambda\left(\sum\limits_{i=1}^{m} x_i\right)^n + (1-\lambda)\sum\limits_{i=1}^{m} x_i^n}{\lambda m^n + (1-\lambda)m}\right)^{\frac{1}{n}} \leq \frac{1}{m}\sum_{i=1}^{m} x_i.$$

I. *Essentially the same solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON; with some detail added by the editor.*

Lemma. Let m and n be positive integers, and let x_1, x_2, $\ldots$, x_n be positive real numbers. Then

$$\left(\sum_{i=1}^{m} x_i\right)^n - \sum_{i=1}^{m} x_i^n \geq (m^n - m)\left(\prod_{i=1}^{m} x_i\right)^{\frac{n}{m}}.$$

[*Ed.* The second solution below gives a reference for this inequality.]

Proof: If we expand $\left(\sum_{i=1}^{m} x_i\right)^n$ completely, without combining like terms, we get

$$\left(\sum_{i=1}^{m} x_i\right)^n = \sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\cdots\sum_{i_n=1}^{m} x_{i_1}x_{i_2}\cdots x_{i_n}\,.$$

The number of elementary terms $x_{i_1}x_{i_2}\cdots x_{i_n}$ on the right side is m^n. Let p denote the total number of times that a given variable, say x_1, occurs as a factor in these terms. (By symmetry, this number is independent of the choice of variable.) Letting $s = \sum_{i=2}^{m} x_i$, we have

$$\left(\sum_{i=1}^{m} x_i\right)^n = (x_1+s)^n = \sum_{k=0}^{n}\binom{n}{k}x_1^{n-k}s^k\,,$$

by the Binomial Theorem. Then p is the total number of times that x_1 occurs as a factor in the terms on the right side of this equation. Since the expansion of s^k contains $(m-1)^k$ elementary terms, we have

$$\begin{aligned} p &= \sum_{k=0}^{n-1}\binom{n}{k}(n-k)(m-1)^k = \sum_{k=0}^{n-1} n\binom{n-1}{k}(m-1)^k \\ &= n\bigl[1+(m-1)\bigr]^{n-1} = nm^{n-1}\,. \end{aligned}$$

We now subtract $\sum_{i=1}^{m} x_i^n$ from the expansion of $\left(\sum_{i=1}^{m} x_i\right)^n$. The number of terms remaining in the expansion is then $m^n - m$. Applying the AM–GM Inequality to these terms, and noting that the total number of times each variable occurs as a factor in the terms is $p - n = nm^{m-1} - n$, we get

$$\frac{1}{m^n-m}\left[\left(\sum_{i=1}^{m} x_i\right)^n - \sum_{i=1}^{m} x_i^n\right] \ge \left(\prod_{i=1}^{m} x_i^{nm^{m-1}-n}\right)^{\frac{1}{m^n-m}} = \left(\prod_{i=1}^{m} x_i\right)^{\frac{n}{m}},$$

which gives the desired result. ∎

Now consider the left inequality in the given problem. By the AM–GM Inequality, we have $\sum_{i=1}^{m} x_i^n \ge m\left(\prod_{i=1}^{m} x_i\right)^{\frac{n}{m}}$. Using this inequality and the lemma, we get

$$\lambda\left[\left(\sum_{i=1}^{m} x_i\right)^n - \sum_{i=1}^{m} x_i^n\right] + \sum_{i=1}^{m} x_i^n \ge \bigl(\lambda(m^n-m)+m\bigr)\left(\prod_{i=1}^{m} x_i\right)^{\frac{n}{m}},$$

which can be rearranged to give the left inequality.

For the right inequality, we start with an application of the Power Mean Inequality:

$$\frac{1}{m}\sum_{i=1}^{m} x_i^n \ge \left(\frac{1}{m}\sum_{i=1}^{m} x_i\right)^n.$$

Since $\lambda \geq 1$, this inequality is equivalent to

$$(1-\lambda)\sum_{i=1}^{m} x_i^n \;\leq\; \frac{1-\lambda}{m^{n-1}}\left(\sum_{i=1}^{m} x_i\right)^n .$$

Adding $\lambda\left(\sum\limits_{i=1}^{m} x_i\right)^n$ to both sides, we get

$$\lambda\left(\sum_{i=1}^{m} x_i\right)^n + (1-\lambda)\sum_{i=1}^{m} x_i^n \;\leq\; \frac{\lambda m^n + (1-\lambda)m}{m^n}\left(\sum_{i=1}^{m} x_i\right)^n ,$$

which can be rearranged to give the right inequality.

II. *Solution by the proposer.*

Let $a = \sum\limits_{i=1}^{m} x_i^n$ and $b = \left(\sum\limits_{i=1}^{m} x_i\right)^n$. Define

$$f(\lambda) \;=\; \frac{a+\lambda(b-a)}{m+\lambda(m^n-m)} .$$

Taking n^{th} powers throughout the proposed inequalities, and using the notation we have just introduced, we obtain the equivalent inequalities

$$\left(\prod_{i=1}^{m} x_i\right)^{\frac{n}{m}} \;\leq\; f(\lambda) \;\leq\; f(1) , \tag{2}$$

which are to be proved for all $\lambda \geq 1$.

The derivative of $f(\lambda)$ is

$$f'(\lambda) \;=\; \frac{m(b-am^{n-1})}{\Big(m+\lambda(m^n-m)\Big)^2} .$$

By the Power Mean Inequality, we have $b \leq m^{n-1}a$. Thus, $f'(\lambda) \leq 0$ for all λ. The right inequality in (2) then follows immediately, for all $\lambda \geq 1$. Furthermore, we must have

$$f(\lambda) \;\geq\; \lim_{\lambda\to\infty} f(\lambda) \;=\; \frac{b-a}{m^n-m} . \tag{3}$$

Now we make use of the following inequality, which is inequality (2.8) in reference [1]:

$$\left(\sum_{i=1}^{m} x_i\right)^n - \sum_{i=1}^{m} x_i^n \;\geq\; (m^n-m)\left(\prod_{i=1}^{m} x_i\right)^{\frac{n}{m}} .$$

[*Ed.* Note that this is the inequality in the lemma in the solution above.]

Rearranging this inequality and using our notation, we obtain

$$\frac{b-a}{m^n-m} \geq \left(\prod_{i=1}^{m} x_i\right)^{\frac{n}{m}} .$$

Combining this with (3), we obtain the left inequality in (2).

Reference

[1] Leng Gangsong, M. Tongyi and Qian Xiangzhong, *Inequalities for a Simplex and an Interior Point*, Geometriae Dedicata 85, pp. 1–10, 2001, Kluwer Academic Publishers, The Netherlands.

There were no other solutions submitted.

KLAMKIN–09. [2005 : 328, 331] *Proposed by Phil McCartney, Northern Kentucky University, Highland Heights, KY, USA.*

For $0 < x < \pi/2$, prove or disprove that

$$\frac{\ln(1-\sin x)}{\ln(\cos x)} < \frac{2+x}{x} .$$

Essentially the same solution by Michel Bataille, Rouen, France; and Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

We prove the claim. Multiplying by the negative number $x\ln(\cos x)$, we see that the proposed inequality is equivalent to $u(x) > 0$, where

$$u(x) = x\ln(1-\sin x) - (x+2)\ln(\cos x) .$$

The first two derivatives of this function are easily obtained:

$$\begin{aligned} u'(x) &= \ln\left(\frac{1-\sin x}{\cos x}\right) - \frac{x}{\cos x} + 2\tan x \\ u''(x) &= \frac{2(1-\cos x) - x\sin x}{\cos^2 x} = \frac{2\sin x}{\cos^2 x}\left(\tan\frac{x}{2} - \frac{x}{2}\right) . \end{aligned}$$

Now, since $\tan\alpha > \alpha$ for $\alpha \in (0,\pi/2)$, we have $u''(x) > 0$ for $x \in (0,\pi/2)$; hence, the function u' is increasing. Since $\lim\limits_{x\to 0} u'(x) = 0$, we have $u'(x) > 0$ for $x \in (0,\pi/2)$. Thus, u is increasing and, since $\lim\limits_{x\to 0} u(x) = 0$, we have $u(x) > 0$ for all $x \in (0,\pi/2)$.

Also solved by JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

Janous noted that the inequality is very sharp near $x = 0$. Indeed, the Taylor expansion about $x = 0$ of the function $u(x)$ in the solution above is $\frac{1}{360}x^6 + O(x^8)$.

KLAMKIN–10. *Proposed by Mihály Bencze, Brasov, Romania.*

Let $P(x) = \sum\limits_{i=0}^{n} a_i x^i$ be a polynomial with real coefficients and simple roots. Prove that

(a) $\displaystyle\sum_{i=1}^{n} \frac{1}{x_i P'(x_i)} \;=\; -\frac{1}{a_0}$, and (b) $\displaystyle\sum_{i=1}^{n} \frac{x_i^{n-1}}{P'(x_i)} \;=\; \frac{1}{a_n}$.

Solution by Joel Schlosberg, Bayside, NY, USA.

For the expressions to make sense, none of the roots can be zero. Consider the function

$$Q(x) \;=\; \sum_i \frac{r_i^k P(x)}{P'(r_i)(x-r_i)} - x^k\,,$$

where $P(x)$ is an n^{th} degree polynomial with simple roots $r_1, r_2, \ldots, r_n$ and k is any integer with $0 \le k \le n-1$. Since $x - r_i \mid P(x)$ for all i, the function $Q(x)$ is a polynomial of degree (at most) $n-1$. For any root r_j

$$\begin{aligned} Q(r_j) \;&=\; \sum_{i\neq j} \frac{r_i^k P(r_j)}{P'(r_i)(r_j-r_i)} + \lim_{x\to r_j} \frac{r_j^k P(x)}{P'(r_j)(x-r_j)} - r_j^k \\ &=\; \lim_{x\to r_j} \frac{r_j^k}{P'(r_j)} \frac{P(x)-P(r_j)}{x-r_j} - r_j^k \;=\; \frac{r_j^k P'(r_j)}{P'(r_j)} - r_j^k \;=\; 0\,. \end{aligned}$$

Therefore, $Q(x) = 0$ for n distinct values $r_1, r_2, \ldots, r_n$, but since $Q(x)$ has degree at most $n-1$, it must be identically zero; thus,

$$\sum_i \frac{r_i^k P(x)}{P'(r_i)(x-r_i)} - x^k \;=\; 0\,,$$

and, if $x \neq r_i$ for all i, we have

$$\sum_i \frac{r_i^k}{P'(r_i)(x-r_i)} \;=\; \frac{x^k}{P(x)}\,. \tag{1}$$

For $k = 0$ in (1), we have

$$\sum_i \frac{1}{P'(r_i)(x-r_i)} \;=\; \frac{1}{P(x)}\,,$$

and, by setting $x = 0$, (a) is proved, since

$$\sum_i \frac{1}{P'(r_i)r_i} \;=\; -\frac{1}{P(0)} \;=\; -\frac{1}{a_0}\,.$$

For $k = n-1$ in (1), we have

$$\sum_i \frac{r_i^{n-1}}{P'(r_i)(x-r_i)} \;=\; \frac{x^{n-1}}{P(x)} \quad\text{and}\quad \sum_i \frac{r_i^{n-1}}{P'(r_i)}\frac{x}{x-r_i} \;=\; \frac{x^n}{P(x)}\,.$$

By taking the infinite limit, (b) is proved, since

$$\lim_{x\to\infty}\sum_i \frac{r_i^{n-1}}{P'(r_i)}\frac{x}{x-r_i} \;=\; \lim_{x\to\infty}\frac{x^n}{P(x)} \quad\text{and}\quad \sum_i \frac{r_i^{n-1}}{P'(r_i)} \;=\; \frac{1}{a_n}\,.$$

Additionally, for $0 \le k \le n-2$ (that is, $1 \le k+1 \le n-1$) in (1), we see that

$$\sum_i \frac{r_i^{k+1}}{P'(r_i)(x-r_i)} \;=\; \frac{x^{k+1}}{P(x)}\,,$$

and for $x = 0$, we have $-\sum_i \frac{r_i^{k+1}}{P'(r_i)(r_i)} = 0$ and $\sum_i \frac{r_i^k}{P'(r_i)} = 0$. We conclude that

$$\sum_i \frac{r_i^k}{P'(r_i)} \;=\; \begin{cases} -\dfrac{1}{a_0} & \text{if } k=-1, \\ 0 & \text{if } 0 \le k \le n-2, \\ \dfrac{1}{a_n} & \text{if } k = n-1. \end{cases}$$

Also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; VEDULA N. MURTY, Dover, PA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and the proposer.

KLAMKIN–11. [2005 : 329, 332] *Proposed by Mohammed Aassila, Strasbourg, France.*

Let P be an interior point of a triangle ABC, and let r_1, r_2, and r_3 be the inradii of the triangles APB, BPC, and CPA, respectively. Prove that

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3} \;\ge\; \frac{6+4\sqrt{3}}{R}\,,$$

where R is the circumradius of triangle ABC. When does equality hold?

Solution by Scott Brown, Auburn University, Montgomery, AL, USA.

According to [1], we have the following:

$$\frac{r}{r_1} \;=\; 1+\frac{1}{\sin C}\,, \qquad \frac{r}{r_2} \;=\; 1+\frac{1}{\sin A}\,, \qquad \frac{r}{r_3} \;=\; 1+\frac{1}{\sin B}\,,$$

where r_1, r_2, and r_3 are the inradii of triangles APB, BPC, and CPA, respectively. Hence,

$$\begin{aligned}\frac{r}{r_1}+\frac{r}{r_2}+\frac{r}{r_3} &= 3+\left(\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\right)\\ &\geq 3+3\cdot\sqrt[3]{\frac{1}{\sin A\sin B\sin C}}\,,\end{aligned}$$

by the AM–GM Inequality. From [2], we have $\sin A\sin B\sin C \leq \frac{3\sqrt{3}}{8}$. It follows that

$$\frac{r}{r_1}+\frac{r}{r_2}+\frac{r}{r_3} \geq 3+3\left(\sqrt[3]{\frac{8}{3\sqrt{3}}}\right) = 3+2\sqrt{3}\,.$$

Since $R \geq 2r$ (see [2]), we have

$$\frac{R}{r_1}+\frac{R}{r_2}+\frac{R}{r_3} \geq 2\left(\frac{r}{r_1}+\frac{r}{r_2}+\frac{r}{r_3}\right) \geq 6+4\sqrt{3}\,.$$

The desired result follows.

Equality occurs when $\triangle ABC$ is equilateral.

References

[1] Matematika Skole, problem 378, pp. 75–76, No. 1, 1968.

[2] O. Bottema, R.Ž. Djordjević, R.R. Janić, D.S. Mitrinović & P.M. Vasić, *Geometric Inequalities*, Groningen, 1969.

Also solved by MIHÁLY BENCZE, Brasov, Romania; and the proposer.

KLAMKIN–12. [2005 : 329, 332] *Proposed by Michel Bataille, Rouen, France.*

Let a, b, c be the sides of a spherical triangle. Show that

$$3\cos a\cos b\cos c \leq \cos^2 a+\cos^2 b+\cos^2 c \leq 1+2\cos a\cos b\cos c\,.$$

Solution by Manuel Benito, Óscar Ciaurri, and Emilio Fernández, Logroño, Spain; and Li Zhou, Polk Community College, Winter Haven, FL, USA.

By the AM–GM Inequality,

$$\begin{aligned}\cos^2 a+\cos^2 b+\cos^2 c &\geq 3\sqrt[3]{\cos^2 a\cos^2 b\cos^2 c}\\ &\geq 3\sqrt[3]{\cos^3 a\cos^3 b\cos^3 c} = 3\cos a\cos b\cos c\,,\end{aligned}$$

which establishes the left inequality.

[*Ed.:* Next, we give the argument of Benito, Ciaurri, and Fernández for the right inequality.] Let A, B, and C be the vertices of the spherical triangle. We use the well-known fundamental formula of spherical trigonometry,

$$\cos a \;=\; \cos b \cos c + \sin b \sin c \cos A\,.$$

We have

$$\begin{aligned} 0 &\le \sin^2 b \sin^2 c \sin^2 A \;=\; \sin^2 b \sin^2 c - \sin^2 b \sin^2 c \cos^2 A \\ &= (1-\cos^2 b)(1-\cos^2 c) - (\cos a - \cos b \cos c)^2 \\ &= 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c\,, \end{aligned}$$

which completes the proof.

[*Ed.:* We now proceed with Zhou's argument for the same inequality.] We may assume that the spherical triangle is spanned by the unit vectors $\overrightarrow{A}$, $\overrightarrow{B}$, and $\overrightarrow{C}$, starting at the centre of the sphere. Let $\overrightarrow{A} = \langle A_1, A_2, A_3\rangle$, $\overrightarrow{B} = \langle B_1, B_2, B_3\rangle$, $\overrightarrow{C} = \langle C_1, C_2, C_3\rangle$, and $M = \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$. Then

$$\begin{aligned} 0 &\le [(\overrightarrow{A}\times\overrightarrow{B})\cdot\overrightarrow{C}]^2 \;=\; (\det M)^2 \;=\; \det(MM^T) \\ &= \begin{vmatrix} \overrightarrow{A}\cdot\overrightarrow{A} & \overrightarrow{A}\cdot\overrightarrow{B} & \overrightarrow{A}\cdot\overrightarrow{C} \\ \overrightarrow{B}\cdot\overrightarrow{A} & \overrightarrow{B}\cdot\overrightarrow{B} & \overrightarrow{B}\cdot\overrightarrow{C} \\ \overrightarrow{C}\cdot\overrightarrow{A} & \overrightarrow{C}\cdot\overrightarrow{B} & \overrightarrow{C}\cdot\overrightarrow{C} \end{vmatrix} \;=\; \begin{vmatrix} 1 & \cos c & \cos b \\ \cos c & 1 & \cos a \\ \cos b & \cos a & 1 \end{vmatrix} \\ &= 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c\,, \end{aligned}$$

completing the proof.

Also solved by the proposer.

KLAMKIN–13. [2005 : 329, 332] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let $\mathcal{C}$ be a smooth closed convex curve in the plane. Theorems in analysis assure us that there is at least one circumscribing triangle $A_0B_0C_0$ to $\mathcal{C}$ having minimum perimeter. Prove that the excircles of $A_0B_0C_0$ are tangent to $\mathcal{C}$.

Similar solutions by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON; and the proposer.

Suppose, to the contrary, that $A_0B_0C_0$ is a circumscribed triangle with the minimum perimeter, but the excircle opposite A_0 does not touch $\mathcal{C}$. Since $\mathcal{C}$ is convex, the common tangent B_0C_0 separates it from the excircle. By means of a dilatation with centre A, we can therefore shrink that circle to a smaller circle Γ that is tangent to $\mathcal{C}$ and to the lines A_0B_0 and A_0C_0. (Note that $\mathcal{C}$ is inside the finite region bounded by Γ and the two lines.) Let the common tangent to $\mathcal{C}$ and Γ at the point where they touch (which is well defined, since $\mathcal{C}$ is convex while Γ is a circle) meet lines A_0B_0 and A_0C_0 at B_1 and C_1. Triangle $A_0B_1C_1$ is then a triangle circumscribed about $\mathcal{C}$, but we have decreased the length of the tangent from A_0 to the opposite excircle. Since that length is equal to the semiperimeter of the triangle (see, for example, Coxeter and Greitzer, *Geometry Revisited*, page 13), we have thereby decreased the perimeter of the triangle. This contradicts the initial assumption that our original triangle had the minimum perimeter. Thus, the excircles of $A_0B_0C_0$ are tangent to $\mathcal{C}$.

Also solved by LI ZHOU, Polk Community College, Winter Haven, FL, USA; the proposer provided a second solution that made use of differential geometry.

KLAMKIN–14. [2005 : 329, 332] *Proposed by Andy Liu, University of Alberta, Edmonton, AB.*

A vertical wall OY meets the horizontal floor OX at the corner O. Initially, a ladder AB is placed so that its bottom B is at O while its apex A is on the wall OY. A cat jumps onto the ladder and clings to the point C where $BC = \lambda AC$ for some real number $0 < \lambda < 1$. This jiggles the ladder so that it begins to slide, with A moving down towards O along YO and B moving away from O along OX, until it comes to rest with A at O and B on OX. What is the curve traced out by the cat?

Solution by Joel Schlosberg, Bayside, NY, USA.

Let C_x and C_y be the projections of C onto OX and OY, respectively. In a Cartesian coordinate system with OX and OY as the x– and y–axes, respectively, let the coordinates of C be (x, y). Let $\angle ABO = \theta$; since $CC_y \parallel BO$, we see that $\angle ACC_y = \angle CBC_x = \theta$. Then $x = AC\cos\theta$, $y = BC\sin\theta$, and

$$\frac{x^2}{AC^2} + \frac{y^2}{BC^2} \;=\; \cos^2\theta + \sin^2\theta \;=\; 1\,,$$

or

$$x^2 + \frac{y^2}{\lambda^2} \;=\; AC^2\,.$$

Therefore, the curve traced out by the cat is the portion of an ellipse in the first quadrant (x, $y > 0$) with centre O, major axis along OX (since $\lambda < 1$), and eccentricity $\sqrt{1-\lambda^2}$.

Also solved by MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

KLAMKIN–15. [2005 : 329, 332] *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

A square $ABCD$ sits in the plane with corners A, B, C, D initially located at positions $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$, respectively. The square is rotated counterclockwise through an angle θ ($0^\circ \le \theta < 360^\circ$) four times, with the centre of rotation at the points A, B, C, D in successive rotations. Suppose point A ends up on the x-axis or y-axis. Find all possible values of θ.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

For $j = 0, 1, 2, 3, 4$, let $z_{j,1}$, $z_{j,2}$, $z_{j,3}$, $z_{j,4}$ be the complex numbers representing the corners A, B, C, D, respectively, after the j^{th} rotation. Then $z_{0,1} = 0$, $z_{0,2} = 1$, $z_{0,3} = 1 + i$, $z_{0,4} = i$, and for $1 \le j,\ k \le 4$, we have

$$z_{j,k} \;=\; z_{j-1,j} + (z_{j-1,k} - z_{j-1,j})e^{\theta i}\,.$$

Hence,

$$\begin{aligned} z_{4,1} &= z_{3,4} + (z_{3,1} - z_{3,4})e^{\theta i} \;=\; z_{2,3} + (z_{2,4} - z_{2,3})e^{\theta i} + (z_{2,1} - z_{2,4})e^{2\theta i} \\ &= z_{1,2} + (z_{1,3} - z_{1,2})e^{\theta i} + (z_{1,4} - z_{1,3})e^{2\theta i} + (z_{1,1} - z_{1,4})e^{3\theta i} \\ &= z_{0,1} + (z_{0,2} - z_{0,1})e^{\theta i} + (z_{0,3} - z_{0,2})e^{2\theta i} \\ &\qquad\qquad + (z_{0,4} - z_{0,3})e^{3\theta i} + (z_{0,1} - z_{0,4})e^{4\theta i} \\ &= e^{\theta i} + ie^{2\theta i} - e^{3\theta i} - ie^{4\theta i} \\ &= \cos\theta - \sin 2\theta - \cos 3\theta + \sin 4\theta \\ &\qquad\qquad + i(\sin\theta + \cos 2\theta - \sin 3\theta - \cos 4\theta)\,. \end{aligned}$$

By the sum-to-product identities,

$$\begin{aligned} \cos\theta - \cos 3\theta - \sin 2\theta + \sin 4\theta &= 2\sin 2\theta \sin\theta + 2\sin\theta\cos 3\theta \\ &= 2\sin\theta\left[\sin 2\theta + \sin\left(\tfrac{\pi}{2} - 3\theta\right)\right] \\ &= 4\sin\theta\sin\left(\tfrac{\pi}{4} - \tfrac{1}{2}\theta\right)\cos\left(-\tfrac{\pi}{4} + \tfrac{5}{2}\theta\right), \end{aligned}$$

which equals 0 when $\theta \in \left\{0,\ \pi,\ \frac{\pi}{2},\ \frac{3\pi}{10},\ \frac{7\pi}{10},\ \frac{11\pi}{10},\ \frac{3\pi}{2},\ \frac{19\pi}{10}\right\}$. Similarly,

$$\begin{aligned} \sin\theta - \sin 3\theta + \cos 2\theta - \cos 4\theta &= -2\sin\theta\cos 2\theta + 2\sin 3\theta\sin\theta \\ &= 2\sin\theta\left[\sin 3\theta - \sin\left(\tfrac{\pi}{2} - 2\theta\right)\right] \\ &= 4\sin\theta\sin\left(-\tfrac{\pi}{4} + \tfrac{5}{2}\theta\right)\cos\left(\tfrac{\pi}{4} + \tfrac{1}{2}\theta\right), \end{aligned}$$

which equals 0 when $\theta \in \left\{0,\ \pi,\ \frac{\pi}{10},\ \frac{\pi}{2},\ \frac{9\pi}{10},\ \frac{13\pi}{10},\ \frac{17\pi}{10}\right\}$.

In conclusion, the set of values of θ is

$$\{0,\ \pi\} \cup \left\{ \frac{(2n+1)\pi}{10} \;\middle|\; n = 0, 1, 2, \ldots, 9 \right\}.$$

Also solved by MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and the proposer. There was one incorrect submission.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

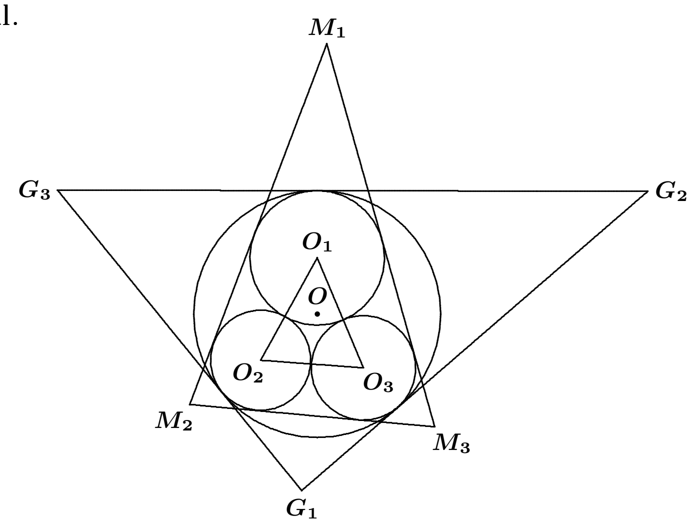
We apologize for omitting the name of Michel Bataille, Rouen, France from the list of solvers of 3028; and for omitting the name of Chip Curtis, Missouri Southern State University, Joplin, MO, USA from the list of solvers of 3039.

1150★. [1986 : 108; 1987 : 264; 1988 : 46] *Proposed by Jack Garfunkel (deceased).*

In the figure, $\triangle M_1M_2M_3$ and the three circles with centres O_1, O_2, O_3 represent the Malfatti configuration. Circle O is externally tangent to these three circles, and the sides of $\triangle G_1G_2G_3$ are each tangent to O and one of the smaller circles. Prove that

$$P(\triangle G_1G_2G_3) \;\geq\; P(\triangle M_1M_2M_3) + P(\triangle O_1O_2O_3)\,,$$

where P stands for perimeter. Equality is attained when $\triangle O_1O_2O_3$ is equilateral.



Solution by Kai-Xian Wang, Qingdao, Shandong, China, summarized by the editor.

Let r be the radius of circle O and let r_i be the radius of circle O_i for $i = 1, 2, 3$. To construct counterexamples, we set $r_2 = r_3 = 2$ and let r_1 vary: set $r_1 = r > 0$. To simplify the notation, we let $g = P(\triangle G_1G_2G_3)$, $m = P(\triangle M_1M_2M_3)$, and $o = P(\triangle O_1O_2O_3)$. Note that $o = 8 + 2r$.

Counterexample 1. When r increases to 4 (see Figure 1), g and o remain bounded while m goes to infinity. Then $g < m + o$, contrary to the conjecture. [*Ed:* This is the essence of the counterexample given in [1988 : 46].]

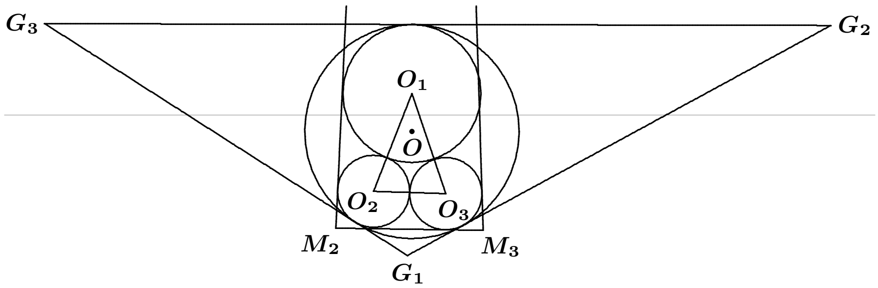


Figure 1: $r = 4 - \varepsilon$, $r_2 = r_3 = 2$

Counterexample 2. When r decreases to $\frac{1}{2}$ (see Figure 2), the radius of circle O becomes infinite as circle O_1 sinks below the horizontal common tangent to circles O_2 and O_3. Here g is bounded near 8 while m goes to infinity; thus, again $g < m + o$, contrary to the conjecture.

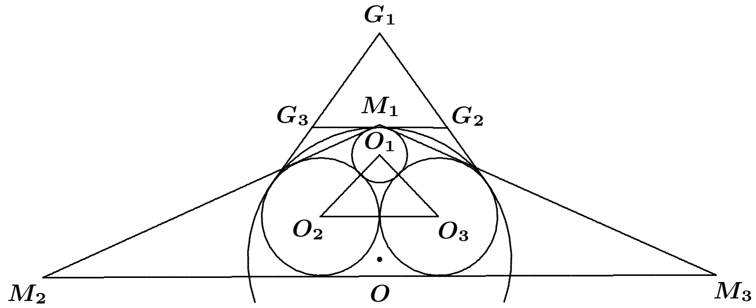


Figure 2: $r = \frac{1}{2} + \varepsilon$, $r_2 = r_3 = 2$

Wang provided explicit details for the case $r = 1$. He showed that $16.9 < g < 17.0$, $42.00 < m < 42.01$, and $o = 10$. The details are straightforward, but the computation requires about a page.

It is clear that there are values of r for which the conjectured inequality holds; for example, $g > m + o$ when the outer circle O has its centre on the line O_2O_3 (since m and o are bounded while g is infinite). Also, it is easily checked that when $r = 2$ (and all three triangles are equilateral), $g = m + o$ since $o = 12$, $m = 12 + 12\sqrt{3}$, and $g = 24 + 12\sqrt{3}$.

The statement of the conjecture contains a small error: it is clear from the figure that circle O should be internally *tangent to the three smaller circles (instead of* externally *tangent as stated). Wang first saw the conjecture among 152 open problems in Kuang Jichang,* Applied Inequalities, *3rd edition (in Chinese), Shandong Science and Technology Press, Jinan, P.R. China (2004), page 706.*

3044. [2005 : 238, 240] *Proposed by José Luis Díaz-Barrero and Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let $\{a_n\}$ be the sequence defined by $a_0 = 1$, $a_1 = 2$, and, for $n \geq 2$, $a_n = a_{n-1} + a_{n-2}$. Find the sum

$$\sum_{n=1}^{\infty} \frac{a_{2n+2}}{a_{n-1}^2 a_{n+1}^2}\,.$$

Essentially the same solution by Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, TX, USA; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

Note that $a_n = f_{n+2}$ for $n \geq 0$, where $\{f_n\}$ is the Fibonacci sequence. It is well known (and easily proved by induction) that for $n \geq 0$

$$\begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n .$$

It follows that

$$\begin{pmatrix} f_{2n+5} & f_{2n+4} \\ f_{2n+4} & f_{2n+3} \end{pmatrix} = \begin{pmatrix} f_{n+2} & f_{n+1} \\ f_{n+1} & f_n \end{pmatrix} \begin{pmatrix} f_{n+4} & f_{n+3} \\ f_{n+3} & f_{n+2} \end{pmatrix} .$$

Equating the upper-right entries, we get

$$\begin{aligned} f_{2n+4} &= f_{n+2}f_{n+3} + f_{n+1}f_{n+2} = f_{n+2}(f_{n+1} + f_{n+3}) \\ &= (f_{n+3} - f_{n+1})(f_{n+1} + f_{n+3}) = f_{n+3}^2 - f_{n+1}^2 . \end{aligned}$$

Thus

$$\frac{a_{2n+2}}{a_{n-1}^2 a_{n+1}^2} = \frac{f_{2n+4}}{f_{n+1}^2 f_{n+3}^2} = \frac{f_{n+3}^2 - f_{n+1}^2}{f_{n+1}^2 f_{n+3}^2} = \frac{1}{f_{n+1}^2} - \frac{1}{f_{n+3}^2} .$$

Therefore

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{a_{2n+2}}{a_{n-1}^2 a_{n+1}^2} &= \lim_{N\to\infty} \sum_{n=1}^{N} \left(\frac{1}{f_{n+1}^2} - \frac{1}{f_{n+3}^2} \right) \\ &= \lim_{N\to\infty} \left(\frac{1}{f_2^2} + \frac{1}{f_3^2} - \frac{1}{f_{N+2}^2} - \frac{1}{f_{N+3}^2} \right) \\ &= \frac{1}{f_2^2} + \frac{1}{f_3^2} = 1 + \frac{1}{4} = \frac{5}{4} . \end{aligned}$$

Also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

Note that the matrix representation of the Fibonacci numbers also occurs in this issue in the solution of KLAMKIN–04 on pages 314–315.

3045. [2005 : 238, 240] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let a, b, c be positive real numbers such that $abc \geq 1$. Prove that

(a) $a^{\frac{a}{b}} b^{\frac{b}{c}} c^{\frac{c}{a}} \geq 1$; (b) $a^{\frac{a}{b}} b^{\frac{b}{c}} c^{c} \geq 1$.

Solution by the proposer.

(a) First we prove the inequality in the case where $abc = 1$. The inequality may be written equivalently as $\frac{a}{b}\ln a + \frac{b}{c}\ln b + \frac{c}{a}\ln c \geq 0$. Since the function $f(x) = x\ln x$ is convex, Jensen's Inequality gives

$$\frac{1}{b}\cdot a\ln a + \frac{1}{c}\cdot b\ln b + \frac{1}{a}\cdot c\ln c \;\geq\; \left(\frac{1}{b}+\frac{1}{c}+\frac{1}{a}\right)\cdot \ln\frac{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}{\frac{1}{b}+\frac{1}{c}+\frac{1}{a}}\,,$$

and it remains to show that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \;\geq\; \frac{1}{b}+\frac{1}{c}+\frac{1}{a}\,.$$

The last inequality is known (see problem 2886 [2003 : 468; 2004 : 518]).

Now we turn to the general case, where $abc \geq 1$. Let $x = ar$, $y = br$ and $z = cr$, where $r = \frac{1}{\sqrt[3]{abc}} \leq 1$. Then $xyz = 1$, and thus $x^{\frac{x}{y}}y^{\frac{y}{z}}z^{\frac{z}{x}} \geq 1$ (applying the special case we have already proved). Then

$$a^{\frac{a}{b}}b^{\frac{b}{c}}c^{\frac{c}{a}} \;\geq\; a^{\frac{a}{b}}b^{\frac{b}{c}}c^{\frac{c}{a}}r^{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}} \;=\; x^{\frac{x}{y}}y^{\frac{y}{z}}z^{\frac{z}{x}} \;\geq\; 1\,.$$

(b) We write the inequality in the form $\frac{a}{b}\ln a + \frac{b}{c}\ln b + c\ln c \geq 0$. As above, by Jensen's Inequality, we get

$$\frac{1}{b}\cdot a\ln a + \frac{1}{c}\cdot b\ln b + c\ln c \;\geq\; \left(\frac{1}{b}+\frac{1}{c}+1\right)\cdot \ln\frac{\frac{a}{b}+\frac{b}{c}+c}{\frac{1}{b}+\frac{1}{c}+1}\,.$$

Thus, it remains to show that

$$\frac{a}{b}+\frac{b}{c}+c \;\geq\; \frac{1}{b}+\frac{1}{c}+1\,,$$

or, equivalently,

$$\frac{ac}{b}+b+c^2 \;\geq\; \frac{c}{b}+1+c\,.$$

Since $ac \geq \frac{1}{b}$, it suffices to show that

$$\frac{1}{b^2}+b+c^2 \;\geq\; \frac{c}{b}+1+c\,.$$

The last inequality can be written as

$$\left(2c-1-\frac{1}{b}\right)^2 + \left(1-\frac{1}{b}\right)^2(4b+3) \;\geq\; 0\,,$$

which is clearly true. This completes the proof.

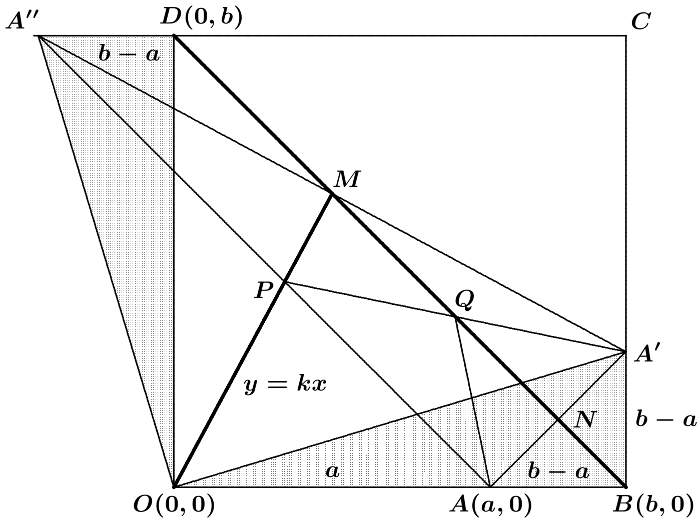
Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and LI ZHOU, Polk Community College, Winter Haven, FL, USA (part (a) only).

3046. [2005 : 238, 240] *Proposed by James T. Bruening, Southeast Missouri State University, Cape Girardeau, MO, USA.*

A mirror is placed in the first quadrant of the xy–plane (perpendicular to the plane) along the straight line joining the points $(b,0)$ and $(0,b)$, for some $b>0$. Another mirror is placed similarly along the line $y=kx$ where $k>1$. A light source at $(a,0)$, $0<a<b$, shoots a beam of light into the first quadrant parallel to the first mirror.

Find k such that when the beam is reflected exactly once by each mirror, it passes through the original light source at $(a,0)$.

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.



Let $OBCD$ be the square whose diagonal is the given mirror that joins $B=(b,0)$ to $D=(0,b)$. Let A' be the mirror image across BD of the light source $A=(a,0)$, and let the line through A parallel to BD intersect CD at A'', while BD intersects $A'A''$ at M and AA' at N. Finally, let $P=AA''\cap OM$ and $Q=PA'\cap BD$.

Since $A''D=AB=A'B=b-a$, the right triangles ODA'' and OBA' are congruent. Hence, $OA''=OA'$ and the lines OA'' and OA' are perpendicular. Since AA'' is parallel to BD and $AN=NA'$, then $A''M=MA'$ and $OM\perp A'A''$. Hence, A'' is the reflection of A' in the mirror OM. Thus, with OM as the initial mirror, the beam AA'' is reflected at P from OM towards A', and reflected again at Q by the mirror BD towards A.

Since $A'A''\perp OM$, the slope k of OM equals the negative reciprocal of the slope of $A''A'$; that is,

$$k \;=\; -\frac{A''C}{A'C} \;=\; \frac{A''D+DC}{BC-BA'} \;=\; \frac{(b-a)+b}{b-(b-a)} \;=\; \frac{2b-a}{a} \;=\; \frac{2b}{a}-1\,.$$

Also solved by MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; Mª JESúS VILLAR RUBIO, Santander, Spain; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

Zhou proved that M is on the bisector of angle PAQ. This implies that the perpendicular bisector of OA passes through M and allows one to calculate the slope of OM easily.

3047. [2005 : 238, 241] *Proposed by Michel Bataille, Rouen, France.*

Let n be a positive integer. Evaluate $\displaystyle\sum_{k=1}^{n} \sec\left(\frac{2k\pi}{2n+1}\right)$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Let $z = e^{i\theta} = x + iy$. Then

$$\begin{aligned}\cos\bigl((2n+1)\theta\bigr) &= \mathfrak{Re}\bigl(z^{2n+1}\bigr) = \sum_{k=0}^{n}\binom{2n+1}{2k}x^{2(n-k)+1}(iy)^{2k} \\ &= \sum_{k=0}^{n}(-1)^k\binom{2n+1}{2k}x^{2(n-k)+1}(1-x^2)^k\,,\end{aligned}$$

which we denote as $f(x)$. It is easy to see that

$$f(x) = a_{2n+1}x^{2n+1} + \cdots + a_1x\,, \tag{1}$$

where

$$a_1 = (-1)^n\binom{2n+1}{2n} = (-1)^n(2n+1)\,. \tag{2}$$

On the other hand, since

$$\begin{aligned}\cos\bigl((2n+1)\theta\bigr) &= \frac{1}{2}\left(z^{2n+1} + \frac{1}{z^{2n+1}}\right) = 1 + \frac{1}{2}\left(z^{2n+1} + \frac{1}{z^{2n+1}} - 2\right) \\ &= 1 + \frac{1}{2z^{2n+1}}\bigl(z^{2n+1}-1\bigr)^2\end{aligned}$$

and

$$\begin{aligned}\bigl(z^{2n+1}-1\bigr)^2 &= \prod_{k=0}^{2n}\left(z - e^{\frac{2k\pi i}{2n+1}}\right)^2 = \prod_{k=0}^{2n}\left(z - e^{\frac{2k\pi i}{2n+1}}\right)\left(z - e^{-\frac{2k\pi i}{2n+1}}\right) \\ &= \prod_{k=0}^{2n}\left(z^2 + 1 - 2z\cos\frac{2k\pi}{2n+1}\right)\,,\end{aligned}$$

we also have

$$f(x) \;=\; 1+\frac{1}{2}\prod_{k=0}^{2n}\left(z+\frac{1}{z}-2\cos\frac{2k\pi}{2n+1}\right)$$

$$=\; 1+2^{2n}\prod_{k=0}^{2n}\left(x-\cos\frac{2k\pi}{2n+1}\right)\,. \qquad (3)$$

Now we compare the expressions (1) and (3) for $f(x)$. We see from (1) that the constant term in (3) is zero. Therefore,

$$2^{2n}\prod_{k=0}^{2n}\cos\left(\frac{2k\pi}{2n+1}\right) \;=\; 1\,.$$

Comparing the coefficients of x in (1) and (3), and dividing by the product above (which equals 1), we get

$$a_1 \;=\; \sum_{k=0}^{2n}\sec\frac{2k\pi}{2n+1} \;=\; 1+2\sum_{k=1}^{n}\sec\frac{2k\pi}{2n+1}\,.$$

Setting this expression for a_1 equal to the expression in (2), we obtain

$$\sum_{k=1}^{n}\sec\frac{2k\pi}{2n+1} \;=\; \frac{(-1)^n(2n+1)-1}{2}\,,$$

which equals n if n is even and $-(n+1)$ if n is odd.

Also solved by ARKADY ALT, San Jose, CA, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

The polynomial $f(x)$ that Zhou used in his solution is closely related to $T_{2n+1}(x)-1$, where T_n is the Chebyshev Polynomial of the first kind, defined by $T_n(\cos\theta) = \cos(n\theta)$. See, for example, `http://mathworld.wolfram.com/ChebyshevPolynomialoftheFirstKind.html`. *Both Alt and the proposer used T_n in their solutions. Howard used a similar polynomial (ascribed to Waring) and provided reference [1]. Janous found the sum in [2].*

References

[1] Aron Pinker, A Generator of Trigonometric Identities. *Two-Year College Math J*, **5**:4 (December 1974), 54–55.

[2] A.P. Prudnikov et al., *Sums and Series (Elementary Functions)* [in Russian], Nauka, Moscow 1981. Page 644, item 4.4.6.2.

3048. [2005 : 238, 241] *Proposed by Gabriel Dospinescu, Paris, France.*

Find all polynomials P with integer coefficients which satisfy the property that, for any relatively prime integers a and b, the sequence $\{P(an+b)\}_{n\geq 1}$ contains an infinite number of terms, any two of which are relatively prime.

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

First, P cannot be a constant polynomial, as $\{P(an+b)\}_{n\geq 1}$ would not contain infinitely many terms.

Suppose that $P(x) = kx^m$ for some integers $k \neq 0$ and $m > 0$. Let a and b be relatively prime integers. Then $P(an+b) = k(an+b)^m$ is divisible by k for all n. Therefore P does not satisfy the required condition if $|k| > 1$. If $k = 1$, then $\{P(an+b)\}_{n\geq 1} = \{(an+b)^m\}_{n\geq 1}$. By Dirichlet's Theorem, the sequence $\{an+b\}_{n\geq 1}$ contains infinitely many primes, which implies that the sequence $\{(an+b)^m\}_{n\geq 1}$ contains infinitely many terms that are pairwise relatively prime. Thus, the polynomial $P(x) = x^m$ is a solution for any positive integer m. Then the polynomial $P(x) = -x^m$ must be a solution as well.

Now suppose that P has at least two terms. Then $P(x) = x^\ell Q(x)$ for some non-negative integer ℓ, where

$$Q(x) \;=\; a_j x^j + a_{j-1}x^{j-1} + \cdots + a_1 x + a_0\,,$$

with $j \geq 1$ and $a_j a_0 \neq 0$. Choose a prime q such that $q \nmid a_0$. Since Q is a nonconstant polynomial, we can choose a sufficiently large positive integer r such that $|Q(q^r)| > 1$. Then we can choose a prime p such that $p \mid Q(q^r)$. Note that $p \neq q$, since $Q(q^r) \equiv a_0 \pmod{q}$ and $q \nmid a_0$.

Let $a = p$ and $b = q^r$. Then a and b are relatively prime, since $p \neq q$. Moreover,

$$P(an+b) \;=\; P(pn+q^r) \;\equiv\; P(q^r) \;=\; q^{r\ell}Q(q^r) \;\equiv\; 0 \pmod{p}\;.$$

Thus, all terms of $\{P(an+b)\}_{n\geq 1}$ are divisible by p. Then there cannot be an infinite number of relatively prime terms, which means that P does not satisfy the required condition.

We conclude that the only solutions are the polynomials $P(x) = \pm x^m$ for $m \in \mathbb{N}$.

Also solved by LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There was one incorrect solution.

3049. [2005 : 239, 241] *Proposed by Óscar Ciaurri, Universidad de La Rioja, Logroño, Spain and José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Given the function $f(x) = \dfrac{x^2}{\sqrt{1+x^2}}\, e^{-\arctan x}$,

(a) find the slant asymptote L in the first quadrant, and

(b) find the area in the first quadrant bounded by the graph of $y = f(x)$ and the line L.

Combination of solutions by Michel Bataille, Rouen, France; and Li Zhou, Polk Community College, Winter Haven, FL, USA; adapted by the editor.

(a) For $x > 1$, we have $\frac{\pi}{2} - \arctan x = \int_x^\infty \frac{1}{1+t^2}\,dt$. Using the substitution $t = 1/w$ we get

$$\begin{aligned} \frac{\pi}{2} - \arctan x &= \sum_{n=0}^{\infty} (-1)^n \int_0^{\frac{1}{x}} w^{2n}\,dw \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)x^{2n+1}} = \frac{1}{x} + O\left(\frac{1}{x^3}\right) . \end{aligned}$$

Hence,

$$e^{\frac{\pi}{2}-\arctan x} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{x} + O\left(\frac{1}{x^3}\right)\right)^n = 1 + \frac{1}{x} + \frac{1}{2x^2} + O\left(\frac{1}{x^3}\right) .$$

Therefore,

$$\begin{aligned} f(x) &= e^{-\frac{\pi}{2}} x \left(1 + x^{-2}\right)^{-\frac{1}{2}} e^{\frac{\pi}{2}-\arctan x} \\ &= e^{-\frac{\pi}{2}} x \left(1 + O(x^{-2})\right)\left(1 + x^{-1} + O(x^{-2})\right) , \end{aligned}$$

which implies that the equation for L is $y = e^{-\frac{\pi}{2}}(x+1)$.

(b) Let $\ell(x) = e^{-\frac{\pi}{2}}(x+1)$, so that the equation for L is $y = \ell(x)$. We claim that $\ell(x) > f(x)$ for $x > 0$. To prove this, we note that the inequality $\ell(x) > f(x)$ is equivalent to

$$-\tfrac{\pi}{2} + \ln(x+1) < 2\ln x - \tfrac{1}{2}\ln(1+x^2) - \arctan x .$$

Replacing x by $1/x$ and using the identity $\arctan x + \arctan(1/x) = \frac{\pi}{2}$, we obtain

$$\ln(x+1) > -\tfrac{1}{2}\ln(1+x^2) + \arctan x . \tag{1}$$

For $x \geq 0$, let

$$\phi(x) = \ln(1+x) + \tfrac{1}{2}\ln(1+x^2) - \arctan x .$$

Then $\phi(0) = 0$ and $\phi'(x) = \frac{2x^2}{(x+1)(1+x^2)} > 0$ for $x > 0$. Therefore, $\phi(x) > 0$ for all $x > 0$. This proves (1) and completes the proof of our claim that $\ell(x) > f(x)$ for $x > 0$.

The desired area is $\int_0^\infty \left(\ell(x) - f(x)\right)dx$. We have

$$\int f(x)\,dx = \int \frac{x^2}{\sqrt{1+x^2}} e^{-\arctan x}\,dx = \int e^{-w} \sec w \tan^2 w\,dw ,$$

where $w = \arctan x$. Integrating by parts yields

$$\begin{aligned}\int f(x)\,dx &= \int e^{-w}\sec w\,d(\tan w) - \int e^{-w}\sec w\,dw \\ &= e^{-w}\sec w\tan w - \int \tan w\,d(e^{-w}) - \int e^{-w}\sec w\,dw \\ &= e^{-w}\sec w\tan w + e^{-w}\sec w - \int e^{-w}\sec w\tan^2 w\,dw \\ &= (x+1)\sqrt{1+x^2}e^{-\arctan x} - \int f(x)\,dx\,.\end{aligned}$$

Thus,

$$\begin{aligned}&\int_0^{\infty}\big(\ell(x)-f(x)\big)\,dx \\ &= \lim_{u\to\infty}\left[e^{-\pi/2}(\tfrac{1}{2}x^2+x) - \tfrac{1}{2}(1+x)\sqrt{1+x^2}\,e^{-\arctan x}\right]\Big|_0^u \\ &= \lim_{u\to\infty}\left[e^{-\pi/2}(\tfrac{1}{2}u^2+u) - \tfrac{1}{2}(1+u)\sqrt{1+u^2}\,e^{-\arctan u}\right] + \frac{1}{2} \\ &= \frac{1}{2} + \frac{1}{2}\lim_{u\to\infty} e^{-\frac{\pi}{2}}u^2\left[+2u^{-1} - \big(1+u^{-1}\big)\big(1+u^{-2}+O(u^{-4})\big)\right. \\ &\qquad \left.\cdot\big(1+u^{-1}+\tfrac{1}{2}u^{-2}+O(u^{-3})\big)\right] \\ &= \tfrac{1}{2} - e^{-\pi/2}\,.\end{aligned}$$

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; HENRY RICARDO, Medgar Evers College (CUNY), Brooklyn, NY, USA; and the proposers. There was one incorrect solution.

3050. [2005 : 239, 241] *Proposed by Christopher J. Bradley, Bristol, UK.*

Let ABC be a triangle with Cevians AX, BY, CZ. Let L, M, N be the mid-points of AX, BY, CZ, respectively. Let AM and AN meet BC at P_1 and P_2, respectively; let BN and BL meet CA at Q_1 and Q_2, respectively; and let CL and CM meet AB at R_1 and R_2, respectively.

Prove that P_1, P_2, Q_1, Q_2, R_1, R_2 lie on a conic.

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

We must assume that the Cevians are concurrent. On the other hand, it is not necessary for L, M, and N to be the mid-points of their respective Cevians. We shall prove that if L, M, N are arbitrary points on the Cevians AX, BY, CZ, respectively, and if $AM \cap BC = P_1$, $BN \cap CA = Q_1$, $CL \cap AB = R_1$, $AN \cap BC = P_2$, $BL \cap CA = Q_2$, and $CM \cap AB = R_2$, then P_1, P_2, Q_1, Q_2, R_1, R_2 lie on a conic if and only if the Cevians AX, BY, and CZ are concurrent.

Let the lines R_1Q_2 and BC meet at P (if the lines are parallel, take P at infinity). Note that X and P are harmonic conjugates with respect to B and C (because B and C are diagonal points of the complete quadrangle AR_1LQ_2, while the diagonals AL and R_1Q_2 meet BC at X and P). In the notation of directed distances, this implies that

$$\frac{BP}{PC} = -\frac{CX}{XB}.$$

Similarly, let P_1R_2 meet CA at Q and let Q_1P_2 meet AB at R. Then, using quadrangles BP_1MR_2 and CQ_1NP_2, we have

$$\frac{CQ}{QA} = -\frac{AY}{YC} \quad \text{and} \quad \frac{AR}{RB} = -\frac{BZ}{ZA}.$$

Multiplying the three equations together, we see that

$$\frac{BP}{PC}\cdot\frac{CQ}{QA}\cdot\frac{AR}{RB} = -\frac{CX}{XB}\cdot\frac{BZ}{ZA}\cdot\frac{AY}{YC}.$$

By the theorems of Ceva and Menelaus applied to triangle ABC, the above equation shows that the lines AX, BY, CZ concur if and only if points P, Q, R are collinear. However, by the converse to Pascal's Theorem, the hexagon $P_1P_2Q_1Q_2R_1R_2$ can be inscribed in a conic if and only if P, Q, R are collinear. Therefore, the points P_1, P_2, Q_1, Q_2, R_1, R_2 lie on a conic if and only if the Cevians AX, BY, CZ are concurrent.

Also solved by MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

The proposer tacitly assumed the existence of a common point as part of the definition of the Cevians (so that each Cevian is determined by that point and a vertex). All solvers except Woo and Zhao went along with that interpretation. By contrast, every reference book on this editor's shelf defines a cevian as in the featured solution—a segment joining a vertex to a point on the opposite side—and they write the word using a lower-case c.

3051. [2005 : 333, 335] *Proposed by Vedula N. Murty, Dover, PA, USA.*

Let x, y, $z \in [0,1)$ such that $x+y+z=1$. Prove that

(a) $\sqrt{\dfrac{x}{x+yz}} + \sqrt{\dfrac{y}{y+zx}} + \sqrt{\dfrac{z}{z+xy}} \le 3\sqrt{\dfrac{3}{2}}$;

(b) $\dfrac{\sqrt{xyz}}{(1-x)(1-y)(1-z)} \le \dfrac{3\sqrt{3}}{8}$.

Similar solutions by Arkady Alt, San Jose, CA, USA; and Peter Y. Woo, Biola University, La Mirada, CA, USA, modified by the editor.

For part (a) we will prove the stronger inequality

$$\sqrt{\frac{x}{x+yz}} + \sqrt{\frac{y}{y+xz}} + \sqrt{\frac{z}{z+xy}} \le \frac{3\sqrt{3}}{2}. \tag{1}$$

We may assume for both parts (a) and (b) that x, y, $z \in (0,1)$, because the inequalities are trivial if one of x, y, z is zero. Our proof will be based on the following inequality, which is an immediate consequence of the convexity of the sine function on $[0,\pi]$:

$$\sin\theta_1 + \sin\theta_2 + \sin\theta_3 \;\leq\; \frac{3\sqrt{3}}{2} \tag{2}$$

for all θ_1, θ_2, $\theta_3 > 0$ such that $\theta_1 + \theta_2 + \theta_3 = \pi$.

Let $a = y + z$, $b = z + x$, and $c = x + y$. Since $x + y + z = 1$, we have $a = 1 - x$, $b = 1 - y$, $c = 1 - z$, and $a + b + c = 2$. The numbers a, b, c satisfy the triangle inequalities $a + b > c$, $b + c > a$, and $c + a > b$. Therefore, a, b, c are the lengths of the sides of a triangle. Let α, β, γ be the angles opposite the sides a, b, c, respectively. Thus, $\alpha + \beta + \gamma = \pi$. Note that the semiperimeter of the triangle is $s = \frac{1}{2}(a + b + c) = 1$.

(a) We have

$$\begin{aligned}\frac{x}{x + yz} &= \frac{1 - a}{1 - a + (1 - b)(1 - c)} = \frac{1 - a}{2 - (a + b + c) + bc} \\ &= \frac{1 - a}{bc} = \frac{s(s - a)}{bc} = \cos^2(\alpha/2)\,.\end{aligned}$$

Similarly, $\dfrac{y}{y + zx} = \cos^2(\beta/2)$ and $\dfrac{z}{z + xy} = \cos^2(\gamma/2)$. Thus, the left side of (1) is equal to $\cos(\alpha/2) + \cos(\beta/2) + \cos(\gamma/2)$. To prove (1), it will be sufficient to prove that

$$\cos(\alpha/2) + \cos(\beta/2) + \cos(\gamma/2) \;\leq\; \frac{3\sqrt{3}}{2}$$

for all α, β, $\gamma > 0$ such that $\alpha + \beta + \gamma = \pi$. But this follows by applying (2) with $\theta_1 = \frac{1}{2}(\pi - \alpha)$, $\theta_2 = \frac{1}{2}(\pi - \beta)$, and $\theta_3 = \frac{1}{2}(\pi - \gamma)$.

(b) Let K and R be the area and circumradius, respectively, of the triangle with sides a, b, c. Then

$$\begin{aligned}\frac{\sqrt{xyz}}{(1 - x)(1 - y)(1 - z)} &= \frac{\sqrt{(s - a)(s - b)(s - c)}}{abc} = \frac{sK}{4KR} = \frac{s}{4R} \\ &= \frac{1}{4}\left(\frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R}\right) \\ &= \frac{1}{4}(\sin\alpha + \sin\beta + \sin\gamma) \;\leq\; \frac{3\sqrt{3}}{8}\,,\end{aligned}$$

where the last step follows by applying (2) with $\theta_1 = \alpha$, $\theta_2 = \beta$, and $\theta_3 = \gamma$.

Also solved by MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; RONGZHENG JIAO, Yangzhou University, Yangzhou, China; PHAM VAN THUAN, Hanoi University of Science, Hanoi, Vietnam; PANOS E. TSAOUSSOGLOU, Athens, Greece; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and the proposer.

All solvers proved the stronger inequality for (a) *which appears in the featured solution.*

3052. [2005 : 333, 335] *Proposed by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.*

Let G be the centroid of $\triangle ABC$, and let A_1, B_1, C_1 be the mid-points of BC, CA, AB, respectively. If P is an arbitrary point in the plane of $\triangle ABC$, show that

$$PA + PB + PC + 3PG \;\geq\; 2(PA_1 + PB_1 + PC_1)\,.$$

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

There is no need to restrict P to the plane of $\triangle ABC$. Indeed, let P be any point of an n–dimensional space that contains $\triangle ABC$. Let $\mathrm{a} = \overrightarrow{PA}$, $\mathrm{b} = \overrightarrow{PB}$, and $\mathrm{c} = \overrightarrow{PC}$. Then $2\overrightarrow{PA_1} = \mathrm{b}+\mathrm{c}$, $2\overrightarrow{PB_1} = \mathrm{c}+\mathrm{a}$, $2\overrightarrow{PC_1} = \mathrm{a}+\mathrm{b}$, and $3\overrightarrow{PG} = \mathrm{a}+\mathrm{b}+\mathrm{c}$. Thus, the inequality under consideration reads

$$|\mathrm{a}| + |\mathrm{b}| + |\mathrm{c}| + |\mathrm{a}+\mathrm{b}+\mathrm{c}| \;\geq\; |\mathrm{a}+\mathrm{b}| + |\mathrm{b}+\mathrm{c}| + |\mathrm{c}+\mathrm{a}|\,.$$

This is the inequality of Hlawka and Hornich. For its proof and for very many extensions of the inequality to various spaces, see D.S. Mitrinović, *Analytic Inequalities*, Springer, Berlin, 1970, pages 170–176; or the more recent D.S. Mitrinović et al., *Classical and New Inequalities in Analysis* (Kluwer Academic Publishers, Dordrecht, 1993), pages 521–534.

The actual inequality from the problem statement can be found in D.S. Mitrinović et al., *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1989, page 410. The reference there is to the 1984 paper by the Romanian mathematicians M. Chiriţâ and R. Constantinescu.

Also solved by MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and the proposer. There was one incorrect submission.

Bataille referred to Problem 2482 [1999 : 430; 2000 : 506]*, where further comments and references can be found. Zhao derived the inequality from Popoviciu's Inequality, referring to the* **CRUX with MAYHEM** *article "Two Generalizations of Popoviciu's Inequality" by Vasile Cîrtoaje* [2005 : 313–318]*. Both Bencze and Janous provided natural generalizations to k points in n–dimensional space.*

3053. [2005 : 333, 335] *Proposed by Avet A. Grigoryan and Hayk N. Sedrakyan, students, A. Shahinyan Physics and Mathematics School, Yerevan, Armenia.*

Let $a_1, a_2, \ldots, a_n$ be non-negative real numbers whose sum is 1. Prove that

$$n-1 \;\leq\; \sqrt{\frac{1-a_1}{1+a_1}} + \sqrt{\frac{1-a_2}{1+a_2}} + \cdots + \sqrt{\frac{1-a_n}{1+a_n}} \;\leq\; n-2+\frac{2}{\sqrt{3}}\,.$$

Solution by the proposers, modified and expanded by the editor.

Let S_n denote the given summation. Note that, for $0 \leq x \leq 1$, we have $1 \geq 1 - x^2$, which implies that $\frac{1-x}{1+x} \geq (1-x)^2$; hence, $\sqrt{\frac{1-x}{1+x}} \geq 1 - x$. Now,

$$S_n \;=\; \sum_{k=1}^{n} \sqrt{\frac{1-a_k}{1+a_k}} \;\geq\; \sum_{k=1}^{n}(1-a_k) \;=\; n - \sum_{k=1}^{n} a_k \;=\; n-1\,.$$

This proves the left inequality.

To prove the right inequality, we first establish two lemmas.

Lemma 1. If $0 \leq x,\, y \leq 1$ such that $x + y = 1$, then

$$\sqrt{\frac{1-x}{1+x}} + \sqrt{\frac{1-y}{1+y}} \;\leq\; \frac{2}{\sqrt{3}}\,.$$

Proof: We have

$$\left(\sqrt{\frac{1-x}{1+x}} + \sqrt{\frac{1-y}{1+y}}\right)^2 \;=\; \frac{1-x}{1+x} + \frac{1-y}{1+y} + 2\sqrt{\frac{(1-x)(1-y)}{(1+x)(1+y)}}\,. \qquad (1)$$

Since

$$\frac{1-x}{1+x} + \frac{1-y}{1+y} \;=\; \frac{2(1-xy)}{(1+x)(1+y)} \;=\; \frac{2(1-xy)}{1+x+y+xy} \;=\; \frac{2(1-xy)}{2+xy}$$

$$\text{and}\quad \frac{(1-x)(1-y)}{(1+x)(1+y)} \;=\; \frac{1-(x+y)+xy}{1+x+y+xy} \;=\; \frac{xy}{2+xy}\,,$$

we have, from (1),

$$\left(\sqrt{\frac{1-x}{1+x}} + \sqrt{\frac{1-y}{1+y}}\right)^2 \;=\; \frac{2}{2+xy}\left(1 - xy + \sqrt{xy(2+xy)}\right)\,. \qquad (2)$$

By the AM–GM Inequality, we have

$$\begin{aligned} \sqrt{xy(2+xy)} \;&=\; \tfrac{1}{3}\sqrt{9xy(2+xy)} \\ &\leq\; \tfrac{1}{6}(9xy + 2 + xy) \;=\; \tfrac{1}{3}(1+5xy)\,. \end{aligned} \qquad (3)$$

Substituting (3) into (2), we then obtain

$$\left(\sqrt{\frac{1-x}{1+x}} + \sqrt{\frac{1-y}{1+y}}\right)^2 \;=\; \frac{2}{2+xy}\left(1 - xy + \frac{1+5xy}{3}\right) \;=\; \frac{4}{3}\,,$$

from which the result follows immediately. ■

Lemma 2. If $x,\, y \geq 0$ such that $x + y \leq \frac{4}{5}$, then

$$\frac{1-x}{1+x} + \frac{1-y}{1+y} \;\leq\; 1 + \sqrt{\frac{1-x-y}{1+x+y}}\,.$$

Proof: If $x = 0$ or $y = 0$, then clearly equality holds. Suppose $xy \neq 0$. By squaring and rearranging, we obtain the equivalent inequality

$$\begin{aligned} & 2\left(\sqrt{\frac{1-x}{1+x}}\cdot\sqrt{\frac{1-y}{1+y}} - \sqrt{\frac{1-x-y}{1+x+y}}\right) \\ & \quad \leq \quad 1 + \frac{1-x-y}{1+x+y} - \left(\frac{1-x}{1+x} + \frac{1-y}{1+y}\right) \\ & \quad = \quad \frac{2}{1+x+y} - \frac{2(1-xy)}{(1+x)(1+y)} \\ & \quad = \quad \frac{2(1+x+y+xy) - 2(1+x+y-xy-xy(x+y))}{(1+x+y)(1+x)(1+y)} \\ & \quad = \quad \frac{4xy + 2xy(x+y)}{(1+x+y)(1+x)(1+y)} \;=\; \frac{2xy(2+x+y)}{(1+x+y)(1+x)(1+y)}\,, \end{aligned}$$

which is equivalent to

$$\begin{aligned} & \frac{1-x}{1+x}\cdot\frac{1-y}{1+y} - \frac{1-x-y}{1+x+y} \\ & \quad \leq \quad \frac{xy(2+x+y)}{(1+x+y)(1+x)(1+y)}\left(\sqrt{\frac{1-x}{1+x}}\sqrt{\frac{1-y}{1+y}} + \sqrt{\frac{1-x-y}{1+x+y}}\right). \end{aligned} \tag{4}$$

Now,

$$\begin{aligned} & \frac{1-x}{1+x}\cdot\frac{1-y}{1+y} - \frac{1-x-y}{1+x+y} \\ & \quad = \quad \frac{(1-x-y+xy)(1+x+y) - (1-x-y)(1+x+y+xy)}{(1+x+y)(1+x)(1+y)} \\ & \quad = \quad \frac{xy(1+x+y) - (1-x-y)xy}{(1+x+y)(1+x)(1+y)} \;=\; \frac{2xy(x+y)}{(1+x+y)(1+x)(1+y)}\,. \end{aligned}$$

Hence, (4) is equivalent to

$$\frac{2(x+y)}{2+x+y} \;\leq\; \sqrt{\frac{1-x}{1+x}}\sqrt{\frac{1-y}{1+y}} + \sqrt{\frac{1-x-y}{1+x+y}}\,. \tag{5}$$

To prove (5), we note that

$$\begin{aligned} \frac{(1-x)(1-y)}{(1+x)(1+y)} \;&=\; \frac{1-x-y+xy}{1+x+y+xy} \;\geq\; \frac{1-x-y}{1+x+y} \\ &=\; -1 + \frac{2}{1+x+y} \;\geq\; -1 + \frac{2}{1+(4/5)} \;=\; \frac{1}{9}\,. \end{aligned}$$

Thus,

$$\sqrt{\frac{1-x}{1+x}}\sqrt{\frac{1-y}{1+y}} \;\geq\; \sqrt{\frac{1-x-y}{1+x+y}} \;\geq\; \frac{1}{3}\,.$$

Hence,

$$\sqrt{\frac{1-x}{1+x}}\sqrt{\frac{1-y}{1+y}} + \sqrt{\frac{1-x-y}{1+x+y}} \;\geq\; 2\sqrt{\frac{1-x-y}{1+x+y}} \;\geq\; \frac{2}{3}\,. \tag{6}$$

We also have

$$\frac{2}{3} > \frac{2(x+y)}{2+x+y}, \tag{7}$$

since $2(2+x+y) - 6(x+y) = 4 - 4(x+y) > 0$. Using (6) and (7), we obtain (5), completing the proof of Lemma 2. ∎

Now we proceed to prove the original right inequality by induction. The case $n = 1$ is trivial, since $a_1 = 1$ implies $S_1 = 0 < -1 + \frac{2}{\sqrt{3}}$. The case $n = 2$ is Lemma 1.

Suppose that, for some $n \geq 3$, we have $S_{n-1} \leq n - 3 + \frac{2}{\sqrt{3}}$ for all non-negative real numbers $a_1, a_2, \ldots, a_n$ that sum to 1. Let $a_1, a_2, \ldots, a_n$ be non-negative real numbers with a sum of 1. Without loss of generality, we may assume that $a_1 \leq a_2 \leq \cdots \leq a_n$. Then

$$2 = (a_1 + a_2) + (a_2 + a_3) + \cdots + (a_n + a_1) \geq n(a_1 + a_2).$$

Thus, $a_1 + a_2 \leq \frac{2}{n} \leq \frac{2}{3} < \frac{4}{5}$. Since $(a_1 + a_2) + \sum\limits_{k=3}^{n} a_k = 1$, we have, by Lemma 2 and the induction hypothesis,

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \sqrt{\frac{1-a_k}{1+a_k}} \leq 1 + \sqrt{\frac{1-a_1-a_2}{1+a_1+a_2}} + \sum_{k=3}^{n} \sqrt{\frac{1-a_k}{1+a_k}} \\ &\leq 1 + (n-3) + \frac{2}{\sqrt{3}} = n - 2 + \frac{2}{\sqrt{3}}, \end{aligned}$$

completing the induction.

Finally, note that equality holds in both inequalities if and only if either $a_1 = a_2 = \cdots = a_{n-1} = 0$ and $a_n = 1$, or $a_1 = a_2 = \cdots = a_{n-2} = 0$ and $a_{n-1} = a_n = \frac{1}{2}$.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and PETER Y. WOO, Biola University, La Mirada, CA, USA. MIHÁLY BENCZE, Brasov, Romania sent in some remarks regarding various upper and lower bounds for S_n under additional assumptions on the quantities a_k. There were also three additional solutions, all of which contained some flaws.

3054. [2005 : 333, 336] *Proposed by Michel Bataille, Rouen, France.*

For $n = 0, 1, 2, \ldots$, let $U_n = \sum\limits_{k=0}^{n} \binom{2k}{k}$ and $V_n = \sum\limits_{k=0}^{n} (-1)^k \binom{2k}{k}$. Evaluate the following in closed form:

(a) $U_n^2 + 2\sum\limits_{k=1}^{n} \binom{2n+2k}{n+k} U_{n-k}$; (b) $V_n^2 + 2\sum\limits_{k=1}^{n} (-1)^{n+k} \binom{2n+2k}{n+k} V_{n-k}$.

Solution by Tom Leong, Brooklyn, NY, USA.

Denote the given expressions in (a) and (b) by A_n and B_n, respectively. It is well known that the generating functions for the sequences $\binom{2k}{k}$ and $(-1)^k\binom{2k}{k}$, for $k = 0, 1, 2, \ldots$, are $\frac{1}{\sqrt{1-4x}}$ and $\frac{1}{\sqrt{1+4x}}$, respectively; that is,

$$\frac{1}{\sqrt{1-4x}} = \sum_{k=0}^{\infty}\binom{2k}{k}x^k \qquad \text{and} \qquad \frac{1}{\sqrt{1+4x}} = \sum_{k=0}^{\infty}(-1)^k\binom{2k}{k}x^k .$$

[See, for example, C.L. Liu, *Introduction to Combinatorial Mathematics*, McGraw-Hill, Inc., 1968.]

(a) We claim that

$$\begin{aligned} A_n &= \sum_{k=0}^{2n}\left(\sum_{i+j=k}\binom{2i}{i}\binom{2j}{j}\right) \\ &= \sum_{i+j=0}\binom{2i}{i}\binom{2j}{j} + \sum_{i+j=1}\binom{2i}{i}\binom{2j}{j} + \cdots + \sum_{i+j=2n}\binom{2i}{i}\binom{2j}{j}, \end{aligned} \qquad (1)$$

where, for each $k = 0, 1, 2, \ldots, 2n$, the summation $\sum\limits_{i+j=k}\binom{2i}{i}\binom{2j}{j}$ is over *all* ordered pairs (i,j) of non-negative integers such that $i+j=k$.

Indeed, the terms $\binom{2i}{i}\binom{2j}{j}$ when $i+j \le n$ are all the terms in the expansion of U_n^2. On the other hand, if either $i \ge n+1$ or $j \ge n+1$, say $i = n+k$ for some k with $1 \le k \le n$, then

$$\binom{2i}{i}\binom{2j}{j} = \binom{2n+2k}{n+k}\binom{2j}{j} .$$

These are all the terms in the expansion of $\binom{2n+2k}{n+k}U_{n-k}$. Hence, (1) is established.

Now we recognize that the terms on the right side of (1) are precisely the coefficients of $x^0, x^1, x^2, \ldots, x^{2n}$, respectively, in the product of $\sum\limits_{k=0}^{\infty}\binom{2k}{k}x^k$ with itself. Since

$$\frac{1}{\sqrt{1-4x}}\cdot\frac{1}{\sqrt{1-4x}} = \frac{1}{1-4x} = \sum_{k=0}^{\infty}(4x)^2 ,$$

we conclude that $A_n = 1 + 4 + 4^2 + \cdots + 4^{2n} = \frac{1}{3}\left(4^{2n+1}-1\right)$.

(b) Using a similar argument, we can show that

$$B_n = \sum_{i+j=0} (-1)^0 \binom{2i}{i}\binom{2j}{j} + \sum_{i+j=1} (-1)^1 \binom{2i}{i}\binom{2j}{j} + \cdots + \sum_{i+j=2n} (-1)^{2n} \binom{2i}{i}\binom{2j}{j}, \qquad (2)$$

and that the coefficients on the right side of (2) are precisely the coefficients of x^0, x^1, x^2, ..., x^{2n} in the product of $\sum_{k=0}^{\infty} (-1)^k \binom{2k}{k} x^k$ with itself. Since

$$\frac{1}{\sqrt{1+4x}} \cdot \frac{1}{\sqrt{1+4x}} = \frac{1}{1+4x} = \sum_{k=0}^{\infty} (-1)^k (4x)^k,$$

we conclude that $B_n = 1 - 4 + 4^2 - \cdots + 4^{2n} = \frac{1}{5}\left(4^{2n+1}+1\right)$.

Also solved by the proposer.

3055. [2005 : 334, 336] *Proposed by Michel Bataille, Rouen, France.*

Let the incircle of an acute-angled triangle ABC be tangent to BC, CA, AB at D, E, F, respectively. Let D_0 be the reflection of D through the incentre of $\triangle ABC$, and let D_1 and D_2 be the reflections of D across the diameters of the incircle through E and F. Define E_0, E_1, E_2 and F_0, F_1, F_2 analogously. Show that

$$[D_0D_1D_2] + [E_0E_1E_2] + [F_0F_1F_2] = [DD_1D_2] = [EE_1E_2] = [FF_1F_2] \le \tfrac{1}{4}[ABC],$$

where $[XYZ]$ denotes the area of $\triangle XYZ$.

Solution by the proposer.

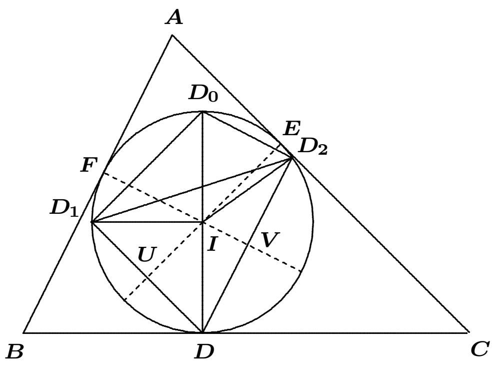
Let I be the incentre of triangle ABC, and let U and V be the midpoints of DD_1 and DD_2, respectively. Since $\angle IEA = \angle IFA = 90°$, we have $\angle UIV = \angle EIF = 180° - A$, and thus, we have

$$\angle D_1DD_2 = \angle UDV = A.$$

Similarly, $\angle EID = 180° - C$, so that $\angle UID = C$. It follows that $\angle DID_1 = 2C$, and that $\angle DD_2D_1 = C$. Consequently, triangles ABC and DD_1D_2 are similar.

The ratio of similarity is the ratio of their circumradii, so that

$$[DD_1D_2] = \left(\frac{r}{R}\right)^2 [ABC].$$



Analogously, we obtain

$$[EE_1E_2] \;=\; [FF_1F_2] \;=\; \left(\frac{r}{R}\right)^2 [ABC]\,.$$

Now, observe that $\angle D_1D_0D = \angle D_1D_2D = C$, from which it follows that $D_0D_1 = 2r\cos C$. Similarly, we have that $D_0D_2 = 2r\cos B$. We now deduce that

$$[D_0D_1D_2] \;=\; \frac{1}{2}\left(4r^2\cos B\cos C\right)\sin A \;=\; \frac{1}{2}r^2(\sin 2B+\sin 2C-\sin 2A)\,.$$

Similar relations hold for $[E_0E_1E_2]$ and $[F_0F_1F_2]$. It follows that

$$\begin{aligned} &[D_0D_1D_2]+[E_0E_1E_2]+[F_0F_1F_2] \\ &\quad=\; \frac{1}{2}(\sin 2B+\sin 2C+\sin 2A) \;=\; 2r^2\sin A\sin B\sin C \\ &\quad=\; 2r^2\left(\frac{a}{2R}\right)\left(\frac{b}{2R}\right)\left(\frac{c}{2R}\right) \;=\; \frac{r^2}{R^2}\,[ABC]\,. \qquad (1) \end{aligned}$$

This completes the proof of the equalities. The known result $R \ge 2r$ provides the desired inequality.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON.

All the solutions were very similar. This editor chose the proposer's solution because he proved equality (1) first. That alone would have been a very interesting problem! However, Janous did comment "A marvellous problem".

3056. [2005 : 334, 336; 2006 : 44, 47] *Proposed by Paul Bracken, University of Texas, Edinburg, TX, USA.*

If $f(x)$ is a non-negative, continuous, concave function on the closed interval $[0,1]$ such that $f(0)=1$, show that

$$2\int_0^1 x^2 f(x)\,dx + \frac{1}{12} \;\le\; \left[\int_0^1 f(x)\,dx\right]^2\,.$$

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

More generally, let p be a positive real number and f a continuous and concave function on [0, 1]. We define

$$M_p \;=\; \int_0^1 x^p f(x)\,dx \qquad \text{and} \qquad M_0 \;=\; \int_0^1 f(x)\,dx\,,$$

and show that

$$\frac{p+2}{2}M_p + \frac{2pf(0)-(p+1)}{4(p+1)} \;\le\; M_0^2\,,$$

with equality if and only if $f(x) = m(x - \frac{1}{2}) + \frac{1}{2}$ for some $m \in \mathbb{R}$.

Integrating by parts, we get

$$\begin{aligned} M_p &= \left[x^p \int_0^x f(t)\,dt\right]_0^1 - \int_0^1 \left[px^{p-1} \int_0^x f(t)\,dt\right] dx \\ &= M_0 - p\int_0^1\!\!\int_0^x x^{p-1} f(t)\,dt\,dx\,. \end{aligned}$$

Since f is concave on [0, 1], we have $f(t) \geq \dfrac{f(x) - f(0)}{x - 0}\,t + f(0)$ for $0 \leq t \leq x \leq 1$. Hence,

$$\begin{aligned} M_p &\leq M_0 - p\int_0^1\!\!\int_0^x \left(x^{p-2}[f(x) - f(0)]\,t + f(0)x^{p-1}\right) dt\,dx \\ &= M_0 - \frac{p}{2}\int_0^1 x^p\left[f(x) + f(0)\right] dx \\ &= M_0 - \frac{p}{2}M_p - \frac{pf(0)}{2(p+1)}\,. \end{aligned}$$

Therefore,

$$\frac{p+2}{2}M_p + \frac{2pf(0) - (p+1)}{4(p+1)} \;\leq\; M_0 - \frac{1}{4} \;=\; M_0^2 - \left(M_0 - \frac{1}{2}\right)^2 \;\leq\; M_0^2\,.$$

Inspecting the proof, we see that equality holds if and only if f is a linear function and $M_0 = \frac{1}{2}$. A short calculation shows that this is true if and only if $f(x) = m(x - \frac{1}{2}) + \frac{1}{2}$ for some $m \in \mathbb{R}$.

Also solved by MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and the proposer. There was one incorrect submission.

All solvers but one used essentially the same argument, but only Zhou replaced 2 *by* p *in the inequality. In particular, they proved (as in our featured solution with* $p = 2$*) that*

$$2\int_0^1 x^2 f(x)\,dx \;\leq\; \int_0^1 f(x)\,dx - \frac{1}{3}\,,$$

which, as Bataille points out, is stronger and more natural. Most mentioned that the problem has also appeared as problem 11133 in The American Mathematical Monthly, **112***:2 (February, 2005), page 180, with the same proposer. Indeed, Zhou submitted our featured solution also to the* Monthly.

3057. [2005 : 334, 336] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let a, b, c be non-negative real numbers, and let $p \geq \dfrac{\ln 3}{\ln 2} - 1$. Prove that

$$\left(\frac{2a}{b+c}\right)^p + \left(\frac{2b}{c+a}\right)^p + \left(\frac{2c}{a+b}\right)^p \;\geq\; 3\,.$$

Solution by the proposer, expanded slightly by the editor.

Let $x = \dfrac{2a}{b+c}$, $y = \dfrac{2b}{c+a}$, and $z = \dfrac{2c}{a+b}$. Then $x \geq 0$, $y \geq 0$, $z \geq 0$, and the given inequality becomes

$$x^p + y^p + z^p \ \geq\ 3 \tag{1}$$

under the following additional constraint:

$$\frac{1}{x+2} + \frac{1}{y+2} + \frac{1}{z+2} \ =\ 1\,. \tag{2}$$

Let $q = \dfrac{\ln 3}{\ln 2} - 1 \approx 0.585$. By the Power-Mean Inequality, we have

$$\left(\frac{x^p + y^p + z^p}{3}\right)^{\frac{1}{p}} \ \geq\ \left(\frac{x^q + y^q + z^q}{3}\right)^{\frac{1}{q}}\,.$$

Hence, to prove (1), it suffices to show that

$$x^q + y^q + z^q \ \geq\ 3\,. \tag{3}$$

Without loss of generality, we may assume that $a = \min\{a,\, b,\, c\}$. Then $x = \dfrac{2a}{b+c} \leq 1$ and $yz = \dfrac{4bc}{(c+a)(a+b)} \geq \dfrac{4bc}{(2c)(2b)} = 1$. Note that (3) can be obtained by adding the following two inequalities:

$$x^q + 2\left(\frac{2}{x+1}\right)^q \ \geq\ 3\,, \tag{4}$$

$$y^q + z^q \ \geq\ 2\left(\frac{2}{x+1}\right)^q\,. \tag{5}$$

We will prove (4) under the constraint $0 \leq x \leq 1$ and (5) under the constraints $yz \geq 1$ and (2). This will suffice to prove (3).

To prove (4), consider the function

$$f(x) \ =\ x^q + 2\left(\frac{2}{x+1}\right)^q\,,$$

for $0 \leq x \leq 1$. Then

$$f'(x) \ =\ qx^{q-1} - q\left(\frac{2}{x+1}\right)^{q+1}\,.$$

Now, for $0 < x \leq 1$, define

$$g(x) \ =\ (q-1)\ln x - (q+1)\ln\left(\frac{2}{x+1}\right)\,.$$

Then $f'(x)$ and $g(x)$ have the same sign on $(0,1]$. Since

$$g'(x) \;=\; \frac{q-1}{x} + (q+1)\left(\frac{1}{x+1}\right) \;=\; \frac{2qx+q-1}{x(x+1)}\,,$$

we have $g'(x) = 0$ for $x = x_0 = (1-q)/(2q) < 1$. Furthermore, $g'(x) < 0$ for $x \in (0, x_0)$, and $g'(x) > 0$ for $x \in (x_0, 1)$. Hence, g is strictly decreasing on $(0, x_0]$ and strictly increasing on $[x_0, 1]$.

Since $g(1) = 0$ and $\lim\limits_{x\to 0+} g(x) = +\infty$, it follows that there exists $x_1 \in (0, x_0)$ such that $g(x_1) = 0$. Furthermore, $g(x) > 0$ for $x \in (0, x_1)$, and $g(x) < 0$ for $x \in (x_1, 1)$. Hence, $f'(x_1) = 0$, $f'(x) > 0$ for $x \in (0, x_1)$, and $f'(x) < 0$ for $x \in (x_1, 1)$. Therefore, f is strictly increasing on $[0, x_1]$ and strictly decreasing on $[x_1, 1]$.

Since $f(0) = 2^{q+1} = 2^{\ln 3/\ln 2} = 2^{\log_2 3} = 3$ and $f(1) = 3$, we conclude that $f(x) \geq 3$ on $[0,1]$, establishing (4).

To prove (5), we first note that $\left(\frac{y^q+z^q}{2}\right)^{\frac{1}{q}} \geq \left(\frac{\sqrt{y}+\sqrt{z}}{2}\right)^2$, by the Power-Mean Inequality, since $q > 1/2$. Therefore, it suffices to show that, for $yz \geq 1$,

$$\left(\frac{\sqrt{y}+\sqrt{z}}{2}\right)^2 \;\geq\; \frac{2}{x+1}\,. \tag{6}$$

From (2), we obtain

$$\frac{1}{x+2} \;=\; 1 - \frac{y+z+4}{(y+2)(z+2)} \;=\; \frac{yz+y+z}{yz+2y+2z+4}\,;$$

whence,

$$x+1 \;=\; \frac{yz+2y+2z+4}{yz+y+z} - 1 \;=\; \frac{y+z+4}{yz+y+z}\,.$$

Hence, (6) is equivalent to the following, in succession:

$$\begin{aligned}
(y+z+2\sqrt{yz})(y+z+4) \;&\geq\; 8(yz+y+z)\,,\\
(y+z)^2 + 2(y+z)(\sqrt{yz}-2) + 8\sqrt{yz} - 8yz \;&\geq\; 0\,,\\
(y+z-2\sqrt{yz})(y+z+4\sqrt{yz}-4) \;&\geq\; 0\,,\\
(\sqrt{y}-\sqrt{z})^2(y+z+4\sqrt{yz}-4) \;&\geq\; 0\,.
\end{aligned}$$

The last inequality is clearly true, since $yz \geq 1$, and this completes the proof.

Note that equality holds if $a = b = c$. In addition, if $p = q$, then equality holds when one of a, b, or c is zero and the other two are equal.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PHAM VAN THUAN, Hanoi University of Science, Hanoi, Vietnam; and PETER Y. WOO, Biola University, La Mirada, CA, USA. MIHÁLY BENCZE, Brasov, Romania sent in six related open questions.

3058. [2005 : 334, 336] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let A, B, C be the angles of a triangle. Prove that

(a) $\dfrac{1}{2-\cos A}+\dfrac{1}{2-\cos B}+\dfrac{1}{2-\cos C} \;\geq\; 2\,;$

(b) $\dfrac{1}{5-\cos A}+\dfrac{1}{5-\cos B}+\dfrac{1}{5-\cos C} \;\leq\; \dfrac{2}{3}\,.$

Solution by Michel Bataille, Rouen, France; Joe Howard, Portales, NM, USA; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; and Vedula N. Murty, Dover, PA, USA.

We use the following well-known identities (see [1], pp. 55–56):

$$\prod_{\text{cyclic}} \cos A \;=\; \frac{s^2-4R^2-4Rr-r^2}{4R^2}\,, \tag{1}$$

$$\sum_{\text{cyclic}} \cos A \;=\; \frac{R+r}{R}\,, \tag{2}$$

$$\sum_{\text{cyclic}} \cos B\cos C \;=\; \frac{r^2+s^2-4R^2}{4R^2}\,, \tag{3}$$

and the best quadratic estimates on s^2 (item 5.9 in [2]):

$$16Rr-5r^2 \;\leq\; s^2 \;\leq\; 4R^2+4Rr+3r^2\,. \tag{4}$$

(a) Multiplying both sides of the given inequality by $\prod_{\text{cyclic}}(2-\cos A)$, we obtain the equivalent inequality

$$2\prod_{\text{cyclic}}\cos A+4\sum_{\text{cyclic}}\cos A \;\geq\; 4+3\sum_{\text{cyclic}}\cos B\cos C\,.$$

Using equations (1), (2), and (3), and simplifying, we obtain

$$s^2 \;\leq\; 4R^2+8Rr-5r^2\,.$$

In the light of inequality (4), it suffices to show that

$$4R^2+4Rr+3r^2 \;\leq\; 4R^2+8Rr-5r^2\,.$$

But this is equivalent to $2r \leq R$, which is the well-known Euler Inequality.

(b) Similarly, we obtain the following equivalent form of the desired inequality:

$$72Rr-9r^2 \;\leq\; 5s^2\,.$$

By inequality (4), it suffices to show that

$$72Rr - 9r^2 \;\le\; 5(16Rr - 5r^2)\,,$$

which is again equivalent to the Euler Inequality $2r \le R$.

References

[1] D.S. Mitrinović et al., *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1989.

[2] O. Bottema et al., *Geometric Inequalities*, Groningen, 1969.

Also solved by MIHÁLY BENCZE, Brasov, Romania; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Janous has proven the following more general results:

(a) $\dfrac{1}{\lambda - \cos A} + \dfrac{1}{\lambda - \cos B} + \dfrac{1}{\lambda - \cos C} \ge \mu$, *if* $2 \le \mu < 6$ *and* $\lambda = \dfrac{\mu + 6}{2\mu}$.

(b) $\dfrac{1}{\lambda - \cos A} + \dfrac{1}{\lambda - \cos B} + \dfrac{1}{\lambda - \cos C} \le \mu$, *if* $0 < \mu \le \dfrac{2}{3}$ *and* $\lambda = \dfrac{\mu + 6}{2\mu}$.