SKOLIAD No. 89

Robert Bilinski

Please send your solutions to the problems in this issue by ***March 1, 2006***. A copy of **MATHEMATICAL MAYHEM Vol. 2** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

Our featured contest this issue is the 1999 New Zealand Junior Mathematics Competition, for which I thank Derek Holton and Warren Palmer, both from the University of Otago in New Zealand.

1999 New Zealand Junior Mathematics Competition
Sponsored by the University of Otago

1. Morris Muddledit multiplies two-digit numbers by multiplying together the ones and tens digit separately and adding the results. Let this erroneous multiplication be noted by $(\times)$. For example:

$$\begin{aligned} 36(\times)47 &= 42+12 = 54\,, \\ 23(\times)40 &= 0+8 = 8\,, \\ \text{and}\quad 65(\times)31 &= 5+18 = 23\,. \end{aligned}$$

Let's call this operation the "Morris product".

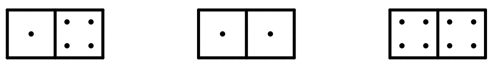
(a) What are the Morris products $11(\times)18$, $91(\times)19$, and $35(\times)62$?

(b) What is the largest possible Morris product of two two-digit numbers?

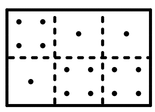
(c) Find all two-digit numbers ab such that $32(\times)ab = 32$.

(d) What is the largest actual product of two two-digit numbers whose Morris product is less than 10?

2. Bored at the beach, Barbara is idly arranging dominoes on the table. Over the years a few dominoes have been lost from the set. In fact, only three are left. These happen to be:



(a) How are the dominoes arranged to give this rectangle?



(b) Give an example of a similar rectangle which can be formed in exactly 2 different ways from these three dominoes.

(c) Give an example of a similar rectangle which can be formed in exactly 3 different ways from these three dominoes.

(d) Are there any rectangles which can be formed in four ways?

3. The mythical country of EnZed is divided into a north and south island. Inhabitants of the north island never tell the truth, while those from the south always do. Furthermore, on the south island they produce and drink a magical brown nectar called Spites. A thirsty traveler once entered a bar seeking a drink of this wondrous brew, only to find three full glasses on the counter, and five people lounging around. Somehow he knew that exactly one of the glasses contained Spites, while the other two contained pale and unappetising imitations. Not unexpectedly, each of the people around the bar made a single statement:

Andy: The left-most glass contains Spites.

Brenda: The right-most glass contains Spites.

Carol: Andy and Brenda are not both from the north island.

Deirdre: Either Andy is from the north island or Brenda is from the south island.

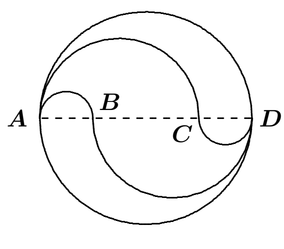
Ed: Either I am from the north island, or Carol and Deirdre are both from the same island.

(a) Remembering that to EnZedians (and to mathematicians everywhere) a statement of the type "Either X or Y" is true if either X or Y or both are true, what can be concluded from Ed's statement?

(b) Which glass (left-most, middle, or right-most) should the traveller take?

4. A circular plate is divided into 20 equal sectors. Ten sectors are painted blue, and ten are painted yellow. Show that somewhere on the plate there must be ten consecutive sectors, five of which are blue and five yellow (no matter how the blue and yellow sectors have been chosen).

5. King Lear, having come to his senses, intends to divide his fortune equally among his three daughters. Among his possessions is a large circular golden disc 1 m in diameter. For aesthetic reasons, he plans to have his goldsmith cut it up into three pieces of equal area using semicircular arcs along a diameter AD as shown below (but not to scale). If AB and CD are to have the same length, what should that length be (exactly)?



(The disc is to be divided along the semicircular solid arcs. The dotted diameter AD is for reference only.)

Compétition Junior de Mathématiques de Nouvelle-Zélande 1999
Organisé par l'Université d'Otago

1. Morris DuConfu multiplie les nombres à 2 chiffres en multipliant ensemble les chiffres des unités et des dizaines séparément puis en additionnant les résultats. On notera cette multiplication erronée par $(\times)$. Par exemple :

$$\begin{aligned} 36(\times)47 &= 42+12 = 54\,, \\ 23(\times)40 &= 0+8 = 8\,, \\ \text{et}\quad 65(\times)31 &= 5+18 = 23\,. \end{aligned}$$

Appelons cette opération le "Morris-produit".

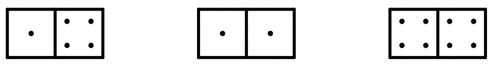
(a) Que valent les Morris-produits $11(\times)18$, $91(\times)19$ et $35(\times)62$?

(b) Quel est le plus grand Morris-produit de 2 nombres à 2 chiffres ?

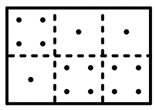
(c) Trouver tous les nombres à 2 chiffres ab tels que $32(\times)ab = 32$.

(d) Quel est le plus grand produit réel de 2 nombres à 2 chiffres dont le Morris-produit est inférieur à 10 ?

2. S'ennuyant à la plage, Barbara arrange des dominos sur la table. Avec les années, quelques dominos ont été perdus à tel point qu'il n'en reste que trois. Notamment :



(a) Comment arranger les dominos pour obtenir le rectangle ?



(b) Donner un exemple d'un rectangle similaire que l'on peut former exactement de 2 manières différentes avec ces trois dominos.

(c) Donner un exemple d'un rectangle similaire que l'on peut former exactement de 3 manières différentes avec ces trois dominos.

(d) Y a-t-il des rectangles que l'on peut former de 4 manières ?

3. Le pays magique de EnZed est formé des îles Nord et Sud. Les habitants de l'île nord ne disent jamais la vérité, alors que ceux du sud le font tout le temps. Sur l'île du sud, ils produisent une potion magique brune appelée Spites. Un voyageur assoiffé entra un jour dans une taverne à la recherche de cette potion. Sur le comptoir il trouva 3 verres pleins et 5 personnes assis autour du bar. Son intuition lui disait que seulement un des verres contenait de la potion, alors que les autres contenaient une imitation peu savoureuse. Sans surprise, chacune des personnes autour du bar ne fit qu'un commentaire :

Andy : Le verre de gauche contient du Spites.

Brenda : Le verre de droite contient du Spites.

Carol : Andy et Brenda ne sont pas de la même île.

Deirdre : Soit Andy est de l'île nord ou Brenda est de l'île sud.

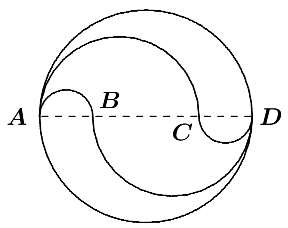
Ed : Soit je suis de l'île nord, ou Carol et Deirdre sont de la même île.

(a) En se rappelant que pour les EnZediens (et pour les mathématiciens partout) une expression du type "Soit X ou Y" est vrai si soit X ou Y ou les deux sont vrais, que peut-on conclure du commentaire à Ed ?

(b) Quel verre (gauche, centre or droite) le voyageur devrait-il prendre ?

4. Une assiette circulaire est divisée en 20 secteurs égaux. Dix secteurs sont peints en bleu, et dix en jaune. Montrez que quelque part sur l'assiette il doit y avoir dix secteurs consécutifs, cinq étant bleus et cinq jaunes (quelque soit la manière que l'on a fait pour choisir les secteurs).

5. Le roi Lear a l'intention de séparer sa fortune également parmi ses trois filles. Parmi ses possessions, on retrouve un large disque doré de 1m de diamètre. Pour des raisons esthétiques, il planifie que son forgeron le coupe en trois morceaux de même aire en utilisant des arcs semi-circulaires le long du diamètre AD comme sur le dessin (pas à l'échelle). Si AB et CD ont la même longueur, quelle devrait être cette longueur (exactement) ?



(Le disque va être partagé selon les lignes pleines. La ligne en pointillés est seulement là pour indiquer le diamètre AD.)

Next we give the solutions to the 4th Annual CNU Regional High School Mathematics Contest [2005 : 129–132].

4th Annual CNU Regional High School Mathematics Contest
Saturday, December 6, 2003

1. (∗) If 64 is divided into three parts proportional to 2, 4, and 6, the smallest part is:

(A) $5\frac{1}{3}$ (B) 11 (C) $10\frac{2}{3}$ (D) none of these

Solution by Karthik Natarajan, student, Edgewater Park Public School, Thunder Bay, ON, modified by the editor.

The answer is C. Since the sum of the integers 2, 4, and 6 in the proportion is 12, we let $x = 64/12 = 16/3$. Then the three parts into which 64 is divided are $2x$, $4x$, and $6x$. The smallest part is then $2x = 32/3 = 10\frac{2}{3}$.

2. ($*$) If, in applying the quadratic formula to a quadratic equation $f(x) = ax^2 + bx + c = 0$, it happens that $c = \dfrac{b^2}{4a}$, then the graph of $y = f(x)$ will certainly:

(A) have a maximum (B) have a minimum
(C) be tangent to the x–axis (D) be tangent to the y–axis

Solution by Alexander Remorov, student, Waterloo Collegiate Institute, Waterloo, ON.

The answer is C. If $c = \dfrac{b^2}{4a}$, then $f(x) = ax^2+bx+\dfrac{b^2}{4a} = a\left(x+\dfrac{b}{2a}\right)^2$. From here, one can see that the vertex of the parabola is $\left(\dfrac{-b}{2a}, 0\right)$; hence, it lies on the x–axis. Because this is a vertex, and we have a parabola, $\left(\dfrac{-b}{2a}, 0\right)$ will be the only point at which the parabola touches the x–axis. Thus, the parabola is tangent to the x–axis.

3. ($*$) Let $\{a_n\}$ be a geometric sequence. If $a_1 = 8$ and $a_7 = 5832$, then a_5 is:

(A) 648 (B) 832 (C) 1168 (D) 1944

Solution by Karthik Natarajan, student, Edgewater Park Public School, Thunder Bay, ON.

The answer is A. Let r be the multiplier in the sequence. Writing the first few terms of the sequence starting with $a_1 = 8$, we get

$$a_2 = a_1 r = 8r\,,\quad a_3 = a_1 r^2 = 8r^2\,,\ \ldots\,,\quad a_7 = a_1 r^6 = 8r^6 = 5832\,.$$

Hence, $r^6 = 5832/8 = 729$. Therefore, $r^3 = 27$, implying that $r = 3$. Then $a_5 = a_1 r^4 = 8 \times 81 = 648$.

Also solved by Alexander Remorov, student, Waterloo Collegiate Institute, Waterloo, ON.

4. ($*$) The area enclosed by $|x| + |y| = 1$ is:

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) 4

Solution by Alexander Remorov, student, Waterloo Collegiate Institute, Waterloo, ON.

The answer is C. For $|x| + |y| = 1$, we must consider four cases (each in its own quadrant).

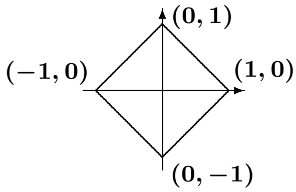
I $x \geq 0$ and $y \geq 0$. This gives us $x + y = 1$ or $y = 1 - x$.

II $x \geq 0$ and $y < 0$. This gives us $x - y = 1$ or $y = x - 1$.

III $x < 0$ and $y \geq 0$. This gives us $-x + y = 1$ or $y = x + 1$.

IV $x < 0$ and $y < 0$. This gives us $-x - y = 1$ or $y = -x - 1$.

Now, we can draw the graph. It is a square consisting of 4 congruent isosceles right triangles of side 1. The area of each of these triangles is $\frac{1\times1}{2} = \frac{1}{2}$. Hence, the square has area 2.



5. (∗) If the graph of $f(x) = \big||x-2| - a\big| - 3$ has exactly three x–intercepts, then a equals:

(A) 3 (B) 4 (C) 0 (D) -3

Solution by Alexander Remorov, student, Waterloo Collegiate Institute, Waterloo, ON, modified by the editor.

The answer is A. An x–intercept occurs when $f(x) = 0$. Thus, we have $\big||x-2| - a\big| = 3$. We know that if the absolute value of a number is k, then the number is either k or $-k$. Therefore, either

$$\begin{array}{rclcrcl} |x-2| - a &=& 3 & \text{or} & |x-2| - a &=& -3\,; \\ |x-2| &=& a+3 & \text{or} & |x-2| &=& a-3\,; \end{array} \qquad (1)$$

Both of the above equations have the form $|x-2| = k$ (in the first equation, $k = a+3$; in the second equation, $k = a-3$). If $k < 0$, the equation $|x-2| = k$ has no solutions; if $k = 0$, it has exactly one solution, $x = 2$; and if $k > 0$, then the equation is equivalent to $x - 2 = \pm k$, which has two solutions, $x = 2 \pm k$.

In order to get exactly 3 solutions for x in (1), we need one of the equations in (1) to have two solutions and the other to have exactly one solution. Thus, we need either $a+3 > 0$ and $a-3 = 0$, or $a+3 = 0$ and $a-3 > 0$. The first of these two alternatives occurs when $a = 3$, while the second is not possible (since $a+3 = 0$ implies that $a = -3$ which does not satisfy $a-3 > 0$). We conclude that $a = 3$.

(The three solutions for x are then $x = 2 \pm (3+3)$ and $x = 2$; that is, $x = 8$, $x = -4$ and $x = 2$.)

6. (∗) If $\dfrac{m}{n} = \dfrac{4}{3}$ and $\dfrac{r}{t} = \dfrac{9}{14}$, then the value of $\dfrac{3mr - nt}{4nt - 7mr}$ is:

(A) $-5\frac{1}{2}$ (B) $-\frac{11}{14}$ (C) $-\frac{2}{3}$ (D) $-1\frac{1}{4}$

Solution by Alexander Remorov, student, Waterloo Collegiate Institute, Waterloo, ON.

The answer is B. We have $m = 4n/3$ and $r = 9t/14$. Substituting this into $\dfrac{3mr - nt}{4nt - 7mr}$, we get

$$\frac{3mr - nt}{4nt - 7mr} = \frac{3\left(\frac{4n}{3}\right)\left(\frac{9t}{14}\right) - nt}{4nt - 7\left(\frac{4n}{3}\right)\left(\frac{9t}{14}\right)} = \frac{\frac{18tn}{7} - nt}{4nt - 6nt}$$
$$= \frac{nt(\frac{18}{7} - 1)}{nt(4 - 6)} = \frac{\frac{11}{7}}{-2} = \frac{-11}{14}.$$

One incomplete solution was also submitted.

7. (∗) Which functions satisfy $f\left(\frac{x+y}{2}\right) = \frac{1}{2}(f(x) + f(y))$?

(A) $\ln x$ (B) $\frac{1}{x}$ (C) $2x$ (D) 2^x

Solution by the editor.

The answer is C. The given equation is supposed to hold for all x and y in the domain of the function f. To show that it does not hold, for some function f, we just have to find one pair of numbers x and y in the domain of f for which it does not hold.

(A) Let $f(x) = \ln x$. Taking $x = \frac{1}{2}$ and $y = \frac{3}{2}$, we get

$$f\left(\frac{x+y}{2}\right) = \ln\left(\frac{\frac{1}{2} + \frac{3}{2}}{2}\right) = \ln 1 = 0$$

and

$$\tfrac{1}{2}(f(x) + f(y)) = \tfrac{1}{2}\left(\ln\tfrac{1}{2} + \ln\tfrac{3}{2}\right) = \tfrac{1}{2}(\ln 3 - \ln 4) \neq 0.$$

Therefore, this function f fails to satisfy the required equation.

(B) Let $f(x) = 1/x$. Taking $x = \frac{1}{2}$ and $y = \frac{3}{2}$, we get

$$f\left(\frac{x+y}{2}\right) = \frac{2}{\frac{1}{2} + \frac{3}{2}} = 1$$

and

$$\tfrac{1}{2}(f(x) + f(y)) = \tfrac{1}{2}\left(2 + \tfrac{2}{3}\right) = \tfrac{4}{3} \neq 1.$$

Therefore, this function f also fails to satisfy the equation.

(C) Let $f(x) = 2x$. Then, for all real numbers x and y,

$$f\left(\frac{x+y}{2}\right) = 2\left(\frac{x+y}{2}\right) = x + y = \tfrac{1}{2}(2x + 2y) = \tfrac{1}{2}(f(x) + f(y)).$$

Therefore, this function f satisfies the equation.

(D) Let $f(x) = 2^x$. Taking $x = 1$ and $y = 3$, we get

$$f\left(\frac{x+y}{2}\right) = 2^{\frac{1+3}{2}} = 2^2 = 4$$

and

$$\tfrac{1}{2}(f(x) + f(y)) = \tfrac{1}{2}(2^1 + 2^3) = \tfrac{1}{2}(10) = 5 \neq 4.$$

Thus, this function f fails to satisfy the required equation.

Note: The reader who is familiar with the graphs of the given functions can see the answer to this problem immediately. Any function whose graph is a straight line (any function of the form $f(x) = mx + b$) satisfies the given equation. If a function f has a graph that is concave up (curving in an upward direction, such as 2^x for all x and $\frac{1}{x}$ for $x > 0$), then

$$f\left(\frac{x+y}{2}\right) < \tfrac{1}{2}\left(f(x) + f(y)\right).$$

If a function f has a graph that is concave down (curving in a downward direction, such as $\ln x$ for all $x > 0$ and $\frac{1}{x}$ for $x < 0$), then

$$f\left(\frac{x+y}{2}\right) > \tfrac{1}{2}\left(f(x) + f(y)\right).$$

These assertions follow from the fact that if $P(x, f(x))$ and $Q(y, f(y))$ are any two points on the graph of f, then the mid-point of the line segment PQ is $\left(\frac{x+y}{2}, \frac{f(x)+f(y)}{2}\right)$.

Also solved by Alexander Remorov, student, Waterloo Collegiate Institute, Waterloo, ON.

8. $(*)$ Let x, $y > 0$, $x > y$, and $z \neq 0$. The inequality which is not always correct is:

(A) $x + z > y + z$ (B) $x - z > y - z$

(C) $xz > yz$ (D) $xz^2 > yz^2$

Solution by the editor.

The answer is C. The inequalities A and B are always true when $x > y$, and D is true when $x > y$ and $z \neq 0$, since $z^2 > 0$ for $z \neq 0$. But C is not always true, because if we take $z < 0$ with $x > y$, we get $xz < yz$.

Also solved by Alexander Remorov, student, Waterloo Collegiate Institute, Waterloo, ON.

9. $(*)$ Si $a^x = c^q = b$ et $c^y = a^z = d$, alors :

(A) $xy = qz$ (B) $x + y = q + z$ (C) $x - y = q - z$ (D) $x^y = q^z$

Solution par le rédacteur.

La réponse est A. Élevons la première équation à la puissance y. On obtient $a^{xy} = c^{qy}$. Élevons la seconde équation à la puissance q. On obtient $c^{yq} = a^{zq}$. Par transitivité de l'égalité, on obtient $a^{xy} = a^{zq}$. Puisque la base est la même des 2 côtés du signe égal, on obtient $xy = qz$.

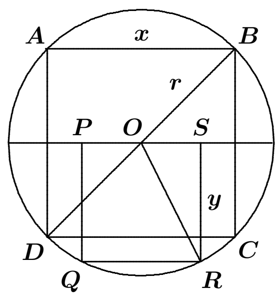
Une solution incorrecte a été soumise.

10. $(*)$ The area of a square inscribed in a semicircle is to the area of the square inscribed in the entire circle as:

(A) $1 : 2$ (B) $2 : 3$ (C) $2 : 5$ (D) $3 : 4$

Solution by Karthik Natarajan, student, Edgewater Park Public School, Thunder Bay, ON, modified by the editor.

The answer is C. In right triangle BCD in the figure to the right, we see that the hypotenuse is $2r$. If we let the two sides have length x and use the Pythagorean Theorem, we get the equation $2x^2 = 4r^2$. Hence, $x^2 = 2r^2$. In right triangle ORS in the figure, the hypotenuse is r, one side is y, and the other side is $y/2$. Using the Pythagorean Theorem, we get $5y^2/4 = r^2$. Thus,



$$y^2 \;=\; \frac{4r^2}{5} \;=\; \frac{2x^2}{5}\,.$$

We know that the ratio of the areas is the same as the ratio of the square of the sides. Thus, the required ratio is $y^2 : x^2 = 2 : 5$.

Also solved by Alexander Remorov, student, Waterloo Collegiate Institute, Waterloo, ON.

11. (*) If $0 < \alpha, \beta < \frac{\pi}{2}$ and $\alpha > \beta$, then:

(A) $\sin(\alpha - \beta) > \sin\alpha - \sin\beta$ (B) $\sin(\alpha - \beta) < \sin\alpha - \sin\beta$
(C) $\sin(\alpha - \beta) = \sin\alpha - \sin\beta$ (D) none of these

Solution by Alexander Remorov, student, Waterloo Collegiate Institute, Waterloo, ON, modified by the editor.

The answer is A. We must compare $\sin(\alpha - \beta)$ and $\sin\alpha - \sin\beta$. But

$$\begin{aligned} \sin(\alpha-\beta) &= 2\cos\left(\frac{\alpha-\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \\ \text{and} \qquad \sin\alpha - \sin\beta &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\,. \end{aligned}$$

Since $0 < \beta < \alpha < \frac{\pi}{2}$, we have $0 < \alpha - \beta < \alpha + \beta < \pi$, and therefore $0 < \frac{\alpha-\beta}{2} < \frac{\alpha+\beta}{2} < \frac{\pi}{2}$. Note that $\sin\left(\frac{\alpha-\beta}{2}\right)$ appears on the right in both identies above, and $\sin\left(\frac{\alpha-\beta}{2}\right) > 0$ since $0 < \frac{\alpha-\beta}{2} < \frac{\pi}{2}$. Hence, we need only compare $\cos\left(\frac{\alpha+\beta}{2}\right)$ and $\cos\left(\frac{\alpha-\beta}{2}\right)$ to get the answer.

Since $0 < \frac{\alpha-\beta}{2} < \frac{\alpha+\beta}{2} < \frac{\pi}{2}$, and since $\cos x$ is strictly decreasing on $\left[0, \frac{\pi}{2}\right]$, we see at once that

$$\cos\left(\frac{\alpha+\beta}{2}\right) \;<\; \cos\left(\frac{\alpha-\beta}{2}\right)\,.$$

It follows that $\sin\alpha - \sin\beta < \sin(\alpha - \beta)$.

12. $(*)$ Let $f(x) = 3^x + 5$. Then the domain of f^{-1} is:

(A) $(0, +\infty)$ (B) $(5, +\infty)$ (C) $(8, +\infty)$ (D) $(-\infty, +\infty)$

Solution by Alexander Remorov, student, Waterloo Collegiate Institute, Waterloo, ON.

The answer is B. From $f(x) = 3^x + 5$, we get $f^{-1}(x) = \log_3(x - 5)$. Hence, we have $x - 5 > 0$ or $x > 5$. The domain is $(5, +\infty)$.

13. $(*)$ A man has a pocket full of change, but can not make change for a dollar. What is the greatest value of coins he could have?

(A) \$0.99 (B) \$1.09 (C) \$1.19 (D) \$1.29

Solution by the editor.

The answer is C. The total of \$1.19 can be obtained with 1 quarter, 9 dimes, and 4 pennies; with 3 quarters, 4 dimes, and 4 pennies; or with 1 fifty-cent piece, 1 quarter, 4 dimes, and 4 pennies. [*Editor:* Obviously, this problem is ignoring the existence of coins of \$1 value or more. Since there is nothing to be gained by allowing a fifty-cent piece in place of two quarters, we assume that the man does not have a fifty-cent piece.]

To show that \$1.19 is the maximum amount the man can have, we first note that the number of quarters he has can be no greater than 3, since 4 quarters makes \$1. Assuming he has 3 quarters, he cannot have more than 4 dimes, because 2 quarters plus 5 dimes makes \$1. If we allow him to have a nickel or 5 pennies, then he can have only 1 dime, because 3 quarters plus 2 dimes plus the 5 cents makes \$1. But, he can have 4 pennies. Therefore, as long as he has 3 quarters, the greatest total he can have is \$1.19.

If he has only 2 quarters, then he still cannot have more than 4 dimes, and the largest total he can have is only \$0.99.

And if he has no quarters at all, then it is easy to see that he can have no more than \$0.99.

Now suppose that he has exactly 1 quarter. He can have 9 dimes, but not 10 dimes, since 10 dimes makes \$1. If we allow him to have a nickel or 5 pennies, then he can have only 6 dimes, because one quarter plus 7 dimes plus 5 cents makes \$1. But he can have 4 pennies. Thus, once again, we obtain \$1.19 for the maximum.

Also solved by Karthik Natarajan, student, Edgewater Park Public School, Thunder Bay, ON; and Alexander Remorov, student, Waterloo Collegiate Institute, Waterloo, ON.

That brings us to the end of another issue.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Shawn Godin (Ottawa Carleton District School Board). ~~The Assistant Mayhem Editor is John Grant McLoughlin~~ (University of New Brunswick). The other staff members are Dan MacKinnon (Ottawa Carleton District School Board), and Ian VanderBurgh (University of Waterloo).

Mayhem Problems

Please send your solutions to the problems in this edition by **1 April 2006**. *Solutions received after this date will only be considered if there is time before publication of the solutions.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

M213. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Set $S = (2+1)(2^2+1)(2^4+1)(2^8+1)\cdots(2^{1024}+1)+1$. Evaluate $S^{\frac{1}{1024}}$ without using a calculator.

M214. *Proposed by Babis Stergiou, Chalkida, Greece.*

Two equilateral triangles ABC and CDE are on the same side of line BCD. If BE intersects AC at K and DA intersects CE at L, prove that KL is parallel to BD.

M215. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Find a rational number s such that s^2+5 and s^2-5 are both squares of rational numbers.

M216. *Proposed by K.R.S. Sastry, Bangalore, India.*

A Heron triangle has integer sides and area. Two sides of a Heron triangle are 442 and 649. If its area is 132396, find its perimeter.

M217. *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

Let a, b, c be integers such that 2005 divides into both $ab+9b+81$ and $bc+9c+81$. Prove that 2005 also divides into $ca+9a+81$.

M218. *Proposed by Neven Jurič, Zagreb, Croatia.*

Compute the sum

$$\sum_{k=1}^{99} \frac{1}{k\sqrt{k+1}+(k+1)\sqrt{k}}\,.$$

....

M213. *Proposé par Edward T.H. Wang, Université Wilfrid Laurier, Waterloo, ON.*

Soit $S=(2+1)(2^2+1)(2^4+1)(2^8+1)\cdots(2^{1024}+1)+1$. Calculer $S^{\frac{1}{1024}}$ sans l'aide d'une calculatrice.

M214. *Proposé par Babis Stergiou, Chalkida, Grèce.*

Deux triangles équilatéraux ABC et CDE sont du même côté de la droite BCD. Si BE coupe AC en K et DA coupe CE en L, montrer que KL est parallèle à BD.

M215. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Trouver un nombre rationnel s tel que s^2+5 et s^2-5 sont tous deux des carrés de nombres rationnels.

M216. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

Un triangle de Héron possède des côtés et une aire mesurés par des nombres entiers. Deux côtés d'un triangle de Héron mesurent 442 et 649. Si son aire est 132396, trouver son périmètre.

M217. *Proposé par Bill Sands, Université de Calgary, Calgary, AB.*

Soit a, b et c des entiers tels que 2005 soit divisible par $ab+9b+81$ et par $bc+9c+81$. Montrer que 2005 est aussi divisible par $ca+9a+81$.

M218. *Proposé par Neven Jurič, Zagreb, Croatie.*

Calculer la somme

$$\sum_{k=1}^{99} \frac{1}{k\sqrt{k+1}+(k+1)\sqrt{k}}\,.$$

Mayhem Solutions

M146. *Proposed by Mohammed Aassila, Strasbourg, France.*

Let a, b, c be three positive numbers satisfying $a+b+c=1$. Prove that

$$(ab)^{5/4}+(bc)^{5/4}+(ca)^{5/4} \;<\; \frac{1}{4}\,.$$

Solution by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.

Cauchy's Inequality gives us

$$\left((ab)^{5/4}+(bc)^{5/4}+(ca)^{5/4}\right)^2 \;\le\; \left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)\left(a^2b^2+b^2c^2+c^2a^2\right).$$

Applying the AM–GM Inequality, we have

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ca} \;\le\; \tfrac{1}{2}(a+b)+\tfrac{1}{2}(b+c)+\tfrac{1}{2}(c+a) \;=\; a+b+c \;=\; 1\,.$$

Therefore, the inequality claimed will be established if we prove that

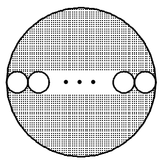
$$a^2b^2+b^2c^2+c^2a^2 \;<\; \frac{1}{16}\,. \tag{1}$$

To prove (1), we may assume without loss of generality that $a\le b\le c$. Using the AM-GM Inequality, we get $\sqrt{(a+b)c} \le \frac{1}{2}\big((a+b)+c\big) = \frac{1}{2}$. Then

$$\frac{1}{16} \;\ge\; c^2(a+b)^2 \;=\; a^2c^2+b^2c^2+2abc^2 \;>\; a^2c^2+b^2c^2+abc^2\,.$$

Since $a\le b\le c$, we have $abc^2>a^2b^2$, and then (1) follows.

M147. *Proposed by the Mayhem staff.*



The diameter of a large circle is broken into n equal parts to construct n smaller circles, as shown. Determine n so that the ratio of the shaded area to the unshaded area in the large circle is $3:1$.

Solution by Gabriel Krimker, grade 10 student, Buenos Aires, Argentina.

Let r be the radius of the large circle. The radius of each smaller circle is $\frac{r}{n}$. The shaded area is $\pi r^2 - n\pi\left(\frac{r}{n}\right)^2 = \pi r^2\left(1-\frac{1}{n}\right)$, and the unshaded area is $n\pi\left(\frac{r}{n}\right)^2 = \pi r^2\left(\frac{1}{n}\right)$. Then

$$3 \;=\; \frac{\pi r^2\left(1-\frac{1}{n}\right)}{\pi r^2\left(\frac{1}{n}\right)} \;=\; n\left(1-\frac{1}{n}\right) \;=\; n-1\,.$$

Hence, $n=4$.

Also solved by Roger He, grade 10 student, Prince of Wales Collegiate, St. John's, NL; Doug Newman, Lancaster, CA, USA; and Luyan Zhong-Qiao, Columbia International College, Hamilton, ON.

M148. *Proposé par Vedula N. Murty, Dover, PA, USA.*

Soit $x > 1$, $y > 1$, $z > 1$ et $x^2 = yz$. Trouver la valeur de

$$\left(\log_{zx} xy^4z\right)\left(\log_{xy} xyz^4\right) .$$

Solution par Houda Anoun, LaBri, Bordeaux, France.

Soient x, y et z des nombres réels tels que $x > 1$, $y > 1$, $z > 1$ et $x^2 = yz$. Posons $f = \log_{zx}(xy^4z)$ et $g = \log_{xy}(xyz^4)$. On a alors

$$(xz)^f \;=\; xy^4z \;=\; x^3y^3 \;=\; (xy)^3 . \qquad (1)$$

D'autre part on a aussi

$$(xy)^g \;=\; xyz^4 \;=\; x^3z^3 \;=\; (xz)^3 . \qquad (2)$$

D'après (1) et (2) on a alors

$$(xz)^{fg} \;=\; \left((xy)^3\right)^g \;=\; \left((xy)^g\right)^3 \;=\; (xz)^9 .$$

Or comme $xz > 1$ donc on en déduit que $fg = 9$.

En outre résolu par Marcie Fairchild, Daniel Mills, Laura Steil et Willie Ward, étudiants, Samford University, Birmingham, Alabama, USA ; Shuang Han, étudiant, $12^{\text{ième}}$ catégorie, Holy Heart of Mary High School, St. John's, NL ; Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentine ; et Luyan Zhong-Qiao, Columbia International College, Hamilton, ON.

M149. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

A right-angled Heron triangle ABC has the following property: the area is λ times the perimeter, where λ is a positive integer. Determine all solutions (a, b, λ). (A Heron triangle is a triangle with integer sides and integer area.)

Solution by Marcie Fairchild, Daniel Mills, Laura Steil, and Willie Ward, students, Samford University, Birmingham, Alabama, USA.

The Heron triangles have to be right-angled with all sides of integer length. Thus, we know that the triangles we are looking for must have sides that make Pythagorean triples. We can list Pythagorean triples by using the following system:

$$\begin{aligned} a &= 2xyt , \\ b &= (x^2 - y^2)t , \\ c &= (x^2 + y^2)t , \end{aligned}$$

where x, y, and t are integers, x and y have opposite parity, $x > y$, and a and b are the legs of the triangle. Using this representation of the triangle sides, and letting P be the perimeter of the triangle and A the area, we have $P = a + b + c = 2xyt + (x^2 - y^2)t + (x^2 + y^2)t = 2xt(x + y)$ and $A = \frac{1}{2}ab = \frac{1}{2}(2xyt)(x^2 - y^2)t = xyt^2(x^2 - y^2)$. Now the problem stipulates that the area is λ times the perimeter, which implies that

$$xyt^2(x^2 - y^2) \;=\; \lambda\bigl(2xt(x + y)\bigr)\,.$$

This equation can be solved for λ to yield $\lambda = \frac{1}{2}y(x - y)t$. Such λ will be an integer unless both t and y are odd. Therefore, all solutions are given by

$$(a, b, c) \;=\; \left(2xyt,\; (x^2 - y^2)t,\; \frac{y(x - y)t}{2}\right)\,,$$

where x and y have opposite parity, $x > y$, and at least one of y and t is even.

One incomplete solution was received.

M150. *Proposed by Arkady Alt, San Jose, CA, USA.*

Let two complex numbers z_1 and z_2 satisfy the conditions

$$\begin{aligned} z_1 + z_2 &= -(i + 1)\,, \\ z_1 \cdot z_2 &= -i\,. \end{aligned}$$

Without calculating z_1 and z_2, find $z_1 \cdot \overline{z_2}$.

Solution by the proposer.

Note that $z_1 \cdot \overline{z_2} = \frac{z_1}{z_2} \cdot |z_2|^2$. From $(z_1 + z_2)^2 = 2i = -2z_1 \cdot z_2$, we immediately obtain $z_1^2 + 4z_1z_2 + z_2^2 = 0$, or equivalently,

$$\left(\frac{z_1}{z_2}\right)^2 + 4\left(\frac{z_1}{z_2}\right) + 1 \;=\; 0\,.$$

Thus, $\frac{z_1}{z_2}$ is real and negative. Therefore, $z_1 \cdot \overline{z_2}$ is also real and negative. Combining this with $|z_1 \cdot \overline{z_2}| = |z_1 \cdot z_2| = 1$, we see that $z_1 \cdot \overline{z_2} = -1$.

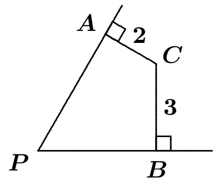
Problem of the Month

Ian VanderBurgh, University of Waterloo

It has been a long time since we have done any geometry! Since contest season is on the horizon, it is probably time to brush up on this aspect of our mathematical repertoire.

Problem (1992 Canadian Invitational Mathematics Challenge, Grade 10)

A point C is situated inside an angle of 60° at a distance of 2 units and 3 units from its sides. Determine the distance from point P to point C.



One of the wonderful things about geometry problems is the many different approaches that can be taken to solve the same problem. Here are three different approaches to this problem. We will keep the really nice approach for last to keep you reading until the end!

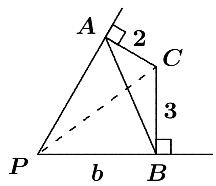
Solution 1: Since $PACB$ is a quadrilateral, the sum of its four interior angles is 360°. Thus, $\angle ACB = 120^\circ$. Join A to B. We can calculate the length of AB using the Cosine Law:

$$AB^2 \;=\; 2^2 + 3^2 - 2(2)(3)\cos 120^\circ \;=\; 4 + 9 - 2(2)(3)\left(-\tfrac{1}{2}\right) \;=\; 19\,.$$

Thus, $AB = \sqrt{19}$.

Now, let $PB = b$. By joining P to C, we form two right triangles sharing the hypotenuse PC. This means that $PA^2 + AC^2 = PC^2 = PB^2 + BC^2$; whence,

$$PA^2 \;=\; PB^2 + BC^2 - AC^2 \;=\; b^2 + 9 - 4 \;=\; b^2 + 5\,.$$



Therefore, $PA = \sqrt{b^2+5}$.

Next, we apply the Cosine Law again, this time in $\triangle PAB$, and solve for b. This requires a bit of patience.

$$\begin{aligned}
AB^2 &= PA^2 + PB^2 - 2(PA)(PB)\cos\angle APB\,,\\
19 &= (b^2+5) + b^2 - 2\left(\sqrt{b^2+5}\right)(b)\left(\tfrac{1}{2}\right),\\
14 - 2b^2 &= -b\sqrt{b^2+5}\,,\\
196 - 56b^2 + 4b^4 &= b^4 + 5b^2 \qquad \text{(squaring to get rid of square roots)},\\
3b^4 - 61b^2 + 196 &= 0\,.
\end{aligned}$$

We have obtained a quartic equation which is really a quadratic in disguise (didn't we see that last month?). We solve for b by factoring:

$$\begin{aligned}
(3b^2 - 49)(b^2 - 4) &= 0\,,\\
b^2 = \tfrac{49}{3} \qquad \text{or} \qquad b^2 &= 4\,.
\end{aligned}$$

Remembering that b must be positive, we get $b = 7/\sqrt{3}$ or $b = 2$. If $b = 2$, then $PA = 3$. Unfortunately, we have to reject this solution (why?). Hence, $b = 7/\sqrt{3}$. (It is interesting to wonder why we obtained the inadmissible solution $b = 2$.)

But we are seeking PC. Well, $PC^2 = PB^2 + BC^2 = \frac{49}{3} + 9 = \frac{76}{3}$, which means that $PC = \sqrt{\frac{76}{3}}$.

That required some persistence, but it worked out in the end. It was interesting that we ended up with some of the same algebraic issues that we discovered last month.

Solution 2: We could use coordinates! I'll get you started on this and leave you to work through the details. Again, it is not very pretty, but it works. And it is a good exercise in analytic geometry, especially since you already know the answer that you should get, which will help you track down any errors you make along the way.

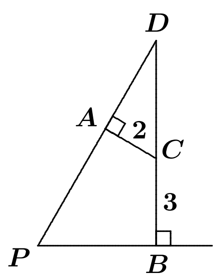
Set point P to be the origin $(0, 0)$, with PB lying along the positive x–axis. Give B coordinates $(b, 0)$. Then C has coordinates $(b, 3)$. Since $\angle APB = 60°$, the slope of the line through P and A is $\tan 60° = \sqrt{3}$. Hence, this line has equation $y = \sqrt{3}x$.

From here, you need to find the equation of the line through A and C (*hint:* it is perpendicular to PA and passes through $(b, 3)$), then find the coordinates of A (these will be in terms of b), and finally use the fact that the distance from A to C is 2. This will allow you to solve for b and then determine PC.

And now for the grand finale! After seeing the next solution, you will probably wish you had not seen either of the previous ones.

Solution 3: Extend the line through B and C up through C until it hits the line through P and A at D.

Since $\triangle DPB$ is right-angled at B, then $\angle PDB = 90° - 60° = 30°$. Thus, $\triangle DPB$ is a $30°$–$60°$–$90°$ triangle. Therefore, $PB = \frac{1}{\sqrt{3}} DB$. But $\triangle DCA$ is also a $30°$–$60°$–$90°$ triangle; whence, $CD = 2CA = 4$, and $DB = 7$. Then $PB = \frac{7}{\sqrt{3}}$. Therefore,



$$PC^2 = PB^2 + BC^2 = \frac{49}{3} + 9 = \frac{76}{3},$$

and $PC = \sqrt{\frac{76}{3}}$.

I think you would agree that this last approach was very nice.

We have seen three very different solutions to the same geometry problem. The last solution involves a nifty construction. Many geometrical problems have really elegant solutions involving a construction, but these solutions are usually very hard to find. What is the best way to find them, you ask? Lots of practice!

Pólya's Paragon

It Ain't So Complex (Part 3)

Shawn Godin

Last month we noticed that when we multiply two complex numbers in polar form, the argument of the product is coterminal with the sum of the arguments; that is,

$$(\cos\theta_1 + i\sin\theta_1)\cdot(\cos\theta_2 + i\sin\theta_2) \;=\; \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\,.$$

If we let $f(\theta) = \cos\theta + i\sin\theta$, then this equation can be rewritten as

$$f(\theta_1)\cdot f(\theta_2) \;=\; f(\theta_1+\theta_2)\,.$$

Thinking of all the functions you know, you might notice that our function f is behaving like an exponential function $f(x) = b^x$ (where $b > 0$ is the base). When $f(x) = b^x$, we have

$$f(\theta_1)f(\theta_2) \;=\; b^{\theta_1}\cdot b^{\theta_2} \;=\; b^{\theta_1+\theta_2} = f(\theta_1+\theta_2)\,.$$

But we defined $f(\theta)$ in terms of $\cos\theta$ and $\sin\theta$. How can f possibly be an exponential function? What is the base b?

Looking to calculus, we have the following power series expansions:

$$\begin{aligned} e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\,, \\ \sin x &= \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\,, \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\,, \end{aligned}$$

where the series are absolutely convergent for all real x. (For those of you who haven't taken calculus, you need to simply accept this.) Recall that $e \approx 2.71828\ldots$ is the base of the natural logarithm.

Let us use the series for $\sin x$ and $\cos x$ to calculate $\cos\theta + i\sin\theta$:

$$\begin{aligned} \cos\theta + i\sin\theta &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\ &= 1 + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \cdots \\ &= 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots \end{aligned}$$

The resulting series above could be obtained by putting $x = i\theta$ in the series for e^x. So we *define* $e^{i\theta} = \cos\theta + i\sin\theta$. Thus, we can write any complex number $z = r(\cos\theta + i\sin\theta)$ in polar form as

$$z \;= re^{i\theta}\,.$$

This exponential notation leads to some interesting results, such as *de Moivre's Formula*, which states that, for any integer n,

$$(\cos\theta + i\sin\theta)^n \;=\; \cos n\theta + i\sin n\theta\,.$$

Using exponential notation, the theorem simply says $\left(e^{i\theta}\right)^n = e^{in\theta}$, which seems quite obvious (using the basic properties of exponents). We can use de Moivre's Formula to work out other trigonometric identities; for example, when $n = 2$, we get

$$\cos 2\theta + i\sin 2\theta \;=\; (\cos\theta + i\sin\theta)^2 \;=\; \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta\,,$$

Equating the real and imaginary parts, we get the double angle formulas for sine and cosine:

$$\cos 2\theta \;=\; \cos^2\theta - \sin^2\theta \qquad \text{and} \qquad \sin 2\theta \;=\; 2\sin\theta\cos\theta\,.$$

That will do for this issue. Next time, we will look at some applications to plane geometry. For homework, try the following:

1. Calculate $\cos 3\theta$ in terms of $\cos\theta$, and calculate $\sin 3\theta$ in terms of $\sin\theta$.
2. By using the exponential notation, obtain the formulas for $\cos(\theta_1 + \theta_2)$ and $\sin(\theta_1 + \theta_2)$ in terms of $\cos\theta_1$, $\cos\theta_2$, $\sin\theta_1$, and $\sin\theta_2$.
3. Use complex numbers to prove the *Triangle Inequality*. That is, for any triangle with sides a, b, c, prove that $c \le a + b$.
4. Use complex numbers to show that the medians of a triangle meet at a common point and determine how to find that point.

Finally, we consider the unanswered questions from last month's homework.

1. Converting the polar form to the exponential form, we have.

$$\begin{aligned}\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} &= \frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} \;=\; \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)} \\ &= \frac{r_1}{r_2}\big(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\big)\,.\end{aligned}$$

3. Considering the equation in exponential form, we get $z^2 = i = e^{i\frac{\pi}{2}}$. If we let $z = re^{i\theta}$, then $z^2 = r^2e^{2i\theta} = e^{i\frac{\pi}{2}}$. The usual convention is that $r > 0$ for all $z \neq 0$; thus, $r = 1$ in our case and one solution is $z_1 = e^{i\frac{\pi}{4}} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$. We approach the second solution by noting the redundancy in the polar form; that is,

$$e^{i\theta} \;=\; e^{i(\theta+2\pi k)}\,,$$

for any integer k. Thus, our other solution comes from considering $z^2 = e^{i\frac{\pi}{2}} = e^{i\frac{5\pi}{2}}$, which yields $z_2 = e^{i\frac{5\pi}{4}} = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$. For further exploration, plot these two solutions on the complex plane; how are they related geometrically? How are the the solutions to $z^3 = i$ related geometrically?

Until next month, happy problem solving.

THE OLYMPIAD CORNER

No. 249

R.E. Woodrow

We begin this number of the *Corner* with the two days of the Bosnia and Herzegovina 7th National Olympiad Selection Test, written May 2002. Thanks go to Bill Sands, Chair of the International Olympiad Committee of the Canadian Mathematical Society, for providing us with a copy of the contest.

BOSNIA AND HERZEGOVINA
7th NATIONAL OLYMPIAD — SELECTION TEST
First Day - May 2002

1. Let x, y, and z be real numbers such that

$$x + y + z = 3 \qquad \text{and} \qquad xy + yz + xz = a$$

(a is a real parameter). Determine the value of the parameter a for which the difference between the maximum and minimum possible values of x equals 8.

2. Triangle ABC is given in a plane. Draw the bisectors of all three of its angles. Then draw the line that connects the points where the bisectors of angles ABC and ACB meet the sides AC and AB, respectively. Through the point of intersection of the bisector of angle BAC and the previously drawn line, draw another line, parallel to the side BC. Let this line intersect the sides AB and AC in points M and N. Prove that $2MN = BM + CN$.

3. If n is a natural number, prove that the number $(n+1)(n+2)\cdots(n+10)$ is not a perfect square.

Second Day - May 2002

4. Let a, b, and c be real numbers such that $a^2 + b^2 + c^2 = 1$. Prove the inequality

$$\frac{a^2}{1+2bc} + \frac{b^2}{1+2ca} + \frac{c^2}{1+2ab} \geq \frac{3}{5}\,.$$

5. Let p and q be different prime numbers. Solve the following system of equations in the set of integers:

$$\begin{aligned} \frac{z+p}{x} + \frac{z-p}{y} &= q\,, \\ \frac{z+p}{y} - \frac{z-p}{x} &= q\,. \end{aligned}$$

6. Let the vertices of the convex quadrilateral $ABCD$ and the intersecting point S of its diagonals be integer points in the plane. Let P be the area of the quadrilateral $ABCD$ and P_1 the area of triangle ABS. Prove the following inequality:

$$\sqrt{P} \;\geq\; \sqrt{P_1} + \frac{\sqrt{2}}{2}\,.$$

Next we give the four problems of the Fourth Hong Kong (China) Mathematical Olympiad, written December 2001. Thanks again go to Bill Sands, Chair of the International Olympiad Committee of the Canadian Mathematical Society, for collecting them for our use.

4th HONG KONG (CHINA) MATHEMATICAL OLYMPIAD
December 22, 2001
Time: 3 hours

1. A triangle ABC is given. A circle Γ passes through vertex A and is tangent to side BC at point P. The circle Γ intersects sides AB and AC at points M and N, respectively. Prove that (minor) arcs MP and NP are equal if and only if Γ is tangent to the circumcircle of $\triangle ABC$ at A.

2. Find all positive integers n such that the equation $x^3+y^3+z^3 = nx^2y^2z^2$ has positive integer solutions. Be sure to give a proof.

3. For each integer $k \geq 4$, prove that if $F(x)$ is a polynomial with integer coefficients which satisfies the condition $0 \;\leq\; F(c) \;\leq\; k$ for every $c = 0, 1, \ldots, k+1$, then $F(0) = F(1) = \cdots = F(k+1)$.

4. There are 212 points inside or on a circle with radius 1. Prove that there are at least 2001 pairs of these points having distances at most 1.

As a final set of questions we give the Fifteenth Irish Mathematical Olympiad, First and Second Paper, written May 2002. My thanks again go to Bill Sands, Chair of the International Olympiad Committee of the Canadian Mathematical Society, for obtaining them for our use.

15th IRISH MATHEMATICAL OLYMPIAD
First Paper
11 May 2002 – morning

1. In a triangle ABC, $AB = 20$, $AC = 21$, and $BC = 29$. The points D and E lie on the line segment BC, with $BD = 8$ and $EC = 9$. Calculate the angle $\angle DAE$.

2. (a) A group of people attends a party. Each person has at most three acquaintances in the group, and if two people do not know each other, then they have a mutual acquaintance in the group. What is the maximum number of people present?

(b) If, in addition, the group contains three mutual acquaintances (that is, three people each of whom knows the other two), what is the maximum number of people?

3. Find all triples of positive integers (p, q, n), with p and q prime, such that

$$p(p+3)+q(q+3) \;=\; n(n+3)\,.$$

4. Let the sequence a_1, a_2, a_3, a_4, $\ldots$ be defined by

$$a_1 \;=\; 1\,, \quad a_2 \;=\; 1\,, \quad a_3 \;=\; 1\,, \quad \text{and} \quad a_{n+1}a_{n-2}-a_na_{n-1} \;=\; 2\,,$$

for all $n \geq 3$. Prove that a_n is a positive *integer* for all $n \geq 1$.

5. Let $0 < a,\, b,\, c < 1$. Prove that

$$\frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c} \;\geq\; \frac{3\sqrt[3]{abc}}{1-\sqrt[3]{abc}}\,.$$

Determine the case of equality.

Second Paper

11 May 2002 – afternoon

6. A $3 \times n$ grid is filled as follows. The first row consists of the numbers from 1 to n arranged from left to right in ascending order. The second row is a cyclic shift of the top row. Thus, the order goes

$$i,\, i+1,\, \ldots,\, n-1,\, n,\, 1,\, 2,\, \ldots,\, i-1$$

for some i. The third row has the numbers 1 to n in some order, subject to the rule that in each of the n columns, the sum of the three numbers is the same.

For which values of n is it possible to fill the grid according to the above rules? For an n for which this is possible, determine the number of different ways of filling the grid.

7. Suppose n is a product of four distinct primes a, b, c, d such that

(a) $a+c=d$;

(b) $a(a+b+c+d)=c(d-b)$;

(c) $1+bc+d=bd$.

Determine n.

8. Denote by $\mathbb{Q}$ the set of rational numbers. Determine all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$f(x + f(y)) \;=\; y + f(x)\,, \quad \text{for all } x,\, y \in \mathbb{Q}.$$

9. For each real number x, define $\lfloor x \rfloor$ to be the greatest integer less than or equal to x. Let $\alpha = 2 + \sqrt{3}$. Prove that

$$\alpha^n - \lfloor \alpha^n \rfloor \;=\; 1 - \alpha^{-n}\,, \quad \text{for } n = 0,\, 1,\, 2,\, \ldots.$$

10. Let ABC be a triangle whose side lengths are all integers, and let D and E be the points at which the incircle of ABC touches BC and AC, respectively. If $|AD^2 - BE^2| \le 2$, show that $AC = BC$.

Before turning to readers' solutions, I want to point out that we are gradually catching up on our backlog of solutions, and I hope that we will soon be in a position that will see readers' solutions appear within a year of publication of the problem sets. This means a bit of a challenge to solvers to send me the solutions within about 8 months to allow the time to review solutions, select and edit them.

The first set of reader's solutions we present are to the selected problems of the Ukrainian Mathematical Olympiad written March 2001, for which problems were given in [2003 : 497–498].

1. (Grade 9) All 5-digit positive integers with digits in increasing order (from left to right) are given. Is it possible to take away one digit from each number so that we obtain all 4-digit positive integers with digits in increasing order?

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

Each positive integer whose digits are strictly increasing is uniquely determined by the set of its digits, which cannot include 0. Thus, each 5–digit positive integer with digits in increasing order corresponds to (and will be identified with) a 5–element subset of $\{1,\, 2,\, \ldots,\, 9\}$, and similarly for 4–digit integers. Let A be the set of all 5-element subsets of $\{1,\, 2,\, \ldots,\, 9\}$ and B the set of all 4-element subsets of $\{1,\, 2,\, \ldots,\, 9\}$. Note that $|A| = |B| = \binom{9}{4}$. The problem asks whether there exists a bijective mapping $f : A \to B$ such that $f(X) \subset X$ for all $X \in A$. We will construct such a mapping.

Let $X \in A$. Write the numbers $1, 2, \ldots, 9$ around a circle, and colour all the numbers red that are not in X. (There will be four red numbers.) Then, for each red number, move clockwise to the next uncoloured number, and colour it blue. The four blue numbers are the set $f(X)$.

The inverse mapping is defined as follows. Let $Y \in B$. Again write the numbers $1, 2, \ldots, 9$ around a circle. Colour all the numbers in Y blue. For each blue number, move counter-clockwise to the next uncoloured number, and colour it red. Then $f^{-1}(Y)$ is the set of all numbers not coloured red.

It is easy to see that f satisfies all conditions, and we are done.

3. (Grade 10) Let $a_1, a_2, \ldots, a_n$ be real numbers such that

$$a_1 + a_2 + \cdots + a_n \;\geq\; n^2 \quad \text{and} \quad a_1^2 + a_2^2 + \cdots + a_n^2 \;\leq\; n^3 + 1\,.$$

Prove that $n - 1 \leq a_k \leq n + 1$ for all k.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON. We give Bataille's solution.

First, note that the inequality $a_1 + a_2 + \cdots + a_n \geq n^2$ can be rewritten as

$$(a_1 - n) + (a_2 - n) + \cdots + (a_n - n) \;\geq\; 0\,. \tag{1}$$

Now,

$$\begin{aligned} &(a_1 - n)^2 + (a_2 - n)^2 + \cdots + (a_n - n)^2 \\ =\;\; & a_1^2 + a_2^2 + \cdots + a_n^2 - n^3 - 2n((a_1 - n) + (a_2 - n) + \cdots + (a_n - n)) \\ \leq\;\; & 1\,, \end{aligned}$$

using (1) and the given inequality $a_1^2 + a_2^2 + \cdots + a_n^2 \leq n^3 + 1$. Thus, we certainly have $(a_k - n)^2 \leq 1$; that is, $n - 1 \leq a_k \leq n + 1$ for all k.

4. (Grade 10) There are n mathematicians in each of three countries. Each mathematician corresponds with at least $n{+}1$ foreign mathematicians. Prove that there exist three mathematicians who correspond with each other.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON. We give Zhao's solution.

Let the three countries be A, B, and C. Let X be the mathematician who corresponds with the greatest number of mathematicians from one country, and let that number be j. Note that $j \leq n$, since there are only n mathematicians in each country. Without loss of generality, let X be in A and let X correspond with j mathematicians from B. Then X corresponds with at least $n + 1 - j > 0$ mathematicians in C. Let Y be a mathematician in C such that X corresponds with Y.

Suppose there do not exist three mathematicians who correspond with each other. Then Y does not correspond with any mathematicians in B who correspond with X; hence, Y corresponds with at most $n{-}j$ mathematicians in B, which implies that Y corresponds with at least $(n{+}1) - (n{-}j) = j{+}1$ mathematicians in A. This contradicts the choice of X since $j + 1 > j$. Therefore, there exist three mathematicians who correspond with each other.

5. (Grade 11) Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ such that for all x, $y \in \mathbb{R}$ the following equality holds?

$$f(xy) \;=\; \max\{f(x), y\} + \min\{f(y), x\}\;.$$

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON. We give Zhao's write-up.

Suppose that there exists such a function. Then, setting $x = y = 1$ in the given equation, we get

$$f(1) \;=\; \max\{f(1), 1\} + \min\{f(1), 1\} \;=\; f(1) + 1\,,$$

a contradiction. Thus, no such function exists.

6. (Grade 11) Positive integers a and n are such that n divides $a^2 + 1$. Prove that there exists a positive integer b such that $n(n^2 + 1)$ divides $b^2 + 1$.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON. We give Zhao's write-up.

Choose b to be $(n^2 + 1)a + n$. Then

$$\begin{aligned} b^2 + 1 \;&=\; \big((n^2+1)a + n\big)^2 + 1 \;\equiv\; a^2 + 1 \;\equiv\; 0 \pmod{n}\;, \\ \text{and}\quad b^2 + 1 \;&=\; \big((n^2+1)a + n\big)^2 + 1 \;\equiv\; n^2 + 1 \;\equiv\; 0 \;\big(\mathrm{mod}\; n^2 + 1\big)\;. \end{aligned}$$

Since n and $n^2 + 1$ are relatively prime, we see that $n(n^2 + 1)$ divides $b^2 + 1$, and the result follows.

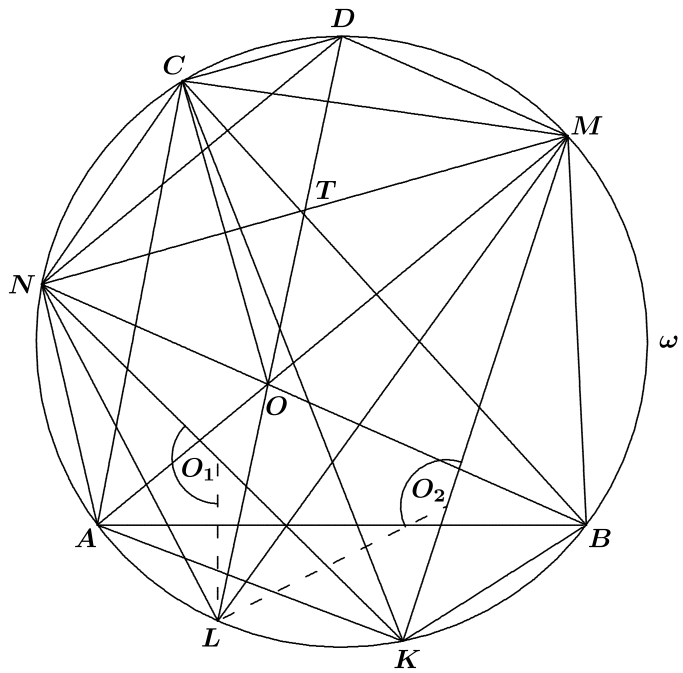
7. (Grade 11) An acute triangle ABC, with $AC \neq BC$, is inscribed in a circle ω. The points A, B, C divide the circle into disjoint arcs $\overparen{AB}$, $\overparen{BC}$, and $\overparen{CA}$. Let M and N be the mid-points of $\overparen{BC}$ and $\overparen{AC}$, respectively, and let K be an arbitrary point of $\overparen{AB}$. Let D be the point of $\overparen{MN}$ such that $CD \parallel NM$. Let O, O_1, O_2 be the incentres of triangles ABC, CAK, CBK, respectively. Let L be the intersection point of the line DO and the circle ω, where $L \neq D$. Prove that the points K, O_1, O_2, L are concyclic.

Solution by Toshio Seimiya, Kawasaki, Japan.

Since M and N are the mid-points of $\overparen{BC}$ and $\overparen{AC}$, respectively, we see that AM and BN are the bisectors of $\angle CAB$ and $\angle CBA$, respectively. Thus, the intersection of AM and BN is the incentre O of $\triangle ABC$. Since O is the incentre of $\triangle ABC$, we have $\angle ACO = \angle BCO$. Hence,

$$\begin{aligned} \angle MOC \;=\; \angle ACO + \angle CAM \;&=\; \angle BCO + \angle BAM \\ &=\; \angle BCO + \angle BCM \;=\; \angle MCO\,. \end{aligned}$$

Thus, $MO = MC = MB$. Similarly, we get $NO = NC = NA$.



Since O_1 and O_2 are the incentres of $\triangle CAK$ and $\triangle CBK$, respectively, we similarly obtain

$$NO_1 \;=\; NC \;=\; NA\,, \qquad \text{and} \qquad MO_2 \;=\; MC \;=\; MB\,.$$

Since $CD \parallel MN$, we have $MD = NC = NO$, and $DN = CM = MO$. Hence, quadrilateral $MDNO$ is a parallelogram. Let T be the intersection of DO and MN. Then $NT = TM$.

Since $\triangle NLT \sim \triangle DMT$, and $\triangle MLT \sim \triangle DNT$, we obtain

$$NL : DM \;=\; LT : MT \;=\; LT : NT \;=\; ML : DN\,.$$

Hence, $NL : ML = DM : DN$. Since $DM = CN = NO_1$, and $DN = CM = MO_2$, we have

$$NL : ML \;=\; NO_1 : MO_2\,. \tag{1}$$

Since $\angle LNO_1 = \angle LNK = \angle LMK = \angle LMO_2$, we get from (1)

$$\triangle NLO_1 \sim \triangle MLO_2\,.$$

Thus, $\angle NO_1L = \angle MO_2L$. Hence,

$$\angle LO_1K \;=\; 180^\circ - \angle NO_1L \;=\; 180^\circ - \angle MO_2L \;=\; \angle LO_2K\,.$$

Therefore, K, O_1, O_2, L are concyclic.

8. (Grade 11) Let a, b, c and α, β, γ be positive real numbers such that $\alpha+\beta+\gamma=1$. Prove the inequality

$$\alpha a+\beta b+\gamma c+2\sqrt{(\alpha\beta+\beta\gamma+\gamma\alpha)(ab+bc+ca)} \;\leq\; a+b+c\,.$$

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON. We give Zhao's write-up.

Since the inequality is homogeneous in a, b, and c, we may assume that $a+b+c=1$ (without loss of generality). Then, using the AM–GM Inequality,

$$\begin{aligned}
&2\sqrt{(\alpha\beta+\beta\gamma+\gamma\alpha)(ab+bc+ca)}\\
&\qquad\leq\; \alpha\beta+\beta\gamma+\gamma\alpha+ab+bc+ca\\
&\qquad=\; \frac{(\alpha+\beta+\gamma)^2-\alpha^2-\beta^2-\gamma^2}{2}+\frac{(a+b+c)^2-a^2-b^2-c^2}{2}\\
&\qquad=\; 1-\frac{a^2+\alpha^2}{2}-\frac{b^2+\beta^2}{2}-\frac{c^2+\gamma^2}{2}\\
&\qquad\leq\; 1-\sqrt{a^2\alpha^2}-\sqrt{b^2\beta^2}-\sqrt{c^2\gamma^2}\\
&\qquad=\; a+b+c-a\alpha-b\beta-c\gamma\,.
\end{aligned}$$

The result follows immediately.

Next we turn to our file of readers' solutions to problems of the 2004 numbers of ***Crux Mathematicorum***, starting with the XVII National Mathematical Contest of Italy [2004 : 18–19].

2. In a basketball tournament, each team played twice against each other team. Two points were awarded for a win and no points for a loss. (No game could finish in a draw.) A single team won the tournament with 26 points, and exactly two teams finished last with 20 points. How many teams participated in the tournament?

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Edward T.H. Wang and Kaiming Zhao, Wilfrid Laurier University, Waterloo, ON. We give the solution by Wang and Zhao.

The answer is 12. Let n be the number of the teams. Then $n\geq 3$. Let s_k be the points scored by team T_k, for $k=1, 2, \ldots, n$. Without loss of generality, we assume that

$$20 \;=\; s_1 \;=\; s_2 \;<\; s_3 \;\leq\; s_4 \;\leq\cdots\leq\; s_{n-1} \;<\; s_n \;=\; 26\,.$$

By assumption, T_n played a total of $2(n-1)$ games and won 13 games; thus, it lost $2n-15$ games. Thus, $2n-15<13$ from which we get $n\leq 13$.

Similarly, T_1 won 10 games and lost $2n - 12$ games; thus, $2n - 12 > 10$ from which we get $n \geq 12$. Hence, $n = 12$ or 13.

Since each s_k is even, it follows that for all $k \in I = \{3, 4, \ldots, n-1\}$, we must have $s_k = 22$ or $s_k = 24$. Suppose $s_k = 22$ for the first ℓ values of $k \in I$, and $s_k = 24$ for the other $n - \ell - 3$ values of $k \in I$, where $0 \leq \ell \leq n - 3$.

Since the total number of games played was $2\binom{n}{2} = n(n-1)$, the total number of points scored was $2n(n-1)$. Hence,

$$20 + 20 + 22\ell + 24(n - \ell - 3) + 26 \;=\; 2n(n-1)\,,$$

which simplifies to $\ell + 3 = n(13 - n)$. Clearly, $n = 13$ is impossible. Therefore, $n = 12$ and $\ell = 9$. That is, the points scored by the 12 teams were 20, 20, 22, $\ldots$, 22, 26. We note that this situation can actually be realized if every team "draws" every other team (that is, wins one game and loses one in the two games played between them), except team T_{12}, which beats T_1 and T_2 in both games.

3. Given the equation $x^{2001} = y^x$,

(a) find all solution pairs (x, y) consisting of positive integers with x prime;

(b) find all solution pairs (x, y) consisting of positive integers.

(Recall that $2001 = 3 \cdot 23 \cdot 29$.)

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Let us solve (b) directly.

If $x = 1$, we find the solution $(x, y) = (1, 1)$. Now, we assume that $x \geq 2$; then $y \geq 2$. Clearly, x and y have the same prime divisors.

Let $x = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ and $y = p_1^{\beta_1} p_2^{\beta_2} \cdots p_n^{\beta_n}$ be the prime decompositions of x and y. Thus, for each i, we have

$$2001\alpha_i \;=\; x\beta_i\,. \tag{1}$$

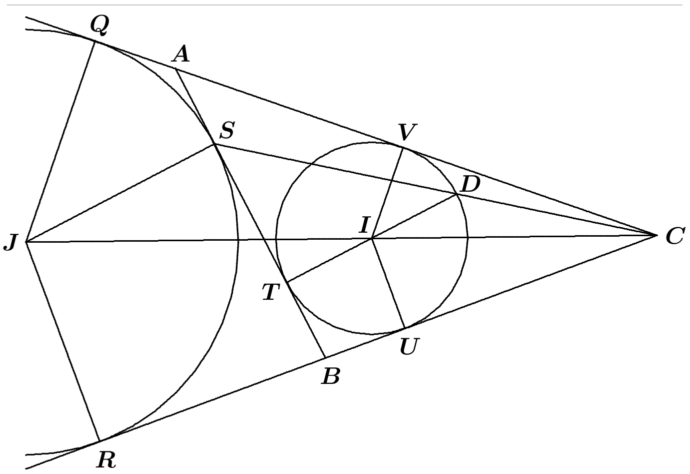
If $\gcd(p_i, 2001) = 1$, then, from (1), we deduce that p_i divides α_i; indeed, we even have $p_i^{\alpha_i}$ divides α_i. But it is easy to prove by induction that $2^k > k$ for each positive integer k. Hence, $p_i^{\alpha_i} \geq 2^{\alpha_i} > \alpha_i$. Thus, for each i, we have $\gcd(p_i, 2001) = p_i$.

Since $2001 = 3 \times 23 \times 29$, it follows that $x = 3^\alpha \times 23^\beta \times 29^\gamma$ for some non-negative integers α, β, γ. From (1), if $\alpha > 0$, we deduce (as above) that $3^{\alpha-1}$ divides α, which leads to $\alpha = 1$. Similarly, we have β, $\gamma \in \{0, 1\}$. Thus, x divides 2001.

Conversely, if $2001 = kx$ for some positive integer k, then the given equation $x^{2001} = y^x$ is equivalent to $y = x^k$. Thus, the set of solutions (x, y) is the set of all ordered pairs $(x, x^{2001/x})$, where x is any positive divisor of 2001; that is, $x \in \{1, 3, 23, 29, 69, 87, 667, 2001\}$.

5. The incircle γ of triangle ABC touches the side AB at T. Let D be the point on γ diametrically opposite to T, and let S be the intersection of the line through C and D with the side AB. Show that $AT = SB$.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; and Toshio Seimiya, Kawasaki, Japan. We give the write-up of Bataille.



Let γ touch BC at U and CA at V. Let I be the incentre and J be the point of intersection of the line CI and the perpendicular to AB at S. Let Q, R be the projections of J onto the lines CA, CB, respectively (see figure). Note that $ID \parallel JS$, $UI \parallel JR$, $IV \parallel JQ$. It follows that $\frac{CI}{CJ} = \frac{IU}{JR} = \frac{ID}{JS}$. Since $IU = ID$ (the inradius), we obtain $JR = JS$. Similarly, $JQ = JS$. Hence, $JQ = JR = JS$ (and clearly $J \neq I$). Thus, J is the excentre in $\angle ACB$, and the excircle with centre J touches CA, AB, CB at Q, S, R, respectively.

Now, it is well known that $AT = BS = s - a$ (with the standard notations) [briefly, $b - a = BR - QA = BS - SA = 2BS - c$ and $b - a = AV - BU = AT - BT = 2AT - c$, from which we obtain $BS = AT = \frac{1}{2}(b + c - a) = s - a$].

Next we turn to solutions of problems of the 52nd Polish Mathematical Olympiad given [2004 : 19].

1. Show that the inequality

$$\sum_{i=1}^{n} ix_i \leq \binom{n}{2} + \sum_{i=1}^{n} x_i^i$$

holds for every integer $n \geq 2$ and all real numbers $x_1, x_2, \ldots, x_n \geq 0$.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON. We give the solution by Bornsztein.

From the AM–GM Inequality, for each non-negative real number x and for each positive integer k, we have

$$x^k + k - 1 \;=\; x^k + 1 + \cdots + 1 \;\geq\; k\sqrt[k]{x^k \times 1 \times \cdots \times 1} \;=\; kx\,.$$

Equality occurs, for $k \geq 2$, if and only if $x = 1$.

Using this for each x_i and summing leads to

$$\sum_{i=1}^{n} ix_i \;\leq\; \sum_{i=1}^{n} \left((i-1) + x_i^i\right) \;=\; \binom{n}{2} + \sum_{i=1}^{n} x_i^i\,,$$

as desired.

Equality occurs if and only if $x_2 = \cdots = x_n = 1$ (and $x_1 \geq 0$ is arbitrary).

2. Consider an arbitrary point P inside the regular tetrahedron with an edge of length 1. Show that the sum of the distances from P to the vertices of the tetrahedron does not exceed 3.

Solved by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

We will first consider a two-dimensional version of the problem.

Lemma. Supose that ABC is an equilateral triangle with side length 1 and that P is a point inside it. Then $PA + PB + PC \leq 2$.

Proof: Consider an ellipse with foci A and B that passes through P. Let the ellipse intersect AC at Q. Then the convexity of the ellipse implies that $PC \leq QC$. Since P and Q both lie on the ellipse, $PA + PB = QA + QB$. Since the longest chord in a triangle is the longest side, we have $QB \leq 1$. Hence,

$$PA + PB + PC \;\leq\; QA + QB + QC \;\leq\; QA + QC + 1 \;=\; 2\,.$$

Returning to the original problem, let the tetrahedron be $ABCD$. Let $\mathcal{N}$ be the ellipsoid with foci A and B which passes through P, and let $\mathcal{M}$ be the ellipsoid with foci C and D which passes through P. Obviously, if a point X lies within the ellipsoid $\mathcal{N}$, then $XA + XB \leq PA + PB$. The inequality is reversed if X is not inside $\mathcal{N}$.

Consider the intersection of the surface of $\mathcal{N}$ with the surface of the tetrahedron. Suppose that the intersection is completely inside the ellipsoid $\mathcal{M}$. Then, due to the convexity of ellipsoids, we see that the surfaces of $\mathcal{N}$ and $\mathcal{M}$ do not meet inside the tetrahedron, implying that P could not exist, a contradiction. Thus, there is a point Q (say on face ABC) such that Q is on $\mathcal{N}$ but not inside $\mathcal{M}$.

Then $PA + PB = QA + QB$ and $PC + PD \le QC + QD$; hence,

$$PA + PB + PC + PD \;\le\; QA + QB + QC + QD\,.$$

By our lemma, we have $QA + QB + QC \le 2$, and clearly $QD \le 1$. Adding the two inequalities yields the result.

3. The sequence x_1, x_2, x_3, ... is defined recursively by

$$x_1 \;=\; a\,,\quad x_2 \;=\; b\,,\quad \text{and}\quad x_{n+2} \;=\; x_{n+1} + x_n \quad \text{for } n = 1,\,2,\,3,\,\ldots,$$

where a and b are real numbers. A number c will be called a *repeated value* if $x_k = x_l = c$ for at least two distinct indices k and l. Prove that the initial terms a and b can be chosen so that there are more than 2000 repeated values, but it is impossible to choose a and b so that there are infinitely many repeated values.

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Then α and β are the zeros of $x^2 - x - 1$ and $\alpha\beta = -1$. Suppose that p is an odd positive integer. Then

$$\begin{aligned}\frac{\alpha^p - \alpha^{8000-p}}{\beta^p - \beta^{8000-p}} &= \frac{\alpha^{8000}(\alpha^p - \alpha^{80000-p})}{\alpha^{8000}\beta^p - \alpha^{8000}\beta^{8000-p}}\\ &= \frac{\alpha^{8000}(\alpha^p - \alpha^{8000-p})}{-\alpha^{8000-p} + \alpha^p} \;=\; \alpha^{8000}\,.\end{aligned}$$

Rearranging the above relationship, we get

$$\alpha^p - \alpha^{8000}\beta^p \;=\; \alpha^{8000-p} - \alpha^{8000}\beta^{8000-p}\,.$$

Define the sequence $\{x_n\}_{n=1}^{\infty}$ by $x_n = \alpha^n - \alpha^{8000}\beta^n$. Then $x_1 = x_{7999}$, $x_3 = x_{7997}$, ..., $x_{3999} = x_{4001}$. Furthermore, x_1, x_2, ..., x_{3999} are distinct, since, if p and q are odd numbers such that $x_p = x_q$, then

$$0 \;=\; x_p - x_q \;=\; \alpha^p + \alpha^{8000-p} - \alpha^q - \alpha^{8000-q} \;=\; (\alpha^p - \alpha^q)(1 - \alpha^{8000-p-q})\,,$$

and hence $p = q$ or $p + q = 8000$; thus, p and q cannot be distinct elements of $\{1, 3, \ldots, 3999\}$. Moreover, by the theory of linear recursive sequences, the sequence $\{x_n\}$ satisfies $x_{n+2} = x_{n+1} + x_n$. Therefore, the chosen sequence has at least 2000 repeated values (namely, x_1, x_3, ..., x_{3999}).

On the other hand, if $\{x_n\}_{n=1}^{\infty}$ is a sequence with the properties given in the problem, then there exist constants a and b such that $x_n = a\alpha^n + b\beta^n$ for all n. We may assume that neither a nor b is zero, for otherwise there would obviously be only a finite number of repeated values. Since $|\alpha| > 1$ and $|\beta| < 1$, the sequence would eventually be strictly monotonic (increasing if $a > 0$, and decreasing if $a < 0$). Hence, there is a value m such that none of the numbers x_m, x_{m+1}, ... appear again in the sequence $\{x_1, x_2, \ldots\}$. Therefore, the number of repeated values must be finite.

4. The integers a and b have the property that, for every non-negative integer n, the number $2^n a + b$ is the square of an integer. Show that $a = 0$.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON. We give Bornsztein's write-up.

Suppose, for a contradiction, that $a \neq 0$. For n sufficiently large, $2^n a + b$ has the same sign as a; thus $a > 0$. For each positive integer n, let x_n be the positive integer such that $x_n^2 = 2^n a + b$.

Suppose first that $b > 0$. Clearly, $\lim\limits_{n\to+\infty} x_n = +\infty$. Thus, there exists n_0 such that, for all $n \geq n_0$, we have $3b - 4x_n + 1 < 0$. Then, for $n \geq n_0$,

$$\begin{aligned}(2x_n-1)^2 &= 4x_n^2 - 4x_n + 1\\ &= 2^{n+2}a + 4b - 4x_n + 1 \;<\; 2^{n+2}a + b \;=\; x_{n+2}^2\,.\end{aligned}$$

Therefore, $2x_n - 1 < x_{n+2}$. On the other hand,

$$(2x_n)^2 \;=\; 4x_n^2 \;=\; 2^{n+2}a + 4b \;>\; 2^{n+2}a + b \;=\; x_{n+2}^2\,,$$

and therefore $2x_n > x_{n+2}$. Thus, we have $2x_n > x_{n+2} > 2x_n - 1$. This is a contradiction, since x_n and x_{n+2} are integers.

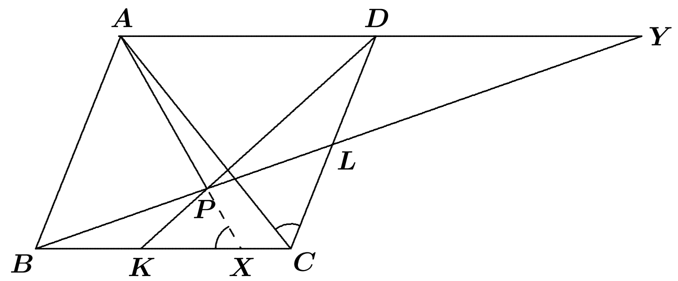
Suppose next that $b < 0$. Similar reasoning leads to a similar contradiction that $2x_n < x_{n+2} < 2x_n + 1$.

Thus, $b = 0$. Let $a = 2^\alpha \beta$, where $\alpha \geq 0$ and $\beta \geq 1$ is odd. Let n have opposite parity to α. Then $n + \alpha$ is odd, and $2^n a = 2^{n+\alpha}\beta$ cannot be a square, a contradiction.

Since we get a contradiction in each case, it follows that $a = 0$.

5. Let $ABCD$ be a parallelogram, and let K and L be points lying on the sides BC and CD, respectively, such that $BK \cdot AD = DL \cdot AB$. The segments DK and BL intersect at P. Show that $\angle DAP = \angle BAC$.

Solved by Toshio Seimiya, Kawasaki, Japan; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON. We give Seimiya's solution.



Let X and Y be the intersections of AP and BL with BC and AD, respectively. Since $BX \parallel AY$, we have

$$BK : BX \;=\; YD : YA\,.$$

Since $DL \parallel AB$, we see that $YD : YA = DL : AB$, and then $BK : BX = DL : AB$; that is,

$$BK : DL \;=\; BX : AB\,. \tag{1}$$

Since $BK \cdot AD = DL \cdot AB$, we have

$$BK : DL \;=\; AB : AD\,. \tag{2}$$

It follows from (1) and (2) that

$$BX : AB \;=\; AB : AD \;=\; DC : AD\,. \tag{3}$$

Since $\angle ABX = \angle ADC$, we have from (3)

$$\triangle ABX \sim \triangle ADC\,.$$

Thus,

$$\angle AXB \;=\; \angle ACD\,. \tag{4}$$

Since $AD \parallel BX$, and $AB \parallel DC$, we have $\angle AXB = \angle DAX = \angle DAP$ and $\angle ACD = \angle BAC$. Therefore, using (4), we get $\angle DAP = \angle BAC$.

6. Let $n_1 < n_2 < \cdots < n_{2000} < 10^{100}$ be positive integers. Prove that the set $\{n_1, n_2, \ldots, n_{2000}\}$ has two non-empty disjoint subsets A and B with equally many elements, equal sums of their elements, and equal sums of the squares of their elements.

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

Let $\mathcal{S}$ be the set of all 1000-element subsets of $\{n_1, n_2, \ldots, n_{2000}\}$. For any set $S \in \mathcal{S}$, we see that the sum of the elements of S must be less than $10^{100} \cdot 1000 = 10^{103}$, and the sum of the squares of the elements of S must be less than $10^{200} \cdot 1000 = 10^{203}$. Let us create $10^{103} \cdot 10^{203} = 10^{306}$ pigeonholes, each representing a distinct combination of a sum and a sum of squares. Now

$$\begin{aligned} |\mathcal{S}| \;&=\; \binom{2000}{1000} \;=\; \prod_{i=1}^{1000} \frac{1000+i}{i} \;=\; \prod_{i=1}^{500} \frac{1000+i}{i} \prod_{i=501}^{1000} \frac{1000+i}{i} \\ &>\; \prod_{i=1}^{500} 3 \prod_{i=501}^{1000} 2 = 6^{500} \;=\; (\sqrt{216})^{1000/3} > 10^{306}\,. \end{aligned}$$

Hence, if we place all the elements of $\mathcal{S}$ into the 10^{306} pigeonholes, there must exist two distinct sets X, Y, belonging to the same pigeonhole. That is, the sum of their elements is equal, and the sum of the squares of their elements is equal. Choose $A = X \setminus Y$ and $B = Y \setminus X$. Then A and B obviously satisfy all the requirements in the problem.

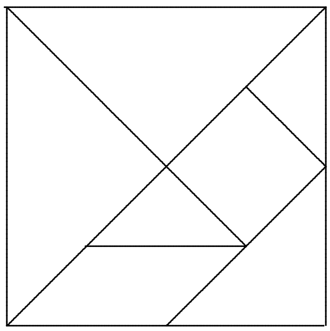
That completes this issue. Please send in solutions to problems from 2004–2005 quickly to help keep our schedule up. Also send your Olympiad materials.

BOOK REVIEW

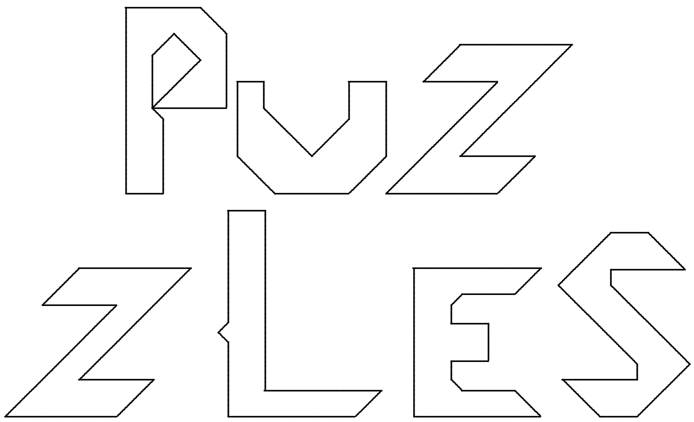
John Grant McLoughlin

The Tangram Book
By Jerry Slocum with Jack Botermans, Dieter Gebhardt, Monica Ma, Xiaohe Ma, Harold Raizer, Dic Sonneveld, and Carla van Splunteren, published by Sterling, New York, 2004
ISBN 1-4027-1688-5, paperback, 192 pages, US$14.95 (CDN$21.95).
Reviewed by ***Andy Liu****, University of Alberta, Edmonton, AB.*

When people are asked to name just one puzzle, it is more than likely that Tangram will be the one mentioned. It is intuitively appealing, and offers endless challenges. By comparison, Ernő Rubik's magic cube is too sophisticated, and Sam Loyd's 14–15 puzzle is too limited in scope. Thus, Tangram is a worthy representative of popular puzzles.

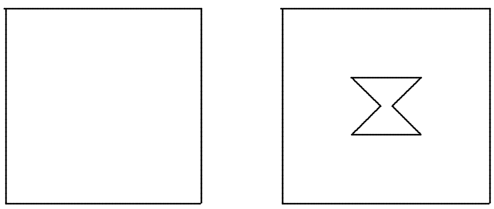


The puzzle consists of seven polygonal pieces which can be assembled into a square, as shown in the diagram above. The challenge is to construct other given figures using these seven pieces. To whet the reader's appetite, construct each of the letters in the diagram below using a complete set of Tangram pieces. This puzzle is taken from the business card of Jerry Slocum, the author of the book under review.



There are many books on Tangram, notably that by Sam Loyd [4]. However, with the publication of this definitive treatise by Jerry Slocum, there is no need to look any further for other references. Jerry is one of the three wise men in the puzzle world (the other two were the late Edward Hordern of the United Kingdom and the late Nobiyuki Yoshigahara of Japan). He is the founder of the International Puzzle Party, an annual gathering of some of the most original minds and the best puzzle designers. Jerry is also the President of the Slocum Puzzle Foundation, an organization which promotes mathematics education via puzzles. He is the author of many other puzzle books, the best known being [5], [6], and [7], which were co-authored by Jack Botermans.

The current volume is beautifully illustrated with striking colours. A quick scan of the book would draw immediate attention to the thousands of puzzles from pages 100 to 173. Those on page 100 are especially appealing. The puzzles come in pairs that are almost identical, but apparently with something missing from one member of the pair. An example is shown in the diagram below.



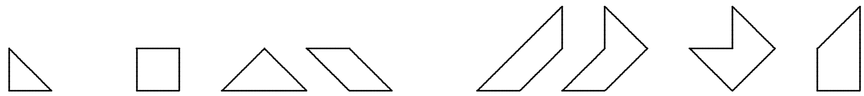
It could not be further from the truth to think of this book as a mere catalogue of puzzles. It is a scholarly work by a noted historian on the subject, going back far in time and spreading wide in geography—a comprehensive research of the origin of Tangram and its name, its twisted history, and its many manifestations. The six pages of bibliographical references alone are a most valuable treasure.

Sterling must be congratulated for publishing such a wonderful volume. This publisher is usually known for its line of books on crossword puzzles, but in the past decade it has ventured into books on mathematical puzzles. Two excellent examples are [2] and [3]. Mathematicians may find many of the other puzzle books wanting, since the puzzles are at a fairly low level, more along the line of I.Q. tests and Mensa stuff. However, if they can arouse the interest of young minds to look further for better puzzle books, they have served their purpose.

Addendum

Tangram is a put-together puzzle, a mathematical version of a jigsaw puzzle without the twisted edges. There are many other similar polygonal puzzles, but none have the enduring popularity of Tangram. Perhaps the main reason is that the Tangram pieces are all members of a family. Martin

Gardner called them polyaboloes in [1] (Chapter 11). However, the pieces formed of half-squares joined edge-to-edge and diagonal-to-diagonal are more commonly known as polytans. There is only one monotan, but there are three ditans and four tritans. These are illustrated in the diagram below.



Kate Jones of Kadon Enterprises has constructed a square consisting of two copies of the monotan, the three ditans and all of the tetratans. She has a larger square consisting of the four tritans and all of the pentatans. Finally, she has an even larger square consisting of the three ditans and all of the hexatans. Visit her website at `http://gamepuzzles.com`.

Bill Ritchie of Binary Arts/Think Fun has marketed a puzzle consisting of two copies of each of the ditans and the tritans. It is called *Shape by Shape*. Visit his website at `http://www.puzzles.com`. This particular puzzle is designed by Nobiyuki Yoshigahara, inventor of numerous intriguing puzzles. He had also authored many wonderful puzzle books, in Japanese. One of them has recently appeared in English [8].

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[1] Martin Gardner, *Mathematical Magic Show*, Mathematical Association of America, Washington, 1990.

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[7] Jerry Slocum and Jack Botermans, *The Book of Ingenious & Diabolical Puzzles*, Times Books, New York, 1994.

[8] Nobiyuki Yoshigahara, *Puzzles 101: A Puzzlemaster's Challenge*, A.K. Peters, Natick, 2004.

Chasing Imaginary Triangles

Ian VanderBurgh and Serge D'Alessio

Problem: A right triangle has a perimeter of 10 and an area of 5. What is the length of its hypotenuse?

This suggests:

Question 1. Given a right triangle with a given area A and a given perimeter P, how can we find the length of its hypotenuse?

Suppose the triangle has hypotenuse c and legs a and b. Then

$$A = \tfrac{1}{2}ab\,, \qquad P = a+b+c\,, \qquad \text{and} \qquad a^2+b^2 = c^2\,.$$

We want to find c in terms of A and P. If we wished to use brute force, we could substitute $c = \sqrt{a^2+b^2}$ into the second equation. This would give us $a+b+\sqrt{a^2+b^2} = P$, which we could solve along with $ab = 2A$ to find a and b, and then we could substitute back to obtain c.

However, there is a more clever way:

$$\begin{aligned} P - c &= a + b\,, \\ P^2 - 2Pc + c^2 &= a^2 + 2ab + b^2 \;=\; c^2 + 4A\,, \\ P^2 - 4A &= 2Pc\,, \\ c &= \frac{P^2 - 4A}{2P}\,. \end{aligned}$$

Wonderful! We were able to solve for the hypotenuse c without solving for a and b (which could potentially be messy).

For mathematical convenience we introduce the semiperimeter s. Then $P = 2s$, and the above expression takes on the simpler form

$$c = \frac{s^2 - A}{s}\,.$$

In the given problem, $c = (5^2 - 5)/5 = 4$.

This leads to:

Question 2. Given a right triangle with a given integral area A and a given integral perimeter P, when is the length of the hypotenuse also an integer?

For c to be an integer, we would like $2P \mid (P^2 - 4A)$. Since $2P$ is even, $P^2 - 4A$ must be even; whence, P must be even. Since $P = 2s$, then s must be an integer, and our condition is equivalent to

$$4s \mid (4s^2 - 4A) \qquad \Longleftrightarrow \qquad s \mid (s^2 - A) \qquad \Longleftrightarrow \qquad s \mid A\,.$$

Hence, any right triangle with integral area and with a semiperimeter which divides the area will have a hypotenuse with integer length.

The story could end there. But, when solving the initial problem, one of us actually tried to calculate the legs a and b. This is not so difficult now that we know the length of the hypotenuse. We have $\frac{1}{2}ab = 5$ and $a+b = 6$, which yields $a(6-a) = 10$, or $a^2-6a+10 = 0$; hence, $a = 3\pm i$. Therefore, this triangle does not actually exist!

Question 3. Given a "right triangle" with a given area A and a given perimeter P, what conditions on A and P guarantee that the "triangle" actually exists?

We go back again to our initial equations,

$$A \;=\; \tfrac{1}{2}ab \qquad \text{and} \qquad P \;=\; a+b+c\,.$$

Using our result for c in terms of s and A in the equation $a+b+c = P$, we get

$$a+b \;=\; 2s - \frac{s^2-A}{s} \;=\; \frac{s^2+A}{s}\,.$$

Solving for b and substituting into the equation $ab = 2A$, we get

$$\begin{aligned} a\left(\frac{s^2+A}{s} - a\right) &= 2A\,, \\ sa^2 - (s^2+A)a + 2As &= 0\,. \end{aligned}$$

For the triangle to exist, the discriminant must be non-negative; that is,

$$\begin{aligned} (s^2+A)^2 - 4(s)(2As) &\geq 0\,, \\ s^4 + 2As^2 + A^2 - 8As^2 &\geq 0\,, \\ s^4 - 6As^2 + A^2 &\geq 0\,. \end{aligned}$$

Although this is a nice result, we can go further. Dividing through by A^2 and letting $x = s^2/A$, we obtain

$$x^2 - 6x + 1 \;\geq\; 0\,.$$

Solving this quadratic inequality, we get

$$x \;\geq\; 3+2\sqrt{2} \qquad \text{or} \qquad x \;\leq\; 3-2\sqrt{2}\,,$$

which is equivalent to

$$s^2 \;\geq\; (3+2\sqrt{2})A \qquad \text{or} \qquad s^2 \;\leq\; (3-2\sqrt{2})A\,;$$

that is,

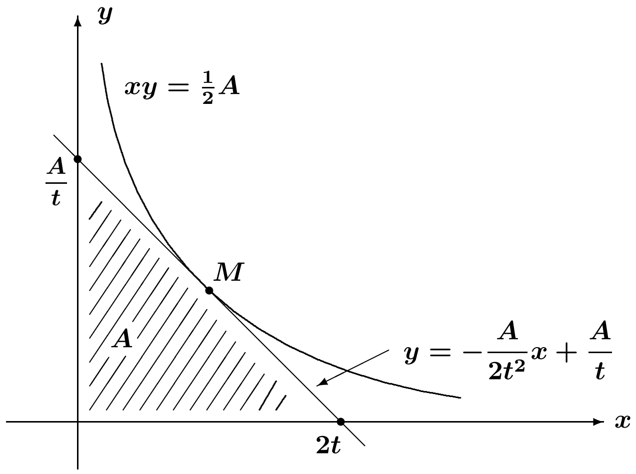
$$s \;\geq\; (1+\sqrt{2})\sqrt{A} \qquad \text{or} \qquad s \;\leq\; (\sqrt{2}-1)\sqrt{A}\,.$$

But we should not forget to check the hypotenuse, which will certainly be a real number (as long as A and s are real), but needs to be positive! For $c > 0$, we need $s^2 - A > 0$; that is, $s > \sqrt{A}$. Hence, the second of the above inequalities yields a triangle with real legs, but negative hypotenuse. (How did that happen?) The first inequality is consistent with $s > \sqrt{A}$.

Therefore, the "right triangle" with area A and semiperimeter s is a *bona fide* triangle if and only if $s \geq (1+\sqrt{2})\sqrt{A}$.

Returning to our original problem where $s = 5$ and $A = 5$, we find that $(1+\sqrt{2})\sqrt{A} \approx 5.4$, which clearly violates the above condition and again confirms that such a triangle does not exist.

We next proceed to offer a geometric interpretation of the condition $s \geq (1+\sqrt{2})\sqrt{A}$. Consider the hyperbola $xy = A/2$ and a line tangent to this curve at point $M\left(t, \frac{A}{2t}\right)$ as shown below. We can quickly determine that the slope of the tangent line is $-\frac{A}{2t^2}$. Thus, the tangent line has equation $y = -\frac{A}{2t^2}x + \frac{A}{t}$, with x–intercept $2t$ and y–intercept $\frac{A}{t}$. We observe that M is the mid-point of the portion of the tangent line that is cut off by the axes. More interesting, though, is the fact that the area of the right triangle formed by the tangent line and the coordinate axes is the same, regardless of where M is located on the hyperbola. As we let M move along the hyperbola, every possible shape of right triangle with area A will be achieved.



Now, the semiperimeter of the above triangle is given by

$$s = \frac{1}{2}\left(2t + \frac{A}{t} + \sqrt{4t^2 + \frac{A^2}{t^2}}\right) = t + \frac{A}{2t} + \sqrt{t^2 + \frac{A^2}{4t^2}}\,.$$

Since we have established that A is constant for all such triangles, we can think of this as a (differentiable) function of a single variable t, where $0 < t < \infty$. Because $s \to \infty$ as $t \to 0$ or $t \to \infty$, and since $s > 0$ for all $t \in (0, \infty)$, it is clear that s must possess an absolute minimum. The minimum value can be found by setting $\frac{ds}{dt} = 0$, where

$$\frac{ds}{dt} = 1 - \frac{A}{2t^2} + \frac{1 - \frac{A^2}{4t^4}}{\sqrt{1 + \frac{A^2}{4t^4}}}\,.$$

This leads to $t = \sqrt{A/2}$ and corresponds to the special case where the right triangle is isosceles. Setting $t = \sqrt{A/2}$ in our formula for s in terms of t and simplifying, we obtain $s_{\min} = (1 + \sqrt{2})\sqrt{A}$, from which the condition $s \geq (1 + \sqrt{2})\sqrt{A}$ follows immediately.

As a final note we wish to make a connection with Heron's formula, which relates the area A, semiperimeter s, and lengths ℓ_1, ℓ_2, ℓ_3 of the sides of an arbitrary triangle according to the expression

$$A = \sqrt{s(s-\ell_1)(s-\ell_2)(s-\ell_3)}\,.$$

In our original problem with $A = 5$ and $s = P/2 = 5$, the above simplifies to $5 = (5-\ell_1)(5-\ell_2)(5-\ell_3)$. Setting $X = 5 - \ell_1$, $Y = 5 - \ell_2$, and $Z = 5 - \ell_3$, it follows that $X + Y + Z = 5$, since $l_1 + l_2 + l_3 = P = 10$.

Thus, we have $XYZ = 5$ and $\frac{1}{3}(X + Y + Z) = \frac{5}{3}$. Enforcing the AM–GM Inequality yields

$$XYZ \leq \left(\frac{X+Y+Z}{3}\right)^3\,,$$

which leads to $5 \leq 125/27$, a contradiction. Hence, such a triangle cannot exist. This reaffirms our earlier finding.

Furthermore, using Heron's formula again for a general triangle, we have

$$A^2/s = (s-\ell_1)(s-\ell_2)(s-\ell_3)\,.$$

If we set $X = s - \ell_1$, $Y = s - \ell_2$, and $Z = s - \ell_3$, then we see that

$$X + Y + Z = s \qquad \text{and} \qquad XYZ = A^2/s\,.$$

Again, by the AM–GM Inequality, it follows that a necessary condition for such a triangle to exist is

$$\frac{A^2}{s} \leq \left(\frac{s}{3}\right)^3$$

(or $s^2 \geq 3\sqrt{3}A$); equality occurs if and only if the triangle is equilateral. Lastly, we point out that this condition applies to all triangles, whereas the previously derived condition, $s \geq \left(1 + \sqrt{2}\right)\sqrt{A}$, applies only to right triangles.

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PROBLEMS

Solutions to problems in this issue should arrive no later than **1 May 2006**. *An asterisk* ($\star$) *after a number indicates that a problem was proposed without a solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

3076. *Proposed by Vedula N. Murty, Dover, PA, USA.*

If x, y, z are non-negative real numbers and a, b, c are arbitrary real numbers, prove that

$$\big(a(y+z)+b(z+x)+c(x+y)\big)^2 \;\geq\; 4(xy+yz+zx)(ab+bc+ca)\,.$$

(Note: If we impose the conditions that $x+y+z=1$ and that a, b, c are positive, then the above is equivalent to

$$ax+by+cz+2\sqrt{(xy+yz+zx)(ab+bc+ca)} \;\leq\; a+b+c\,,$$

which is problem #8 of the 2001 Ukrainian Mathematical Olympiad, given in the December 2003 issue of ***CRUX with MAYHEM*** [2003 : 498]. The solution of the Ukrainian problem appears on page 443.)

3077. *Proposed by Arkady Alt, San Jose, CA, USA.*

In $\triangle ABC$, we denote the sides BC, CA, AB as usual by a, b, c, respectively. Let h_a, h_b, h_c be the lengths of the altitudes to the sides a, b, c, respectively. Let d_a, d_b, d_c be the signed distances from the circumcentre of $\triangle ABC$ to the sides a, b, c, respectively. (The distance d_a, for example, is positive if and only if the circumcentre and vertex A lie on the same side of the line BC.)

Prove that
$$\frac{h_a+h_b+h_c}{3} \;\leq\; d_a+d_b+d_c\,.$$

3078. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let ABC be a triangle with $a>b$. Let D be the foot of the altitude from A to the line BC, let E be the mid-point of AC, and let CF be an external bisector of $\angle BCA$ with F on the line AB. Suppose that D, E, F are collinear.

(a) Determine the range of $\angle BCA$.

(b) Show that $c>b$.

(c) If $c^2=ab$, determine the measures of the angles of $\triangle ABC$, and show that $\sin B=\cos^2 B$.

3079. *Proposed by Mihály Bencze, Brasov, Romania.*

Let x_1, x_2, $\ldots$, x_n be real numbers such that $x_1 \leq x_2 \leq \cdots \leq x_n$. Prove that

$$\left(\sum_{i,j=1}^{n} |x_i - x_j|\right)^4 \leq \frac{8(n-1)^2(n+1)(2n^2-3)}{15} \sum_{i,j=1}^{n} (x_i - x_j)^4 .$$

3080. *Proposed by Christopher J. Bradley, Bristol, UK.*

Let ABC be a triangle, and let U and V be any two points in the plane of the triangle, but not on the sides of the triangle. Let $L = BU \cap CV$, $L' = BV \cap CU$, $M = CU \cap AV$, $M' = CV \cap AU$, $N = AU \cap BV$, $N' = AV \cap BU$. Prove that the triangles ABC, LMN are in perspective, as are triangles ABC, $L'M'N'$ and triangles LMN, $L'M'N'$. If the centres of these three perspectivities are P, P', P'', prove that P, P', P'' are collinear. Prove further that if U is the centroid G of $\triangle ABC$, then P'' is the mid-point of PP'.

[The proposer based this problem on an item he found in a century-old issue of the *Educational Times*, in which U and V are the centroid and the symmedian point of $\triangle ABC$. He verified that the result generalized to U and V being isogonal conjugates, and, using Cabri, he also found the result to be true for any two points in the plane of the triangle not on the sides. He adds "It is unlikely to be original, but is the sort of result that should not be lost to the world, and which solvers should enjoy."]

3081. *Proposed by Christopher J. Bradley, Bristol, UK.*

Let ABC be an acute-angled triangle and let the altitudes from A, B, C to the opposite sides have lengths d, e, f, respectively. The circle centred at A with radius d meets the line segments AB and AC at P and U, respectively, and it meets the rays BA and CA at P' and U', respectively. Similarly, we define the points Q, V, Q', V' using the circle centred at B with radius e, and we define the points R, W, R', W' using the circle centred at C with radius f (where Q, Q', W, W' lie on the line BC). Let A' be the intersection of PW and QU, let B' be the intersection of QU and RV, and let C' be the intersection of RV and PW. Further, let A'' be the intersection of $P'W'$ and $Q'U'$, B'' the intersection of $Q'U'$ and $R'V'$, and C'' the intersection of $R'V'$ and $P'W'$.

Prove that the triangles ABC, $A'B'C'$, and $A''B''C''$ have a common incentre.

3082. *Proposed by J. Walter Lynch, Athens, GA, USA.*

Suppose that four consecutive terms of a geometric sequence with common ratio r are the sides of a quadrilateral. What is the range of all possible values for r?

3083. *Proposed by Edward T.H. Wang and Kaiming Zhao, Wilfrid Laurier University, Waterloo, ON.*

Let n be a natural number, and suppose that the generalized Newton's binomial coefficient $\binom{\frac{1}{2}}{n}$ is written as a reduced fraction p/q where p and q are integers with $q > 0$. Show that $q = 2^k$ for some k with $0 \le k \le 2n-1$.

3084. *Proposed by Mihály Bencze, Brasov, Romania.*

Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying

$$\sum_{k=1}^{n} x_k \;=\; 0 \qquad \text{and} \qquad \sum_{k=1}^{n} x_k^4 \;=\; 1\,.$$

Prove that

$$\left(\sum_{k=1}^{n} kx_k\right)^4 \;\le\; \frac{n^3(n^2-1)(3n^2-7)}{240}\,.$$

3085. *Proposed by Neven Jurič, Zagreb, Croatia.*

A *magic square of order* n is an $n \times n$ array containing the integers from 1 to n^2 such that the sum of the elements in each row, in each column, and on each of the two diagonals is the same.

Let $\mathfrak{M}$ be a magic square of odd order $n \ge 3$. Increase the values of all the entries in $\mathfrak{M}$ by $2n+2$ to get a new $n \times n$ array, say M_1. Place M_1 in the interior of an $(n+2) \times (n+2)$ array M'. Show that the border rows and columns of this can be filled in with the unused integers between 1 and $(n+2)^2$ to create a new magic square $\mathfrak{M}'$ of order $n+2$.

3086. *Proposed by Mihály Bencze, Brasov, Romania.*

If $a_k > 0$ for $k = 1, 2, \ldots, n$, prove that

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right) \;\ge\; \frac{1}{n}\left(\sqrt[3]{\frac{a_1}{a_2}} + \sqrt[3]{\frac{a_2}{a_3}} + \cdots + \sqrt[3]{\frac{a_n}{a_1}}\right)^3 \;\ge\; n^2\,.$$

3087. *Proposed by Mihály Bencze, Brasov, Romania.*

Let ABC be a triangle with sides a, b, c opposite the angles A, B, C, respectively. If R is the circumradius and r the inradius of $\triangle ABC$, prove that:

(a) $\dfrac{3R}{r} \;\ge\; \dfrac{a+c}{b} + \dfrac{b+a}{c} + \dfrac{c+b}{a} \;\ge\; 6$;

(b) $\left(\dfrac{R}{r}\right)^3 \;\ge\; \left(\dfrac{a}{b}+\dfrac{b}{a}\right)\left(\dfrac{b}{c}+\dfrac{c}{b}\right)\left(\dfrac{a}{c}+\dfrac{c}{a}\right) \;\ge\; 8$.

(Both (a) and (b) are refinements of Euler's Inequality, $R \ge 2r$.)

3076. *Proposé par Vedula N. Murty, Dover, PA, USA.*

Si x, y et z sont des nombres réels non négatifs et si a, b et c sont des nombres réels arbitraires, montrer que

$$\big(a(y+z)+b(z+x)+c(x+y)\big)^2 \;\geq\; 4(xy+yz+zx)(ab+bc+ca)\,.$$

(Note : Si l'on impose les conditions que $x+y+z=1$ et que a, b et c sont positifs, alors l'inégalité ci-dessus est équivalente à

$$ax+by+cz+2\sqrt{(xy+yz+zx)(ab+bc+ca)} \;\leq\; a+b+c\,,$$

qui est en fait le problème #8 de l'Olympiade Mathématique d'Ukraine de 2001, paru dans le numéro de décembre 2003 de ***CRUX with MAYHEM*** [2003 : 498]. La solution du problème d'Ukraine est à la page 443.)

3077. *Proposé par Arkady Alt, San Jose, CA, USA.*

Dans le triangle ABC, on désigne les côtés BC, CA et AB respectivement par a, b et c. Soit h_a, h_b et h_c les longueurs des hauteurs respectives abaissées sur les côtés a, b et c. Soit d_a, d_b et d_c les distances respectives (affectées d'un signe) du centre du cercle circonscrit du triangle aux côtés a, b et c. (Par exemple, la distance d_a est positive si et seulement si ce centre et le sommet A sont situés du même côté de la droite BC.)

Montrer que

$$\frac{h_a+h_b+h_c}{3} \;\leq\; d_a+d_b+d_c\,.$$

3078. *Proposé par D.J. Smeenk, Zaltbommel, Pays-Bas.*

Soit ABC un triangle avec $a>b$. Soit D le pied de la perpendiculaire abaissée de A sur la droite BC, soit E le point milieu de AC, et soit CF une bissectrice extérieure de l'angle BCA avec F sur la droite AB. Supposons que D, E et F sont colinéaires.

(a) Déterminer le domaine de variation de l'angle BCA.

(b) Montrer que $c>b$.

(c) Si $c^2=ab$, trouver les mesures des angles du triangle ABC et montrer que $\sin B=\cos^2 B$.

3079. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit $x_1, x_2, \ldots, x_n$ des nombres réels tels que $x_1\leq x_2\leq\cdots\leq x_n$. Montrer que

$$\Bigg(\sum_{i,j=1}^{n}|x_i-x_j|\Bigg)^4 \;\leq\; \frac{8(n-1)^2(n+1)(2n^2-3)}{15}\sum_{i,j=1}^{n}(x_i-x_j)^4\,.$$

3080. *Proposé par Christopher J. Bradley, Bristol, GB.*

Dans le plan, on considère le triangle ABC et deux points U et V non situés sur les côtés du triangle. Soit $L = BU \cap CV$, $L' = BV \cap CU$, $M = CU \cap AV$, $M' = CV \cap AU$, $N = AU \cap BV$, $N' = AV \cap BU$. Montrer que les triangles ABC, LMN sont en perspective, ainsi que les triangles ABC, $L'M'N'$ et les triangles LMN, $L'M'N'$. Si les centres de ces trois perspectivités sont P, P' et P'', montrer que P, P', P'' sont colinéaires. Montrer de plus que si U est le centre de gravité G du triangle ABC, alors P'' est le point milieu de PP'.

[Dans ce problème, le proposeur dit s'être inspiré d'un résultat paru dans un numéro centenaire du journal *Educational Times*, dans lequel U et V sont le centre de gravité et le symédian du triangle ABC. Il a vérifié que ce résultat se généralise au cas où U sont V sont des isogonales conjuguées et, à l'aide de Cabri, il a aussi trouvé que ce résultat demeure vrai pour deux points quelconques non situés sur les côtés du triangle. Il ajoute : "Il n'est peut-être pas original, mais il est de cette sorte de résultat qui, non seulement, ne devrait pas sombrer dans l'oubli, mais aussi faire la joie des solutionneurs."]

3081. *Proposé par Christopher J. Bradley, Bristol, GB.*

Soit ABC un triangle acuteangle et soit d, e et f les longueurs des hauteurs respectives issues des sommets A, B et C. Le cercle centré en A de rayon d coupe respectivement les segments AB et AC en P et U, et il coupe respectivement les rayons BA et CA en P' et U'. De manière analogue, on définit les points Q, V, Q', V' en utilisant le cercle centré en B de rayon e et on définit les points R, W, R', W' en utilisant le cercle centré en C de rayon f (où Q, Q', W, W' sont sur la droite BC). Soit A' l'intersection de PW et QU, soit B' celle de QU et RV, et soit C' celle de RV et PW. Finalement, soit A'' l'intersection de $P'W'$ et $Q'U'$, soit B'' celle de $Q'U'$ et $R'V'$, et soit C'' celle de $R'V'$ et $P'W'$.

Montrer que les triangles ABC, $A'B'C'$ et $A''B''C''$ ont même centre du cercle inscrit.

3082. *Proposé par J. Walter Lynch, Athens, GA, USA.*

On suppose que quatre termes consécutifs d'une suite géométrique de raison r sont les côtés d'un quadrilatère. Quelle est la gamme des valeurs possibles pour r ?

3083. *Proposé par Edward T.H. Wang et Kaiming Zhao, Université Wilfrid Laurier, Waterloo, ON.*

Soit n un nombre naturel, et supposons que le coefficient binomial généralisé de Newton $\binom{\frac{1}{2}}{n}$ est écrit sous la forme d'une fraction réduite p/q où p et q sont des entiers avec $q > 0$. Montrer que $q = 2^k$ pour un certain k avec $0 \leq k \leq 2n - 1$.

3084. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit x_1, x_2, ..., x_n des nombres réels satisfaisant

$$\sum_{k=1}^{n} x_k = 0 \quad \text{and} \quad \sum_{k=1}^{n} x_k^4 = 1 .$$

Montrer que

$$\left(\sum_{k=1}^{n} kx_k\right)^4 \leq \frac{n^3(n^2-1)(3n^2-7)}{240} .$$

3085. *Proposé par Neven Jurič, Zagreb, Croatie.*

Un *carré magique d'ordre* n est un tableau $n \times n$ contenant les entiers de 1 à n^2 tel que la somme des éléments de chaque ligne, de chaque colonne et de chacune des deux diagonales soit la même.

Soit $\mathfrak{M}$ un carré magique d'ordre impair $n \geq 3$. On augmente la valeur de tous les éléments de $\mathfrak{M}$ de $2n+2$ pour obtenir un nouveau tableau $n \times n$, disons M_1. Insérer M_1 à l'intérieur d'un tableau $(n+2) \times (n+2)$, disons M'. Montrer que les lignes et les colonnes de M' peuvent être remplies avec les nombres entre 1 et $(n+2)^2$ non utilisés pour créer un nouveau carré magique $\mathfrak{M}'$ d'ordre $n+2$.

3086. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Si $a_k > 0$ pour $k = 1, 2, \ldots, n$, montrer que

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right) \geq \frac{1}{n}\left(\sqrt[3]{\frac{a_1}{a_2}} + \sqrt[3]{\frac{a_2}{a_3}} + \cdots + \sqrt[3]{\frac{a_n}{a_1}}\right)^3 \geq n^2 .$$

3087. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit ABC un triangle de côtés a, b et c respectivement opposés aux sommets A, B et C. Si R est le rayon du cercle circonscrit au triangle et r celui de son cercle inscrit, montrer que :

(a) $\dfrac{3R}{r} \geq \dfrac{a+c}{b} + \dfrac{b+a}{c} + \dfrac{c+b}{a} \geq 6$;

(b) $\left(\dfrac{R}{r}\right)^3 \geq \left(\dfrac{a}{b}+\dfrac{b}{a}\right)\left(\dfrac{b}{c}+\dfrac{c}{b}\right)\left(\dfrac{a}{c}+\dfrac{c}{a}\right) \geq 8.$

((a) et (b) sont toutes deux des raffinements de l'Inégalité d'Euler, $R \geq 2r$.)

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2963. [2004 : 367, 370; 2005 : 350–352] *Proposed by Mihály Bencze, Brasov, Romania.*

Let ABC be any acute-angled triangle. Let r and R be the inradius and circumradius, respectively, and let s be the semiperimeter; that is, $s = \frac{1}{2}(a+b+c)$. Let m_a be the length of the median from A to BC, and let w_a be the length of the internal bisector of $\angle A$ from A to the side BC. We define m_b, m_c, w_b and w_c similarly. Prove that

(a) $\displaystyle \frac{3s^2-r^2-4Rr}{8sRr} \le \sum_{\text{cyclic}} \frac{m_a}{aw_a} \le \frac{s^2-r^2-4Rr}{7sRr}$;

(b) $\displaystyle \frac{3}{4} \le \sum_{\text{cyclic}} \frac{m_a^2}{b^2+c^2} \le \frac{4R+r}{4R}$.

The editor apologizes for misplacing the solution of Li Zhou, Polk Community College, Winter Haven, FL, USA. In his solution Zhou actually proves that the lower bound of $3/4$ in part (b) *can be increased to 1, which was conjectured by Walther Janous, Ursulinengymnasium, Innsbruck, Austria, as mentioned in the comments following the featured solutions to this problem* [2005 : 350, 352]. *We present Zhou's proof below.*

We prove that the lower bound of $\frac{3}{4}$ in inequality (b) can be increased to 1. Since ABC is an acute-angled triangle, we have $b^2+c^2 > a^2$, so that $b^2+c^2 > \frac{1}{2}(a^2+b^2+c^2)$. Hence,

$$\begin{aligned}\sum_{\text{cyclic}} \frac{m_a^2}{b^2+c^2} &= \sum_{\text{cyclic}} \frac{2(b^2+c^2)-a^2}{4(b^2+c^2)} \\ &= \frac{3}{2} - \frac{1}{4}\sum_{\text{cyclic}} \frac{a^2}{b^2+c^2} > \frac{3}{2} - \frac{1}{4}\sum_{\text{cyclic}} \frac{2a^2}{a^2+b^2+c^2} = 1,\end{aligned}$$

which completes the proof.

2972. [2004 : 369, 372] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

(a) Prove that if $0 \le \lambda \le 4$, then, for all positive real numbers x, y, z, t,

$$\begin{aligned}&(t^2+1)(x^3+y^3+z^3)+3(1-t^2)xyz \\ &\quad \ge (1+\lambda t)(x^2y+y^2z+z^2x)+(1-\lambda t)(xy^2+yz^2+zx^2).\end{aligned}$$

(b) For $t = \frac{1}{4}$ and $\lambda = 4$, the above inequality becomes

$$17(x^3 + y^3 + z^3) + 45xyz \;\geq\; 32(x^2y + y^2z + z^2x)\,.$$

Find all positive values of δ such that the inequality

$$x^3 + y^3 + z^3 + 3\delta xyz \;\geq\; (1+\delta)(x^2y + y^2z + z^2x)$$

holds for all x, y, z which are: (i) positive real numbers; (ii) side lengths of a triangle.

Composite of solutions by Li Zhou, Polk Community College, Winter Haven, FL, USA and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

(a) Without loss of generality, assume $x \leq y$ and $x \leq z$. Let $p = y - x$ and $q = z - x$. Then

$$\begin{aligned} D &= (1+t^2)(x^3+y^3+z^3) + 3(1-t^2)xyz \\ &\quad - (1+\lambda t)(x^2y+y^2z+z^2x) - (1-\lambda t)(xy^2+yz^2+zx^2) \\ &= (1+3t^2)(p^2-pq+q^2)x + (p^3+q^3)t^2 \\ &\quad - \lambda pq(p-q)t + (p+q)(p-q)^2\,. \end{aligned}$$

The first term of this expression is clearly non-negative, and the last three terms form a quadratic function of t with discriminant

$$\begin{aligned} &\lambda^2p^2q^2(p-q)^2 - 4(p^3+q^3)(p+q)(p-q)^2 \\ &\qquad \leq\; 4(p-q)^2\big(4p^2q^2 - (p^3+q^3)(p+q)\big) \\ &\qquad =\; -4(p-q)^4(p^2+3pq+q^2) \;\leq\; 0\,. \end{aligned}$$

Hence, $D \geq 0$.

(b)(i) Similarly, assume that $p = y - x \geq 0$ and $q = z - x \geq 0$. Then

$$\begin{aligned} D &= x^3+y^3+z^3+3\delta xyz - (1+\delta)(x^2y+y^2z+z^2x) \\ &= (2-\delta)(p^2-pq+q^2)x + p^3 - (1+\delta)p^2q + q^3\,. \end{aligned}$$

Since x can be arbitrarily close to 0, a necessary condition for $D \geq 0$ is

$$p^3 - (1+\delta)p^2q + q^3 \;\geq\; 0\,.$$

Define $f(r) = r^3 - (1+\delta)r^2 + 1$ for $r \geq 0$. Then $f'(r) = 3r^2 - 2(1+\delta)r = 0$ at $r = \frac{2}{3}(1+\delta)$. Setting $f\left(\frac{2}{3}(1+\delta)\right) \geq 0$, we get $-\frac{4}{27}(1+\delta)^3 + 1 \geq 0$; that is, $\delta \leq \frac{3\sqrt[3]{2}}{2} - 1 \approx 0.88988$.

Conversely, suppose $0 \leq \delta \leq \frac{3\sqrt[3]{2}}{2} - 1$. Since $(2-\delta)(p^2-pq+q^2)x \geq 0$, the condition $p^3 - (1+\delta)p^2q + q^3 \geq 0$ is also sufficient for $D \geq 0$. Observe that $p^3 - (1+\delta)p^2q + q^3 \geq p^3 - \frac{3\sqrt[3]{2}}{2}p^2q + q^3$, and, by the AM–GM Inequality,

$$p^3 + q^3 \;=\; \tfrac{1}{2}p^3 + \tfrac{1}{2}p^3 + q^3 \;\geq\; \tfrac{3\sqrt[3]{2}}{2}p^2q\,.$$

(b)(ii) Let $u = \frac{1}{2}(z+x-y)$, $v = \frac{1}{2}(x+y-z)$, and $w = \frac{1}{2}(y+z-x)$. Then $x = u+v$, $y = v+w$, $z = w+u$, and u, v, $w \geq 0$. Hence,

$$\begin{aligned} D &= x^3+y^3+z^3+3\delta xyz-(1+\delta)(x^2y+y^2z+z^2x) \\ &= (1-\delta)(u^3+v^3+w^3)-6uvw+(1+\delta)(u^2v+v^2w+w^2u)\,. \end{aligned}$$

Assume similarly that $p=v-u\geq 0$ and $q=w-u\geq 0$. Then

$$D = 2(2-\delta)(p^2-pq+q^2)u+(1-\delta)p^3+(1+\delta)p^2q+(1-\delta)q^3\,.$$

Again, a necessary condition for $D\geq 0$ is

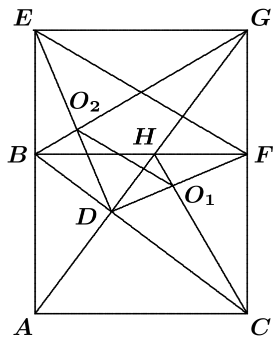
$$(1-\delta)p^3+(1+\delta)p^2q+(1-\delta)q^3 \;\geq\; 0\,,$$

which forces $\delta\leq 1$. Conversely, if $0\leq\delta\leq 1$, then $D\geq 0$.

Also solved by the proposer.

2973. [2004 : 369, 372] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain (dedicated to Toshio Seimiya).*

Let ABC be a non-isosceles right triangle with right angle at A and $AC>AB$. Let D be the foot of the altitude from A to the side BC. Let G be the point of intersection of the line AD (extended) with the line through C which is parallel to AB. Let E be the point such that $ACGE$ is a rectangle, and let F be the point such that $BFGE$ is a rectangle. Let H be the intersection of AG and BF. Let O_1 be the intersection of the diagonals of the quadrilateral $CDHF$, and let O_2 be the intersection of the diagonals of the quadrilateral $BDGE$.



Prove that the triangles ABC, DFE, and DO_1O_2 are similar.

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

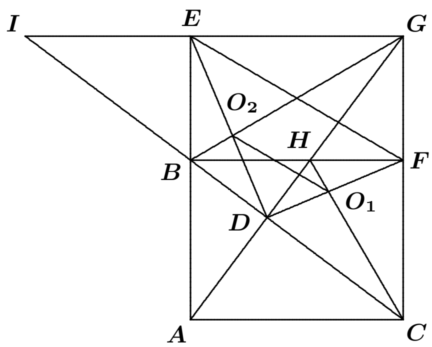
Since $\angle BEG=\angle BDG=\angle BFG=90^\circ$, the points B, E, G, F, and D are concyclic, implying that $\angle EDF=\angle EBF=\angle BAC$ and that $\angle DEF=\angle DBF=\angle BCA$. Thus, $\triangle ABC\sim\triangle DEF$, proving the first part.

Extend CB to meet line EG at the point I. Since BG is a transversal of $\triangle DEI$, we have, by Menelaus' Theorem,

$$\frac{DO_2}{O_2E}\cdot\frac{EG}{GI}\cdot\frac{IB}{BD} \;=\; -1\,.$$

Similarly, since HC is a transversal of $\triangle DFG$, we have

$$\frac{DO_1}{O_1F}\cdot\frac{FC}{CG}\cdot\frac{GH}{HD} \;=\; -1\,.$$



Thus,

$$\frac{DO_2}{O_2E} \cdot \frac{EG}{GI} \cdot \frac{IB}{BD} \;=\; \frac{DO_1}{O_1F} \cdot \frac{FC}{CG} \cdot \frac{GH}{HD}\,. \tag{1}$$

Since $\triangle CFB \sim \triangle CGI$, we have

$$\frac{EG}{GI} \;=\; \frac{BF}{GI} \;=\; \frac{FC}{CG}\,.$$

Since $BH \parallel IG$, we have $\frac{IB}{BD} = \frac{GH}{HD}$. Therefore, equation (1) reduces to

$$\frac{DO_2}{O_2E} \;=\; \frac{DO_1}{O_1F}\,,$$

implying that $O_1O_2 \parallel EF$. Thus, $\triangle DO_1O_2 \sim \triangle DEF$, and we are done.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (two solutions); MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Other solvers used a variety of methods including coordinates and vectors. Woo emphasized that his proof needed no trigonometry. In fact, only Zvonaru used trigonometry, and only one application of the Cosine Rule at that. Janous commented: "a lovely problem".

2974. [2004 : 369, 372] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let P be any point on the line BC in $\triangle ABC$. Let A_1 be the intersection of AP (possibly extended) with the line through B which is parallel to AC, and let A_2 be the intersection of AP (possibly extended) with the line through C which is parallel to AB.

Prove that the area of $\triangle ABC$ is the geometric mean of the areas of $\triangle A_1BC$ and $\triangle A_2BC$.

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.

Let $[ABC]$ denote the area of triangle ABC. Let points H, H_1, and H_2 be the feet of the perpendiculars to the line BC from points A, A_1, and A_2, respectively. From the pairs of similar triangles A_1BP, ACP, and ABP, A_2CP, we have

$$\frac{AH}{A_1H_1} \;=\; \frac{CP}{BP} \qquad \text{and} \qquad \frac{AH}{A_2H_2} \;=\; \frac{BP}{CP}\,.$$

Multiplying these two equalities, we find

$$(AH)^2 \;=\; A_1H_1 \cdot A_2H_2\,. \tag{1}$$

Since AH, A_1H_1, and A_2H_2 are just the altitudes to the side BC of triangles ABC, A_1BC, and A_2BC, respectively, and the areas of triangles with equal bases are proportional to the altitudes of the triangles, the equality (1) yields

$$[ABC]^2 = [A_1BC] \cdot [A_2BC] ,$$

which is the desired result.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (2 solutions); MICHEL BATAILLE, Rouen, France; KIN FUNG CHUNG, student, University of Toronto, Toronto, ON; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD B. EDEN, Ateneo de Manila University, The Philippines; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; ANDY PHAM, California State University, Fullerton, CA, USA; BOB SERKEY, Leonia, NJ, USA; D.J. SMEENK, Zaltbommel, the Netherlands; GEORGE TSAPAKIDIS, Agrinio, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

2975. [2004 : 370, 372] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Given an inscribed convex quadrilateral with sides of length m, n, p, q, taken in order around the quadrilateral, and diagonals of length d and d', prove that $\sqrt{mp+nq} \le \frac{1}{2}(d+d')$.

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Michel Bataille, Rouen, France; Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain; Kin Fung Chung, student, University of Toronto, Toronto, ON; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Richard B. Eden, Ateneo de Manila University, The Philippines; John G. Heuver, Grande Prairie, AB; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; Michael Parmenter, Memorial University of Newfoundland, St. John's, NL; Victor Pambuccian, Arizona State University West, Phoenix, AZ, USA; Bogdan Suceavă, California State University, Fullerton, CA, USA; Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany; Peter Y. Woo, Biola University, La Mirada, CA, USA; Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON; and Li Zhou, Polk Community College, Winter Haven, FL, USA.

By Ptolemy's Theorem and the AM–GM Inequality, we have

$$\sqrt{mp+nq} = \sqrt{dd'} \le \tfrac{1}{2}(d+d') .$$

Equality occurs if and only if $d = d'$; that is, if and only if the inscribed quadrilateral is an isosceles trapezium.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GEORGE TSAPAKIDIS, Agrinio, Greece; and the proposer.

2976. [2004 : 429, 432] *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Let a, b, $c \in \mathbb{R}$. Prove that

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \;\geq\; (ab+bc+ca)^3\,.$$

Solution by Kee-Wai Lau, Hong Kong, China.

It can be readily checked that

$$\begin{aligned}&(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)-(ab+bc+ca)^3\\ &\quad=\;\frac{1}{6}\Big[2(ab+bc+ca)^2\sum_{\text{cyclic}}(a-b)^2+(a+b+c)^2\sum_{\text{cyclic}}a^2(b-c)^2\Big]\,.\end{aligned}$$

Clearly, the last expression is non-negative, and the result is immediate.

Also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania (two solutions); VASILE CÎRTOAJE, University of Ploiesti, Romania; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria (two solutions); ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; TITU ZVONARU, Comăneşti, Romania; and the proposer.

Bencze and Cîrtoaje proved the two stronger inequalities

$$\begin{aligned}(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) &\geq \frac{27}{64}(a+b)^2(b+c)^2(a+b)^2\\ &\geq (ab+bc+ca)^3\,,\end{aligned}$$

from which the given inequality follows. Bencze also gave a generalization and stated some related problems, while Cîrtoaje mentioned that he had proposed this problem earlier and gave a reference to the book of L. Panaitopol, V. Băndilă and M. Lascu, Inequalities, *GIL, Zalău (Romania) 1995, p. 147.*

2977. [2004 : 429, 432] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let a_1, a_2, $\ldots$, a_n be positive real numbers, let $r=\sqrt[n]{a_1a_2\cdots a_n}$, and let

$$E_n \;=\; \frac{1}{a_1(1+a_2)}+\frac{1}{a_2(1+a_3)}+\cdots+\frac{1}{a_n(1+a_1)}-\frac{n}{r(1+r)}\,.$$

(a) Prove that $E_n \geq 0$ for

- (a_1) $n=3$;
- (a_2) $n=4$ and $r\leq 1$;
- (a_3) $n=5$ and $\frac{1}{2}\leq r\leq 2$;
- (a_4) $n=6$ and $r=1$.

(b)★ Prove or disprove that $E_n \geq 0$ for

- (b_1) $n=5$ and $r>0$;
- (b_2) $n=6$ and $r\leq 1$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

(a) For convenience, we take all indices modulo n. Let

$$F_n \;=\; \sum_{i=1}^{n} \frac{1}{a_i(1+a_{i+1})}\,.$$

Put $k \;=\; \sum\limits_{i=1}^{n} \sqrt[n]{a_i a_{i+1}^2 \cdots a_{i+n-2}^{n-1}}$ and $x_i \;=\; \frac{1}{k}\sqrt[n]{a_i a_{i+1}^2 \cdots a_{i+n-2}^{n-1}}$ for $1 \le i \le n$. Then

$$\sum_{i=1}^{n} x_i \;=\; 1 \qquad \text{and} \qquad F_n \;=\; \sum_{i=1}^{n} \frac{x_i}{r(x_{i+1}+rx_{i+2})}\,.$$

Applying Jensen's Inequality to the convex function $1/t$ on $(0,\infty)$, we get

$$F_n \;\ge\; \frac{1}{r(S+rT)}\,,$$

where $S = \sum\limits_{i=1}^{n} x_i x_{i+1}$ and $T = \sum\limits_{i=1}^{n} x_i x_{i+2}$.

(a_1) When $n=3$, it is easy to see that

$$S \;=\; T \;\le\; \tfrac{1}{3}\left(\sum_{i=1}^{3} x_i\right)^2 \;=\; \tfrac{1}{3}\,.$$

Hence, $F_3 \ge \dfrac{3}{r(1+r)}$.

(a_2) For $n=4$, let $M = (x_1+x_2)(x_3+x_4)+(x_2+x_3)(x_4+x_1)$ and $N = (x_1+x_3)(x_2+x_4)$. Then $S = N$ and $T = M - N$. By the AM–GM Inequality,

$$\begin{aligned} M &\;\le\; 2\left(\tfrac{1}{2}(x_1+x_2+x_3+x_4)\right)^2 \;=\; \tfrac{1}{2} \\ \text{and} \qquad N &\;\le\; \left(\tfrac{1}{2}(x_1+x_2+x_3+x_4)\right)^2 \;=\; \tfrac{1}{4}\,. \end{aligned}$$

Hence, for $r \le 1$, we have

$$F_4 \;\ge\; \frac{1}{r((1-r)N+rM)} \;\ge\; \frac{4}{r((1-r)+2r)} \;=\; \frac{4}{r(1+r)}\,.$$

(a_3) We improve this part to $E_5 \ge 0$ for $\frac{3}{7} \le r \le \frac{7}{3}$. Let

$$Q \;=\; \sum_{i=1}^{5} x_i^2\,, \qquad M \;=\; \sum_{i=1}^{5}\left(x_i+x_{i+1}+\tfrac{1}{2}x_{i+2}\right)\left(\tfrac{1}{2}x_{i+2}+x_{i+3}+x_{i+4}\right)\,,$$

$$\text{and} \qquad N \;=\; \sum_{i=1}^{5}\left(x_i+x_{i+2}+\tfrac{1}{2}x_{i+4}\right)\left(\tfrac{1}{2}x_{i+4}+x_{i+6}+x_{i+8}\right)\,.$$

Then it is purely computational to verify that

$$1 \;=\; \left(\sum_{i=1}^{5} x_i\right)^2 \;=\; Q+2S+2T\,,$$

$$M \;=\; \tfrac{1}{4}Q+2S+4T\,, \qquad \text{and} \qquad N \;=\; \tfrac{1}{4}Q+4S+2T\,.$$

Solving the system, we get

$$S \;=\; \tfrac{1}{20}(7N-3M-1) \qquad \text{and} \qquad T \;=\; \tfrac{1}{20}(7M-3N-1)\,.$$

Also, by the AM–GM Inequality,

$$M \;\le\; \sum_{i=1}^{5}\left(\tfrac{1}{2}(x_i+x_{i+1}+x_{i+2}+x_{i+3}+x_{i+4})\right)^2 \;=\; \tfrac{5}{4}\,.$$

Likewise, $N \le \frac{5}{4}$. Hence,

$$\begin{aligned} S+rT &= \tfrac{1}{20}\big((7r-3)M+(7-3r)N-(r+1)\big) \\ &\le \tfrac{1}{80}\big(5(7r-3)+5(7-3r)-4(r+1)\big) \;=\; \tfrac{1}{5}(1+r)\,. \end{aligned}$$

Thus, $F_5 \ge \dfrac{5}{r(1+r)}$ for $\frac{3}{7} \le r \le \frac{7}{3}$.

(a$_4$) By the AM–GM Inequality,

$$\begin{aligned} 1 \;=\; \left(\sum_{i=1}^{6} x_i\right)^2 &= (x_1+x_4)^2+(x_2+x_5)^2+(x_3+x_6)^2+2(S+T) \\ &\ge (x_1+x_4)(x_2+x_5)+(x_2+x_5)(x_3+x_6) \\ &\qquad +(x_3+x_6)(x_1+x_4)+2(S+T) \\ &= 3(S+T)\,. \end{aligned}$$

Hence, $F_6 \ge \dfrac{1}{S+T} \ge 3$ for $r=1$.

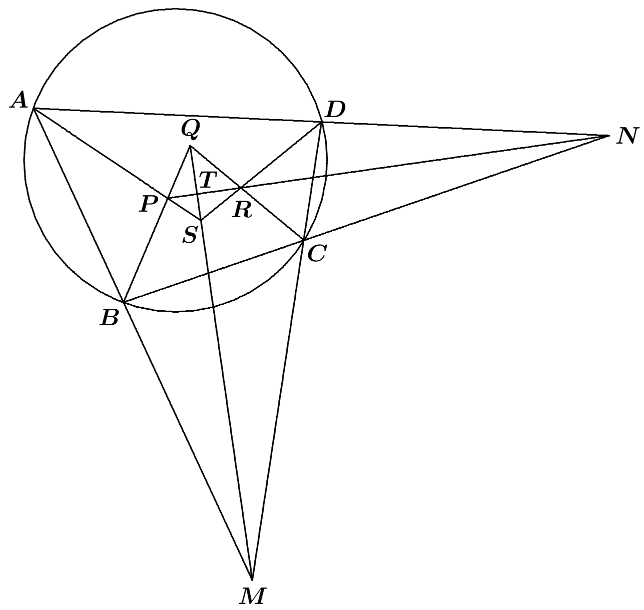
Also solved by the proposer. Part (a$_1$) *alone was solved by MIHÁLY BENCZE, Brasov, Romania and ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany. The starred part (part* (b)) *remain open.*

2978★. [2004 : 429, 432] *Proposed by Christopher J. Bradley, Bristol, UK.*

Let $ABCD$ be a cyclic quadrilateral. The internal bisectors of angles A and B meet at P. Points Q, R, S are similarly defined by a cyclic change of letters. It is easy to show that $PQRS$ is a cyclic quadrilateral. Suppose that the circles $ABCD$ and $PQRS$ have centres O and X, respectively. Let AC meet BD at E. Prove that O, E, and X are collinear. Prove also that $PR \perp QS$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA, with some additional detail taken from the solutions marked with an asterisk $(*)$ *in the list of solvers below.*

We assume that the quadrilateral $ABCD$ is convex. If $AB \parallel CD$ or $BC \parallel AD$, then $ABCD$ is an isosceles trapezoid, and O, E, and X lie on its axis of symmetry. Thus, we may suppose that the lines AB and CD meet at a point M and the lines BC and AD meet at a point N (see the diagram on the next page). We denote the internal angles at the vertices of the quadrilaterals $ABCD$ and $PQRS$ by the same symbols as the vertices themselves. We also denote the angles AMD and BNA simply by M and N, respectively.



To show that the quadrilateral $PQRS$ is cyclic, we check that the opposite angles Q and S are supplementary:

$$Q + S \;=\; \left(180^\circ - \tfrac{1}{2}C - \tfrac{1}{2}B\right) + \left(180^\circ - \tfrac{1}{2}A - \tfrac{1}{2}D\right) \;=\; 180^\circ\,,$$

since $A + B + C + D = 360^\circ$. This shows that $PQRS$ is cyclic whether or not $ABCD$ is cyclic.

Next, we note that Q and S both lie on the internal bisector of angle AMD, because Q is an excentre of $\triangle BMC$ while S is the incentre of $\triangle AMD$. (The internal and external bisectors of the three angles of a triangle meet by threes in four points, which are the incentre and the three excentres of the triangle.) Similarly, P and R both lie on the bisector of $\triangle BNA$.

Now,

$$\begin{aligned} \angle SDC \;=\; \tfrac{1}{2}D \;&=\; \tfrac{1}{2}\,(180^\circ - A - M) \;=\; \tfrac{1}{2}(C - M) \\ &=\; \angle QCD - \angle QMD \;=\; \angle MQC\,. \end{aligned}$$

It follows that opposite angles in the quadrilateral $QCDS$ are supplementary; hence, this quadrilateral is cyclic. Then $MC \cdot MD = MQ \cdot MS$. Similarly, $NC \cdot NB = NP \cdot NR$. Therefore, MN is the radical axis of the circles $ABCD$ and $PQRS$, which implies that $OX \perp MN$. Moreover, MN is the polar of E with respect to the circle $ABCD$, and consequently, $OE \perp MN$. Thus, O, X, and E are collinear.

Let T be the point at which PR intersects QS. Then

$$\begin{aligned}\angle MTN &= \angle AMT + A + \angle ANT \;=\; \tfrac{1}{2}(M+A) + \tfrac{1}{2}(A+N) \\ &= \tfrac{1}{2}(180^\circ - D) + \tfrac{1}{2}(180^\circ - B) \;=\; 90^\circ\,,\end{aligned}$$

since $B + D = 180^\circ$. Thus, $PR \perp QS$.

*Also solved by *MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; *VÁCLAV KONEČNÝ, Big Rapids, MI, USA; and DAVID MONK, Edinburgh, Scotland, UK.*

2979. [2004 : 430, 432] *Proposed by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.*

If $e_n = \left(1+\frac{1}{n}\right)^n$, find $\displaystyle\lim_{n\to\infty}\left(\frac{2n(e-e_n)}{e}\right)^n$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Since $n\ln\left(1+\frac{1}{n}\right) = 1 - \frac{1}{2n} + \frac{1}{3n^2} + O\left(\frac{1}{n^3}\right)$, we obtain

$$e_n \;=\; \left(1+\frac{1}{n}\right)^n \;=\; e^{n\ln\left(1+\frac{1}{n}\right)} \;=\; e^{1-\frac{1}{2n}+\frac{1}{3n^2}+O\left(\frac{1}{n^3}\right)}\,.$$

Then

$$\begin{aligned}\frac{e-e_n}{e} &= 1 - \frac{e_n}{e} \;=\; 1 - e^{-\frac{1}{2n}+\frac{1}{3n^2}+O\left(\frac{1}{n^3}\right)} \\ &= -\left(-\frac{1}{2n} + \frac{1}{3n^2} + O\left(\frac{1}{n^3}\right)\right) \\ &\qquad - \frac{1}{2}\left(-\frac{1}{2n} + \frac{1}{3n^2} + O\left(\frac{1}{n^3}\right)\right)^2 + O\left(\frac{1}{n^3}\right) \\ &= \frac{1}{2n} - \frac{11}{24n^2} + O\left(\frac{1}{n^3}\right)\,.\end{aligned}$$

Hence,

$$\begin{aligned}n\ln\frac{2n(e-e_n)}{e} &= n\ln\left(1 - \frac{11}{12n} + O\left(\frac{1}{n^2}\right)\right) \;=\; n\left(-\frac{11}{12n} + O\left(\frac{1}{n^2}\right)\right) \\ &= -\frac{11}{12} + O\left(\frac{1}{n}\right)\,,\end{aligned}$$

and therefore,

$$\lim_{n\to\infty}\left(\frac{2n(e-e_n)}{e}\right)^n \;=\; \lim_{n\to\infty} e^{n\ln\frac{2n(e-e_n)}{e}} \;=\; e^{-\frac{11}{12}}\,.$$

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; HENRY RICARDO, Medgar Evers College (CUNY), Brooklyn, NY, USA; JOEL SCHLOSBERG, Bayside, NY, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and the proposer.

2980. [2004 : 430, 432] *Proposed by Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Guang Dong Province, China.*

Let Γ be a semicircle with centre O on diameter AB. Let C be the mid-point of the semicircular arc $\widehat{AB}$. Let P be an arbitrary point on the semicircle different from both A and B.

Determine all points Q on the semicircle such that if the lines BP and AQ intersect at a point S, then C is the orthocentre of $\triangle SPQ$.

Solution by Michel Bataille, Rouen, France.

Since Q is defined such that QC is an altitude in $\triangle SPQ$, Q must be the point where the perpendicular to BP through C intersects Γ. Conversely, suppose first that P is on the arc $\widehat{AC}$ of Γ; then the perpendicular to BP from C meets the line segment AB and does not meet Γ, in which case there exists no suitable point Q. Suppose now that P is on the arc $\widehat{BC}$ of Γ (B and C excluded) and let Q and M be the points of intersection of the perpendicular to BP through C with Γ and BP, respectively. Being perpendicular to BP, AP and CM are parallel, implying that $\angle PCM = \angle APC = 45^\circ$. If N is the intersection point of PC and AQ, then $BQ \perp QN$ and $\angle BQC = 45^\circ$, implying that $\angle CQN = 45^\circ$. Since we also have $\angle QCN = \angle PCM = 45^\circ$, it follows that $\angle QNC = 90^\circ$; that is, $PC \perp AQ$. The lines PC and QC are two altitudes of the triangle PSQ, which makes C its orthocentre. Thus, for a point P situated on the arc $\widehat{BC}$, there is one suitable point Q, namely the intersection of Γ with the perpendicular to BP through C.

Also solved by JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and LI ZHOU, Polk Community College, Winter Haven, FL, USA. There was one incomplete solution.

Solvers provided two other characterizations for the point Q: For each point P on arc $\widehat{BC}$, Q is the unique point of arc $\widehat{AC}$ for which $\angle POQ = 90^\circ$; also, Q is the unique point of arc $\widehat{AC}$ for which QC is parallel to AP. Both these characterizations follow easily from the featured solution. Also, from the known angles at N, C, and M, we find that $\angle ASB$ $(= \angle NSM) = 45^\circ$. It follows that as P moves along arc $\widehat{BC}$ of Γ, the point S traces out an arc of the circle, say Ω, whose chord AB subtends an angle of 45°. This observation provides yet another characterization of the point Q: for each point P, define S to be the point where the line BP intersects Ω, in which case Q is the point of intersection of SA with Γ.

As a final observation, note that there is no reason to restrict Γ to a semicircle: Were P allowed to move about the entire circle ABC, all our characterizations of Q remain valid, and C continues to be the orthocentre of triangle PQS except when the triangle degenerates (when P coincides with A or when Q coincides with B). The only surprise is that when P and Q both lie on the half of circle ABC opposite point C, $\angle BSA = 135^\circ$; thus, in the more general problem, the locus of S consists of two arcs of a single circle on chord AB.

2981★. [2004 : 430, 433] *Proposed by Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Guang Dong Province, China.*

Find all pairs of positive integers a and b such that a divides b^2+b+1, and b divides a^2+a+1.

Solution by Charles R. Diminnie, Angelo State University, San Angelo, TX, USA.

Define a sequence $\{a_n\}$ by $a_0 = a_1 = 1$ and $a_{n+2} = 5a_{n+1} - a_n - 1$ for $n \geq 0$. It is easily shown by induction that $a_{n+1} > a_n \geq 1$ for all $n \geq 1$. We claim that $\{a_n\}$ satisfies the following recurrence relation:

$$a_{n-1}a_{n+1} \;=\; a_n^2 + a_n + 1 \tag{1}$$

for all $n \geq 1$. Since $a_0 = a_1 = 1$ and $a_2 = 3$, equation (1) is true for $n = 1$. Suppose that (1) holds for some $n \geq 1$. Then

$$\begin{aligned} a_n a_{n+2} &= a_n(5a_{n+1} - a_n - 1) \;=\; 5a_n a_{n+1} - a_n^2 - a_n \\ &= 5a_n a_{n+1} - a_{n-1}a_{n+1} + 1 \;=\; (5a_n - a_{n-1})a_{n+1} + 1 \\ &= (a_{n+1}+1)a_{n+1} + 1 \;=\; a_{n+1}^2 + a_{n+1} + 1\,, \end{aligned}$$

completing the induction.

As a result, a_n divides $a_{n+1}^2 + a_{n+1} + 1$ and a_{n+1} divides $a_n^2 + a_n + 1$ for all $n \geq 1$. Since a_0 clearly divides $a_1^2 + a_1 + 1$ and a_1 divides $a_0^2 + a_0 + 1$, we conclude that the set

$$\mathbb{S} \;=\; \{(a_n, a_{n+1}) \mid n \geq 0\} \cup \{(a_{n+1}, a_n) \mid n \geq 0\}$$

provides infinitely many solutions.

We now prove that $\mathbb{S}$ actually contains all the solutions.

Suppose that a and b are positive integers satisfying the given conditions. If $a = b$, then the given conditions imply that $a = b = 1$. Hence, $(a,b) = (a_0, a_1) \in \mathbb{S}$.

If $a \neq b$, we may assume without loss of generality that $a < b$. Note that a and b must be relatively prime, since if $k = \gcd(a,b)$, then k divides both a and a^2+a+1, which clearly implies that $k = 1$. From the given conditions, there exist positive integers x_1 and y_1 such that $a^2+a+1 = bx_1$ and $b^2+b+1 = ay_1$.

If $x_1 > a$, then $x_1 \geq a+1$, which, together with $b \geq a+1$, imply that $a^2+a+1 = bx_1 \geq (a+1)^2$, a contradiction (since $a > 0$).

If $x_1 = a$, then $a^2+a+1 = ba$, which implies that $a = 1$. Since b divides a^2+a+1, it follows that $b = 3$ and we have $(a,b) = (a_1, a_2) \in \mathbb{S}$.

Thus, we are only left with the case $x_1 < a$.

Note that

$$\begin{aligned} & b^2(x_1^2 + x_1 + 1) \\ &\quad= (a^2+a+1)^2 + b(a^2+a+1) + b^2 \\ &\quad= a^2(a+1)^2 + 2a(a+1) + ba(a+1) + b^2 + b + 1\,. \end{aligned} \tag{2}$$

Since a divides $b^2 + b + 1$ and a and b are relatively prime, we deduce from (2) that a divides $x_1^2 + x_1 + 1$. It follows that x_1 and a also satisfy the given conditions.

If $x_1 > 1$, then we may repeat this procedure to find a positive integer x_2 with $x_2 < x_1$ such that x_1 divides $x_2^2 + x_2 + 1$ and $x_1^2 + x_1 + 1 = ax_2$.

Continuing, we obtain positive integers x_1, x_2, x_3, ... with $x_{i+1} < x_i$, such that x_i divides $x_{i+1}^2 + x_{i+1} + 1$, and $x_i^2 + x_i + 1 = x_{i-1}x_{i+1}$ for all $i = 1, 2, 3, \ldots$ where $x_0 = a$. Since $a > x_1 > x_2 > \cdots \geq 1$, the process must terminate at $x_m = 1 = a_1$, for some m.

Since $x_{m-1} > x_m$ and x_{m-1} divides $x_m^2 + x_m + 1$, we see that $x_{m-1} = 3 = a_2$. Also,

$$a_1 x_{m-2} \;=\; x_m x_{m-2} \;=\; x_{m-1}^2 + x_{m-1} + 1 \;=\; a_2^2 + a_2 + 1 \;=\; a_1 a_2\,,$$

which implies that $x_{m-2} = a_3$. Continuing this argument, we then obtain $x_2 = a_{m-1}$, $x_1 = a_m$, and $a = a_{m+1}$.

Finally, from $a_m b = x_1 b = a^2 + a + 1 = a_{m+1}^2 + a_{m+1} + 1 = a_m a_{m+2}$, we conclude that $b = a_{m+2}$.

Therefore, $(a, b) = (a_{m+1}, a_{m+2}) \in \mathbb{S}$, and the proof is complete.

Also solved by MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; IRVINE ROBINSON, Math Challenge at Western, London, Ontario; MARIAN TETIVA, Bîrlad, Romania; and YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON. There was also one incorrect and two incomplete solutions.

Tetiva informed us that this problem has appeared before in the September, 1988 issue of Komal Magazine *(vol. 48, no. 6), and it was proposed by Ervin Fried of Budapest.*

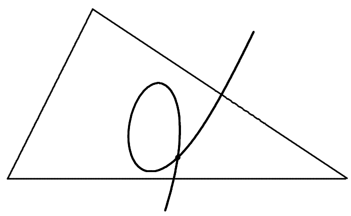
2982★. [2004 : 430, 433] *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

In a given triangle ABC, points D, E, F are taken on the sides BC, CA, AB, respectively, such that

$$BD : DC \;=\; CE : EA \;=\; AF : FB \;=\; \frac{1-\lambda}{\lambda}\,,$$

where λ is a constant. (If $0 < \lambda < 1$, then the points are interior to the sides; if $\lambda < 0$ or $\lambda > 1$, then the points are exterior to the sides; if $\lambda = 0$ or $\lambda = 1$, then the points are coincident with the vertices A, B, C.)

It is easy to see that the centroid of $\triangle DEF$ is a fixed point as λ varies. The curve in the figure is the locus of the circumcentre of $\triangle DEF$ as λ varies. Determine this curve.



Solution by Michel Bataille, Rouen, France.

Let $BC = a$, $CA = b$, $AB = c$, $EF = d$, $FD = e$, $DE = f$. From the hypotheses, we have $\overrightarrow{AF} = (1-\lambda)\overrightarrow{AB}$ and $\overrightarrow{AE} = \lambda\overrightarrow{AC}$, so that

$$\begin{aligned} d^2 &= \left(\overrightarrow{AF} - \overrightarrow{AE}\right)^2 = AF^2 + AE^2 - 2\overrightarrow{AF}\cdot\overrightarrow{AE} \\ &= (1-\lambda)^2c^2 + \lambda^2b^2 - 2\lambda(1-\lambda)\overrightarrow{AB}\cdot\overrightarrow{AC}\,. \end{aligned}$$

Since $2\overrightarrow{AB}\cdot\overrightarrow{AC} = b^2 + c^2 - a^2$, we easily obtain that

$$d^2 = \lambda^2\left(2b^2 + 2c^2 - a^2\right) + \lambda\left(a^2 - b^2 - 3c^2\right) + c^2\,.$$

Similar relations (with cyclic permutation of the letters a, b, c) hold for e^2 and f^2.

Now, the circumcentre, Ω, of $\triangle DEF$ has areal coordinates (α, β, γ) with respect to (D, E, F) with

$$\begin{aligned} \alpha &= \alpha(\lambda) = d^2\left(e^2 + f^2 - d^2\right)\,, \\ \beta &= \beta(\lambda) = e^2\left(f^2 + d^2 - e^2\right)\,, \\ \gamma &= \gamma(\lambda) = f^2\left(d^2 + e^2 - f^2\right)\,. \end{aligned}$$

Note that $\alpha+\beta+\gamma = 16[DEF]^2$, where $[X]$ denotes the area of polygon X.

From

$$D = \lambda B + (1-\lambda)C\,, \quad E = \lambda C + (1-\lambda)A\,, \quad F = \lambda A + (1-\lambda)B\,,$$

we deduce two results. First, the ratio $\dfrac{[DEF]}{[ABC]}$, being the modulus of the determinant

$$\begin{vmatrix} 0 & 1-\lambda & \lambda \\ \lambda & 0 & 1-\lambda \\ 1-\lambda & \lambda & 0 \end{vmatrix}\,,$$

is $3\lambda^2 - 3\lambda + 1$. Secondly, the areal coordinates of Ω with respect to (A, B, C) are

$$\left((1-\lambda)\beta + \lambda\gamma,\ \lambda\alpha + (1-\lambda)\gamma,\ (1-\lambda)\alpha + \lambda\beta\right)\,.$$

As a result, we have

$$16K^2\left(3\lambda^2 - 3\lambda + 1\right)^2\overrightarrow{a\Omega} = \left(\lambda\alpha + (1-\lambda)\gamma\right)\overrightarrow{AB} + \left((1-\lambda)\alpha + \lambda\beta\right)\overrightarrow{AC}\,,$$

where $K = [ABC]$.

In the system of oblique axes with origin A, x–axis AB (with unit direction vector $\overrightarrow{AB}$) and y–axis AC (with unit direction vector $\overrightarrow{AC}$), the coordinates of Ω are given by

$$x(\lambda) = \frac{\lambda\alpha(\lambda) + (1-\lambda)\gamma(\lambda)}{16K^2\left(3\lambda^2 - 3\lambda + 1\right)^2}\,, \quad y(\lambda) = \frac{(1-\lambda)\alpha(\lambda) + \lambda\beta(\lambda)}{16K^2\left(3\lambda^2 - 3\lambda + 1\right)^2}\,. \qquad (1)$$

A lengthy (but easy) calculation yields

$$\begin{aligned}
\alpha(\lambda) &= d^2\left(e^2+f^2-d^2\right) \\
&= \Big(\lambda^2\left(\lambda 2b^2+2c^2-a^2\right)+\lambda\left(a^2-b^2-3c^2\right)+c^2\Big) && (2)\\
&\quad \cdot\Big(\lambda^2\left(5a^2-b^2-c^2\right)+\lambda\left(3c^2-b^2-5a^2\right)+a^2+b^2-c^2\Big),\\
\beta(\lambda) &= d^2\left(f^2+d^2-e^2\right) \\
&= \Big(\lambda^2\left(\lambda 2c^2+2a^2-b^2\right)+\lambda\left(b^2-c^2-3a^2\right)+a^2\Big) && (3)\\
&\quad \cdot\Big(\lambda^2\left(5b^2-c^2-a^2\right)+\lambda\left(3a^2-c^2-5b^2\right)+b^2+c^2-a^2\Big),\\
\alpha(\lambda) &= d^2\left(e^2+f^2-d^2\right) \\
&= \Big(\lambda^2\left(\lambda 2a^2+2b^2-c^2\right)+\lambda\left(c^2-a^2-3b^2\right)+b^2\Big) && (4)\\
&\quad \cdot\Big(\lambda^2\left(5c^2-a^2-b^2\right)+\lambda\left(3b^2-a^2-5c^2\right)+c^2+a^2-b^2\Big),
\end{aligned}$$

and the desired locus of Ω is the parametrized curve (with parameter λ) defined by $x = x(\lambda)$, $y = y(\lambda)$ (given by (1) with $\alpha(\lambda)$, $\beta(\lambda)$, $\gamma(\lambda)$ given by (2), (3), (4), respectively).

In particular, if $\triangle ABC$ is isosceles, right angled at A with $b = c = 1$ (so that $a^2 = 2$ and $K = \frac{1}{2}$), we obtain that

$$\begin{aligned}
x(\lambda) &= \frac{3\lambda^5+3\lambda^4-14\lambda^3-6\lambda+1}{2\left(3\lambda^2-3\lambda+1\right)^2}\,,\\
y(\lambda) &= \frac{-3\lambda^5+18\lambda^4-28\lambda^3+20\lambda^2-7\lambda+1}{2\left(3\lambda^2-3\lambda+1\right)^2}\,.
\end{aligned}$$

Plotting this using MAPLE® gives a curve that looks like the proposer's.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria (who, however, did not check that his curve looked like the proposer's).

2983. [2004 : 430, 433] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let $a_1, a_2, \ldots, a_n < 1$ be non-negative real numbers satisfying

$$a = \sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}} \geq \frac{\sqrt{3}}{3}\,.$$

Prove that

$$\frac{a_1}{1-a_1^2}+\frac{a_2}{1-a_2^2}+\cdots+\frac{a_n}{1-a_n^2} \geq \frac{na}{1-a^2}\,.$$

Solution by Michel Bataille, Rouen, France.

The inequality to be proved can be rewritten as

$$\frac{a_1^2}{na^2}f(a_1) + \frac{a_2^2}{na^2}f(a_2) + \cdots + \frac{a_n^2}{na^2}f(a_n) \;\geq\; f(a)\,, \tag{1}$$

where f denotes the function defined on $[0,1)$ by $f(x) = \frac{1}{x(1-x^2)}$. The first two derivatives of f are given by

$$f'(x) \;=\; \frac{3x^2-1}{\big(x(1-x^2)\big)^2} \qquad \text{and} \qquad f''(x) \;=\; \frac{2(6x^4-3x^2+1)}{\big(x(1-x^2)\big)^3}\,,$$

showing that $f'(x) \geq 0$ for $x \in \big[\sqrt{3}/3, 1\big)$ and $f''(x) > 0$ for $x \in [0,1)$. From the latter, f is convex. Noticing that $\sum\limits_{k=1}^{n} \frac{a_k^2}{na^2} = 1$, we apply Jensen's Inequality to get

$$\frac{a_1^2}{na^2}f(a_1) \;+\; \frac{a_2^2}{na^2}f(a_2) \;+\; \cdots \;+\; \frac{a_n^2}{na^2}f(a_n) \;\geq\; f\left(\frac{a_1^3+a_2^3+\cdots+a_n^3}{na^2}\right).$$

From the Power Mean Inequality, we have

$$\left(\frac{a_1^3+a_2^3+\cdots+a_n^3}{n}\right)^{1/3} \;\geq\; \left(\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}\right)^{1/2},$$

and from the hypothesis on a, we obtain

$$\frac{a_1^3+a_2^3+\cdots+a_n^3}{na^2} \;\geq\; a \;\geq\; \frac{\sqrt{3}}{3}\,.$$

Since f is increasing on $\big[\sqrt{3}/3, 1\big)$, we thus have

$$f\left(\frac{a_1^3+a_2^3+\cdots+a_n^3}{na^2}\right) \;\geq\; f(a)\,.$$

Using this result above, we obtain (1).

Also solved by ARKADY ALT, San Jose, CA, USA; MIHÁLY BENCZE, Brasov, Romania; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; B.J. VENKATACHALA, Indian Institute of Science, Bangalore, India; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2984. [2004 : 431, 433] *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij(i+j)} \;=\; 2\sum_{n=1}^{\infty}\frac{1}{n^3}\,.$$

I. *Solution by Roger Zarnowski, Angelo State University, San Angelo, TX, USA.*

Denoting the expression on the left by S, we have

$$S = \sum_{i=1}^{\infty}\left(\frac{1}{i^2}\sum_{j=1}^{\infty}\left(\frac{1}{j} - \frac{1}{i+j}\right)\right) = \sum_{i=1}^{\infty}\left(\frac{1}{i^2}\sum_{j=1}^{i}\frac{1}{j}\right).$$

Hence,

$$\begin{aligned} 2S &= 2\left(1 + \sum_{i=2}^{\infty}\left(\frac{1}{i^2}\sum_{j=1}^{i}\frac{1}{j}\right)\right) = 2\left(1 + \sum_{i=2}^{\infty}\left(\frac{1}{i^2}\left(\frac{1}{i} + \sum_{j=1}^{i-1}\frac{1}{j}\right)\right)\right) \\ &= 2\left(1 + \sum_{i=2}^{\infty}\frac{1}{i^3} + \sum_{i=2}^{\infty}\left(\frac{1}{i^2}\sum_{j=1}^{i-1}\frac{1}{j}\right)\right) \\ &= 2\sum_{i=1}^{\infty}\frac{1}{i^3} + 2\sum_{i=2}^{\infty}\left(\frac{1}{i^2}\sum_{j=1}^{i-1}\frac{1}{j}\right). \qquad (1) \end{aligned}$$

Next, we set $k = i+j$ in the original double summation. Since k ranges from 2 to infinity and, for each fixed k, j ranges from 1 to $k-1$ in order for $i = k - j$ to remain positive, we have

$$\begin{aligned} S &= \sum_{k=2}^{\infty}\sum_{j=1}^{k-1}\frac{1}{(k-j)jk} = \sum_{k=2}^{\infty}\left(\frac{1}{k^2}\sum_{j=1}^{k-1}\left(\frac{1}{k-j} + \frac{1}{j}\right)\right) \\ &= \sum_{k=2}^{\infty}\left(\frac{2}{k^2}\sum_{j=1}^{k-1}\frac{1}{j}\right) = 2\sum_{i=2}^{\infty}\left(\frac{1}{i^2}\sum_{j=1}^{i-1}\frac{1}{j}\right). \qquad (2) \end{aligned}$$

Substituting (2) into (1), the result follows immediately.

II. *Composite of essentially the same solutions by Arkady Alt, San Jose, CA, USA; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Henry Ricardo, Medgar Evers College (CUNY), Brooklyn, NY, USA; and Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Let S denote the given double summation. Then

$$\begin{aligned} S &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij}\int_0^1 x^{i+j-1}\,dx = \int_0^1\left(\frac{1}{x}\left(\sum_{i=1}^{\infty}\frac{x^i}{i}\right)\left(\sum_{j=1}^{\infty}\frac{x^j}{j}\right)\right)dx \\ &= \int_0^1 \frac{\ln^2(1-x)}{x}\,dx. \qquad (1) \end{aligned}$$

Changing variable via $t = -\ln(1-x)$, we have $x = 1-e^{-t}$ and $dx = e^{-t}dt$; whence,

$$\int_0^1 \frac{\ln^2(1-x)}{x}\,dx = \int_0^\infty \frac{t^2e^{-t}}{1-e^{-t}}\,dt = \int_0^\infty t^2\left(\sum_{n=1}^\infty e^{-nt}\,dt\right)$$
$$= \sum_{n=1}^\infty \int_0^\infty t^2e^{-nt}\,dt\,. \qquad (2)$$

Applying the usual integration by parts twice, we find, after some routine computations involving improper integrals, that

$$\int_0^\infty t^2e^{-nt}\,dt = \frac{2}{n^3}\,. \qquad (3)$$

The desired result now follows from (1), (2), and (3).

Also solved by MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

Both Curtis and Janous pointed out that this problem is not new. Curtis cited the book The Red Book of Mathematical Problems *by K.S. Williams and K. Hardy, Dover, 1996; and Janous gave the reference* Mathematical Constants *by Steven R. Finch, Cambridge University Press, 2003.*

Alt obtained the following identity as a by-product:

$$\sum_{n=1}^\infty \frac{H_n}{n^2} = 2\sum_{n=1}^\infty \frac{1}{n^3}\,,$$

where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

The proposer gave the following comments: if we let

$$P(k) = \sum_{i_1=1}^\infty \cdots \sum_{i_k=1}^\infty \frac{1}{i_1i_2\cdots i_k(i_1+i_2+\cdots+i_k)}\,,$$

for $k = 1, 2, 3, \ldots$, *then clearly,* $P(1) = \zeta(2)$, *and the current problem shows that* $P(2) = 2\zeta(3)$, *where* $\zeta(s) = \sum\limits_{n=1}^\infty \frac{1}{n^s}$ *denotes the Riemann Zeta function. He offered the conjecture that* $\zeta(k) = k\zeta(k+1)$ *for all* $k \in \mathbb{N}$. [Ed: *Here,* $P(1)$ *is interpreted to be* $\sum\limits_{i_1=1}^\infty \frac{1}{i_1^2}$.]