SKOLIAD No. 88

Robert Bilinski

Please send your solutions to the problems in this issue by ***February 1, 2006***. A copy of MATHEMATICAL MAYHEM Vol. 4 will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

In this issue we feature the 2004 W.J. Blundon Mathematics Contest, for which I thank Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.

21ième Concours de Mathématiques W.J. Blundon
Commandité par la SMC et le département de mathématiques de l'Université Mémorial, 18 Février 2004

1. Un fermier dépense exactement $100\,\$$ pour acheter 100 animaux. Une vache coûte $10\,\$$, un mouton $3\,\$$ et un cochon 50 cents. Combien de chaque sorte a-t-il acheté ?

2. Montrer que si un nombre à trois chiffres est divisible par 3, alors la somme de ses chiffres est divisible par 3.

3. Considérer les points $A(1,0)$, $B(3,0)$, $C(3,5)$ et $D(1,4)$. Trouver l'équation de la droite passant par l'origine qui coupe le quadrilatère $ABCD$ en deux parties de même aire.

4. Trouver toutes les solutions réelles de l'équation $1+x+x^2+x^3 = x^4+x^5$.

5. Trouver l'aire exacte de l'octogone régulier formé en coupant des triangles rectangles isocèles congrus des coins d'un carré dont les côtés mesurent une unité.

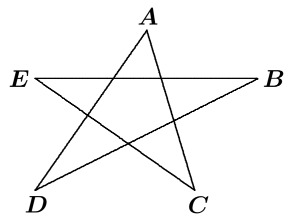
6. Si A, B et C sont les angles d'un triangle, montrer que

$$\cos C \;=\; \sin A \sin B - \cos A \cos B\,.$$

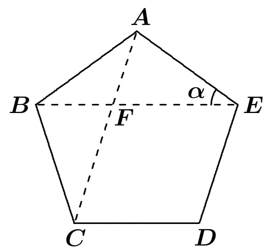
7. Si $a+b+c=0$ et $abc=4$, trouver $a^3+b^3+c^3$.

8. (a) Si $\log_{10} 2 = a$ et $\log_{10} 3 = b$, trouver $\log_5 12$.
(b) Résoudre $x^{\log_{10} x} = 100x$.

9. Dans la figure suivante, trouver la somme des angles A, B, C, D et E.



10. Soit $ABCDE$ le pentagone régulier de côté 1. La longueur de BE est τ, et l'angle FEA est α. Trouver τ et $\cos\alpha$.



The 21st W.J. Blundon Mathematics Contest
Sponsored by the CMS and the Mathematics Department at Memorial University, February 18, 2004

1. A farmer spent exactly \$100 to buy 100 animals. Cows cost \$10, sheep \$3 and pigs 50 cents each. How many of each did he buy?

2. Show that if a three-digit number is divisible by 3, then the sum of its digits is divisible by 3.

3. Consider the points $A(1,0)$, $B(3,0)$, $C(3,5)$, and $D(1,4)$. Find an equation of the line through the origin that divides the quadrilateral $ABCD$ into two parts of equal area.

4. Find all real solutions to the equation $1+x+x^2+x^3=x^4+x^5$.

5. Find the exact area of the regular octagon formed by cutting equal isosceles right triangles from the corners of a square with sides of length one unit.

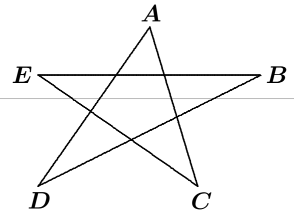
6. If A, B, and C are angles of a triangle, prove that

$$\cos C \;=\; \sin A \sin B - \cos A \cos B\,.$$

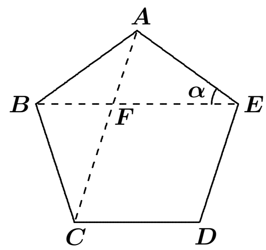
7. If $a+b+c=0$ and $abc=4$, find $a^3+b^3+c^3$.

8. (a) If $\log_{10} 2 = a$ and $\log_{10} 3 = b$, find $\log_5 12$.
(b) Solve $x^{\log_{10} x} = 100x$.

9. In the figure below, find the sum of the angles A, B, C, D, and E.



10. Let $ABCDE$ be a regular pentagon with each side of length 1. The length of BE is τ, and the angle FEA is α. Find τ and $\cos\alpha$.



Next we give the solutions to the South African Interprovincial Mathematics Olympiad 2004 [2005 : 65–68].

South African Interprovincial Mathematics Olympiad 2004

Team Paper: Juniors, 60 minutes allowed

1. ($*$) Five bags of rice are weighed two at a time, in all possible combinations. The ten weights are 72, 73, 76, 77, 79, 80, 81, 83, 84, and 87. What are the weights of the five bags?

Solution by the editor.

First note that no two bags can have the same weight, because all of the weights given in the problem are different. For example, if bag #1 had the same weight as bag #2, then the total weight of bag #1 and bag #3 would equal that of bag #2 and bag #3.

Let a, b, c, d, and e represent the weights of the bags from lightest to heaviest. Thus, $a < b < c < d < e$. The lightest pair of bags is a and b, and the next lightest pair is a and c. Therefore, $a + b = 72$ and $a + c = 73$. Similarly, we must have $d+e = 87$ and $c+e = 84$. Since $a+c = (a+b)+1$ and $d + e = (c + e) + 3$, we have $c = b + 1$ and $d = c + 3$.

The next smallest sum, after $a + b$ and $a + c$ is either $a + d$ or $b + c$. Therefore, $\{b+c, a+d\} = \{76, 77\}$. Suppose that $b+c = 76$. Together with $a + b = 72$ and $a + c = 73$, this yields $a = 34.5$, $b = 37.5$, and $c = 38.5$. Since $a + d = 77$, we also have $d = 42.5$, which contradicts $d = c + 3$. Therefore, we must have $b + c = 77$ and $a + d = 76$. A similar argument then yields $a = 34$, $b = 38$, $c = 39$, $d = 42$. From this, we easily obtain $e = 45$. A quick check shows that this is, indeed, a solution.

Hence, the five bags weigh 34, 38, 39, 42, and 45.

There were three solutions submitted which found the correct answer but assumed that the weights were all integers. It is true that the weights turn out to be integers, but this cannot be assumed in advance; for example, suppose the original ten weights were 69, 72, 73, 74, 75, 77, 78, 79, 82, *and* 87.

2. $(*)$ Delete 60 digits from the number $1\,2\,3\,4\,5\,6\,\ldots\,38\,39\,40$ in such a way as to make the resulting number as small as possible.

[*Ed:* Two different interpretations were given to the question. In the first solution below, the aim was the absolute lowest number possible and thus could start with 0, while in the second, the lowest number had to have 11 digits and thus start with a 1.]

I. *Solution par Marianella Ouellet, catégorie* 12*, Collège Montmorency, Laval, QC.*

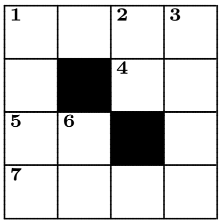
Tout d'abord, il y a 71 chiffres dans cette série. On sait alors que si on enlève 60 chiffres de la liste, on obtient un nombre de 11 chiffres de long. Ensuite, on doit garder tous les zéros car ceux-ci ne peuvent que rapetisser le résultat final. De plus, on doit éliminer tous les chiffres de 1 jusqu'à 30 en excluant évidemment les zéros, car malgré que l'on élimine des 1, on se rend compte que ceux-ci donneraient une valeur au zéros qui les suivent. Pour finir, il s'agit de prendre les plus petits chiffres dans l'ordre et on obtient un résultat de 00012333330.

II. *Solution by Geoffrey Siu, grade* 12*, London Central Secondary School, London, ON.*

There are $9 + 31 \times 2 = 71$ digits. We will have $71 - 60 = 11$ digits left. For the smallest number, we want the first digit to be as small as possible; the first digit must be 1. Then we want as many 0s as possible. We can have $10\,00x\,xxx\,xxx$ by using the 0s from 10, 20, and 30. But we cannot have another 0 (there are not enough numbers). Now, we use 1, 2, and 3 (from 31, 32, and 33), more 3s (from the numbers 34 through 39), then 0 (from 40), obtaining the smallest possible digit at each step. The smallest number is therefore 10001233330.

Also solved by Alexander Remorov, grade 9, Waterloo Collegiate Institute, Waterloo, ON. One incorrect solution was submitted.

3. Solve the crossnumber puzzle:



Across	**Down**
1. Cube of a prime	1. Square of a prime
4. Square	2. Three times cube root of 1 Across
5. Square	3. Square of a prime
7. Cube	6. Twice cube root of 7 Across

Solution by Geoffrey Siu, grade 12, London Central Secondary School, London, ON, modified by the editor.

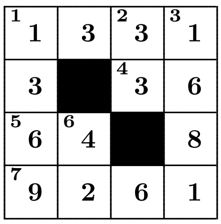
Since $10^3 = 1000$ and $22^3 = 10648 > 9999$, then #1 Across is the cube of one of 11, 13, 17, and 19. Since the third digit of #1 Across is the first digit of #2 Down, only $11^3 = 1331$ and $19^3 = 6859$ are acceptable for #1 Across. But from #4 Across and the fact that there are no squares starting with 7, we see that #1 Across is 1331. Then #2 Down is 33 and #4 Across is 36.

Now #3 Down is the square of a prime and starts with 16. The only possibility is $41^2 = 1681$.

Since #7 Across is a cube between 10^3 and 21^3 ending with 1, it is either $11^3 = 1331$ or $21^3 = 9261$. But #6 Down is an even number. Hence, we conclude that #7 Across is 9261. Thus, #6 Down is 42.

Then #5 Across is a 2-digit square ending in 4, which leaves us only one choice, namely 64.

Now #1 Down has the form 1?69 and it is the square of a prime. By trial and error, #1 Down must be $37^2 = 1369$.



Also solved by Bobby Xiao, grade 10, Walter Murray Collegiate Institute, Saskatoon, SK. One solver simply supplied the answer.

4. Find the sum of the digits of $10^{2004} - 2004$.

Solution by Alexander Remorov, grade 9, Waterloo Collegiate Institute, Waterloo, ON.

Note that 10^{2004} has 2005 digits. When we substract 2004 from 10^{2004}, we get a number with 2004 digits. This number has the form $999\ldots997996$. Because this number has 2004 digits, the number of nines in the beginning of the number is $2004 - 4 = 2000$. Thus, the sum of the digits of this number is $9 \times 2000 + 7 + 9 + 9 + 6 = 18031$.

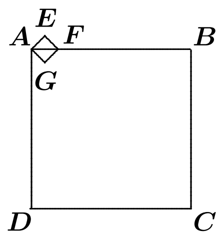
Also solved by Bobby Xiao, grade 10*, Walter Murray Collegiate Institute, Saskatoon, SK; and Geoffrey Siu, grade* 12*, London Central Secondary School, London, ON.*

5. An urn contains 100 balls of different colours: namely, 10 white, 10 black, 12 yellow, 14 blue, 24 green, and 30 red. What is the minimum number of balls that must be drawn from the urn without looking if you want to be certain that at least 15 of the balls drawn are of the same colour?

Identical solutions by Alexander Remorov, grade 9, Waterloo Collegiate Institute, Waterloo, ON; Bobby Xiao, grade 10*, Walter Murray Collegiate Institute, Saskatoon, SK; and Geoffrey Siu, grade* 12*, London Central Secondary School, London, ON.*

In the worst case, you draw all the white, black, yellow, and blue balls, 14 of the green balls, and 14 of the red balls, before you get 15 of any colour. Then you have drawn $10+10+12+14+14+14 = 74$ balls. There are then only red and green balls left, and any draw will give you 15 of one colour. Hence, 75 balls drawn guarantees 15 of the same colour.

6. In the diagram at right, square $ABCD$ has side 24 cm and square $AEFG$ has side 2 cm. What is the length of CE in cm?



I. *Similar solutions by Bobby Xiao, grade* 10*, Walter Murray Collegiate Institute, Saskatoon, SK; and Geoffrey Siu, grade* 12*, London Central Secondary School, London, ON.*

Extend EG to cross DC at I. Let H be the intersection of EG and AB. Since EG and AF are diagonals of the square $AEFG$ with F on AB, we have $EG \perp AB$. Since $AB \parallel DC$, we see that $EG \perp DC$ and $\triangle EIC$ has a right angle at I. We have $EH = AH = HF = HG = \sqrt{2}$ (since each side of $AEFG$ has length 2). Thus, we have $EI = EH + HI = 24 + \sqrt{2}$. Now $HBCI$ is a rectangle; whence, $CI = HB = AB - AH = 24 - \sqrt{2}$. Applying the Pythagorean Theorem to $\triangle EIC$, we have

$$CE = \sqrt{\left(24+\sqrt{2}\right)^2 + \left(24-\sqrt{2}\right)^2} = 34 \text{ cm}.$$

II. *Solution by Alexander Remorov, grade 9, Waterloo Collegiate Institute, Waterloo, ON.*

Applying the Pythagorean Theorem to $\triangle ADC$, we find that $AC = \sqrt{AD^2 + DC^2} = 24\sqrt{2}$ cm. Since AF is a diagonal of the square $AEFG$, we have $\angle FAG = 45^\circ$. Since AC is the diagonal of the square $ABCD$, we get $\angle BAC = 45^\circ$, and G lies on AC. Hence, $\triangle EAC$ has a right angle at A. Thus, by the Pythagorean Theorem in $\triangle EAC$, we have $EC = \sqrt{AC^2 + EA^2} = \sqrt{(24\sqrt{2})^2 + 2^2} = 34$ cm.

7. All the positive integers, starting with 1, are written in order, namely,

$$12345678910111213141516\ldots$$

Find the digit appearing in the 206 788$^{\text{th}}$ position.

Solution by Alexander Remorov, grade 9, Waterloo Collegiate Institute, Waterloo, ON.

There are 9 one-digit integers, 90 two-digit integers, 900 three-digit integers, 9000 four-digit integers. These numbers are the integers from 1 to 9999, and the total number of digits in them is $9 \times 1 + 90 \times 2 + 900 \times 3 + 9000 \times 4 = 38889$.

After that come the five-digit numbers. Between positions 38889 and 206788, there are $\left\lfloor \frac{206788 - 38889}{5} \right\rfloor = 33579$ complete five-digit numbers. The last one of them will be $9999 + 33579 = 43578$. Number 43579 will start in position 206785. Thus, 7 will be in position 206788.

There were two solutions submitted which had small errors in them.

8. How many times does the number 2 appear when the product

$$1002 \cdot 1003 \cdot 1004 \cdots 2004$$

is expanded into its prime factors?

Solution by the editor.

Since the given product can be expressed as $\frac{2004!}{1001!}$, it is sufficient to calculate the number of times that the prime 2 appears in 2004! and subtract the number of times it appears in 1001!.

Let n be any positive integer. We will now examine the number of times a prime p appears in $n!$. We note first that every multiple of p between 1 and n will contribute a factor of p. This generates $\left\lfloor \frac{n}{p} \right\rfloor$ factors (where $\lfloor x \rfloor$ is the greatest integer not exceeding x). However, multiples of p^2 also contribute a second factor of p. This generates $\left\lfloor \frac{n}{p^2} \right\rfloor$ more factors of p. Similarly, by considering the multiples of p^3 we get $\left\lfloor \frac{n}{p^3} \right\rfloor$ more factors of p. And so on. Thus the number of factors of the prime p in $n!$ is given by

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots + \left\lfloor \frac{n}{p^k} \right\rfloor + \cdots$$

(The above summation has only finitely many non-zero terms, since p^k will eventually exceed n.)

Using this result, we see that the number of 2s in 2004! is

$$\left\lfloor \frac{2004}{2} \right\rfloor + \left\lfloor \frac{2004}{4} \right\rfloor + \left\lfloor \frac{2004}{8} \right\rfloor + \cdots + \left\lfloor \frac{2004}{1024} \right\rfloor$$
$$= \; 1002 + 501 + 250 + 125 + 62 + 31 + 15 + 7 + 3 + 1 \; = \; 1997\,.$$

Similarly, the number of 2s in 1001! is

$$\left\lfloor \frac{1001}{2} \right\rfloor + \left\lfloor \frac{1001}{4} \right\rfloor + \left\lfloor \frac{1001}{8} \right\rfloor + \cdots + \left\lfloor \frac{1001}{512} \right\rfloor$$
$$= \; 500 + 250 + 125 + 62 + 31 + 15 + 7 + 3 + 1 \; = \; 994\,.$$

Thus, the number of 2s in the given product is simply $1997 - 994 = 1003$. (One could also observe that the last 8 terms in the two summations are the same, and could simply be cancelled, leaving a much simpler computation.)

Also solved by Geoffrey Siu, grade 12, London Central Secondary School, London, ON; and Bobby Xiao, grade 10, Walter Murray Collegiate Institute, Saskatoon, SK. One incorrect solution was submitted.

9. In the addition below, digits have been replaced by letters in a one-to-one fashion. Given that $D = 5$, work out the original numbers.

$$\begin{array}{cccccc} \mathrm{D} & \mathrm{O} & \mathrm{N} & \mathrm{A} & \mathrm{L} & \mathrm{D} \\ \mathrm{G} & \mathrm{E} & \mathrm{R} & \mathrm{A} & \mathrm{L} & \mathrm{D} \\ \hline \mathrm{R} & \mathrm{O} & \mathrm{B} & \mathrm{E} & \mathrm{R} & \mathrm{T} \end{array}$$

Solution by Bobby Xiao, grade 10, Walter Murray Collegiate Institute, Saskatoon, SK.

We are given that $D = 5$; since $5 + 5 = 10$, we have $T = 0$ with a carry of 1 into the tens column. Now R must be odd, since $2L + 1 = R$. From $O + E = O$ (with a possible carry), we have either $E = 0$ or $E = 9$, depending on whether there is a carry into that column. But $E \neq T = 0$, which means that $E = 9$, and this addition has a carry, as does $N + R = B$. Since $5 + G + 1 = R$ does not carry, we see that $G < 4$ and $R > 6$, since $G \neq 0$. But R is odd; hence, $(G, R) = (1, 7)$ or $(G, R) = (3, 9)$. But $R \neq E = 9$; this implies that $(G, R) = (1, 7)$. Since $2A \neq E = 9$, there must be a carry into the hundreds column. Hence, $2L + 1 = 17$, which gives us $L = 8$, and $2A + 1 = 9$ gives us $A = 4$. Only N, B, and O remain with 2, 3, and 6 left to assign. Now $N + 7 = 10 + B$ simplifies to $N = B + 3$ and $(N, B) = (6, 3)$ is the only fit among the remaining values. Hence, $O = 2$, since it is the only remaining value. The addition was thus:

$$\begin{array}{cccccc} 5 & 2 & 6 & 4 & 8 & 5 \\ 1 & 9 & 7 & 4 & 8 & 5 \\ \hline 7 & 2 & 3 & 9 & 7 & 0 \end{array}$$

Also solved by Alexander Remorov, grade 9, Waterloo Collegiate Institute, Waterloo, ON; and Geoffrey Siu, grade 12*, London Central Secondary School, London, ON.*

10. Consider a square having 16 cells each containing a plus sign or a minus sign. Suppose we change all the signs in a given row (or column), doing this several times until the number of minus signs is a minimum. A table that has the property that any such change does not decrease the number of minus signs is called a *minimal table*, and the number of minus signs in a minimal table is called the *characteristic* of the table. Find all possible values of the characteristic.

Solution by Bobby Xiao, grade 10*, Walter Murray Collegiate Institute, Saskatoon, SK.*

There cannot be more than 2 minus signs in any row or column of a minimal table because if there were 3 or 4 minus signs, then changing all the signs in that row or column would reduce the overall number of minus signs. There cannot be more than 8 minus signs in a minimal table, since there are only 4 rows and each can hold at most 2 minus signs. Thus, the largest possible characteristic is 8. Examples of minimal tables with characteristics from 0 to 8 are shown below:

+	+	+	+
+	+	+	+
+	+	+	+
+	+	+	+

-	+	+	+
+	+	+	+
+	+	+	+
+	+	+	+

-	+	+	+
+	-	+	+
+	+	+	+
+	+	+	+

-	+	+	+
+	-	+	+
+	+	-	+
+	+	+	+

-	+	+	+
+	-	+	+
+	+	-	+
+	+	+	-

-	+	+	+
-	-	+	+
+	+	-	+
+	+	+	-

-	+	+	+
-	-	+	+
+	-	-	+
+	+	+	-

-	+	+	+
-	-	+	+
+	-	-	+
+	+	-	-

-	+	+	-
-	-	+	+
+	-	-	+
+	+	-	-

Also solved by Alexander Remorov, grade 9, Waterloo Collegiate Institute, Waterloo, ON. One incorrect solution was submitted.

That brings us to the end of another Skoliad.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **premier mars 2006**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

M207. *Proposé par Edward J. Barbeau, Université de Toronto, Toronto, ON.*

A midi, Iphigénie quitte la maison pour faire une promenade à bicyclette, maintenant une moyenne de 20 km/h sur un sentier agréablement plat. Un peu plus tard, sa mère se rend compte qu'elle a oublié son pique-nique et envoie Electre le lui porter en vélo. Electre arrive à maintenir une vitesse de 30 km/h. Mais voilà que le ciel s'assombrit et que l'orage menace. Si bien qu'exactement une demi-heure après le départ de Electre, on envoie Oreste pour amener à ses deux soeurs de quoi se protéger contre la pluie. Oreste arrive à maintenir une vitesse de 40 km/h, si bien que les trois enfants, ayant suivi le même chemin, se rencontrent exactement au même moment. A quelle heure la rencontre a-t-elle eu lieu ?

M208. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

Déterminer tous les triangles distincts ayant un côté de longueur 6, les deux autres côtés étant des entiers et le périmètre étant numériquement égal à la surface.

M209. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Montrer que $3x^2 + 4y^2$ et $4x^2 + 3y^2$ ne peuvent être simultanément des carrés parfaits pour tous les entiers positifs x et y.

M210. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Une grille 9×9 est subdivisée en neuf sous-grilles de 3×3, appelées boîtes. Chaque ligne et chaque colonne de la grille 9×9 de même que chaque boîte 3×3 doivent contenir les chiffres de 1 à 9.

Compléter la grille ci-contre.

4				9			8	
			5			7		
6	2	3	7				4	
	4	9					7	3
7	6					9	2	
	3				2	4	1	5
		2			6			
	1			5				7

M211. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Deux cercles de rayon r sont tangents extérieurement. Ils sont aussi intérieurement tangents aux côtés d'un triangle rectangle de côtés 3, 4 et 5, l' hypoténuse du triangle étant tangente aux deux cercles. Déterminer r.

M212. *Proposé par Robert Bilinski, Collège Montmorency, Laval, QC.*

Dans le programme d'ordinateur Excel, les colonnes sont indiquées par des lettres. Les 26 premières colonnes comportent les lettres de A à Z. La $27^{\text{ième}}$ colonne est intitulée AA ; la $28^{\text{ième}}$ colonne est intitulée AB.

(a) Quel est le numéro de la colonne intitulée DXA ?

(b) Quel est l'indication de la 2005-ième colonne ?

. . . .

M207. *Proposed by Edward J. Barbeau, University of Toronto, Toronto, ON.*

At noon, Iphigenia set off on a bike ride from her home in Saskatoon, maintaining a leisurely pace of 20 km/h on the pleasantly level terrain. Later, her mother noticed that she had forgotten her lunch, and sent Electra off on her bike to meet her; Electra maintained a steady pace of 30 km/h. But then the sky darkened and the storm clouds gathered. So, exactly a half hour after Electra left, Orestes was sent off to meet the others with rain gear. Orestes rode at a steady pace of 40 km/h. All three followed the same route. As it happened, the three siblings met at exactly the same time. What time was that?

M208. *Proposed by K.R.S. Sastry, Bangalore, India.*

Determine all distinct triangles having one side of length 6, with the other two sides being integers, and the perimeter numerically equal to the area.

M209. *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that $3x^2 + 4y^2$ and $4x^2 + 3y^2$ cannot be simultaneously perfect squares for all x, y positive integers.

M210. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

A 9×9 grid is subdivided into nine 3×3 smaller grids, called boxes. Each row and each column of the 9×9 grid, and each 3×3 box, must contain each of the digits 1 through 9.

Complete the grid on the right.

4				9			8	
			5			7		
6	2	3	7				4	
	4	9					7	3
7	6					9	2	
	3				2	4	1	5
		2			6			
	1			5				7

M211. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Two circles of radius r are externally tangent. They are also internally tangent to the sides of a right triangle of sides 3, 4, and 5, with the hypotenuse of the triangle being tangent to both circles. Determine r.

M212. *Proposed by Robert Bilinski, Collège Montmorency, Laval, QC.*

In the computer program Excel, the columns are labelled with letters. The first 26 columns are labelled with the letters A to Z. The 27^{th} column is labelled AA; the 28^{th} column is labelled AB.

(a) What is the number of the column labelled DXA?

(b) What label appears on the 2005^{th} column?

Mayhem Solutions

M141. *Proposed by the Mayhem Staff.*

Create a list of perfect squares in which all of the digits are perfect squares (that is, 0, 1, 4, 9).

Solution by Doug Newman, Lancaster, CA, USA, modified by the editor.

We want to create a list of perfect squares, n^2, whose digits come from the set $\{0, 1, 4, 9\}$. To make the task manageable, the number of values, n, to be checked must be pared down. The following are strategies to do this:

1. Only check n if the least significant digit is 0, 1, 2, 3, 7, 8, or 9. These will give an n^2 with least significant digit of 0, 1, 4, or 9.

2. Limit the ranges of values of n such that the most significant digits of n^2 are some combination of 0, 1, 4, and/or 9. (For example, since $\sqrt{1.4} = 1.1832\ldots$ and $\sqrt{1.5} = 1.2247\ldots$, if the most significant digits of n are between $11832\ldots$ and $12247\ldots$, then the most significant digits of n^2 will be $14\ldots$. Hence, when checking 3 digit numbers, you need only check 119, 120, 121, and 122 for candidates.)

3. When a value for n is rejected, then subsequent values of n that have the same leading digits as the rejected number are also rejected. For example, since 83^2 does not have the desired property, all n starting with 83 are rejected. [*Ed:* This may lead to missing some values. For example, since $139^2 = 19321$, we would reject 139; thus, any number starting with $139\ldots$ would be rejected, but $1393^2 = 1940449$, which has the desired property.]

Checking $1 \leq n \leq 10\,000$, we get the following (clearly if n has the desired property, then $10n$ also has the property; such numbers are not listed below; that is, $2^2 = 4$ is listed but $20^2 = 400$ is not listed, but is nevertheless considered part of the list):

n	n^2	n	n^2	n	n^2	n	n^2
1	1	21	441	138	19044	1002	1004004
2	4	38	1444	201	40401	2001	4004001
3	9	97	9409	212	44944	7001	49014001
7	49	102	10404	701	491401	9997	99949991
12	144	107	11449	997	994009		

From this point on, it is clear that patterns are present for the values of n

- 1002, 1020, 1200
- 2001, 2010, 2100
- 7001, 7010
- 9700, 9970, 9997

These can be used to get larger values for n^2.

[*Ed:* The patterns above can be proved quite easily. If $n = 10^a + 2 \times 10^b$ with $a \neq b$, then $n^2 = 10^{2a} + 4 \times 10^{a+b} + 4 \times 10^{2b}$, which demonstrates that the first two patterns generalize. The third can be proved in a similar way. Note also that the solver did not find solutions in some of his ranges, but that does not mean that there are not others to be found. The problem remains open for other solutions.]

M142. *Proposed by Ali Feizmohammadi, University of Toronto, Toronto, ON.*

For every natural number n, define $S(n)$ to be the unique integer m (if it exists) which satisfies the equation

$$n = \lfloor m \rfloor + \left\lfloor \frac{m}{2!} \right\rfloor + \left\lfloor \frac{m}{3!} \right\rfloor + \cdots + \left\lfloor \frac{m}{k!} \right\rfloor + \ldots ,$$

where $\lfloor x \rfloor$ is the greatest integer not exceeding x.

(a) Find $S(3438)$.

(b) Does there exist a number k such that, for any non-negative integer n, at least one of $S(n+1)$, $S(n+2)$, $\ldots$, $S(n+k)$ exists?

Solution by the proposer.

(a) First note that any integer m can be uniquely written in the form $m = a_1 \cdot 1! + a_2 \cdot 2! + \cdots + a_k \cdot k! + \ldots$, where $0 \le a_j \le j$. Now notice that

$$\sum_{j=1}^{\infty} \left\lfloor \frac{m}{j!} \right\rfloor = \sum_{j=1}^{\infty} \left\lfloor \sum_{i=1}^{\infty} \frac{a_i \cdot i!}{j!} \right\rfloor ,$$

but

$$\left\lfloor \sum_{i=1}^{\infty} \frac{a_i \cdot i!}{j!} \right\rfloor = \left\lfloor \sum_{i=1}^{j-1} \frac{a_i \cdot i!}{j!} + \sum_{i=j}^{\infty} \frac{a_i \cdot i!}{j!} \right\rfloor .$$

Since

$$\sum_{i=1}^{j-1} \frac{a_i \cdot i!}{j!} \le \sum_{i=1}^{j-1} \frac{i \cdot i!}{j!} \le \frac{j! - 1}{j!} < 1 ,$$

and since the second sum on the right consists only of integers, the first sum on the right contributes nothing to the value of the right side; thus,

$$\left\lfloor \sum_{i=1}^{\infty} \frac{a_i \cdot i!}{j!} \right\rfloor = \left\lfloor \sum_{i=j}^{\infty} \frac{a_i \cdot i!}{j!} \right\rfloor = a_j + a_{j+1} \cdot (j+1) + a_{j+2} \cdot (j+2)(j+1) + \cdots$$

Now let us consider $m + \left\lfloor \frac{m}{2!} \right\rfloor + \cdots = 3438$. Obviously, $m < 7!$; hence, $m = a \cdot 6! + b \cdot 5! + c \cdot 4! + d \cdot 3! + e \cdot 2 + f$. By the above argument, we have to solve for:

$$a(1 + 6 + 6 \times 5 + \cdots + 6!) + b(1 + 5 + 5 \times 4 + \cdots + 5!) + \cdots + f = 3438 ,$$

which is equivalent to

$$1237a + 206b + 41c + 10d + 3e + f = 3438 .$$

One can easily verify that $a = 2$, $b = 4$, $c = 3$, $d = 1$, $e = 2$, $f = 1$ works. But the answer is unique (since $S(n)$ is a strictly increasing integer-valued function). Therefore, the solution is $m = 2003$.

(b) Obviously not! Let m be such that $S(m) = k! - 1$. Then

$$\lfloor k + 1 \rfloor + \cdots + \left\lfloor \frac{k+1}{j!} \right\rfloor - \lfloor k \rfloor - \left\lfloor \frac{k}{2!} \right\rfloor - \cdots \ge k .$$

Taking k to be sufficiently large we can get blocks of consecutive integers of arbitrary length, none of which belong to range of S.

Also solved by Doug Newman, Lancaster, CA, USA.

M143. *Proposed by the Mayhem Staff.*

Find the equation(s) of the line(s) through the point $(2, 5)$ for which the y–intercept is a prime number and the x–intercept is an integer.

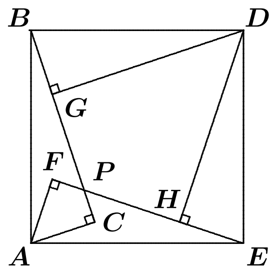
Solution by Luyan Zhong-Qiao, Columbia International College, Hamilton, ON.

Let k be the slope of a line through the point $(2, 5)$. The line then has equation $y - 5 = k(x - 2)$. To find the x-intercept, set $y = 0$, which yields $x = 2 - \frac{5}{k}$. Similarly, the y-intercept is given by $y = -2k + 5$. Since the intercepts are integers, we must have $k = \pm 1$ (with y-intercepts of 3 or 7), or $k = \pm\frac{1}{2}$ (with y-intercepts of 4 or 6), $k = \pm\frac{5}{2}$ (with y-intercepts of 0 or 10), or $k = \pm 5$ (with y-intercepts of -5 or 15). Since the only prime numbers among this list of y-intercepts are 3 and 7, we must have $k = \pm 1$, and the equations of the lines are $y - 5 = \pm(x - 2)$.

Also solved by Doug Newman, Lancaster, CA, USA.

M144. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

A square $ABDE$ is drawn on the hypotenuse AB of right triangle ABC so that C lies in the interior of the square. A directly similar right triangle BDG is drawn so that G lies in the interior of the square. Indirectly similar right triangles EDH and AEF are drawn so that H and F lie in the interior of the square. Let BC and EF intersect at P. Determine the area of quadrilateral $DGPH$ in terms of the legs CA and CB of the original right triangle.



Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

Since triangles ABC and AFE are congruent, triangles AFP and ACP are also congruent (since they are right-angled, $AF = AC$, and AP is a common hypotenuse). Then $\angle FAP = \angle PAC$. Since $\angle CAE = \angle FAB$, we have $\angle PAE = \angle PAB = 45^\circ$. Hence, P lies over AD.

Triangles GPD and CPA are similar, because, from $GD \parallel AC$, we have $\angle GDP = \angle PAC$.

Therefore,

$$\frac{GD}{AC} = \frac{GP}{PC} = \frac{PD}{AP}$$

$$\text{and} \quad \frac{BC}{AC} = \frac{GP}{PC} = \frac{PD}{AP} \qquad (1)$$

because $GD = BC$.

From (1) we have

$$GP = \frac{BC \cdot PC}{AC} \qquad (2)$$

Since $BG = AC$, and $GP + PC = BC - AC$, from (2) we have

$$\frac{BC \cdot PC}{AC} + PC = BC - AC$$

$$PC = \frac{(BC - AC)AC}{BC + AC} \tag{3}$$

It is easy to prove that triangles GDP and DPH are congruent. Therefore, from (2) and (3) we get

$$[DGHP] = GD \cdot GP = BC \cdot GP = \frac{(BC - AC) \cdot BC^2}{BC + AC}.$$

($[DGHP]$ denotes the area of the quadrilateral $DGHP$.)

Also solved by Doug Newman, Lancaster, CA, USA.

M145. *Proposé par Ovidiu-Gabriel Dinu, Balcesti-Valcea, Roumanie.*

Trouver tous les nombres naturels n pour lesquels n, $n+2$, $n+6$, $n+8$ et $n+14$ sont premiers.

Solution par Houda Anoun, LaBri, Bordeaux, France.

Soit n un entier naturel, et soit $S = \{n, n+2, n+6, n+8, n+14\}$. C'est facile de vérifier que

$$\exists k \in S : \; k \equiv 0 \pmod 5 .$$

En effet, on a ce qui suit :

$$\begin{aligned} n &\equiv n \pmod 5 \\ n+2 &\equiv n+2 \pmod 5 \\ n+6 &\equiv n+1 \pmod 5 \\ n+8 &\equiv n+3 \pmod 5 \\ n+14 &\equiv n+4 \pmod 5 . \end{aligned}$$

Parce que n, $n+1$, $n+2$, $n+3$, $n+4$ sont 5 nombres consécutifs, il existe forcément un qui est divisible par 5.

Soit k le nombre divisible par 5 qui fait partie de S. On a $\forall i \in S$, i est premier donc k est lui aussi premier. Ainsi, $k = 5$.

D'autre part on a ce qui suit : $n + 14 > n + 8 > n + 6 > 5$. Il reste donc deux possibilités à savoir $k = n = 5$ ou $k = n + 2 = 5$. Le premier cas permet d'engendrer la liste suivante $\{5, 7, 11, 13, 19\}$ dont les éléments sont tous des nombres premiers, en revanche, le second cas engendre $\{3, 5, 9, 11, 17\}$ qui ne constitue pas une liste de nombre premiers.

Seule la première solution est retenue alors. Donc $n = 5$.

En outre résolu par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne ; Roger He, étudiant de la catégorie 10, Prince of Wales Collegiate, St. John's, NL ; et Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentine.

Problem of the Month

Ian VanderBurgh, University of Waterloo

In last month's Problem of the Month, we looked for integer solutions to an equation. This month, we will look at a problem where we have to find all *real* solutions to a nasty-looking equation.

Problem (1991 Austrian Mathematical Olympiad, Round 2)
Determine all real numbers x which satisfy the equation

$$\frac{1}{x}+\frac{1}{x+2}-\frac{1}{x+4}-\frac{1}{x+6}-\frac{1}{x+8}-\frac{1}{x+10}+\frac{1}{x+12}+\frac{1}{x+14} \;=\; 0\,.$$

This equation does not look particularly easy to solve. (Can you see one solution by inspection, using some sort of symmetry?) One approach would be to multiply the equation by the product of the eight denominators on the left side. This would give, on the left side, eight terms with seven factors each, which could then be expanded and simplified to yield an enormous polynomial, which we could then attempt to factor. I would not recommend this approach.

Solution: First we will make a substitution to take advantage of the symmetry in the equation. The denominators form an arithmetic sequence x, $x+2$, $x+4$, $x+6$, $x+8$, $x+10$, $x+12$, $x+14$. Rewrite these in terms of the "middle" term (okay, there are eight terms, so there is no middle term, but we can use the average, $x+7$). By making the substitution $y=x+7$, these terms become $y-7$, $y-5$, $y-3$, $y-1$, $y+1$, $y+3$, $y+5$, $y+7$, and the equation becomes

$$\frac{1}{y-7}+\frac{1}{y-5}-\frac{1}{y-3}-\frac{1}{y-1}-\frac{1}{y+1}-\frac{1}{y+3}+\frac{1}{y+5}+\frac{1}{y+7} \;=\; 0\,.$$

This looks a bit more appealing and approachable. (You may have seen this idea in other places. For example, if you are trying to write an arithmetic sequence with five terms, it is sometimes more convenient to use $a-2d$, $a-d$, a, $a+d$, $a+2d$ instead of a, $a+d$, $a+2d$, $a+3d$, $a+4d$.)

At some point, we must start combining the fractions. But how shall we do this in a way that is as painless as possible? Well, we should try to use the symmetry. After fiddling around a bit, we might notice that

$$\frac{1}{y-c}+\frac{1}{y+c} \;=\; \frac{y+c+y-c}{(y-c)(y+c)} \;=\; \frac{2y}{y^2-c^2}\,.$$

Using this, we can regroup our terms and then simplify as follows:

$$\left(\frac{1}{y-7}+\frac{1}{y+7}\right)+\left(\frac{1}{y-5}+\frac{1}{y+5}\right)-\left(\frac{1}{y-3}+\frac{1}{y+3}\right)-\left(\frac{1}{y-1}+\frac{1}{y+1}\right)$$
$$=\;\frac{2y}{y^2-49}+\frac{2y}{y^2-25}-\frac{2y}{y^2-9}-\frac{2y}{y^2-1}\,.$$

Aha! All of the terms now have a common factor $2y$. Thus, either $y = 0$, or we can factor out $2y$ and obtain

$$\frac{1}{y^2 - 49} + \frac{1}{y^2 - 25} - \frac{1}{y^2 - 9} - \frac{1}{y^2 - 1} = 0 .$$

What about the solution $y = 0$? This solution makes perfect sense. If we substitute $y = 0$ into our first equation involving y, the terms cancel each other in $+/-$ pairs. This solution also corresponds to $x = -7$, which of course yields the same type of cancellation back in the original equation. (This is the "symmetrical" solution at which I hinted earlier.)

Where to now? Again, we should probably combine pairs of the four terms that are left. But which pairs? Well, if we combine a "$+$" term with a "$-$" term, then the resulting combination will have no y^2 term in the numerator—watch! First, we will rearrange:

$$\begin{aligned} \frac{1}{y^2 - 49} - \frac{1}{y^2 - 1} &= -\frac{1}{y^2 - 25} + \frac{1}{y^2 - 9}, \\ \frac{48}{(y^2 - 49)(y^2 - 1)} &= \frac{-16}{(y^2 - 9)(y^2 - 25)}, \\ 3(y^2 - 9)(y^2 - 25) &= -(y^2 - 49)(y^2 - 1), \\ 4y^4 - 152y^2 + 724 &= 0, \\ y^4 - 38y^2 + 181 &= 0 . \end{aligned}$$

We have ended up with a quartic polynomial which is actually a quadratic polynomial in y^2. Hence, we can solve it using the quadratic formula to obtain $y^2 = 19 \pm 6\sqrt{5}$. Since these values for y^2 are both positive, we can continue by taking square roots to obtain four possible values of y, namely $y = \pm\sqrt{19 + 6\sqrt{5}}$ and $y = \pm\sqrt{19 - 6\sqrt{5}}$.

Now, we have five values for y which solve the equation. (We must not forget about our earlier discovery that $y = 0$ is a solution.) We need to translate these back into values of x, remembering that $x = y - 7$. Then we get the values of x which are solutions:

$$-7, \quad -7 \pm \sqrt{19 + 6\sqrt{5}}, \quad -7 \pm \sqrt{19 - 6\sqrt{5}} .$$

Normally at this stage, I would recommend that we go back and check these solutions by substituting into the original equation... Maybe this is one place where a calculator would come in handy!

We have solved a tough equation here. Along the way, we have used a lot of different techniques—making a substitution motivated by symmetry, combining terms in thoughtful ways, solving a special quartic polynomial by noticing that it is really a quadratic polynomial in disguise...the list goes on!

Pólya's Paragon

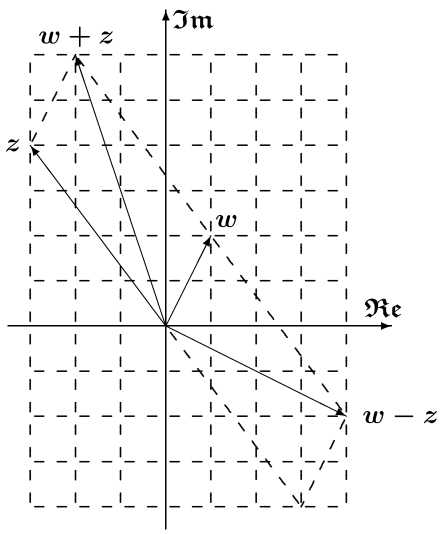
It Ain't So Complex (Part 2)

Shawn Godin

Last month, we started exploring the wonderful world of complex numbers. You were left with the assignment of looking at these numbers geometrically by assigning the complex number $z = a + bi$ to the point with co-ordinates (a, b).

Let us now try to find a geometric interpretation for addition and subtraction. Let $w = 1 + 2i$ and $z = -3 + 4i$. Then, as we saw last month,

$$\begin{aligned} w + z &= -2 + 6i \\ \text{and} \quad w - z &= 4 - 2i\,. \end{aligned}$$

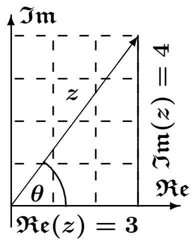


The complex numbers w, z, $w + z$, and $w - z$ are shown in the diagram to the right. Notice that $w + z$ is the diagonal of a parallelogram formed using w and z as sides.

Similarly, to interpret $w - z$, we just think of this subtraction as the addition $w + (-z)$, and note that $-z$ is represented by an arrow with the same length as z, but in the opposite direction.

(When you study *vectors*, you will recognize this geometric way of adding. In essence, we are treating the complex numbers as vectors.)

If you look at those arrows, it should become evident that we could define the arrow (complex number) by the co-ordinates of its tip (its real and imaginary parts) *or* by its length and direction. If we define the direction by the angle the arrow makes with the positive real axis, we have a second way to reference the complex number z.



For example, let $z = 3 + 4i$ (as shown in the diagram to the right). The length of z (called the *modulus* of z and denoted by $|z|$) is $\sqrt{3^2 + 4^2} = 5$. The angle θ (called the *argument* of z) satisfies $\tan\theta = \frac{4}{3}$.

Thus, we can express any complex number z in two forms: $z = a + bi$ (the *rectangular* form) and $z = r(\cos\theta + i\sin\theta)$ (the *polar* form).

Let us now go back and look at the multiplication example from last month:

$$(1 + 2i) \times (-3 + 4i) \;=\; -11 - 2i\,.$$

Putting each of these numbers in polar form, we get

$$\begin{aligned} w \;=\; 1 + 2i &\;=\; \sqrt{5}(\cos\theta_1 + i\sin\theta_1)\,, \\ z \;=\; -3 + 4i &\;=\; 5(\cos\theta_2 + i\sin\theta_2)\,, \\ w \times z \;=\; -11 - 2i &\;=\; 5\sqrt{5}(\cos\theta_3 + i\sin\theta_3)\,, \end{aligned}$$

where θ_1, θ_2, and θ_3 are the arguments of the three complex numbers.

Notice that the length of $w \times z$ is $|w \times z| = 5\sqrt{5}$, which is the product of $|w| = \sqrt{5}$ and $|z| = 5$. What about the argument? If you calculate the arguments θ_1, θ_2, θ_3, you will find that θ_3 is coterminal with $\theta_1 + \theta_2$. (Try it!) Thus, when we multiply complex numbers, we *add* their arguments. We will explore this idea and its implications next month.

For homework, you have a couple of tasks:

1. Work out a rule for dealing with division of complex numbers in polar form; that is, find how r_3 and θ_3 are related to r_1, r_2, θ_1 and θ_2, if

$$r_3(\cos\theta_3 + i\sin\theta_3) \;=\; [r_1(\cos\theta_1 + i\sin\theta_1)] \div [r_2(\cos\theta_2 + i\sin\theta_2)]\,.$$

2. Design another mathematical form for the expression $\cos\theta + i\sin\theta$ which suggests the rule for multiplication as a natural consequence. *Hint:* What function converts addition to multiplication?

3. Solve the equation $z^2 = i$.

Finally, from last month's homework, we were looking at generalizing division of complex numbers. If we look at $1 \div (c + di)$, we want a complex number, $x + yi$ such that $(x + yi)(c + di) = 1$. Following the method presented last month, you should get $x + yi = \dfrac{c - di}{c^2 + d^2}$. Thus, $\dfrac{1}{z} = \dfrac{\overline{z}}{|z|^2}$ (which you could see if you multiply the numerator and denominator by $\overline{z}$). This gives us a quicker method of computing division. Thus, the general division question gives us:

$$(a + bi) \div (c + di) \;=\; \frac{ac + bd}{c^2 + d^2} + \frac{-ad + bc}{c^2 + d^2}i\,.$$

Happy problem solving; see you next month.

THE OLYMPIAD CORNER

No. 248

R.E. Woodrow

We begin this number of the *Corner* with the problems of the 2002 Yugoslav Mathematical Olympiad. Thanks go to Bill Sands, Chair of the International Olympiad Committee of the Canadian Mathematical Society, for obtaining them for our use.

2002 YUGOSLAV MATHEMATICAL OLYMPIAD

1. Let a, b, and c be positive numbers, and let n and k be positive integers. Prove the inequality:

$$\frac{a^{n+k}}{b^n} + \frac{b^{n+k}}{c^n} + \frac{c^{n+k}}{a^n} \;\geq\; a^k + b^k + c^k\,.$$

2. Let (f_n) be a sequence defined by: $f_1 = f_2 = 1$ and $f_{n+2} = f_{n+1} + f_n$, for $n \geq 1$. Prove that the area of the triangle with side lengths $\sqrt{f_{2n+1}}$, $\sqrt{f_{2n+2}}$, and $\sqrt{f_{2n+3}}$ equals $\frac{1}{2}$.

3. Let $ABCD$ be a rhombus with $\angle BAD = 60^\circ$. Points S and R lie inside triangles ABD and DBC, respectively, such that $\angle SBR = \angle RDS = 60^\circ$. Prove that $SR^2 \geq AS \cdot CR$.

4. Does there exist a positive integer k such that the digits 3, 4, 5, and 6 do not appear in the decimal representation of the number $2002! \cdot k$?

Next we give the Yugoslav Qualification for IMO 2002, First and Second Rounds. Thanks again go to Bill Sands, Chair of the International Olympiad Committee of the Canadian Mathematical Society, for obtaining them for us.

YUGOSLAV QUALIFICATION FOR IMO 2002

First Round

1. A man standing at the point $(1,1)$ in the coordinate plane wants to find an object that lies at some point (α,β), where $\alpha \in \{1,\,2,\,\ldots,\,m\}$, and $\beta \in \{1,\,2,\,\ldots,\,n\}$. After finding the object, he will return to the starting point. Find the minimal worst case time needed for doing this job, if he does not know exactly at which point the object lies, and if he can move in any direction with velocity not greater than one.

2. Let p be the semiperimeter of the triangle ABC. Let the points E and F lie on the line AB such that $CE = CF = p$. Prove that the circumcircle of the triangle EFC and the circle that touches the side AB and the extension of the sides AC and BC of the triangle ABC meet in one point.

3. Let $\{x_n\}_{n\geq 2}$, be a sequence such that $x_2 = 1$, $x_3 = 1$, and, for $n \geq 3$,

$$(n+1)(n-2)x_{n+1} \;=\; n(n^2-n-1)x_n - (n-1)^3 x_{n-1}\,.$$

Prove that x_n is an integer if and only if n is a prime.

Second Round

1. What is the maximal value of the expression $a+b+c+abc$, if a, b, c are non-negative numbers such that $a^2+b^2+c^2+abc \leq 4$?

2. Let $ABCD$ be a convex quadrilateral with $\angle DAB = \angle ABC = \angle BCD$. Let O and H be the circumcentre and the orthocentre, respectively, of triangle ABC. Prove that the points O, H, and D are collinear.

3. For any positive integer n, let $f(n)$ denote the number of distinct possible choices for plus and minus signs such that $\pm 1 \pm 2 \pm 3 \pm \cdots \pm n = 0$ holds true. Prove that:

(a) $f(n) = 0$, for $n \equiv 1,\, 2 \pmod 4$;

(b) $f(n) \geq 2^{\frac{n}{2}-1}$, for $n \equiv 0,\, 3 \pmod 4$.

To round out the contests presented, we give the problems of the Midi Finale 2002 and the Maxi Finale 2002 of the 27ième Olympiade Mathématique Belge. Thanks again go to Bill Sands, Chair of the International Olympiad Committee of the Canadian Mathematical Society, for obtaining these problems.

VINGT-SEPTIÈME OLYMPIADE MATHÉMATIQUE BELGE

Midi Finale

Mercredi 24 avril 2002

1. Soit M le milieu de la base $[AB]$ d'un trapèze isocèle $ABCD$ et E le point d'intersection de MD et de AC. Si M est le centre du cercle circonscrit au trapèze et si $[AD]$ et $[DE]$ ont la même longueur, déterminer l'amplitude de l'angle $\angle DAB$.

2. La somme de quatre nombres réels est nulle ; la somme de leurs cubes est également nulle. Est-il vrai qu'alors deux de ces quatre nombres sont nécessairement opposés ?

3. (a) Existe-t-il quatre nombres naturels distincts non nuls tels que la somme de trois quelconques d'entre eux soit toujours un nombre premier ?

(b) Existe-t-il cinq nombres naturels distincts non nuls tels que la somme de trois quelconques d'entre eux soit toujours un nombre premier ?

4. Soit un rectangle $ABCD$, P un point situé sur un des côtés de ce rectangle, E et F les pieds des hauteurs abaissées de P sur les diagonales du rectangle. Démontrer que la somme $|PE| + |PF|$ reste constante lorsque P parcourt le périmètre de $ABCD$.

Maxi Finale

Mercredi 24 avril 2002

1. Soit la suite $(a_n)_{n \in N}$ telle que $a_n = n + \lfloor \sqrt{n} \rfloor$ pour tout $n \in \mathbb{N}$. Déterminer le plus petit entier naturel k pour lequel $a_k, a_{k+1}, \ldots, a_{k+2001}$ constituent une suite de 2002 entiers consécutifs. (Note : $\lfloor x \rfloor$ désigne le plus grand entier plus petit ou égal à x.)

2. (a) Dans le plan, soient $AB_1C_1D_1$ et $AB_2C_2D_2$ deux carrés ayant un sommet commun (les sommets sont cités dans le même sens). Si B, C et D sont respectivement les milieux des segments $[B_1B_2]$, $[C_1C_2]$ et $[D_1D_2]$, le quadrilatère $ABCD$ est-il aussi un carré ?

(b) Qu'en est-il si les sommets des carrés $AB_1C_1D_1$ et $AB_2C_2D_2$ sont cités en sens opposés ?

3. Voici une vue partielle d'une table de multiplication dans laquelle un tableau rectangulaire a été sélectionné.

1	2	3	4	5	6	$\cdots$
2	4	6	8	10	12	$\cdots$
3	6	9	12	15	18	$\cdots$
4	8	12	16	20	24	$\cdots$
5	10	15	20	25	30	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Pour chaque tableau dont l'élément du coin supérieur gauche et celui du coin inférieur droit sont respectivement 1 et 2002, on calcule la somme de tous ses éléments. Quelle est la plus petite des sommes ainsi obtenues ?

4. Trouver tous les nombres premiers a et b tels que $a^{a+1} + b^{b+1}$ est aussi un nombre premier.

Next we turn to solutions by our readers to the problems of the 32nd Austrian Mathematics Olympiad given in the October 2003 number of the *Corner* [2003 : 374–375].

1. Prove that
$$\frac{1}{25}\sum_{k=0}^{2001}\left\lfloor\frac{2^k}{25}\right\rfloor$$
is a natural number, where $\lfloor x\rfloor$ denotes the greatest whole number less than or equal to x.

Solved by Christopher J. Bradley, Bristol, UK; Pierre Bornsztein, Maisons-Laffitte, France; Mike Spivey, Samford University, Birmingham, AL, USA; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON. We give Spivey's write-up.

Since $\lfloor 2^k/25\rfloor$ is the quotient when 2^k is divided by 25, we have, by the Division Algorithm, $2^k = 25\lfloor 2^k/25\rfloor + r_k$, where r_k is the unique integer such that $r_k \equiv 2^k \pmod{25}$ and $0 \le r_k < 25$. Therefore,
$$25\sum_{k=0}^{2001}\left\lfloor\frac{2^k}{25}\right\rfloor = \sum_{k=0}^{2001}\left(2^k - r_k\right) = 2^{2002} - 1 - \sum_{k=0}^{2001} r_k\,. \tag{1}$$

If n is a natural number such that 2 and n are relatively prime, then, by Euler's Theorem, $2^{\phi(n)} \equiv 1 \pmod{n}$, where ϕ is Euler's totient function. In particular, taking $n = 25$ and noting that $\phi(25) = \phi\left(5^2\right) = 5^2 - 5 = 20$, we find that $2^{20} \equiv 1 \pmod{25}$. Then $2^{2000} = \left(2^{20}\right)^{100} \equiv 1 \pmod{25}$ and $2^{2001} = 2\cdot 2^{2000} \equiv 2 \pmod{25}$. Thus, $r_{2000} = 1$ and $r_{2001} = 2$.

As k runs through any 20 consecutive natural numbers, r_k runs through the 20 natural numbers that are less than 25 and relatively prime to 25. The sum of these 20 values for r_k is $\sum\limits_{i=1}^{25} i - (5+10+15+20+25) = 250$. Therefore,
$$\begin{aligned}\sum_{k=0}^{2001} r_k &= r_{2000} + r_{2001} + \sum_{k=0}^{1999} r_k = 1 + 2 + \frac{2000}{20}(250)\\ &\equiv 1 + 2 + 25000 \equiv 3 \pmod{625}\,.\end{aligned}$$

Applying Euler's Theorem again, we see that $2^{\phi(625)} \equiv 1 \pmod{625}$. Since $\phi(625) = \phi\left(5^4\right) = 5^4 - 5^3 = 500$, we have $2^{500} \equiv 1 \pmod{625}$. Then $2^{2000} = \left(2^{500}\right)^4 \equiv 1 \pmod{625}$, and hence $2^{2002} \equiv 4 \pmod{625}$.

Using our results in (1), we obtain
$$25\sum_{k=0}^{2001}\left\lfloor\frac{2^k}{25}\right\rfloor \equiv 4 - 1 - 3 = 0 \pmod{625}\,.$$

Then $\sum\limits_{k=0}^{2001}\left\lfloor\frac{2^k}{25}\right\rfloor$ is divisible by 25, which gives the desired result.

2. Determine all triplets of positive real numbers x, y, and z solving the system of equations

$$\begin{aligned} x+y+z &= 6\,, \\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z} &= 2-\frac{4}{xyz}\,. \end{aligned}$$

Solved by Michel Bataille, Rouen, France; Robert Bilinski, Collège Montmorency, Laval, QC; Pierre Bornsztein, Maisons-Laffitte, France; Christopher J. Bradley, Bristol, UK; Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA; Skotidas Sotirios, Karditso, Greece; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON. We give Zhao's write-up.

Suppose that x, y, z satisfy the first equation, $x+y+z=6$. Then, applying the AM–HM Inequality to $1/x$, $1/y$, $1/z$, we have

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \;\geq\; \frac{9}{x+y+z} \;=\; \frac{3}{2}\,.$$

By applying the GM–HM Inequality to the same set of variables, we get

$$\frac{1}{xyz} \;\geq\; \left(\frac{3}{x+y+z}\right)^3 \;=\; \frac{1}{8}\,.$$

It follows that

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \;\geq\; \frac{3}{2} \;=\; 2-4\left(\frac{1}{8}\right) \;\geq\; 2-\frac{4}{xyz}\,,$$

with equality if and only if $x=y=z=2$. Comparing the above inequality with the second equation in the system, we see that $(2,2,2)$ is the only solution to the system.

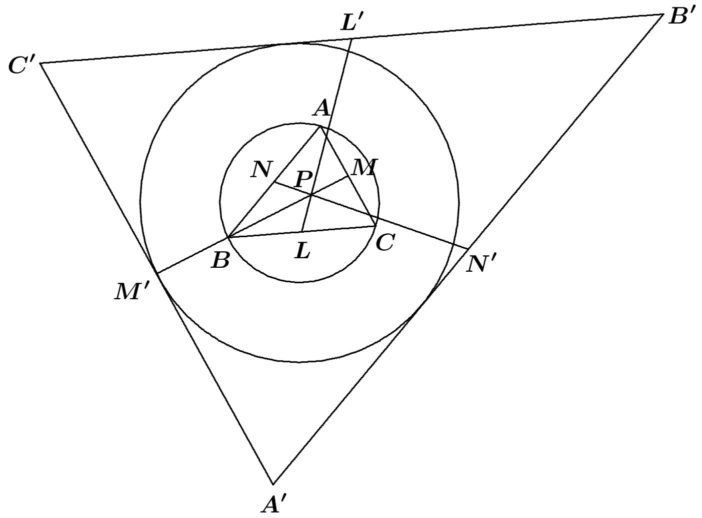
3. We are given a triangle ABC having $k(U,r)$ as its circumcircle. Next we construct the 'doubled' circle $k(U,2r)$ and its two tangents parallel to $c=AB$. Among them we select the one (and designate it c') for which C lies between c and c'. In a similar way we get the tangents a' and b'.

Let $A'B'C'$ be the triangle having its sides on a', b', and c', respectively. Prove: The lines joining the mid-points of corresponding sides of the two triangles intersect in a single point.

Solved by Christopher J. Bradley, Bristol, UK; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON. We give Bradley's write-up.

Triangles ABC and $A'B'C'$ have equal corresponding angles and are therefore similar. Corresponding sides of these triangles are parallel, which means that the triangles are homothetic. It follows that AA', BB', CC' are concurrent at a point P. Likewise, if L, M, N are the mid-points of ABC

and L', M', N' are the mid-points of $A'B'C'$, then (L, L'), (M, M'), and (N, N') are pairs of corresponding points; that is, LL', MM', NN' pass through P.



4. Determine all functions $f : \mathbb{R} \mapsto \mathbb{R}$, such that for all real numbers x and y the functional equation $f\big(f(x)^2 + f(y)\big) = x \cdot f(x) + y$ is satisfied.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON. We give Zhao's version.

Suppose f is a function $f : \mathbb{R} \to \mathbb{R}$ that satisfies

$$f\big(f(x)^2 + f(y)\big) \;=\; x \cdot f(x) + y\,. \tag{1}$$

By setting $x = 0$ in (1), we get

$$f\big(f(0)^2 + f(y)\big) \;=\; y\,.$$

Thus, f can take any value in $\mathbb{R}$; hence, it is surjective. Also, if $f(a) = f(b)$, then $a = f\big(f(0)^2 + f(a)\big) = f\big(f(0)^2 + f(b)\big) = b$. Therefore, f is injective. Thus, we see that f must be a bijective function.

Let h be a number such that $f(h) = 0$. By setting $x = y = h$ in (1), we get $f(0) = h$. Then, putting $x = y = 0$ in (1) gives $f(h^2 + h) = 0$. Since f is injective and $f(h^2 + h) = f(h)$, we must have $h^2 + h = h$; hence, $h = 0$. Thus, $f(0) = 0$.

By setting $x = 0$ in (1), we obtain

$$f\big(f(y)\big) \;=\; y\,. \tag{2}$$

And, by setting $y = 0$ in (1), we get

$$f\big(f(x)^2\big) \;=\; x \cdot f(x)\,. \tag{3}$$

Replacing x by $f(x)$ in (3) gives $f\Big(f\big(f(x)\big)^2\Big) \;=\; f(x) f\big(f(x)\big)$. Using (2), we simplify this to $f\left(x^2\right) = x \cdot f(x)$. By comparing this with (3), we get $f(f(x)^2) = f\left(x^2\right)$. Hence (since f is injective), $f(x)^2 = x^2$ for all x.

Suppose that there exist a and b in $\mathbb{R} \setminus \{0\}$ such that $f(a) = a$ and $f(b) = -b$. Setting $x = a$ and $y = b$ in (1) gives $f\left(a^2 - b\right) = a^2 + b$, which does not obey $f(x)^2 = x^2$. Therefore, either $f(x) = x$ for all x, or $f(x) = -x$ for all x.

5. Determine all whole numbers m for which all solutions of the equation $3x^3 - 3x^2 + m = 0$ are rational numbers.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Christopher J. Bradley, Bristol, UK; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON. We give Bradley's solution.

If $m = 0$, the given equation reduces to $x^3 - x^2 = 0$. The solutions of this equation are 0 (repeated) and 1, both of which are rational. Conversely, we will show that if all solutions of the equation are rational, then $m = 0$.

Suppose that all solutions are rational. Then there is a rational solution of the form $x = a/b$, where a and b are integers, $b > 0$, and a and b are relatively prime. Substituting such a solution into the equation, we get

$$3a^3 - 3a^2b + b^3m \;=\; 0\,.$$

Since b divides the last two terms, it follows that $b \mid 3a^3$. Then, since a and b are relatively prime, we must have $b = 1$ or $b = 3$. If $b = 3$, then $3a^3 - 9a^2 + 27m = 0$, or $a^3 - 3a^2 + 9m = 0$; hence, $3 \mid a$. This is a contradiction, since a and b are relatively prime. Therefore, $b = 1$. We conclude that $m = 3a^2(1 - a)$.

Setting $m = 3a^2(1 - a)$ in the given equation, the equation becomes $x^3 - x^2 - a^3 + a^2 = 0$, or

$$(x - a)\left(x^2 + ax + a^2 - x - a\right) \;=\; 0\,.$$

The discriminant of the quadratic factor is $(1 - a)(3a + 1)$. Recalling that a is an integer, we see that the discriminant is non-negative only when $a = 0$ or $a = 1$. In both of these cases, we have $m = 0$.

6. We are given a semicircle s with diameter AB. On s we choose any two points C and D such that $AC = CD$. The tangent at C intersects line BD in a point E. Line AE intersects s at point F.

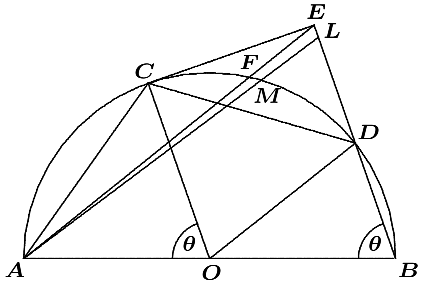
Prove that $CD < FD$.

Correction and solution by Christopher J. Bradley, Bristol, UK.

Presumably, the question should read: Prove that $CF < FD$ (and not $CD < FD$).

Take rectangular Cartesian coordinates with origin $O(0,0)$ at the centre of S and with $A(-1,0)$ and $B(1,0)$. Let θ be the angle subtended at O by AC. Then $0 < \theta < \frac{\pi}{2}$, and $C(-\cos\theta, \sin\theta)$ and $D(-\cos 2\theta, \sin 2\theta)$.

Letting M be the mid-point of the arc CD, we have $M\left(-\cos\left(\frac{3}{2}\theta\right), \sin\left(\frac{3}{2}\theta\right)\right)$.



Since $AC = CD$, the angle subtended at B by AD is equal to θ. Hence, BD is parallel to OC and has equation $x\sin\theta + y\cos\theta = \sin\theta$. The equation of the tangent at C is $-x\cos\theta + y\sin\theta = 1$. Letting E be the point at which these two lines meet, we find that the corrdinates of E are

$$x = 1 - (1+\cos\theta)\cos\theta \qquad \text{and} \qquad y = (1+\cos\theta)\sin\theta\,.$$

It follows that $\overrightarrow{BE} = \left[-(1+\cos\theta)\cos\theta, (1+\cos\theta)\sin\theta\right]$.

Now, $\overrightarrow{AM} = \left[1-\cos\left(\frac{3}{2}\theta\right), \sin\left(\frac{3}{2}\theta\right)\right]$. The slope of AM is therefore

$$\frac{\sin\left(\frac{3}{2}\theta\right)}{1-\cos\left(\frac{3}{2}\theta\right)} = \frac{2\sin\left(\frac{3}{4}\theta\right)\cos\left(\frac{3}{4}\theta\right)}{2\sin^2\left(\frac{3}{4}\theta\right)} = \cot\left(\tfrac{3}{4}\theta\right),$$

and the equation of AM is $y\sin\left(\frac{3}{4}\theta\right) = (x+1)\cos\left(\frac{3}{4}\theta\right)$.

Let L be the point at which AM meets BD. The coordinates of L are $x = -\dfrac{\cos\left(\frac{7}{4}\theta\right)}{\cos\left(\frac{1}{4}\theta\right)}$ and

$$\begin{aligned} y &= \frac{2\cos\left(\frac{3}{4}\theta\right)\sin\theta}{\cos\left(\frac{1}{4}\theta\right)} = \frac{2\left(4\cos^3\left(\frac{1}{4}\theta\right) - 3\cos\left(\frac{1}{4}\theta\right)\right)\sin\theta}{\cos\left(\frac{1}{4}\theta\right)} \\ &= 2\left(4\cos^2\left(\tfrac{1}{4}\theta\right) - 3\right)\sin\theta\,. \end{aligned}$$

We aim to show that $BL < BE$, which implies that $CF < FD$ (see the diagram). It is thus sufficient to prove that

$$2\left(4\cos^2\left(\tfrac{1}{4}\theta\right) - 3\right)\sin\theta < (1+\cos\theta)\sin\theta\,.$$

Since $\theta \neq 0$, this is true if and only if

$$8\cos^2\left(\tfrac{1}{4}\theta\right) < 7 + \cos\theta\,. \tag{1}$$

Now consider $f(\theta) = 7 + \cos\theta - 8\cos^2\left(\frac{1}{4}\theta\right)$. We have $f(0) = 0$, and

$$\begin{aligned} \frac{df}{d\theta} &= -\sin\theta + 4\cos\left(\tfrac{1}{4}\theta\right)\sin\left(\tfrac{1}{4}\theta\right) = -\sin\theta + 2\sin\left(\tfrac{1}{2}\theta\right) \\ &= -2\sin\left(\tfrac{1}{2}\theta\right)\cos\left(\tfrac{1}{2}\theta\right) + 2\sin\left(\tfrac{1}{2}\theta\right) = 2\sin\left(\tfrac{1}{2}\theta\right)\left(1-\cos\left(\tfrac{1}{2}\theta\right)\right), \end{aligned}$$

which is positive for $0 < \theta \leq \frac{\pi}{2}$. Therefore, (1) holds, and the inequality is proved.

Next we turn to the November 2003 *Corner* and solutions to problems of the 14th Nordic Mathematical Contest given [2003 : 435].

1. In how many ways can the number 2000 be written as a sum of three positive, not necessarily different, integers? (Sums like $1+2+3$ and $3+1+2$, etc. are considered to be the same.)

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Bristol, UK. We use Bataille's solution.

We are required to find the number of triples (n_1, n_2, n_3) of integers such that $1 \le n_1 \le n_2 \le n_3$ and $n_1 + n_2 + n_3 = 2000$. This number is 333333, the integer nearest to $2000^2/12$, as we shall now prove.

More generally, let n be any integer with $n \ge 3$, and let t_n be the number of triples (n_1, n_2, n_3) with $1 \le n_1 \le n_2 \le n_3$ and $n_1+n_2+n_3 = n$. For $n \ge 2$, we similarly denote by d_n the number of pairs (n_1, n_2) such that $1 \le n_1 \le n_2$ and $n_1 + n_2 = n$.

Clearly, $d_2 = d_3 = 1$, and, more generally, it is readily seen that $d_n = n/2$ if n is even and $d_n = (n-1)/2$ if n is odd. Returning to t_n, we find that $t_3 = t_4 = 1$ and $t_5 = 2$. For $n \ge 6$, we have $t_n = d_{n-1} + t_{n-3}$, because there are d_{n-1} triples for which $n_1 = 1$, and the set of suitable triples with $n_1 \ge 2$ is obviously in bijection with the set of triples $(n_1 - 1, n_2 - 1, n_3 - 1)$ summing to $n - 3$.

Now, assume that $n = 6k + 2$ for some positive integer k. Then

$$\begin{aligned} t_n &= (t_{6k+2} - t_{6k-1}) + (t_{6k-1} - t_{6k-4}) + \cdots \\ &\qquad\qquad + (t_{11} - t_8) + (t_8 - t_5) + t_5 \\ &= d_{6k+1} + d_{6k-2} + d_{6k-5} + \cdots + d_7 + t_5 \\ &= 3k + (3k-1) + (3k-3) + (3k-4) + \cdots + 6 + 5 + 3 + 2 \\ &= \frac{k(3k+3)}{2} + \frac{k(3k+1)}{2} \;=\; k(3k+2) \;=\; \frac{n^2}{12} - \frac{1}{3}, \end{aligned}$$

which is the integer nearest to $n^2/12$. For $n = 2000$ this gives $t_n = 333333$. (Examining the cases $n = 6k$, $6k+1$, $6k+3$, $6k+4$, $6k+5$ in the same way, it is easy to show that t_n is always the integer nearest to $n^2/12$.)

2. The persons P_1, P_2, ..., P_{n-1}, P_n sit around a table, in this order, and each one has a number of coins. At the start, P_1 has one coin more than P_2, P_2 has one coin more than P_3, etc., up to P_{n-1}, who has one coin more than P_n. Now P_1 gives one coin to P_2, who in turn gives two coins to P_3, etc., up to P_n, who gives n coins to P_1. The process continues in the same way: P_1 gives $n+1$ coins to P_2, P_2 gives $n+2$ coins to P_3, and so on. The transactions go on until someone has not enough coins to give away one coin more than he just received. At the moment when the process comes to an end in this manner, it turns out that there are two neighbours at the table one of whom has exactly five times as many coins as the other. Determine the number of persons and the number of coins circulating around the table.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

For $i = 1, 2, \ldots, n$ and $j \geq 0$, let $x_i(j)$ be the number of coins owned by P_i at the end of his j^{th} transaction. Since this transaction consists of P_i giving away one more coin than he has just received, we see that

$$x_i(j) \;=\; x_i(j-1) - 1\,. \qquad (1)$$

Let a be the number of coins that P_n has at the beginning. Then $x_i(0) = a + n - i$ for each i, and the total number of coins is

$$S \;=\; \sum_{i=1}^{n}(a+n-i) \;=\; an + \frac{n(n-1)}{2}\,.$$

It follows from equation (1) that the process will end when we have had $x_i(j) = 0$ for some i and j, and P_i is in the midst of his $(j+1)^{\text{st}}$ transaction (unable to give away one more coin than he has just received). In fact, since $x_1(0) > x_2(0) > \cdots > x_n(0) = a$, the process will end when P_n has to give to P_1 for the $(a+1)^{\text{th}}$ time. At this moment, using (1) again, we have $x_i(a+1) = n - i - 1$ for $i \leq n-1$, and P_n has $na + n - 1$ coins (having just received all of these from P_{n-1}).

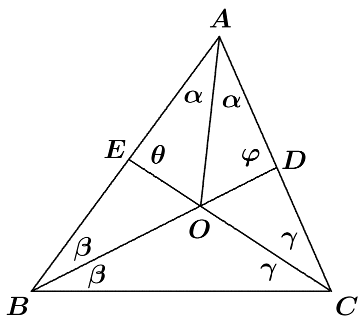
Since there are two neighbours at the table one of whom has exactly five times as many coins as the other, and since we may easily verify that $n - i \neq 5(n-i-1)$, we conclude that $na + n - 1 = 5x_1(a+1) = 5(n-2)$. This yields $9 = n(4-a)$. Thus, either $n = 3$ and $a = 1$, or $n = 9$ and $a = 3$. That is,

$$\left\{\begin{array}{l} n \;=\; 3 \\ S \;=\; 6 \end{array}\right. \quad\text{or}\quad \left\{\begin{array}{l} n \;=\; 9 \\ S \;=\; 63\,. \end{array}\right.$$

(Obviously, the argument is "if and only if".)

3. In the triangle ABC, the bisector of angle B meets AC at D, and the bisector of angle C meets AB at E. The bisectors intersect at O, and $OD = OE$. Prove that either $\triangle ABC$ is isosceles or $\angle BAC = 60^\circ$.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; Christopher J. Bradley, Bristol, UK; and Geoffrey A. Kandall, Hamden, CT, USA. We give the solution by Kandall.



The bisector of $\angle BAC$ is AO. Let α, β, γ, θ, and φ be angles as shown in the diagram. Since

$$\frac{\sin\theta}{\sin\alpha} \;=\; \frac{AO}{OE} \;=\; \frac{AO}{OD} \;=\; \frac{\sin\varphi}{\sin\alpha}\,,$$

we have $\sin\theta = \sin\varphi$. Thus, either $\theta = \varphi$, or θ and φ are supplementary.

(a) If $\theta = \varphi$, then $2\beta + \gamma = \beta + 2\gamma$. Consequently, $\beta = \gamma$ and $\triangle ABC$ is isosceles.

(b) If θ and φ are supplementary, then $(2\beta + \gamma) + (\beta + 2\gamma) = 180^\circ$. It follows that $\beta + \gamma = 60^\circ$. Hence, $\angle BAC = 60^\circ$.

4. The real-valued function f is defined for $0 \le x \le 1$, and satisfies $f(0) = 0$, $f(1) = 1$, and

$$\frac{1}{2} \le \frac{f(z) - f(y)}{f(y) - f(x)} \le 2\,,$$

for all $0 \le x < y < z \le 1$ with $z - y = y - x$. Prove that

$$\frac{1}{7} \le f\left(\frac{1}{3}\right) \le \frac{4}{7}\,.$$

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Bristol, UK. We give Bornsztein's solution.

Let $f\left(\frac{1}{3}\right) = a$ and $f\left(\frac{2}{3}\right) = b$. Setting $x = 0$, $y = \frac{1}{3}$, and $z = \frac{2}{3}$, we get

$$\frac{1}{2} \le \frac{b-a}{a} \le 2\,. \tag{1}$$

Setting $x = \frac{1}{3}$, $y = \frac{2}{3}$, and $z = 1$, we obtain

$$\frac{1}{2} \le \frac{1-b}{b-a} \le 2\,. \tag{2}$$

Suppose that $a < 0$. From (1), we deduce that $b - a < 0$; then $b < 0$, and hence $1 - b > 0$. Then, from (2), we get $b - a > 0$, a contradiction. Thus, $a > 0$.

Using (1), we deduce that $b - a > 0$. Then (1) can be rewritten as

$$b \le 3a \le 2b\,, \tag{3}$$

and (2) can be rewritten as

$$1 + 2a \le 3b \le 2 + a\,, \tag{4}$$

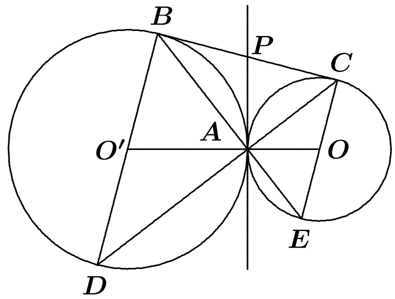
From the inequalities on the right in (3) and (4), we get $2 + a \ge 3b \ge \frac{9}{2}a$, and then $a \le \frac{4}{7}$. From the inequalities on the left, we get $1 + 2a \le 3b \le 9a$, and then $a \ge \frac{1}{7}$. Thus, $\frac{1}{7} \le a \le \frac{4}{7}$, and we are done.

Next we turn to solutions to problems of the Finnish High School Mathematics Competition 2000 given in [2003 : 436].

1. Two circles touch each other externally at A. A common tangent touches one circle at B and the other at C ($B \ne C$). The segments BD and CE are diameters of the circles. Prove that D, A, and C are collinear.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Bristol, UK. We first give Bradley's write-up.

Let the tangent at A meet BC at P. Then $PA = PB = PC$ (since these three segments are tangents from the external point P). Hence, P is the centre of the circle BAC. Thus, BC is a diameter of this circle, and $\angle BAC = 90^\circ$. Similarly, $\angle BAD = 90^\circ$, since BD is a diameter of the circle BAD. Therefore,



$$\begin{aligned} \angle DAC &= \angle BAD + \angle BAC \\ &= 90^\circ + 90^\circ = 180^\circ\,. \end{aligned}$$

Hence, D, A, C are collinear.

Next we give the presentation by Bataille.

Consider the homothety h with centre A which transforms the circle ACE into the circle ABD. Since the lines CE and BD are both perpendicular to the line BC, we have $CE \parallel BD$. In addition, CE and BD pass through the centres O and O' of ACE and ABD (respectively) and $h(O) = O'$; whence, the image of the line CE under h is the line BD. Therefore $h(C)$ is a point on the line BD and on the circle ABD. But $h(C)$ cannot be B (since C, A, B are not collinear); thus, we must have $h(C) = D$, and D, A, C are collinear.

2. Prove that the integer part of $(3+\sqrt{5})^n$ is odd for every positive integer n.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Bristol, UK. We give Bornsztein's solution.

Let n be a positive integer. Let

$$S_n = (3+\sqrt{5})^n + (3-\sqrt{5})^n\,.$$

Using the Binomial Theorem, we have

$$S_n = \sum_{k=0}^{n} \binom{n}{k} 3^{n-k} \left((\sqrt{5})^k + (-\sqrt{5})^k \right) = 2 \sum_{k \geq 0} \binom{n}{2k} 3^{n-2k} 5^k\,,$$

with the usual convention that $\binom{n}{p} = 0$ for $p > n$. Then S_n is an even integer.

Moreover, since $0 < 3 - \sqrt{5} < 1$, we deduce that $0 < (3-\sqrt{5})^n < 1$. It follows that $S_n - 1 < (3+\sqrt{5})^n < S_n$. Thus, $\left\lfloor (3+\sqrt{5})^n \right\rfloor = S_n - 1$, which is odd.

3. Determine all positive integers n such that $n! > \sqrt{n^n}$.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Christopher J. Bradley, Bristol, UK; and Yuming Chen and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give the solution by Chen and Wang.

Clearly, $n! = \sqrt{n^n}$ for $n = 1$ and $n = 2$. We show by induction that $n! > \sqrt{n^n}$, or equivalently, $(n!)^2 > n^n$, for all $n \geq 3$. This is clearly true for $n = 3$. Suppose $(n!)^2 > n^n$ for some $n \geq 3$. Then

$$\big((n+1)!\big)^2 \;=\; (n+1)^2(n!)^2 \;>\; (n+1)^2 n^n\,. \tag{1}$$

Since $(\frac{n+1}{n})^n = (1 + \frac{1}{n})^n < e \leq 3 < n+1$, we have $n^n > (n+1)^{n-1}$. Hence,

$$(n+1)^2 n^n \;>\; (n+1)^{n+1}\,. \tag{2}$$

From (1) and (2), it follows that $\big((n+1)!\big)^2 > (n+1)^{n+1}$, completing the induction.

4. There are seven points in the plane, no three of which are collinear. Every point is joined to every other by either a blue or a red line segment. Prove that there are at least four monochromatic triangles in the figure.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

More generally, we will prove that if $n \geq 3$ points are pairwise joined by a red or a blue line segment, then the minimum number of monochromatic triangles is $f(n)$, with:

$$f(2a{+}2) \;=\; \frac{a(a-1)(a+1)}{3} \qquad\text{and}\qquad f(2a{+}1) \;=\; \left\lceil \frac{a(2a+1)(a-2)}{6} \right\rceil,$$

(where $\lceil x \rceil$ denotes the least integer greater than or equal to x).

Let A_1, A_2, $\ldots$, A_n be the given points. For i, j, k pairwise distinct in $\{1, 2, \ldots, n\}$, let $p_i(j,k) = 2$ if the line segments A_iA_j and A_iA_k have the same color, and let $p_i(j,k) = -1$ otherwise. Note that if the triangle $A_iA_jA_k$ is monochromatic then $p_i(j,k) + p_j(k,i) + p_k(i,j) = 6$, and $p_i(j,k) + p_j(k,i) + p_k(i,j) = 0$ otherwise.

Let $p_i = \sum\limits_{\substack{1 \leq j < k \leq n \\ j,\,k \neq i}} p_i(j,k)$ and $p = \sum\limits_{i=1}^{n} p_i$. Let b and r denote the number of blue and red monochromatic triangles, respectively.

Since each pair of adjacent line segments belongs to only one triangle, we then have

$$p \;=\; 6(b+r)\,. \tag{1}$$

On the other hand, for a given i, let k be the number of red line segments with end-point A_i. Thus, there are $n-1-k$ blue segments with end-point A_i.

We then have:

$$
\begin{aligned}
p_i &= 2\binom{k}{2} + 2\binom{n-1-k}{2} - \binom{k}{1}\binom{n-1-k}{1} \\
&= n^2 - 3nk - 3n + 3k^2 + 3k + 2 \\
&= 3\left(k - \frac{n-1}{2}\right)^2 + \frac{(n-5)(n-1)}{4}\,. \qquad (2)
\end{aligned}
$$

Case (i). $n = 2a+2$ is even.

From (2), we see that $p_i \geq \big(3 + (n-5)(n-1)\big)/4 = a(a-1)$ for each i. Using (1), it follows that $b + r \geq \frac{1}{6}na(a-1) = a(a+1)(a-1)/3$. Since $f(2a+2)$ and $a(a+1)(a-1)/3$ are integers, we have

$$
f(2a+2) \;\geq\; \frac{a(a+1)(a-1)}{3}\,.
$$

To establish the opposite inequality, we consider the complete graph whose vertices are the $2a+2$ given points. Now colour the edge A_iA_j red if $i \equiv j \pmod 2$ and blue otherwise. It is easy to verify that there are exactly $2\binom{a+1}{3} = a(a+1)(a-1)/3$ monochromatic [*Ed:* actually red] triangles, which proves that $f(2a+2) \leq a(a+1)(a-1)/3$.

Thus, $f(2a+2) = a(a+1)(a-1)/3$, as claimed.

Case (ii). $n = 2a+1$ is odd.

From (2), we see that $p_i \geq (n-5)(n-1)/4 = a(a-2)$, for each i. Using (1), it follows that $b + r \geq \frac{1}{6}na(a-2) = a(2a+1)(a-2)/6$. Since $f(2a+1)$ is an integer, we deduce that

$$
f(2a+1) \;\geq\; \left\lceil \frac{a(2a+1)(a-2)}{6} \right\rceil\,.
$$

If a is even, this inequality is equivalent to

$$
f(4m+1) \;\geq\; \frac{2m(4m+1)(m-1)}{3}\,, \qquad (3)
$$

which is an integer. If a is odd, it is equivalent to

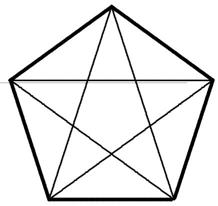
$$
f(4m+3) \;\geq\; \frac{(2m+1)(4m+3)(2m-1)}{6} + \frac{1}{2}\,, \qquad (4)
$$

since $(2m+1)(4m+3)(2m-1)/6$ is half an integer.

To establish the opposite inequality, we will again use graph theory terminology.

Case (a). $n = 4m+1$.

Our proof will be by induction on m. First, for $m = 1$, we consider the complete graph K_5 where the edges are coloured as in the diagram below right (the thick lines represent red edges, the thin lines blue).



This graph has no monochromatic triangles, from which we deduce that $f(5) = 0$. Note that in this graph, at each vertex there are the same number of red and blue edges. We call such a graph *balanced*. Now, assume that for a given $m \geq 2$ we have a $K_{4(m-1)+1}$ balanced edge-bicoloured complete graph. Now let us consider a complete graph K_{4m+1}, and colour the edges of that graph in red and blue as follows:

First, construct a balanced $K_{4(m-1)+1}$ for the subgraph whose vertices are A_5, A_6, $\ldots$, A_{4m+1} (which is possible by the induction hypothesis). Then, colour the edges of the subgraph having vertices A_1, A_2, A_3, A_4, A_5 as in the K_5 above. Last, for each $i \geq 6$, give the colour of A_5A_i to A_1A_i and A_2A_i, and the opposite one to A_3A_i and A_4A_i. It is easy to verify that the bicolouration of the edges of the K_{4m+1} gives a balanced graph.

It follows that for each $m \geq 1$, we may consider a balanced K_{4m+1}. For such a complete graph, for each i the number of red edges from A_i is equal to $k = 2m$, from which (using (2)) we deduce that $p_i = (n-5)(n-1)/4$. From (1), we then have $b + r = \frac{1}{6}n(n-5)(n-1)/4 = 2m(4m+1)(m-1)/3$. Thus, $f(4m+1) \leq 2m(4m+1)(m-1)/3$.

Combining this with (3), we are done.

Case (b). $n = 4m + 3$.

Let us colour the edges of the complete graph K_{4m+3} as follows: First colour the edges of the subgraph K_{4m+1} having vertices A_3, A_4, $\ldots$, A_{4m+3} to obtain a balanced graph as above.

Then, for each $i \geq 4$, give the colour of A_3A_i to A_1A_i and the opposite one to A_2A_i. Now give colour blue to A_1A_2, and red to A_1A_3 and A_2A_3. It is easy to verify that for each $i \neq 3$, we have exactly $k = 2m + 1$ red edges from A_i, so that (using (2)) $p_i = (n-5)(n-1)/4$. Moreover, the number of red edges from A_3 is $2m$, so that $p_3 = 3 + \big((n-5)(n-1)\big)/4$. It follows that $p = 3 + \big(n(n-5)(n-1)\big)/4$ and

$$b + r \;=\; \frac{(2m+1)(4m+3)(2m-1)}{6} + \frac{1}{2}\,.$$

Thus,

$$f(4m+3) \;\leq\; \frac{(2m+1)(4m+3)(2m-1)}{6} + \frac{1}{2}\,,$$

and we are done.

For the special case $n = 7$, we have $f(7) \geq \left\lceil \frac{3(2 \times 3 + 1)(3-2)}{6} \right\rceil = 4$, which solves the given problem.

That completes this number of the *Corner* and the solutions on file for the 2003 numbers. Send me your nice solutions and generalizations.

BOOK REVIEWS

John Grant McLoughlin

The Mathematial Century: The 30 Greatest Problems of the Last 100 Years
By Piergiorgio Odifreddi, translated by Arturo Sangalli, published by Princeton Press, 2004
ISBN 0-691-09294-X, cloth, xvi+204 pages, US$29.95.
Reviewed by **Peter A. Fillmore**, *Dalhousie University, Halifax, NS.*

The task that Odifreddi, a professor of mathematical logic at Turin and a frequent contributor to La Repubblica, has set himself in this book—to tell for the educated layman the story of twentieth century mathematics—is a daunting one. The edifice that is mathematics is now so vast that it is quite impossible for any single individual to be familiar in any detail with all of it. As Odifreddi says in his introduction, John von Neumann (1903–57) was perhaps the last of the "universal mathematicians", who could "dominate the entire landscape of the mathematics of their time".

But writing about mathematics is not the same as doing it. By selecting as his guides, firstly, David Hilbert's influential address to the 1900 International Mathematical Congress, "indicating probable directions for mathematics in the new century", together with the work of the Fields, Wolf, Turing, and relevant Nobel Prize winners, he has produced an admirable account that both professionals and laymen can learn from. Not only this, it is brief, readable, and informed by its placement in the intellectual and philosophical landscape.

The book opens with a superb foreword by Freeman Dyson, musing on the dual nature of science: Baconian ("all depends on keeping the eye steadily fixed on the facts of nature") and Cartesian ("I think, therefore I am"), neither of which "has the power to elucidate Nature's secrets by itself". Dyson finds the book too Cartesian for his taste, with not enough emphasis on the jokes and surprises that Nature springs on us (he cites the unexpected relevance of the imaginary unit in physics, the linearity of quantum mechanics, and quasicrystals).

At the heart of the book are two chapters titled Pure Mathematics and Applied Mathematics, the first consisting of discussions, chronologically arranged and each four or five pages in length, from fifteen problem areas, and the second from ten areas. These range from the problem of area and Lebesgue measure (1902) to the solution of Kepler's sphere-packing problem (1998), and from crystallography and Bieberbach's symmetry groups (1910) to the Jones invariant and knot theory (1984). These chapters are preceded by one on the evolution of the foundations of mathematics during the century, and followed by shorter chapters on the influence of the computer on mathematics (brief essays on Turing machines, artificial intelligence, chaos theory, computer-assisted proofs, fractals) and on open problems (odd perfect numbers, the Riemann Hypothesis, the Poincaré Conjecture, the $P = NP$ problem).

The book is not without its blemishes. It is a translation (from the Italian original of 2000) and there are verbal infelicities, mostly a result of literal translation: maximum [great] circle, second world conflict [war], moonlight [moonshine] conjecture. "Circle" sometimes means just the circumference and sometimes the whole disc. There are a few misprints (homotyopy; Kevin for Kelvin), and a description of which homotopy groups of spheres are infinite omits the obvious ones. But these are minor points; I enjoyed reading this book and recommend it highly.

Puzzles 101: A Puzzlemaster's Challenge
By Nobuyuki Yoshigahara, published by A.K. Peters, 2004
ISBN 1-56881-206-X, paperback, x+121 pages, US$14.00.
Reviewed by **John Grant McLoughlin**, *University of New Brunswick, Fredericton, NB.*

A gem indeed! This is a delightful collection of mathematical puzzles. The style is familiar enough. Numerous puzzles—101 to be precise—are presented sequentially much like a book of brainteasers. A book of brainteasers is common, but this collection is different. Intrinsically, the book captured me as a reader and solver. Perhaps its familiar appearance suggested to me a more-of-the-same sort of expectation. The surprise came while playing with the puzzles, many of which represent twists on familiar forms. I was pleasantly surprised by the quality of the puzzles that constitute the collection.

What makes this book different? Almost all of the puzzles contain some form of visual element. The visuals suggest many different forms of problems, ranging from grids with missing numbers to trisection puzzles to representations of graphs to examples of games with rules. The visuals invite a level of engagement in which one becomes curious to read the text accompanying the diagrams. The style proved to be effective. I found myself accepting many invitations implicitly placed before me.

The book offers puzzles for a wide range of levels. Solutions are provided. This book is highly recommended for armchair puzzlers who enjoy shapes and numbers. *Puzzles 101* will make a nice addition to the collections of recreational mathematicians. Furthermore, the book would be a wonderful one for starting off a budding mathematical enthusiast. The accessibility of the material and the cost are assets that merit mention.

This book introduced me to the work of the late Nobuyuki Yoshigahara. Recently Yoshigahara's name and work have become familiar to me through other contexts. He has been recognized world-wide as a masterful puzzler. His contributions will be missed.

A Right-to-Left Division Algorithm

N.H. Guersenzvaig and G.S. Krimker

Introduction.

In her remarkable paper, published in the April 2003 issue of Crux (see [1, pp. 170–173]), H. Havens (who was a young high school student when she wrote the paper) gives a criterion for divisibility by numbers ending in 9. She also shows that a similar algorithm works for numbers ending in 3 and asked if there are other similar criteria that work in bases different from 10.

Reviewing the existing literature, we found that Havens rediscovered part of a general criterion established by N.N. Vorobiov in [2, §4, Th. 24, p. 47] (it appears there is no English version available), and later on, although independently, by J. Whittaker in [3]. We will prove this fact describing precisely Vorobiov's algorithm. Furthermore, we will establish a dual result to that of Vorobiov which can be presented as a division algorithm that proceeds from right to left. In order to appropriately state these facts, we need some terminology.

Let F be a given function of $\mathbb{N}_0$ to $\mathbb{N}_0$, where $\mathbb{N}_0 = \{0, 1, 2, \ldots, \}$. As usual, the composite function $F^m : \mathbb{N}_0 \to \mathbb{N}_0$ is recursively defined by $F^0(n) = n$ and $F^k(n) = F\big(F^{k-1}(n)\big)$ for $k \geq 1$. Let d be any integer greater than 1. Partly following Vorobiov, we will say (cf. [2, §3, pp. 29–43]) that F is an *algorithm for divisibility by d* if the following three conditions are satisfied:

(A) $F(n)$ is completely determined by n for each $n \in \mathbb{N}_0$.

(B) For each integer $n \geq d$, there exists $k \in \mathbb{N}_0$ such that $F^k(n) < d$.

(C) For each $n \in \mathbb{N}_0$, we have $d \mid n$ if and only if $d \mid F(n)$.

Vorobiov's Algorithm.

For convenience, we will consider an arbitrary base $b \geq 2$. For each positive integer n, we denote by n_0 the rightmost digit of the representation of n in base b. Let us also suppose that d and b are relatively prime (that is, that d_0 and b are relatively prime). [This is equivalent to the existence of integers s and t such that $tb+sd = 1$.] Let t_0 be the least positive solution of the congruence $bx \equiv 1 \pmod{d}$ and let $r_d(n)$ be the remainder on division of n by d. Let $n = (n_k \ldots n_1 n_0)_{(b)} = n_k b^k + \cdots + n_1 b + n_0$ in base b. The algorithm of Vorobiov is based on the following fact:

We have $(n_k \ldots n_2 n_1)_{(b)} b + n_0 = \lfloor n/d \rfloor d + r_d(n)$ from the division algorithm. Hence,

$$(n_k \ldots n_2 n_1)_{(b)} + t_0 n_0 \;\equiv\; t_0 r_d(n) \pmod{d}\,.$$

The left side of this congruence constitutes the recursive part of the algorithm F_V defined by

$$F_V(n) = \begin{cases} \dfrac{n-n_0}{b} + t_0 n_0 & \text{if } n > (b-1)d, \\ r_d(n) & \text{otherwise.} \end{cases}$$

Now suppose $b = 10$ (which is the case considered by Vorobiov). From [1], we can define Havens' algorithms for cases $d_0 = 9$ and $d_0 = 3$ by

$$F_H(n) = \begin{cases} \dfrac{n-n_0}{10} + y(d_0)n_0 & \text{if } n \geq d, \\ n & \text{otherwise,} \end{cases}$$

where

$$y(d_0) = \begin{cases} (d+1)/10 & \text{if } d_0 = 9, \\ (3d+1)/10 & \text{if } d_0 = 3. \end{cases}$$

Then note that $F_H(n) = F_V(n)$ for $n > 9d$, because

$$t_0 = \begin{cases} y(d_0) & \text{if } d_0 \in \{3, 9\}, \\ (9d+1)/10 & \text{if } d_0 = 1, \\ (7d+1)/10 & \text{if } d_0 = 7. \end{cases}$$

The Dual to Vorobiov's Algorithm.

Let s_0 be the smallest positive solution of the congruence $dx \equiv 1 \pmod{b}$ and let $w_b(n)$ be the number of meaningful digits of the representation of n in base b (that is, $w_b(n) = \lfloor \log_b(n) \rfloor + 1$). Our algorithm is based on the following observation:

Writing $\lfloor n/d \rfloor$ in base b, say $\lfloor n/d \rfloor = (a_t \ldots a_2 a_1 a_0)_{(b)}$ with $a_t \neq 0$, we obtain $n - r_d(n) = (a_t \ldots a_2 a_1)_{(b)} bd + a_0 d$ from the division algorithm. Hence,

$$a_0 \equiv s_0(n - r_d(n)) \pmod{b} .$$

Let us now note that, if $d \mid n$, we have

$$\frac{n}{d} = (a_t \ldots a_1 a_0)_{(b)}, \quad a_0 = (s_0 n_0)_0, \quad \text{and} \quad \frac{n - a_0 d}{b} = (a_t \ldots a_2 a_1)_{(b)} d,$$

which tells us that in this case the representation of n/d in base b may be obtained in exactly $w_b(n/d) - 1 = \lfloor \log_b(n/d) \rfloor$ steps.

Consequently, we define $F : \mathbb{N}_0 \to \mathbb{N}_0$ by

$$F(n) = \begin{cases} \dfrac{n - (s_0 n_0)_0 d}{b} & \text{if } n > (b-1)d, \\ r_d(n) & \text{otherwise.} \end{cases}$$

In order to prove that F is an algorithm for divisibility by d, we will show first that F is well-defined; that is, that $F(n) \in \mathbb{N}_0$ for each $n \in \mathbb{N}_0$. From the definition of F, it will be enough to prove that $F(n) \in \mathbb{N}$ if $n > (b-1)d$. Indeed, in such a case we have

$$\begin{aligned} 0 &\le (b-1)d - (s_0 n_0)_0 d < bF(n) = n - (s_0 n_0)_0 d \\ &\equiv n_0 - n_0 s_0 d_0 \equiv 0 \pmod{b}. \end{aligned}$$

Condition (A) clearly follows from the definition of F. On the other hand, condition (B) holds because $F(n) < n$ for $n \ge d$, while condition (C) is satisfied because b and d are relatively prime.

As an example, we have $s_0 = 1$ if and only if $d_0 = 1$. Furthermore, it is easy to check that when $b = 10$ we have

$$s_0 = \begin{cases} d_0 & \text{if } d_0 \in \{1, 9\}, \\ 10 - d_0 & \text{if } d_0 \in \{3, 7\}. \end{cases}$$

Next we establish the connection between F and F_V. To this end we first prove that

$$1 + bd = s_0 d + t_0 b.$$

From $ds_0 \equiv 1 \pmod{b}$, we have $1 = s_0 d - kb$ for some positive integer k. Hence $k < d$, because otherwise we get the contradiction $s_0 d = 1 + kb > db$. Letting $k' = d - k$, we have $0 < d - k' < d$ and $1 + bd = s_0 d + k'b$; whence, k' is the smallest positive solution of $bx \equiv 1 \pmod{d}$; that is, $k' = t_0$.

Now we can prove that $F(n) \le F_V(n)$ for $n > (b-1)d$, where equality holds if and only if $n_0 = 0$. More precisely, for such $n \in \mathbb{N}$, we have

$$\begin{aligned} F(n) &= \frac{n - n_0}{b} + \frac{n_0 - (s_0 n_0)_0 d}{b} = F_V(n) + \frac{-t_0 n_0 b + n_0 - (s_0 n_0)_0 d}{b} \\ &= F_V(n) + \frac{-n_0\big(1 + (b - s_0)d\big) + n_0 - (s_0 n_0)_0 d}{b} \\ &= F_V(n) + \frac{\big(-n_0 b + s_0 n_0 - (s_0 n_0)_0\big) d}{b} \\ &= F_V(n) + \left\lfloor -\frac{(b - s_0) n_0}{b} \right\rfloor d. \end{aligned}$$

The following theorem establishes the main properties of F.

Theorem 1. Let b and d be arbitrary relatively prime integers greater than 1. Let n be any integer, $n \ge d$, and let $m = \min\{k \in \mathbb{N}_0 : F^k(n) \le (b-1)d\}$. For $k = 0, 1, \ldots, m$, we define

$$q_k = \begin{cases} \big(s_0 (F^k(n))_0\big)_0 & \text{if } 0 \le k < m, \\ \lfloor F^m(n)/d \rfloor & \text{if } k = m. \end{cases}$$

(a) i) The numbers $q = (q_m \ldots q_1 q_0)_{(b)} = q_m b^m + \cdots + q_1 b + q_0$ and $r = F^{m+1}(n)$ are the unique integers that satisfy

$$n = qd + rb^m \quad \text{and} \quad 0 \le r < d. \tag{1}$$

ii) $d \mid n \iff r = 0 \iff q = \frac{n}{d}$.

iii) If d is a prime number, then

$$r \equiv \pm r_d(n) \pmod{d} \iff b^m \equiv \pm 1 \pmod{d} .$$

(b) i) $w_b(n) - w_b(d) - 1 \le w_b(\lfloor n/d \rfloor) - 1 \le m$
$\le w_b(\lfloor n/d \rfloor) \le w_b(n) - w_b(d) + 1.$

ii) $m = w_b(\lfloor n/d \rfloor) - 1$ if and only if $F^k(n)/b^{m-k} \ge d$ for some $k \in \{0, \dots, m\}$.

iii) Suppose $b > 2$. Then

$$m = w_b(\lfloor n/d \rfloor) - 1 \iff F^m(n) \ge d \text{ or } a \ne b - 1,$$

where a denotes the left-most non-zero digit of the representation of $\lfloor n/d \rfloor$ in base b.

iv) $w_b(\lfloor n/d \rfloor) = w_b(n) - w_b(d) + 1$ if and only if we have either $m = w_b(n) - w_b(d) + 1$ or both $m = w_b(n) - w_b(d)$ and $F^k(n)/b^{m-k} \ge d$ for some $k \in \{0, 1, \dots, m\}$.

Proof: (a) i) In case $m = 0$, we have $n \le (b-1)d$. Thus, $q = q_0 = \lfloor n/d \rfloor$ and $r = F(n) = r_d(n)$; whence, $n = qd + rb^0$. Next suppose $m \ge 1$. It follows easily by induction that

$$n = (q_{k-1}b^{k-1} + \cdots + q_0)d + F^k(n)b^k , \quad k = 1, \dots, m. \tag{2}$$

Therefore, (1) holds because

$$\begin{aligned} r = F\big(F^m(n)\big) &= r_d\big(F^m(n)\big) = F^m(n) - q_m d \\ &= \frac{n - (q_m b^m + \cdots + q_0)d}{b^m} . \end{aligned}$$

In order to prove the uniqueness of q and r, we suppose that (1) is satisfied by integers q' and r' as well as. Because d and b are relatively prime and $(q - q')d = (r' - r)b^m$, it follows that $d \mid (r' - r)$. Hence, $r' = r$ (because $|r' - r| < d$) and $q = q'$.

ii) This follows at once from (1), because b and d are relatively prime.

iii) From (1), we get $r = r_d(t_0^m n)$, where t_0 denotes the smallest positive solution of $bx \equiv 1 \pmod{d}$. Then it is easy to see that our assertion is a particular case of the following result:

$$r \equiv \pm r_d(n) \pmod{d} \iff r(b^m \mp 1) \equiv 0 \pmod{d} .$$

(b) i) The extreme inequalities are well-known and follow at once from

$$b^{w_b(n)-w_b(d)-1} = \frac{b^{w_b(n)-1}}{b^{w_b(d)}} < \frac{n}{d} < \frac{b^{w_b(n)}}{b^{w_b(d)-1}} = b^{w_b(n)-w_b(d)+1} .$$

On the other hand, the central inequalities are immediate consequences of

$$(b-1)b^{m-1} \;<\; \left(\frac{F^{m-1}(n)}{d}\right) b^{m-1} \;\le\; \frac{n}{d} \;<\; (q_m \cdots q_0)_{(b)} + b^m\,. \qquad (3)$$

ii) ($\Rightarrow$) This is clear, since $w_b(\lfloor n/d \rfloor) = m+1$ implies that $n/d \ge b^m$.

($\Leftarrow$) Suppose $k \in \{0, 1, \ldots, m\}$ and $F^k(n)/b^{m-k} \ge d$. Hence, from (2), we get $n/d = (q_{k-1} \cdots q_1 q_0)_{(b)} + b^k F^k(n)/d \ge b^m$. Thus, $w_b(\lfloor n/d \rfloor) - 1 \ge m$. Then, from i), we have $w_b(\lfloor n/d \rfloor) - 1 = m$.

iii) ($\Rightarrow$) Suppose $w_b(\lfloor n/d \rfloor) = m+1$ and $F^m(n) < d$. Then $a \ne b-1$, because otherwise, from $a > 1$ and (3), we have the contradiction

$$\lfloor n/d \rfloor \;<\; (q_{m-1} \cdots q_1 q_0)_{(b)} + b^m\,.$$

($\Leftarrow$) Case $F^m(n) \ge d$ follows directly from ii). Then, in case $a \ne b-1$, we have $w_b(\lfloor n/d \rfloor) - 1 = m$, because otherwise from $a < b-1$ and (3), we obtain the contradiction

$$\lfloor n/d \rfloor \;>\; (b-1)b^{m-1}\,.$$

iv) ($\Leftarrow$) This follows directly from i) and ii).

($\Rightarrow$) Suppose that $w_b(\lfloor n/d \rfloor) = w_b(n) - w_b(d) + 1$ and that $m \ne w_b(n) - w_b(d) + 1$. Note that $m \ne w_b(n) - w_b(d) - 1$, because otherwise we have the contradiction $w_b(\lfloor n/d \rfloor) = m+2$. Thus, from i), we have $m = w_b(n) - w_b(d)$. From ii), it only remains to prove that $m = w_b(\lfloor n/d \rfloor)$. But this equality holds, because otherwise we have $m = w_b(\lfloor n/d \rfloor) - 1$; whence, we obtain the contradiction $w_b(\lfloor n/d \rfloor) = w_b(n) - w_b(d)$. ∎

Miscellaneous remarks.

(I) For each $k \in \{1, \ldots, m\}$, the numbers

$$Q_k = (q_{k-1} \cdots q_1 q_0)_{(b)} = q_{k-1} b^{k-1} + \cdots + q_1 b + q_0 \quad \text{and} \quad F_k = F^k(n)$$

are the unique integers which satisfy

$$n = Q_k d + F_k b^k \quad \text{and} \quad 0 \le Q_k < b^k\,.$$

(II) Suppose $n = (n_k \ldots n_1 n_0)_{(b)}$ in base b. The following very well-known facts follow at once from (1) and the fact that d and b are relatively prime:

$$b \equiv 1 \pmod{d} \implies \Big[d \mid n \iff d \mid (n_0 + n_1 + \cdots + n_k)\Big]$$

$$b \equiv -1 \pmod{d} \implies \Big[d \mid n \iff d \mid (n_0 - n_1 + \cdots + (-1)^k n_k)\Big]\,.$$

(III) Let $\overline{q} = b^m - (q_{m-1} \cdots q_0)_{(b)} = (\overline{q}_{m-1} \cdots \overline{q}_0)_{(b)}$, where the base b digits $\overline{q}_k$, for $k = 0, \ldots, m-1$, are defined by

$$\overline{q}_k = \begin{cases} 0 & \text{if } (q_k \ldots q_0)_{(b)} = 0, \\ b - q_k & \text{if } q_k \ne 0 \text{ and } (q_{k-1} \ldots q_0)_{(b)} = 0, \\ b - 1 - q_k & \text{otherwise.} \end{cases}$$

Now from (2) (with $k = m$), it follows that

$$m \;=\; w_b(\lfloor n/d \rfloor) - 1 \Longleftrightarrow F^m(n) \geq (\overline{q}/b^m)d\,.$$

(IV) Statement iv) of (b) is logically equivalent to $w_b(\lfloor n/d \rfloor) = w_b(n) - w_b(d)$ if and only if $m = w_b(n) - w_b(d) - 1$ or both $m = w_b(n) - w_b(d)$ and $F^k(n)/b^{m-k} < d$ for $k = 0, 1, \ldots, m$.

(V) In base b^k one simply replaces m by $[m/k]$, r by $\tau_d\left(rb^{\tau_k(m)}\right)$, and q by $q + \lfloor rb^{\tau_k(m)}/d \rfloor$.

(VI) Algorithm F and Theorem 1 also make sense, mutatis mutandis, for elements in an arbitrary euclidean domain (for example, for polynomials in one indeterminate with coefficients in an arbitrary field).

Finally, we give an example of each one of the four possible cases of the algorithm F.

Example 1. We consider $b = 10$, $n = 9999989$ and $d = 1001$. Thus, $s_0 = 1$. The algorithm proceeds from the right to the left as follows:

k	$F^k(n)$	q_k	$q_k d$
0	9999989	9	9009
	-9009		
1	999098	8	8008
	-8008		
2	99109	9	9009
	-9009		
3	9010	0	0
	-0		
$4 = m$	$901 = r$	$0989 = q$	

Then $9999989 = 989 \cdot 1001 + 901 \cdot 10^4$ and (since $m = w_{10}(n) - w_{10}(d) + 1$) we have $w_{10}(\lfloor n/d \rfloor) = w_{10}(n) - w_{10}(d) + 1 = 4$.

Example 2. Let $n = 101011$ and $d = 101$ in base $b = 2$. Obviously $s_0 = 1$.

k	$F^k(n)$	q_k	$q_k d$
0	101011	1	101
	-101		
1	10011	1	101
	-101		
2	0111	1	101
	-101		
$3 = m$	$001 = r$	$111 = q$	

Then $101011 = 111 \cdot 101 + 1 \cdot 10^3$ and (since $m = w_2(n) - w_2(d)$ and $n/2^m \geq d$) we have $w_2(\lfloor n/d \rfloor) = w_2(n) - w_2(d) + 1 = 4$.

Example 3. Let $n = 100111$ and $d = 101$ in base $b = 2$. Thus $s_0 = 1$.

k	$F^k(n)$	q_k	$q_k d$
0	100111	1	101
	-101		
1	10001	1	101
	-101		
2	0110	0	0
	-0		
$3 = m$	$011 = r$	$011 = q$	

Then $100111 = 011 \cdot 101 + 11 \cdot 10^3$ and $w_2(\lfloor n/d \rfloor) = w_2(n) - w_2(d) = 3$ (because $m = w_2(n) - w_2(d)$ and $F^k(n)/2^{m-k} < d$ for $k = 0, 1, 2, 3$).

Example 4. We consider $b = 10$, $n = 15354363$ and $d = 97$. Hence, $s_0 = 3$.

k	$F^k(n)$	q_k	$q_k d$
0	15354363	9	873
	-873		
1	1535349	7	679
	-679		
2	153467	1	97
	-97		
3	15337	1	97
	-97		
4	1524	2	194
	-194		
$5 = m$	$d \leq 133 \leq (b-1)d$	1	97
	97		
	$36 = r$	$121179 = q$	

It follows that $15354363 = 121179 \cdot 97 + 36 \cdot 10^5$, and we obtain $w_{10}(\lfloor n/d \rfloor) = w_{10}(n) - w_{10}(d) = 6$ (because $m = w_{10}(n) - w_{10}(d) - 1$).

Acknowledgement. The authors would like to express their appreciation to the referee for helpful suggestions.

References.

[1] Havens, H., Divisibility by numbers ending in nine, *Crux Mathematicorum with Mathematical Mayhem* 29 (2003), 170–173.

[2] Vorobiov, N.N., *Criterios de divisibilidad*, Mir, Moscow, 1975.

[3] Whittaker, J., A New Divisibility Criterion, *The College Math. Journal*, Vol. 16, No. 4 (1985), 268–276.

Natalio H. Guersenzvaig
Universidad CAECE
Buenos Aires
Argentina
nguersenz@fibertel.com.ar

Gustavo S. Krimker
Universidad CAECE
Buenos Aires
Argentina
guskrimker@fullzero.com.ar

PROBLEMS

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1er avril 2006**. *Une étoile* ($\star$) *après le numéro indique que le problème a été soumis sans solution.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

3064. *Proposé par J. Chris Fisher, Université de Regina, Regina, SK.*

(a) Soit A, B, C et D les sommets d'un quadrilatère. Soit P, Q, R et S les points milieu respectifs de AB, CD, AC et BD. Soit L le point d'intersection de AQ et DP et M celui de BR et CS. Montrer que le point milieu de BC est situé sur la droite LM si et seulement si $AD \parallel BC$.

(b) Soit A_0, A_1, A_2, A_3 et A_4 les sommets d'un pentagone non dégénéré. On appelle *médian* une droite joignant un sommet A_j soit au milieu du côté opposé $A_{j+2}A_{j-2}$ soit au milieu de la diagonale opposée $A_{j+1}A_{j-1}$ (les indices étant pris modulo 5). Montrer que le pentagone est affinement régulier si et seulement si les dix médians sont concourants.

Le résultat s'appuie sur un théorème de Zvonco Čerin, *Journal of Geometry*, **77** (2003), 22–34.

Note : On dit qu'un pentagone est *affinement régulier* s'il est l'image par une transformation linéaire d'un pentagone régulier ou d'un pentagramme régulier.

3065. *Proposé par Gabriel Dospinescu, Onesti, Roumanie.*

Soit ABC un triangle acutangle, et soit M un point intérieur de ce triangle. Montrer que

$$\frac{1}{MA}+\frac{1}{MB}+\frac{1}{MC} \;\geq\; 2\left(\frac{\sin\angle AMB}{AB}+\frac{\sin\angle BMC}{BC}+\frac{\sin\angle CMA}{CA}\right)\,.$$

3066. *Proposé par Gabriel Dospinescu, Onesti, Roumanie.*

Un entier $n > 2$ étant donné, soit $A_1, A_2, \ldots, A_n$ et $B_1, B_2, \ldots, B_n$ des sous-ensembles de $S = \{1, 2, \ldots, n\}$ avec la propriété que pour tout i, $j \in S$, les sous-ensembles A_i et B_j ont exactement un élément en commun. Montrer que s'il y a au moins deux sous-ensembles distincts parmi $B_1, B_2, \ldots, B_n$, alors il existe un sous-ensemble non vide $T \subseteq S$ ayant un nombre pair d'éléments en commun avec chacun des sous-ensembles $A_1, A_2, \ldots, A_n$.

3067. *Proposé par Gabriel Dospinescu, Onesti, Roumanie.*

Trouver toutes les fonctions $f : (0, \infty) \to (0, \infty)$ telles que

1. $f(f(f(x))) + 2x = f(3x)$ pour tout $x > 0$, et
2. $\lim\limits_{x \to \infty} \big(f(x) - x\big) = 0$.

3068. *Proposé par Vasile Cîrtoaje, Université de Ploiesti, Roumanie.*

Soit a, b et c trois nombres réels non négatifs dont deux au moins sont non nuls. Montrer que

$$\sqrt{1 + \frac{48a}{b+c}} + \sqrt{1 + \frac{48b}{c+a}} + \sqrt{1 + \frac{48c}{a+b}} \;\geq\; 15\,,$$

et déterminer quand il y a égalité.

3069. *Proposé par Cristinel Mortici, Valahia Université de Targoviste, Roumanie.*

Soit A et $B \in M_2(\mathbb{C})$ telles que $(AB)^2 = A^2B^2$. Montrer que

$$\det(I + AB - BA) \;=\; 1\,.$$

3070. *Proposé par Zhang Yun, High School attached to Xi′ An Jiao Tong University, Xi′ An City, Shan Xi, Chine.*

Soit x_1, x_2, …, x_n des nombres réels positifs tels que

$$x_1 + x_2 + \cdots + x_n \;\geq\; x_1x_2\cdots x_n\,.$$

Montrer que

$$(x_1x_2\cdots x_n)^{-1}\left(x_1^{n-1} + x_2^{n-1} + \cdots + x_n^{n-1}\right) \;\geq\; \sqrt[n-1]{n^{n-2}}\,,$$

et déterminer quand il y a égalité.

3071. *Proposé par Arkady Alt, San Jose, CA, USA.*

Soit $k > -1$ un nombre réel donné. Soit a, b et c des nombres réels non négatifs tels que $a + b + c = 1$ et $ab + bc + ca > 0$. Trouver

$$\min\left\{\frac{(1+ka)(1+kb)(1+kc)}{(1-a)(1-b)(1-c)}\right\}\,.$$

3072. *Proposé par Mohammed Aassila, Strasbourg, France.*

Trouver la plus petite constante k telle que, pour tous nombres réels positifs a, b et c, on ait

$$abc\left(a^{125} + b^{125} + c^{125}\right)^{16} \;\leq\; k\left(a^{2003} + b^{2003} + c^{2003}\right)\,.$$

3073. *Proposé par Zhang Yun, High School attached to Xi′ An Jiao Tong University, Xi′ An City, Shan Xi, Chine.*

Soit x, y et z trois nombres réels positifs. Montrer que

$$\frac{1}{x+y+z+1} - \frac{1}{(x+1)(y+1)(z+1)} \;\leq\; \frac{1}{8}\,,$$

et déterminer quand il y a égalité.

3074. *Proposé par Cristinel Mortici, Valahia Université de Targoviste, Roumanie.*

Soit $f\,:\,\left[0,\,\frac{1}{2005}\right] \to \mathbb{R}$ une fonction telle que

$$f\left(x+y^2\right) \;\geq\; y + f(x)\,,$$

pour tous les x et y réels avec $x \in \left[0,\,\frac{1}{2005}\right]$ et $x+y^2 \in \left[0,\,\frac{1}{2005}\right]$? Donner un exemple d'une telle function, ou montrer qu'il n'en existe pas.

3075. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Résoudre l'équation suivante, où x est un nombre réel positif :

$$(8^x-5^x)\,(7^x-2^x)\,(6^x-4^x)+(9^x-4^x)\,(8^x-3^x)\,(5^x-2^x) \;=\; 105^x\,.$$

. . . .

3064. *Proposed by J. Chris Fisher, University of Regina, Regina, SK.*

(a) Starting with four points A, B, C, D in the plane, no three of which are collinear, let P, Q, R, S be the mid-points of AB, CD, AC, BD, respectively. Let L be the point of intersection of AQ and DP, and let M be the point of intersection of BR and CS. Prove that the mid-point of BC lies on the line LM if and only if $AD \parallel BC$.

(b) Let A_0, A_1, A_2, A_3, and A_4 be the vertices of a non-degenerate pentagon. Define a *median* to be a line that joins a vertex A_j either to the mid-point of the opposite side $A_{j+2}A_{j-2}$ or to the mid-point of the opposite diagonal $A_{j+1}A_{j-1}$ (where subscripts are taken modulo 5). Prove that the pentagon is affinely regular if and only if the ten medians are concurrent.

The result is based on a theorem of Zvonco Čerin, *Journal of Geometry*, 77 (2003), 22–34.

Note: A pentagon is said to be *affinely regular* if it is the image under a linear transformation of a regular pentagon or a regular pentagram.

3065. *Proposed by Gabriel Dospinescu, Onesti, Romania.*

Let ABC be an acute-angled triangle, and let M be an interior point of the triangle. Prove that

$$\frac{1}{MA}+\frac{1}{MB}+\frac{1}{MC} \;\geq\; 2\left(\frac{\sin\angle AMB}{AB}+\frac{\sin\angle BMC}{BC}+\frac{\sin\angle CMA}{CA}\right)\,.$$

3066. *Proposed by Gabriel Dospinescu, Onesti, Romania.*

Given an integer $n > 2$, let A_1, A_2, $\dots$, A_n and B_1, B_2, $\dots$, B_n be subsets of $S = \{1, 2, \dots, n\}$ with the property that for all i, $j \in S$, the subsets A_i and B_j have exactly one element in common. Prove that, if there are at least two distinct subsets among B_1, B_2, $\dots$, B_n, then there exists a non-empty subset $T \subseteq S$ that has an even number of elements in common with each of the subsets A_1, A_2, $\dots$, A_n.

3067. *Proposed by Gabriel Dospinescu, Onesti, Romania.*

Find all functions $f : (0, \infty) \to (0, \infty)$ such that

1. $f\big(f(f(x))\big) + 2x = f(3x)$ for all $x > 0$, and
2. $\lim\limits_{x \to \infty} \big(f(x) - x\big) = 0$.

3068. *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let a, b, c be non-negative real numbers, no two of which are zero. Prove that

$$\sqrt{1 + \frac{48a}{b+c}} + \sqrt{1 + \frac{48b}{c+a}} + \sqrt{1 + \frac{48c}{a+b}} \;\geq\; 15\,,$$

and determine when there is equality.

3069. *Proposed by Cristinel Mortici, Valahia University of Targoviste, Romania.*

Let A, $B \in M_2(\mathbb{C})$ be such that $(AB)^2 = A^2B^2$. Prove that

$$\det(I + AB - BA) \;=\; 1\,.$$

3070. *Proposed by Zhang Yun, High School attached to Xi′ An Jiao Tong University, Xi′ An City, Shan Xi, China.*

Let x_1, x_2, $\dots$, x_n be positive real numbers such that

$$x_1 + x_2 + \cdots + x_n \;\geq\; x_1x_2\cdots x_n\,.$$

Prove that

$$(x_1x_2\cdots x_n)^{-1}\left(x_1^{n-1} + x_2^{n-1} + \cdots + x_n^{n-1}\right) \;\geq\; \sqrt[n-1]{n^{n-2}}\,,$$

and determine when there is equality.

3071. *Proposed by Arkady Alt, San Jose, CA, USA.*

Let $k > -1$ be a fixed real number. Let a, b, and c be non-negative real numbers such that $a + b + c = 1$ and $ab + bc + ca > 0$. Find

$$\min\left\{\frac{(1+ka)(1+kb)(1+kc)}{(1-a)(1-b)(1-c)}\right\}\,.$$

3072. *Proposed by Mohammed Aassila, Strasbourg, France.*

Find the smallest constant k such that, for any positive real numbers a, b, c, we have

$$abc\left(a^{125}+b^{125}+c^{125}\right)^{16} \;\le\; k\left(a^{2003}+b^{2003}+c^{2003}\right)\,.$$

3073. *Proposed by Zhang Yun, High School attached to Xi′ An Jiao Tong University, Xi′ An City, Shan Xi, China.*

Let x, y, z be positive real numbers. Prove that

$$\frac{1}{x+y+z+1}-\frac{1}{(x+1)(y+1)(z+1)} \;\le\; \frac{1}{8}\,,$$

and determine when there is equality.

3074. *Proposed by Cristinel Mortici, Valahia University of Targoviste, Romania.*

Let $f:\left[0,\frac{1}{2005}\right]\to\mathbb{R}$ be a function such that

$$f\left(x+y^2\right) \;\ge\; y+f(x)\,,$$

for all real x and y with $x\in\left[0,\frac{1}{2005}\right]$ and $x+y^2\in\left[0,\frac{1}{2005}\right]$? Give an example of such a function, or show that no such function exists.

3075. *Proposed by Mihály Bencze, Brasov, Romania.*

Solve the following equation where x is a positive real number:

$$\left(8^x-5^x\right)\left(7^x-2^x\right)\left(6^x-4^x\right)+\left(9^x-4^x\right)\left(8^x-3^x\right)\left(5^x-2^x\right) \;=\; 105^x\,.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2958. [2004 : 297, 300] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let ABC be a right triangle with right angle at A. Let $ACDE$, $BAFG$, and $CBIJ$ be squares mounted externally on the sides of $\triangle ABC$. Let H be the intersection of the interior angle bisector of angle A (extended) with the line segment EF, and let A' be the point outside the square $CBIJ$ such that $\triangle A'JI$ is directly congruent to $\triangle ABC$.

Show that $A'DHG$ is a cyclic quadrilateral.

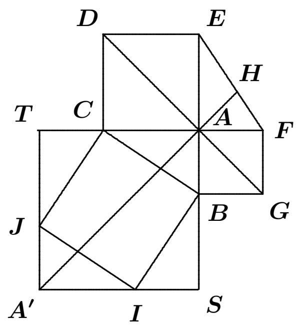
Composite solution extracted from the solutions of several solvers marked with ∗ below.

Complete the square $ATA'S$ as shown, and join DAG. Clearly, DAG and HAA' are straight lines. Note that

$$EA = AC = TJ = A'I = SB = b\,,$$

and

$$AF = AB = SI = A'J = TC = c\,.$$



The length of the angle bisector AH is well known to be $\frac{2bc}{b+c}\cos\left(\frac{A}{2}\right) = \frac{\sqrt{2}\,bc}{b+c}$. It is easy to see that $AA' = \sqrt{2}\,(b+c)$, $DA = \sqrt{2}\,b$ and $AG = \sqrt{2}\,c$. Thus, $DA \cdot AG = A'A \cdot AH$, proving that $HDA'G$ is a cyclic quadrilateral.

Solved by ∗MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; ∗FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Bristol, UK; ∗KIN FUNG CHUNG, student, University of Toronto, Toronto, ON; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; ∗TIMOTHY DILEO, student, California State University, Fullerton, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; ∗TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; ∗ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; ∗MIHAELA VÂJIAC, Chapman University, Orange, CA, USA, and BOGDAN SUCEAVĂ, California State University, Fullerton, CA, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; ∗YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; ∗LI ZHOU, Polk Community College, Winter Haven, FL, USA; ∗TITU ZVONARU, Comănești, Romania; and the proposer.

Other solvers used Ptolemy's Theorem, trigonometry, vectors, and/or coordinates.

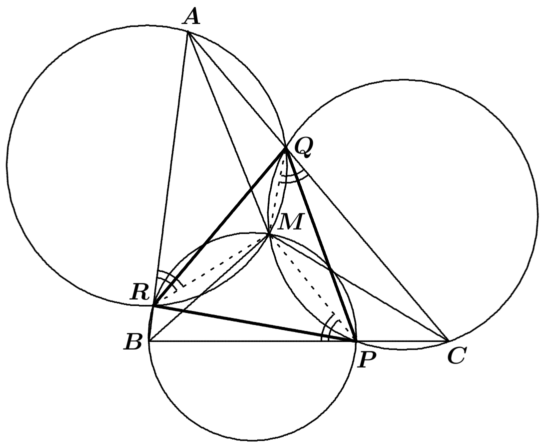
2959. [2004 : 297, 300] *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Given a non-isosceles triangle ABC, prove that there exists a unique inscribed equilateral triangle PQR of minimal area, with P, Q, R on BC, CA, and AB, respectively. Construct it by straightedge and compass.

Solution by Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.

Let PQR be an equilateral triangle inscribed in triangle ABC. Then, by Miquel's Theorem, the circumcircles of triangles QAR, RBP, and PCQ intersect in the Miquel point M. It is well-known that

$$\angle BMC = \angle CAB + \angle RPQ = A + 60^\circ\,, \tag{1}$$
$$\angle CMA = \angle ABC + \angle PQR = B + 60^\circ\,, \tag{2}$$
$$\angle AMB = \angle BCA + \angle QRP = C + 60^\circ\,. \tag{3}$$



Thus, for any equilateral triangle PQR inscribed in triangle ABC, the point M is characterized as the point that views the sides of triangle ABC under the above fixed angles. Furthermore, since the quadrilaterals $MQAR$, $MRBP$, and $MPCQ$ are cyclic,

$$\angle MRA = \angle MPB = \angle MQC\,.$$

Hence, if we rotate the lines MP, MQ, and MR about point M at the same angle, then their intersections with the sides of $\triangle ABC$ still form an equilateral triangle, because the rotation does not affect equations (1)–(3). In order to obtain a triangle of minimum area (which, for an equilateral triangle, translates into minimum perimeter as well), we need to minimize the distances MP, MQ, and MR. This is clearly achieved when the three angles MRA, MPB, and MQC are 90°; that is, when $\triangle PQR$ is the (unique) *pedal triangle* of the point M.

To construct it by straightedge and compass, first we construct the Miquel point as the intersection of two sets: the first one is the set of points viewing the side BC under angle $A+60°$, and the second is the set of points viewing the side CA under angle $B+60°$. It remains to find the feet of the perpendiculars from point M to the sides of $\triangle ABC$. Clearly, both steps can be performed by straightedge and compass.

Also solved by CHRISTOPHER J. BRADLEY, Bristol, UK; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Specht mentioned several published problems related to the construction at hand: Problems 623 and 624c [1981 : 116] *and Problem 5 from the 1999 Brazilian Mathematical Olympiad, published in the book* Mathematical Olympiads: Problems and Solutions From Around the World 1999–2000, *by T. Andreescu and Z. Fang, MAA, 2002, p. 28.*

2964. [2004 : 367, 370] *Proposed by Joe Howard, Portales, NM, USA.* (Inspired by Problem 80.D, Math. Gazette 80 (489) (1996) p. 606.)

Let $x \in \left(0, \frac{\pi}{2}\right)$. Show that:

(a) $\displaystyle \left[\frac{2+\cos x}{3}\right]\left[\frac{2(1-\cos x)}{x^2}\right] > \frac{1+\cos x}{2}$;

(b) $\displaystyle \frac{2+\cos x}{3} < \sqrt{\frac{1+\cos x}{2}} < \frac{2(1-\cos x)}{x^2}$.

Solution by Michel Bataille, Rouen, France.

(a) We will prove that the proposed inequality actually holds for all $x \in (0,\pi)$.

Let $x \in (0,\pi)$. The inequality can be rewritten as

$$\frac{1+2\cos^2\left(\frac{x}{2}\right)}{3} \cdot \frac{4\sin^2\left(\frac{x}{2}\right)}{x^2} > \cos^2\left(\frac{x}{2}\right),$$

$$\text{or} \qquad \tan^2 u + 2\sin^2 u > 3u^2, \qquad (1)$$

where $u = x/2 \in (0,\pi/2)$.

In order to prove (1), we first observe that,

$$3\sin^2 u + u^4 > 3u^2, \qquad \text{for } u > 0. \qquad (2)$$

Indeed, for $u > 0$, the function $f(u) = 3\sin^2 u + u^4 - 3u^2$ is increasing, since its derivative is $f'(u) = 3\big(\sin(2u) - 2u + \frac{(2u)^3}{3!}\big) > 0$. Since $f(0) = 0$, we see that $f(u) > 0$ for $u > 0$.

Similarly, by considering the function $g(u) = \sin u \tan u - u^2$, we find that

$$\sin u \tan u > u^2, \qquad \text{for } u \in \left(0, \frac{\pi}{2}\right). \qquad (3)$$

Indeed, we have $g(0) = 0$ and, for $u \in (0,\pi/2)$,

$$\begin{aligned} g'(u) &= \sin u \sec^2 u + \cos u \tan u - 2u \\ &= (\tan u)\left(\frac{1}{\cos u} + \cos u\right) - 2u \geq 2(\tan u - u) > 0, \end{aligned}$$

since $\frac{1}{\cos u} + \cos u \geq 2$ and $\tan u > u$.

Thus, for $u \in (0, \pi/2)$,

$$\begin{aligned} \tan^2 u + 2\sin^2 u &= \sin^2 u(\sec^2 u + 2) \\ &= \sin^2 u(3 + \tan^2 u) \\ &> \sin^2 u\left(3 + \frac{u^4}{\sin^2 u}\right) \qquad \text{by (3)} \\ &= 3\sin^2 u + u^4 \ > \ 3u^2 \qquad \text{by (2)}. \end{aligned}$$

This proves (1) (and hence the proposed inequality).

Note that we have proved (1) for all $x \in (0, \pi)$.

(b) This is problem 80.D in Math. Gazette alluded to in the statement of the problem.

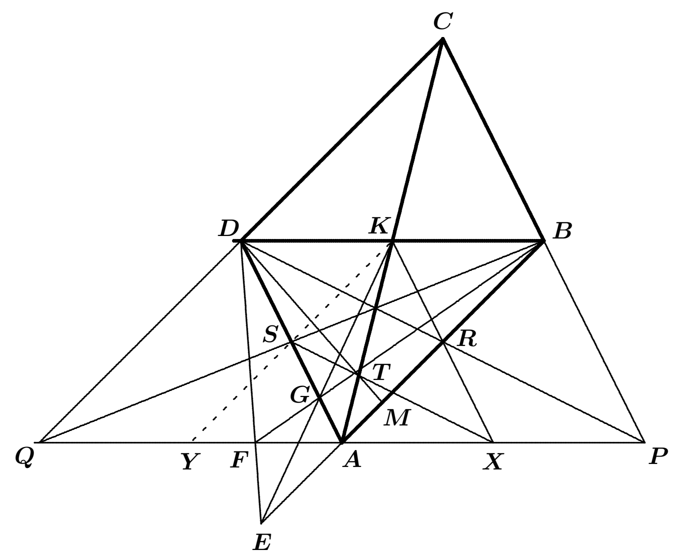
Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There was one partly incorrect solution.

As pointed out by Bataille, part (b) *is the same as problem 80.D that appeared in Math. Gazette in 1996. The proposer's original submission was actually the inequality in part* (a) *only, but he mentioned that it was motivated by the problem in Math. Gazette, which somehow was inadvertently included as part* (b) *of the current problem.*

2965. [2004 : 367, 370] *Proposed by Titu Zvonaru, Bucharest, Romania.*

Let $ABCD$ be a parallelogram. Using only an unmarked straightedge, find a point M on AB such that $AM = \frac{1}{5}AB$.

I. *Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.*



Draw $K = AC \cap BD$. We now construct the line PAQ parallel to BD as follows: Pick any point E on line AB. Construct $G = EK \cap AD$ and $F = ED \cap BG$. From Ceva's Theorem applied to $\triangle EBD$, we have

$$\frac{EA}{AB} \cdot \frac{BK}{KD} \cdot \frac{DF}{FE} = 1,$$

which implies that $EA/AB = EF/FD$. Hence, FA is parallel to DB. Next, define $P = FA \cap CB$ and $Q = AF \cap CD$, and draw $R = AB \cap PD$ and $S = AD \cap BQ$. Then R and S are the centres of the parallelograms $APBD$ and $AQDB$. Draw $X = KR \cap AP$ and $T = SX \cap KA$. Then T is the centroid of $\triangle KXY$, where Y is the point where KS meets AQ (but Y need not be drawn); thus, $AT = AK/3$. Finally, draw $M = DT \cap AB$. By Menelaus's Theorem applied to $\triangle ABK$ and transversal MD, we have

$$\frac{AM}{MB} = \frac{AT}{TK} \cdot \frac{KD}{BD} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

We conclude that $AM = AB/5$, as required.

Conclusion: Counting AC, BD, EK, ED, BG, AF, DP, BQ, KR, SX, DT, we found $AB/5$ by drawing 11 lines.

II. *Comment by Victor Pambuccian, Phoenix, AZ.*

The theory of straightedge constructions was worked out by Jacob Steiner (*Geometrische Konstruktionen, ausgeführt mittels der geraden Linie und eines festen Kreises*, Berlin, 1833). His solution to our problem (more precisely, to the problem of dividing a given segment into n equal parts by straightedge given a line parallel to it) was reproduced in A. Adler, *Theorie der geometrischen Konstruktionen*, Leipzig, 1906, pages 76-77. See `http://historical.library.cornell.edu/cgi-bin/cul.math/docviewer?did=05310001&seq=90&frames=0&view=50`.

Also solved by MICHEL BATAILLE, Rouen, France; TOSHIO SEIMIYA, Kawasaki, Japan; SOUTHWEST MISSOURI STATE UNIVERSITY PROBLEM-SOLVING GROUP; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2966. [2004 : 368, 370] *Proposed by Mikhael Kotchetov, Memorial University of Newfoundland, St. John's, NL.*

Consider non-intersecting and non-congruent circles Γ_1 and Γ_2 with centres O_1 and O_2, respectively. Let Q be the point of intersection of the two common tangents, t_1 and t_2, which do not intersect the line segment O_1O_2. A common tangent, t_c, which intersects the segment O_1O_2 meets the tangents t_1 and t_2 at E_1 and E_2, respectively.

Let P be the mid-point of the line segment O_1O_2. Prove that P, Q, E_1, and E_2 are concyclic.

I. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Suppose that Γ_1 is smaller than Γ_2. Since O_1E_1 and O_2E_1 are internal and external bisectors of $\angle QE_1E_2$, $O_1E_1 \perp O_2E_1$. Likewise, $O_1E_2 \perp O_2E_2$. Hence, O_1, E_1, O_2, E_2 lie on a circle whose diameter is O_1O_2; thus, its centre is P. Thus, $\angle QPE_2 = \angle O_1PE_2 = 2\angle O_1E_1E_2 = \angle QE_1E_2$, completing the proof.

II. *Solution by Michel Bataille, Rouen, France.*

Project O_1, O_2, P orthogonally onto t_1 at U_1, U_2, P_1, respectively. Since P is the mid-point of O_1O_2, we see that P_1 is the mid-point of U_1U_2; thus, $P_1U_1^2 = P_1U_2^2$. Since U_1 and U_2 are the points where t_1 is tangent to Γ_1 and Γ_2, it follows that P_1 has the same power with respect to Γ_1 and Γ_2. Therefore, P_1 lies on the radical axis ℓ of Γ_1 and Γ_2. Similarly, using the projections P_2 and P_c of P onto t_2 and t_c, we see that P_1, P_2, P_c are all on ℓ. Since the projections of P on the sides of the triangle E_1QE_2 are collinear, P is on the circumcircle of $\triangle E_1QE_2$ (and ℓ is the Simson line of P). The result follows.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and the proposer.

Several readers pointed out that since O_1 and O_2 are the incentre and excentre of $\triangle E_1QE_2$, Kotchetov's problem is a known theorem: The circumcircle of a triangle contains the mid-point of each segment joining the incentre to an excentre. *See, for example, Nathan Altshiller Court,* College Geometry, *Theorem 122, page 76, or Roger A. Johnson,* Advanced Euclidean Geometry, *Section 292, page 185.*

2967. [2004 : 368, 371] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let a_1, a_2, $\ldots$, a_n be positive real numbers, and let

$$E_n = \sum_{i=1}^{n}\left(\sum_{j=0}^{n-1} a_i^j\right)^{-1}.$$

If $r = \sqrt[n]{a_1a_2\cdots a_n} \geq 1$, prove that $E_n \geq n\left(\sum_{j=0}^{n-1} r^j\right)^{-1}$ for:

(a) $n = 2$, (b) $n = 3$, (c)★ $n \geq 4$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

(a) Let $x = \dfrac{\sqrt{a_2}}{\sqrt{a_2}+\sqrt{a_1}}$ and $y = \dfrac{\sqrt{a_1}}{\sqrt{a_2}+\sqrt{a_1}}$. Then $x + y = 1$, and $E_2 = \dfrac{x}{x+ry} + \dfrac{y}{y+rx}$. Applying Jensen's Inequality to the convex function

$1/t$ on $(0,\infty)$, we see that

$$E_2 \;\geq\; \frac{1}{x(x+ry)+y(y+rx)} \;=\; \frac{1}{(x+y)^2+2(r-1)xy}\,.$$

By the AM–GM Inequality, we have $xy \leq \left(\frac{x+y}{2}\right)^2 = \frac{1}{4}$. Hence,

$$E_2 \;\geq\; \frac{1}{1+\frac{1}{2}(r-1)} \;=\; \frac{2}{1+r}\,.$$

(b) Let $k = \sum\limits_{i=1}^{3} \sqrt[3]{a_i^2 a_{i+1}^2}$, and let $x_i = \frac{1}{k}\sqrt[3]{a_{i+1}^2 a_{i+2}^2}$ for $i = 1,\,2,\,3$, where all subscripts are taken modulo 3. Then $x_1+x_2+x_3 = 1$, and

$$E_3 \;=\; \sum_{i=1}^{3} \frac{x_i}{x_i + r\sqrt{x_{i+1}x_{i+2}} + r^2\left(\frac{x_{i+1}x_{i+2}}{x_i}\right)}\,.$$

Applying Jensen's Inequality to the convex function $1/t$ on $(0,\infty)$, we see that

$$\begin{aligned} E_3 \;&\geq\; \frac{1}{\sum\limits_{i=1}^{3} x_i\left(x_i + r\sqrt{x_{i+1}x_{i+2}} + r^2\left(\frac{x_{i+1}x_{i+2}}{x_i}\right)\right)} \\ &=\; \frac{1}{\left(\sum\limits_{i=1}^{3} x_i\right)^2 + r\sum\limits_{i=1}^{3} x_i\sqrt{x_{i+1}x_{i+2}} + (r^2-2)\sum\limits_{i=1}^{3} x_i x_{i+1}}\,. \end{aligned}$$

By the AM–GM Inequality, we have

$$\begin{aligned} \sum_{i=1}^{3} x_i\sqrt{x_{i+1}x_{i+2}} \;&\leq\; \sum_{i=1}^{3} x_i x_{i+1} \;\leq\; \frac{1}{3}\left(\sum_{i=1}^{3} x_i^2 + 2\sum_{i=1}^{3} x_i x_{i+1}\right) \\ &=\; \tfrac{1}{3}(x_1+x_2+x_3)^2 \;=\; \tfrac{1}{3}\,. \end{aligned}$$

Hence,

$$\begin{aligned} E_3 \;&\geq\; \frac{1}{1+(r^2+r-2)\left(\sum\limits_{i=1}^{3} x_i x_{i+1}\right)} \\ &\geq\; \frac{1}{1+\frac{1}{3}(r^2+r-2)} \;=\; \frac{3}{1+r+r^2}\,. \end{aligned}$$

Also solved by ARKADY ALT, San Jose, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

Part (c) remains open. Janous provided a partial solution where each $a_i \geq 1$.

2968. [2004 : 368, 371] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let a_1, a_2, ..., a_n be positive real numbers, and let

$$E_n = \frac{1+a_1a_2}{1+a_1} + \frac{1+a_2a_3}{1+a_2} + \cdots + \frac{1+a_na_1}{1+a_n}.$$

Let $r = \sqrt[n]{a_1a_2\cdots a_n} \geq 1$,.

(a) Prove that $E_n \geq \dfrac{n\left(1+r^2\right)}{1+r}$ for $n=3$ and $n=4$.

(b)★ Prove or disprove that $E_n \geq \dfrac{n\left(1+r^2\right)}{1+r}$ for $n=5$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

(a) Case $n=3$.

Let $k = \sum\limits_{i=1}^{3} \sqrt[3]{a_ia_{i+1}^2}$, and let $x_i = \frac{1}{k}\sqrt[3]{a_{i+1}a_{i+2}^2}$ for $i=1$, 2, 3, where all subscripts are taken modulo 3. Then $x_1+x_2+x_3=1$ and

$$E_3 = \sum_{i=1}^{3} \frac{x_i + r^2x_{i+2}}{x_i + rx_{i+1}}.$$

Applying Jensen's Inequality to the convex function $1/t$ on $(0,\infty)$, we see that

$$E_3 \geq \frac{\left(\sum\limits_{i=1}^{3}\left(x_i+r^2x_{i+2}\right)\right)^2}{\sum\limits_{i=1}^{3}\left(x_i+r^2x_{i+2}\right)\left(x_i+rx_{i+1}\right)} = \frac{\left(1+r^2\right)^2}{(x_1+x_2+x_3)^2 + (r^3+r^2+r-2)\sum\limits_{i=1}^{3} x_ix_{i+1}}.$$

Now

$$1 = (x_1+x_2+x_3)^2 = \sum_{i=1}^{3} x_i^2 + 2\sum_{i=1}^{3} x_ix_{i+1} \geq 3\sum_{i=1}^{3} x_ix_{i+1};$$

thus,

$$E_3 \geq \frac{\left(1+r^2\right)^2}{1+\frac{1}{3}\left(r^3+r^2+r-2\right)} = \frac{3\left(1+r^2\right)}{1+r}.$$

Case $n=4$.

Let $k = \sum\limits_{i=1}^{4} \sqrt[4]{a_ia_{i+1}^2a_{i+2}^3}$, and let $x_i = \frac{1}{k}\sqrt[4]{a_{i+1}a_{i+2}^2a_{i+3}^3}$ for $i=1$, 2, 3, 4, where all subscripts are taken modulo 4. Then $x_1+x_2+x_3+x_4=1$, and

$$E_4 \;=\; \sum_{i=1}^{4} \frac{x_i + r^2 x_{i+2}}{x_i + r x_{i+1}}\,.$$

Applying Jensen's Inequality to the convex function $1/t$ on $(0,\infty)$, we see that

$$\begin{aligned} E_4 \;&\geq\; \frac{\left(\sum\limits_{i=1}^{4}\left(x_i + r^2 x_{i+2}\right)\right)^2}{\sum\limits_{i=1}^{4}\left(x_i + r^2 x_{i+2}\right)\left(x_i + r x_{i+1}\right)} \\ &=\; \frac{\left(1+r^2\right)^2}{(x_1^2 + x_2^2 + x_3^2 + x_4^2) + (r^3 + r)\left(\sum\limits_{i=1}^{4} x_i x_{i+1}\right) + r^2(2x_1x_3 + 2x_2x_4)}\,. \end{aligned}$$

Now

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 + x_4^2 \;&=\; (x_1+x_2+x_3+x_4)^2 - (x_1+x_2)(x_3+x_4) \\ &\qquad - (x_1+x_3)(x_2+x_4) - (x_1+x_4)(x_2+x_3)\,, \\ \sum_{i=1}^{4} x_i x_{i+1} \;&=\; (x_1+x_3)(x_2+x_4)\,, \\ 2x_1x_3 + 2x_2x_4 \;&=\; (x_1+x_2)(x_3+x_4) + (x_1+x_4)(x_2+x_3) \\ &\qquad - (x_1+x_3)(x_2+x_4)\,. \end{aligned}$$

Hence,

$$E_4 \;\geq\; \frac{\left(1+r^2\right)^2}{\begin{array}{c}(x_1+x_2+x_3+x_4)^2 + \left(r^2+1\right)(r-1)(x_1+x_3)(x_2+x_4) \\ +\;\left(r^2-1\right)\left[(x_1+x_2)(x_3+x_4) + (x_1+x_4)(x_2+x_3)\right]\end{array}}\,.$$

Finally, by the AM–GM Inequality,

$$(x_1+x_3)(x_2+x_4) \;\leq\; \tfrac{1}{4}\left[(x_1+x_3)+(x_2+x_4)\right]^2 \;\leq\; \tfrac{1}{4}\,.$$

Similarly, we have $(x_1+x_2)(x_3+x_4) \leq \frac{1}{4}$ and $(x_1+x_4)(x_2+x_3) \leq \frac{1}{4}$. Thus,

$$E_4 \;\geq\; \frac{\left(1+r^2\right)^2}{1 + \frac{1}{4}\left(r^3 - r^2 + r - 1\right) + \frac{1}{2}\left(r^2-1\right)} \;=\; \frac{4\left(1+r^2\right)}{1+r}\,.$$

Also solved by MIHÁLY BENCZE, Brasov, Romania; and the proposer.
Part (b) remains open.

2969. [2004 : 368, 371] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let a, b, c, d, and r be positive real numbers such that $r = \sqrt[4]{abcd} \geq 1$. Prove that

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} \;\geq\; \frac{4}{(1+r)^2}\,.$$

Solution by Arkady Alt, San Jose, CA, USA.

I suggest the following generalization. For any natural $n \geq 2$, let $a_1, a_2, \ldots, a_n > 0$ such that $a_1 a_2 \cdots a_n = r^n$. Then

$$\frac{1}{(1+a_1)^2} + \frac{1}{(1+a_2)^2} + \cdots + \frac{1}{(1+a_n)^2} \;\geq\; \frac{n}{(1+\sqrt[n]{a_1 a_2 \cdots a_n})^2}$$

if and only if $r \geq \sqrt{n} - 1$.

[*Ed.:* In fact, the condition is not sufficient when $n = 2$. It is possible to find $\varepsilon > 0$ such that $a_1 = \sqrt{2} - 1$, $a_2 = a_1 + \varepsilon$, and $r > \sqrt{2} - 1$, but the inequality fails. The slightly stronger condition $r \geq 0.5$ is sufficient when $n = 2$. Moreover, the inductive step still holds for $n = 2$ using this stronger condition. That is, for $n > 2$, $r \geq \sqrt{n} - 1$ is sufficient for the inequality to hold. The editor has not determined the minimum sufficient value of r in the case $n = 2$.]

We begin with necessity. From the supposition that the inequality holds for all $a_1, a_2, \ldots, a_n > 0$ with $a_1 a_2 \cdots a_n = r^n$, and by setting $a_1 = a_2 = \cdots = a_{n-1} = m$, $a_n = \frac{r^n}{m^{n-1}}$, for $m \in \mathbb{R}^+$, we obtain

$$\frac{n-1}{(1+m)^2} + \frac{m^{2(n-1)}}{\left(m^{n-1}+r^n\right)^2} \;\geq\; \frac{n}{(1+r)^2}\,,$$

which holds for all positive m. Thus,

$$\lim_{m\to\infty}\left(\frac{n-1}{(1+m)^2} + \frac{m^{2(n-1)}}{\left(m^{n-1}+r^n\right)^2}\right) \;=\; 1 \;\geq\; \frac{n}{(1+r)^2}\,,$$

which implies $r \geq \sqrt{n} - 1$.

We prove sufficiency by mathematical induction on $n \geq 2$.

Let $n = 2$ and a, $b > 0$ such that $ab = r^2$ with $r \geq 0.5$.

Set $x = a + b$. Then $x \geq 2r$. Since

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} \;=\; \frac{2 + 2(a+b) + (a+b)^2 - 2ab}{(1+a+b+ab)^2}\,,$$

the inequality can rewritten in the form:

$$\frac{2 + 2x + x^2 - 2r^2}{\left(1+r^2+x\right)^2} \;\geq\; \frac{2}{(1+r)^2}$$

$$(1+r)^2\left(2 + 2x + x^2 - 2r^2\right) \;\geq\; 2\left(1+x+r^2\right)^2\,.$$

This inequality holds if and only if

$$\begin{aligned} 0 &\le (1+r)^2\left(2+2x+x^2-2r^2\right)-2\left(1+x+r^2\right)^2 \\ &= x^2\left((1+r)^2-2\right)-2x\left(2\left(1+r^2\right)-(1+r)^2\right) \\ &\qquad +\left(2-2r^2\right)(1+r)^2-2\left(1+r^2\right)^2 \\ &= x^2\left(r^2+2r-1\right)-2x\left(r^2-2r+1\right)-4r^4-4r^3-4r^2+4r \\ &= (x-2r)\left(x\left(r^2+2r-1\right)+2\left(r^3+r^2+r-1\right)\right) . \end{aligned}$$

Since $r^2+2r-1 \ge 0$ (this follows from $r \ge \sqrt{2}-1$) and $x \ge 2r$, we have

$$\begin{aligned} & x\left(r^2+2r-1\right)+2\left(r^3+r^2+r-1\right) \\ &\qquad \ge 2r\left(r^2+2r-1\right)+2r^3+2r^2+2r-2 \\ &\qquad = 2\left(2r^3+3r^2-1\right) = 2(r+1)^2(2r-1) \ge 0 . \end{aligned}$$

Thus,

$$(x-2r)\left(x\left(r^2+2r-1\right)+2\left(r^3+r^2+r-1\right)\right) \ge 0 .$$

Let $a_1,\ a_2,\ \ldots,\ a_n,\ a_{n+1} > 0$ and $a_1a_2\cdots a_{n+1} = r^{n+1}$, where $r \ge \sqrt{n+1}-1$. Due to symmetry of the inequality, we can suppose that $a_1 \ge a_2 \ge \cdots \ge a_n \ge a_{n+1} > 0$.

Set $x = \sqrt[n]{a_1a_2\cdots a_n}$; then $a_{n+1} = \dfrac{r^{n+1}}{x^n}$. Since

$$x \ge a_{n+1} \iff x^{n+1} \ge r^{n+1} \iff x \ge r ,$$

we have $x \ge r$.

Given $x \ge \sqrt{n+1}-1 > \sqrt{n}-1$ and the induction hypothesis, we obtain the inequality:

$$\frac{1}{(1+a_1)^2}+\frac{1}{(1+a_2)^2}+\cdots+\frac{1}{(1+a_n)^2} \ge \frac{n}{(1+x)^2} .$$

[*Ed.:* Note that, for $n=2$, we have $x \ge \sqrt{3}-1 > 0.5$; hence, the inequality does indeed hold. For $n > 2$, $x > \sqrt{n}-1$.]

Therefore,

$$\begin{aligned} & \frac{1}{(1+a_1)^2}+\frac{1}{(1+a_2)^2}+\cdots+\frac{1}{(1+a_n)^2}+\frac{1}{(1+a_{n+1})^2} \\ &\qquad \ge \frac{n}{(1+x)^2}+\frac{x^{2n}}{\left(x^n+r^{n+1}\right)^2} , \end{aligned}$$

and it is enough to prove that, for all $x \ge r \ge \sqrt{n+1}-1$,

$$\frac{n}{(1+x)^2}+\frac{x^{2n}}{\left(x^n+r^{n+1}\right)^2} \ge \frac{n+1}{(1+r)^2} .$$

Let $h(x) = \dfrac{n}{(1+x)^2} + \dfrac{x^{2n}}{(x^n + r^{n+1})^2}$. Then

$$h'(x) \;=\; \frac{2n\left(x^{n+1} - r^{n+1}\right)\left(x^{n+1}r^{n+1} + 3x^n r^{n+1} + r^{2n+2} - x^{2n-1}\right)}{(1+x)^3\left(x^n + r^{n+1}\right)^3}\,.$$

Now everything depends on the behaviour of the polynomial

$$P_n(x) \;=\; x^{n+1}r^{n+1} + 3x^n r^{n+1} + r^{2n+2} - x^{2n-1}\,.$$

Note that

$$\begin{aligned} x^{n+1}r^{n+1} + 3x^n r^{n+1} + r^{2n+2} - x^{2n-1} &\;=\; 0 \\ \text{or} \qquad r^{n+1} + \frac{3r^{n+1}}{x} + \frac{r^{2n+2}}{x^{n+1}} - x^{n-2} &\;=\; 0\,. \end{aligned}$$

Set $\phi(x) = r^{n+1} + \dfrac{3r^{n+1}}{x} + \dfrac{r^{2n+2}}{x^{n+1}} - x^{n-2}$.
Since $r \geq \sqrt{n+1} - 1 > \frac{1}{2}$ for $n \geq 2$, we have

$$\begin{aligned} P_n(r) &\;=\; 2r^{2n+2} + 3r^{2n+1} - r^{2n-1} \\ &\;=\; r^{2n-1}\left(2r^3 + 3r^2 - 1\right) \\ &\;=\; r^{2n-1}(r+1)^2(2r-1) \;>\; 0 \\ \Longleftrightarrow \qquad \phi(r) &\;>\; 0\,. \end{aligned}$$

Since $\phi(x)$ is continuous on $(0,\infty)$, $\phi(x)$ strictly decreases on $[r,\infty)$, and $\phi(\infty)\phi(r) < 0$, there is only one point, x_0, in (r,∞) such that $\phi(x_0) = 0$, or equivalently $P_n(x_0) = 0$.

Moreover, $\phi(x) > \phi(x_0) = 0$ is equivalent to $P_n(x) > 0$ for all $x \in [r, x_0)$, and $0 = \phi(x_0) > \phi(x)$ is equivalent to $P_n(x) < 0$ for all $x \in (x_0, \infty)$.

Since

$$\min_{x\in[r,x_0]} h(x) \;=\; h(r) \;=\; \frac{n}{(1+r)^2} + \frac{r^{2n}}{\left(r^n + r^{n+1}\right)^2} \;=\; \frac{n+1}{(1+r)^2}\,,$$

and, for any $x \in [x_0, \infty)$,

$$h(x) \;>\; \lim_{x\to\infty} h(x) \;=\; 1 \;\geq\; \frac{n+1}{(1+r)^2} \;=\; h(r)\,,$$

we obtain

$$\min_{x\in[r,\infty)} h(x) \;=\; h(r) \;=\; \frac{n+1}{(1+r)^2}\,.$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MIHÁLY BENCZE, Brasov, Romania; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2970. [2004 : 368, 371] *Proposed by Titu Zvonaru, Comăneşti, Romania.*

If m and n are positive integers such that $m \geq n$, and if a, b, $c > 0$, prove that

$$\frac{a^m}{b^m + c^m} + \frac{b^m}{c^m + a^m} + \frac{c^m}{a^m + b^m} \geq \frac{a^n}{b^n + c^n} + \frac{b^n}{c^n + a^n} + \frac{c^n}{a^n + b^n}.$$

I. *Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA, modified slightly by the editor.*

Without loss of generality, we may assume that $a \geq b \geq c$. Let $x = a^n$, $y = b^n$, $z = c^n$, and set $r = \frac{m}{n}$. Then $x \geq y \geq z$ and $r \geq 1$. Let $F = \sum\limits_{\text{cyclic}} \frac{a^m}{b^m + c^m} - \sum\limits_{\text{cyclic}} \frac{a^n}{b^n + c^n}$. Then $F = \sum\limits_{\text{cyclic}} \frac{x^r}{y^r + z^r} - \sum\limits_{\text{cyclic}} \frac{x}{y + z}$.

We have

$$\frac{x^r}{y^r + z^r} - \frac{x}{y+z} = \frac{xy\left(x^{r-1} - y^{r-1}\right)}{(y+z)\left(y^r + z^r\right)} + \frac{xz\left(x^{r-1} - z^{r-1}\right)}{(y+z)\left(y^r + z^r\right)}. \qquad (1)$$

Similarly,

$$\frac{y^r}{z^r + x^r} - \frac{y}{z+x} = \frac{yz\left(y^{r-1} - z^{r-1}\right)}{(z+x)\left(z^r + x^r\right)} + \frac{yx\left(y^{r-1} - x^{r-1}\right)}{(z+x)\left(z^r + x^r\right)} \qquad (2)$$

and

$$\frac{z^r}{x^r + y^r} - \frac{z}{x+y} = \frac{zx\left(z^{r-1} - x^{r-1}\right)}{(x+y)\left(x^r + y^r\right)} + \frac{zy\left(z^{r-1} - y^{r-1}\right)}{(x+y)\left(x^r + y^r\right)}. \qquad (3)$$

Adding (1), (2), and (3), we obtain

$$\begin{aligned} F = {} & xy\left(x^{r-1} - y^{r-1}\right)\left(\frac{1}{(y+z)\left(y^r + z^r\right)} - \frac{1}{(z+x)\left(z^r + x^r\right)}\right) \\ & + yz\left(y^{r-1} - z^{r-1}\right)\left(\frac{1}{(z+x)\left(z^r + x^r\right)} - \frac{1}{(x+y)\left(x^r + y^r\right)}\right) \\ & + xz\left(x^{r-1} - z^{r-1}\right)\left(\frac{1}{(y+z)\left(y^r + z^r\right)} - \frac{1}{(x+y)\left(x^r + y^r\right)}\right). \end{aligned}$$

Since $x \geq y \geq z$ and $r \geq 1$, we see that all the terms on the right side of the last expression above are non-negative. Hence, $F \geq 0$.

II. *Composite of similar solutions by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON; and Li Zhou, Polk Community College, Winter Haven, FL, USA.*

We show that the result actually holds for all reals m and n with $m \geq n \geq 0$. Due to symmetry, we may assume that $a \geq b \geq c > 0$.

Let $x_m = a^m/(a^m + b^m + c^m)$, $y_m = b^m/(a^m + b^m + c^m)$, and $z_m = c^m/(a^m + b^m + c^m)$. Then $x_m \geq y_m \geq z_m$ and $x_n \geq y_n \geq z_n$.

Furthermore,

$$x_m - x_n = \frac{a^n b^n \left(a^{m-n} - b^{m-n}\right) + a^n c^n \left(a^{m-n} - c^{m-n}\right)}{(a^m + b^m + c^m)(a^n + b^n + c^n)} \ge 0$$

$$(x_m + y_m) - (x_n + y_n)$$
$$= \frac{a^n c^n \left(a^{m-n} - c^{m-n}\right) + b^n c^n \left(b^{m-n} - c^{m-n}\right)}{(a^m + b^m + c^m)(a^n + b^n + c^n)} \ge 0,$$

and $(x_m + y_m + z_m) - (x_n + y_n + z_n) = 1 - 1 = 0$. Thus, (x_m, y_m, z_m) majorizes (x_n, y_n, z_n).

Consider the function $f(t) = t/(1-t)$. It is easy to check that f is continuous and convex on $(0,1)$. By the Majorization Inequality, we have

$$f(x_m) + f(y_m) + f(z_m) \ge f(x_n) + f(y_n) + f(z_n),$$

which is equivalent to the given inequality.

[*Ed:* Zhao uses the term Karamata's Majorization Inequality. However, this inequality is usually attributed to Hardy, Pólya, and Littlewood. See, for example, Proposition B1 on page 108 of the book *Inequalities: Theory of Majorization and its Applications* by Albert W. Marshall and Ingram Olkin, Academic Press, 1979.]

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; VASILE CÎRTOAJE, University of Ploiesti, Romania; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VEDULA N. MURTY, Dover, PA, USA; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was one incomplete solution.

2971. [2004 : 369, 371] *Proposed by Michel Bataille, Rouen, France.*

For a, b, $c \in (0,1)$, find the least upper bound and the greatest lower bound of $a + b + c + abc$, subject to the constraint $ab + bc + ca = 1$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Note first that

$$\begin{aligned} a + b + c + abc &= 1 + ab + bc + ca - (1-a)(1-b)(1-c) \\ &= 2 - (1-a)(1-b)(1-c) \le 2. \end{aligned}$$

If we let $a = b = t$ and $c = \dfrac{1-t^2}{2t}$, for $t \in (0,1)$, then

$$ab + bc + ca = (a+b)c + ab = 1 - t^2 + t^2 = 1.$$

Since $(1-a)(1-b)(1-c) \to 0^+$ as $t \to 1^-$, we conclude that the required least upper bound is 2, which is not attainable.

On the other hand,

$$(a+b+c)^2 = \tfrac{1}{2}\left((a-b)^2+(b-c)^2+(c-a)^2+6(ab+bc+ca)\right) \geq 3\,.$$

Thus, by the AM–GM Inequality, we have

$$(1-a)(1-b)(1-c) \leq \left(1-\frac{a+b+c}{3}\right)^3 \leq \left(1-\frac{\sqrt{3}}{3}\right)^3 = 2-\frac{10\sqrt{3}}{9}\,.$$

Hence, the required greatest lower bound is $\dfrac{10\sqrt{3}}{9}$, which is attained when $a=b=c=\sqrt{3}/3$.

Also solved by ANGELO STATE UNIVERSITY PROBLEM GROUP, San Angelo, TX, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VINAYAK MURALIDHAR, student, Corona del Sol High School, Tempe, AZ, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and the proposers. There were two incomplete solutions and one partly incorrect solution.

The Angelo State University Problem Group showed that the following result is an immediate consequence of the answers to the current problem: if $\triangle ABC$ is an acute-angled triangle, then

$$\frac{10\sqrt{3}}{9} \leq \tan\left(\frac{A}{2}\right)+\tan\left(\frac{B}{2}\right)+\tan\left(\frac{C}{2}\right)+\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right) < 2\,,$$

with equality in the left inequality if and only if the triangle is equilateral.