SKOLIAD No. 86

Robert Bilinski

Please send your solutions to the problems in this edition by ***1 August, 2005***. A copy of **MATHEMATICAL MAYHEM Vol. 4** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

We will only print solutions to problems marked with an asterisk (*) if we receive them from students in grade 10 or under (or equivalent), or if we receive a unique solution or a generalization.

Our items come, once again, from the 4th annual CNU Regional Mathematics Contest. We are presenting only a selection of the remaining problems. Thanks go to R. Porsky, C.N.U., Newport News, VA.

4e Concours Annuel CNU Régional de Mathématique du Secondaire
Samedi, le 6 Décembre 2003

17. Le nombre de solutions entières de $2(x+y) = xy + 7$ est

(A) 1 (B) 2 (C) 3 (D) 4

20. Le minimum de $S = x^2 + 2xy + 3y^2 + 2x + 6y + 4$ est

(A) 4 (B) 1 (C) 0 (D) -1

22. Si $3\sin\theta + 4\cos\theta = 5$, alors $\tan\theta$ vaut

(A) 1 (B) -1 (C) $\frac{3}{4}$ (D) $\frac{4}{3}$

24. Si $f(x+y) = f(xy)$ et $f(7) = 7$, alors $f(49) =$

(A) 49 (B) 14 (C) 7 (D) 1

26. Soit $a = 1! + 2! + 3! + \cdots + 2003!$. Le chiffre des unités de a vaut

(A) 9 (B) 5 (C) 3 (D) 0

27. Trouver la somme de tous les entiers positifs non-multiples de 3 inférieurs à 45.

(A) 600 (B) 625 (C) 650 (D) 675

28. Quel est le plus petit entier k tel que $2x(kx-4)-x^2+6=0$ n'ait pas de solution réelle ?

(A) -1 (B) 2 (C) 3 (D) 4

29. Le nombre de facteurs entiers distincts de 30^4 est

(A) 100 (B) 125 (C) 123 (D) 30

30. La somme des racines de $f(x)=x(2x+3)(4x+5)+(6x+7)(8x+9)$ est

(A) $\frac{-35}{4}$ (B) $\frac{35}{4}$ (C) -70 (D) 70

32. Une ligne L a une pente de -2 et passe par le point $(r,-3)$. Une seconde ligne K, perpendiculaire à L en (a,b), passe par le point $(6,r)$. Le valeur de a est

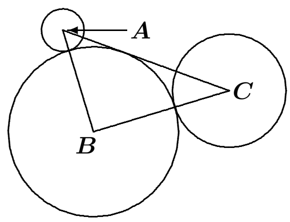
(A) r (B) $\frac{2r}{5}$ (C) $2r-3$ (D) 1

38. Une boîte contient 11 balles, numérotées $1, 2, \ldots, 11$. Si 6 balles sont pigées simultanement au hasard, quelle est la probabilité que la somme des nombres pigés soit impaire ?

(A) $\frac{100}{231}$ (B) $\frac{115}{231}$ (C) $\frac{1}{2}$ (D) $\frac{118}{231}$

3. (Questions en équipe) Trouver le maximum de $f(x)=(\frac{1}{2})^{x^2-2x}$.

10. (Questions en équipe) Sur le dessin, le cercle centré en A a un rayon de 1 et le gros cercle centré en B a un rayon de 4. Le troisième cercle est centré en C, et le gros cercle touche les deux autres cercles. De plus, $\angle ABC$ est un angle droit et la ligne AC touche le gros cercle. Trouver le rayon du cercle centré en C.



11. (Questions en équipe) Un homme voyage en automobile à une vitesse moyenne de 50 miles par heure. Il revient par le même chemin à une vitesse moyenne de 30 miles par heure. Quelle est la vitesse moyenne pour le voyage ?

12. (Questions en équipe) Résoudre algébriquement pour x :

$$\left(\log_{10} x^2\right)^2 = \log_{10}(x^4)\,.$$

4^{th} Annual CNU Regional High School Mathematics Contest
Saturday December 6, 2003

17. The number of positive integer solutions for $2(x+y) = xy + 7$ is

(A) 1 (B) 2 (C) 3 (D) 4

20. The minimum of $S = x^2 + 2xy + 3y^2 + 2x + 6y + 4$ is

(A) 4 (B) 1 (C) 0 (D) -1

22. If $3\sin\theta + 4\cos\theta = 5$, then $\tan\theta$ is

(A) 1 (B) -1 (C) $\frac{3}{4}$ (D) $\frac{4}{3}$

24. If $f(x+y) = f(xy)$ and $f(7) = 7$, then $f(49) =$

(A) 49 (B) 14 (C) 7 (D) 1

26. Let $a = 1! + 2! + 3! + \cdots + 2003!$. Then the units digit of a is

(A) 9 (B) 5 (C) 3 (D) 0

27. Find the sum of all the positive integers less than 45 that are not divisible by 3.

(A) 600 (B) 625 (C) 650 (D) 675

28. What is the smallest integer k such that $2x(kx-4) - x^2 + 6 = 0$ has no real solutions?

(A) -1 (B) 2 (C) 3 (D) 4

29. The number of distinct positive integer factors of 30^4 is

(A) 100 (B) 125 (C) 123 (D) 30

30. The sum of the zeros of $f(x) = x(2x+3)(4x+5) + (6x+7)(8x+9)$ is

(A) $\frac{-35}{4}$ (B) $\frac{35}{4}$ (C) -70 (D) 70

32. A line L has a slope of -2 and passes through the point $(r, -3)$. A second line K is perpendicular to L at (a, b) and passes through the point $(6, r)$. The value of a is

(A) r (B) $\frac{2r}{5}$ (C) $2r-3$ (D) 1

38. A box contains 11 balls, numbered 1, 2, ..., 11. If 6 balls are drawn simultaneously at random, what is the probability that the sum of the numbers drawn is odd?

(A) $\frac{100}{231}$ (B) $\frac{115}{231}$ (C) $\frac{1}{2}$ (D) $\frac{118}{231}$

3. (Team Round) Find the maximum value of $f(x) = (\frac{1}{2})^{x^2-2x}$.

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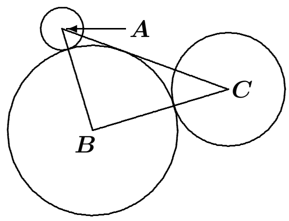
10. (Team Round) In the diagram, the circle with centre A has radius 1, and the big circle with centre B has radius 4. The third circle has centre C, and the big circle touches the two other circles. Also, $\angle ABC$ is a right angle and the line AC touches the big circle. Find the radius of the circle with centre C.



11. (Team Round) A man makes a trip by automobile at an average speed of 50 miles per hour. He returns over the same route at an average speed of 30 miles per hour. What is his average speed for the entire trip?

12. (Team Round) Solve algebraically for x:

$$\left(\log_{10} x^2\right)^2 = \log_{10}(x^4) .$$

Next we give the solutions to the 1993–1994 Newfoundland and Labrador Teachers Association, Senior Mathematics League Game 4 [2004 : 449–451].

1993–1994 Newfoundland and Labrador Teachers Association
Senior Mathematics League Game 4

1. (∗) If n is a positive integer then $n!$ (read "n factorial") is defined to be

$$n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdots 3 \cdot 2 \cdot 1 .$$

For example $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$ and $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$.

Determine the positive integer m such that the number of seconds in a year is between $m!$ and $(m+1)!$

Solution by the editor.

The number of seconds in a year is $S = 365 \times 24 \times 3600 = 2^7 \cdot 3^3 \cdot 5^3 \cdot 73$. If we organize the factors of S to resemble a factorial, we get:

$$\begin{aligned} S &= 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 8 \cdot 3 \cdot 5^2 \cdot 73 \\ &= 10! \cdot \frac{5 \cdot 73}{2 \cdot 3 \cdot 7} = 10! \cdot \frac{365}{42} . \end{aligned}$$

However, $1 < \frac{365}{42} < 11$. Thus, $10! < S < 11!$, which implies that $m = 10$ is the integer we seek.

(Even if we had assumed a leap year and used 366 instead of 365, the answer would have remained the same.)

Also solved by Alan Guo, grade 10 student, O'Neill Collegiate and Vocational Institute, Oshawa, ON. There was one incorrect solution submitted.

2. $(*)$ A movie showing was attended by 500 people. Adults paid \$10 each and children \$4 each. The total amount taken for tickets was \$4,160. How many children attended?

Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.

An adult pays \$6 more than a child; thus, there are $4(500) = 2000$ dollars in sales plus an extra $\$6a$ in sales, where a is the number of adults in the group. We must have $2000 + 6a = 4160$. Solving for a, we obtain $a = \dfrac{4160 - 2000}{6} = 360$. Hence, there were $500 - 360 = 140$ children.

Also solved by Alan Guo, grade 10 student, O'Neill Collegiate and Vocational Institute, Oshawa, ON.

3. $(*)$ Assume that the earth is a sphere with circumference 40,250 km and that a belt is placed around the equator, one metre above the earth's surface at all points. How much greater than the circumference of the earth would the length of the belt be? Would this difference be:

(a) 2π metres,

(b) 40,250 metres,

(c) $40{,}250\pi$ metres,

(d) 40,250 kilometres, or

(e) none of the above?

Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.

Suppose the radius of the earth is r km. Then the circumference of the earth is $C = 40{,}250 = 2\pi r$ km, while the circumference of the belt in km is clearly $2\pi(r + 0.001) = C + 0.002\pi$. Thus, the difference is 0.002π km, or 2π metres.

Also solved by Alan Guo, grade 10 student, O'Neill Collegiate and Vocational Institute, Oshawa, ON.

4. $(*)$ Let a, b and c be integers. You are given that $a \star b$ is defined to be $ab - 2a - 2b + 6$. Compute

$$(a \star b) \star c - a \star (b \star c)\,.$$

I. *Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.*

Equivalently, $a \star b$ is defined as $(a-2)(b-2)+2$. Then

$$\begin{aligned}(a\star b)\star c &= \big((a\star b)-2\big)(c-2)+2\\ &= \big((a-2)(b-2)+2-2\big)(c-2)+2\\ &= (a-2)(b-2)(c-2)+2\,.\end{aligned}$$

Similarly, $a\star(b\star c) = (a-2)(b-2)(c-2)+2$. Thus, the difference is zero.

II. *Solution by Alan Guo, grade 10 student, O'Neill Collegiate and Vocational Institute, Oshawa, ON.*

$$\begin{aligned}&(a\star b)\star c - a\star(b\star c)\\ &\quad= (ab-2a-2b+6)\star c - a\star(bc-2b-2c+6)\\ &\quad= \big[(ab-2a-2b+6)c-2(ab-2a-2b+6)-2c+6\big]\\ &\qquad\quad -\big[a(bc-2b-2c+6)-2a-2(bc-2b-2c+6)+6\big]\\ &\quad= (abc-2ac-2bc+6c-2ab+4a+4b-12-2c+6)\\ &\qquad\quad -(abc-2ab-2ac+6a-2a-2bc+4b+4c-12+6)\\ &\quad= 0\,.\end{aligned}$$

5. ($*$) A cube with sides of length 3 cm is painted red and then cut into $3\times3\times3=27$ cubes with sides of length 1 cm. If a denotes the number of small cubes (that is, 1 cm $\times$ 1 cm $\times$ 1 cm cubes) that are not painted at all, b the number painted on one side, c the number painted on two sides, and d the number painted on three sides, determine $a-b-c+d$.

Solution by Alan Guo, grade 10 student, O'Neill Collegiate and Vocational Institute, Oshawa, ON.

The diagram at right shows one face of the painted cube, where the letter on a square shows to which count that small cube contributes. (Remember that each square on the face represents a small cube.)

d	c	d
c	b	c
d	c	d

Clearly $a=1$, since only the small cube in the core of the big cube receives no paint. Also, $b=6$ since there are six faces as in the diagram each contributing 1 to b. There are 4 squares labelled c in the diagram, but each of the corresponding small cubes is counted twice when we sum over all six faces of the big cube; thus, $c=\dfrac{4\times6}{2}=12$. Finally, there are 4 squares labelled d in the diagram, and each of the corresponding small cubes is counted three times when we sum; hence, $d=\dfrac{4\times6}{3}=8$. Consequently, $a-b-c+d=1-6-12+8=-9$.

6. For which value or values of k, if any, is x^2+k a factor of

$$x^4-3x^3+6x^2-3kx+8\,?$$

Solution by Eric Zhang, grade 11*, Lisgar Collegiate, Ottawa, ON.*

Suppsoe that $x^2 + k$ is a factor. Then the other factor must be of the form $x^2 + ax + b$, and we have

$$\begin{aligned} x^4 - 3x^3 + 6x^2 - 3kx + 8 &= (x^2 + k)(x^2 + ax + b) \\ &= x^4 + ax^3 + (b+k)x^2 + akx + kb\,. \end{aligned}$$

Equating coefficients on the left and right sides, we obtain the system of equations:

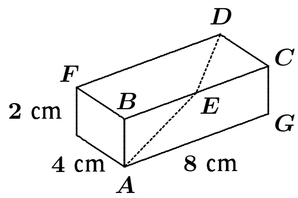
$$\begin{aligned} a &= -3\,, & (1)\\ b + k &= 6\,, & (2)\\ kb &= 8\,. & (3) \end{aligned}$$

From (2), we get $b = 6 - k$; substituting this into (3), we obtain $k(6-k) = 8$; that is, $(k-4)(k-2) = 0$. Therefore, $k = 2$ or $k = 4$.

Also solved by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.

7. An ant wishes to travel from A to D on the surface of a small wooden block with dimensions 2 cm by 4 cm by 8 cm, as shown on the right. The shortest such route involves crossing the edge BC at a point E.

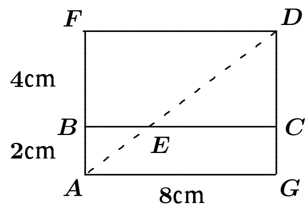
Find the distance BE.



Solution by Eric Zhang, grade 11*, Lisgar Collegiate, Ottawa, ON.*

By unfolding the block as shown, we see that for DEA to be the shortest route, it must be a straight line. Observe that the two right triangles DAF and EAB are similar. Thus, $\frac{BE}{FD} = \frac{AB}{AF}$, which implies that

$$BE = \frac{AB}{AF} \times FD = \frac{2}{6} \times 8 = \frac{8}{3} \text{ cm}.$$



Also solved by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.

8. A typical large hamburger has 427 calories, 48% of them from fat. The same hamburger with cheese has 31 grams of fat, 53% of its calories coming from fat. Regular French fries have 220 calories in total and 12 grams of fat. A popular sundae has 360 calories, and 28% of these are fat calories.

A high school student has a meal consisting of this hamburger with double cheese, an order of regular French fries and the popular sundae. What percentage of calories in the meal are fat calories?

You need to know that 1 gram of fat has 9 calories. Give your answer to the nearest whole number percentage.

Solution by Eric Zhang, grade 11*, Lisgar Collegiate, Ottawa, ON, modified by the editor.*

Let the pair (t, f) represent t total calories and f fat calories. Let HB be the calorie pair for a hamburger. In the same manner, let CB represent a Cheeseburger, FF an order of regular French fries, S a sundae, C a piece of cheese, and DCB a double cheeseburger. We have

$$\begin{aligned} HB &= (427, 48\% \times 427) = (427, 204.96)\,, \\ CB &= \left(\frac{31 \times 9}{53\%}, 31 \times 9\right) \approx (526.42, 279)\,, \\ C &= CB - HB \approx (99.42, 74.04)\,, \\ DCB &= CB + C \approx (625.83, 353.04)\,, \\ FF &= (220, 12 \times 9) = (220, 108)\,, \\ S &= (360, 28\% \times 360) = (360, 100.8)\,. \end{aligned}$$

Hence, if M represnts the calorie pair for the meal in question, then

$$M = DCB + FF + S \approx (1205.83, 561.84)\,.$$

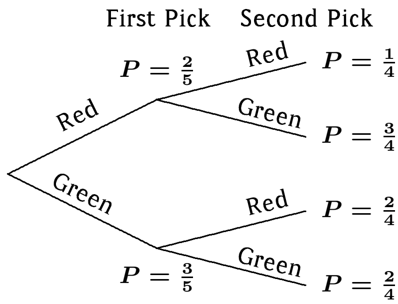
Therefore, the fraction of calories which are fat calories in the meal is $561.84/1205.83$, which is approximately 47%.

One incorrect solution was received.

9. A bag contains 2 red cabbages and 3 green cabbages. Tracy, who is blindfolded, randomly selects one of the cabbages and places it in an empty pan. Then she randomly selects a second cabbage from those remaining in the bag and also places that in the pan. What is the percentage likelihood that, of the two cabbages that are now in the pan, one is red and the other is green?

Composite of solutions by Alex Wice, grade 11 student, Leaside High School, Toronto, ON; and Eric Zhang, grade 11*, Lisgar Collegiate, Ottawa, ON.*

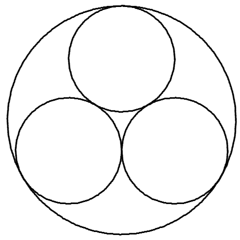
We can achieve the desired outcome by selecting either red followed by green or green followed by red (see the diagram). The first possibility yields the probability of success as $\frac{2}{5} \cdot \frac{3}{4} = \frac{3}{10}$; the second possibility gives the probability of success as $\frac{3}{5} \cdot \frac{2}{4} = \frac{3}{10}$. Summing gives the total probability as $\frac{6}{10}$, which is a 60% likelihood.



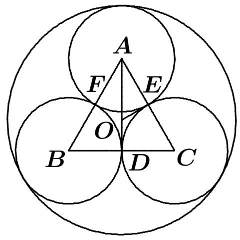
[*Ed.:* An alternate method that avoids decision trees is also available. The total number of possible ways to pick two cabbages from the bag is $\binom{5}{2} = 10$. Tracy needs to pick one of the two red cabbages, which can be done in $\binom{2}{1} = 2$ ways, and one of the three green cabbages, which can be done in $\binom{3}{1} = 3$ ways. Thus, the probability of success is $\frac{2 \cdot 3}{10}$, or 60%.]

10. Three small circles each of radius 1 cm and one larger circle are located as indicated on the right. Determine the area of the larger circle.

Your answer should be expressed in the form $\left(\frac{a+b\sqrt{3}}{c}\right)\pi$, where a, b, and c are integers.



Solution by Eric Zhang, grade 11*, Lisgar Collegiate, Ottawa, ON.*



Let A, B, C be the centres of the three interior circles, let D, E, F be the points at which these three circles touch, and let O be the centre of the large circle as shown.

We denote the radius of the large circle by r. Note that $r = AO + 1$. Therefore, our primary intention is to determine the length of AO.

By Pythagoras' Theorem, we have

$$AD \;=\; \sqrt{AC^2 - DC^2} \;=\; \sqrt{4-1} \;=\; \sqrt{3}\,.$$

Furthermore, since $\triangle AEO \sim \triangle ADC$, we get $\frac{AO}{AE} = \frac{AC}{AD}$; that is,

$$AO \;=\; \frac{AC}{AD} \times AE \;=\; \frac{2}{\sqrt{3}}\,.$$

Hence, $r = \frac{2}{\sqrt{3}} + 1$. Now the area of the large circle is

$$\pi r^2 \;=\; \pi\left(\frac{2}{\sqrt{3}}+1\right)^2 \;=\; \pi\left(\frac{7+4\sqrt{3}}{3}\right)\,.$$

One incorrect solution was received.

Relay

R1. The sum of five consecutive numbers is 130. Call the smallest of these numbers A.

Write the value of A in Box #1 of the Relay Answer Sheet.

Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.

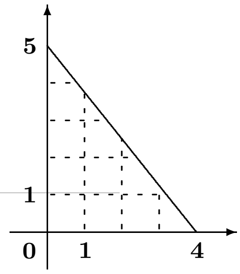
Let the five consecutive numbers be $x-2$, $x-1$, x, $x+1$, $x+2$. Their sum is $5x = 130$, implying that $x = 26$. The smallest is $A = x - 2 = 24$.

R2. A triangle has vertices at $(0,0)$, $(A/6,0)$, and $(0,5)$. How many points with integer coordinates lie inside the triangle?

Write your answer, B, in Box #2 of the Relay Answer Sheet.

Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.

The number of lattice points on the boundary of the triangle is clearly 10, since the hypotenuse does not intersect any lattice points other than its end-points. Let E denote the number of lattice points on the sides of the triangle. The area of the triangle is $\Delta = 10$. By Pick's Theorem, the number of lattice points inside the triangle is



$$\Delta - E/2 + 1 \;=\; 10 - 5 + 1 \;=\; 6\,.$$

[*Ed.:* With an accurate diagram such as the one above, we can simply count the number of lattice points in the interior.]

R3. Determine c if

$$\begin{aligned}(B+18)c + 7d &= 4\,,\\ d + e &= 20\,,\\ e + f &= 36\,,\\ f + 5c &= 15\,.\end{aligned}$$

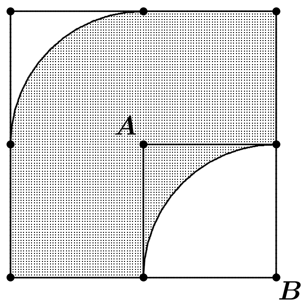
Write the value of c in Box #3 of the Relay Answer Sheet.

Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.

Working backwards, we have $f = 15 - 5c = 36 - e$, which implies that $5c + 21 = e = 20 - d$, further implying that $d = -(5c+1)$. Thus, the first equation becomes $24c - 7(5c+1) = 4$; whence, $-11c = 11$, or $c = -1$.

R4. The sides of the large square in the diagram are twice the length of the sides of the small square. The two arcs are portions of circles with radii equal to the length of the sides of the small square and with centres at the points A and B. If the area of the hatched region is c^2, determine the length of the sides of the small square.

Write this value in Box #4 of the Relay Answer Sheet and hand it to your proctor.



Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.

Split the large square into 4 small squares. Notice that the area missing in the top-left and bottom-right squares sum to one small square. Thus, we have 3 small squares as the shaded area. Suppose the sidelength of the small square is x. Then $3x^2 = c^2 = 1$, yielding $x = \frac{1}{\sqrt{3}}$.

That brings us to the end of another issue.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **premier octobre 2005**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

M194. *Proposé par Équipe de Mayhem.*

On suppose que $n-1$ et $n+1$ sont des premiers jumeaux, où $n \in \mathbb{N}$ et $n \geq 3$. Montrer que 1, 2, 3, $\ldots$, n peuvent être arrangés dans une liste de telle sorte que la somme de deux éléments consécutifs quelconques est un nombre premier. (Par exemple, si $n = 6$, un tel arrangement est 6, 5, 2, 1, 4, 3.)

M195. *Proposé par J. Walter Lynch, Athens, GA, USA.*

On divise un fil de longueur 1 en trois morceaux qu'on déforme pour en faire un carré, un cercle et un triangle équilatéral ayant les trois la même aire. Trouver la longueur de chacun des morceaux de fil.

M196. *Proposé par Équipe de Mayhem.*

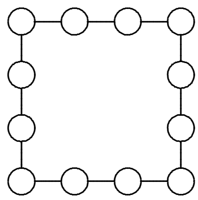
On cherche à former des comités à partir d'un groupe de personnes. Montrer que le nombre de comités possibles comportant un nombre impair de membres est exactement le même que le nombre de comités possibles comportant un nombre pair de membres.

M197. *Proposé par Neven Jurič, Zagreb, Croatie.*

Douze bateaux, occupant chacun trois cases horizontales ou verticales dans une grille 10×10, sont désignés par les lettres de A à L comme indiqué dans la figure. Chaque bateau contient un certain nombre de passagers. Les nombres qui figurent à droite de la dernière colonne et en-dessous de la dernière ligne indiquent le total des passagers de tous les bateaux touchés par la ligne ou la colonne correspondante. Par exemple, les deux bateaux B et L de la dernière colonne contiennent en tout 9 passagers. Quel est le nombre de passagers de chacun des douze bateaux, sachant que deux d'entre eux n'en contiennent aucun et que les dix restants en ont 1, 2, 3, 4, 5, 6, 7, 8, 9 et 10 ?

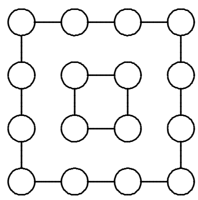
A	A	A							B	
					C	C	C		B	
	E	D	D	D					B	17
	E							J		
	E					I		J		15
						I		J		
F	F	F				I	K			19
		G					K		L	
		G	H	H	H		K		L	
		G							L	6
8		21			6				9	

M198. *Proposé par Équipe de Mayhem.*



On doit placer chacun des entiers de 1 à 12 dans un des cercles de la figure de telle sorte que, sur chaque côté de la figure, la somme des entiers soit 25. Trouver la somme des quatre entiers placés dans les coins.

M199. *Proposé par Équipe de Mayhem.*



Ce problème est une modification du problème précédent. Ici, on demande d'utiliser chacun des entiers de 1 à 16 de telle sorte que les quatre entiers des coins extérieurs aient la même somme que les quatre entiers formant la figure intérieure. Quelle est la somme maximale qu'on puisse obtenir, si c'est possible ?

M200. *Proposé par Équipe de Mayhem.*

Dans un carré d'aire unitaire on dessine deux droites perpendiculaires passant par son centre, le divisant ainsi en 4 morceaux. Quelle est l'aire maximale possible pour l'un quelconque d'entre eux ? Justifiez votre réponse.

M194. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Suppose $n-1$ and $n+1$ are twin primes where $n \in \mathbb{N}$ with $n \geq 3$. Show that 1, 2, 3, $\dots$, n can be arranged in a row so that the sum of any two consecutive numbers is prime. (For example, when $n = 6$, one such arrangement is 6, 5, 2, 1, 4, 3.)

M195. *Proposed by J. Walter Lynch, Athens, GA, USA.*

A wire of unit length is divided into three pieces, which are used to construct a square, a circle, and an equilateral triangle such that each of them has the same area. Find the length of each of the three pieces of wire.

M196. *Proposed by the Mayhem Staff.*

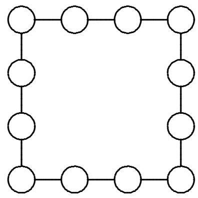
Committees are to be formed from a group of people. Show that the number of possible committees that can be formed with an odd number of members is exactly the same as the number of possible committees that can be formed with an even number of members.

M197. *Proposed by Neven Jurič, Zagreb, Croatia.*

There are twelve ships situated on a 10×10 grid. The ships are denoted by the letters A through L, and each ship consists of three cells of the grid in either a horizontal or a vertical line, as shown in the diagram. Each ship contains a certain number of passengers. There are also some numbers in the last row and the last column of the diagram. These numbers represent the total number of passengers on all the ships intersected by that row or column. For example, the two ships B and L in the last (right-most) column together contain 9 passengers. How many passengers does each of the twelve ships contain, if there are no passengers on two of the ships and the remaining ten ships contain 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10 passengers?

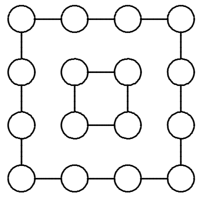
A	A	A							B	
					C	C	C		B	
	E	D	D	D					B	17
	E							J		
	E					I		J		15
						I		J		
F	F	F				I	K			19
		G					K		L	
		G	H	H	H		K		L	
		G							L	6
8		21			6				9	

M198. *Proposed by the Mayhem Staff.*



Each of the integers from 1 to 12 is to be placed in one of the circles in the figure so that the sum of the integers along each side of the figure is 25. Determine the sum of the four integers placed in the corners.

M199. *Proposed by the Mayhem Staff.*



This is a modification of the previous problem. In this case, the requirement is to use all the integers from 1 to 16 once each so that the integers along each of the four outer edges of the large figure and the four integers that make up the inner figure have identical sums. What is the largest sum, if any, that can be obtained?

M200. *Proposed by the Mayhem Staff.*

Two perpendicular lines are drawn through the centre of a square with area 1 square unit, cutting the square into 4 pieces. What is the largest possible area for any of the pieces? Justify your answer.

Problem of the Month

Ian VanderBurgh, University of Waterloo

Problem (2004 United Kingdom Mathematics Trust Junior Math Olympiad) At a summer camp, five students, called A, B, C, D, and E, each take part in five events, called V, W, X, Y, and Z. In each competition scores of 5, 4, 3, 2, and 1 are awarded for 1^{st}, 2^{nd}, 3^{rd}, 4^{th}, and 5^{th}, respectively. There are no ties. Student A scores a total of 24 points, student C scores the same in each of four events, student D scores 4 in competition V, and student E scores 5 in W and 3 in X. Surprisingly, their overall positions are in alphabetical order. There are no ties in the final standings. Show that this information is enough to find all the scores, and that there is only one solution.

Trying to solve this problem is a good exercise in logic, as well as numerical manipulation. There is no high-level mathematics involved. It is a problem which students of all ages can approach. It is a bit reminiscent of logic problems we tried when we were in elementary school: "Al, Betty, and Charles have three different kinds of pets and live in three different coloured houses"—and then you are given a bunch of seemingly unconnected statements and asked who owns the schnauzer.

Solution. The maximum score any student can have over the five events is $5 \times 5 = 25$ points. Since A scores 24 points in total, she must have scored 5 in four events and 4 in the other event. Since E scores 5 in event W, then A must score 4 in W and 5 in the rest of the events.

At this stage, it is a good idea to make a table:

	V	W	X	Y	Z	Total
A	5	4	5	5	5	24
B						
C						
D	4					
E		5	3			

So far it does not look good! However, there is an unusual approach—to look at the total scores. Note that the total of the five students' total scores must be $5(5 + 4 + 3 + 2 + 1) = 75$.

What is the minimum possible total score for E? Since E must get at least 1 in each of the three unspecified events, this minimum must be $1 + 5 + 3 + 1 + 1 = 11$. Now, could E have scored 12 or more points? If E scores 12, then D scores at least 13, C at least 14, and B at least 15, since there are no ties. But then the total of the five students' totals is at least $24 + 15 + 14 + 13 + 12 = 78$, which is impossible.

Therefore, E must have a total of 11 (hence, 1 in each of the three remaining events). Thus, E and A together score 35 points, leaving 40 points for B, C, and D, with each of these students scoring at least 12. But since none of the these totals can be equal, they must be 12, 13, and 15. (Try fiddling around with this to convince yourself!)

Remarkably, we now know the total scores for each student, without knowing all of the scores in the individual events. Thus, we now have:

	V	W	X	Y	Z	Total
A	5	4	5	5	5	24
B						15
C						13
D	4					12
E	1	5	3	1	1	11

At this stage, we should look back at the information we were given to see what we have not yet used. We still have not used the fact that C scores the same score in four different events. This score cannot be 1, 4, or 5, since we already have at least two of each of these scores in our table. If C scored 2 in four different events, then to get a total of 13, he must have scored 5 in the remaining event, which is impossible, since each event already has a score of 5 accounted for. Thus, C scores 3 in four different events (and 1 in the other event, since C's total must be 13). Looking at the entries already in the table, C must score 1 in event X. This gives us:

	V	W	X	Y	Z	Total
A	5	4	5	5	5	24
B						15
C	3	3	1	3	3	13
D	4					12
E	1	5	3	1	1	11

Now D scores 8 points in the last four events, and must score 1 or 2 in event W, and 2 or 4 in events X, Y, and Z. Since D cannot have only one odd score and an even total, then she must score 2 in event W, and thus she scores 2 in the rest of the events to get a total of 12. (A score of 4 would push her above this total score.) We can then quickly fill in B's scores by a process of elimination:

	V	W	X	Y	Z	Total
A	5	4	5	5	5	24
B	2	1	4	4	4	15
C	3	3	1	3	3	13
D	4	2	2	2	2	12
E	1	5	3	1	1	11

We have figured out all of the scores. These are the only possible scores because, in each step of our argument, there was only one possibility.

As this is the last Problem of the Month before summer, I would like to thank Shawn Godin for asking me to write this column. I hope that you have enjoyed the problems this year, and I look forward to continuing in the fall. Have a good summer!

Pólya's Paragon

Fun With Numbers (Part 4)

Shawn Godin

Last issue we introduced the idea of modular arithmetic, and we looked at using digital sums to check calculations. The digital sum of a number is actually a single digit congruent to the number modulo 9. For example, from last issue, $43\,658\,912 \equiv 38 \equiv 11 \equiv 2 \pmod 9$. When we check a calculation using digital sums, we are really checking whether our answer is correct modulo 9.

How can we show this congruence between a number and its digital sum modulo 9? It becomes rather simple when we remember that an n–digit number $d_{n-1}d_{n-2}\ldots d_2d_1d_0$ is really the number

$$d_{n-1} \times 10^{n-1} + d_{n-2} \times 10^{n-2} + \cdots + d_2 \times 10^2 + d_1 \times 10 + d_0 .$$

Now we simply have to notice that $10 \equiv 1 \pmod 9$, which implies that $10^k \equiv 1 \pmod 9$ for any k. Hence,

$$\begin{aligned} & d_{n-1} \times 10^{n-1} + d_{n-2} \times 10^{n-2} + \cdots + d_2 \times 10^2 + d_1 \times 10 + d_0 \\ & \quad \equiv \; d_{n-1} \times 1^{n-1} + d_{n-2} \times 1^{n-2} + \cdots + d_2 \times 1^2 + d_1 \times 1 + d_0 \\ & \quad \equiv \; d_{n-1} + d_{n-2} + \cdots + d_2 + d_1 + d_0 \pmod 9 . \end{aligned}$$

The process of calculating the digital sum of a number has often been called "*casting out nines*", because when you calculate the digital sum you can ignore multiples of 9. For example, if you were to look at $43\,658\,912$, your first step would be to get rid of the 9s to get $43\,658\,\not{9}12$. Then, since $4+5=9$, $3+6=9$, and $8+1=9$, we get

$$\not{4}\not{3}\;\not{6}\not{5}\not{8}\;\not{9}\not{1}2 .$$

Therefore, the digital sum is 2, and we must have $43\,658\,912 \equiv 2 \pmod 9$. Use this trick to amaze your parents and impress your peers!

This can be turned into a divisibility test, since $9 \mid a$ if and only if $a \equiv 0 \pmod 9$ (why?).

Divisibility by 9: A number is divisible by 9 if and only if its digital sum is divisible by nine.

We can extend this idea a little further. By noting that $10^2 = 100$, we can convert a number from base ten to base hundred quite easily by looking at blocks of two digits starting from the right. For example,

$$\begin{aligned} 43\,658\,912 &= 4 \times 10^7 + 3 \times 10^6 + 6 \times 10^5 + 5 \times 10^4 \\ &\qquad + 8 \times 10^3 + 9 \times 10^2 + 1 \times 10 + 2 \\ &= 43 \times 100^3 + 65 \times 100^2 + 89 \times 100 + 12 \\ &= 43{,}65{,}89{,}12_{100} , \end{aligned}$$

where the commas separate the "digits" and the subscript 100 means we are working in base hundred instead of base ten. Then, since $100 \equiv 1 \pmod{99}$, we can see that the digital sum will be equivalent to the number modulo 99. That is, $43 + 65 + 89 + 12 = 209$ and $2 + 09 = 11$. This implies that $43\,658\,912 \equiv 11 \pmod{99}$.

So what? you say. How often do you want to divide something by 99? The real bonus here is that $43\,658\,912 \equiv 11 \pmod{99}$, which tells us that $43\,658\,912 = 99k + 11$ for some integer k. Since we know that $99 = 9 \times 11$, we see that $99 \equiv 0 \pmod 9$ and $99 \equiv 0 \pmod{11}$, which gives us

$$43\,658\,912 \equiv 11 ,$$

using either $\bmod 9$ or $\bmod 11$. This means, in effect, that we have come up with one test that tests for divisibility by *both* 9 and 11.

We can use the structure of our number system to come up with other divisibility rules. The next two should be well known:

Divisibility by 2: A number is divisible by 2 if and only if its last digit is 0, 2, 4, 6, or 8.

Divisibility by 5: A number is divisible by 5 if and only if its last digit is either 0 or 5.

We can justify each of these two rules by noting that $10 = 2 \times 5$. Thus, $10 \equiv 0 \pmod{2}$ and $10 \equiv 0 \pmod{5}$. As a result, we have

$$d_{n-1}d_{n-2}\ldots d_2d_1d_0 \;\equiv\; d_0\,,$$

using either $\bmod\ 2$ or $\bmod\ 5$. We can extend this idea by noting that $4 = 2^2$. Then $100 = 10^2 \equiv 0 \pmod{4}$. Similarly, for $25 = 5^2$ we get $100 = 10^2 \equiv 0 \pmod{25}$. Thus,

$$\begin{aligned} d_{n-1}d_{n-2}\ldots d_2d_1d_0 &\equiv d_1 \times 10 + d_0 \pmod{4} \\ \text{and} \quad d_{n-1}d_{n-2}\ldots d_2d_1d_0 &\equiv d_1 \times 10 + d_0 \pmod{25}\;. \end{aligned}$$

From this we get

Divisibility by 4: A number is divisible by 4 if and only if the two-digit number formed by its last two digits is divisible by 4.

Divisibility by 25: A number is divisible by 25 if and only if the two-digit number formed by its last two digits is divisible by 25 (that is, 00, 25, 50, or 75).

This should be enough to get you going. Here are some things to try for homework.

1. Modify the divisibility test for 9 to get another test for 3.

2. Modify the divisibility test for 9 to get another test for 11. *Hint:* Note that $10 \equiv -1 \pmod{11}$. (Why?)

3. Construct divisibility tests for 2^n and 5^n for any integer $n > 1$.

4. Develop divisibility tests for 101, 1001, 10001, $\ldots$ and 999, 9999, 99999, $\ldots$, and see what else comes out of it.

One last note: the divisibility rule for a composite number is just a combination of the rules for the prime powers which are factors of the number. For example, to test divisibility by 12, you would just test for divisibility by 3 and 2^2, since a number is divisible by 12 if and only if it is divisible by both 3 and 2^2.

Have a great summer! See you again in September!

Misère Games

Arthur Holshouser and Harold Reiter

Abstract.

The theory of last-player-winning counter-pickup games is well known. See [1] and [2]. The corresponding *misère* games in which the last player loses are less well understood. In this note, we define a special class of combinatorial games and find the winning strategies for all composite games with these special games as components. In the first section we recall the method of using Nim values of component games to solve a composite game. In the second section, we define *special* games and find winning strategies for misère games.

Section 1: Terminology

Definition. A finite impartial game G played under normal rules of play is called a *regular* game. This means that

1. two players alternate moving,
2. there is no infinite sequence of moves,
3. both players have the same moves available, and
4. the winner is the last player to make a move.

Such a game can be thought of as a directed acyclic graph. Each *vertex* of the graph corresponds to a position in the game and each *directed edge* corresponds to a move. The *followers* of a vertex are those positions joined to it by an outgoing edge. We will briefly say that G is a regular impartial game.

Nim Values. The minimum excluded value, denoted mex, of a finite set of non-negative integers is the least non-negative integer not in the set. For example, $\mathrm{mex}\{1, 2, 4, 0\} = 3$, $\mathrm{mex}\{2, 4, 5\} = 0$, $\mathrm{mex}\{\} = 0$. The *Nim value* of a position, denoted by $g(n)$, is the mex of the Nim values of its followers. A position with no followers (that is, a *terminal* position) has Nim value 0. It is easy to see that the winning strategy is to move to a position with Nim value 0, for then the opponent either has no move at all and loses immediately, or must move to a position with Nim value greater than 0 and hence must eventually lose.

Composite Games. *Composite games*, denoted $G = G_1 \oplus G_2 \oplus \cdots \oplus G_k$, are games that have several components. Two players alternate moves. Each player on his turn selects a component game G_i in which a legal move can be made and makes a legal move in that game. The winner is the last player to move. We define the Nim value of the composite game as the Nim sum (denoted by $\oplus$) of the Nim values of each of the component games. The Nim sum is obtained by writing the integers in binary and adding modulo 2 without carrying. For example, $6 \oplus 3 = 110_{(2)} \oplus 11_{(2)} = 101_{(2)} = 5_{(10)}$, since, by considering the digits in the summation from left to right, we get $1 \equiv 1 \pmod 2$, $1 + 1 \equiv 0 \pmod 2$, and $0 + 1 \equiv 1 \pmod 2$.

Strategy. The *balanced* positions are those positions whose Nim values are 0. The *unbalanced* positions are those positions whose Nim values are not zero.

If a position is balanced, it will always become unbalanced after the moving player moves. This follows from the definition of mex since if a player moves from n_i to m_i in G_i, then $g(n_i) \neq g(m_i)$.

Also, if a position is unbalanced, the moving player can always move to a balanced position. Such a winning move can always be selected from the component G_i that contributes the left-most 1 in the Nim sum of the component values. This follows from the definition of mex since if $g(n_i) \geq 1$ in game G_i, the moving player can move in G_i to a vertex m_i having any of the values $\{0, 1, 2, \ldots, g(n_i) - 1\}$. In particular, the moving player can move to a position m_i whose value is the sum of the Nim values of the other components. Of course, all terminal positions have a Nim value of $0 \oplus 0 \oplus \cdots \oplus 0 = 0$, which is balanced.

Section 2

***Misère* version of a game**. The misère version of a regular impartial game G_i is played by the same rules as G_i except the loser is the player who makes the last move.

The misère version of a composite game $G_1 \oplus G_2 \oplus \cdots \oplus G_k$ is played by the same rules as $G_1 \oplus G_2 \oplus \cdots \oplus G_k$ except the loser is the last player to move.

Special Games Suppose G is a regular impartial game. We say that G_i is *special* if, for each position n in G, when $g(n) = 0$, we have either (i) n is a terminal position or (ii) there exists a follower m of n such that $g(m) = 1$.

Problem 1. Suppose $G_1, G_2, \ldots, G_k$ are special, regular impartial games. Find a strategy for playing the misère version of $G_1 \oplus G_2 \oplus \cdots \oplus G_k$.

Solution. Let $(n_1, n_2, \ldots, n_k)$ denote an arbitrary position in the composite game $G_1 \oplus G_2 \oplus \cdots \oplus G_k$. We will first define the balanced positions.

A. If each $g(n_i) \in \{0, 1\}$, then $(n_1, n_2, \ldots, n_k)$ is balanced if and only if $g(n_1) \oplus g(n_2) \oplus \cdots \oplus g(n_k) = 1$.

B. If at least one $g(n_i) \notin \{0, 1\}$, then $(n_1, n_2, \ldots, n_k)$ is balanced if and only if $g(n_1) \oplus g(n_2) \oplus \cdots \oplus g(n_k) = 0$.

Let B, U denote the balanced and unbalanced positions respectively.

We note that all terminal positions, which we denote 0, are unbalanced. We will prove the following which we have illustrated in Figure 1.

(1) If $(n_1, n_2, \ldots, n_k)$ is balanced, then all moves must be to an unbalanced position.

(2) If $(n_1, n_2, \ldots, n_k)$ is unbalanced and non-terminal, then there exists a move to a balanced position.

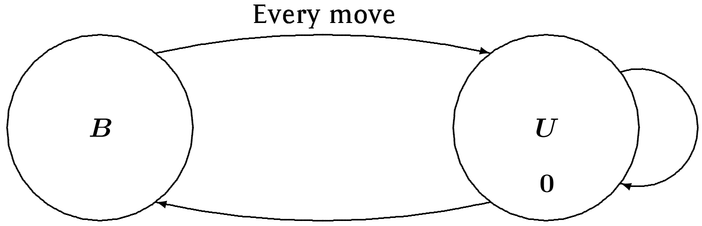


Figure 1

From (1) and (2) it follows that if $(n_1, n_2, \ldots, n_k)$ is the initial position in the game, then

(a) if $(n_1, n_2, \ldots, n_k)$ is balanced, the first player to move will lose if the opposing player plays perfectly.

(b) if $(n_1, n_2, \ldots, n_k)$ is unbalanced, then the first player to move will win with perfect play.

We now prove (1) and (2).

Proof of (1). For the balanced position $(n_1, n_2, \ldots, n_k)$, we consider two cases.

Case (a). $g(n_i) \in \{0, 1\}$ for all i.

Since the position is balanced, we have $g(n_1) \oplus g(n_2) \oplus \cdots \oplus g(n_k) = 1$. Without loss of generality, we may assume the player to move chooses to make a move in game G_1, which must be non-terminal, of course.

If $g(n_1) = 0$, then, by the definition of mex, the player to move must move to m_1 with $g(m_1) = 1$ or $g(m_1) \geq 2$. In either case, the new position, $(m_1, n_2, n_3, \ldots, n_k)$, is unbalanced.

If $g(n_1) = 1$, then, by the definition of mex, the player to move must move to m_1 with $g(m_1) = 0$ or $g(m) \geq 2$. In either case, the new position, $(m_1, n_2, n_3, \ldots, n_k)$, is unbalanced.

Case (b). $g(n_i) \notin \{0, 1\}$ for some i.

Then $g(n_1) \oplus g(n_2) \oplus \cdots \oplus g(n_k) = 0$, which implies there must also be a $j \neq i$ such that $g(n_j) \notin \{0, 1\}$. Now after the next move, there must still exist a game G_j such that $g(n_j) \notin \{0, 1\}$. By the definition of mex, after this next move it will be impossible for $g(\overline{n_1}) \oplus g(\overline{n_2}) \oplus \cdots \oplus g(\overline{n_k}) = 0$ where $(\overline{n_1}, \overline{n_2}, \ldots, \overline{n_k})$ is the new position. Therefore, $(\overline{n_1}, \overline{n_2}, \ldots, \overline{n_k})$ is unbalanced.

Proof of (2). For the unbalanced non-terminal position $(n_1, n_2, \ldots, n_k)$, we consider the two cases.

Case (a). $g(n_i) \notin \{0, 1\}$ for some i.

(i) Only one $g(n_i) \notin \{0, 1\}$. Since $g(n_i) \geq 2$, by the definition of mex, the player to move can move to an m_i such that $g(m_i) = 0$ and move to an $\overline{m_i}$ such that $g(\overline{m_i}) = 1$. This easily implies that he can move to a balanced position.

(ii) Two or more $g(n_i) \notin \{0, 1\}$. By the definition of mex, the player to move (as in Bouton's Nim) moves to a position $(\overline{n_1}, \overline{n_2}, \ldots, \overline{n_k})$ such that $g(\overline{n_1}) \oplus g(\overline{n_2}) \oplus \cdots \oplus g(\overline{n_k}) = 0$, which is a balanced position.

Case (b). $g(n_i) \in \{0, 1\}$ for all i.

Since the position $(n_1, n_2, \ldots, n_k)$ is unbalanced, it follows that $g(n_1) \oplus g(n_2) \oplus \cdots \oplus g(n_k) = 0$. Now, since $(n_1, n_2, \ldots, n_k)$ is non-terminal, let n_i be a non-terminal vertex in a game G_i. If $g(n_i) = 1$, by the definition of mex, the player to move can move to m_i with $g(m_i) = 0$, which balances the game. If $g(n_i) = 0$, by the definition of a special game, the player to move can move to m_i with $g(m_i) = 1$, which again balances the game. ∎

References

[1] Berlekamp, Conway, and Guy, *Winning Ways*, Academic Press, New York, 1982.

[2] Richard K. Guy, *Fair Game*, 2nd ed., COMAP, New York, 1989.

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THE OLYMPIAD CORNER

No. 246

R.E. Woodrow

We begin this number with problems of the Singapore Mathematical Olympiad 2002 (Open Section), written 30 May 2002. My thanks go to Bill Sands, Chair of the International Olympiad Committee of the Canadian Mathematical Society, for obtaining them for our use.

SINGAPORE MATHEMATICAL OLYMPIAD 2002

Open Section

May 30, 2002—Part A

No calculators are allowed. Each question carries a weight of 4 marks. No steps are needed to justify your answers.

1. Let $f(x)$ be a function which satisfies

$$f(29+x) \;=\; f(29-x)\,,$$

for all values of x. If $f(x)$ has exactly three real roots α, β, and γ, determine the value of $\alpha+\beta+\gamma$.

2. John left town A at x minutes past 6:00 pm and reached town B at y minutes past 6:00 pm the same day. He noticed that at both the beginning and the end of the trip, the minute hand made the same angle of 110 degrees with the hour hand on his watch. How many minutes did it take John to go from town A to town B?

3. Let $x_1 = \frac{1}{2002}$. For $n \geq 1$, define $nx_{n+1} = (n+1)x_n + 1$. Find x_{2002}.

4. For integers $n \geq 1$, let $a_n = n^2 + 500$ and $d_n = \gcd(a_n, a_{n+1})$. Determine the largest value of d_n.

5. It is given that the polynomial $p(x) = x^3 + ax^2 + bx + c$ has three distinct positive integer roots and $p(2002) = 2001$. Let $q(x) = x^2 - 2x + 2002$. It is also given that the polynomial $p\big(q(x)\big)$ has no real roots. Determine the value of a.

6. Find the largest positive integer N such that $N!$ ends with exactly twenty-five "zero" digits.

7. A circle passes through the vertex C of a rectangle $ABCD$ and touches its sides AB and AD at points M and N, respectively. Suppose the distance from C to MN is 2 cm. Find the area of $ABCD$ in cm^2.

8. Let $m = 144^{\sin^2 x} + 144^{\cos^2 x}$. How many such m's are integers?

9. Evaluate $\sum_{k=1}^{2002} \frac{k \cdot k!}{2^k} - \sum_{k=1}^{2002} \frac{k!}{2^k} - \frac{2003!}{2^{2002}}$.

10. How many ways are there to arrange 5 identical red, 5 identical blue, and 5 identical green marbles in a straight line such that every marble is adjacent to at least one marble of the same colour as itself?

Part B

Each question carries a weight of 15 marks. Show the steps in your calculations.

1. In the plane, Γ is a circle with centre O and radius r, P and Q are distinct points on Γ, A is a point outside Γ, M and N are the mid-points of PQ and AO, respectively. Suppose $OA = 2a$ and $\angle PQA$ is a right angle. Find the length of MN in terms of r and a. Express your answer in its simplest form, and justify your answer.

2. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers between 1001 and 2002 inclusive. Suppose $\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} b_i^2$. Prove that

$$\sum_{i=1}^{n} \frac{a_i^3}{b_i} \geq \frac{17}{10} \sum_{i=1}^{n} a_i^2 .$$

Determine when equality holds.

3. Let n be a positive integer. Determine the smallest possible sum

$$a_1 b_1 + a_2 b_2 + \cdots + a_{2n+2} b_{2n+2} ,$$

where $a_1, a_2, \ldots, a_{2n+2}$ and $b_1, b_2, \ldots, b_{2n+2}$ are rearrangements of the binomial coefficients

$$\binom{2n+1}{0}, \quad \binom{2n+1}{1}, \ldots, \binom{2n+1}{2n+1} .$$

Justify your answer.

4. Find all real-valued functions $f : \mathbb{Q} \longrightarrow \mathbb{R}$ defined on the set of all rational numbers $\mathbb{Q}$ satisfying the conditions

$$f(x+y) = f(x) + f(y) + 2xy ,$$

for all x, y in $\mathbb{Q}$ and $f(1) = 2002$. Justify your answers.

The final problems we offer for the April 2005 corner is the XVIII Italian Mathematical Olympiad, Cesanatico, May 3rd, 2002. Thanks again go to Bill Sands for obtaining them.

XVIII ITALIAN MATHEMATICAL OLYMPIAD
Cesenatico, Italy
May 3, 2002

1. Find all 3–digit positive integers that are 34 times the sum of their digits.

2. The plan of a house has the shape of a capital L, obtained by suitably placing side-by-side four squares whose sides are 10 metres long. The external walls of the house are 10 metres high. The roof of the house has six faces, starting at the top of the six external walls, and each face forms an angle of 30° with respect to a horizontal plane.

Determine the volume of the house (that is, of the solid delimited by the six external walls, the six faces of the roof, and the base of the house).

3. Let A and B be two points of the plane, and let M be the mid-point of AB. Let r be a line, and let R and S be the projections of A and B onto r. Assuming that A, M, and R are not collinear, prove that the circumcircle of triangle AMR has the same radius as the circumcircle of BSM.

4. Find all values of n for which all solutions of the equation $x^3-3x+n=0$ are integers.

5. Prove that, if $m=5^n+3^n+1$ is prime, then 12 divides n.

6. We are given a chessboard with 100 rows and 100 columns. Two squares of the board are said to be *adjacent* if they have a common side. Initially, all squares are white.

(a) Is it possible to colour an odd number of squares in such a way that each coloured square has an odd number of adjacent coloured squares?

(b) Is it possible to colour some squares in such a way that an odd number of them have exactly 4 adjacent coloured squares and all the remaining coloured squares have exactly 2 adjacent coloured squares?

(c) Is it possible to colour some squares in such a way that an odd number of them have exactly 2 adjacent coloured squares and all the remaining coloured squares have exactly 4 adjacent coloured squares?

Next we give a comment on a problem from the XV Gara Nazionale di Matematica 1999 [2002 : 481; 2005 : 37].

2. A natural number is said to be *balanced* if the number of its decimal digits equals the number of its distinct prime factors (for instance 15 is balanced, whereas 49 is not balanced). Prove that there are only finitely many balanced numbers.

Comment by Stan Wagon, Macalester College, St. Paul, MN, USA.

This problem makes me wonder what is the largest balanced number. I believe it is 9592993410, a product of the primes with indices 1, 2, 3, 4, 5, 6, 7, 8, 9, and 14. This number can be found by an easy search once it is known that the answer has at most 10 digits, which can be gleaned from the published solution.

Now, how many balanced numbers are there?

We turn to readers' solutions to problems of the 2000 Hungarian Mathematical Olympiad, given in April, 2003 [2003 : 150].

1. Consider the number of positive even divisors for each of the first n positive integers, and form the sum of these numbers. Form a similar sum of the numbers of positive odd divisors of the first n positive integers. Prove that the two sums differ by at most n.

Solution by Mohammed Aassila, Strasbourg, France.

We will prove that $0 \le \sum\limits_{k=1}^{n} o(k) - \sum\limits_{k=1}^{n} e(k) \le n$, where $o(k)$ and $e(k)$ denote the number of positive divisors of k which are odd and the number which are even, respectively.

We know that the number of integers divisible by d among $1, 2, \dots, n$ is $\left\lfloor \frac{n}{d} \right\rfloor$. Hence,

$$\sum_{k=1}^{n} o(k) = \sum_{d \text{ odd}} \left\lfloor \frac{n}{d} \right\rfloor = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{2i-1} \right\rfloor$$

$$\text{and} \quad \sum_{k=1}^{n} e(k) = \sum_{d \text{ even}} \left\lfloor \frac{n}{d} \right\rfloor = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{2i} \right\rfloor .$$

Since $\left\lfloor \frac{n}{d} \right\rfloor \ge \left\lfloor \frac{n}{d+1} \right\rfloor$, we have

$$\sum_{k=1}^{n} o(k) - \sum_{k=1}^{n} e(k) = \sum_{i=1}^{\infty} \left(\left\lfloor \frac{n}{2i-1} \right\rfloor - \left\lfloor \frac{n}{2i} \right\rfloor \right) \ge 0$$

$$\text{and} \quad \sum_{k=1}^{n} o(k) - \sum_{k=1}^{n} e(k) = \left\lfloor \frac{n}{1} \right\rfloor - \sum_{i=1}^{\infty} \left(\left\lfloor \frac{n}{2i} \right\rfloor - \left\lfloor \frac{n}{2i+1} \right\rfloor \right) \le n .$$

2. Construct the point P inside a given triangle such that the feet of the perpendiculars from P to the sides of the triangle determine a triangle whose centroid is P.

Solution by Michel Bataille, Rouen, France.

Let the given triangle be ABC, and let D, E, and F denote the feet of the perpendiculars from P to the sides BC, CA, and AB, respectively.

First, suppose that P is the centroid of $\triangle DEF$. Then the areas $[PEF]$, $[PFD]$, and $[PDE]$ are equal. Hence,

$$PE \cdot PF \cdot \sin A \;=\; PF \cdot PD \cdot \sin B \;=\; PD \cdot PE \cdot \sin C\,.$$

(Note that $\angle EPF = 180^\circ - A$, since A, P, E, F all lie on the circle with diameter AP, and similarly, $\angle DPF = 180^\circ - B$ and $\angle EPD = 180^\circ - C$.) It follows that

$$\frac{PE}{PD} \;=\; \frac{\sin B}{\sin A} \;=\; \frac{b}{a} \quad\text{and}\quad \frac{PF}{PE} \;=\; \frac{\sin C}{\sin B} \;=\; \frac{c}{b}\,,$$

where, as usual, $a = BC$, $b = CA$, $c = AB$.

Thus, $\frac{PD}{a} = \frac{PE}{b} = \frac{PF}{c}$, and P is the point inside $\triangle ABC$ such that the distances $d(P,BC)$, $d(P,CA)$, $d(P,AB)$ are proportional to a, b, c, respectively. This point is the well-known Lemoine point of $\triangle ABC$.

Conversely, if P is the Lemoine point of $\triangle ABC$, then we have $\frac{PD}{a} = \frac{PE}{b} = \frac{PF}{c} = \lambda$, which implies that

$$[PEF] \;=\; PE \cdot PF \cdot \sin A \;=\; \lambda^2 bc\,\frac{a}{2R} \;=\; \frac{\lambda^2}{2R}\,abc$$

(where R is the circumradius of $\triangle ABC$). Thus, $[PEF] = [PFD] = [PDE]$, and P is the centroid of $\triangle DEF$.

To construct P, note that any point M on the median AA', where A' is the mid-point of BC, satisfies

$$\frac{d(M,AB)}{d(M,AC)} \;=\; \frac{d(A',AB)}{d(A',AC)} \;=\; \frac{b}{c}\,.$$

(The latter follows because $[A'AB] = [A'AC]$).

Thus, if S_A is the reflection of AA' in the internal bisector of $\angle BAC$, then, for any point M' of S_A, we have $\frac{d(M',AB)}{c} = \frac{d(M',AC)}{b}$. The line S_A is the symmedian through A. Constructing similarly the symmedian S_B through B, we obtain P at the intersection of S_A and S_B.

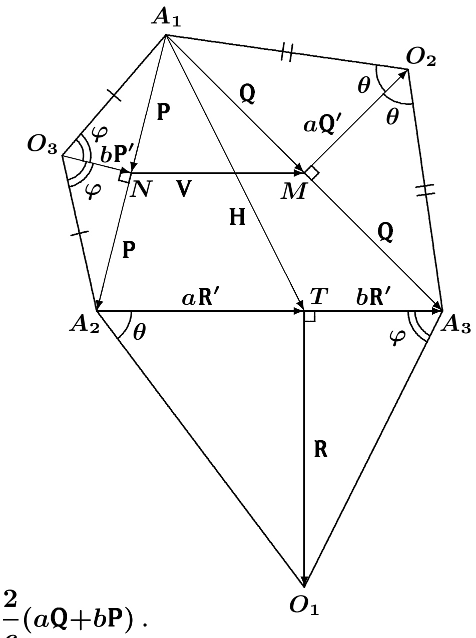
Now we look at solutions from our readers to problems of the 2000 Iranian Mathematical Olympiad given [2003 : 150–151].

2. Triangles $A_3A_1O_2$ and $A_1A_2O_3$ are constructed outside triangle $A_1A_2A_3$, with $O_2A_3 = O_2A_1$ and $O_3A_1 = O_3A_2$. A point O_1 is outside $A_1A_2A_3$ such that $\angle O_1A_3A_2 = \frac{1}{2}\angle A_1O_3A_2$ and $\angle O_1A_2A_3 = \frac{1}{2}\angle A_1O_2A_3$, and T is the foot of the perpendicular from O_1 to A_2A_3. Prove that:

(a) A_1O_1 is perpendicular to O_2O_3;

(b) $\dfrac{A_1O_1}{O_2O_3} = 2\dfrac{O_1T}{A_2A_3}$.

Solution by Geoffrey A. Kandall, Hamden, CT, USA.



Let $X \mapsto X'$ be the linear transformation that rotates a vector X counterclockwise by 90°. Let M and N be the mid-points of A_1A_3 and A_1A_2, respectively. Let $\theta = \angle O_1A_2A_3$ and $\varphi = \angle O_1A_3A_2$. Then $\angle A_3O_2M = \angle A_1O_2M = \theta$ and $\angle A_2O_3N = \angle A_1O_3N = \varphi$. Let $a = \cot\theta$, $b = \cot\varphi$, and $c = a+b$. Let $\mathbf{P} = \overrightarrow{A_1N} = \overrightarrow{NA_2}$, $\mathbf{Q} = \overrightarrow{A_1M} = \overrightarrow{MA_3}$, $\mathbf{R} = \overrightarrow{TO_1}$, $\mathbf{H} = \overrightarrow{A_1T}$, and $\mathbf{V} = \overrightarrow{NM}$.

Now $\overrightarrow{MO_2} = a\mathbf{Q}'$, $\overrightarrow{O_3N} = b\mathbf{P}'$, $\overrightarrow{A_2T} = a\mathbf{R}'$, and $\overrightarrow{TA_3} = b\mathbf{R}'$. Note that $(a+b)\mathbf{R}' = 2\mathbf{V}$; that is, $\mathbf{V} = \frac{c}{2}\mathbf{R}'$. Also, since $A_2T : TA_3 = a : b$, we have

$$\mathbf{H} \;=\; \frac{a}{a+b}(2\mathbf{Q}) + \frac{b}{a+b}(2\mathbf{P}) \;=\; \frac{2}{c}(a\mathbf{Q}+b\mathbf{P})\,.$$

Thus, $a\mathbf{Q}+b\mathbf{P} = \frac{c}{2}\mathbf{H}$.

Now, $\overrightarrow{A_1O_1} = \mathbf{H} + \mathbf{R}$ and

$$\overrightarrow{O_3O_2} = b\mathbf{P}' + \mathbf{V} + a\mathbf{Q}' \;=\; (a\mathbf{Q}+b\mathbf{P})' + \mathbf{V} \;=\; \frac{c}{2}\mathbf{H}' + \frac{c}{2}\mathbf{R}' = \frac{c}{2}\left(\overrightarrow{A_1O_1}\right)'\,.$$

This proves assertion (a).

Also, $A_1O_1 : O_2O_3 = 2 : c$ and $2O_1T : A_2A_3 = 2|\mathbf{R}| : c|\mathbf{R}'| = 2 : c$, which proves (b).

5. Suppose a, b, and c are real numbers such that for any positive real numbers x_1, x_2, $\ldots$, x_n,

$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^a \cdot \left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right)^b \cdot \left(\frac{1}{n}\sum_{i=1}^{n} x_i^3\right)^c \;\geq\; 1\,.$$

Prove that the vector (a,b,c) can be represented as a non-negative linear combination of the vectors $(-2,1,0)$ and $(1,-2,1)$.

Solved by Mohammed Aassila, Strasbourg, France; Robert Bilinski, Outremont, QC; and Pierre Bornsztein, Maisons-Laffitte, France. We give Aassila's write-up.

First set $n=1$. Then $x_1^{a+2b+3c} = (x_1)^a(x_1^2)^b(x_1^3)^c \geq 1$ for all $x_1 > 0$. Since this is true for both $x_1 > 1$ and $x_1 < 1$, we must have $a+2b+3c = 0$. Now,

$$(b+2c)(-2,1,0) + c\,(1,-2,1) \;=\; (-2b-3c,b,c) \;=\; (a,b,c)\,.$$

It remains to show that $b+2c \geq 0$ and $c \geq 0$.

To prove that $b+2c \geq 0$, set $n=2$, $x_1 = 1$, and $x_2 = t > 0$. The given inequality becomes

$$\left(\frac{1+t}{2}\right)^a \left(\frac{1+t^2}{2}\right)^b \left(\frac{1+t^3}{2}\right)^c \;\geq\; 1\,.$$

Letting $t \to 0$, we obtain $\frac{1}{2^{a+b+c}} \geq 1$. Hence, $a+b+c \leq 0$ and

$$b+2c \;=\; (a+2b+3c)-(a+b+c) \;\geq\; 0\,.$$

To prove that $c \geq 0$, set $n = k+1$, $x_1 = x_2 = \cdots = x_k = 1-t$, and $x_{k+1} = 1+kt$, where k is a positive integer and $t \in (0,1)$. Then

$$\begin{aligned}
\sum_{i=1}^{n} x_i &= k+1\,,\\
\sum_{i=1}^{n} x_i^2 &= (k+1)(1+kt^2)\,,\\
\sum_{i=1}^{n} x_i^3 &= (k+1)\big(1+3kt^2+(k^2-k)t^3\big)\,.
\end{aligned}$$

The given inequality becomes

$$1^a(1+kt^2)^b\big(1+3kt^2+k(k-1)t^3\big)^c \;\geq\; 1\,.$$

Now, take $t = 1/\sqrt{k}$. Then $2^b\left(4+\frac{k-1}{\sqrt{k}}\right)^c \geq 1$. Since this inequality holds for all positive integers k, and $\lim\limits_{k\to+\infty}\left(4+\frac{k-1}{\sqrt{k}}\right) \;=\; +\infty$, then c must be non-negative, and the proof is complete.

6. Prove that for every positive integer n, there exists a polynomial $p(x)$ with integer coefficients such that $p(1)$, $p(2)$, ..., $p(n)$ are distinct powers of 2.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

We will prove a stronger statement: For every positive integer n, there exists a polynomial $p(x)$ with integer coefficients and degree at most n such that $p(0)$, $p(1)$, ..., $p(n)$ are distinct powers of 2.

Define an $(n+1)\times(n+1)$ matrix M as follows:

$$M = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2^2 & \dots & 2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & n & n^2 & \dots & n^n \end{pmatrix} .$$

For convenience, we will index the rows and columns of our matrices and vectors starting with 0 rather than 1. Then the entries of M are $m_{ij} = i^j$, for $i = 0, 1, 2, \ldots, n$ and $j = 0, 1, 2, \ldots, n$ (with $0^0 = 1$).

Let d denote the determinant of M. Since M is a Van der Monde matrix, we have $d = \prod\limits_{0 \le i < j \le n} (i - j)$, by the well-known formula for a Van der Monde determinant. It follows that d is a non-zero integer and M is invertible. The entries of M^{-1} are of the form t_{ij}/d, where t_{ij} is an integer.

Since

$$M \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} ,$$

we must have

$$M^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} .$$

Therefore,

$$\sum_{j=0}^{n} \frac{t_{ij}}{d} = \begin{cases} 1\,, & \text{if } i = 0, \\ 0\,, & \text{if } i > 0. \end{cases}$$

Note in particular that the above sum is an integer for all i.

Let $d = 2^a b$, where a, $b \ge 0$ are integers and b is odd. Let ω be the order of 2 in the ring of integers modulo b. Then $2^\omega \equiv 1 \pmod{b}$ and $\omega \ge 1$. For every integer $k \ge 0$, we have $2^{k\omega} \equiv 1 \pmod{b}$, and hence there exists an integer c_k such that $2^{k\omega} = 1 + c_k b$.

Let a_0, a_1, ..., a_n be such that

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = M^{-1} \begin{pmatrix} 2^a \\ 2^{a+\omega} \\ 2^{a+2\omega} \\ \vdots \\ 2^{a+n\omega} \end{pmatrix} .$$

Then, for all i,

$$a_i = \sum_{j=0}^{n} \frac{t_{ij}}{d} 2^{a+j\omega} = \sum_{j=0}^{n} \frac{t_{ij}}{d} 2^a(1 + c_j b) = 2^a \sum_{j=0}^{n} \frac{t_{ij}}{d} + \sum_{j=0}^{n} t_{ij} c_j .$$

Since $\sum\limits_{j=0}^{n} \frac{t_{ij}}{d}$ is an integer, we see that a_i is an integer for each i.

Therefore, the polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients. Its degree is at most n, clearly. Moreover, since

$$M \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 2^a \\ 2^{a+\omega} \\ 2^{a+2\omega} \\ \vdots \\ 2^{a+n\omega} \end{pmatrix} ,$$

we have $p(i) = 2^{a+i\omega}$ for each $i = 0$, 1, ..., n, and we are done.

Now we turn to the May 2003 number of the *Corner* and readers' solutions to problems proposed and shortlisted for the 2000 International Olympiad in Korea given [2003 : 215–216]. George Evagelopoulos, Athens, Greece has supplied the following solutions, some or all of which may be official solutions. My thanks go to him for providing them.

1. (*Brazil*) Determine all triples of positive integers (a, m, n) such that $a^m + 1$ divides $(a + 1)^n$.

Solution supplied by George Evagelopoulos, Athens, Greece.

It is clear that a triple of positive integers (a, m, n) is a solution if $a = 1$ or $m = 1$. We now look for solutions with $a > 1$ and $m > 1$. We will make use of the following fact (a consequence of the unique prime factorization of positive integers): for any positive integers u and v,

$$u \mid v^\ell \Longrightarrow u \mid \big(\gcd(u, v)\big)^\ell . \qquad (1)$$

Let us first prove that, if $a > 1$ and $a^m + 1$ divides $(a+1)^n$, then m must be odd. Indeed, if m is even, then $a+1$ divides $a^m - 1$, and hence, $\gcd(a^m+1, a+1)$ must be 1 or 2. It follows from (1) that $a^m + 1$ divides 2^n and, hence, $a^m + 1$ is a power of 2, say $a^m + 1 = 2^s$. We have $s \geq 2$, because $a > 1$. Then $a^m = 2^s - 1 \equiv -1 \pmod 4$, which is impossible, since a^m is a perfect square.

Now suppose that $a > 1$ and m is an odd integer greater than 1. Then $n > 1$. Let p be a prime that divides m, and let $m = pr$ and $b = a^r$. Since r is odd, we see that $a+1$ divides $b+1$. Thus, $b^p + 1 = a^m + 1$ divides $(b+1)^n$. Therefore, the number $B = (b^p+1)/(b+1)$ divides $(b+1)^{n-1}$. In view of (1), we see that B divides $\big(\gcd(B, b+1)\big)^{n-1}$. Since $B > 1$, it follows that $\gcd(B, b+1) > 1$.

Since p is odd, the Binomial Theorem gives

$$B \;=\; \frac{b^p+1}{b+1} \;=\; \frac{(b+1-1)^p+1}{b+1} \;\equiv\; p \pmod{(b+1)}\;.$$

Thus, $\gcd(B, b+1)$ divides the prime p; whence $\gcd(B, b+1) = p$. Then B divides p^{n-1}, implying that B is a power of p. In particular, p divides $b^p + 1$, which in turn divides $(b+1)^n$. Therefore, p divides $b+1$, say $b = kp - 1$. Using the Binomial Theorem again, we have

$$\begin{aligned} b^p + 1 \;&=\; (kp-1)^p + 1 \;=\; \left((kp)^p - \cdots - \binom{p}{2}(kp)^2 + kp^2 - 1\right) + 1 \\ &\equiv\; kp^2 \pmod{(kp)^2}\;. \end{aligned}$$

Then

$$B \;=\; \frac{b^p+1}{b+1} \;=\; \frac{b^p+1}{kp} \;\equiv\; p \pmod{(kp)^2}\;,$$

showing that B is divisible by p but not by p^2. Since B is a power of p, it follows that $B = p$.

Now, if $p \geq 5$, we have

$$\begin{aligned} \frac{b^p+1}{b+1} \;&=\; b^{p-1} - b^{p-2} + \cdots - b + 1 \;>\; b^{p-1} - b^{p-2} \\ &=\; (b-1)b^{p-2} \;\geq\; 2^{p-2} \;>\; p\,. \end{aligned}$$

Thus, we must have $p = 3$, and $B = p$ translates into $b^2 - b + 1 = 3$. This gives $a^r = b = 2$, leading to $a = 2$ and $m = p = 3$. It is immediate that $(2, 3, n)$ is a solution for each $n > 2$.

The set of solutions is, therefore,

$$\{(a, m, n) \mid a = 1 \text{ or } m = 1 \text{ or } (a = 2, m = 3 \text{ and } n \geq 2)\}\,.$$

2. (*Bulgaria*) Prove that there exist infinitely many positive integers n such that $p = nr$, where p and r are respectively the semiperimeter and the inradius of a triangle with integer side lengths.

Solved by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; and by George Evagelopoulos, Athens, Greece. We give the write-up by Diaz-Barrero, modified by the editor.

Consider an arbitrary triangle with sides a, b, and c, semiperimeter $p = \frac{1}{2}(a+b+c)$, and inradius r. Let $x = b+c-a$, $y = a-b+c$, and $z = a+b-c$. Note that $x = 2(p-a)$, $y = 2(p-b)$, and $z = 2(p-c)$. The area of the triangle is given by $\sqrt{p(p-a)(p-b)(p-b)}$ (Heron's Formula) and also by pr. Therefore, $pr^2 = (p-a)(p-b)(p-c)$; that is, $8pr^2 = xyz$. Hence, $(x+y+z)^3 = (p/r)^2 xyz$. Thus, for any integer n, the relation $p = nr$ is equivalent to

$$(x+y+z)^3 \;=\; n^2 xyz\,. \tag{1}$$

Now let $n \in \mathbb{N}$ such that there exist positive integers x, y, and z that satisfy equation (1). We can assume that x, y, and z are even, because the equation is homogeneous in x, y, and z. Letting $a = \frac{1}{2}(y+z)$, $b = \frac{1}{2}(x+z)$, and $c = \frac{1}{2}(x+y)$, we see that a, b, and c are positive integers which are the sides of a triangle such that $p = nr$. Thus, the statement in the problem will be proved if we show that equation (1) has positive integer solutions (x, y, z) for infinitely many positive integers n.

We restrict our search for solutions by assuming that $z = k^2(x+y)$ and $n = 3(k^2+1)$ for some $k \in \mathbb{N}$. Then equation (1) becomes

$$(x+y)^2(k^2+1) \;=\; 9k^2xy\,. \tag{2}$$

Setting $u = x/y$, we obtain $(u+1)^2(k^2+1) = 9k^2u$, or

$$(k^2+1)u^2 - (7k^2-2)u + (k^2+1) \;=\; 0\,.$$

The discriminant of the quadratic on the left side above is

$$\Delta \;=\; (7k^2-2)^2 - 4(k^2+1) \;=\; 9k^2(5k^2-4)\,.$$

Now u will be rational if and only if Δ is a square, and this will be true if and only if there is some $\ell \in \mathbb{N}$ such that

$$5k^2 - 4 \;=\; \ell^2\,. \tag{3}$$

We claim that equation (3) has solutions (k_j, ℓ_j), for $j \in \mathbb{N}$, defined recursively by $(k_1, \ell_1) = (1, 1)$ and, for $j \geq 1$,

$$(k_j, \ell_j) \;=\; \left(\frac{3k_{j-1} + \ell_{j-1}}{2}\,,\; \frac{5k_{j-1} + 3\ell_{j-1}}{2}\right)\,.$$

In fact, $(1, 1)$ is a solution, and it is easy to check that (k_j, ℓ_j) is a solution whenever (k_{j-1}, ℓ_{j-1}) is a solution. Thus, the claim follows by induction.

An easy induction shows that $k_{j-1} < k_j$ for all j. Thus, the sequence $\{k_j\}_{j=1}^{\infty}$ is strictly increasing. For each integer k in this sequence, there is a rational number $u = x/y$ such that equation (2) is satisfied. Then, for $n = 3(k^2+1)$, we have a solution (x, y, z) for equation (2). We have obtained solutions for infinitely many values of n.

3. (*Colombia*) Let $n \geq 4$ be a fixed integer. A set $S = \{P_1, \ldots, P_n\}$ of n points is given in the plane such that no three are collinear and no four concyclic. For $1 \leq t \leq n$, let a_t be the number of circles $P_iP_jP_k$ that contain P_t in their interior, and let $m(S) = a_1 + a_2 + \cdots + a_n$. Prove that there exists a positive integer $f(n)$, depending only on n, such that the points of S are the vertices of a convex polygon if and only if $m(S) = f(n)$.

Solution supplied by George Evagelopoulos, Athens, Greece.

We prove that $m(S) \leq 2\binom{n}{4}$, with equality if and only if the points of S are the vertices of a convex polygon. The proof is by induction on n.

First we prove the assertion for the case $n = 4$. We consider two cases.

Case (i). The points are not the vertices of a convex quadrilateral.

In this case, we may assume that P_4 lies inside the triangle $P_1P_2P_3$. Then $a_1 = a_2 = a_3 = 0$ and $a_4 = 1$; whence, $m(S) = 1 < 2\binom{4}{4}$.

Case (ii). The points P_1, P_2, P_3, and P_4 are consecutive vertices of a convex quadrilateral.

This quadrilateral is not cyclic (according to one of the restrictions on S). If $\angle P_1 + \angle P_3 > 180°$, then $\angle P_2 + \angle P_4 < 180°$ and it is easily checked that $a_1 = a_3 = 1$ and $a_2 = a_4 = 0$, giving $m(S) = 2 = 2\binom{4}{4}$. Otherwise, we must have $\angle P_1 + \angle P_3 < 180°$ and $\angle P_2 + \angle P_4 > 180°$, which again gives us $m(S) = 2 = 2\binom{4}{4}$.

Now assume that the claim is true for $n = k \geq 4$, and consider a set $S = \{P_1, P_2, \ldots, P_{k+1}\}$ of $k+1$ points in general position in the plane. For each $i = 1, \ldots, k+1$, let $S_i = S \setminus \{P_i\}$. We compute in two different ways the number N of possible choices of five distinct points P_a, P_b, P_c, P_d, P_e from S such that the circle $P_bP_cP_d$ contains P_a.

There are $m(S)$ choices for the pair $(P_a, P_bP_cP_d)$. Once we have made this choice, there are $k-3$ choices for P_e. Hence, $N = (k-3)m(S)$. On the other hand, choosing P_e first, we have $m(S_e)$ choices for the pair $(P_a, P_bP_cP_d)$. Hence, $N = m(S_1) + m(S_2) + \cdots + m(S_{k+1})$. Thus,

$$\begin{aligned} m(S) &= \frac{m(S_1) + m(S_2) + \cdots + m(S_{k+1})}{k-3} \\ &\leq \frac{(k+1)2\binom{k}{4}}{k-3} = 2\binom{k+1}{4}. \end{aligned}$$

Here, we have used the induction hypothesis for each of the k-element sets S_1, S_2, $\ldots$, S_{k+1}. Equality holds if and only if the k points of each set S_i form a convex k-gon. It is easily seen that this is true if and only if the points of S form a convex $(k+1)$-gon. This completes the proof.

4. (*Czech Republic*) Let n and k be positive integers such that $\frac{n}{2} < k \leq \frac{2n}{3}$. Determine the least positive integer m for which it is possible to place each of m pawns on a square of an $n \times n$ chessboard so that no column or row contains a block of k adjacent unoccupied squares.

Solution supplied by George Evagelopoulos, Athens, Greece.

Let us say that a placement of pawns on the board is *good* if there is no $k \times 1$ or $1 \times k$ block of unoccupied squares. Label the rows and columns 0 through $n - 1$.

A standard good placement is obtained by putting pawns on all squares (i, j) such that $i + j + 1$ is divisible by k. This pawn pattern consists of diagonal lines of pawns across the board. Note that $i + j + 1 \leq 2n - 1$, and hence, since $n < 2k$, we have $i + j + 1 < 4k - 1$. Therefore, when $i + j + 1$ is divisible by k, we must have $i + j + 1 \in \{k,\ 2k,\ 3k\}$. Thus, there are at most three lines of pawns. One line (where $i + j + 1 = k$) contains k pawns, a second (where $i + j + 1 = 2k$) contains $2n - 2k$ pawns, and a third (where $i + j + 1 = 3k$) contains $2n - 3k$ pawns (this line vanishes in the extreme case where $3k = 2n$). In all cases, the total number of squares occupied by pawns is $4(n - k)$.

We now show that $4(n - k)$ is actually the least possible number of pawns in a good placement. Suppose we have a good placement of m pawns. Partition the board into nine rectangular regions

$$\begin{matrix} A & B & C \\ D & E & F \\ G & H & I \end{matrix}$$

so that the corner regions A, C, G, and I are $(n - k) \times (n - k)$ squares, B and H are $(n - k) \times (2k - n)$ rectangles, and D and F are $(2k - n) \times (n - k)$ rectangles. This is possible, since $2k - n > 0$. Assume that there are exactly b rows free of pawns in the rectangle B, h rows free in H, d columns free in D, and f columns free in F.

Take any one of the b rows that are pawn-free in B, and extend it to the left and to the right across the whole board. The portion of the extended row within rectangle A must contain at least one occupied square; otherwise the placement would not be good. The same can be said about its portion within C. Choose one occupied square in A and one occupied square in C (in that row) and put markers on those two squares.

Do the same for each of the b rows that are pawn-free in B and the h rows that are pawn-free in H. Perform a similar operation on each of the d columns that are pawn-free in D and the f columns that are pawn-free in F. At each step, we have put markers on two squares in $A \cup C \cup G \cup I$. In total, we have distributed $2(b + h + d + f)$ markers. A square could have been marked at most twice. Thus, the number of marked squares is at least $b + h + d + f$, and there is a pawn on each of them. Hence, we have at least that many pawns in $A \cup C \cup G \cup I$.

Moreover, there are at least $n - k - b$ pawns in B (since there is at least one pawn in each of the $n - k$ rows of B that are not pawn-free in B). Likewise, there are at least $n - k - h$ pawns in H, at least $n - k - d$ in D, and at least $n - k - f$ in F. Hence, $m \geq 4(n - k)$. Thus, $4(n - k)$ is the minimum sought.

5. (*France*) Let p and q be relatively prime positive integers. Determine the number of subsets S of $\{0, 1, 2, \dots\}$ such that $0 \in S$ and, for each element $n \in S$, the integers $n + p$ and $n + q$ belong to S.

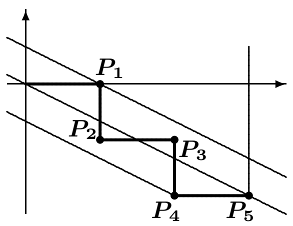
Solution supplied by George Evagelopoulos, Athens, Greece.

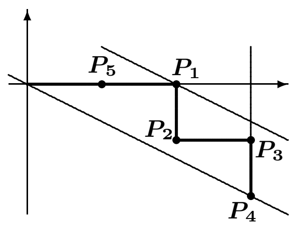
Every integer z has a unique representation $z = px + qy$ with integer coefficients x, y such that $0 \le x < q$. This inequality defines a vertical strip in $\mathbb{R}^2$ in which every lattice point (x, y) corresponds bijectively to an integer $px + qy$. For each such lattice point, we write the corresponding integer $px + qy$ in the unit square $[x, x+1) \times [y, y+1)$. Then every integer appears exactly once, and the non-negative integers are in the squares that correspond to lattice points (x, y) on or above the line $px + qy = 0$.

Let S be a set as described in the problem. We will call any such set *ideal*. All the elements of S have been written in some squares (in the strip in question). Put markers in those squares. We are given that if $n \in S$, then $n + p \in S$ and $n + q \in S$. It follows by induction that if a square Q is marked, then all squares that lie above Q or to the right of Q are also marked. In particular, since $0 \in S$, all squares in the half-strip $0 \le x < q$, $y \ge 0$ are marked.

It remains only to consider the squares associated with lattice points in the right triangle Δ defined by $px+qy > 0$, $y < 0$, $x < q$. If any such square Q is marked, then the whole portion of Δ upwards and rightwards of Q has to be marked. The border between the marked part of Δ and the unmarked part must be a polygonal path which follows gridlines between lattice points, running south and east from $(0, 0)$ to $(q, -p)$ without ever crossing the line $px + qy = 0$. We call such a path an *ideal path*. All that remains is to count the ideal paths.

Let Γ denote the set of all gridline paths of length $p + q$ from $(0, 0)$ to $(q, -p)$. Then the number of members of Γ equals $\binom{p+q}{p}$. Let E and S denote the unit moves east and south, respectively. Each path $\gamma \in \Gamma$ gives rise to a sequence D_1, D_2, $\dots$, D_{p+q}, where $D_i \in \{E, S\}$, such that q of the D_is are E and p of the D_is are S. For a path $\gamma = D_1, D_2, \dots, D_{p+q}$, and for $i = 1, 2, \dots, p+q$, let P_i be the point, called a *vertex* of γ, reached after tracing D_1, D_2, $\dots$, D_i from $(0, 0)$, and let ℓ_i be the line that is parallel to $px + qy = 0$ and passes through P_i. Since the correspondence between values of $px + qy$ and lattice points within the strip $0 \le x < q$ is bijective, we see that the lines ℓ_1, $\dots$, ℓ_{p+q} are distinct.





Two paths are said to be equivalent if one is obtained from the other by a circular shift of the coding sequence $D_1, D_2, \ldots, D_{p+q}$. For $\gamma \in \Gamma$, the equivalence class containing γ has $p+q$ elements, because γ admits precisely $p+q$ cyclic shifts and all of them are distinct. If $\gamma = D_1, D_2, \ldots, D_{p+q}$, let m be such that ℓ_m is the lowest among the ℓ_i's. Since the lines ℓ_i are distinct, such an m is unique. Then the path

$$D_m, \ldots, D_{p+q}, D_1, \ldots, D_{m-1}$$

is above the line $px+qy=0$. Every other cyclic shift gives rise to a path of which at least one vertex is below the line $px+qy=0$. Thus, each equivalence class contains exactly one ideal path, implying that the number of ideal paths equals $\dfrac{1}{p+q}\dbinom{p+q}{p}$.

6. (*France*) For a positive integer n, let $d(n)$ be the number of positive divisors of n. Find all positive integers such that $d(n)^3 = 4n$.

Solution supplied by George Evagelopoulos, Athens, Greece.

Let n be a positive integer such that $d(n)^3 = 4n$. For each prime number p, let a_p denote the exponent of p in the prime factorization of n. Since $4n$ is a cube, we must have $a_2 = 1 + 3\beta_2$ and $a_p = 3\beta_p$ for $p \geq 3$, where each β_p is a non-negative integer. Now we use the well-known result $d(n) = \prod\limits_p (1 + a_p)$, where the product is taken over all primes p. We obtain

$$d(n) \;=\; (2 + 3\beta_2) \prod_{p \geq 3} (1 + 3\beta_p)\,.$$

This relation shows that 3 does not divide $d(n)$. Since $d(n)^3 = 4n$, we see that 3 does not divide n. Thus, $\beta_3 = 0$, and the equation $d(n)^3 = 4n$ becomes

$$\frac{2+3\beta_2}{2^{1+\beta_2}} \;=\; \prod_{p \geq 5} \frac{p^{\beta_p}}{1+3\beta_p}\,. \tag{1}$$

.

For $p \geq 5$, we have $p^{\beta_p} \geq 5^{\beta_p} = (1+4)^{\beta_p} \geq 1 + 4\beta_p$. It follows that the right side of (1) is at least 1, and equality holds if and only if $\beta_p = 0$ for all $p \geq 5$. We deduce that $2 + 3\beta_2 \geq 2^{1+\beta_2}$. Hence,

$$2 + 3\beta_2 \;\geq\; 2(1+1)^{\beta_2} \;\geq\; 2\left(1 + \binom{\beta_2}{1} + \binom{\beta_2}{2}\right) \;=\; 2 + \beta_2 + \beta_2^2\,.$$

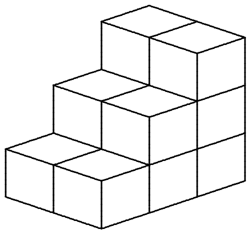
This gives $2\beta_2 \geq \beta_2^2$, implying that $\beta_2 \leq 2$. Thus, $\beta_2 \in \{0, 1, 2\}$.

If $B_2 = 0$ or $B_2 = 2$, then $\dfrac{2+3\beta_2}{2^{1+\beta_2}} = 1$ and both sides of equation (1) are equal to 1. Hence, 2 is the only prime divisor of n. Thus, we obtain the solutions $n = 2$ and $n = 2^7 = 128$.

If $\beta_2 = 1$, then $\frac{2+3\beta_2}{2^{1+\beta_2}} = \frac{5}{4}$, and there must be a factor of 5 on the right side of equation (1); that is, $\beta_5 > 0$. We cannot have $\beta_5 > 1$, because this would give $\frac{5^{\beta_5}}{1+3\beta_5} > \frac{5}{4}$. Therefore, $\beta_5 = 1$. Then $\frac{5^{\beta_5}}{1+3\beta_5} = \frac{5}{4}$. Substituting in (1) gives $\beta_p = 0$ for all $p \geq 7$. Thus, we obtain the solution $n = 2^4 \cdot 5^3 = 2000$.

In conclusion, the solutions are 2, 128, and 2000.

7. (*Italy*) The diagram shows a staircase-brick with 3 steps of width 2, made of 12 unit cubes. Determine all positive integers n for which it is possible to build an $n \times n \times n$ cube using such bricks.



Solution supplied by George Evagelopoulos, Athens, Greece.

The volume of a single brick is equal to 12. Thus, a necessary condition is that the side of the cube is a multiple of 6. Two bricks fit together to make a parallelepiped of size $2 \times 3 \times 4$, and such parallelepipeds can be stacked to form a cube of side 12, as well as any multiple of 12. We claim that this last condition is also necessary; that is, a cube of side $n = 6\ell$ can be constructed only if ℓ is even.

Now suppose a cube is constructed using $m = n^3/12 = 18\ell^3$ bricks, and that it is positioned in the octant with x, y, $z \geq 0$ with a vertex at the origin $O = (0,0,0)$. Colour each unit cube $[i,i+1] \times [j,j+1] \times [k,k+1]$ with one of eight colours, depending on the parity of i, j, and k in the triple (i,j,k). In each brick, all the eight colours are present, six of them appearing on only one unit cube and each of the remaining two colours appearing on three unit cubes.

Choose one of the eight colours, and let p be the number of bricks in which this colour appears three times. The number of bricks in which this colour appears only once is then $m - p$. Thus, this colour appears in the cube a total of $3p + (m - p) = m + 2p$ times. On the other hand, the eight colours are equally distributed in the cube, since the cube has sides of even length. Thus, each colour appears exactly $12m/8$ times. It follows that $m + 2p = 12m/8$; whence, $m = 4p$. Therefore, m is a multiple of 4, and ℓ must be even.

That completes the *Corner* for this issue. I wish you good problem solving over the summer. Send me your nice solutions and generalizations.

BOOK REVIEWS

John Grant McLoughlin

Mathematical Miniatures
By Svetoslav Savchev and Titu Andreescu, published by the Mathematical Association of America, 2003
ISBN 0-88385-645-X (paper), 230 pages, US$31.50.
Reviewed by ***Stan Wagon****, Macalester College, St. Paul, MN, USA.*

This book is a collection of problems from a variety of sources, such as national Olympiads and the Russian journal *Kvant*. The overall level is that of Olympiad problems, but many easier problems are included. Most importantly, the problems are chosen for their broad appeal and are grouped together according to technique or theme-and-variations. The style is nicely educational, and I found the book a pleasure to read.

One of my favorites in the book is Problem 23: Arrange the integers from 1 to n in a row so that the average of any two does not lie between them. The authors call this "an old and popular problem", but I had not heard of it before. It is challenging but not too difficult. What is noteworthy is that a Bulgarian high-school teacher came up with a two-dimensional variation. It is not obvious how to phrase such an extension. The result that was finally proved is that there are infinitely many integers n such that the integers from 1 to n^2 can be arranged in a square array so that the average of any two is not contained in the rectangle spanned by the two. The book is full of such extensions, which makes for interesting reading.

We all know that problem posers often have only one solution to a problem, but then learn of a much simpler solution from a student or journal contributor. A nice example occurs in Chapter 42, where it is related that Sergei Konyagin found a remarkably simple construction of an infinite set of positive integers such that the sum of any finite subset is not a perfect power. The problem poser had a quite complicated solution.

There is one problem in the book that I would consider... well... problematic. Problem 5 asks for a certain real solution to the equation $6x + 8\sqrt{1-x^2} = 5\sqrt{1+x} + \sqrt{1-x}$. But algorithms for doing this sort of thing have been known for centuries, and now such problems can be done in an eyeblink by a computer algebra system. Some of these algorithms are very sophisticated (for example, using cylindric algebras to verify inequalities). Knowledge of them (and their limitations) is critical to appreciating what is possible in modern mathematics. Thus, it seems sad that the problem community ignores their existence.

The book includes nine sets of three "Coffee Break Problems". These are not as hard as the main problems, and all have quite short solutions. But there are some gems in this group as well.

In all, this is an excellent collection, and any problemist will enjoy studying the contents and sharing them with his or her students.

Divisibility Relations between Fibonacci and Lucas Numbers

Peter Hilton and Jean Pedersen

1. Introduction

Over the years, the authors have published many papers on the arithmetical properties of Fibonacci and Lucas numbers (see [2], [3], [4], and [5]). Somer [7] had already obtained deep divisibility results, of a more general nature than ours and has, just recently, carried the story even further in [8]. We gave special attention to those numbers, called Lucasian, that are divisors of Lucas numbers, but not necessarily Lucas numbers themselves.

In this paper we concentrate on divisibility relations entirely within the domain of Fibonacci numbers F_m and Lucas number L_n. Thus, we have four questions to study: for what values of m and n is F_m a divisor of F_n, L_m a divisor of L_n, F_m a divisor of L_n, and L_n a divisor of F_m. To answer these questions, we include some already published results on the gcd of corresponding pairs of numbers; that is,

$$\gcd(F_m, F_n)\,, \quad \gcd(L_m, L_n)\,, \quad \gcd(F_m, L_n)\,,$$

that are contained in Section 3. Armed with these statements, in Section 4 we obtain complete characterizations of the required divisibility results. It turns out that, in all 4 cases, there is an *initial* condition and a *general* situation. Thus, for example, in Theorem 5 we answer the question: when is F_m a divisor of F_n? The answer is that this occurs precisely in the initial situation $m = 2$ and in the general situation $m \mid n$.

Our results are stated for m and n positive. One can extend them to encompass the possibility of m or n being negative (or zero) in light of (4).

Section 2 collects together the elementary properties of Fibonacci and Lucas numbers required in the last two sections.

2. Elementary Properties

The elementary linear properties are

$$L_n \;=\; F_{n-1} + F_{n+1} \tag{2}$$

and

$$5F_n \;=\; L_{n-1} + L_{n+1}\,. \tag{3}$$

Negative values of n are included through the rules

$$F_{-n} \;=\; (-1)^{n-1}F_n\,, \quad L_{-n} \;=\; (-1)^n L_n\,, \tag{4}$$

which follow immediately from Binet's formulas:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n = \alpha^n + \beta^n, \tag{5}$$

where

$$\alpha + \beta = 1, \quad \alpha\beta = -1. \tag{6}$$

From (5), we immediately deduce

$$F_{2n} = F_n L_n. \tag{7}$$

A simple induction, based on the recurrence relation $F_{n+2} = F_n + F_{n+1}$, shows that

$$\gcd(F_m, F_{m-1}) = 1, \quad \text{for all } m \geq 1. \tag{8}$$

The Fibonacci and Lucas sequences modulo 2, starting with $m = 0$, both read

$$0, 1, 1, 0, 1, 1, \ldots$$

so that

$$F_m \text{ even} \iff L_m \text{ even} \iff 3 \mid m. \tag{9}$$

From (8) and (9), we infer that

Theorem 1 Either $\gcd(F_m, L_m) = 1$ or $\gcd(F_m, L_m) = 2$.

Remark. Obviously, here and subsequently, we have $\gcd(F_m, L_m) = 2$ if and only if $3 \mid m$.

Proof:

$$\begin{aligned} \gcd(F_m, L_m) &= \gcd(F_m, F_{m-1} + F_{m+1}), \quad \text{by (2)} \\ &= \gcd(F_m, F_m + 2F_{m-1}), \quad \text{by the recurrence relation} \\ &= \gcd(F_m, 2F_{m-1}) \\ &= 1 \text{ or } 2, \quad \text{by (8) and (9).} \end{aligned}$$

■

Finally, Binet's formulas, together with (6), make it plain that

$$F_m \mid F_n \quad \text{if } m \mid n \tag{10}$$

and

$$L_m \mid L_n \quad \text{if } m \mid n \text{ oddly.} \tag{11}$$

(Recall that $a \mid b$ oddly means that b/a is an odd integer.)

3. The gcd Theorems

In this section we recall the three key gcd theorems relating to the sequences $\{F_n\}$ and $\{L_n\}$. Throughout the rest of this paper m and n are positive integers, $d = \gcd(m, n)$, and $|m|_2 = \max\{\mu : 2^\mu \mid m\}$. We will allow ourselves to consider F_m and L_m with negative (or zero) values of m, but never in the statements of our theorems. The three theorems are:

Theorem 2 $\gcd(F_m, F_n) = F_d$.

Theorem 3 $\gcd(L_m, L_n) = \begin{cases} L_d & \text{if } |m|_2 = |n|_2, \\ 1 \text{ or } 2 & \text{otherwise.} \end{cases}$

Theorem 4 $\gcd(F_m, L_n) = \begin{cases} L_d & \text{if } |m|_2 > |n|_2, \\ 1 \text{ or } 2 & \text{otherwise.} \end{cases}$

An elementary proof of Theorem 2 may be found in [1], Theorem 13.3. We do not know of any elementary proof of Theorem 3 or Theorem 4, whose statements are obviously more subtle than that of Theorem 2. Proofs of Theorems 3 and 4 may be found in [2] and [6].

4. The Divisibility Properties

Theorem 5 $F_m \mid F_n$ if and only if $m = 2$ or $m \mid n$.

Proof: The divisibility statement that $F_m \mid F_n$ if $m \mid n$ is (10). Since $F_2 = 1$, it is trivial that $F_m \mid F_n$ if $m = 2$. Now suppose $F_m \mid F_n$. We will assume that $m \geq 3$, since $F_1 = F_2 = 1$, and it is obvious that $m \mid n$ if $m = 1$.

By Theorem 2, $F_m = F_d$ if $F_m \mid F_n$. But F_m strictly increases with m if $m \geq 2$; thus, if $m \geq 3$ and $F_m \mid F_n$, then $m = d$; that is, $m \mid n$. ∎

Theorem 6 $L_m \mid L_n$ if and only if $m = 1$ or $m \mid n$ oddly.

Proof: The statement that $L_m \mid L_n$ if $m \mid n$ oddly is (11). Since $L_1 = 1$, it is obvious that $L_m \mid L_n$ if $m = 1$. Now suppose that $L_m \mid L_n$ and $m \neq 1$. Then $\gcd(L_m, L_n) = L_m$ and $L_m \geq 3$. Hence, by Theorem 3, $|m|_2 = |n|_2$ and $L_m = L_d$. But L_m is strictly increasing with m (for m positive). Hence, $m = d$; that is, $m \mid n$. But since $|m|_2 = |n|_2$, it follows that $m \mid n$ oddly. ∎

We now come to the most interesting, and surprising, results.

Theorem 7 $F_m \mid L_n$ if and only if one of the following holds:

(i) $m = 1$,

(ii) $m = 2$,

(iii) $m = 3$ and $3 \mid n$, or

(iv) $m = 4$ and $n \equiv 2 \pmod 4$.

Proof: We first show that, if $F_m \geq 3$ and F_m divides some L_n, then $F_m = 3$. To prove this, we start by making the stronger hypothesis that $F_m \geq 3$ and $F_m = L_n$, for some n. Then $\gcd(F_m, L_n) = F_m = L_n \geq 3$. Thus, by Theorem 4, we have $|m|_2 > |n|_2$ and $F_m = L_n = L_d$. Since L_n is strictly increasing, we see that $n = d$, implying that $n \mid m$. But since $|m|_2 > |n|_2$, one has $2n \mid m$. Thus,

$$L_n \mid F_{2n} \mid F_m \,.$$

But $L_n = F_m$. Then $L_n = F_{2n}$; whence $F_n = 1$, implying that $n = 1$ or 2. But it is not possible that $n = 1$, since $L_1 = 1 < 3$. Therefore, $n = 2$ and $F_m = L_2 = 3$.

Now assume only that $F_m \mid L_n$ with $F_m \geq 3$. Then we have $\gcd(F_m, L_n) = F_m \geq 3$. Hence, by Theorem 4, we see that $|m|_2 > |n|_2$ and $F_m = L_d$. But we have already shown that $F_m = L_d \geq 3$ implies that $F_m = 3$.

Thus, the condition $F_m \mid L_n$ implies that $F_m \leq 3$; that is, $m \leq 4$. Of course, $F_m \mid L_n$ if $m = 1$ or 2. Assume now that $m = 3$. Then $F_3 = 2$. Hence, $F_3 \mid L_n$ if and only if L_n is even; that is, $3 \mid n$.

Finally, suppose $m = 4$. Since $F_4 = 3$, we know that $F_m \mid L_n$ if and only if $3 \mid L_n$. But the Lucas sequence modulo 3, starting with $n = 1$, reads

$$1\,,\,0\,,\,1\,,\,1\,,\,2\,,\,0\,,\,2\,,\,2\,,\,1\,,\,0\,,\;\ldots$$

Therefore, $3 \mid L_n$ if and only if $n \equiv 2 \pmod 4$.

This completes the proof of Theorem 7. ∎

Our final theorem is

Theorem 8 $L_n \mid F_m$ if and only if $n = 1$ or $2n \mid m$.

Proof: If $n = 1$, then $L_n = 1$, and $L_n \mid F_m$. If $2n \mid m$, then $L_n \mid F_{2n} \mid F_m$ by (7) and (10). Conversely, suppose $L_n \mid F_m$, for $n \neq 1$. Then $L_n \geq 3$ and $\gcd(F_m, L_n) = L_n$. It follows from Theorem 4 that $L_n = L_d$ and $|m|_2 > |n|_2$. Since L_n is a strictly increasing function of n, we have $n = d$; that is, $n \mid m$. But $|m|_2 > |n|_2$, so, in fact, $2n \mid m$. ∎

Theorems 5 and 8 have the following rather striking consequence.

Corollary 1 $L_n \mid F_m$ if and only if $F_{2n} \mid F_m$.

Proof: Both divisibility conditions are equivalent to $n = 1$ or $2n \mid m$. ∎

Of course, it is obvious from (7) that $L_n \mid F_m$ if $F_{2n} \mid F_m$; it is the converse implication which is somewhat surprising. Indeed, the corollary itself has the consequence that $F_n \mid F_m$ if $L_n \mid F_m$, itself rather striking since, by Theorem 1, F_n and L_n are 'nearly' coprime.

This may also be seen by noting that $3 = L_2$ and we know, by Theorem 6, that $L_2 \mid L_n$ if and only if $2 \mid n$ oddly; that is, $n \equiv 2 \pmod 4$.

References

[1] Burton, David, *Elementary Number Theory*, 5th edition, McGraw-Hill, 2002.

[2] Hilton, Peter, and Jean Pedersen, Fibonacci and Lucas numbers in teaching and research, *Journées Mathématiques & Informatique*, Vol. 3 (1991–1992), 36–57.

[3] Hilton, Peter, Derek Holton, and Jean Pedersen, *Mathematical Reflections – In a Room with Many Mirrors*, Undergraduate Texts in Mathematics, Springer–Verlag, New York, 1997, 351 pp. (Second printing, 1998.)

[4] Hilton, Peter, Derek Holton, and Jean Pedersen, *Mathematical Vistas – In a Room with Many Mirrors*, Undergraduate Texts in Mathematics, Springer–Verlag, New York, 2002, 335 pp.

[5] Hilton, Peter, Jean Pedersen, and Lawrence Somer, On Lucasian numbers, *The Fibonacci Quarterly*, 35(1), (1997), 43–47.

[6] McDaniel, Wayne, The gcd in Lucas sequences and Lehmer number sequences, *The Fibonacci Quarterly*, 29(1), (1991), 24–29.

[7] Somer, Lawrence, Divisibility of terms in Lucas sequences by their subscripts, G.E. Bergum et al. (eds), *Applications of Fibonacci Numbers*, 5, Kluwer Academic Publishers (1993), 515–525.

[8] Somer, Lawrence, A further note on Lucasian numbers, F.T. Howard (ed.), *Applications of Fibonacci Numbers*, 9: Proceedings of the Tenth International Research Conference on Fibonacci numbers and their Applications. Kluwer Academic Publishers (2004), 225–234.

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PROBLEMS

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1er décembre 2005**. *Une étoile* ($\star$) *après le numéro indique que le problème a été soumis sans solution.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

3039. *Proposé par Dorin Mărghidanu, Colegiul Naţional "A.I. Cuza", Corabia, Roumanie.*

Soit a et b deux nombres réels non nuls donnés. Trouver toutes les fonctions $f:\mathbb{R}\to\mathbb{R}$ telles que, pour tout $x\in\mathbb{R}$,

$$f\left(x-\frac{b}{a}\right)+2x \;\le\; \frac{a}{b}x^2+2\frac{b}{a} \;\le\; f\left(x+\frac{b}{a}\right)-2x\,.$$

3040. *Proposé par Dorin Mărghidanu, Colegiul Naţional "A.I. Cuza", Corabia, Roumanie.*

Montrer que, pour trois nombres naturels a, b, c arbitraires mais distincts et plus grands que 1,

$$\left(1+\frac{1}{a}\right)\left(2+\frac{1}{b}\right)\left(3+\frac{1}{c}\right) \;\le\; \frac{91}{8}\,.$$

3041. *Proposé par Ovidiu Furdui, étudiant, Western Michigan University, Kalamazoo, MI, USA.*

Montrer que

(a) $\sin x = 2^{n-1}\displaystyle\sum_{k=0}^{n-1}\sin\left(x+\frac{k\pi}{n}\right)$ pour tout $x\in\mathbb{R}$ et tout entier $n\ge 1$;

(b) $n\cot nx = \displaystyle\sum_{k=0}^{n-1}\cot\left(x+\frac{k\pi}{n}\right)$ pour $x\in\left(0,\frac{\pi}{n}\right)$.

3042. *Proposé par Vasile Cîrtoaje, Université de Ploiesti, Roumanie.*

Soit x_1, x_2, $\ldots$, x_n des nombres positifs tels que $x_1x_2\cdots x_n = 1$. Si $n\ge 3$ et $0<\lambda\le(2n-1)/(n-1)^2$, montrer que

$$\frac{1}{\sqrt{1+\lambda x_1}}+\frac{1}{\sqrt{1+\lambda x_2}}+\cdots+\frac{1}{\sqrt{1+\lambda x_n}} \;\le\; \frac{n}{\sqrt{1+\lambda}}\,.$$

3043. *Proposé par Ovidiu Furdui, étudiant, Western Michigan University, Kalamazoo, MI, USA.*

Pour tout quadrilatère convexe $ABCD$, montrer que

$$\begin{aligned} 1 - \cos(A+B)\cos(A+C)\cos(A+D) \\ \leq\ 2M\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B+C}{2}\right)\sin\left(\frac{C+A}{2}\right), \end{aligned}$$

où $M = \max\{\sin A, \sin B, \sin C, \sin D\}$.

3044. *Proposé par José Luis Díaz-Barrero et Juan José Egozcue, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit $\{a_n\}$ la suite définie par $a_0 = 1$, $a_1 = 2$ et, pour $n \geq 2$, $a_n = a_{n-1} + a_{n-2}$. Trouver la somme

$$\sum_{n=1}^{\infty} \frac{a_{2n+2}}{a_{n-1}^2 a_{n+1}^2}.$$

3045. *Proposé par Vasile Cîrtoaje, Université de Ploiesti, Roumanie.*

Soit a, b et c des nombres réels positifs tels que $abc \geq 1$. Montrer que

(a) $a^{\frac{a}{b}} b^{\frac{b}{c}} c^{\frac{c}{a}} \geq 1$;

(b) $a^{\frac{a}{b}} b^{\frac{b}{c}} c^{c} \geq 1$.

3046. *Proposé par James T. Bruening, Southeast Missouri State University, Cape Girardeau, MO, USA.*

On place un miroir dans le premier quadrant du plan des xy, perpendiculairement à ce plan et suivant la droite joignant les points $(b, 0)$ et $(0, b)$, pour un certain $b > 0$. Un autre miroir est placé de façon analogue, mais le long de la droite $y = kx$ où $k > 1$. Une source lumineuse placée en $(a, 0)$, $0 < a < b$, émet un rayon de lumière dans le premier quadrant parallèlement au premier miroir.

Trouver k tel que, lorsque le rayon est réfléchi une seule fois par chacun des miroirs, il revient à la source lumineuse en $(a, 0)$.

3047. *Proposé par Michel Bataille, Rouen, France.*

Soit n un entier positif. Calculer $\displaystyle\sum_{k=1}^{n} \sec\left(\frac{2k\pi}{2n+1}\right)$.

3048. *Proposé par Gabriel Dospinescu, Onesti, Roumanie.*

Trouver tous les polynômes P à coefficients entiers satisfaisant la propriété que, pour tous entiers a et b relativement premiers, la suite $\{P(an+b)\}_{n\geq 1}$ contient un nombre infini de termes relativement premiers deux à deux.

3049. *Proposé par Óscar Ciaurri, Université de La Rioja, Logroño, Espagne et José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

On donne la fonction $f(x) = \dfrac{x^2}{\sqrt{1+x^2}}e^{-\arctan x}$,

(a) trouver l'asymptote oblique L dans le premier quadrant, et

(b) trouver dans le premier quadrant l'aire limitée par le graphe de $y = f(x)$ et la droite L.

3050. *Proposé par Christopher J. Bradley, Bristol, GB.*

Soit ABC un triangle avec les Céviennes AX, BY et CZ. Soit respectivement L, M et N les points milieu de AX, BY et CZ. On suppose que AM et AN coupent respectivement BC en P_1 et P_2 ; que BN et BL coupent respectivement CA en Q_1 et Q_2 et finalement que CL et CM coupent respectivement AB en R_1 et R_2.

Montrer que P_1, P_2, Q_1, Q_2, R_1, R_2 sont sur une conique.

. . . .

3039. *Proposed by Dorin Mărghidanu, Colegiul Naţional "A.I. Cuza", Corabia, Romania.*

Let a, b be fixed non-zero real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all $x \in \mathbb{R}$,

$$f\left(x - \frac{b}{a}\right) + 2x \;\le\; \frac{a}{b}x^2 + 2\frac{b}{a} \;\le\; f\left(x + \frac{b}{a}\right) - 2x\,.$$

3040. *Proposed by Dorin Mărghidanu, Colegiul Naţional "A.I. Cuza", Corabia, Romania.*

Prove that, for any three distinct natural numbers a, b, c greater than 1,

$$\left(1 + \frac{1}{a}\right)\left(2 + \frac{1}{b}\right)\left(3 + \frac{1}{c}\right) \;\le\; \frac{91}{8}\,.$$

3041. *Proposed by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.*

Prove that

(a) $\sin x = 2^{n-1}\displaystyle\sum_{k=0}^{n-1}\sin\left(x + \frac{k\pi}{n}\right)$ for all $x \in \mathbb{R}$ and all integers $n \ge 1$;

(b) $n\cot nx = \displaystyle\sum_{k=0}^{n-1}\cot\left(x + \frac{k\pi}{n}\right)$ for $x \in \left(0, \frac{\pi}{n}\right)$.

3042. *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let x_1, x_2, $\ldots$, x_n be positive numbers such that $x_1x_2\cdots x_n = 1$. For $n \geq 3$ and $0 < \lambda \leq (2n-1)/(n-1)^2$, prove that

$$\frac{1}{\sqrt{1+\lambda x_1}} + \frac{1}{\sqrt{1+\lambda x_2}} + \cdots + \frac{1}{\sqrt{1+\lambda x_n}} \;\leq\; \frac{n}{\sqrt{1+\lambda}}\,.$$

3043. *Proposed by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.*

For any convex quadrilateral $ABCD$, prove that

$$\begin{aligned} 1 - \cos(A+B)\cos(A+C)\cos(A+D) & \\ \leq\; 2M\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B+C}{2}\right)\sin\left(\frac{C+A}{2}\right)\,, \end{aligned}$$

where $M = \max\{\sin A,\, \sin B,\, \sin C,\, \sin D\}$.

3044. *Proposed by José Luis Díaz-Barrero and Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let $\{a_n\}$ be the sequence defined by $a_0 = 1$, $a_1 = 2$, and, for $n \geq 2$, $a_n = a_{n-1} + a_{n-2}$. Find the sum

$$\sum_{n=1}^{\infty} \frac{a_{2n+2}}{a_{n-1}^2 a_{n+1}^2}\,.$$

3045. *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let a, b, c be positive real numbers such that $abc \geq 1$. Prove that

(a) $a^{\frac{a}{b}} b^{\frac{b}{c}} c^{\frac{c}{a}} \geq 1$;   (b) $a^{\frac{a}{b}} b^{\frac{b}{c}} c^{c} \geq 1$.

3046. *Proposed by James T. Bruening, Southeast Missouri State University, Cape Girardeau, MO, USA.*

A mirror is placed in the first quadrant of the xy–plane (perpendicular to the plane) along the straight line joining the points $(b,0)$ and $(0,b)$, for some $b > 0$. Another mirror is placed similarly along the line $y = kx$ where $k > 1$. A light source at $(a,0)$, $0 < a < b$, shoots a beam of light into the first quadrant parallel to the first mirror.

Find k such that when the beam is reflected exactly once by each mirror, it passes through the original light source at $(a,0)$.

3047. *Proposed by Michel Bataille, Rouen, France.*

Let n be a positive integer. Evaluate $\sum_{k=1}^{n} \sec\left(\frac{2k\pi}{2n+1}\right)$.

3048. *Proposed by Gabriel Dospinescu, Onesti, Romania.*

Find all polynomials P with integer coefficients which satisfy the property that, for any relatively prime integers a and b, the sequence $\{P(an+b)\}_{n\geq 1}$ contains an infinite number of terms, any two of which are relatively prime.

3049. *Proposed byÓscar Ciaurri, Universidad de La Rioja, Logroño, Spain and José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Given the function $f(x) = \frac{x^2}{\sqrt{1+x^2}} e^{-\arctan x}$,

(a) find the slant asymptote L in the first quadrant, and

(b) find the area in the first quadrant bounded by the graph of $y = f(x)$ and the line L.

3050. *Proposed by Christopher J. Bradley, Bristol, UK.*

Let ABC be a triangle with Cevians AX, BY, CZ. Let L, M, N be the mid-points of AX, BY, CZ, respectively. Let AM and AN meet BC at P_1 and P_2, respectively; let BN and BL meet CA at Q_1 and Q_2, respectively; and let CL and CM meet AB at R_1 and R_2, respectively.

Prove that P_1, P_2, Q_1, Q_2, R_1, R_2 lie on a conic.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2608★. [2001 : 49; 2002 : 181–184] *Proposed by Faruk Zejnulahi and Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Suppose that x, y, $z \geq 0$ and $x^2+y^2+z^2=1$. Prove or disprove that

(a) $1 \;\leq\; \dfrac{x}{1-yz}+\dfrac{y}{1-zx}+\dfrac{z}{1-xy} \;\leq\; \dfrac{3\sqrt{3}}{2}$;

(b) $1 \;\leq\; \dfrac{x}{1+yz}+\dfrac{y}{1+zx}+\dfrac{z}{1+xy} \;\leq\; \sqrt{2}$.

Solution by B.J. Venkatachala, Indian Institute of Science, Bangalore, India.

For the inequality on the right in part (b), the only solution given in [2002 : 183–184] used the method of Lagrange Multipliers. We prove it using only elementary algebra.

Without loss of generality, we assume that $x \leq z$ and $y \leq z$. Then

$$\sum_{\text{cyclic}} \frac{x}{1+yz} \;\leq\; \frac{x+y+z}{1+xy},$$

and it is sufficient to prove that

$$x+y+z \;\leq\; \sqrt{2}(1+xy),$$

for $0 \leq x \leq z$, $0 \leq y \leq z$, and $x^2+y^2+z^2=1$. That is,

$$x+y+\sqrt{1-x^2-y^2} \;\leq\; \sqrt{2}(1+xy).$$

This is further equivalent to

$$1-x^2-y^2 \;\leq\; \left(\sqrt{2}(1+xy)-x-y\right)^2.$$

Setting $\alpha = x+y$ and $\beta = xy$, this inequality becomes

$$1-\alpha^2+2\beta \;\leq\; \left(\sqrt{2}(1+\beta)-\alpha\right)^2.$$

This simplifies to

$$\left(\sqrt{2}\alpha-\beta-1\right)^2+\beta^2 \;\geq\; 0.$$

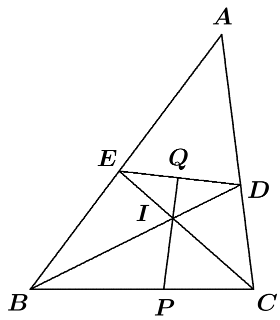
Since the left side is a sum of squares, the result follows.

We observe that equality occurs only when $\beta = 0$ and $\sqrt{2}\alpha - \beta - 1 = 0$; that is, $xy = 0$ and $x + y = 1/\sqrt{2}$. We conclude that we get equality only when $(x, y, z) = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ or $(x, y, z) = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$. By removing the artificially imposed restrictions that $x \le z$ and $y \le z$, we obtain a third (symmetric) possibility, namely $(x, y, z) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$.

2939. [2004 : 229, 232] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that $\triangle ABC$ has incentre I and that BI, CI meet AC, AB at D, E, respectively. Suppose further that the bisector of $\angle BIC$ meets BC and DE at P and Q, respectively, and that $PI = 2QI$. Prove that $\angle BAC = 60^\circ$.

I. *Solution by Michel Bataille, Rouen, France.*



With the usual notations, the area of $\triangle ABC$ is $\frac{1}{2}bc\sin A$. This area is also $\frac{1}{2}w_a c\sin\left(\frac{1}{2}A\right) + \frac{1}{2}w_a b\sin\left(\frac{1}{2}A\right)$, where w_a is the length of the internal bisector of $\angle BAC$. It follows that $\left(\frac{1}{b} + \frac{1}{c}\right) w_a = 2\cos\left(\frac{1}{2}A\right)$.

Applying this result to $\triangle BIC$ and $\triangle DIE$ in place of $\triangle ABC$, we obtain

$$\begin{aligned}\left(\frac{1}{BI} + \frac{1}{CI}\right) IP &= 2\cos\left(\tfrac{1}{2}\angle BIC\right) \\ &= 2\cos\left(\tfrac{1}{2}\angle DIE\right) \;=\; \left(\frac{1}{DI} + \frac{1}{EI}\right) IQ\,.\end{aligned}$$

By hypothesis, we have $PI = 2QI$. Hence,

$$2\left(\frac{1}{BI} + \frac{1}{CI}\right) = \left(\frac{1}{DI} + \frac{1}{EI}\right)\,. \tag{1}$$

Letting r denote the inradius, we have

$$\frac{1}{BI} + \frac{1}{CI} = \frac{\sin(\frac{1}{2}B) + \sin(\frac{1}{2}C)}{r}$$

and

$$\begin{aligned}\frac{1}{DI} + \frac{1}{EI} &= \frac{\sin(\angle ADI) + \sin(\angle AEI)}{r} \\ &= \frac{\sin(A + \frac{1}{2}B) + \sin(A + \frac{1}{2}C)}{r}\,.\end{aligned}$$

Therefore, equation (1) may be written successively as

$$\begin{aligned}2\left(\sin(\tfrac{1}{2}B) + \sin(\tfrac{1}{2}C)\right) &= \sin(A + \tfrac{1}{2}B) + \sin(A + \tfrac{1}{2}C)\,, \\ 2\sin\left(\frac{B+C}{4}\right)\cos\left(\frac{C-B}{4}\right) &= \sin\left(A + \frac{B+C}{4}\right)\cos\left(\frac{C-B}{4}\right)\,.\end{aligned}$$

Note that $\cos\left(\frac{C-B}{4}\right) \neq 0$, since $\frac{C-B}{4} \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$. Hence,

$$\begin{aligned} 2\sin\left(\frac{\pi - A}{4}\right) &= \sin\left(\frac{\pi + 3A}{4}\right), \\ \sin\left(\frac{\pi - A}{4}\right) &= \sin\left(\frac{\pi - 3A}{4}\right) - \sin\left(\frac{\pi - A}{4}\right) \\ &= 2\sin\left(\frac{A}{2}\right)\sin\left(\frac{\pi - A}{4}\right). \end{aligned}$$

Note that $\sin\left(\frac{\pi-A}{4}\right) \neq 0$, since $\frac{\pi - A}{4} \in \left(0, \frac{\pi}{4}\right)$. Hence, $1 = 2\sin\left(\frac{1}{2}A\right)$, and the result follows immediately.

II. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

As usual, let $a = BC$, $b = CA$, $c = AB$, and $s = \frac{1}{2}(a+b+c)$. Then it is well known that $BD = \frac{2\sqrt{acs(s-b)}}{a+c}$. Also, $\frac{BI}{ID} = \frac{BC}{DC} = \frac{a+c}{b}$. Hence,

$$BI = \sqrt{\frac{ac(s-b)}{s}} \qquad \text{and} \qquad ID = \frac{b}{a+c}\sqrt{\frac{ac(s-b)}{s}},$$

Similarly,

$$CI = \sqrt{\frac{ab(s-c)}{s}} \qquad \text{and} \qquad IE = \frac{c}{a+b}\sqrt{\frac{ab(s-c)}{s}}.$$

Now, let $\theta = \angle BIP$. Then

$$\tfrac{1}{2}BI \cdot CI \sin(2\theta) = [BIC] = [BIP] + [PIC] = \tfrac{1}{2}PI(BI + CI)\sin\theta.$$

Hence, $PI = \frac{2BI \cdot CI \cos\theta}{BI + CI}$. Likewise, $IQ = \frac{2ID \cdot IE \cos\theta}{ID + IE}$. Substituting the above expressions for BI, CI, ID and IE, and setting $PI = 2IQ$, we get

$$\frac{1}{\sqrt{c(s-b)} + \sqrt{b(s-c)}} = \frac{2\sqrt{bc}}{(a+b)\sqrt{b(s-b)} + (a+c)\sqrt{c(s-c)}},$$

which can be transformed into

$$\left(2\sqrt{(s-b)(s-c)} - \sqrt{bc}\right)\left(\sqrt{b(s-c)} + \sqrt{c(s-b)}\right) = 0.$$

Hence, $4(s-b)(s-c) = bc$. Then $\cos A = 1 - \frac{2(s-b)(s-c)}{bc} = \frac{1}{2}$, so that $A = 60^\circ$.

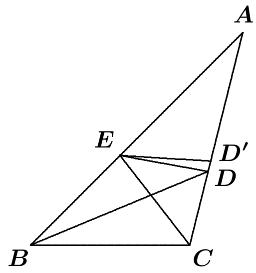
Comment: As a consequence, $\triangle DEP$ is equilateral with I as its centroid.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

2940. [2004 : 229, 232] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

In $\triangle ABC$, the bisectors of $\angle ABC$ and $\angle ACB$ meet AC and AB at D and E, respectively, and $\angle ADE - \angle AED = 60^\circ$. Prove that $\angle ACB = 120^\circ$.

Composite of similar solutions by Peter Y. Woo, Biola University, La Mirada, CA, USA; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.



We will prove the stronger result that $\angle ADE - \angle AED = 60^\circ$ if and only if $\angle ACB = 120^\circ$.

Since BD bisects $\angle B$, we get $\dfrac{DC}{DA} = \dfrac{BC}{BA}$. Using the Sine Law in $\triangle AEC$ and then in $\triangle ABC$, we get

$$\begin{aligned}\frac{EC}{AE} &= \frac{\sin\angle EAC}{\sin\angle ECA} = \frac{\sin\angle A}{\sin\left(\frac{1}{2}\angle C\right)} = 2\cos\left(\tfrac{1}{2}\angle C\right)\frac{\sin\angle A}{\sin\angle C}\\ &= 2\cos\left(\tfrac{1}{2}\angle C\right)\frac{BC}{BA} = 2\cos\left(\tfrac{1}{2}\angle C\right)\frac{DC}{DA}.\end{aligned}$$

Let the angle bisector of $\angle AEC$ meet AC at D'. Then

$$\frac{D'C}{D'A} = \frac{EC}{AE} = 2\cos\left(\tfrac{1}{2}\angle C\right)\frac{DC}{DA}.$$

Suppose that $\angle C < 120^\circ$. Then $2\cos\left(\frac{1}{2}\angle C\right) > 1$, and thus we have $\dfrac{D'C}{D'A} > \dfrac{DC}{DA}$. It follows that the points C, D, D', and A lie in that order on AC. (This is the case shown in the diagram.) Thus,

$$\angle AED > \angle AED' = \angle CED' > \angle CED.$$

Hence,

$$\angle ADE - \angle AED < \angle ADE - \angle CED = \angle DCE = \tfrac{1}{2}\angle C < 60^\circ.$$

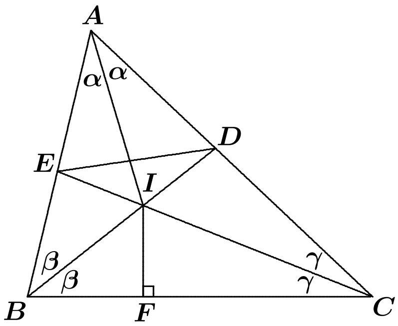
If $\angle C > 120^\circ$, then all the inequalities in the paragraph above are reversed, giving $\angle ADE - \angle AED > 60^\circ$. Finally, if $\angle C = 120^\circ$, then $D' = D$, and we find that $\angle ADE - \angle AED = 60^\circ$. The result follows.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; KIN FUNG CHUNG, student, University of Toronto, Toronto, ON; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; D.J. SMEENK, Zaltbommel, the Netherlands; IAN VANDERBURGH, University of Waterloo, Waterloo, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer. There was one incomplete solution.

2941. [2004 : 229, 232] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

In $\triangle ABC$, the bisectors of $\angle ABC$ and $\angle ACB$ meet AC and AB at D and E, respectively. Let I be the intersection of BD and CE, and let F be the foot of the perpendicular from I to BC. Prove that if $\angle ADE = \angle BIF$, then $\angle AED = \angle CIF$.

Composite of almost identical solutions by John G. Heuver, Grande Prairie, AB; Li Zhou, Polk Community College, Winter Haven, FL, USA; and the proposer.



Let $\alpha = \angle A/2$, $\beta = \angle B/2$, and $\gamma = \angle C/2$. Then $\alpha + \beta + \gamma = 90^\circ$. Note that

$$\angle ADE \;=\; \angle CED + \angle DCE \;=\; \angle CED + \gamma\,.$$

Also, using the hypothesis that $\angle ADE = \angle BIF$, we have

$$\angle ADE \;=\; \angle BIF \;=\; 90^\circ - \beta \;=\; \alpha + \gamma\,.$$

It follows that $\angle CED = \alpha$. Thus, $\angle IED = \angle IAD$. This implies that the points A, E, I, and D are concyclic. Then

$$\angle AED \;=\; \angle AID \;=\; \angle BAI + \angle ABI \;=\; \alpha + \beta \;=\; 90^\circ - \gamma \;=\; \angle CIF\,.$$

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; D.J. SMEENK, Zaltbommel, the Netherlands; IAN VANDERBURGH, University of Waterloo, Waterloo, ON; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and TITU ZVONARU, Comăneşti, Romania. Most solvers used trigonometry, the Law of Sines in particular.

Other consequences that may be deduced here are that $DI = EI$ and $\angle A = 60^\circ$.

2942. [2004 : 229, 232] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given $\triangle ABC$ with $\angle ABC = 2\angle ACB$, suppose that D is a point on the ray CB such that $\angle ADC = \frac{1}{2}\angle BAC$. Prove that

$$\frac{1}{CD} \;=\; \frac{1}{AB} - \frac{1}{AC}\,.$$

Solution by Richard B. Eden, Ateneo de Manila University, The Philippines.

Let $\angle ACB = 2\theta$. Then $\angle ABC = 4\theta$, $\angle BAC = 180^\circ - 6\theta$, and $\angle ADC = 90^\circ - 3\theta$. Now, $\angle DAB = 4\theta - (90^\circ - 3\theta) = 7\theta - 90^\circ$, so that $\angle DAC = (7\theta - 90^\circ) + (180^\circ - 6\theta) = \theta + 90^\circ$. Using the Sine Law for triangles DAC and BAC, we obtain

$$\begin{aligned}\frac{1}{CD} + \frac{1}{AC} &= \frac{\sin(90^\circ - 3\theta)}{AC\sin(90^\circ + \theta)} + \frac{1}{AC} \\ &= \frac{1}{AC}\left(\frac{\cos 3\theta}{\cos\theta} + 1\right) \\ &= \frac{\sin 2\theta}{AB\sin 4\theta}\left(\frac{\cos 3\theta + \cos\theta}{\cos\theta}\right) \\ &= \frac{1}{2AB\cos 2\theta} \cdot \frac{2\cos 2\theta\cos\theta}{\cos\theta} = \frac{1}{AB},\end{aligned}$$

which completes the proof.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; KIN FUNG CHUNG, student, University of Toronto, Toronto, ON; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; BOB SERKEY, Leonia, NJ, USA; D.J. SMEENK, Zaltbommel, the Netherlands; IAN VANDERBURGH, University of Waterloo, Waterloo, ON; Mª JESÚS VILLAR RUBIO, Santander, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Note that the featured solution is valid only if $\theta \geq 90^\circ/7$. Then $4\theta \geq 90^\circ - 3\theta$, which implies that the point D is on the ray CB, beyond the point B (or it coincides with B). If $\theta < 90^\circ/7$, then the solution is similar; the only difference is that $\angle DAB$ is $90^\circ - 7\theta$, rather than $7\theta - 90^\circ$. In this case, $4\theta < 90^\circ - 3\theta < 90^\circ$, meaning that the point D lies on the segment CB. Villar Rubio was the only solver who considered both cases.

2943. [2004 : 230, 232] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given $\triangle ABC$, let D be the point on AB produced beyond B such that $BD = BC$, and let E be the point on AC produced beyond C such that $CE = BC$. Let P be the intersection of BE and CD, and suppose that $\frac{DP}{BE} + \frac{EP}{CD} = 2\sin\left(\frac{\angle BAC}{2}\right)$. Prove that $\angle BAC = 90^\circ$.

Solution by Michel Bataille, Rouen, France.

Let $AB = c$, $BC = a$ and $CA = b$. Since triangles DBC and ECB are isosceles, and since $\angle BCE = 180^\circ - C$ and $\angle CBD = 180^\circ - B$, we have $\angle CBP = \angle CEP = \frac{1}{2}C$ and $\angle BDP = \angle BCP = \frac{1}{2}B$. It follows that

$$\begin{aligned}\angle PCE &= 180^\circ - C - \tfrac{1}{2}B = A + \tfrac{1}{2}B, \\ \angle PBD &= 180^\circ - B - \tfrac{1}{2}C = A + \tfrac{1}{2}C, \\ \text{and} \quad \angle BPD &= \angle CPE = \tfrac{1}{2}(B+C) = 90^\circ - \tfrac{1}{2}A.\end{aligned}$$

Using the Sine Law for triangles BCE and CBD, we obtain

$$BE \;=\; 2a\cos\left(\tfrac{1}{2}C\right) \quad \text{and} \quad CD \;=\; 2a\cos\left(\tfrac{1}{2}B\right) .$$

Using the Sine Law for triangles DBP and ECP, we then obtain

$$DP \;=\; \frac{a}{\cos\left(\frac{1}{2}A\right)}\sin\left(A+\tfrac{1}{2}C\right) \quad \text{and} \quad EP \;=\; \frac{a}{\cos\left(\frac{1}{2}A\right)}\sin\left(A+\tfrac{1}{2}B\right) .$$

Thus, the hypothesis $\dfrac{DP}{BE}+\dfrac{EP}{CD} \;=\; 2\sin\left(\dfrac{\angle BAC}{2}\right)$ can be written as

$$\begin{aligned}
\sin A \;&=\; \frac{1}{2}\left(\frac{\sin\left(A+\frac{1}{2}C\right)}{\cos\left(\frac{1}{2}C\right)}+\frac{\sin\left(A+\frac{1}{2}B\right)}{\cos\left(\frac{1}{2}B\right)}\right) \\
&=\; \sin A+\frac{1}{2}\cos A\left(\frac{\sin\left(\frac{1}{2}C\right)\cos\left(\frac{1}{2}B\right)+\sin\left(\frac{1}{2}B\right)\cos\left(\frac{1}{2}C\right)}{\cos\left(\frac{1}{2}B\right)\cos\left(\frac{1}{2}C\right)}\right) , \\
&=\; \sin A+\frac{1}{2}\cos A\left(\frac{\cos\left(\frac{1}{2}A\right)}{\cos\left(\frac{1}{2}B\right)\cos\left(\frac{1}{2}C\right)}\right) .
\end{aligned}$$

It follows that $\cos A = 0$ and, therefore, $\angle BAC = A = 90^\circ$.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD B. EDEN, Ateneo de Manila University, The Philippines; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

2944. [2004 : 230, 232] *Proposed by Václav Konečný, Big Rapids, MI, USA.*

Given an ellipse with foci F_1 and F_2, minor vertices V_1' and V_2', a line ℓ, and a point P not on ℓ. Construct, with straightedge alone, the line through P which is

(a) parallel to ℓ;

(b) perpendicular to ℓ.

The constructions are well known, if a circle with its centre is given instead of an ellipse and its foci (Poncelet–Steiner Construction Theorem).

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

(a) We freely use the following two constructions (that use unmarked straightedge only).

1. Given two points A and B and a line parallel to AB, we can locate the mid-point of AB.
2. Given A and B and their mid-point, we can draw a line parallel to AB through any point P.

[*Editor's comment*. Constructions by straightedge alone are the subject of problems 2694–2696 [2001 : 535; 2002 : 553–557], 2740 [2002 : 245; 2003 : 246], and 2741 [2002 : 245; 2003 : 325–328]. Further references are provided with the first set on page 557. The above constructions 1 and 2 can be found in the solution to 2695 [2002 : 553–554] and to 2741 [2003 : 326, construction II(b)].]

The given line ℓ must intersect one of F_1V_1' or F_1V_2'. Without loss of generality, assume that ℓ intersects F_1V_1' at M_1 and $V_2'F_2$ at M_2. Since F_1V_2' is parallel to $V_1'F_2$, using (1) we can construct the mid-points N_1 and N_2 of F_1V_2' and $V_1'F_2$, respectively. Then N_1N_2 intersects ℓ at the mid-point M of M_1M_2. The construction is then completed by (2).

(b) By (a) we can construct line m through F_2 and parallel to ℓ. By solution II to 2741 cited above, we can construct line n through F_2 and perpendicular to m. By (a) we can construct the line through P and parallel to n.

Also solved by Peter Y. Woo, Biola University, La Mirada, CA, USA; and the proposer.

2945. [2004 : 230, 233] *Proposed by Michel Bataille, Rouen, France.*

Let G be the centroid of $\triangle A_1A_2A_3$. For $j = 1, 2, 3$, a circle is tangent to A_jA_{j+1} at T_j and to A_jA_{j+2} at U_j, so that G lies on the line segment T_jU_j (subscripts are taken modulo 3). Prove that

$$|GT_1| \cdot |GT_2| \cdot |GT_3| \;=\; |GU_1| \cdot |GU_2| \cdot |GU_3| \,.$$

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

In fact, the theorem is true if G is any point inside $\triangle ABC$. Using the Law of Sines, we have, for each i,

$$GT_i \;=\; A_iT_i\frac{\sin\angle GA_iT_i}{\sin\angle A_iGT_i} \quad\text{and}\quad GU_i \;=\; A_iU_i\frac{\sin\angle GA_iU_i}{\sin\angle A_iGU_i}\,.$$

Noting that $A_iT_i = A_iU_i$ and $\angle A_iGT_i + \angle A_iGU_i = \pi$, we get

$$\frac{GT_i}{GU_i} \;=\; \frac{\sin\angle GA_iT_i}{\sin\angle GA_iU_i} \;=\; \frac{\sin\angle GA_iA_{i+1}}{\sin\angle GA_iA_{i+2}}\,.$$

Hence,

$$\begin{aligned}\prod_{i=1}^{3}\frac{GT_i}{GU_i} \;&=\; \prod_{i=1}^{3}\frac{\sin\angle GA_iA_{i+1}}{\sin\angle GA_iA_{i+2}} \\ &=\; \prod_{i=1}^{3}\frac{\sin\angle GA_iA_{i+1}}{\sin\angle GA_{i+1}A_i} \;=\; \prod_{i=1}^{3}\frac{GA_{i+1}}{GA_i} \;=\; 1\,.\end{aligned}$$

The result follows immediately.

Also solved by CHRISTOPHER J. BRADLEY, Bristol, UK; KIN FUNG CHUNG, student, University of Toronto, Toronto, ON; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

2946. [2004 : 230, 233] *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*

Let x, y, z be positive real numbers satisfying $x^2+y^2+z^2=1$. Prove that

(a) $\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)-(x+y+z)\geq 2\sqrt{3}$.

(b) $\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)+(x+y+z)\geq 4\sqrt{3}$.

I. *Solution by Arkady Alt, San Jose, CA, USA.*

(a) Note first that, for any $u>0$, the inequality $\frac{1}{u}-u\geq\frac{4\sqrt{3}}{3}-2\sqrt{3}u^2$ is equivalent to each of the following:

$$\begin{aligned} 6\sqrt{3}u^3-3u^2-4\sqrt{3}u+3 &\geq 0\,,\\ 2(\sqrt{3}u)^3-(\sqrt{3}u)^2-4(\sqrt{3}u)+3 &\geq 0\,,\\ (\sqrt{3}u-1)^2(2\sqrt{3}u+3) &\geq 0\,. \end{aligned}$$

The last inequality is clearly true. Hence,

$$\begin{aligned} \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)-(x+y+z) &= \sum_{\text{cyclic}}\left(\frac{1}{x}-x\right)\\ &\geq \sum_{\text{cyclic}}\left(\frac{4\sqrt{3}}{3}-2\sqrt{3}x^2\right) = 2\sqrt{3}\,. \end{aligned}$$

(b) For any $u>0$, the inequality $\frac{1}{u}+u\geq\frac{5\sqrt{3}}{3}-\sqrt{3}u^2$ is equivalent to each of the following:

$$\begin{aligned} 3\sqrt{3}u^3+3u^2-5\sqrt{3}u+3 &\geq 0\,,\\ (\sqrt{3}u)^3+(\sqrt{3}u)^2-5(\sqrt{3}u)+3 &\geq 0\,,\\ (\sqrt{3}u-1)^2(\sqrt{3}u+3) &\geq 0\,. \end{aligned}$$

The last inequality is clearly true. Hence,

$$\begin{aligned} \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)+(x+y+z) &= \sum_{\text{cyclic}}\left(\frac{1}{x}+x\right)\\ &\geq \sum_{\text{cyclic}}\left(\frac{5\sqrt{3}}{3}-\sqrt{3}x^2\right) = 4\sqrt{3}\,. \end{aligned}$$

II. *Solution by Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.*

(a) By the Root–Mean–Square Inequality, we have

$$\frac{x+y+z}{3} \;\le\; \sqrt{\frac{x^2+y^2+z^2}{3}} \;=\; \frac{1}{\sqrt{3}}$$

and hence,

$$x+y+z \;\le\; \sqrt{3}\,. \tag{1}$$

By the AM–HM Inequality, we have

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \;\ge\; \frac{9}{x+y+z} \;\ge\; 3\sqrt{3}\,. \tag{2}$$

From (1) and (2), the claim follows.

(b) We apply the AM–GM Inequality twice:

$$\begin{aligned}
&\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+(x+y+z)\\
&\qquad\ge\; 4\left(\frac{x+y+z}{xyz}\right)^{1/4} \;=\; 4\left(\frac{(x+y+z)(x^2+y^2+z^2)}{xyz}\right)^{1/4}\\
&\qquad=\; 4\left(\frac{x^2}{yz}+\frac{y}{z}+\frac{z}{y}+\frac{y^2}{zx}+\frac{z}{x}+\frac{x}{z}+\frac{z^2}{xy}+\frac{x}{y}+\frac{y}{x}\right)^{1/4}\\
&\qquad\ge\; 4\left(9\sqrt[9]{\frac{x^4y^4z^4}{x^4y^4z^4}}\right)^{1/4} \;=\; 4\sqrt[4]{9} \;=\; 4\sqrt{3}\,.
\end{aligned}$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; CHRISTOPHER J. BRADLEY, Bristol, UK; KIN FUNG CHUNG, student, University of Toronto, Toronto, ON; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD B. EDEN, Ateneo de Manila University, The Philippines; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; VEDULA N. MURTY, Dover, PA, USA; GEORGE TSAPAKIDIS, Agrinio, Greece; IAN VANDERBURGH, University of Waterloo, Waterloo, ON; PETER Y. WOO, Biola University, La Mirada, CA, USA; ROGER ZARNOWSKI, Angela State University, San Angela, TX, U.S.A., YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer. About two thirds of the submitted solutions used one or more of the following: AM–GM Inequality, AM–HM Inequality, AM–RMS Inequality, Cauchy–Schwarz Inequality. The rest used calculus, convexity and Jensen's Inequality. One solver used the method of Lagrange's Multipliers.

Many solvers remarked that equality holds in either of the two inequalities if and only if $x=y=z=1/\sqrt{3}$.

Bencze obtained the generalization that if $x_k > 0$ (for $k = 1, 2, \dots, n$) such that $\sum\limits_{k=1}^{n} x_k^2 = 1$, then for all $a, b > 0$,

$$a\left(\sum_{k=1}^{n}\frac{1}{x_k}\right) \pm b\left(\sum_{k=1}^{n} x_k\right) \;\ge\; (an \pm b)\sqrt{n}\,.$$

Chung and Janous gave a similar comment which is the special case when $a = b = 1$. Indeed, Janous generalized even further: if $\lambda \ge \mu > 0$ and if $x_k > 0$ (for $k = 1, 2, \ldots, n$) such that $\sum_{k=1}^{n} x_k^\lambda = 1$, then

$$\left(\sum_{k=1}^{n} \frac{1}{x_k^\mu}\right) \pm \left(\sum_{k=1}^{n} x_k^\mu\right) \ \ge\ n^{(\lambda+\mu)/\lambda} \pm n^{(\lambda-\mu)/\lambda} .$$

2947★. [2004 : 230, 233] *Proposed by Abbas Mehrabian, student, Tehran, Iran.*

The featured solution to problem 2149 [1996 : 171; 1997 : 306–308] is missing a step, which we remedy by means of the following problem. Let $A'B'C'D'$ be a quadrilateral with an inscribed circle centred at O. For any point P inside $A'B'C'D'$, define $ABCD$ to be the convex quadrilateral whose sides pass through the vertices of $A'B'C'D'$ and are perpendicular at the vertex to the line joining it to P. Prove that P is the intersection point of the diagonals AC and BD if and only if $P = O$.

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

Let us label the points so that A', B', C', D' lie on DA, AB, BC, CD, respectively.

Claim 1. $\angle A'OB' = \angle A'D'O + \angle OC'B'$.

Proof: We have [using the fact that the incentre lies on the angle bisectors],

$$\begin{aligned}
&\angle A'D'O + \angle OC'B' - \angle A'OB' \\
&\quad= \angle A'D'O + \angle OC'B' + \angle OB'A' + \angle B'A'O - 180^\circ \\
&\quad= \tfrac{1}{2}\left(\angle C'D'A' + \angle B'C'D' + \angle A'B'C' + \angle D'A'B'\right) - 180^\circ \\
&\quad= 180^\circ - 180^\circ = 0^\circ . \qquad \blacksquare
\end{aligned}$$

Suppose now that $P = O$. Using Claim 1 along with the fact that $OD'DA'$ and $OB'BC'$ are cyclic, we have

$$\begin{aligned}
&\angle DOA' + \angle A'OB' + \angle B'OB \\
&\quad= 90^\circ - \angle A'DO + \angle A'OB' + 90^\circ - \angle OBB' \\
&\quad= \angle A'OB' - \angle A'D'O - \angle OC'B' + 180^\circ \\
&\quad= \angle A'OB' - \angle A'D'O - \angle D'C'O + 180^\circ \ =\ 180^\circ .
\end{aligned}$$

Hence, B, O, and D are collinear. Similarly, A, O, and C are collinear. It follows that $O = P$ is the intersection of AC and BD.

Conversely, suppose that $P \ne O$. We may assume, without loss of generality, that P is in or on the triangle $C'D'O$.

Claim 2. $\angle A'PB' < \angle A'OB'$.

Proof: Consider the tangent to the circle $A'OB'$ at O. From Claim 1 we have $\angle A'OD' = \angle A'B'O + \angle OC'D' > \angle A'B'O$; it follows that the tangent goes through the interior of $\angle A'OD'$ and, similarly, through the interior of $\angle B'OC'$. Hence, circle $A'OB'$ intersects triangle $C'D'O$ only at the point O. Thus, P is outside circle $A'OB'$, and $\angle A'PB' < \angle A'OB'$ as claimed. ∎

Assuming, for the moment, that the quadrangles $C'PB'B$ and $D'PA'D$ are convex, we have

$$\begin{aligned}
&\angle DPA' + \angle A'PB' + \angle B'PB \\
&\qquad = 90^\circ - \angle A'DP + \angle A'PB' + 90^\circ - \angle PBB' \\
&\qquad = \angle A'PB' - \angle A'D'P - \angle PC'B' + 180^\circ \\
&\qquad < \angle A'OB' - \angle A'D'O - \angle OC'B' + 180^\circ \\
&\qquad = 180^\circ .
\end{aligned}$$

Hence, B, P, and D are not collinear. Thus, P is not the intersection of AC and BD, and we are done.

Editor's comment: Neither of the submitted solutions addressed the possibility that $C'PB'B$ or $D'PA'D$ might not be convex. For the former, non-convexity means that B and P are in the same half-plane determined by $C'B'$; that is, $\angle PC'B' > 90^\circ$ (because $\angle PC'B = 90^\circ$). The above argument becomes valid with only minor modifications: we must replace $\angle B'PB$ in the sum on the left by $-\angle BPB'$, which equals $-\angle BC'B'$ (angles inscribed in a circle are equal), which in turn equals $-(\angle PC'B' - 90^\circ)$, as in the third line of the final set of equations in the earlier argument. A similar modification works for the expressions involving D and D'.

Also solved by PETER Y. WOO, Biola University, La Mirada, CA, USA.

2948★. [2004 : 231, 233] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Prove or disprove that the unique solution of the system of equations

$$\begin{gathered}
bb' + cc' = aa' - rr' , \\
a^2 = b^2 + c^2 , \qquad a'^2 = b'^2 + c'^2 , \\
2r = b + c - a , \qquad 2r' = b' + c' - a' ,
\end{gathered}$$

among Heron right triangles, where r and r' are their associated inradii, is given by

$$a = 5 ,\ b = 4 ,\ c = 3 ,\ r = 1 ; \quad \text{and} \quad a' = 13 , b' = 12 ,\ c' = 5 ,\ r' = 2 .$$

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

The system does not have a unique solution, since dilating either or both of the triangles by an integer factor will produce another solution.

This tells us that the problem can be reduced to finding all pairs of primitive Pythagorean triangles that satisfy $bb' + cc' = aa' - rr'$. It is well known that b and c are the lengths of the legs of a primitive Pythagorean triangle if and only if there exist relatively prime positive integers m, n, with different parity, such that $m > n$ and $\{b,\, c\} = \{m^2 - n^2,\, 2mn\}$.

Thus, let $\{b,\, c\} = \{m^2 - n^2,\, 2mn\}$, where m and n are two relatively prime positive integers of different parity and $m > n$. Similary, let $\{b',\, c'\} = \{x^2 - y^2,\, 2xy\}$, where x and y are two relatively prime positive integers of different parity and $x > y$. Then $a = m^2 + n^2$, $r = mn - n^2$, $a' = x^2 + y^2$, $r' = xy - y^2$.

There are two essentially different cases that we need to consider.

Case 1: $(b, c, b', c') = (m^2 - n^2, 2mn, x^2 - y^2, 2xy)$.

Then the equation $bb' + cc' = aa' - rr'$ is equivalent to

$$(m^2 - n^2)(x^2 - y^2) + 4mnxy \;=\; (m^2 + n^2)(x^2 - y^2) - (mn - n^2)(xy - y^2)\,.$$

When factored, the above equation becomes

$$(nx + ny - 2my)(my + ny - 2nx) \;=\; 0\,.$$

Hence, either $nx + ny = 2my$ or $my + ny = 2nx$. Since these two cases are symmetric, we will assume that the former is true. Thus,

$$n(x + y) \;=\; 2my\,.$$

Since $\gcd(m, n) = \gcd(x+y, 2y) = 1$, we must have $n = 2y$ and $m = x+y$. Therefore, $(m, n, x, y) = (x+y, 2y, x, y)$ (note that $m > n$, $\gcd(m, n) = 1$, and $m + n \equiv 1 \pmod 2$).

Case 2: $(b, c, b', c') = (m^2 - n^2, 2mn, 2xy, x^2 - y^2)$.

Then $bb' + cc' = aa' - rr'$ is equivalent to

$$2(m^2 - n^2)xy + 2mn(x^2 - y^2) \;=\; (m^2 + n^2)(x^2 - y^2) - (mn - n^2)(xy - y^2)\,,$$

which factors into

$$(3ny + my + nx - mx)(mx - my - nx) \;=\; 0\,.$$

If $3ny + my + nx - mx = 0$, then $4ny = (m-n)(x-y)$, which is impossible since the left side is even and the right side is odd. Hence, $mx - my - nx = 0$, which is equivalent to

$$(m - n)(x - y) \;=\; ny\,.$$

Again, since $\gcd(m - n, n) = \gcd(x - y, y) = 1$, we have $m - n = y$ and $x - y = n$. Thus, $(m, n, x, y) = (x, x - y, x, y)$ (note that x must be even).

All solutions to the original problem can be generated from the aforementioned results, with trivial permutations and dilations. [*Ed.:* For any primitive Pythagorean triangle, calculate the corresponding x and y. Find the second triangle using the results. All such pairs are thus constructed.]

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and LI ZHOU, Polk Community College, Winter Haven, FL, USA.

2949★. [2004 : 231, 234] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $n \geq 3$ be an *odd* natural number. Determine the smallest number $\mu = \mu(n)$ such that the entries of any row and of any column of the matrix

$$\begin{pmatrix} 1 & a_{1,2} & \cdots & a_{1,\mu} \\ 2 & a_{2,2} & \cdots & a_{2,\mu} \\ \vdots & \vdots & \ddots & \vdots \\ n & a_{n,2} & \cdots & a_{n,\mu} \end{pmatrix}$$

are distinct numbers from the set $\{1, 2, \ldots, n-1, n\}$, and the numbers in each row sum to the same value.

Editor's Comment:

No solutions were received for this problem. However, we were informed by Edward T.H. Wang and Kaiming Zhao, Wilfrid Laurier University, Waterloo, ON, that they have obtained some partial results. Specifically, they have proved that:

1. $\mu(n) \geq 3$ for all odd $n \geq 3$;

2. $\mu(3) = \mu(7) = \mu(9) = \mu(11) = 3$;

3. $\mu(5) = 5$;

4. $\mu(n) = 3$ for all odd n which are divisible by 3.

Proofs will not be given yet, since Wang and Zhao are continuing their work on the problem.

The proposer, in his submission of the problem, suggested that perhaps $\mu(n) = n$ for all odd n. The partial results are enough to show that this is false. Wang and Zhao believe that $\mu(n) = 3$ for all odd n except $n = 5$.

2950★. [2004 : 231, 234] *Proposed by Walther Janous, Ursulinen-gymnasium, Innsbruck, Austria.*

Let ABC be a triangle whose largest angle does not exceed $2\pi/3$. For λ, $\mu \in \mathbb{R}$, consider inequalities of the form

$$\cos\left(\frac{A}{2}\right)\cdot\cos\left(\frac{B}{2}\right)\cdot\cos\left(\frac{C}{2}\right) \;\geq\; \lambda+\mu\cdot\sin\left(\frac{A}{2}\right)\cdot\sin\left(\frac{B}{2}\right)\cdot\sin\left(\frac{C}{2}\right)\,.$$

(a) Prove that $\lambda_{\text{max}} \geq \frac{2\sqrt{3}-1}{8}$.

(b) Prove or disprove that

$$\lambda \;=\; \frac{2\sqrt{3}-1}{8} \qquad \text{and} \qquad \mu \;=\; 1+\sqrt{3}$$

yield the best inequality in the sense that λ cannot be increased. Determine also the cases of equality.

Editor's remark: There were no solutions submitted for this problem. As a result, it remains open.