# SKOLIAD No. 78

## Shawn Godin

Please send your solutions to the problems in this edition by *1 November, 2004*. A copy of **MATHEMATICAL MAYHEM Vol. 4** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

We will only print solutions to problems marked with an asterisk ($*$) if we receive them from students in grade 10 or under (or equivalent), or if we receive a unique solution or a generalization.

This month's questions are drawn from the first annual Hypatia contest for grade 11 students run by the Canadian Mathematics Competition. Thanks to Ian VanderBurgh and Peter Crippin for making these questions available.

## 2003 Concours Hypatie (11$^e$ année - Sec. V au Québec) mercredi 16 avril 2003

**1**. (a) Carl a un certain nombre de tuiles carrées mesurant chacune 1 cm sur 1 cm. Il les place de manière à former un grand carré dont les côtés mesurent $n$ cm et constate qu'il reste 92 tuiles non utilisées. S'il avait allongé les côtés du grand carré jusqu'à $(n+2)$ cm, il lui aurait manqué 100 tuiles pour réussir à former le grand carré. Combien de tuiles Carl a-t-il ?

(b) Diane, l'amie de Carl, arrive avec une grosse pile de blocs, chacun étant un cube dont les arêtes mesurent 1 cm. Carl prend une partie des blocs et Diane garde le reste. Carl utilise ses blocs pour tenter de former un gros cube dont les arêtes mesurent 8 cm, mais il constate qu'il lui manque 24 blocs. Diane réussit à former un gros cube en utilisant tous ses blocs. S'ils utilisent tous les blocs que Diane a apportés, ils peuvent former un grand cube dont les arêtes ont 2 cm de plus que celles du grand cube de Diane. Combien y a-t-il de cubes en tout ?

*Prolongement du Problème 1 :* Comme dans la question 1 (a), Carl place ses tuiles de manière à former un grand carré et il lui reste 92 tuiles non utilisées. Pour former un carré plus grand, il ajoute un certain nombre de tuiles à chaque côté du carré précédent et constate qu'il lui manque 100 tuiles pour compléter le carré. Combien de nombres différents de tuiles Carl peut-il avoir ?

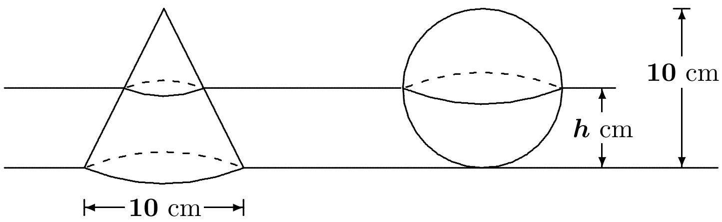
**2**. Xavier et Yvonne participent à un jeu. Au départ, il y a un certain nombre de pièces de monnaie placées en piles. Xavier joue toujours le premier. À tour de rôle, chacun enlève au moins une pièce d'une seule pile. La personne qui enlève la dernière pièce gagne.

(a) S'il y a deux piles contenant chacune trois pièces de monnaie, démontrer qu'Yvonne peut s'assurer de toujours gagner.

(b) S'il y a une pile de 1 pièce, une pile de 2 pièces et une pile de 3 pièces, démontrer qu'Yvonne peut s'assurer de toujours gagner.

*Prolongement du Problème 2 :* S'il y a trois piles, contenant 2, 4 et 5 pièces, quel joueur gagnera si chacun fait toujours le meilleur choix? Expliquer la stratégie gagnante.

**3**. Une sphère a un diamètre de 10 cm. Un cône droit a une hauteur de 10 cm et sa base est un cercle dont le diamètre mesure 10cm. Les deux solides reposent sur une surface horizontale. Si un plan horizontal coupe la sphère et le cône, la coupe transversale est un cercle dans les deux cas, comme l'indique le diagramme. Déterminer la hauteur du plan qui forme deux cercles de même aire.




*Prolongement du Problème 3 :* Une sphère de diamètre $d$ et un cône circulaire droit, dont la base a un diamètre $d$, reposent sur une surface horizontale. Dans ce cas, la hauteur du cône est égale au rayon de la sphère. Démontrer que si un plan horizontal coupe les deux solides, la somme de l'aire des coupes transversales est toujours constante.

**4**. Le carré $ABCD$ a pour sommets $A(1,4)$, $B(5,4)$, $C(5,8)$, et $D(1,8)$. Depuis un point $P$ à l'extérieur du carré, on dit qu'un sommet du carré est visible si on peut le relier à $P$ au moyen d'un segment de droite qui ne passe pas à travers le carré. Ainsi depuis un point $P$ à l'extérieur du carré, il y a toujours deux ou trois sommets du carré qui sont visibles. L'aire visible de $P$ est l'aire du triangle ou la somme de l'aire des deux triangles formés en reliant $P$ aux deux ou trois sommets visibles du carré.

(a) Démontrer que l'aire visible du point $P(2,-6)$ est égale à 20 unités carrées.

(b) Démontrer que l'aire visible du point $Q(11,0)$ est aussi égale à 20 unités carrées.

(c) L'ensemble des points $P$ qui ont une aire visible de 20 unités carrées est appelé l'ensemble 20/20. Cet ensemble a la forme d'un polygone. Déterminer le périmètre de l'ensemble 20/20.

*Prolongement du Problème 4 :* Depuis un point quelconque $P$, à l'extérieur d'un cube unitaire, 4, 6 ou 7 sommets du cube sont visibles dans le même sens que pour le carré. Si on relie le point $P$ à chacun de ces sommets, on obtient 1, 2 ou 3 pyramides à base carrée qui forment le volume visible de $P$. L'ensemble 20/20 est l'ensemble de tous les points P qui ont un volume visible de 20. Il a la forme d'un polyèdre. Quelle est l'aire totale de ce polyèdre ?

## 2003 Hypatia Contest (Grade 11)
## Wednesday, April 16, 2003

**1**. (a) Quentin has a number of square tiles, each measuring 1 cm by 1 cm. He tries to put these small square tiles together to form a larger square of side length $n$ cm, but finds that he has 92 tiles left over. If he had increased the side length of the larger square to $(n+2)$ cm, he would have been 100 tiles short of completing the larger square. How many tiles does Quentin have?

(b) Quentin's friend Rufus arrives with a big pile of identical blocks, each in the shape of a cube. Quentin takes some of the blocks and Rufus takes the rest. Quentin uses his blocks to try to make a large cube with 8 blocks along each edge, but finds that he is 24 blocks short. Rufus, on the other hand, manages to exactly make a large cube using all of his blocks. If they use all of their blocks together, they are able to make a complete cube which has a side length that is 2 blocks longer than Rufus' cube. How many blocks are there in total?

*Extension to #1:* As in Question #1 (a), Quentin tries to make a large square out of square tiles and has 92 tiles left over. In an attempt to make a second square, he increases the side length of this first square by an unknown number of tiles and finds that he is 100 tiles short of completing the square. How many different numbers of tiles is it possible for Quentin to have?

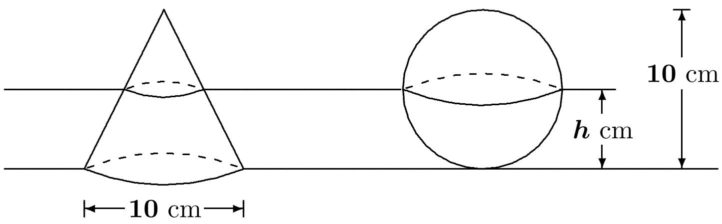
**2**. Xavier and Yolanda are playing a game starting with some coins arranged in piles. Xavier always goes first, and the two players take turns removing one or more coins from any one pile. The player who takes the last coin wins.

(a) If there are two piles of coins with 3 coins in each pile, show that Yolanda can guarantee that she always wins the game.

(b) If the game starts with piles of 1, 2, and 3 coins [*Ed:* three piles altogether], explain how Yolanda can guarantee that she always wins the game.

*Extension to #2:* If the game starts with piles of 2, 4, and 5 coins, which player wins if both players always make their best possible move? Explain the winning strategy.

**3**. In the diagram, the sphere has a diameter of 10 cm. Also, the right circular cone has a height of 10 cm, and its base has a diameter of 10 cm. The sphere and cone sit on a horizontal surface. If a horizontal plane cuts both the sphere and the cone, the cross-sections will both be circles, as shown. Find the height of the horizontal plane that gives circular cross-sections of the sphere and cone of equal area.




*Extension to #3:* A sphere of diameter $d$ and a right circular cone with a base of diameter $d$ stand on a horizontal surface. In this case, the height of the cone is equal to the radius of the sphere. Show that, for any horizontal plane that cuts both the cone and the sphere, the sum of the areas of the circular cross-sections is always the same.

**4**. Square $ABCD$ has vertices $A(1,4)$, $B(5,4)$, $C(5,8)$, and $D(1,8)$. From a point $P$ outside the square, a vertex of the square is said to be visible if it can be connected to $P$ by a straight line that does not pass through the square. Thus, from any point $P$ outside the square, either two or three of the vertices of the square are visible. The visible area of $P$ is the area of the one triangle or the sum of the areas of the two triangles formed by joining $P$ to the two or three visible vertices of the square.

(a) Show that the visible area of $P(2,-6)$ is 20 square units.

(b) Show that the visible area of $Q(11,0)$ is also 20 square units.

(c) The set of points $P$ for which the visible area equals 20 square units is called the 20/20 set, and is a polygon. Determine the perimeter of the 20/20 set.

*Extension to #4:* From any point $P$ outside a unit cube, 4, 6, or 7 vertices are visible in the same sense as in the case of the square. Connecting point $P$ to each of these vertices gives 1, 2, or 3 square-based pyramids, which make up the visible volume of $P$. The 20/20 set is the set of all points $P$ for which the visible volume is 20, and is a polyhedron. What is the surface area of this 20/20 set?

# MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Shawn Godin (Ottawa Carleton District School Board). ~~The Assistant Mayhem Editor is John Grant McLoughlin~~ (University of New Brunswick). The other staff members are Larry Rice (University of Waterloo) and Dan MacKinnon (Ottawa Carleton District School Board).

## Mayhem Problems

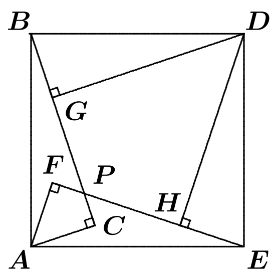
*Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le* **premier novembre 2004**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

*Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.*

*La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.*

**M144**. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Sur l'hypoténuse $AB$ d'un triangle rectangle $ABC$ on dessine un carré $ABDE$ de manière que $C$ en soit un point intérieur. On dessine ensuite un triangle rectangle directement semblable $BDG$ de sorte que $G$ soit aussi un point intérieur du carré. On dessine finalement deux triangles rectangles indirectement semblables $EDH$ et $AEF$, tels que $H$ et $F$ soient des points intérieurs du carré. Soit $P$ le point d'intersection de $BC$ et $EF$. Déterminer l'aire du quadrilatère $DGPH$ en fonction des côtés $CA$ et $CB$ du triangle rectangle original.




**M145**. *Proposé par Ovidiu-Gabriel Dinu, Balcesti-Valcea, Roumanie.*

Trouver tous les nombres naturels $n$ pour lesquels $n$, $n+2$, $n+6$, $n+8$ et $n+14$ sont premiers.

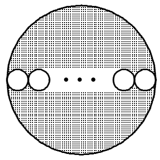
**M146**. *Proposé par Mohammed Aassila, Strasbourg, France.*

Soit $a$, $b$ et $c$ trois nombres positifs satisfaisant $a+b+c=1$. Montrer que

$$(ab)^{5/4}+(bc)^{5/4}+(ca)^{5/4} \;<\; \frac{1}{4}\,.$$

**M147**. *Proposé par l'Equipe de Mayhem.*

Le diamètre d'un grand cercle est divisé en $n$ parties égales pour construire $n$ cercles plus petits, comme dans la figure. Trouver $n$, sachant que le rapport entre l'aire hachurée et l'aire non hachurée du grand cercle est $3:1$.



**M148**. *Proposé par Vedula N. Murty, Dover, PA, USA.*

Soit $x>1$, $y>1$, $z>1$ et $x^2=yz$. Trouver la valeur de

$$\left(\log_{zx} xy^4z\right)\left(\log_{xy} xyz^4\right)\,.$$

**M149**. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Soit $ABC$ un triangle rectangle de Héron possédant la propriété suivante : son aire est $\lambda$ fois son périmètre, avec $\lambda$ un entier positif. Trouver toutes les solutions $(a,b,\lambda)$. (Un triangle de Héron est un triangle dont la longueur des côtés et l'aire sont des nombres entiers.)

**M150**. *Proposé par Arkady Alt, San Jose, CA, USA.*

Soit deux nombres complexes $z_1$ et $z_2$ satisfaisant les conditions

$$\begin{aligned} z_1+z_2 &= -(i+1)\,,\\ z_1\cdot z_2 &= -i\,. \end{aligned}$$

Sans calculer $z_1$ et $z_2$, trouver $z_1\cdot\overline{z_2}$.

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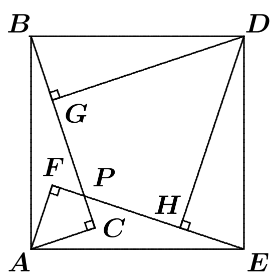
**M144**. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

A square $ABDE$ is drawn on the hypotenuse $AB$ of right triangle $ABC$ so that $C$ lies in the interior of the square. A directly similar right triangle $BDG$ is drawn so that $G$ lies in the interior of the square. Indirectly similar right triangles $EDH$ and $AEF$ are drawn so that $H$ and $F$ lie in the interior of the square. Let $BC$ and $EF$ intersect at $P$. Determine the area of quadrilateral $DGPH$ in terms of the legs $CA$ and $CB$ of the original right triangle.



**M145**. *Proposed by Ovidiu-Gabriel Dinu, Balcesti-Valcea, Romania.*

Find all natural numbers $n$ for which $n$, $n+2$, $n+6$, $n+8$, and $n+14$ are prime numbers.

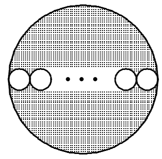
**M146**. *Proposed by Mohammed Aassila, Strasbourg, France.*

Let $a$, $b$, $c$ be three positive numbers satisfying $a+b+c=1$. Prove that

$$(ab)^{5/4}+(bc)^{5/4}+(ca)^{5/4} \;<\; \frac{1}{4}\,.$$

**M147**. *Proposed by the Mayhem staff.*

The diameter of a large circle is broken into $n$ equal parts to construct $n$ smaller circles, as shown. Determine $n$ so that the ratio of the shaded area to the unshaded area in the large circle is $3:1$.



**M148**. *Proposed by Vedula N. Murty, Dover, PA, USA.*

Let $x>1$, $y>1$, $z>1$ and $x^2=yz$. Determine the value of

$$\left(\log_{zx} xy^4z\right)\left(\log_{xy} xyz^4\right)\,.$$

**M149**. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

A right-angled Heron triangle $ABC$ has the following property: the area is $\lambda$ times the perimeter, where $\lambda$ is a positive integer. Determine all solutions $(a,b,\lambda)$. (A Heron triangle is a triangle with integer sides and integer area.)

**M150**. *Proposed by Arkady Alt, San Jose, CA, USA.*

Let two complex numbers $z_1$ and $z_2$ satisfy the conditions

$$\begin{aligned} z_1+z_2 &= -(i+1)\,,\\ z_1\cdot z_2 &= -i\,. \end{aligned}$$

Without calculating $z_1$ and $z_2$, find $z_1\cdot\overline{z_2}$.

# Pólya's Paragon

## What's the difference? (Part 2)

Shawn Godin

When last we met, in the March issue [2004 : 77], we began our analysis of sequences by looking at the sequences of differences, second differences, and so on. Let's search for a relationship between the differences and the original sequence.

Consider the sequence $\{t_n\}$. In March, we numbered the terms in our sequences starting with $n = 1$, but now we would like to start with $n = 0$ (since this will make our formulas look nicer). We recall the notation $\{{}_1d_n\}$ for the sequence of first differences, $\{{}_2d_n\}$ for the sequence of second differences, and so on.

Try to extend the following table of differences until a pattern emerges. Do **not** read on until you see the pattern.

$$
\begin{array}{llll}
t_n & {}_1d_n & {}_2d_n & \cdots \\
\hline
t_0 & & & \\
 & -t_0 + t_1 & & \\
t_1 & & t_0 - 2t_1 + t_2 & \\
 & -t_1 + t_2 & & \cdots \\
t_2 & & t_1 - 2t_2 + t_3 & \\
 & -t_2 + t_3 & & \cdots \\
t_3 & & t_2 - 2t_3 + t_4 & \\
 & -t_3 + t_4 & & \\
t_4 & & & \\
\vdots & \vdots & \vdots &
\end{array}
$$

If you have extended the table to the $5^{\text{th}}$ differences (try it now if you haven't already!), you should have found that those entries are

$$
\begin{array}{c}
-t_0 + 5t_1 - 10t_2 + 10t_3 - 5t_4 + t_5 \\
-t_1 + 5t_2 - 10t_3 + 10t_4 - 5t_5 + t_6 \\
\vdots
\end{array}
$$

Each $n^{\text{th}}$ difference seems to be an alternating sum of multiples of $n+1$ consecutive terms from the original sequence. The multiples are just the binomial coefficients. Thus,

$$
{}_kd_n \;=\; \sum_{i=0}^{k} (-1)^i \binom{k}{i}\, t_{n+k-i}\,.
$$

We can use the binomial inversion formula given by Hathout in the September 2003 issue [2003 : 275] to get

$$t_{n+k} = \sum_{i=0}^{n} \binom{n}{i} \, {}_{i}d_{m} \, .$$

If the given sequence $\{t_n\}$ is really a polynomial in disguise, then at some point the differences will all be zero (as we saw last time), giving us

$$t_n = \sum_{i=0}^{n} \binom{n}{i} \, {}_{i}d_{0} = \sum_{i=0}^{r} \binom{n}{i} \, {}_{i}d_{0} \, ,$$

where $r$ is the degree of the polynomial. Note that the numbers ${}_{i}d_{0}$ in this formula are the top entries in the columns of the table of differences.

Let's return to the problem that we started with back in March:

*"Little Johnnie encounters the following list of numbers* $12$, $49$, $62$, $57$, $40$, $17$, $\ldots$. *What is the next number in the list?"*

Constructing a difference table for this problem yields:

| $t_n$ | ${}_{1}d_n$ | ${}_{2}d_n$ | ${}_{3}d_n$ | ${}_{4}d_n$ |
|---|---|---|---|---|
| 12 | | | | |
| | 37 | | | |
| 49 | | −24 | | |
| | 13 | | 6 | |
| 62 | | −18 | | 0 |
| | −5 | | 6 | |
| 57 | | −12 | | 0 |
| | −17 | | 6 | |
| 40 | | −6 | | |
| | −23 | | | |
| 17 | | | | |

We see that the original sequence was a cubic relation ($r = 3$) and that

$$\begin{aligned} t_n &= \sum_{i=0}^{r} \binom{n}{i} \, {}_{i}d_{0} \\ &= 12\binom{n}{0} + 37\binom{n}{1} - 24\binom{n}{2} + 6\binom{n}{3} \\ &= 12 + 37n - 24 \cdot \frac{n(n-1)}{2} + 6 \cdot \frac{n(n-1)(n-2)}{6} \\ &= 12 + 37n - 12n^2 + 12n + n^3 - 3n^2 + 2n \\ &= n^3 - 15n^2 + 51n + 12 \, . \end{aligned}$$

Hence, we see that the next number in Little Johnny's list is $t_6 = -6$.

We can use these relationships also in cases where the sequence of $n^{\text{th}}$ differences is known but not just constant like in the polynomial case. We will examine some such cases (including the homework from the March issue) in our next installment.

# THE OLYMPIAD CORNER

No. 238

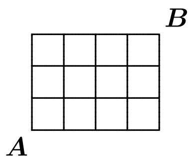
R.E. Woodrow

The first problem set we give this issue is the National Round of the XXXVI Spanish Mathematical Olympiad of the Real Sociedad Matemática Española. Thanks go to Christopher Small, Canadian Team Leader to the 42nd IMO, for collecting these problems for our use.

## XXXVI SPANISH MATHEMATICAL OLYMPIAD
### National Round
#### First Day

**1**. Let $P(x) = x^4 + ax^3 + bx^2 + cx + 1$ and $Q(x) = x^4 + cx^3 + bx^2 + ax + 1$, with $a$, $b$, $c$ real numbers and $a \neq c$. Find conditions on $a$, $b$, and $c$ so that $P(x)$ and $Q(x)$ have two common roots. In this case, solve the equations $P(x) = 0$, $Q(x) = 0$.

**2**. The figure shows a street plan of twelve square blocks. A person $P$ goes from point $A$ to point $B$, and a second person $Q$ goes from $B$ to $A$. Both of them ($P$ and $Q$) leave at the same time with the same speed, following shortest paths on the grid. At each corner they choose among the possible streets with equal probability. What is the probability that $P$ meets $Q$?




**3**. Circles $C_1$ and $C_2$ intersect at points $A$ and $B$. A line $r$ through $B$ intersects $C_1$ and $C_2$ again at points $P_r$ and $Q_r$, respectively. Prove that there is a point $M$, which depends only on $C_1$ and $C_2$, such that the perpendicular bisector of $P_rQ_r$ passes through $M$.

#### Second Day

**4**. For any integer $x$, let $\lfloor x \rfloor$ denote the integer part of $x$. Find the largest integer $N$ satisfying the following conditions:

(a) $\lfloor \frac{N}{3} \rfloor$ has three identical digits, and

(b) $\lfloor \frac{N}{3} \rfloor$ is the sum of $n$ consecutive positive integers starting at 1; that is, there is a positive integer $n$ such that

$$\left\lfloor \frac{N}{3} \right\rfloor = 1 + 2 + \cdots + (n-1) + n\,.$$

**5**. Four points are placed in a square of side $1$. Show that the distance between some two of them is less than or equal to $1$.

**6**. Show that there is no function $f : \mathbb{N} \to \mathbb{N}$ such that $f(f(n)) = n + 1$.

As a second set this issue we give the 2000 Taiwan Mathematical Olympiad. Thanks again go to Christopher Small.

## TAIWAN (ROC) MATHEMATICAL OLYMPIAD
### Part I – April 7, 2000
**Time: 4.5 hours**

**1**. Find all pairs $(x, y)$ of positive integers such that $y^{x^2} = x^{y+2}$.

**2**. In an acute triangle $ABC$ with $|AC| > |BC|$, let $M$ be the mid-point of $AB$. Let $AP$ be the altitude from $A$ and $BQ$ be the altitude from $B$. These altitudes meet at $H$, and the lines $AB$ and $PQ$ meet at $R$. Prove that the two lines $RH$ and $CM$ are perpendicular.

**3**. Let $S = \{1, 2, 3, \ldots, 100\}$, and let $P$ denote the family of all subsets $T$ of $S$ such that $|T| = 49$. We assign to each set $T$ in $P$ a label from the set $\{1, 2, \ldots, 100\}$, chosen at random. Show that there exists a subset $M$ of $S$, with $|M| = 50$, such that for each $x \in M$, the label of $M\backslash\{x\}$ is not equal to $x$.

### Part 2 — April 29, 2000
**Time: 4.5 hours**

**4**. For each positive integer $k$, let $\varphi(k)$ denote the number of positive integers $n$ satisfying $\gcd(n, k) = 1$ and $n \leq k$. Suppose that $\varphi(5^m - 1) = 5^n - 1$ for some positive integers $m$, $n$. Prove that $\gcd(m, n) > 1$.

**5**. Let $A = \{1, 2, 3, \ldots, n\}$, where $n$ is a positive integer. A subset of $A$ is *connected* if it consists of one element or some consecutive integers. Determine the greatest integer $k$ for which $A$ contains $k$ distinct subsets $A_1$, $A_2, \ldots, A_k$ such that the intersection of any two sets $A_i$ and $A_j$ is connected.

**6**. Let $f : \mathbb{N} \to \mathbb{N} \cup \{0\}$ be defined by

$$f(1) = 0, \quad \text{and} \quad f(n) = \max_{1 \leq j \leq \lfloor \frac{n}{2} \rfloor} \{f(j) + f(n-j) + j\}, \quad \forall n \geq 2.$$

Determine $f(2000)$.

As a third set of problems, we give the first and final rounds of the 2000 Hungarian National Olympiad for specialized math classes. Thanks go to Christopher Small, Canadian Team Leader to the 42$^{\text{nd}}$ IMO, for collecting them for our use.

# 2000 HUNGARIAN NATIONAL OLYMPIAD
## Specialized Math Classes

### First Round

**1**. Let $x$, $y$, and $z$ denote positive real numbers, each less than $4$. Prove that at least one of the numbers $\frac{1}{x}+\frac{1}{4-y}$, $\frac{1}{y}+\frac{1}{4-z}$, and $\frac{1}{z}+\frac{1}{4-x}$ is greater than or equal to $1$.

**2**. Find the integer solutions of $5x^2-14y^2=11z^2$.

**3**. Find the triangles for which the median and altitude starting from the same vertex are symmetrical to the angle bisector starting from the same vertex.

**4**. If $1\le m\le n$, prove that $m$ is a divisor of

$$n\left(\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots+(-1)^{m-1}\binom{n}{m-1}\right).$$

**5**. Find the smallest real number $c$ with the following property: On the perimeter of any triangle, there are two points, separated by a distance of at most $c$ times the perimeter, that divide the perimeter into two equal parts.

### Final Round

**1**. Let $c$ denote a positive integer, and let $c_1$, $c_3$, $c_7$, and $c_9$ be the number of divisors of $c$ which have last digit $1$, $3$, $7$, and $9$, respectively (in the decimal system). Prove that $c_3+c_7\le c_1+c_9$.

**2**. Circles $k_1$ and $k_2$ and a point $P$ are given in a plane. Construct a line passing through $P$ which meets the circle $k_i$ at $A_i$ and $B_i$ in such a way that there exist points $C_i$ on $k_i$ such that $A_1C_1=B_1C_1=A_2C_2=B_2C_2$. (It is not necessary to find the number of solutions, nor the condition for the existence of such a line.)

**3**. We have integers greater than $1$, denoted by $a_1$, $\ldots$, $a_k$, $b_1$, $\ldots$, $b_m$. Every $a_i$ is the product of an even number of (not necessarily distinct) primes. We have chosen some integers from the $k+m$ given integers (possibly none or all of them) such that every $b_i$ has an even number of divisors among the chosen integers. In how many ways can we make such a choice?

As a final set of problems for your puzzling pleasure over the (Canadian) summer, we give the problems of the 2000 Kürschák Contest from Hungary. Thanks again go to Christopher Small for providing these problems.

## 2000 KÜRSCHÁK CONTEST

**1**. For a positive integer $n$, consider the square in the Cartesian plane whose vertices are $A(0,0)$, $B(n,0)$, $C(n,n)$ and $D(0,n)$. The grid points of the integer lattice inside or on the boundary of this square are coloured either red or green in such a way that every unit square in the lattice has exactly two red vertices. How many such colourings are possible?

**2**. Let $T$ be a point in the plane of the non-equilateral triangle $ABC$ which is different from the vertices of the triangle. Let the lines $AT$, $BT$, and $CT$ meet the circumcircle of the triangle at $A_T$, $B_T$, and $C_T$, respectively. Prove that there are exactly two points $P$ and $Q$ in the plane for which the triangles $A_PB_PC_P$ and $A_QB_QC_Q$ are equilateral. Prove, furthermore, that the line $PQ$ passes through the circumcentre of the triangle $ABC$.

**3**. Let $k$ denote a non-negative integer. Assume that the integers $a_1, \ldots, a_n$ give at least $2k$ different remainders when divided by $n+k$. Prove that some of the integers add up to a number divisible by $n+k$.

An error has been pointed out by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. In the statement of Problem #6 of the 2001 Hungary-Israel Binational Mathematics Competition [2004 : 82], the word "one" should be "none", since the disjoint subsets in the problem do not exist if any of the numbers is greater than $60$.

We now turn our attention to readers' solutions to problems of the February 2002 number of the *Corner*. First we have solutions to problems of the Vietnamese Mathematical Olympiad 1999, Category A [2002 : 5–6].

**1**. Solve the system of equations

$$\left\{\begin{array}{rcl} (1+4^{2x-y})5^{1-2x+y} & = & 1+2^{2x-y+1} \\ y^3+4x+1+\ln(y^2+2x) & = & 0 \end{array}\right.$$

*Solved by Mohammed Aassila, Strasbourg, France; and Pavlos Maragoudakis, Lefkogia, Crete, Greece. We give the write-up of Maragoudakis.*

Letting $t = 2x-y$, the first equation becomes $f(t) = 0$, where

$$f(t) \;=\; 1+2\cdot 2^t - 5\cdot\left(\tfrac{1}{5}\right)^t - 5\cdot\left(\tfrac{4}{5}\right)^t\,.$$

Since the function $f$ is strictly increasing, the obvious solution $t = 1$ is the only solution. Thus, $2x-y=1$; that is, $y = 2x-1$.

The second equation then becomes $g(x) = 0$, where

$$g(x) \;=\; (2x-1)^3 + 4x + 1 + \ln(4x^2 - 2x + 1)\,.$$

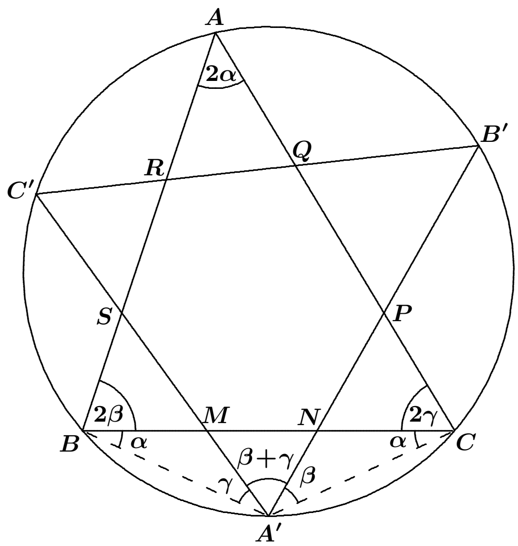
We easily find

$$g'(x) \;=\; 6(2x-1)^2 + \frac{16x^2+2}{3x^2+(x-1)^2} \;>\; 0\,.$$

Therefore, $g$ is strictly increasing, and the obvious solution $x = 0$ is the only solution. Finally, $(x,y) = (0,-1)$.

**2**. Let $A'$, $B'$, $C'$ be the respective mid-points of the arcs $BC$, $CA$, $AB$, not containing points $A$, $B$, $C$, respectively, of the circumcircle of the triangle $ABC$. The sides $BC$, $CA$, and $AB$ intersect the pairs of segments $(C'A', A'B')$, $(A'B', B'C')$, and $(B'C', C'A')$ at the pairs of points $(M,N)$, $(P,Q)$, and $(R,S)$, respectively. Prove that $MN = PQ = RS$ if and only if the triangle $ABC$ is equilateral.

*Solved by Christopher J. Bradley, Bristol, UK; Geoffrey A. Kandall, Hamden, CT, USA; and Toshio Seimiya, Kawasaki, Japan. We give Seimiya's write-up.*




We put $\angle BAC = 2\alpha$, $\angle ABC = 2\beta$, and $\angle ACB = 2\gamma$. Then $\alpha+\beta+\gamma = \frac{\pi}{2}$. Since $A'$ is the mid-point of the arc $BC$, we have

$$\angle A'AB \;=\; \angle A'CB \;=\; \angle A'BC \;=\; \angle A'AC \;=\; \alpha\,.$$

Also, $\angle B'A'C = \angle B'BC = \beta$ and $\angle BA'C' = \angle BCC' = \gamma$. Hence,

$$\angle A'NM \;=\; \angle A'CB + \angle B'A'C \;=\; \alpha + \beta\,,$$

and

$$\angle A'MN = \angle A'BC + \angle BA'C' = \alpha + \gamma.$$

Then

$$\begin{aligned}\angle MA'N &= 180^\circ - \angle A'NM - \angle A'MN \\ &= 180^\circ - (2\alpha + \beta + \gamma) = \beta + \gamma.\end{aligned}$$

Let $R$ be the circumradius of $\triangle ABC$. Then

$$BA' = 2R\sin\angle A'CB = 2R\sin\alpha. \tag{1}$$

Applying the Law of Sines to $\triangle A'BM$ and $\triangle A'MN$, we get

$$\frac{MA'}{BA'} = \frac{\sin\angle A'BM}{\sin\angle A'MB} = \frac{\sin\alpha}{\sin(\pi - \alpha - \gamma)} = \frac{\sin\alpha}{\sin(\alpha + \gamma)} = \frac{\sin\alpha}{\cos\beta},$$

and

$$\frac{MN}{MA'} = \frac{\sin\angle MA'N}{\sin\angle A'NM} = \frac{\sin(\beta + \gamma)}{\sin(\alpha + \beta)} = \frac{\cos\alpha}{\cos\gamma}.$$

Hence,

$$\frac{MN}{BA'} = \frac{MN}{MA'} \cdot \frac{MA'}{BA'} = \frac{\cos\alpha}{\cos\gamma} \cdot \frac{\sin\alpha}{\cos\beta}. \tag{2}$$

It follows from (1) and (2) that

$$MN = \frac{\sin\alpha\cos\alpha}{\cos\beta\cos\gamma} \cdot BA' = 2R\frac{\sin^2\alpha\cos\alpha}{\cos\beta\cos\gamma} = \frac{R(\sin 2\alpha)^2}{2\cos\alpha\cos\beta\cos\gamma}.$$

Similarly, we have

$$PQ = \frac{R(\sin 2\beta)^2}{2\cos\alpha\cos\beta\cos\gamma} \quad \text{and} \quad RS = \frac{R(\sin 2\gamma)^2}{2\cos\alpha\cos\beta\cos\gamma}.$$

Thus, $MN = PQ = RS$ if and only if $(\sin 2\alpha)^2 = (\sin 2\beta)^2 = (\sin 2\gamma)^2$, which is true if and only if $\sin 2\alpha = \sin 2\beta = \sin 2\gamma$. This is equivalent to $\alpha = \beta = \gamma$, which means that $\triangle ABC$ is equilateral.

**3**. Let $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be two sequences defined recursively as follows:

$$\begin{aligned}&x_0 = 1, \quad x_1 = 4, \quad x_{n+2} = 3x_{n+1} - x_n, \\ &y_0 = 1, \quad y_1 = 2, \quad y_{n+2} = 3y_{n+1} - y_n,\end{aligned}$$

for all $n = 0, 1, 2, \ldots$.

(a) Prove that

$$x_n^2 - 5y_n^2 + 4 = 0$$

for all non-negative integers $n$.

(b) Suppose that $a$, $b$ are two positive integers such that $a^2 - 5b^2 + 4 = 0$. Prove that there exists a non-negative integer $k$ such that $x_k = a$ and $y_k = b$.

*Solved by Michel Bataille, Rouen, France; Robert Bilinski, Outremont, QC; Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Bristol, UK. We first give Bataille's treatment of the problem.*

(a) The sequences $\{x_n\}$, $\{y_n\}$ may be defined equivalently by

$$x_0 \;=\; y_0 \;=\; 1\,, \quad \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} \;=\; A \begin{pmatrix} x_n \\ y_n \end{pmatrix}\,, \tag{1}$$

where $A$ denotes the matrix $\begin{pmatrix} 3/2 & 5/2 \\ 1/2 & 3/2 \end{pmatrix}$. Indeed, (1) gives $x_1 = 4$, $y_1 = 2$,

$$\begin{aligned} x_{n+2} &= \tfrac{3}{2}x_{n+1} + \tfrac{5}{2}y_{n+1} \;=\; \tfrac{3}{2}x_{n+1} + \tfrac{5}{2}\left(\tfrac{1}{2}x_n + \tfrac{3}{2}y_n\right) \\ &= \tfrac{3}{2}x_{n+1} + \tfrac{5}{4}x_n + \tfrac{3}{2}\left(x_{n+1} - \tfrac{3}{2}x_n\right) \;=\; 3x_{n+1} - x_n\,, \end{aligned}$$

and $y_{n+2} \;=\; 3y_{n+1} - y_n$ (similarly). Now, the required result follows by induction, using the relations $x_0^2 - 5y_0^2 + 4 = 0$ and

$$\begin{aligned} x_{n+1}^2 - 5y_{n+1}^2 + 4 &= \left(\tfrac{3}{2}x_n + \tfrac{5}{2}y_n\right)^2 - 5\left(\tfrac{1}{2}x_n + \tfrac{3}{2}y_n\right)^2 + 4 \\ &= x_n^2 - 5y_n^2 + 4\,. \end{aligned}$$

(b) Let $a$ and $b$ be positive integers such that $a^2 - 5b^2 + 4 = 0$. Clearly, $a = b = 1$ if $a = 1$ or $b = 1$. Also, we see that $a$ and $b$ have the same parity. Now, suppose $a > 1$ and $b > 1$. Let $\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = A^{-1}\begin{pmatrix} a \\ b \end{pmatrix}$; that is,

$$a_1 \;=\; \frac{3a - 5b}{2} \quad \text{and} \quad b_1 \;=\; \frac{-a + 3b}{2}\,.$$

Then $a_1$ and $b_1$ are integers, and $a_1^2 - 5b_1^2 + 4 = a^2 - 5b^2 + 4 = 0$. We have $a - a_1 = \dfrac{5b - a}{2}$ and $b - b_1 = \dfrac{a - b}{2}$. The following calculations show that $a_1$, $b_1$, $a - a_1$, and $b - b_1$ are all positive:

$$\begin{aligned} (3a - 5b)(a + 3b) &= 4(ab - 3) \;>\; 0\,, \\ (3b - a)(3a + 5b) &= 4(ab + 3) \;>\; 0\,, \\ (5b - a)(5b + a) &= 25b^2 - a^2 \;=\; 4(a^2 + 5) \;>\; 0\,, \\ (a - b)(a + b) &= a^2 - b^2 \;=\; 4(b^2 - 1) \;>\; 0\,. \end{aligned}$$

If $a_1 = 1$ or $b_1 = 1$, then $a_1 = b_1 = 1 = x_0 = y_0$ and

$$\begin{pmatrix} a \\ b \end{pmatrix} \;=\; A\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \;=\; \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\,.$$

Otherwise, we iterate the process until we get $a_k \;=\; 1 \;=\; b_k$ (which will necessarily occur, since we have decreasing sequences of positive integers). Then

$$\begin{pmatrix} a \\ b \end{pmatrix} \;=\; A^k\begin{pmatrix} a_k \\ b_k \end{pmatrix} \;=\; A^k\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \;=\; \begin{pmatrix} x_k \\ y_k \end{pmatrix}\,;$$

that is, $a = x_k$ and $b = y_k$.

*Next we give Bornsztein's write-up, which points out some interesting connections.*

(a) Solving the recurrence relations, we find that, for all $n = 1, 2, \ldots,$

$$\begin{aligned} x_n &= \varphi^{2n-1} + \left(-\frac{1}{\varphi}\right)^{2n-1} \\ y_n &= \frac{1}{\sqrt{5}}\varphi^{2n-1} - \frac{1}{\sqrt{5}}\left(-\frac{1}{\varphi}\right)^{2n-1}, \end{aligned}$$

where $\varphi = \dfrac{1+\sqrt{5}}{2}$ is the Golden Ratio. Using Binet's Formula, we deduce that

$$x_n = L_{2n-1} \qquad \text{and} \qquad y_n = f_{2n-1}\,,$$

where $\{f_n\}_{n=1}^{\infty}$ is the Fibonacci sequence (defined by $f_1 = 1, f_2 = 1$, and $f_{n+2} = f_{n+1} + f_n$, for all $n = 1, 2, \ldots$), and $\{L_n\}_{n=1}^{\infty}$ is the Lucas sequence (defined by $L_1 = 1$, $L_2 = 3$, and $L_{n+2} = L_{n+1} + L_n$, for all $n = 1, 2, \ldots$).

Using Binet's Formula, it is easy to verify that, for all $n = 1, 2, \ldots,$

$$5f_n^2 + 4(-1)^n = L_n^2\,. \tag{1}$$

It follows that $x_n^2 - 5y_n^2 + 4 = 0$.

(b) The following theorem is known (see [1]):

**Theorem**. Either $5x^2 + 4 = y^2$ or $5x^2 - 4 = y^2$ has a solution $(x, y)$ in positive integers if and only if $(x, y) = (f_n, L_n)$ for some $n$..

From (1), we have $5f_{2n}^2 + 4 = L_{2n}^2$. Consequently, the solutions of $5x^2 - 4 = y^2$ in positive integers are the pairs $(x, y) = (f_{2n-1}, L_{2n-1})$, where $n = 1, 2, \ldots$. Then the solutions of $a^2 - 5b^2 + 4 = 0$ in positive integers are the pairs $(a, b) = (x_n, y_n)$, where $n = 1, 2, \ldots$.

**Reference**

[1] R. Honsberger, *Mathematical Gems III*, MAA, p. 115.

**4**. Let $a$, $b$, $c$ be real positive numbers such that $abc + a + c = b$. Determine the greatest possible value of the following expression

$$P = \frac{2}{a^2+1} - \frac{2}{b^2+1} + \frac{3}{c^2+1}\,.$$

*Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Christopher J. Bradley, Bristol, UK; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We give Bataille's solution.*

Let $a$, $b$, $c$ be real positive numbers such that $abc + a + c = b$. Let $\alpha = \tan^{-1} a$ and $\beta = \tan^{-1} b$. Then $\alpha, \beta \in (0, \frac{\pi}{2})$ and $\alpha < \beta$ (since $a < b$). Thus, we have $a = \tan\alpha$, $b = \tan\beta$, and $c = \dfrac{b-a}{1+ab} = \tan(\beta - \alpha)$.

We may rewrite $P$ as

$$\begin{aligned} P &= 2\cos^2\alpha - 2\cos^2\beta + 3\cos^2(\beta-\alpha) \\ &= 3\cos^2(\beta-\alpha) + 2\sin(\beta+\alpha)\sin(\beta-\alpha)\,. \\ &\le 3\cos^2(\beta-\alpha) + 2\sin(\beta-\alpha) \;=\; f(\beta-\alpha)\,, \end{aligned}$$

where $f(x) = 3\cos^2 x + 2\sin x$. Since $f'(x) = 2\cos x(1 - 3\sin x)$, the maximum value of $f$ on the interval $(0, \frac{\pi}{2})$ is attained at $x_0 = \sin^{-1}(\frac{1}{3})$. The maximum is $f(x_0) = 3(1-\frac{1}{9}) + \frac{2}{3} = \frac{10}{3}$. It follows that $P \le \frac{10}{3}$, the value $\frac{10}{3}$ being attained when $\beta + \alpha = \frac{\pi}{2}$ and $\beta - \alpha = x_0$.

These conditions on $\alpha$ and $\beta$ yield $2\alpha = \frac{\pi}{2} - x_0$ and $2\beta = \frac{\pi}{2} + x_0$. Hence,

$$\frac{1-\tan^2\alpha}{1+\tan^2\alpha} \;=\; \cos 2\alpha \;=\; \cos\left(\frac{\pi}{2} - x_0\right) \;=\; \sin x_0 \;=\; \frac{1}{3}\,,$$

from which it is easy deduce that $\tan\alpha = \dfrac{\sqrt{2}}{2}$; that is, $a = \dfrac{\sqrt{2}}{2}$. Then $b = \tan\left(\dfrac{\pi}{2} - \alpha\right) = \cot\alpha = \sqrt{2}$, and $c = \tan(\beta - \alpha) = \dfrac{\sqrt{2}}{4}$.

In conclusion, the maximum value of $P$, under the specified conditions, is $\dfrac{10}{3}$, which is attained when $a = \dfrac{\sqrt{2}}{2}$, $b = \sqrt{2}$, and $c = \dfrac{\sqrt{2}}{4}$.

**5**. In three-dimensional space, let $Ox$, $Oy$, $Oz$, $Ot$ be four non-planar distinct rays such that the angles between any two of them have the same measure.

(a) Determine this common measure.

(b) Let $Or$ be another ray different from the above four rays. Let $\alpha$, $\beta$, $\gamma$, $\delta$ be the angles formed by $Or$ with $Ox$, $Oy$, $Oz$, $Ot$, respectively. Put

$$p \;=\; \cos\alpha + \cos\beta + \cos\gamma + \cos\delta\,,$$

$$q \;=\; \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta\,.$$

Prove that $p$ and $q$ are invariant when $Or$ rotates about the point $O$.

*Solved by Christopher J. Bradley, Bristol, UK; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We give Klamkin's solution and generalization.*

(a) This is a known result which has even appeared in this journal for $n+1$ rays in $E^n$.

Let $XYZT$ be a regular tetrahedron with centre $O$ and circumradius 1. Let the four unit vectors from $O$ to $X$, $Y$, $Z$, and $T$ be denoted by $\mathrm{x}$, $\mathrm{y}$, $\mathrm{z}$, and $\mathrm{t}$, respectively. These four vectors make equal angles with each other. Also, $\mathrm{x}+\mathrm{y}+\mathrm{z}+\mathrm{t} = 0$, since $O$ is the centroid. Expanding $|\mathrm{x}+\mathrm{y}+\mathrm{z}+\mathrm{t}|^2 = 0$, we get $4 + 12\cos\theta = 0$, or $\theta = \cos^{-1}(-1/3)$, where $\theta$ is the angle between any two of the vectors.

(b) Let $\mathrm{v}$ be the vector of the form $c_0\mathrm{x} + c_1\mathrm{y} + c_2\mathrm{z} + c_3\mathrm{t}$ such that $\mathrm{v}$ is parallel to the ray $Or$ and $\sum_{k=0}^{3} c_k = 1$. It will suffice to show that $p$ and $q$ are independent of $c_0$, $c_1$, $c_2$, and $c_3$. We have

$$\cos\alpha \;=\; \mathrm{x}\cdot\frac{\mathrm{v}}{|\mathrm{v}|} \;=\; \frac{c_0 - \frac{1}{3}(c_1+c_2+c_3)}{|\mathrm{v}|} \;=\; \frac{4c_0 - 1}{3|\mathrm{v}|}\,,$$

with similar expressions for $\cos\beta$, $\cos\gamma$, and $\cos\delta$. Also,

$$\begin{aligned} |\mathrm{v}|^2 &= \sum_{k=0}^{3} c_k^2 - \frac{2}{3}\sum_{i<j} c_i c_j \\ &= \sum_{k=0}^{3} c_k^2 - \frac{1}{3}\left(1 - \sum_{k=0}^{3} c_k^2\right) \;=\; \frac{1}{3}\left(4\sum_{k=0}^{3} c_k^2 - 1\right)\,. \end{aligned}$$

Hence, $p = \dfrac{1}{3|\mathrm{v}|}\sum_{k=0}^{3}(4c_k - 1) = 0$, and

$$\begin{aligned} q &= \frac{1}{9|\mathrm{v}|^2}\sum_{k=0}^{3}(4c_k-1)^2 \;=\; \frac{1}{9|\mathrm{v}|^2}\left(16\sum_{k=0}^{3} c_k^2 - 8\sum_{k=0}^{3} c_k + 4\right) \\ &= \frac{4}{9|\mathrm{v}|^2}\left(4\sum_{k=0}^{3} c_k^2 - 1\right) \;=\; \frac{4}{3}\,. \end{aligned}$$

These results generalize to $E^n$, where we have $n+1$ concurrent rays such that the angle between any two of them has the same measure. We start with $n+1$ unit vectors, $\mathrm{x_0}, \mathrm{x_1}, \ldots, \mathrm{x_n}$, from the centroid to the vertices of a regular simplex $X_0X_1\cdots X_n$. Expanding $|\mathrm{x_0}+\mathrm{x_1}+\cdots+\mathrm{x_n}|^2 = 0$ gives

$$n + 1 + 2\binom{n+1}{2}\cos\theta \;=\; 0\,,$$

which yields $\cos\theta = -1/n$, or $\theta = \cos^{-1}(-1/n)$.

Next, let $\mathrm{v} = c_0\mathrm{x_0} + c_1\mathrm{x_1} + \cdots + c_n\mathrm{x_n}$ such that the angle between $\mathrm{v}$ and $\mathrm{x_k}$ is $\alpha_k$ and $\sum_{k=0}^{n} c_k = 1$. Then

$$\cos\alpha_k \;=\; \mathrm{x_k}\cdot\frac{\mathrm{v}}{|\mathrm{v}|} \;=\; \frac{(n+1)c_k - 1}{n|\mathrm{v}|} \quad\text{and}\quad n|\mathrm{v}|^2 \;=\; (\mathrm{n}+1)\sum_{\mathrm{k}=0}^{\mathrm{n}} \mathrm{c_k^2} - 1\,.$$

Hence, $\sum_{k=0}^{n}\cos\alpha_k = 0$ and

$$\sum_{k=0}^{n}\cos^2\alpha_k \;=\; \frac{(n+1)\left((n+1)\sum_{k=0}^{n} c_k^2 - 1\right)}{n^2|\mathrm{v}|^2} \;=\; \frac{n+1}{n}\,.$$

Next we turn to problems of the Vietnamese Mathematical Olympiad 1999 Category B, given [2002 : 7–8].

**1**. Let $\{u_n\}_{n=1}^{\infty}$ be a sequence defined by

$$u_1 \;=\; 1, \quad u_2 \;=\; 2 \quad \text{and} \quad u_{n+2} \;=\; 3u_{n+1} - u_n$$

for all $n = 1, 2, \ldots$. Prove that

$$u_{n+2} + u_n \;\geq\; 2 + \frac{u_{n+1}^2}{u_n}$$

for all $n = 1, 2, \ldots$.

*Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Christopher J. Bradley, Bristol, UK; and Vedula N. Murty, Dover, PA, USA. We give Bataille's write-up.*

Let $\{f_n\}$ be the usual Fibonacci sequence, defined by $f_1 = f_2 = 1$ and $f_{n+2} = f_{n+1} + f_n$ for all positive integers $n$. Clearly, $u_1 = f_1$ and $u_2 = f_3$. Suppose that $u_k = f_{2k-1}$ and $u_{k+1} = f_{2k+1}$ for some integer $k \geq 1$. Then

$$\begin{aligned} u_{k+2} &= 3u_{k+1} - u_k \;=\; 3f_{2k+1} - f_{2k-1} \\ &= 2f_{2k+1} + f_{2k} \;=\; f_{2k+1} + f_{2k+2} \;=\; f_{2k+3}\,. \end{aligned}$$

Thus, $u_n = f_{2n-1}$ for all $n \geq 1$ (by induction).

Now, for any positive integer $n$,

$$\begin{aligned} u_{n+1}^2 - u_n u_{n+2} &= f_{2n+1}^2 - f_{2n-1}f_{2n+3} \\ &= f_{2n+1}^2 - f_{2n-1}f_{2n+1} - f_{2n-1}f_{2n+2} \\ &= f_{2n+1}f_{2n} - f_{2n-1}f_{2n+2} \\ &= f_{2n}^2 + f_{2n-1}f_{2n} - f_{2n-1}f_{2n+2} \\ &= f_{2n}^2 - f_{2n-1}f_{2n+1} \;=\; (-1)^{2n+1} = -1\,, \end{aligned}$$

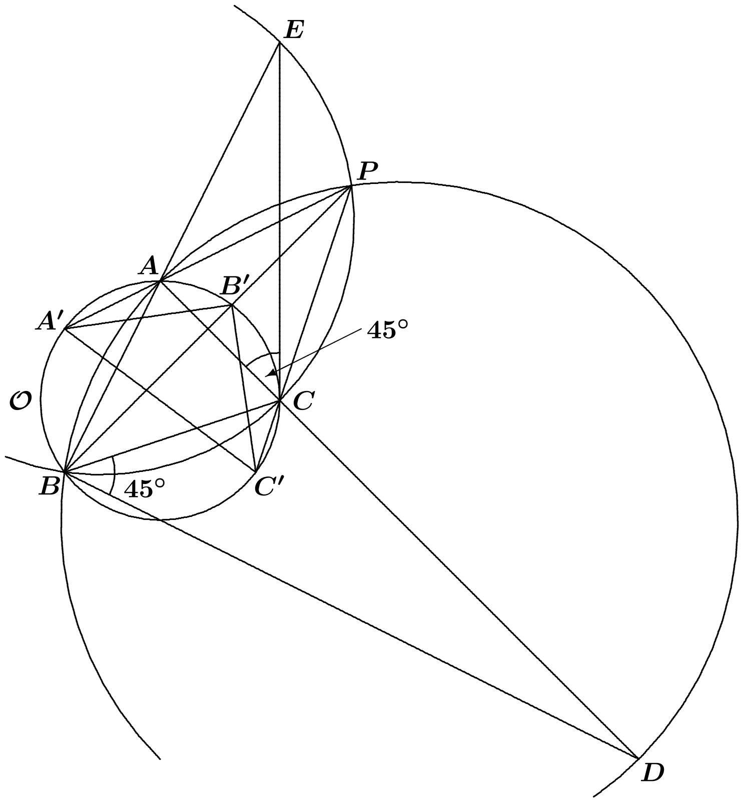
where we have used the well-known relation $f_m^2 - f_{m-1}f_{m+1} = (-1)^{m+1}$, valid for all $m \geq 2$. As a result,

$$u_{n+2} + u_n - \frac{u_{n+1}^2}{u_n} \;=\; u_n + \frac{1}{u_n}\,.$$

The desired inequality follows, since $x + \frac{1}{x} \geq 2$ for all positive $x$.

**2**. Let $ABC$ be a triangle inscribed in the circle $\mathcal{O}$. Locate the position of the points $P$, not lying in the circle $\mathcal{O}$, of the plane $(ABC)$ with the property that the lines $PA$, $PB$, $PC$ intersect the circle $\mathcal{O}$ again at points $A'$, $B'$, $C'$ such that $A'B'C'$ is a right-angled isosceles triangle with $\angle A'B'C' = 90^\circ$.

*Solved by Christopher J. Bradley, Bristol, UK; and Toshio Seimiya, Kawasaki, Japan. We give Seimiya's solution.*




In the following solution we assume that $\angle A > 45^\circ$ and $\angle C > 45^\circ$. But the following construction and proof work in other cases with minor changes.

Let $D$ be the point on $AC$ produced beyond $C$ such that $\angle CBD = 45^\circ$, and let $E$ be the point on $AB$ produced beyond $A$ such that $\angle ACE = 45^\circ$. Let $P$ be the intersection of the circumcircles of $\triangle ABD$ and $\triangle BCE$ other than $B$.

**Claim**. Then $P$ is the point we are looking for.

*Proof.* Let $A'$, $B'$, and $C'$ be the points of intersection of $PA$, $PB$, and $PC$ with circle $O$ other than $A$, $B$, and $C$. Then

$$\begin{array}{rrcl} & \angle B'A'C' & = & \angle B'CP \;=\; \angle BB'C - \angle BPC \\ & & = & \angle BAC - \angle BEC \;=\; \angle ACE \;=\; 45^\circ\,, \\ \text{and} & \angle B'C'A' & = & \angle B'AP \;=\; \angle AB'B - \angle APB \\ & & = & \angle ACB - \angle ADB \;=\; \angle CBD \;=\; 45^\circ\,. \end{array}$$

Hence, $\triangle A'B'C'$ is a right-angled isosceles triangle with $\angle A'B'C' = 90^\circ$.

**3**. Consider real numbers $a$, $b$ such that all roots of the equation

$$ax^3 - x^2 + bx - 1 = 0$$

are real and positive.

Determine the smallest possible value of the following expression:

$$P = \frac{5a^2 - 3ab + 2}{a^2(b-a)}.$$

*Solution by Vedula N. Murty, Dover, PA, USA.*

Let $r_1$, $r_2$, $r_3$ be the roots of the given equation. Then

$$r_1 + r_2 + r_3 = \frac{1}{a}, \tag{1}$$
$$r_1r_2 + r_2r_3 + r_3r_1 = \frac{b}{a}, \tag{2}$$
$$r_1r_2r_3 = \frac{1}{a}. \tag{3}$$

Suppose $r_1$, $r_2$, $r_3$ are real and positive. This implies that $a > 0$ and $b > 0$. Using the AM–GM Inequality with equations (1) and (3), we obtain

$$\frac{1}{a} \geq 3\sqrt{3}. \tag{4}$$

Using the inequality $3(r_1r_2 + r_2r_3 + r_3r_1) \leq (r_1 + r_2 + r_3)^2$ with equations (1) and (2), we obtain $3ab \leq 1$, and hence,

$$P = \frac{5a^2 - 3ab + 2}{a^2(b-a)} \geq \frac{5a^2 + 1}{a^2(b-a)}.$$

The inequality $(r_2 + r_3 - r_1)(r_3 + r_1 - r_2)(r_1 + r_2 - r_3) \leq r_1r_2r_3$ gives us

$$\left(\frac{1}{a} - 2r_1\right)\left(\frac{1}{a} - 2r_2\right)\left(\frac{1}{a} - 2r_3\right) \leq \frac{1}{a}.$$

This, with equations (1), (2), and (3), yields $9a^2 - 4ab + 1 \geq 0$; that is,

$$5a^2 + 1 \geq 4a(b-a).$$

Then

$$P \geq \frac{4a(b-a)}{a^2(b-a)} = \frac{4}{a} \geq 12\sqrt{3},$$

where the last step follows by (4).

Equality is attained when $a = \frac{\sqrt{3}}{9}$ and $b = \sqrt{3}$ (in which case the original equation is $\left(\frac{x}{\sqrt{3}} - 1\right)^3 = 0$). We conclude that the smallest value of $P$ is $12\sqrt{3}$.

**4**. Let $f(x)$ be a continuous function defined on $[0,1]$ such that

(i) $f(0) = f(1) = 0$,

(ii) $2f(x) + f(y) = 3f\left(\frac{2x+y}{3}\right) \quad \forall\, x,\, y \in [0,1]$.

Prove that $f(x) = 0$ for all $x \in [0,1]$.

*Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Pavlos Maragoudakis, Lefkogia, Crete, Greece. We give Bataille's solution, modified slightly by the editor.*

Since $|f|$ is continuous on the closed and bounded interval $[0,1]$, there exists a real number $a \in [0,1]$ at which $|f|$ attains its maximum $M$; that is, $|f(x)| \leq |f(a)| = M$ for all $x \in [0,1]$. We will show that $M = 0$, which implies that $f$ is the zero function.

**Case 1**: $0 \leq a \leq \frac{1}{2}$.

Let $a_1 = 2a/3$ and $b_1 = 5a/3$. Then $0 \leq a_1 \leq a \leq b_1 < 1$ and $a = \dfrac{2a_1 + b_1}{3}$. Hence, using (ii),

$$\begin{aligned} M &= |f(a)| = \left|f\left(\frac{2a_1+b_1}{3}\right)\right| = \left|\tfrac{2}{3}f(a_1) + \tfrac{1}{3}f(b_1)\right| \\ &\leq \tfrac{2}{3}|f(a_1)| + \tfrac{1}{3}|f(b_1)| \leq \tfrac{2}{3}M + \tfrac{1}{3}M = M\,, \end{aligned}$$

from which we deduce that $|f(a_1)| = M$.

Iterating, we construct a sequence $\{a_n\}_{n=1}^{\infty}$ such that $a_n = (\frac{2}{3})^n a$ and $|f(a_n)| = M$ for all positive integers $n$. Since $\{a_n\}$ converges to 0, the continuity of $f$ implies that $M = \lim\limits_{n\to\infty} |f(a_n)| = |f(0)| = 0$.

**Case 2**: $\frac{1}{2} < a \leq 1$.

Let $g(x) = f(1-x)$. Then $g$ satisfies all the given conditions on $f$, and the maximum of $|g|$ is $M$, attained at $1-a$. Since $0 \leq 1-a < \frac{1}{2}$, we may apply Case 1 to the function $g$ to deduce that $M = 0$.

**5**. The base side and the altitude of a regular hexagonal prism $ABCDEF$, $A'B'C'D'E'F'$ are equal to $a$ and $h$, respectively. Prove that six planes $(AB'F)$, $(CD'B)$, $(EF'D)$, $(D'EC)$, $(F'AE)$ and $(B'CA)$ are tangent to the same sphere. Determine the centre and the radius of this sphere.

*Solution by Christopher J. Bradley, Bristol, UK, modified by the editor.*

First, note the misprint in the question: the second of the six planes should be $(CD'B)$, not $(CD'B')$. [*Ed.* This has been corrected already in the problem statement above.]

We introduce rectangular coordinates $(x,y,z)$ such that the hexagons $ABCDEF$ and $A'B'C'D'E'F'$ lie in the planes $z=0$ and $z=h$, respectively, and the coordinates of $A$, $B$, $C$, $D$, $E$, $F$ are as follows:

$$A = a\left(-\frac{1}{2},\, \frac{\sqrt{3}}{2},\, 0\right)\,, \quad B = a\left(\frac{1}{2},\, \frac{\sqrt{3}}{2},\, 0\right)\,, \quad C = a(1,0,0)\,,$$

$$D = a\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0\right), \quad E = a\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0\right), \quad F = a(-1,0,0).$$

(The coordinates of $A'$, $B'$, $C'$, $D'$, $E'$, $F'$ are the same except that their $z$-coordinates have the value $h$ instead of 0.)

Note that the three planes $(AB'F)$, $(CD'B)$, and $(EF'D)$ are placed symmetrically around the prism under a rotation of $120°$, as are the three planes $(D'EC)$, $(F'AE)$, and $(B'CA)$. Therefore, if there is a sphere with centre on the $z$-axis which is tangent to the planes $(AB'F)$ and $(D'EC)$, then this sphere will be tangent to all six planes.

The planes $(AB'F)$ and $(D'EC)$ have the respective equations

$$\frac{x}{a} - \frac{1}{\sqrt{3}}\frac{y}{a} - \frac{z}{h} + 1 = 0 \qquad \text{and} \qquad \frac{x}{a} - \sqrt{3}\,\frac{y}{a} - \frac{z}{h} - 1 = 0.$$

We now look for a point $(0,0,k)$ which is the same distance $r$ from these two planes. This condition gives us

$$r = \frac{a}{\mu}\left|1 - \frac{k}{h}\right| = \frac{a}{\nu}\left|1 + \frac{k}{h}\right|,$$

where

$$\mu = \sqrt{\frac{4}{3} + \left(\frac{a}{h}\right)^2} \qquad \text{and} \qquad \nu = \sqrt{4 + \left(\frac{a}{h}\right)^2}.$$

Solving for $k$, we obtain two solutions, with corresponding values for $r$:

$$k = \left(\frac{\nu-\mu}{\nu+\mu}\right)h = \frac{3}{8}(\nu-\mu)^2 h, \qquad r = \frac{3}{4}a(\nu-\mu);$$

$$k = \left(\frac{\nu+\mu}{\nu-\mu}\right)h = \frac{3}{8}(\nu+\mu)^2 h, \qquad r = \frac{3}{4}a(\nu+\mu).$$

We have found *two* spheres that meet the requirements in the problem. Their centres are on the axis of symmetry of the prism at $(0,0,k)$, where the two values of $k$ and the corresponding values of the radius $r$ are given by the equations above.

**6**. Two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are determined recursively by

$$x_1 = 1, \qquad y_1 = 2, \qquad \text{and}$$
$$x_{n+1} = 22y_n - 15x_n,$$
$$y_{n+1} = 17y_n - 12x_n,$$

for all $n = 1, 2, \ldots$.

(a) Prove that

(i) $\{x_n\}$ and $\{y_n\}$ are not equal to zero for all $n = 1, 2, \ldots$.

(ii) The sequences $\{x_n\}$ and $\{y_n\}$ contain infinitely many positive terms and infinitely many negative terms.

(b) Are the $(1999^{1945})^{\text{th}}$ terms of the sequence $\{x_n\}$ and the sequence $\{y_n\}$ divisible by 7 or not?

*Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Bristol, UK. We give Bornsztein's solution.*

(a) (i) From the recurrence relations, we get $17x_{n+1} - 22y_{n+1} = 9x_n$, for all $n = 1, 2, 3, \ldots$. Then, for all $n = 2, 3, \ldots$,

$$x_{n+1} \;=\; 22y_n - 15x_n \;=\; 2x_n - 9x_{n-1}\,. \tag{1}$$

Similarly,

$$y_{n+1} \;=\; 2y_n - 9y_{n-1}\,. \tag{2}$$

Note that $x_1 = 1$, $x_2 = 29$, and $y_1 = 2$, $y_2 = 22$. We deduce easily by induction that $x_n$ is odd and $y_n \equiv 2 \bmod 4$, for all $n = 1, 2, \ldots$. It follows that $x_n$ and $y_n$ are not equal to 0.

(ii) From (1), for all $n = 2, 3, \ldots$, we have

$$x_{n+2} \;=\; -5x_n - 18x_{n-1}\,.$$

If $x_n$ and $x_{n-1}$ have the same sign, then $x_{n+2}$ and $x_n$ have opposite signs. Thus, in every four consecutive terms of the sequence $\{x_n\}$, there are always two which have opposite signs. It follows that the sequence $\{x_n\}$ contains infinitely many positive terms and infinitely many negative terms.

We prove the same result in the same way for the sequence $\{y_n\}$.

(b) From (1), we have $x_{n+1} \equiv 2(x_n - x_{n-1}) \pmod 7$, for all $n \geq 2$. Suppose that, for some $n \geq 2$, we have $x_n \equiv x_{n-1} \not\equiv 0 \pmod 7$. Then $x_{n+1} \equiv 0 \pmod 7$, $x_{n+2} \equiv 5x_n \not\equiv 0 \pmod 7$, $x_{n+3} \equiv 3x_n \not\equiv 0 \pmod 7$, and $x_{n+4} \equiv 3x_n \not\equiv 0 \pmod 7$. Since, $x_1 \equiv x_2 \equiv 1 \pmod 7$, we deduce by induction that $x_n \equiv 0 \pmod 7$ if and only if $n \equiv 3 \pmod 4$.

Now, since $3^2 \equiv 1 \pmod 4$, we have

$$1999^{1945} \;\equiv\; 3^{1945} \;\equiv\; 3 \pmod 4\,.$$

Thus, $x_n$ is divisible by 7 when $n \equiv 1999^{1945}$.

From (2), we have $y_{n+1} \equiv 2(y_n - y_{n-1}) \pmod 7$, for all $n \geq 2$. Suppose $y_{n+1} \equiv 0 \pmod 7$ for some $n \geq 4$. Then, $y_n \equiv y_{n-1} \pmod 7$. Since $y_n \equiv 2(y_{n-1} - y_{n-2}) \pmod 7$, we get $y_{n-1} \equiv 2y_{n-2} \pmod 7$. Then, since $y_{n-1} \equiv 2(y_{n-2} - y_{n-3}) \pmod 7$, we deduce that $y_{n-3} \equiv 0 \pmod 7$. Thus, if $y_{n+1} \equiv 0 \pmod 7$, then $y_{n-3} \equiv 0 \pmod 7$.

By induction, it follows that at least one of the first five terms of the sequence $(y_n)$ must be divisible by 7. But it is easy to verify that this is not the case.

Thus, $y_n \not\equiv 0 \pmod 7$ for all $n \geq 1$. In particular, $y_n \not\equiv 0 \pmod 7$ for $n = 1999^{1945}$.

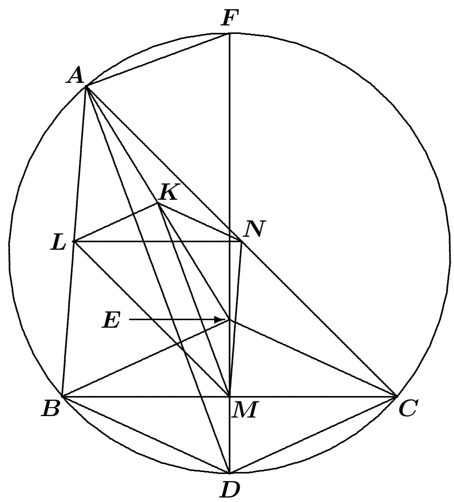
Next we turn to solutions to problems of the 16th Balkan Mathematical Olympiad, given [2002 : 8].

**1**. Given an acute-angled triangle $ABC$, let $D$ be the mid-point of the arc $BC$ of the circumcircle of $ABC$ not containing $A$. The points which are symmetric to $D$ with respect to the line $BC$ and the centre of the circumcircle are denoted by $E$ and $F$, respectively. Finally, let $K$ stand for the mid-point of $[EA]$. Prove that:

(a) the circle passing through the mid-points of the edges of the triangle $ABC$, also passes through $K$;

(b) the line passing through $K$ and the mid-point of $[BC]$ is perpendicular to $AF$.

*Solved by Christopher J. Bradley, Bristol, UK; and Toshio Seimiya, Kawasaki, Japan. We give Seimiya's account.*




(a) Letting $L$, $M$, and $N$ be the mid-points of $AB$, $BC$, and $CA$, respectively, we have $LM \parallel AC$ and $MN \parallel AB$. Thus,

$$\angle LMN = \angle LAN = \angle BAC .$$

Since $K$ is the mid-point of $AE$, we see that $LK \parallel BE$ and $KN \parallel EC$. Thus, $\angle LKN = \angle BEC$. Since $E$ is the reflection of $D$ with respect to $BC$, we have $\angle BEC = \angle BDC$, and then $\angle LKN = \angle BDC$. Hence,

$$\angle LKN + \angle LMN = \angle BDC + \angle BAC = 180^\circ .$$

Therefore, $L$, $M$, $N$, and $K$ are concyclic.

(b) Since $D$ is the mid-point of the arc $BC$, we have $BD = DC$. Thus, $BDCE$ is a rhombus, and $M$ is the mid-point of $DE$. Since $K$ and $M$ are mid-points of $AE$ and $DE$, respectively, we get $KM \parallel AD$.

Since $F$ is symmetric to $D$ with respect to the centre of the circumcircle of $\triangle ABC$, we see that $DF$ is a diameter of this circle. Thus, $\angle DAF = 90°$; that is, $AD \perp AF$. Since $KM \parallel AD$, we obtain $KM \perp AF$.

**2**. Let $p > 2$ be a prime number such that 3 divides $p - 2$. Let

$$S = \{y^2 - x^3 - 1 \mid x,\, y \text{ are integers, } 0 \leq x,\, y \leq p - 1\}\,.$$

Prove that at most $p - 1$ elements of the set $S$ are divisible by $p$.

*Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Bristol, UK. We give Bradley's write-up.*

Since $p$ is a prime of the form $3k + 2$, we have $k^{3k+1} \equiv 1 \pmod p$ for $k \not\equiv 0 \pmod p$, by Fermat's Theorem. Since $3 \nmid (3k + 1)$, we see that $h^3 \equiv 1 \pmod p$ implies that $h \equiv 1 \pmod p$. Suppose now that $s^3 \equiv t^3 \pmod p$ and $s,\, t \not\equiv 0 \pmod p$. Then $(t^{-1}s)^3 \equiv 1 \pmod p$ implies that $t \equiv s \pmod p$. It follows that the cubic residues modulo $3k + 2$ are $1, 2, 3, \ldots, p - 1$. We also have $0^3 \equiv 0 \pmod p$. Hence, $x^3 \equiv 0, 1, 2, 3, \ldots, p - 1 \pmod p$ once each as $x$ ranges from 0 to $p - 1$.

Then, for each value of $y^2 - 1$ where $0 \leq y \leq p - 1$, there is one and only one value of $x^3$ such that $y^2 - x^3 - 1$ is divisible by $p$. At first sight it appears as though there may be as many as $p$ elements of $S$ divisible by $p$. However, at least two of them, namely $1^2 - 0^3 - 1$ and $3^2 - 2^3 - 1$, are equal to 0. Hence, there are at most $p - 1$ elements of $s$ that are divisible by $p$.

**3**. Let $ABC$ be an acute-angled triangle; $M$, $N$ and $P$ are the feet of the perpendiculars from the centroid $G$ of the triangle upon its sides $AB$, $BC$ and $CA$ respectively. Prove that

$$\frac{4}{27} < \frac{\text{area}(MNP)}{\text{area}(ABC)} \leq \frac{1}{4}\,.$$

*Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Christopher J. Bradley, Bristol, UK; and by Murray S. Klamkin, University of Alberta, Edmonton, AB. We give Bataille's solution.*

As usual, let $a = BC$, $b = CA$, and $c = AB$, and let $[XYZ]$ denote the area of $\triangle XYZ$. Let $\rho = \dfrac{[MNP]}{[ABC]}$. We will prove that $\frac{2}{9} < \rho \leq \frac{1}{4}$, which is slightly stronger than requested.

First recall that $[GBC] = [GAC] = [GAB] = \frac{1}{3}[ABC]$. This follows at once be remarking that, for instance, $GN$ is a third of the altitude from $A$ in $\triangle ABC$. Now, since $\angle MGN = 180° - \angle B$,

$$\begin{aligned} 2[GMN] &= GM \cdot GN \cdot \sin B \\ &= \frac{2[GAB]}{c} \cdot \frac{2[GBC]}{a} \sin B \;=\; \frac{4}{9}\frac{[ABC]^2}{ac} \sin B\,. \end{aligned}$$

Similar results hold for $[GNP]$ and $[GMP]$. Thus,

$$[MNP] \;=\; \frac{2[ABC]^2}{9}\left(\frac{\sin A}{bc}+\frac{\sin B}{ca}+\frac{\sin C}{ab}\right)\,.$$

Since $[ABC]=\frac{1}{2}ab\sin C=\frac{1}{2}bc\sin A=\frac{1}{2}ca\sin B$, it follows that

$$\rho \;=\; \tfrac{1}{9}(\sin^2 A+\sin^2 B+\sin^2 C)\,.$$

By the usual trigonometric formulas, we get

$$\begin{aligned}\sin^2 A+\sin^2 B+\sin^2 C &= \tfrac{3}{2}-\tfrac{1}{2}(\cos 2A+\cos 2B+\cos 2C)\\ &= 2(1+\cos A\cos B\cos C)\,.\end{aligned}$$

[*Ed.* This calculation is given in more detail in the solution to problem 2676 [2002 : 475].] Since $\triangle ABC$ is acute-angled, $\cos A\cos B\cos C > 0$, and hence $\rho > \frac{2}{9}$. Furthermore, by the Law of Cosines,

$$\cos A\cos B\cos C \;=\; \frac{(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)}{8a^2b^2c^2}\,.$$

By the AM–GM Inequality,

$$\begin{aligned}\sqrt{(b^2+c^2-a^2)(c^2+a^2-b^2)} &\le \tfrac{1}{2}(2c^2) \;=\; c^2\,,\\ \sqrt{(c^2+a^2-b^2)(a^2+b^2-c^2)} &\le a^2\,,\\ \sqrt{(a^2+b^2-c^2)(b^2+c^2-a^2)} &\le b^2\,.\end{aligned}$$

Multiplying these inequalities gives

$$(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2) \;\le\; a^2b^2c^2\,,$$

which implies that $\cos A\cos B\cos C \le \frac{1}{8}$. The inequality $\rho \le \frac{1}{4}$ follows immediately.

*Klamkin comments on the inequalities* $2 < \sin^2 A+\sin^2 B+\sin^2 C \le \frac{9}{4}$*, which Bataille has proved above:*

The upper bound of $\frac{9}{4}$ is known to hold for all triangles, with equality if and only if the triangle is equilateral (see [1]). The lower bound of $2$ is known to be the best possible for acute triangles (again, see [1]). One can get arbitrarily close to this lower bound for triangles of angles $2\varepsilon$, $90-\varepsilon$, $90-\varepsilon$ where $\varepsilon$ is arbitrarily small.

**Reference**:

[1] O. Bottema, R.Ž. Djordjević, R.R. Janić, D.S. Mitrinović & P.M. Vasić, *Geometric Inequalities*, Groningen, 1969, p. 18.

That completes this number of the *Corner*. Send me your nice solutions and generalizations.

# BOOK REVIEW

## John Grant McLoughlin

*A Friendly Mathematics Competition: 35 Years of Teamwork in Indiana*
Edited by Rick Gillman, published by Mathematics Association of America, 2003
ISBN 0-88385-808-8, paperbound, 196 pages, US$29.95.
*Reviewed by* **John Grant McLoughlin**, *University of New Brunswick, Fredericton, NB.*

"The Friendly Exam" is the popular nickname of an annual undergraduate mathematics competition that began in 1966 at Wabash College, Indiana. The competition has grown in scope from its inception as a college competition to include participation from larger state universities throughout Indiana.

In the opening pages of the book, the history of the competition is synopsized neatly by Paul T. Mielke (a builder of the competition) and extended upon by Rick Gillman (an enthusiastic supporter since being introduced to the competition in 1987 as a faculty member at Valparaiso University). The historical perspective and the tone of presentation combine to offer a valuable addition to the subsequent collection of problems and solutions. As a reviewer, it was evident to me that the competition represented a significant mathematical community event.

The core of the book is a presentation of the "Exams" (#1 to #35) and the solutions for 1966 to 2000 inclusive. The contest is team-based in that teams of three students collaborate upon the questions posed in the contests. Each contest consists of 6 to 8 problems. The presentation of the problems for each exam is preceded in the book by a paragraph identifying the host institution, names of winning team members, and any instructions pertinent to the particular paper. The inclusion of such information reinforces the editorial commitment to sharing a story rather than simply presenting old papers. A total of 43 pages of problems is followed by 130 pages of detailed solutions. Each solution concludes with a statement such as "Look under Algebraic Structures in the Index for similar problems".

The final segment of the book is entitled Index by Problem Type. Each problem has been classified into one of the forty categories. Some categories such as Arclength with one problem seem unnecessary, particularly, given that a small number of problems have been classified Miscellaneous. However, the Index serves a beneficial role in providing a survey of the content and as a source of problems from particular areas of mathematics. For example, there range from 10 to 33 problems in each of the following categories: Analytic Geometry; Enumeration; Geometry; Integration; Limit Evaluation; Matrix Algebra; Number Theory; Probability; Polynomials; and Real-Valued Functions.

It is hard to identify the level of difficulty of problems. One of the introductory remarks suggests that the level is a notch below the Putnam Contest. A basic working knowledge of calculus and linear algebra appears to be an expectation. Further, some questions address more specific concepts concerning topics in algebra, calculus, or probability. The range and quality of problems and solutions offer a rich resource for undergraduates or senior high school students interested in competitions. The problems are also valuable for high school teachers or university faculty, whether working with such students or simply seeking mathematical problems for solving or sharing with others.

Three problems are provided here to offer a snapshot:

P1975-2. A polygon having all its angles equal and an odd number of vertices is inscribed in a circle. Prove that it must be regular.

P1988-7. A fair coin is tossed ten times. Find the probability that two tails do not appear in succession.

P2000-2. Call a number $N$ *fortunate* if it can be written with four equal digits in some base $b \in \mathbb{Z}^+$.

(a) Clearly $2222$ is fortunate; why is $2000$ fortunate?

(b) Find the greatest fortunate number less than $2000$.

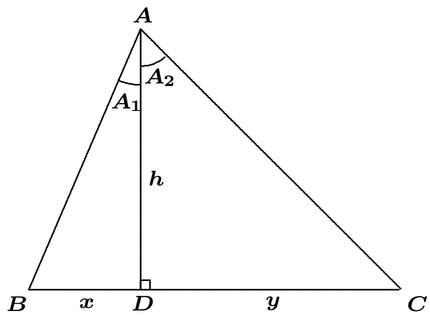
The organization of the problems from past contests into a book form is not unusual. In fact, Canadian initiatives such as the Cariboo College Mathematics Contest, the Newfoundland and Labrador Senior High School Mathematics League, and various Canadian Mathematics Competition contests offer such examples. Such books should be encouraged, as they extend the work and efforts of collectives into a broader domain while offering insight into the existence of other initiatives at regional, national, or international levels.

In summary, the Mathematical Association of America should be commended for publishing a collection of problems and solutions. The publication is strengthened by the inclusion of a historical context to supplement the material. *A Friendly Mathematics Competition: 35 Years of Teamwork in Indiana* would be a welcome addition to the shelves of libraries, departmental lounges, or those of the individual problem solvers themselves.

# A Maximum Vertical Angle I

G.D. Chakerian and M.S. Klamkin

A problem proposed by C.N. Schmall [1] was to show that among all spherical triangles (convex) having the same base and equal altitudes, the isosceles triangle has the greatest vertical angle, and also to show that this was true for planar triangles. The solution by W.J. Thome was obtained via calculus. It turns out that, for the spherical case, there was the following prior non-calculus solution by W. Nicolls:




From the right triangle $ADB$, we get $\cot A_1 = \sin h \cot x$, and from right triangle $ADC$, we get $\cot A_2 = \sin h \cot y$. Thus,

$$\cot A \;=\; \cot\left(\cot^{-1}(\sin h \cot x) + \cot^{-1}(\sin h \cot y)\right) \,.$$

Since $\cot(u+v) = (\cot u \cot v - 1)/(\cot u + \cot v)$, we have

$$\cot A \;=\; \frac{\sin^2 h \cot x \cot y - 1}{\sin h(\cot x + \cot y)} \;=\; \frac{\sin^2 h \cos x \cos y - \sin x \sin y}{\sin h \sin a} \,,$$

where we have set $a = x + y$, the length of the fixed base. Using the identities

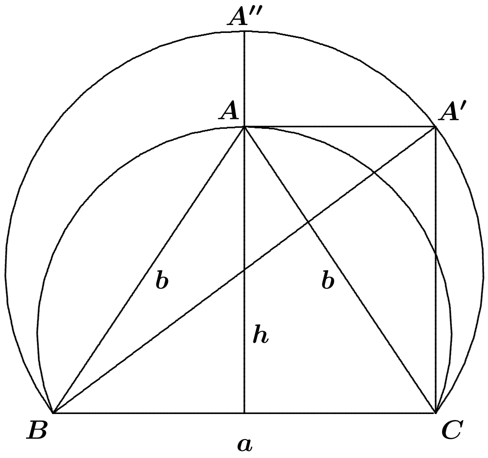
$$\begin{aligned} 2\cos x \cos y &= \cos(x-y) + \cos(x+y) \,, \\ 2\sin x \sin y &= \cos(x-y) - \cos(x+y) \,, \end{aligned}$$

we get

$$\cot A \;=\; \frac{(\cos a)(1+\sin^2 h) - \cos(x-y)\cos^2 h}{2\sin h \sin a} \,.$$

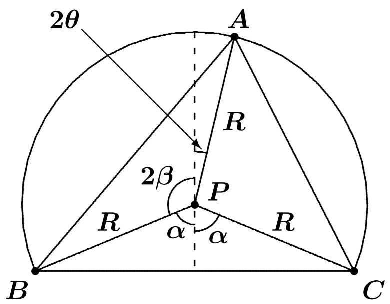
For $A$ to be a maximum, we must have $\cot A$ a minimum; this occurs when $x = y$. Therefore, the triangle with the greatest vertical angle is isosceles.

We now give a solution that is guided by the following approach to the planar triangle case.




In the figure above, $ABC$ is the isosceles triangle with given base $a$ and altitude $h$. Let $A'BC$ be another triangle with the same base and altitude, and with $A'$ on the same side of $BC$ as $A$. Then the part of the circumcircle of $A'BC$ on the same side of $BC$ as $A'$ contains the part of the circumcircle of $ABC$ on that side of $BC$. Since angles inscribed in a given segment are equal, $\angle A > \angle A'' = \angle A'$.

Applying this approach to the spherical case requires knowing how the angles inscribed in the same segment of a small circle behave. Consider now the above figure as the spherical case. First note that $\triangle A''BC$ is isosceles. The fact that $\angle A'' < \angle A$ follows from $\cot(A/2) = \sin h \cot(a/2)$ and $\cot(A''/2) = \sin h' \cot(a/2)$, where $h'$ denotes the altitude of $A''BC$ and is less than $\pi/2$. Finally, we show that $\angle A' < \angle A''$.




$$\begin{aligned} \angle APB &= 2\beta + 2\theta \\ \angle APC &= 2\beta - 2\theta \\ \angle BPC &= 2\alpha \end{aligned}$$

In the figure above, $ABC$ is a triangle inscribed in a segment of a small circle, where $\alpha + 2\beta = \pi$. We show that $\angle A$ is a decreasing function of $\theta$. Since, in a right triangle with base angles $x$ and $\alpha$ and hypotenuse $R$, one has $\cos R = \cot \alpha \cot x$, then, by dropping perpendiculars from $P$ to the sides $AB$ and $AC$, we see that

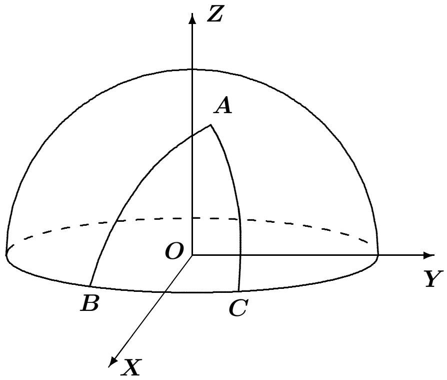
$$\cos R = \cot(\beta + \theta) \cot A_1 = \cot(\beta - \theta) \cot A_2 \,.$$

Hence,

$$\begin{aligned}
\cot A &= \cot\left(\cot^{-1}\left(\frac{\cos R}{\cot(\beta+\theta)}\right) + \cot^{-1}\left(\frac{\cos R}{\cot(\beta-\theta)}\right)\right) \\
&= \frac{\cos^2 R - \cot(\beta+\theta)\cot(\beta-\theta)}{\cos R\big(\cot(\beta+\theta)+\cot(\beta-\theta)\big)} \\
&= \frac{(\cos^2 R)\sin(\beta+\theta)\sin(\beta-\theta) - \cos(\beta+\theta)\cos(\beta-\theta)}{\cos R \sin 2\beta} \\
&= \frac{-(\cos 2\beta)(1+\cos^2 R) - (\cos 2\theta)(\sin^2 R)}{2(\cos R)(\sin 2\beta)}
\end{aligned}$$

As $\theta$ increases from 0, the value of $-\cos 2\theta$ increases, implying that $A$ decreases.

To better understand what is going on here, we use coordinates. We position the spherical triangle on the sphere $x^2+y^2+z^2=1$, as shown below, in such a way that $C=(a,b,0)$, $B=(a,-b,0)$, and $A=(x,y,z)$, where $a$, $b>0$ and $a^2+b^2=1$. Using standard spherical coordinates, we have $x=\cos\theta\sin\phi$, $y=\sin\theta\sin\phi$, and $z=\cos\phi$. If the angle between $OB$ and $OC$ is $2\alpha$, then $a=\cos\alpha$ and $b=\sin\alpha$.




The formula used in the above solution by Nicolls takes the form

$$\cot A = \frac{\cos^2\phi\cos(\alpha+\theta)\cos(\alpha-\theta) - \sin(\alpha+\theta)\sin(\alpha-\theta)}{\cos\phi\sin 2\alpha}.$$

This can be simplified to

$$\cot A = \frac{y^2+(a^2z^2-b^2)}{2abz}. \tag{1}$$

The latter formula can also be established directly with some vector algebra. It is clear from (1) that, with $z>0$ kept constant (so that spherical triangle $ABC$ has constant altitude), $\cot A$ is minimized when $y=0$; whence, $\angle A$ is maximized when $ABC$ is isosceles.

The formula (1) also gives some insight into the behavior of $\angle A$ as the point $A(x, y, z)$ varies over the sphere with $B$ and $C$ fixed. For instance, the locus of points $A$ such that $\angle A$ is constant satisfies

$$y^2 + a^2z^2 - b^2 \;=\; 2abkz\,,$$

where $x^2 + y^2 + z^2 = 1$ and $k = \cot A$ is a constant. The projection of this spherical curve onto the $yz$–plane is an arc of an ellipse centered at $(0, bk/a)$ and having semi-minor axis $b\sqrt{1+k^2}$ and semi-major axis $b\sqrt{1+k^2}/a$. This ellipse passes through $(b, 0)$ and $(-b, 0)$.

The locus of points $A$ for which $\angle A = 90^\circ$ is of interest. In this case, we have $k = \cot A = 0$, and the projection onto the $yz$–plane has the equation $y^2+a^2z^2 = b^2$, an ellipse centered at $(0, 0)$ with semi-minor axis $b$ and semi-major axis $b/a$. In particular, if $a = b = 1/\sqrt{2}$, the projection is the ellipse $2y^2 + z^2 = 2$. Note that in this case the locus itself on the sphere consists of two perpendicular arcs of great circles joining $B$ and $C$ to the north pole $(0, 0, 1)$!

As an exercise, the reader may verify that for $\angle A = 90^\circ$, the projection of the locus onto the $xy$–plane is an arc of the hyperbola having equation

$$a^2x^2 - b^2y^2 \;=\; a^2 - b^2\,.$$

When $a = b = 1/\sqrt{2}$, this degenerates into a pair of perpendicular straight lines.

**Remark:** The referee has noted some previously unpublished work of Dieter Ruoff and J. Chris Fisher in which our result is proved as one in absolute geometry. Ruoff and Fisher's proof follows immediately as part II.

**Reference:**

[1] C.N. Schmall, Problem 414, Amer. Math. Monthly **24:3** (1917), 185–186.

G.D. Chakerian
Department of Mathematics
University of California
Davis, CA 95616
USA

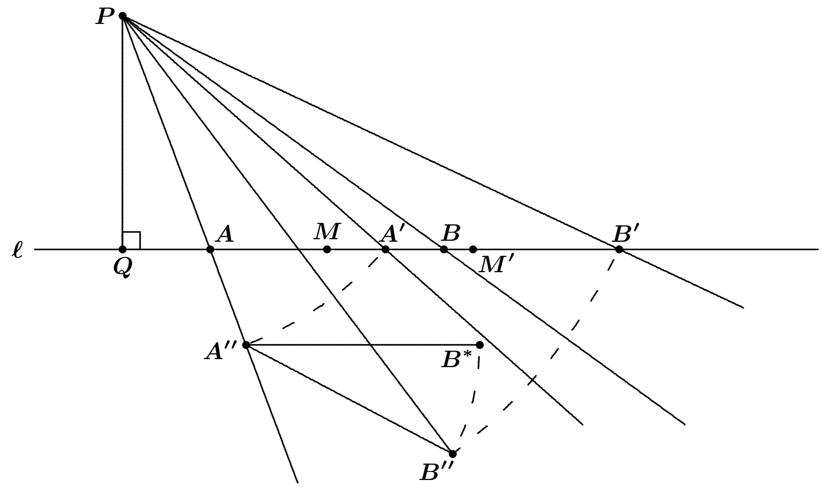
M.S. Klamkin
Department of Mathematics
University of Alberta
Edmonton, AB T6G 2G1
klamkin@ualberta.ca

# A Maximum Vertical Angle II

D. Ruoff and J.C. Fisher

**Theorem 1** On a line $\ell$ consider the congruent segments $AB$ and $A'B'$ with mid-points $M$ and $M'$, respectively. Let $P$ be a point not on $\ell$, and let $Q$ be the foot of the perpendicular from $P$ to $\ell$. Then $QM < QM'$ implies that $\angle A'PB' < \angle APB$.

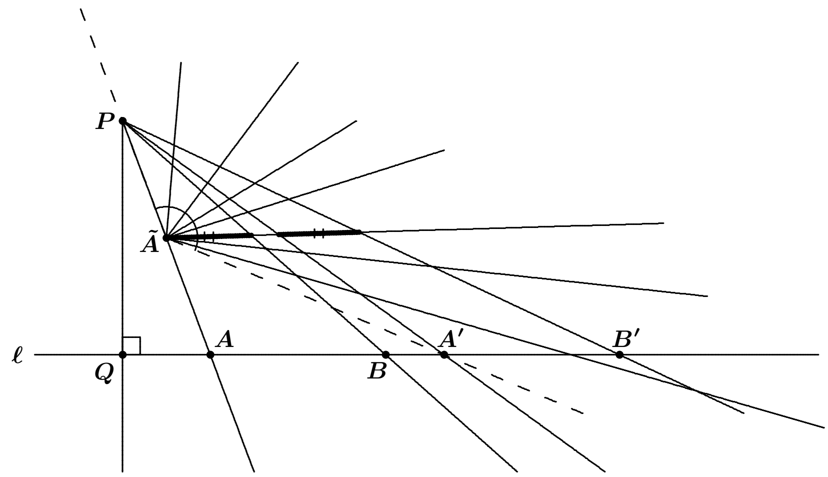
*Proof.* We first consider the case that $AB$ and $A'B'$ belong to the same ray from $Q$ with $QA < QB$ and $QA' < QB'$. We rotate $\triangle PA'B'$ about $P$ into $\triangle PA''B''$, with $A''$ on the ray $\overrightarrow{PA}$.




Since $\angle A''PB'' = \angle A'PB'$, the proof is reduced to showing that $B''$ lies in the interior of $\angle APB$ (since that implies that $\angle APB > \angle A''PB''$). To this end, we translate segment $AB$ along ray $\overrightarrow{PA}$ into $A''B^*$. The sides of an angle grow further apart as the distance from the vertex $P$ increases, in both hyperbolic and Euclidean geometry. The same is true in spherical geometry for the first half of the distance from $P$ to its antipodal point. For now, let us assume that $A''B^*$ does indeed lie sufficiently close to $P$; at the end we shall see that this assumption comes without loss of generality. Thus, in all the classical geometries, $B^*$ lies in the interior of $\angle APB$. Note that with our assumption on $Q$, both $\angle PA''B^*$ ($= \angle PAB$) and $\angle PA''B''$ ($= \angle PA'B'$) are at least $\pi/2$ with $\angle PA''B'' > \angle PA''B^*$. That, together with $A''B'' = A''B^*$, means that $B''$, just as $B^*$, belongs to the interior of $\angle APB$, as desired.

For the remaining cases we may assume without loss of generality that $AB$ and $A'B'$ are disjoint—any overlap could be subtracted from both angles for the purpose of comparing them. Two situations have to be considered: that of $Q$ between $A$ and $B$, and that of $Q$ between $B$ and $A'$. (Note that we still assume $QA < QB$ and $QA' < QB'$.) Both cases are quickly reduced to the one that has already been handled. In the latter case, simply reflect $A'B'$ in the line $PQ$ to the other side (so that $AB$ and the image of $A'B'$ belong to the same ray from $Q$). In the former case, define $B'''$ to be the point on $\overrightarrow{QB}$ that satisfies $QB''' = AB$, then prove that $\angle APB > \angle QPB''' > \angle A'PB'$.

Finally, we return to the question of the distance from the vertex $P$ to the segment $A'B'$. Fix the sides of $\angle APB$ and $\angle A'PB'$ and choose a new position $\tilde{A}$ for $A$ between $A$ and $P$ that is arbitrarily close to $P$. Consider the pencil of rays which pass through this new point $\tilde{A}$ and lie within $\angle A'\tilde{A}P$.




The portion of each line intercepted by the sides of $\angle A'PB'$ shrinks to zero from a segment longer than $A'B'$ $(= AB)$. Somewhere along the way, equal segments will be intercepted by both $\angle APB$ and $\angle A'PB'$. We began our proof with these two segments in place of the original pair.

Dieter Ruoff
Department of Mathematics
and Statistics
University of Regina
Regina, SK, S4S 0A2
ruoff@math.uregina.ca

J. Chris Fisher
Department of Mathematics
and Statistics
University of Regina
Regina, SK, S4S 0A2
fisher@math.uregina.ca

# PROBLEMS

*Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le* **1er décembre 2004**. *Une étoile* ($\star$) *après le numéro indique que le problème a été soumis sans solution.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

**2939**. *Proposé par Toshio Seimiya, Kawasaki, Japon.*

Supposons que $I$ soit le centre du cercle inscrit du triangle $ABC$ et que $BI$ et $CI$ coupent respectivement $AC$ et $AB$ en $D$ et $E$. Supposons de plus que la bissectrice de l'angle $BIC$ coupe $BC$ respectivement $DE$ en $P$ et $Q$, et que finalement $PI = 2QI$. Montrer que l'angle $BAC$ est égal à $60^\circ$.

**2940**. *Proposé par Toshio Seimiya, Kawasaki, Japon.*

On donne un triangle $ABC$ dans lequel les bissectrices des angles $ABC$ and $ACB$ coupent $AC$ et $AB$ en $D$ et $E$ ; de plus, la différence entre l'angle $ADE$ et l'angle $AED$ est $60^\circ$. Montrer que l'angle $ACB$ est égal à $120^\circ$.

**2941**. *Proposé par Toshio Seimiya, Kawasaki, Japon.*

Soit $ABC$ un triangle tel que les bissectrices des angles $ABC$ et $ACB$ coupent respectivement $AC$ et $AB$ en $D$ et $E$. Soit $I$ l'intersection de $BD$ et $CE$, et soit $F$ le pied de la perpendiculaire abaissée de $I$ sur $BC$. Montrer que l'égalité des angles $ADE$ et $BIF$ entraîne celle des angles $AED$ et $CIF$.

**2942**. *Proposé par Toshio Seimiya, Kawasaki, Japon.*

Soit $ABC$ un triangle tel que l'angle $ABC$ soit le double de l'angle $ACB$. De plus, soit $D$ un point sur le rayon $CB$ tel que l'angle $ADC$ soit la moitié de l'angle $BAC$. Montrer que

$$\frac{1}{CD} = \frac{1}{AB} - \frac{1}{AC}.$$

**2943**. *Proposé par Toshio Seimiya, Kawasaki, Japon.*

Dans un triangle donné $ABC$, soit $\alpha$ l'angle $BAC$, $D$ le point sur $AB$ au-delà de $B$ tel que $BD = BC$, et $E$ le point sur $AC$ au-delà de $C$ tel que $CE = BC$. De plus, on suppose que $P$ est l'intersection de $BE$ et $CD$, et que $\frac{DP}{BE} + \frac{EP}{CD} = 2\sin\left(\frac{\alpha}{2}\right)$. Montrer que $\alpha = 90^\circ$.

**2944**. *Proposé par Václav Konečný, Big Rapids, MI, USA.*

On donne une ellipse de foyers $F_1$ et $F_2$, ainsi que les sommets $V_1'$ et $V_2'$ du petit axe. Soit $P$ un point et $\ell$ une droite ne passant pas par $P$. Avec une équerre seulement, construire la droite par $P$ qui est

(a) parallèle à $\ell$ ;

(b) perpendiculaire à $\ell$.

Les constructions sont bien connues si l'on donne un cercle avec son centre au lieu d'une ellipse avec ses foyers (Théorème de Poncelet–Steiner).

**2945**. *Proposé par Michel Bataille, Rouen, France.*

Soit $G$ le centre de gravité du triangle $A_1A_2A_3$. Pour $j = 1, 2, 3$, on considère un cercle tangent à $A_jA_{j+1}$ en $T_j$ et à $A_jA_{j+2}$ en $U_j$, de telle sorte que $G$ soit sur le segment $T_jU_j$ (les indices sont comptés modulo 3). Montrer que

$$|GT_1| \cdot |GT_2| \cdot |GT_3| = |GU_1| \cdot |GU_2| \cdot |GU_3| .$$

**2946**. *Proposé par Panos E. Tsaoussoglou, Athènes, Grèce.*

Soit $x$, $y$ et $z$ des nombres réels positifs satisfaisant $x^2 + y^2 + z^2 = 1$. Montrer que

(a) $\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) - (x + y + z) \geq 2\sqrt{3}$.

(b) $\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) + (x + y + z) \geq 4\sqrt{3}$.

**2947**★. *Proposé par Abbas Mehrabian, étudiant, Téhéran, Iran.*

La solution proposée pour le problème 2149 [1996 : 171 ; 1997 : 306–308] a une lacune que nous proposons de combler par le problème suivant. Soit $A'B'C'D'$ un quadrilatère avec un cercle inscrit centré en $O$. Pour tout point $P$ à l'intérieur de $A'B'C'D'$, on définit $ABCD$ comme étant le quadrilatère dont les côtés passent par les sommets de $A'B'C'D'$ et sont perpendiculaires au sommet à la droite joignant celui-ci à $P$. Montrer que $P$ est le point d'intersection des diagonales $AC$ et $BD$ si et seulement si $P = O$.

**2948★**. *Proposé par Juan-Bosco Romero Márquez, Université de Valladolid, Valladolid, Espagne.*

Confirmer ou infirmer qu'il existe une unique solution du système d'équations

$$\begin{array}{rcl} & bb' + cc' \;=\; aa' - rr' \,, & \\ a^2 \;=\; b^2 + c^2 \,, & & a'^2 \;=\; b'^2 + c'^2 \,, \\ 2r \;=\; b + c - a \,, & & 2r' \;=\; b' + c' - a' \,, \end{array}$$

(où $r$ et $r'$ sont les rayons intérieurs des triangles rectangles de Héron associés) donnés par

$$a = 5\,,\;\; b = 4\,,\;\; c = 3\,,\;\; r = 1\,; \quad \text{et} \quad a' = 13\,, b' = 12\,,\;\; c' = 5\,,\;\; r' = 2\,.$$

**2949★**. *Proposé par Walther Janous, Ursulinengymnasium, Innsbruck, Autriche.*

Soit $n \geq 3$ un nombre naturel *impair*. Trouver le plus petit nombre $\mu = \mu(n)$ tel que les éléments de chaque ligne et de chaque colonne de la matrice

$$\begin{pmatrix} 1 & a_{1,2} & \cdots & a_{1,\mu} \\ 2 & a_{2,2} & \cdots & a_{2,\mu} \\ \vdots & \vdots & \ddots & \vdots \\ n & a_{n,2} & \cdots & a_{n,\mu} \end{pmatrix}$$

sont des éléments différents pris dans la liste $\{1, 2, \ldots, n-1, n\}$, et que la somme des éléments de chaque ligne est la même.

**2950★**. *Proposé par Walther Janous, Ursulinengymnasium, Innsbruck, Autriche.*

Soit $ABC$ un triangle dont le plus grand angle ne dépasse pas $2\pi/3$. Avec $\lambda$, $\mu \in \mathbb{R}$, on considère des inégalités de la forme

$$\cos\left(\frac{A}{2}\right) \cdot \cos\left(\frac{B}{2}\right) \cdot \cos\left(\frac{C}{2}\right) \;\geq\; \lambda + \mu \cdot \sin\left(\frac{A}{2}\right) \cdot \sin\left(\frac{B}{2}\right) \cdot \sin\left(\frac{C}{2}\right)\,.$$

(a) Montrer que $\lambda_{\mathrm{max}} \geq \dfrac{2\sqrt{3}-1}{8}$.

(b) Montrer ou réfuter l'assertion suivante :

$$\lambda \;=\; \frac{2\sqrt{3}-1}{8} \qquad \text{et} \qquad \mu \;=\; 1 + \sqrt{3}$$

donne la meilleure inégalité, dans le sens que $\lambda$ ne peut pas être augmenté. Déterminer aussi les cas d'égalité.

**2939**. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that $\triangle ABC$ has incentre $I$ and that $BI$, $CI$ meet $AC$, $AB$ at $D$, $E$, respectively. Suppose further that the bisector of $\angle BIC$ meets $BC$ and $DE$ at $P$ and $Q$, respectively, and that $PI = 2QI$. Prove that $\angle BAC = 60^\circ$.

**2940**. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

In $\triangle ABC$, the bisectors of $\angle ABC$ and $\angle ACB$ meet $AC$ and $AB$ at $D$ and $E$, respectively, and $\angle ADE - \angle AED = 60^\circ$. Prove that $\angle ACB = 120^\circ$.

**2941**. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

In $\triangle ABC$, the bisectors of $\angle ABC$ and $\angle ACB$ meet $AC$ and $AB$ at $D$ and $E$, respectively. Let $I$ be the intersection of $BD$ and $CE$, and let $F$ be the foot of the perpendicular from $I$ to $BC$. Prove that if $\angle ADE = \angle BIF$, then $\angle AED = \angle CIF$.

**2942**. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given $\triangle ABC$ with $\angle ABC = 2\angle ACB$, suppose that $D$ is a point on the ray $CB$ such that $\angle ADC = \frac{1}{2}\angle BAC$. Prove that

$$\frac{1}{CD} = \frac{1}{AB} - \frac{1}{AC}.$$

**2943**. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given $\triangle ABC$, let $D$ be the point on $AB$ produced beyond $B$ such that $BD = BC$, and let $E$ be the point on $AC$ produced beyond $C$ such that $CE = BC$. Let $P$ be the intersection of $BE$ and $CD$, and suppose that $\frac{DP}{BE} + \frac{EP}{CD} = 2\sin\left(\frac{\angle BAC}{2}\right)$. Prove that $\angle BAC = 90^\circ$.

**2944**. *Proposed by Václav Konečný, Big Rapids, MI, USA.*

Given an ellipse with foci $F_1$ and $F_2$, minor vertices $V_1'$ and $V_2'$, a line $\ell$, and a point $P$ not on $\ell$. Construct, with straightedge alone, the line through $P$ which is

(a) parallel to $\ell$;

(b) perpendicular to $\ell$.

The constructions are well known, if a circle with its centre is given instead of an ellipse and its foci (Poncelet–Steiner Construction Theorem).

**2945**. *Proposed by Michel Bataille, Rouen, France.*

Let $G$ be the centroid of $\triangle A_1A_2A_3$. For $j = 1, 2, 3$, a circle is tangent to $A_jA_{j+1}$ at $T_j$ and to $A_jA_{j+2}$ at $U_j$, so that $G$ lies on the line segment $T_jU_j$ (subscripts are taken modulo 3). Prove that

$$|GT_1| \cdot |GT_2| \cdot |GT_3| \;=\; |GU_1| \cdot |GU_2| \cdot |GU_3| \,.$$

**2946**. *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*

Let $x$, $y$, $z$ be positive real numbers satisfying $x^2 + y^2 + z^2 = 1$. Prove that

(a) $\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) - (x + y + z) \geq 2\sqrt{3}$.

(b) $\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) + (x + y + z) \geq 4\sqrt{3}$.

**2947★**. *Proposed by Abbas Mehrabian, student, Tehran, Iran.*

The featured solution to problem 2149 [1996 : 171; 1997 : 306–308] is missing a step, which we remedy by means of the following problem. Let $A'B'C'D'$ be a quadrilateral with an inscribed circle centred at $O$. For any point $P$ inside $A'B'C'D'$, define $ABCD$ to be the convex quadrilateral whose sides pass through the vertices of $A'B'C'D'$ and are perpendicular at the vertex to the line joining it to $P$. Prove that $P$ is the intersection point of the diagonals $AC$ and $BD$ if and only if $P = O$.

**2948★**. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Prove or disprove that the unique solution of the system of equations

$$\begin{aligned} bb' + cc' &= aa' - rr' \,, \\ a^2 = b^2 + c^2 \,, \qquad a'^2 &= b'^2 + c'^2 \,, \\ 2r = b + c - a \,, \qquad 2r' &= b' + c' - a' \,, \end{aligned}$$

among Heron right triangles, where $r$ and $r'$ are their associated inradii, is given by

$$a = 5\,,\; b = 4\,,\; c = 3\,,\; r = 1\,; \quad \text{and} \quad a' = 13\,, b' = 12\,,\; c' = 5\,,\; r' = 2\,.$$

**2949★**. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $n \geq 3$ be an *odd* natural number. Determine the smallest number $\mu = \mu(n)$ such that the entries of any row and of any column of the matrix

$$\begin{pmatrix} 1 & a_{1,2} & \cdots & a_{1,\mu} \\ 2 & a_{2,2} & \cdots & a_{2,\mu} \\ \vdots & \vdots & \ddots & \vdots \\ n & a_{n,2} & \cdots & a_{n,\mu} \end{pmatrix}$$

are distinct numbers from the set $\{1, 2, \ldots, n-1, n\}$, and the numbers in each row sum to the same value.

**2950★**. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $ABC$ be a triangle whose largest angle does not exceed $2\pi/3$. For $\lambda$, $\mu \in \mathbb{R}$, consider inequalities of the form

$$\cos\left(\frac{A}{2}\right) \cdot \cos\left(\frac{B}{2}\right) \cdot \cos\left(\frac{C}{2}\right) \;\geq\; \lambda + \mu \cdot \sin\left(\frac{A}{2}\right) \cdot \sin\left(\frac{B}{2}\right) \cdot \sin\left(\frac{C}{2}\right) .$$

(a) Prove that $\lambda_{\text{max}} \geq \dfrac{2\sqrt{3}-1}{8}$.

(b) Prove or disprove that

$$\lambda \;=\; \frac{2\sqrt{3}-1}{8} \qquad \text{and} \qquad \mu \;=\; 1+\sqrt{3}$$

yield the best inequality in the sense that $\lambda$ cannot be increased. Determine also the cases of equality.

# SOLUTIONS

*No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.*

**2548★**. [2000 : 238] *Proposed by Clark Kimberling, University of Evansville, Evansville, Indiana, USA.*

Let $a(1) = 1$ and, for $n \geq 2$, define $a(n) = \lfloor a(n-1)/2 \rfloor$, if this is not in $\{0, a(1), \ldots, a(n-1)\}$, and $a(n) = 3a(n-1)$ otherwise.

(a) Does any positive integer occur more than once in this sequence?

(b) Does every positive integer occur in this sequence?

*Solution by Mateusz Kwaśnicki, student, Wroclaw University of Technology, Poland.*

Every positive integer occurs exactly once in the sequence. We will prove that, more generally, when the numbers $2$ and $3$ in the definition of the sequence are replaced by arbitrary integers $p$ and $q$ greater than $1$, then, in the sequence,

(a) (Uniqueness) No positive integer occurs more than once, and

(b) (Existence) Every positive integer occurs if and only if the following condition is satisfied: $\log_q p$ is irrational, or equivalently,

$$p^n = q^m \text{ for integers } m, n \quad \Longrightarrow \quad m = n = 0\,.$$

Note that the condition that $\log_q p$ is irrational is satisfied, in particular, when $p$ and $q$ are relatively prime.

Let $p, q$ be any integers greater that $1$. Define the *multiplying-dividing (M-D) sequence* $(a_n)$ for the multiplier $p$ and divisor $q$ recursively as follows. Begin with $a_1 = 1$, and take $a_{n+1}$ equal to $\lfloor a_n/q \rfloor$ if this number is positive and has not yet occured in the sequence, or equal to $pa_n$ otherwise.

For instance, if $p = 3$ and $q = 2$ (the original problem), then the corresponding M-D sequence is $1$, $3$, $9$, $4$, $2$, $6$, $18$, $54$, $27$, $13$, $39$, $19$, $57$, $28$, $14$, $7$, $21$, $\ldots$.

As usual, let $\mathbb{Z}$, $\mathbb{Z}^+$, and $\mathbb{R}$ denote the sets of integers, positive integers, and real numbers, respectively. We define

$$A_n = \{0\} \cup \{a_m : m \leq n\} \quad \text{and} \quad A = \{a_m : m \in \mathbb{Z}^+\}\,.$$

Then the definition of the M-D sequence $(a_n)$ may be expressed as follows:

$$\begin{aligned} a_1 &= 1\,, \\ a_{n+1} &= \begin{cases} pa_n & \text{if } \lfloor a_n/q \rfloor \in A_n\,, \\ \lfloor a_n/q \rfloor & \text{if } \lfloor a_n/q \rfloor \notin A_n\,. \end{cases} \end{aligned}$$

**Lemma 1**. Let $x \in \mathbb{R}$, $u \in \mathbb{Z}$, and $p \in \mathbb{Z}^+$. If $pu = \lfloor px \rfloor$, then $u = \lfloor x \rfloor$.

*Proof*. If $pu = \lfloor px \rfloor$, then

$$u = \lfloor u \rfloor = \left\lfloor \frac{pu}{p} \right\rfloor = \left\lfloor \frac{\lfloor px \rfloor}{p} \right\rfloor = \left\lfloor \frac{px}{p} \right\rfloor = \lfloor x \rfloor .$$

□

**Lemma 2**. For any $n \in \mathbb{Z}^+$, if $\lfloor a_n/q \rfloor \in A_n$, then $\lfloor a_n/q^k \rfloor \in A_n$, for all $k \in \mathbb{Z}^+$.

*Proof*. Fix $n \in \mathbb{Z}^+$ such that $\lfloor a_n/q \rfloor \in A_n$. We apply induction. Assume that $\lfloor a_n/q^k \rfloor \in A_n$ for some $k \in \mathbb{Z}^+$. If $\lfloor a_n/q^k \rfloor = 0$, then we have $\lfloor a_n/q^{k+1} \rfloor = 0 \in A_n$. If $\lfloor a_n/q^k \rfloor \neq 0$, then $\lfloor a_n/q^k \rfloor = a_m$ for some $m \le n$. Note that $m$ cannot equal $n$, since $\lfloor a_n/q^k \rfloor < a_n$; thus, $m < n$. It follows that either $\lfloor a_m/q \rfloor \in A_m \subset A_n$ or $\lfloor a_m/q \rfloor = a_{m+1} \in A_n$. Hence, $\lfloor a_n/q^{k+1} \rfloor = \lfloor a_m/q \rfloor \in A_n$, which completes the induction. □

**Proof of Uniqueness**. Assume, on the contrary, that $a_i = a_j$ for some $i \neq j$. Let $n$ be the smallest integer such that $a_n = a_m$ for some $m$, $0 < m < n$. Then $a_n \in A_{n-1}$. Hence, $a_n \neq \lfloor a_{n-1}/q \rfloor$, and so $a_n = pa_{n-1}$. Let $i \le m$ be such that

$$a_m = \left\lfloor \frac{a_{m-1}}{q} \right\rfloor, \quad a_{m-1} = \left\lfloor \frac{a_{m-2}}{q} \right\rfloor, \quad \ldots , \quad a_{i+1} = \left\lfloor \frac{a_i}{q} \right\rfloor, \quad a_i = pa_{i-1} .$$

(Note that such $i$ exists, since $a_2 = pa_1$.) Let $k = m - i$. Then

$$pa_{n-1} = a_n = a_m = \left\lfloor \frac{a_i}{q^k} \right\rfloor = \left\lfloor \frac{pa_{i-1}}{q^k} \right\rfloor .$$

Hence, $a_{n-1} = \lfloor a_{i-1}/q^k \rfloor$, using Lemma 1. Now, since $a_i = pa_{i-1}$, we have $\lfloor a_{i-1}/q \rfloor \in A_{i-1}$. By Lemma 2, it follows that $a_{n-1} = \lfloor a_{i-1}/q^k \rfloor \in A_{i-1}$. Then $a_{n-1} = a_j$ for some $j \le i - 1 < n - 1$, contradicting our choice of $n$. Hence, our assumption was false, and thus we have proved uniqueness. □

**Proof of Existence – Part I**. We will prove the necessity of the condition that $\log_q p$ is irrational. Suppose that $\log_q p$ is rational, say $\log_q p = m/n$, for some relatively prime integers $m$, $n > 0$. Then $q^m = p^n$. Hence, $q^{1/n} = p^{1/m}$ is an integer greater than 1; denote it by $d$. Induction shows that $a_n = d^{k_n}$ for some non-negative integer $k_n$. In particular, there are positive integers which do not occur in the sequence $(a_n)$. □

To prove the sufficiency of the condition that $\log_q p$ is irrational, we will need to probe a little deeper, using some ideas from topological dynamics. The quotient group of the additive (topological) groups $\mathbb{R}$ and $\mathbb{Z}$ (which is isomorphic to $[0, 1)$ with addition 'modulo 1') will be denoted by $\mathbb{T}$. Elements of $\mathbb{T}$ are cosets $[a] = a + \mathbb{Z} = \{a + n : n \in \mathbb{Z}\}$, where $a \in \mathbb{R}$. Let $\kappa : \mathbb{R} \to \mathbb{T}$ be the canonical homomorphism defined by $\kappa(a) = [a]$. We recall that a subset $U$ of $\mathbb{T}$ is open if and only if $\kappa^{-1}(U)$ is an open subset of $\mathbb{R}$ (with the usual topology).

**Lemma 3**. The canonical homomorphism $\kappa : \mathbb{R} \to \mathbb{T}$ is an open mapping.

*Proof.* Let $V$ be an open subset of $\mathbb{R}$, and define $U = \kappa(V)$. Then

$$\begin{aligned}\kappa^{-1}(U) = \{a : [a] \in U\} &= \{a : \exists\, v \in V, a \in [v]\} \\ &= \{a : \exists\, v \in V, \exists\, n \in \mathbb{Z}, a = v + n\} \\ &= \{a : \exists\, n \in \mathbb{Z}, a \in V + n\} = \bigcup_{n \in \mathbb{Z}} (V + n),\end{aligned}$$

where $V + n = \{v + n : v \in V\}$. This implies that $\kappa^{-1}(U)$ is open, since $V + n$ is open for every $n \in \mathbb{Z}$. Hence, $U$ is open, which proves that $\kappa$ is an open mapping. □

**Lemma 4**. If $b \in \mathbb{Z}^+ \backslash A$, then $q^k b + l \in \mathbb{Z}^+ \backslash A$ for any $k \in \mathbb{Z}^+$ and $l \in \mathbb{Z}$ such that $0 \le l < q^k$.

*Proof.* Let $a \in A$. Then $a = a_n$ for some $n$. Either $\lfloor a_n/q \rfloor = a_{n+1} \in A$ or $\lfloor a_n/q \rfloor \in A_n \subset A \cup \{0\}$. Hence, $\lfloor a/q \rfloor \in A$ or $\lfloor a/q \rfloor = 0$. By induction, we get $\lfloor a/q^k \rfloor \in A$ or $\lfloor a/q^k \rfloor = 0$. Writing $b = \lfloor a/q^k \rfloor$, we get the desired result. □

**Lemma 5** (Topological Dynamics Lemma). Let $a$ be an irrational number, and let $U \subset \mathbb{T}$ be a nonempty open set. Then there exists $N \in \mathbb{Z}^+$ such that, for every $x \in \mathbb{R}$, there exists $n$, $0 \le n \le N$, such that $[x + na] \in U$.

*Proof.* Fix $x \in \mathbb{R}$, and let $A = \{[x + na] : n \ge 0\}$. Then $A$ is dense in $\mathbb{T}$. (Since the proof of this statement is well known and long, we omit it.) Hence, $[x + na] \in U$ for some $n$. Therefore, $[x] \in \bigcup_{n=0}^{\infty} (U - [na])$. Since $x$ was arbitrary, $\bigcup_{n=0}^{\infty} (U - [na]) = \mathbb{T}$. By the compactness of $\mathbb{T}$, there exists $N \in \mathbb{Z}^+$ such that $\bigcup_{n=0}^{N} (U - [na]) = \mathbb{T}$.

Take any $x \in \mathbb{R}$. Then $[x] \in U - [na]$ for some $n$, $0 \le n \le N$. For this $n$, we have $[x + na] \in U$. □

**Proof of Existence – Part II**. Assume that $\log_q p$ is irrational. We intend to show that $\mathbb{Z}^+ \backslash A = \emptyset$. Assume, to the contrary, that $\mathbb{Z}^+ \backslash A$ is not empty, and let $a$ be any element of this set. We will show that $q^t a + s \in A$ for some $t$, $s \in \mathbb{Z}^+$, $0 \le s < q^t$, which contradicts Lemma 4.

The main idea is that the difference between $\log_q a$ and $\log_q(q^t a + s)$ is nearly an integer. This means that the distance between $[\log_q a]$ and $[\log_q(q^t a + s)]$ in $\mathbb{T}$ is small. Hence, it is sufficient to show that the set of cosets $[\log_q a_n]$ cannot be separated from $[\log_q a]$ for all $n$. It will turn out that the sequence of $[\log_q a_n]$ has much in common with rotations, well studied transformations on $\mathbb{T}$.

The uniqueness proved above implies that $\lim_{n \to \infty} a_n = \infty$. It follows that there are infinitely many $n$ such that $a_{n+1} = p a_n$; let $k_1$, $k_2$, … be the increasing sequence of all those $n$. Thus,

$$a_{k_{n+1}} \;=\; \left\lfloor \frac{a_{k_n+1}}{q^{k_{n+1}-k_n-1}} \right\rfloor , \qquad a_{k_n+1} \;=\; p a_{k_n} \,. \tag{1}$$

Let $b_n = \log_q a_n$ and $\alpha = \log_q p$ (which, we recall, is irrational). Define

$$\epsilon_n \;=\; \log_q \frac{a_{k_n+1}}{q^{k_{n+1}-k_n-1}} - \log_q a_{k_{n+1}} \;=\; b_{k_n+1} - b_{k_{n+1}} - k_{n+1} + k_n + 1 \,.$$

Using (1), and noting that $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$ for $x \in \mathbb{R}$, we obtain $\epsilon_n \ge 0$ and

$$\epsilon_n \;<\; \log_q(a_{k_{n+1}}+1) - \log_q a_{k_{n+1}} \;=\; \log_q\left(1 + \frac{1}{a_{k_{n+1}}}\right) \;<\; \frac{1}{(\ln q) a_{k_{n+1}}} \,.$$

Thus,

$$0 \;\le\; \epsilon_n \;<\; \frac{1}{(\ln q) a_{k_{n+1}}} \,. \tag{2}$$

By (1) and the definition of $\epsilon_n$,

$$\begin{aligned} [b_{k_{n+1}}] \;&=\; [b_{k_n+1} - \epsilon_n] \,, \\ [b_{k_n+1}] \;&=\; [\log_q(p a_{k_n})] = [b_{k_n} + \alpha] \,. \end{aligned} \tag{3}$$

This is what we needed: $[b_{k_{n+1}}]$ is very close to $[b_{k_n+1}]$, which is an irrational rotation of $[b_{k_n}]$. Hence, we can apply methods of topological dynamics.

Before we proceed with some technical details, let us recall that we are looking for $a_m$ such that $a_m = q^t a + s$ for some $t,\ s \in \mathbb{Z}^+$, $0 \le s < q^t$, which is equivalent to $\log_q a_m \in \big(\log_q a + t \,,\ \log_q(a+1) + t\big)$.

Define

$$\delta \;=\; \log_q(a + \tfrac{1}{2}) - \log_q a \,, \quad V \;=\; \big(\log_q(a+\tfrac{1}{2}), \log_q(a+1)\big), \quad U \;=\; \kappa(V) \,.$$

Since $U$ is a nonempty open subset of $\mathbb{T}$ and $\alpha$ is irrational, Lemma 5 implies that there exists $L \in \mathbb{Z}^+$ such that for every $x \in \mathbb{R}$ we have $[x + l\alpha] \in U$ for some $l$, $0 \le l \le L$.

Let $M$ be large enough so that $L < M \delta \ln q$. Since $\lim_{n\to\infty} a_n = \infty$, there exists $N$ such that $a_n > M$ for $n \ge N$. Fix any $n$ such that $k_n \ge N$. For some $l$, $0 \le l \le L$, we have $[b_{k_n} + l\alpha] \in U$. Equivalently, for some $i \in \mathbb{Z}$,

$$b_{k_n} + l\alpha + i \in V \,.$$

By definition of $V$,

$$\log_q(a + \tfrac{1}{2}) \;<\; b_{k_n} + l\alpha + i \;<\; \log_q(a+1) \tag{4}$$

Let $m = n + l$. Using (3) and a simple induction argument, we get

$$[b_{k_m}] \;=\; [b_{k_n} \;+\; l\alpha \;-\; \epsilon_n \;-\; \epsilon_{n+1} \;-\; \ldots \;-\; \epsilon_{m-1}] \,.$$

Again this is equivalent to

$$b_{k_m} \;=\; b_{k_n} \,+\, l\alpha \,-\, \epsilon_n \,-\, \epsilon_{n+1} \,-\, \ldots \,-\, \epsilon_{m-1} \,+\, j$$

for some $j \in \mathbb{Z}$. Recall that if $n \le \nu < m$, then $k_{\nu+1} > k_n \ge N$ and so $a_{k_{\nu+1}} > M$. Then, according to (2), we have $0 \le \epsilon_\nu < (M \ln q)^{-1}$, so that

$$b_{k_n} \,+\, l\alpha \,+\, j \,-\, \frac{m-n}{M \ln q} \;<\; b_{k_m} \;\le\; b_{k_n} \,+\, l\alpha \,+\, j\,.$$

But $M$ was defined so that

$$\frac{m-n}{M \ln q} \;=\; \frac{l}{M \ln q} \;\le\; \frac{L}{M \ln q} \;<\; \delta\,.$$

Hence,

$$b_{k_n} \,+\, l\alpha \,+\, j \,-\, \delta \;<\; b_{k_m} \le b_{k_n} \,+\, l\alpha \,+\, j\,.$$

Together with (4) and the definition of $\delta$, this leads us to

$$\begin{aligned}\log_q(a+\tfrac{1}{2}) \,+\, j \,-\, i \,-\, \left(\log_q(a+\tfrac{1}{2}) - \log_q a\right) \qquad\qquad \\ <\; b_{k_m} \;<\; \log_q(a+1) \,+\, j \,-\, i\,.\end{aligned}$$

Finally, we get

$$\log_q a \,+\, j \,-\, i \;<\; b_{k_m} \;<\; \log_q(a+1) \,+\, j \,-\, i\,.$$

In terms of $a_\nu$, this means $q^{j-i}a \;<\; a_{k_m} \;<\; q^{j-i}(a+1)$. It follows that $j - i > 0$ and $a_{k_m} = q^t a + s$ for $t = i - j$ and some $s$ such that $0 < s < q^t$, which contradicts Lemma 4.

This proves that $\mathbb{Z}^+ \backslash A$ is empty, or in other words, that every positive integer occurs in the sequence. □

**2799**★. [2002 : 536, 2003 : 47, 2003 :531–534] *Proposed Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Prove or disprove the inequality

$$\sum_{\substack{i,\,j \,\in\, \{1,\,2,\,\ldots\,,n\} \\ 1 \le i < j \le n}} \frac{1}{1 - x_i x_j} \;\le\; \binom{n}{2} \frac{1}{1 - \frac{1}{n^2}}\,,$$

where $\sum\limits_{j=1}^{n} x_j = 1$, $x_j \ge 0$.

[*Ed:* The solution to this problem, originally published in [2003 : 531–534], was incorrect. The error was identified by Vasile Cîrtoaje. Below is his generalization of the problem and its solution.]

*Generalization and solution by Vasile Cîrtoaje, University of Ploiesti, Romania.*

We prove a slightly more general result:

**Theorem**. If $0 < s \leq 1$ and $x_1,\, x_2,\, \ldots,\, x_n \geq 0$ are real numbers such that $x_1 + x_2 + \cdots + x_n = s$, then

$$\sum_{1\leq i<j\leq n} \frac{1}{1 - x_i x_j} \;\leq\; \binom{n}{2}\frac{n^2}{n^2 - s^2}\,.$$

*Proof.* Since the inequality is clearly true for $n = 2$, we will assume $n \geq 3$. Also, we suppose (without loss of generality) that $x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$. The set $\{(x_1, x_2, \ldots, x_n) : x_1 + x_2 + \cdots + x_n = s\}$ is a compact set in $\mathbb{R}^n$, and the function

$$F(x_1, x_2, \cdots, x_n) \;=\; \sum_{1\leq i<j\leq n} \frac{1}{1 - x_i x_j}$$

is continuous on this set. Consequently, $F$ attains a maximum value at one or more points of the set. We will prove, in two steps, that the maximum occurs only at the point $\left(\frac{s}{n}, \frac{s}{n}, \cdots, \frac{s}{n}\right)$, whence the desired inequality follows.

In the first step, we will show that for $s \leq \sqrt{3}$, if the function $F$ is maximal at $(x_1, x_2, \ldots, x_n)$, then $x_1 = x_2$.

In the second step, we will show that if the function $F$ is maximal at $(x_1, x_2, \ldots, x_n)$ with $x_1 = x_2 = \cdots = x_i$, where $i \in \{2,\, 3,\, \ldots,\, n-1\}$, then $x_{i+1} = x_i$. These statements will complete the proof.

**Step 1.**

To prove the claimed statement, we suppose that $x_1 > x_2$ and show that $F$ is not maximal at $(x_1, x_2, \ldots, x_n)$. More precisely, we show that

$$F(x_1, x_2, \ldots, x_n) \;<\; F\left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}, x_3, \ldots, x_n\right)\,.$$

For convenience, we replace $x_1$, $x_2$, and $\frac{x_1 + x_2}{2}$ by $x$, $y$, and $t$, respectively. Thus, the inequality can be written in the form

$$\sum_{j=3}^{n}\left(\frac{1}{1 - x x_j} + \frac{1}{1 - y x_j} - \frac{2}{1 - t x_j}\right) \;<\; \frac{1}{1 - t^2} - \frac{1}{1 - xy}\,.$$

After combining terms and dividing by the positive factor $t^2 - xy$, we get

$$\sum_{j=3}^{n}\left(\frac{2x_j^2}{(1 - x x_j)(1 - y x_j)(1 - t x_j)}\right) \;<\; \frac{1}{(1 - t^2)(1 - xy)}\,. \qquad (1)$$

The condition $s \leq \sqrt{3}$ is necessary. Indeed, letting $x_4 = \cdots = x_n = 0$ and $x = y = x_3 = \frac{s}{3}$, we get $x < \frac{1}{\sqrt{3}}$ from (1), which implies $s < \sqrt{3}$.

Now, we notice that

$$\begin{aligned}\sum_{j=3}^{n}\left(\frac{2x_j^2}{(1-xx_j)(1-yx_j)(1-tx_j)}\right) &\leq \frac{2(x_3^2+\cdots+x_n^2)}{(1-xx_3)(1-yx_3)(1-tx_3)} \\ &\leq \frac{2(x_3^2+\cdots+x_n^2)}{(1-xy)(1-y^2)(1-t^2)}\,.\end{aligned}$$

Thus, to prove (1), it suffices to show that

$$2(x_3^2+\cdots+x_n^2)+y^2 \;<\; 1\,. \tag{2}$$

We introduce the notation $r = x_3 + \cdots + x_n$, and consider two cases for $r$.

**Case 1**. $r \leq y$. Since $x_3^2 + \cdots + x_n^2 \leq r^2$, it is sufficient to show that

$$2r^2 + y^2 \;<\; 1\,.$$

From $x > y \geq r$ and $x + y + r = s$, we obtain $r < \frac{s}{3}$ and $2y + r < s$. Thus,

$$\begin{aligned}2r^2+y^2-1 &< 2r^2+\left(\frac{s-r}{2}\right)^2-1 \\ &= \frac{1}{4}\left(r-\frac{s}{3}\right)(s+9r)+\left(\frac{s^2}{3}-1\right) \;<\; 0\,.\end{aligned}$$

**Case 2**. $r > y$. The condition $r > y$ implies $n \geq 4$, (since in the case $n = 3$ we have $r = x^3 \leq y$) and $y > 0$ (since $y = 0$ implies $r = 0$). From the order relation $x > y \geq x_3 \geq \cdots \geq x_n$, we get $y < r \leq (n-2)y$. Thus, we may conclude there is $k \in \{1, 2, \ldots, n-3\}$ such that

$$ky \;<\; r \;\leq\; (k+1)y\,. \tag{3}$$

According to the Majorization Inequality (of Karamata) applied to the convex function $f(x) = x^2$, the following inequality holds

$$x_3^2+\cdots+x_n^2 \;\leq\; ky^2+(r-ky)^2+(n-k-3)\cdot 0^2\,.$$

Consequently, to prove (2) it suffices to show that

$$(2k+1)y^2+2(r-ky)^2 \;<\; 1\,. \tag{4}$$

[*Ed:* Karamata's Majorization Theorem is as follows. If two vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ having $n$ components, $a_i$ and $b_i$, $i = 1, 2, \ldots, n$, arranged in non-increasing order of magnitude, satisfy

$$\sum_{i=1}^{k} a_i \;\geq\; \sum_{i=1}^{k} b_i\,, \qquad k = 1, 2, \ldots, n-1,$$

$$\text{and} \qquad \sum_{i=1}^{n} a_i \;=\; \sum_{i=1}^{n} b_i\,,$$

we say that $\overrightarrow{A}$ *majorizes* $\overrightarrow{B}$, which we denote as $\overrightarrow{A} \succ \overrightarrow{B}$. For a convex function $F(x)$, we then have

$$F(a_1) + F(a_2) + \cdots + F(a_n) \;\geq\; F(b_1) + F(b_2) + \cdots + F(b_n)\,.$$

[1] Klamkin, Murray S., On a "Problem of the Month", [2002 : 86–87].

[2] Klamkin, Murray S., Quickie Inequalities, *π in the sky*, September 2003, 26–29.

[3] Marshall, A. M. and Olkin, I., *Inequalities: Theory of Majorization and Its Applications,* Academic Press, New York, 1979.]

Letting $x = (p-k-1)y$ and $y = qs$, we have

$$r \;=\; s - x - y \;=\; \left(\frac{1}{q} - p + k\right) y\,.$$

Therefore,

$$r - ky \;=\; \left(\frac{1}{q} - p\right) y\,.$$

From this result and (3), we get

$$\frac{1}{p+1} \;\leq\; q \;<\; \frac{1}{p}\,.$$

Also, the condition $x > y$ implies that

$$p \;>\; k+2\,. \tag{5}$$

The inequality (4) is equivalent to

$$2k + 1 + 2\left(\frac{1}{q} - p\right)^2 \;<\; \frac{1}{y^2}\,.$$

Since $y = qs \leq q\sqrt{3}$, it suffices to prove that

$$3(2k+1)q^2 + 6(1-pq)^2 \;<\; 1\,. \tag{6}$$

In order to prove (6), we consider the function

$$f(u) \;=\; 3(2k+1)u^2 + 6(1-pu)^2 - 1\,.$$

We have to show that $f(u) < 0$ for $\frac{1}{p+1} \leq u < \frac{1}{p}$. Since $f(u)$ is a convex function, it suffices to show that $f\left(\frac{1}{p+1}\right) < 0$ and $f\left(\frac{1}{p}\right) \leq 0$. Indeed, according to (5), we have

$$f\left(\frac{1}{p+1}\right) \;=\; \frac{6k+9}{(p+1)^2} - 1 \;<\; \frac{6k+9}{(k+3)^2} - 1 \;=\; \frac{-k^2}{(k+3)^2} \;<\; 0\,,$$

and

$$f\left(\frac{1}{p}\right) \;=\; \frac{3(2k+1)}{p^2} - 1 \;<\; \frac{3(2k+1)}{(k+2)^2} - 1 \;=\; \frac{-(k-1)^2}{(k+2)^2} \;\leq\; 0\,.$$

**Step 2.**

To prove the claim, we proceed the same way as in Step 1. We consider $x_i > x_{i+1}$ for $2 \le i \le n-1$, and then show that $F$ is not maximal at $(x_1, x_2, \dots, x_n)$. Using the substitutions $x_i = x$, $x_{i+1} = y$, and $t = \dfrac{x+y}{2}$, the inequality corresponding to (1) has the form

$$\frac{2(i-1)x^2}{(1-x^2)(1-yx)(1-tx)} + \sum_{j=i+2}^{n} \frac{2x_j^2}{(1-xx_j)(1-yx_j)(1-tx_j)} < \frac{1}{(1-t^2)(1-xy)} .$$

Since $x > y \ge x_{i+2} \ge \cdots \ge x_n \ge 0$, it suffices to show that

$$\frac{2(i-1)x^2}{(1-x^2)(1-yx)(1-x^2)} + \frac{\sum_{j=i+2}^{n} 2x_j^2}{(1-x^2)(1-yx)(1-x^2)} < \frac{1}{1-xy} .$$

This inequality can be written as

$$2ix^2 + 2(x_{i+2}^2 + \cdots + x_n^2) < 1 + x^4 . \qquad (7)$$

Since $x^4 > 0$, it suffices to show that

$$2ix^2 + 2(x_{i+2}^2 + \cdots + x_n^2) \le 1 . \qquad (8)$$

**Case 1.** $n = 3$. We have $i = 2$ and $x_{i+2}^2 + \cdots + x_n^2 = 0$. The inequality (8) becomes $4x^2 \le 1$. From $2x + y = s$, we get $x \le \frac{s}{2} \le \frac{1}{2}$, therefore, $4x^2 \le 1$.

**Case 2.** $n > 3$. Let $r = x_{i+2} + \cdots + x_n$. From $x > y \ge x_{i+2} \ge \cdots \ge x_n \ge 0$ and $ix + y + r = s \le 1$, we obtain

$$4\left(\sum_{j=i+2}^{n} x_j^2\right) \le 4\left(\sum_{j=i+2}^{n} x_j y\right) = 4ry \le (y+r)^2 \le (1-ix)^2 .$$

Thus, to prove (8), it suffices to show that $4ix^2 + (1-ix)^2 \le 2$. Indeed, we have

$$\begin{aligned} 4ix^2 + (1-ix)^2 - 2 &= (i^2+4i)x^2 - 2ix - 1 \\ &= (ix-1)(3ix+1) - 2i(i-2)x^2 \le 0 . \end{aligned}$$

This completes the proof. □

**Remark.** The statement of the theorem is valid for a larger range of $s$. In our proof, the range is restricted by (8). Really, the inequality (7) allows us to increase the range of $s$ to

$$0 < s \le \sqrt{6} - \sqrt{2} \approx 1.035 .$$

However, we conjecture the following more general statement.

**Conjecture.** Let $n \geq 3$, $x_1, x_2, \ldots, x_n \geq 0$, and $s = x_1 + x_2 + \cdots + x_n$.

(a) If $s \leq \sqrt{\dfrac{2n}{n+1}}$, then

$$\sum_{1 \leq i < j \leq n} \frac{1}{1 - x_i x_j} \leq \frac{n^3(n-1)}{2(n^2 - s^2)};$$

(b) If $\sqrt{\dfrac{2n}{n+1}} \leq s < 2$, then

$$\sum_{1 \leq i < j \leq n} \frac{1}{1 - x_i x_j} \leq \frac{n(n-1)}{2} + \frac{s^2}{4 - s^2}.$$

The inequalities can be written as follows:

(a) $F(x_1, x_2, \ldots, x_n) \leq F\left(\frac{s}{n}, \frac{s}{n}, \cdots, \frac{s}{n}\right)$;

(b) $F(x_1, x_2, \ldots, x_n) \leq F\left(\frac{s}{2}, \frac{s}{2}, 0, \cdots, 0\right)$.

**2839**. [2003 : 239; corrected 2003 : 315] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, AB.*

Suppose that $x$, $y$, and $z$ are real numbers. Prove that

$$\left(x^3 + y^3 + z^3\right)^2 + 3(xyz)^2 \geq 4\left(y^3 z^3 + z^3 x^3 + x^3 y^3\right).$$

Determine the cases of equality.

I *Solution by Charles R. Diminnie, Angelo State University, San Angelo, TX, USA.*

If $z = 0$, then

$$\begin{aligned}
&(x^3 + y^3 + z^3)^2 + 3(xyz)^2 - 4(y^3 z^3 + z^3 x^3 + x^3 y^3) \\
&= (x^3 + y^3)^2 - 4x^3 y^3 = (x^3 - y^3)^2 \geq 0,
\end{aligned}$$

with equality if and only if $x = y$.

If $z \neq 0$, let $u = \dfrac{x}{z}$ and $v = \dfrac{y}{z}$. Then (with considerable help from MAPLE) we have

$$\begin{aligned}
&\frac{1}{z^6}\left((x^3 + y^3 + z^3)^2 + 3(xyz)^2 - 4(y^3 z^3 + z^3 x^3 + x^3 y^3)\right) \\
&= (u^3 + v^3 + 1)^2 + 3(uv)^2 - 4(u^3 v^3 + u^3 + v^3) \\
&= (u^2 + u + uv + v^2 + v + 1) \\
&\quad \cdot (u^4 - u^3 v - uv^3 + v^4 - u^3 + u^2 v + uv^2 - v^3 + uv - u - v + 1).
\end{aligned}$$

Note that $u^2+u+uv+v^2+v+1=\frac{1}{2}\left((u+v+1)^2+u^2+v^2+1\right)>0$ and

$$\begin{aligned} & u^4-u^3v-uv^3+v^4-u^3+u^2v+uv^2-v^3+uv-u-v+1 \\ &= \ \tfrac{1}{4}\left[\left((u-v)^2+u+v-2\right)^2+3(u-v)^2(u+v-1)^2\right] \ \geq \ 0\,. \end{aligned}$$

Hence, $(u^3+v^3+1)^2+3(uv)^2-4(u^3v^3+u^3+v^3)\geq 0$.

Equality occurs if and only if $(u-v)^2+u+v-2=0$ and either $u=v$ or $u+v=1$. In the first case, we have $u=v$ and $u+v=2$; that is, $u=v=1$. In the second case, we have $u+v=1$ and $(u-v)^2=1$; that is, $(u,v)=(0,1)$ or $(u,v)=(1,0)$. In terms of $x$, $y$, and $z$, these conditions yield three solutions: (i) $x=y=z$, (ii) $x=0$, $y=z$, and (iii) $y=0$, $x=z$.

Therefore, in all cases,

$$(x^3+y^3+z^3)^2+3(xyz)^2 \ \geq \ 4(y^3z^3+z^3x^3+x^3y^3)\,,$$

with equality if and only if one of the following occurs: (i) $x=y=z$, (ii) $x=0$ and $y=z$, (iii) $y=0$ and $x=z$, or (iv) $z=0$ and $x=y$.

II *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Let

$$\begin{aligned} f(x,y,z) &= (x^3+y^3+z^3)^2+3(xyz)^2-4(y^3z^3+z^3x^3+x^3y^3) \\ &= x^6+(y^3-z^3)^2+3(xyz)^2-2z^3x^3-2x^3y^3\,. \end{aligned}$$

Clearly, $f(0,y,z)\geq 0$ with equality if and only if $y=z$. Symmetrically, $f(x,0,z)\geq 0$ and $f(x,y,0)\geq 0$ with equalities if and only if $z=x$ and $x=y$, respectively.

Assume henceforth that $xyz\neq 0$. Then $f(x,y,z)>0$ if both $zx<0$ and $xy<0$. Symmetrically, $f(x,y,z)>0$ for the other two cases where two of $x$, $y$, $z$ have the same sign while the third has the opposite sign. Since $f(-x,-y,-z)=f(x,y,z)$ and since $f(x,y,z)$ is symmetric in its variables, we may now assume, without loss of generality, that $0<x\leq y\leq z$. Then

$$\begin{aligned} f(x,y,z) \ &= \ \tfrac{1}{16}\big[x^2(2x-y-z)^2(4x^2+3y^2+3z^2+4xy+6yz+4zx) \\ &\quad +(y-z)^2(16y^4+32y^3z+48y^2z^2+32yz^3+16z^4 \\ &\quad -3x^2y^2-18x^2yz-3z^2x^2-24x^3y-24zx^3)\big] \ \geq \ 0\,. \end{aligned}$$

[*Ed*: This is true because $16y^4>3x^2y^2$, $32y^3z>24zx^3$, $48y^2z^2>18x^2yz$, $32yz^3>24x^3y$, and $16z^4>3z^2x^2$.]

Equality holds above if and only if $2x-y-z=0=y-z$; that is, if and only if either $x=y=z$ or two of $x$, $y$, $z$ are equal while the third is 0.

*Also solved by MICHEL BATAILLE, Rouen, France; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VEDULA N. MURTY, Dover, PA, USA; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; LI ZHOU, Polk Community College, Winter Haven, FL, USA (a second solution); and the proposer. There were seven incomplete* *or partially incorrect solutions, most of which either made no mention of the cases for equality or claimed by mistake that "equality holds if and only if $x = y = z$." Some correspondents simply pointed out that the originally posed problem was incorrect.*

*All the correct solutions with the exception of the two featured above and the ones by Hess and Janous, invoked Schur's Inequality or other known results. Zhou, in his second solution, remarked that the essential case when $x$, $y$, $z > 0$ has been proved in the book Mathematical Miniatures (MAA, 2003, Chap. 6, pp. 19–20) by S. Savchev and T. Andreescu, as an application of Popoviciu's Inequality: if $f$ is a convex real-valued function defined on an interval $I$, then*

$$f(x) + f(y) + f(z) + 3f\left(\frac{x+y+z}{3}\right) \ \geq\ 2\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right]$$

*for all $x$, $y$, $z$ in $I$.*

**2840**. [2003 : 239] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let $A'$ be an interior point of the line segment $BC$ in $\triangle ABC$. The interior bisectors of $\angle BA'A$ and $\angle CA'A$ intersect $AB$ and $CA$ at $D$ and $E$, respectively. Prove that $AA'$, $BE$, and $CD$ are concurrent.

*Solution by almost everyone: Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain; Jacques Choné, Nancy, France; Chip Curtis, Missouri Southern State College, Joplin, MO, USA; Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA; John G. Heuver, Grande Prairie, AB; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; Geoffrey A. Kandall, Hamden, CT, USA; Murray S. Klamkin, University of Alberta, Edmonton, AB; Václav Konečný, Big Rapids, MI, USA; David Loeffler, student, Trinity College, Cambridge, UK; Gottfried Perz, Pestalozzigymnasium, Graz, Austria; Toshio Seimiya, Kawasaki, Japan; Bob Serkey, Leonia, NJ, USA; D.J. Smeenk, Zaltbommel, the Netherlands; Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany; Mihaï Stoënescu, Bischwiller, France; Peter Y. Woo, Biola University, La Mirada, CA, USA; Li Zhou, Polk Community College, Winter Haven, FL, USA; Titu Zvonaru, Bucharest, Romania; and the proposer.*

An internal angle bisector of a triangle divides the opposite side in the ratio of the two other sides. Applying this to triangles $AA'B$ and $AA'C$ we get

$$\frac{AD}{DB} = \frac{|AA'|}{|BA'|} \quad\text{and}\quad \frac{CE}{EA} = \frac{|CA'|}{|AA'|}\,.$$

Hence,

$$\frac{AD}{DB}\cdot\frac{BA'}{A'C}\cdot\frac{CE}{EA} = \frac{AA'}{BA'}\cdot\frac{BA'}{A'C}\cdot\frac{A'C}{AA'} = 1\,.$$

From the converse to Ceva's Theorem, we see that $AA'$, $BB'$, and $CC'$ are parallel or concurrent. As $A'$, $D$, and $E$ are interior points of the sides of triangle $ABC$, concurrency is the only possibility, since these three cevians must intersect one another in the interior of the triangle.

*Also solved by MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain, whose equally nice solution used Cartesian coordinates.*

*Loeffler reported that the locus of the intersection point as $A'$ varies on $BC$ is a smooth quartic curve passing through $B$ and $C$ and tangent to the angle bisectors of the given triangle at these points. [The graphics program* Cinderella™ *suggests that if $A$ is allowed to move along the entire line $BC$, then $B$ and $C$ are nodes of the quartic where it crosses itself at right angles.] In the case where $\triangle ABC$ is equilateral, the locus has equation $u^2(v^2+vw+w^2)=v^2w^2$ in trilinear coordinates $u$, $v$, $w$.*

**2841**. [2003 : 239] Corrected. *Proposed by Mihály Bencze, Brasov, Romania.*

Prove the following inequalities:

$$\begin{aligned}
&\frac{\pi}{2}\left(1-\frac{1}{4n}+\frac{5}{32n^2}-\frac{11}{128n^3}\right)\\
&\qquad\le\ \left(\frac{(2n)!!}{(2n-1)!!}\right)^2\frac{1}{2n+1}\\
&\qquad\le\ \frac{\pi}{2}\left(1-\frac{1}{4n}+\frac{5}{32n^2}-\frac{11}{128n^3}+\frac{83}{2048n^4}\right).
\end{aligned}$$

*Solution by Natalio H. Guersenzvaig, Universidad CAECE, Buenos Aires, Argentina.*

Since $(2n-1)!!=\dfrac{(2n)!}{(2n)!!}$ and $(2n)!!=2^n n!$, we can apply the well-known Stirling formula, $n!=n^n\sqrt{2\pi n}e^{-n+O(\frac{1}{n})}$, to obtain

$$\frac{(2n)!!}{(2n-1)!!}=\frac{(2^n n!)^2}{(2n)!}=\frac{2^{2n}n^{2n}(2\pi n)e^{-2n+O(\frac{1}{n})}}{(2n)^{2n}\sqrt{4\pi n}\,e^{-2n+O(\frac{1}{n})}}=\sqrt{\pi n}\,e^{O(\frac{1}{n})}.$$

Then

$$\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\frac{1}{2n+1}=\frac{\pi n}{2n+1}e^{O(\frac{1}{n})};$$

whence,

$$\lim_{n\to\infty}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\frac{2}{2n+1}=\pi.$$

Let

$$\begin{aligned}
f(n) &= \left(\frac{(2n)!!}{(2n-1)!!}\right)^2\frac{2}{2n+1},\\
g_1(n) &= 1-\frac{1}{4n}+\frac{5}{32n^2}-\frac{11}{128n^3}+\frac{83}{2048n^4},\\
\text{and}\quad g_2(n) &= 1-\frac{1}{4n}+\frac{5}{32n^2}-\frac{11}{128n^3}.
\end{aligned}$$

Since $\lim_{n\to\infty} g_1(n) = \lim_{n\to\infty} g_2(n) = 1$, we have

$$\lim_{n\to\infty} \frac{f(n)}{g_1(n)} = \lim_{n\to\infty} \frac{f(n)}{g_2(n)} = \pi\,. \tag{1}$$

Now, $\dfrac{f(n+1)}{f(n)} = \dfrac{4(n+1)^2}{(2n+1)(2n+3)} = \dfrac{4n^2+8n+4}{4n^2+8n+3}$, so that, using the inequalities

$$\begin{aligned}(4n^2+8n+4)g_1(n) - (4n^2+8n+3)g_1(n+1) \\ = \frac{715n^4+1712n^3+2036n^2+1288n+332}{2048n^4(n+1)^4} > 0\end{aligned}$$

and

$$\begin{aligned}(4n^2+8n+3)g_2(n+1) - (4n^2+8n+4)g_2(n) \\ = \frac{83n^3+168n^2+140n+44}{128n^3(n+1)^3} > 0\,,\end{aligned}$$

we obtain $\dfrac{f(n+1)}{f(n)} > \dfrac{g_1(n+1)}{g_1(n)}$ and $\dfrac{g_2(n+1)}{g_2(n)} > \dfrac{f(n+1)}{f(n)}$. Therefore, the sequence $\left\{\dfrac{f(n)}{g_1(n)}\right\}$ is strictly increasing, and the sequence $\left\{\dfrac{f(n)}{g_2(n)}\right\}$ is strictly decreasing. Since $g_1(n) > g_2(n)$, we have $\dfrac{f(n)}{g_1(n)} < \dfrac{f(n)}{g_2(n)}$. Finally, using the limits (1) and the fact that $\pi$ is irrational, we obtain

$$\frac{\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \frac{2}{2n+1}}{1-\frac{1}{4n}+\frac{5}{32n^2}-\frac{11}{128n^3}+\frac{83}{2048n^4}} < \pi < \frac{\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \frac{2}{2n+1}}{1-\frac{1}{4n}+\frac{5}{32n^2}-\frac{11}{128n^3}}\,,$$

which completes the proof.

*Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and LI ZHOU, Polk Community College, Winter Haven, FL, USA. All solvers have noticed that the original inequality is incorrect as stated and suggested the following correction: The original term $3/(32n^2)$ should have been $5/(32n^2)$ in both expressions in which it occurred. (This correction has been made in the problem statement above.) Then they proceeded with solving the corrected version.*

**2842**. [2003 : 240]

Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Prove that

(a) $$\frac{\sum_{k=1}^{n} x_k^n}{n\prod_{k=1}^{n} x_k} + \frac{n\left(\prod_{k=1}^{n} x_k\right)^{\frac{1}{n}}}{\sum_{k=1}^{n} x_k} \geq 2,$$

(b) $$\frac{\sum_{k=1}^{n} x_k^n}{\prod_{k=1}^{n} x_k} + \frac{\left(\prod_{k=1}^{n} x_k\right)^{\frac{1}{n}}}{\sum_{k=1}^{n} x_k} \;\geq\; 1.$$

*Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

(a) By the Power-Mean Inequality, we have

$$\frac{1}{n}\sum_{k=1}^{n} x_k^n \;\geq\; \left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)^n,$$

and by the AM–GM Inequality, we get

$$t \;=\; \frac{\sum_{k=1}^{n} x_k}{\left(\prod_{k=1}^{n} x_k\right)^{\frac{1}{n}}} \;\geq\; n\,.$$

Thus,

$$\frac{\sum_{k=1}^{n} x_k^n}{n\prod_{k=1}^{n} x_k} + \frac{n\left(\prod_{k=1}^{n} x_k\right)^{\frac{1}{n}}}{\sum_{k=1}^{n} x_k} \;\geq\; \left(\frac{t}{n}\right)^n + \frac{n}{t} \;\geq\; 2\left(\frac{t}{n}\right)^{\frac{n-1}{2}} \;\geq\; 2\,.$$

Equality holds if and only if either $n = 1$ or $x_1 = x_2 = \cdots = x_n$ for $n \geq 2$.

(b) Similarly,

$$\begin{aligned}\frac{\sum_{k=1}^{n} x_k^n}{\prod_{k=1}^{n} x_k} + \frac{\left(\prod_{k=1}^{n} x_k\right)^{\frac{1}{n}}}{\sum_{k=1}^{n} x_k} \;&\geq\; n\left(\frac{t}{n}\right)^n + \frac{1}{t} \;\geq\; (n+1)\left(\frac{t}{n}\right)^{\frac{n^2}{n+1}}\left(\frac{1}{t}\right)^{\frac{1}{n+1}} \\ &=\; \frac{n+1}{n^{1/(n+1)}}\left(\frac{t}{n}\right)^{n-1} \;\geq\; \frac{n+1}{n^{1/(n+1)}}\,,\end{aligned}$$

where the second inequality makes use of the weighted AM–GM Inequality. Equality holds if and only if $n = 1$. Note also that

$$n\left(\frac{t}{n}\right)^n + \frac{1}{t} \;>\; n\,,$$

which is a better bound than $\frac{n+1}{n^{1/(n+1)}}$ when $n \geq 4$.

[*Ed.* It is easy to verify that $n\left(\frac{t}{n}\right)^n + \frac{1}{t} \geq n + \frac{1}{n}$, a lower bound which several solvers discovered.]

*Also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; PIERRE BORNSZTEIN, Maisons-Laffitte, France; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.*

**2843**. [2003 : 240; corrected 2003 : 463] *Proposed by Bektemirov Baurjan, student, Aktobe, Kazakstan.*

Suppose that $2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = 4+\frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}$ for positive real $x$, $y$, $z$. Prove that

$$(1-x)(1-y)(1-z) \;\leq\; \frac{1}{64}\,.$$

[*Editor's note*: The first printed version of this problem [2003 : 240] contained a misprint. Unfortunately, the problem intended by the proposer, which later appeared as a "correction" [2003 : 463], was still incorrect. Almost all the solvers who submitted solutions pointed this out by giving simple counterexamples, two of which are now given below.]

I *Solution by Michel Bataille, Rouen, France; Christopher J. Bradley, Bristol, UK; James T. Bruening, Southeast Missouri State University, Cape Girardeau, MO, USA; and Vedula N. Murty, Dover, PA, USA.*

As stated, the problem is wrong. Let $x=y=\frac{1}{4}$ and $z=\frac{3}{4}$. Then

$$2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \;=\; 4+\frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}\,,$$

while $(1-x)(1-y)(1-z)=\frac{9}{64}>\frac{1}{64}$.

II *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria; and Murray S. Klamkin, University of Alberta, Edmonton, AB.*

The problem is incorrect. Let $x=y=\varepsilon$, where $\varepsilon>0$ is arbitrarily small. Then the given condition is satisfied if we choose $z=4\varepsilon(1-\varepsilon)$. However, $(1-x)(1-y)(1-z)=(1-\varepsilon)^2(1-2\varepsilon)^2\to 1$ as $\varepsilon\to 0$.

*Also solved by ∗ARKADY ALT, San Jose, CA, USA; ∗MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; NEVEN JURIČ, Zagreb, Croatia; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; and LI ZHOU, Polk Community College, Winter Haven, FL, USA. (An asterisk ∗ indicates that the solver provided a counterexample to the first printed version only.) There were two incorrect solutions.*

*Alt remarked that if, in the corrected version, we add the condition that $x$, $y$, and $z$ are the sides of a triangle, then the conclusion is true.*

**2844**. [2003 : 241] *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that the sequence $\{x_n\}$ satisfies $\sum\limits_{k=1}^{n} \frac{1}{k} - \ln(n + x_n) = \gamma$, where $\gamma$ is Euler's constant.

(a) Prove that $\{x_n\}$ is convergent and that $\lim\limits_{n\to\infty} x_n = \frac{1}{2}$.

(b) Determine an asymptotic approximation for the general term $x_n$, with an error that is $O\left(\frac{1}{n^2}\right)$.

*Solution by Roger Zarnowski, Angelo State University, San Angelo, TX, USA.*

For each $n = 1, 2, \ldots$, let

$$\epsilon_n = \sum_{k=1}^{n} \frac{1}{k} - \ln n - \gamma .$$

The given condition that defines the sequence $\{x_n\}$ may be rewritten as $\ln(1 + x_n/n) = \epsilon_n$. Solving for $n$, we get

$$x_n = n(e^{\epsilon_n} - 1) .$$

For part (a), we use the following inequalities from [1]:

$$\frac{1}{2(n+1)} < \epsilon_n < \frac{1}{2n} .$$

These imply that

$$n\left(e^{\frac{1}{2(n+1)}} - 1\right) < x_n < n\left(e^{\frac{1}{2n}} - 1\right) .$$

Expanding the exponentials, we obtain

$$n\left(\frac{1}{2(n+1)} + O\left(\frac{1}{(n+1)^2}\right)\right) < x_n < n\left(\frac{1}{2n} + O\left(\frac{1}{n^2}\right)\right) .$$

Thus, $x_n$ is sandwiched between expressions which approach $\frac{1}{2}$ as $n \to \infty$. Therefore, the sequence $\{x_n\}$ is convergent, and $\lim\limits_{n\to\infty} x_n = \frac{1}{2}$.

For part (b), we use the well-known asymptotic expansion

$$\epsilon_n = \frac{1}{2n} - \frac{1}{12n^2} + O\left(\frac{1}{n^4}\right) .$$

See, for example, [2] (*Ed.* or [3] or [4]). We have

$$\begin{aligned} x_n = n(e^{\epsilon_n} - 1) &= n\left(\epsilon_n + \tfrac{1}{2}\epsilon_n^2 + O\left(\epsilon_n^3\right)\right) \\ &= n\left(\frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{2}\left(\frac{1}{2n}\right)^2 + O\left(\frac{1}{n^3}\right)\right) \\ &= \frac{1}{2} + \frac{1}{24n} + O\left(\frac{1}{n^2}\right) . \end{aligned}$$

[*Editor*: The result in part (a) is an immediate consequence of this asymptotic approximation for $x_n$. Therefore the proof of (a) given above is redundant. However, that proof has the advantage that it avoids using an asymptotic expansion for $\epsilon_n$. The inequalities from [1] are, by comparsion, quite simple, and their proof in [1] is elementary.]

**References**

[1] R.M. Young, Euler's Constant, *Mathematical Gazette*, Vol. 75, No. 472 (June, 1991), pp. 187–190.

[2] Julian Havil, *Gamma*, Princeton Univ. Press, Princeton, NJ, 2003, p. 89.

[3] Ronald L. Graham, Donald E. Knuth and Oren Patashnik, *Concrete Mathematics*, Addison-Wesley, Boston, MA, formula (9.28), 1st ed. (1989), p. 438, or 2nd ed. (1994), p. 452.

[4] Vito Lampret, The Euler-Maclauin and Taylor Formulas: Twin, Elementary Derivations, *Mathematics Magazine*, vol. 74, No. 2 (April, 2001), pp. 109–122.

*Also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA* (*part* (a) *only*)*; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; MIKE SPIVEY, Samford University, Birmingam, AL, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. One solution was spoiled by computational errors.*

*Loeffler notes that* $x_n = e^{\psi(n+1)} - n$*, where* $\psi(x)$ *is the digamma function, defined by* $\psi(x) = \frac{d}{dx}\Gamma(x)$*. He then bases his proof on the asymptotic expansion*

$$\psi(x) \;=\; \ln x - \frac{1}{2x} - \frac{1}{12x^2} + O\left(\frac{1}{x^4}\right),$$

*for which he gives the reference* `http://functions.wolfram.com`.

*Janous conjectures that the sequence* $\{x_n\}$ *is strictly decreasing.* (*Can anyone supply a proof of this?*) *He heartily recommends the book* [*2*].

*Reference* [*4*] *was given by Bataille.*

**2845**. [2003 : 241] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let $Q$ be a square of side length 1, and let $S$ be a set consisting of a finite number of squares such that the sum of their areas is $\frac{1}{2}$.

Prove that the set $S$ can be packed inside the square $Q$.

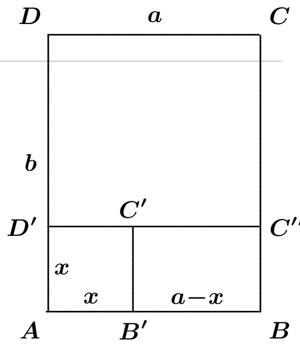
*Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

This result can be generalized as follows:

**Theorem**. *Given a finite set* $S$ *of square tiles and any rectangle* $R$*, then* $R$ *can be packed with tiles from* $S$ *up to at least half the area of* $R$*, so long as there are sufficient tiles in* $S$ *small enough to fit into* $R$*.*

*Proof.* We shall use induction on the number $n$ of tiles in $S$ that can fit into $R$.

For $n = 1$, the theorem is true.




Assume that the theorem is true for $n = 1, 2, \ldots, k-1$. For case $k$, assume that $S$ has $k$ tiles that can fit into $R$. Let the vertices of $R$ be $A$, $B$, $C$, $D$, and let $AB = a \leq b = AD$. Ignoring the tiles in $S$ that will not fit, pick the largest tile and place it into $R$ at vertex $A$. Call this $AB'C'D'$ where $D'$ lies on $AD$, and let $AD' = x$. Now, consider the rectangle $R' = BC''C'B'$. It is large enough for any of the remaining $k-1$ tiles having size less than or equal to $x$. Thus, by the inductive assumption, it can be filled with tiles from $S$ up to $\frac{1}{2}x(a-x)$ units of area. Again, because of the inductive assumption, the rectangle $D'DCC''$ can be tiled up to $\frac{1}{2}a(b-x)$ units of area, as long as there are sufficient remaining tiles. Therefore, as long as there are enough tiles of size less than or equal to $a$ in $S$, the number of tiles used so far is $x^2$ plus half the remaining area of $ABCD$ not used up by $AB'C'D'$. This sum exceeds half of the area of $ABCD$.

Case $k$ of the theorem is now proved, so that the theorem is true by induction.

The proposed problem follows as a corollary.

*Also solved by NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; and the proposer.*

*Guersenzvaig also proved the more general result.*

**2846**. [2003 : 241] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

A regular simplex $S_n = A_1A_2A_3\ldots A_{n+1}$ is inscribed in the unit sphere $\Sigma$ in $\mathbb{E}^n$. Let $O$ be the origin in $\mathbb{E}^n$, $M \in \Sigma$, $u_k = \overrightarrow{OA_k}$ and $v = \overrightarrow{OM}$.

Find the maximum value of $\sum\limits_{k=1}^{n+1} |u_k \cdot v|$.

*Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

By symmetry, $\sum\limits_{k=1}^{n+1} u_k = 0$ and $u_i \cdot u_j = u_k \cdot u_l$ whenever $i \neq j$ and $k \neq l$. Thus,

$$0 \;=\; u_1 \cdot \sum_{k=1}^{n+1} u_k \;=\; ||u_1||^2 + \sum_{k=2}^{n+1} (u_1 \cdot u_k) \;=\; 1 + nu_1 \cdot u_2\,.$$

Hence, $u_i \cdot u_j = -\dfrac{1}{n}$ if $i \neq j$. Without loss of generality, we may assume that, for some $m \in \{1, \ldots, n\}$, we have

$$\begin{aligned}\sum_{k=1}^{n+1} |u_k \cdot v| &= \left(\sum_{k=1}^{m} u_k - \sum_{k=m+1}^{n+1} u_k\right) \cdot v = \left(2\sum_{k=1}^{m} u_k\right) \cdot v \\ &\le 2\left\|\sum_{k=1}^{m} u_k\right\|,\end{aligned}$$

with equality if and only if $v$ is in the direction of $\sum\limits_{k=1}^{m} u_k$. Now

$$\begin{aligned}\left\|\sum_{k=1}^{m} u_k\right\|^2 &= \left(\sum_{k=1}^{m} u_k\right) \cdot \left(\sum_{k=1}^{m} u_k\right) = m + 2\binom{m}{2}\left(-\frac{1}{n}\right) \\ &= \frac{m(n-m+1)}{n},\end{aligned}$$

which achieves a maximum at $m = \left\lfloor \frac{n+1}{2} \right\rfloor$. It follows that the desired maximum value of $\sum\limits_{k=1}^{n+1} |u_k \cdot v|$ is

$$2\sqrt{\frac{\lfloor\frac{n+1}{2}\rfloor\left(n - \lfloor\frac{n+1}{2}\rfloor + 1\right)}{n}} = \begin{cases} \dfrac{n+1}{\sqrt{n}} & \text{if } n \text{ is odd,} \\ \sqrt{n+2} & \text{if } n \text{ is even.} \end{cases}$$

*Also solved by MICHEL BATAILLE, Rouen, France; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; and the proposer.*

**2847**. [2003 : 242] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

The *inscircle* inscribed in a tetrahedron is a circle of maximum radius inscribed in the tetrahedron, considering every possible orientation in $\mathbb{E}^3$.

Find the radius of the inscircle of a regular tetrahedron.

*Comment by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

The problem of determining the inscircle was solved around the end of the 19th century in an applied (technical) context, as far as I can remember. Maybe a reader can supply a reference. The desired radius equals

$$\frac{e}{2\sqrt{3}},$$

where $e$ denotes the edge length of the regular tetrahedron. [Note that if Janous has correctly remembered the radius, it follows that the inscircle is inscribed in a face of the tetrahedron.]

*Question for further research*: What is the radius of the largest $k$–dimensional ball that can be inscribed in a regular simplex of $n$–dimensional space, $n \geq 4$ and $1 \leq k \leq n-1$?

**2848**. [2003 : 242] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, AB and K.R.S. Sastry, Bangalore, India.*

Suppose that $A$, $B$, and $C$ are the angles of $\triangle ABC$ and that $\omega$ is its Crelle-Brocard angle. Prove that $A+\omega = \frac{\pi}{2}$ if and only if $\tan\ C$, $\tan\ A$, $\tan\ B$ are in geometric progression in that order.

*Solution by David Loeffler, student, Trinity College, Cambridge, UK.*

The Crelle-Brocard angle satisfies $\cot\omega = \cot A + \cot B + \cot C$. [Some explanation is given at the end.] Since the given criterion $A{+}\omega = \pi/2$ is equivalent to $\cot\omega = \tan A$, we have

$$\tan A \;=\; \cot A + \cot B + \cot C\,. \tag{1}$$

But we know that, for angles $A$, $B$, $C$ satisfying $A+B+C=\pi$, we have

$$\frac{\tan A\ \tan B\ \tan C}{\tan A + \tan B + \tan C} \;=\; 1\,.$$

Let us therefore multiply the right side of equation (1) by this quantity. Our given equation $A+\omega = \pi/2$ holds if and only if

$$\tan A \;=\; \frac{\tan A\ \tan B + \tan B\ \tan C + \tan C\ \tan A}{\tan A + \tan B + \tan C}\,,$$

which happens if and only if

$$\begin{aligned}\tan A\ \tan B + \tan B\ \tan C + \tan C\ \tan A&\\ &= \tan A(\tan A + \tan B + \tan C)\,,\end{aligned}$$

and this occurs if and only if

$$\tan B\ \tan C \;=\; \tan^2 A\,.$$

This last equation is equivalent to $\tan C$, $\tan A$, $\tan B$ being in geometric progression, as required.

*Also solved by MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; NEVEN JURIČ, Zagreb, Croatia; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposers.*

*The* Crelle-Brocard points *(often called simply the* Brocard points*) are the pair of points denoted by $\Omega$ and $\Omega'$ that lie inside a triangle $ABC$ and satisfy*

$$\angle\Omega AB \;=\; \angle\Omega BC \;=\; \angle\Omega CA\,, \quad \text{and} \quad \angle\Omega' BA \;=\; \angle\Omega' AC \;=\; \angle\Omega' CB\,.$$

*One easily shows that these points are unique, that both triples of angles have the same measure $\omega$, and that $\omega$ satisfies $\cot\omega = \cot A + \cot B + \cot C$. A couple of solvers supplied proofs, four appealed to references (listed below), and the others accepted the properties as common knowledge. We thank Bataille, Janous, Jurić, and Zhou for supplying references.*

*Our featured solution provides no hint at whether any triangles exist that satisfy $A + \omega = \pi/2$. Jurić included an explicit example with his solution:*

$$a \;=\; 2\,, \quad b \;=\; \sqrt{2}\,, \quad c \;=\; \sqrt{2+\sqrt{8}}\,.$$

*One computes here that $\tan A = \sqrt{1+\sqrt{8}}$, $\tan B = (\sqrt{2}-1)\sqrt{1+\sqrt{8}}$, and $\tan C = (\sqrt{2}+1)\sqrt{1+\sqrt{8}}$. To answer the question of existence in general, substitute*

$$\tan C \;=\; -\tan(A+B) \;=\; \frac{\tan A + \tan B}{\tan A\,\tan B - 1}$$

*(which is a simple consequence of $A+B+C=\pi$) into our criterion $\tan^2 A = \tan B\,\tan C$ to get a quadratic equation for $\tan B$ in terms of $\tan A$. One finds a single solution when $A = B = C = \pi/3$, a pair of solutions when $\pi/3 < A < \pi/2$, and no solution for other values of $A$. In other words, if we take $B \le C$, then there is a unique triangle $ABC$ that satisfies the given criterion when the measure of $A$ lies in the interval $\pi/3 \le A < \pi/2$, and no triangle otherwise.*

## References


[1] R.A. Johnson, *Advanced Euclidean Geometry*, Dover, page 266, Thm. 434.

[2] D.S. Mitrinović et al., *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1989, pp. 59 (item (74)) and 360.

[3] Dominik Palman, article in *Matematika* (Zagreb), (July/September, 1984) p. 39.

[4] V.V. Prasolov, *Essays on Numbers and Figures*, AMS, 1997, Chapter 14.