SKOLIAD No. 70

Shawn Godin

Solutions may be sent to Shawn Godin, 2191 Saturn Cres., Orleans, ON, K4A 3T6, or emailed to

mayhem-editors@cms.math.ca.

We are especially looking for solutions from high school students. Please include your name, school or other affiliation (if applicable), city, province or state, and country on any correspondence. High school students should also include their grade in school. Please send your solutions to the problems in this edition by *1 November 2003*. A copy of **MATHEMATICAL MAYHEM Vol. 3** will be presented to the pre-university reader(s) who send in the best solutions before the deadline. The decision of the editor is final.

We will only print solutions to problems marked with an asterisk ($*$) if we receive them from students in grade 10 or under (or equivalent), or if we receive a unique solution or a generalization.

The first item in this issue comes from the 2002 Christopher Newport University Regional High School Mathematics Contest. My thanks go to Ron Persky of Christopher Newport University in Newport News, Virginia for forwarding the material, and to Joanne Longworth and Robert Woodrow of the University of Calgary for sending it to me.

The Third Annual CNU Regional High School Mathematics Contest
TEAM ROUND
Saturday, November 16, 2002

1. ($*$) Randy and Hannah are eating at a restaurant. The items ordered by Randy cost twice as much as the items ordered by Hannah. Randy leaves a tip of 15% of the price of what he has ordered. Hannah leaves a tip of 20% of her items. The total, including tips, paid by the pair is \$70.00. How much was the cost of the items Hannah ordered?

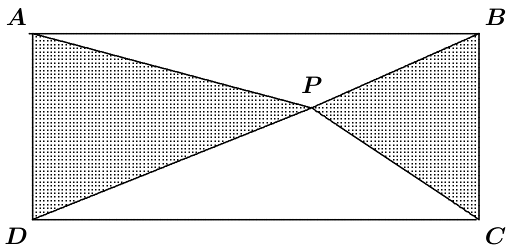
2. ($*$) Solve $x^2 - |x| - 1 = 0$.

3. ($*$) Let $\{a_n\}$ be an arithmetic sequence. If $a_p = q$ and $a_q = p$, find a_{p+q}.

4. ($*$) A five-digit number is called a *mountain number* if the first three digits are increasing and the last three are decreasing. For example, 34541 is a mountain number, but 34534 is not. How many mountain numbers are greater than 70000?

5. ($*$) Each day, Hai and Wai separately play a video game and compare scores. Hai's score on Tuesday was 10% less than his score on Monday, while Wai's score on Tuesday was 20 points higher than on Monday. However, on Wednesday, Hai's score was 10 points higher than on Tuesday, while Wai's score on Wednesday was 20% less than his score on Tuesday. Strangely, Hai's score plus Wai's score turned out to be the same on all three days. What were their scores on Wednesday?

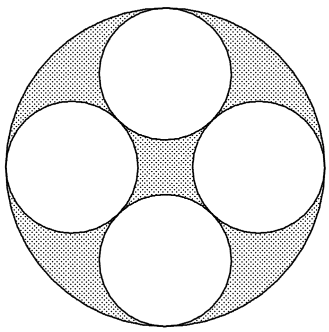
6. ($*$) A point P is given in the interior of rectangle $ABCD$ with $AB = CD = 24$ and $AD = BC = 5$. What is the total area of the two triangles $\triangle PAD$ and $\triangle PBC$ (shaded in the figure)?



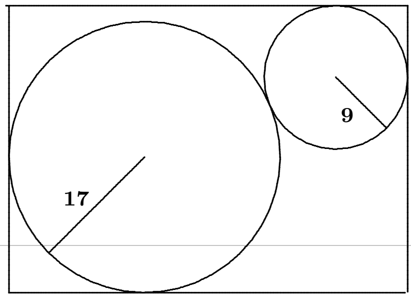
7. ($*$) Samantha bought a stock for \$1000 whose price then doubled every year for the next n years. In the year after that, the stock price fell by 99%. Nevertheless, the stock was still worth more than \$1000. What is the smallest whole number of years n for which this is possible?

8. ($*$) In $\triangle ABC$, $\cos(A-B)+\sin(A+B) = 2$. Determine the shape of the triangle.

9. ($*$) Four small circles of radius 1 are tangent to each other and to a larger circle containing them, as shown in the figure. What is the area of the region inside the larger circle, but outside all the smaller circles?



10. ($*$) Two circles of radii 9 and 17 centimetres are enclosed in a rectangle with one side of length 50 centimetres. The two circles touch each other, and each touches two adjacent sides of the rectangle, as indicated. Find the area of the rectangle.



11. (∗) Find three *different* prime numbers a, b, and c, so that their sum $a + b + c$ and their product abc both end in the digit 7.

12. (∗) Karen ran a 42-kilometre marathon in 3 hours, 49 minutes. She did this by running for 10 minutes, walking for 2 minutes, then running for 10 minutes, walking for 2 minutes, and so on until she crossed the finish line. She runs twice as fast as she walks. What is her average speed, in kilometres per hour, while running?

Next, we present solutions to the 2002 British Columbia Colleges Junior High School Mathematics Contest, Final Round, Part B that appeared in [2002 : 448–449].

1. $PQRS$ is a rectangle inscribed in a circle. The circle has centre O and radius r. The angle POQ is 120°.

(a) Draw a diagram.

(b) Find the degree measure of angle POS.

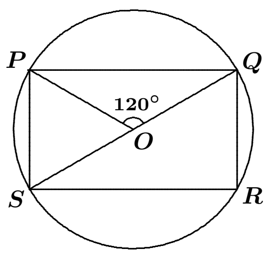
(c) Find the length of side PS in terms of r.

(d) Find the length of side PQ in terms of r.

(e) Find the ratio of the circumference of the circle to the perimeter of the rectangle.

Solution by Jefferson Lin, grade 10 *student, Stuyvesant High School, New York, NY, USA.*

(a) The diagram:



(b) Since $\angle POQ$ and $\angle POS$ are supplementary,

$$\angle POS = 180^\circ - 120^\circ = 60^\circ .$$

(c) Since $OS = OP = r$ and $\angle POS = 60^\circ$, we see that

$$\angle OPS = \angle OSP = \frac{180^\circ - 60^\circ}{2} = 60^\circ .$$

Thus, $\triangle OPS$ is equilateral, and $PS = r$.

(d) If we drop a perpendicular from O to meet PQ at X, we see that $\triangle OPX$ and $\triangle OQX$ are congruent 30°–60°–90° triangles with hypotenuse r. Thus, $PX = XQ = \frac{r\sqrt{3}}{2}$, and $PQ = r\sqrt{3}$.

(e) We have

$$\begin{aligned} C : P &= 2\pi r : 2r + 2r\sqrt{3} \\ &= 2\pi r : 2r(1+\sqrt{3}) \\ &= \pi : 1 + \sqrt{3} \end{aligned}$$

Also solved by Robert Bilinski, Outremont, QC; and Zack Wolske, grade 11 *student, Anderson CVI, Whitby, ON.*

2. Write 58 as the sum of four positive integers so that if 1 is added to the first number, 2 is subtracted from the second, 3 multiplied by the third, and 4 divided into the fourth, the results are all equal. Find the four numbers.

Solution by Robert Bilinski, Outremont, QC.

Let w, x, y, and z be the four numbers. Thus, $w + x + y + z = 58$. If we call n the result of the given operations, then

$$w + 1 = x - 2 = 3y = \frac{z}{4} = n .$$

Solving for w, x, y, and z in terms of n and plugging into the first equation, we get $n - 1 + n + 2 + \frac{n}{3} + 4n = 58$. Hence, $n = 9$, and the four numbers are $w = 8$, $x = 11$, $y = 3$, and $z = 36$.

Also solved by Jefferson Lin, grade 10 *student, Stuyvesant High School, New York, NY, USA; and Zack Wolske, grade* 11 *student, Anderson CVI, Whitby, ON.*

3. Two circles have radii m and n, where $m > n$. The distance between their centres is k.

(a) State the condition(s) relating m, n, and k that guarantee that the circles intersect in exactly one point.

(b) State the condition(s) relating m, n, and k that guarantee that the circles intersect in exactly two points.

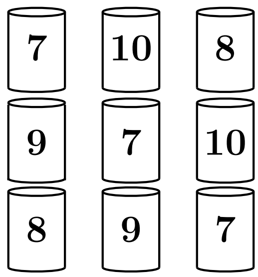
Identical solutions by Robert Bilinski, Outremont, QC; and Zack Wolske, grade 11 *student, Anderson CVI, Whitby, ON.*

(a) If the two circles are tangent externally, then $k = m + n$; if internally, then $k = m - n$.

(b) The two circles have two intersections if $m - n < k < m + n$.

One incorrect solution was received.

4. At a carnival game, you see nine paint cans stacked and numbered as shown. You get three throws, and you must knock down one, and only one, can per throw. Further, a can may only be knocked down after the one(s) directly above it have been knocked down on a previous throw. Your first throw scores the number on that can, the second throw scores twice the number on that can, and the third throw scores triple the number on that can. To win a prize you must score exactly 50 points, no more, no less. Determine all of the possible combinations of throws that win a prize.



Solution by Zack Wolske, grade 11 *student, Anderson CVI, Whitby, ON.*

The first throw can score either 7, 10, or 8 points. The possible scores are shown below.

First throw	7	7	7	7	7	7	7	7	7
Second throw	18	18	18	20	20	20	16	16	16
Third throw	24	30	24	27	21	24	27	30	30
Total	49	55	49	54	48	51	50	53	53

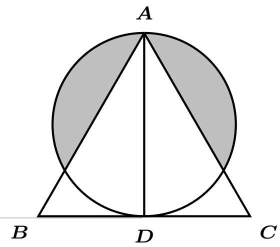
First throw	10	10	10	10	10	10	10	10	10
Second throw	14	14	14	14	14	14	16	16	16
Third throw	27	21	24	21	27	24	21	21	30
Total	51	45	48	45	51	48	47	47	56

First throw	8	8	8	8	8	8	8	8	8
Second throw	14	14	14	20	20	20	20	20	20
Third row	27	30	30	21	21	30	21	30	21
Total	49	52	52	49	49	58	49	58	49

The only way to win a prize is to hit 7 first, then 8, and finally 9.

Also solved by Robert Bilinski, Outremont, QC; and Jefferson Lin, grade 10 *student, Stuyvesant High School, New York, NY, USA.*

5. The diameter AD of a circle is perpendicular to side BC of the equilateral triangle ABC with D lying on BC. If the length of BC is 4, find the area of the shaded part of the circle that is outside the triangle.



Solution by Zack Wolske, grade 11 *student, Anderson CVI, Whitby, ON.*

Triangle ABD is right-angled with $AB = 4$ and $BD = 2$. Using the Pythagorean Theorem, $AD = 2\sqrt{3}$. Thus, $OD = OA = \sqrt{3}$, where O is the centre of the circle. If we let E and F be the points of intersection of the circle with AB and AC, respectively, we get $OF = OE = OD = OA = \sqrt{3}$.

Drop perpendiculars from O to AB and AC, and name the feet of the perpendiculars G and H, respectively. Since $\angle BAC = 60^\circ$, we get

$$\begin{aligned} \angle GAO &= \angle HAO = \angle GEO = \angle HFO = 30^\circ\,, \\ \angle AOE &= \angle AOF = 120^\circ\,, \\ \angle EOD &= \angle FOD = 60^\circ\,. \end{aligned}$$

The area of sector EOF will then be $\frac{1}{3}\pi\left(\sqrt{3}\right)^2 = \pi$.

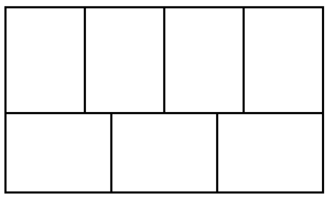
Since $\triangle AOG$, $\triangle EOG$, $\triangle AOH$, and $\triangle FOH$ are 30°–60°–90° triangles, $AG = AH = \frac{\sqrt{3}}{2} \cdot \sqrt{3} = \frac{3}{2}$, and $OG = OH = \frac{\sqrt{3}}{2}$. Thus, $AE = AF = 3$, and the areas of $\triangle OAE$ and $\triangle OAF$ are each equal to $\frac{1}{2} \cdot 3 \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4}$. The shaded area is the area of the circle minus the areas of sector EOF, $\triangle OAE$, and $\triangle OAF$. That is,

$$\begin{aligned} \text{AREA} &= \pi\left(\sqrt{3}\right)^2 - \left(\pi + \frac{3\sqrt{3}}{4} + \frac{3\sqrt{3}}{4}\right) \\ &= 3\pi - \pi - \frac{3\sqrt{3}}{2} = 2\pi - \frac{3\sqrt{3}}{2}\,. \end{aligned}$$

Also solved by Jefferson Lin, grade 10 *student, Stuyvesant High School, New York, NY, USA. One incorrect solution was received.*

Now, we present solutions to the 2002 British Columbia Colleges Senior High School Mathematics Contest, Final Round, Part B that appeared in the [2002 : 449–450].

1. In the figure the seven rectangles are congruent and form a larger rectangle whose area is $336\ \text{m}^2$. What is the perimeter of the large rectangle?



Solution by Jefferson Lin, grade 10 *student, Stuyvesant High School, New York, NY, USA.*

Let x and y be the length and width, respectively, of the small rectangles. From the diagram, we have

$$3x(x+y) = 336 \text{ m}^2 \qquad (1)$$

Also, $3x = 4y$, and therefore, $y = \frac{3x}{4}$. Substituting into (1) yields

$$\begin{aligned} 3x\left(x+\frac{3x}{4}\right) &= 336 \text{ m}^2 \\ 3x\left(\frac{7x}{4}\right) &= 336 \text{ m}^2 \\ \frac{7x^2}{4} &= 112 \text{ m}^2 \\ x &= \pm 8 \text{ m}\,. \end{aligned}$$

Then $y = \pm 6$ m. Of course, since these are lengths, they must be positive. Thus, the perimeter is $P = 5x + 6y = 76$ m.

Also solved by Robert Bilinski, Outremont, QC; and Zack Wolske, grade 11 *student, Anderson CVI, Whitby, ON.*

2. Two players, A and B, play a game in which each chooses, alternately, a positive integer between 1 and 6, inclusive. After each number is chosen the cumulative sum is computed. The player who chooses a number that makes the cumulative sum equal to 22 wins the game. For example,

A chooses 2	sum = 2	
B chooses 6	sum = 8	
A chooses 3	sum = 11	
B chooses 6	sum = 17	
A chooses 5	sum = 22	A wins.

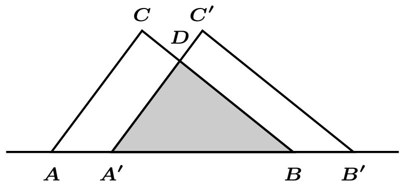
Suppose that A starts the game. Show that A has a winning strategy; that is, the player that starts the game can always win. What number must A choose on the first turn in order to have a winning strategy?

Solution by Zack Wolske, grade 11 *student, Anderson CVI, Whitby, ON.*

Player A must have a total that is any number from 16 to 21 inclusive before his last turn. To reach that sum, B must have a sum of 15 before choosing. To give B a sum of 15, A must have any number from 9 to 14 inclusive before choosing. To get that, B must be given a sum of 8. A must have any number from 2 to 7 inclusive before choosing. To get that, B needs to have been given 1 before choosing. Therefore, A's first choice should be 1, and each subsequent choice should be calculated by subtracting B's previous choice from 7.

Also solved by Jefferson Lin, grade 10 *student, Stuyvesant High School, New York, NY, USA.*

3. Triangle ABC has dimensions: $AB = 10$, $AC = 7$, and $BC = 8$. How far do you need to slide it along side AB so that the area of the overlapping region (the shaded triangle $A'BD$ in the diagram) is one-half the area of triangle ABC?



Solution by Jefferson Lin, grade 10 *student, Stuyvesant High School, New York, NY, USA.*

Since $\triangle A'DB$ is created by the intersection of the sliding triangles, it is similar to the original triangle [*Ed*: since two of the angles are from the original triangle, the third must also be the same]. In order for the ratio of the areas to be $\frac{1}{2}$, the ratio of the sides must be $1/\sqrt{2}$. Therefore,

$$\begin{aligned} \frac{A'B}{AB} &= \frac{1}{\sqrt{2}} \\ \frac{A'B}{10} &= \frac{1}{\sqrt{2}} \\ A'B &= 5\sqrt{2} \end{aligned}$$

Thus, the shift must be $10 - 5\sqrt{2}$.

Also solved by Robert Bilinski, Outremont, QC; and Zack Wolske, grade 11 *student, Anderson CVI, Whitby, ON.*

4. In the addition below each of the letters stands for a distinct decimal digit:

$$\begin{array}{r} PACIFIC \\ BALTIC \\ +\ \ ARCTIC \\ \hline CCCCCCC \end{array}$$

Find the decimal digit corresponding to each of the letters. Show all of your work.

Solution by Robert Bilinski, Outremont, QC.

The unit digit of $3C$ is C. Thus, $C = 0$ or $C = 5$. But $C = 0$ leads to all letters equal to 0, which is not allowed; so, $C = 5$, and we carry 1 to the second column. Now, in the second column, $3I + 1$ finishes with a 5, which means that $3I$ finishes with a 4. The only possibility is $I = 8$, and we need to carry 2 to the third column. From the third column, we gather that $2T + F$ ends with a 3, which implies that the couple (T, F) is one of the following: $(0,3)$, $(2,9)$, $(3,7)$, $(6,1)$, or $(7,9)$.

If $(T, F) = (7,9)$, then $2T + F = 23$, and we carry 2 to the fourth column. This implies that $L = 0$, and that we have to carry 1 to the fifth column. In turn, this means that $A + R = 9$, so that A and R are 3 and 6, though we cannot yet tell which is which. In either case, we carry 1 to the sixth column. Hence, $2A + B$ ends with 4. If $A = 3$, then $B = 8$. But

$B \neq 8$, since $I = 8$. Therefore, $A = 6$ and $B = 2$ (and also, $R = 3$). We must carry 1 to the last column, where we get $P = 4$. We have obtained the solution $(A, B, C, F, I, L, P, R, T) = (6, 2, 5, 9, 8, 0, 4, 3, 7)$, giving the following sum:

$$\begin{array}{r} 4658985 \\ 260785 \\ +\ 635785 \\ \hline 5555555 \end{array}$$

The question seems to imply that the solution is unique. If we look at the other cases, we see that we get no new solutions.

Also solved by Jefferson Lin, grade 10 student, Stuyvesant High School, New York, NY, USA; and Zack Wolske, grade 11 student, Anderson CVI, Whitby, ON.

5. For this problem we define a *decreasing number* as a positive integer with two or more digits, all of which are written in strictly decreasing order from left to right.

1. How many three-digit decreasing numbers are there?
2. What is the largest number of digits that a decreasing number can have?
3. How many decreasing numbers are there?

Solution by Geneviève Lalonde, Massey, ON.

Note that we can create a decreasing number by starting with the number 9876543210 and removing any digit(s). All decreasing numbers can be created this way.

(a) To create a 3-digit number, we need to remove 7 digits. We can do this in $\binom{10}{7}$ ways.

(b) A decreasing number can have at most 10 digits (every digit used once).

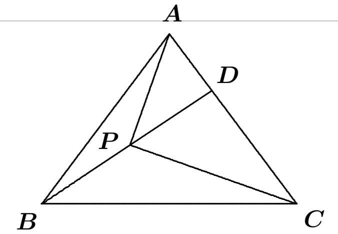
(c) There are decreasing numbers with anywhere from 2 to 10 digits. The total number of these numbers is

$$\binom{10}{2} + \binom{10}{3} + \binom{10}{4} + \cdots + \binom{10}{10} = 1013\,.$$

Also solved by Robert Bilinski, Outremont, QC; Jefferson Lin, grade 10 student, Stuyvesant High School, New York, NY, USA; and Zack Wolske, grade 11 student, Anderson CVI, Whitby, ON.

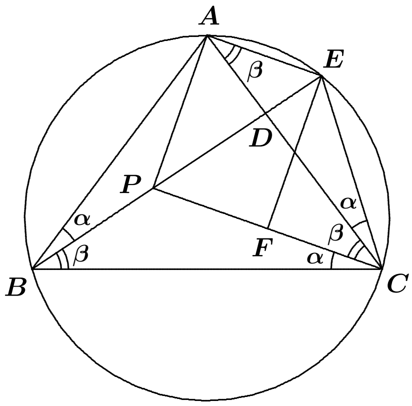
Finally, we have an alternate solution to question 4, part B of the Canadian Open Mathematics Challenge that appeared in [2002 : 243]. One solution was given already in [2003 : 11].

4. Triangle ABC is isosceles with $AB = AC = 5$ and $BC = 6$. Point D lies on AC, and P is the point on BD so that $\angle APC = 90^\circ$. If $\angle ABP = \angle BCP$, determine the ratio $AD : DC$.



Solution by Geoffrey A. Kandall, Hamden, CT, USA.

Let $\alpha = \angle ABP = \angle BCP$ and $\beta = \angle PBC = \angle ACP$. Extend BD to meet the circumcircle of $\triangle ABC$ at E. Draw AE and EC. Then $\angle ECA = \angle EBA = \alpha$, and $\angle EPC = \angle ECP = \alpha + \beta$. Therefore, $\triangle EPC$ is isosceles. Also, $\angle EAC = \angle EBC = \beta$, which implies that $AE \parallel PC$. Let F be the foot of the perpendicular from E onto PC. [*Ed*: Since $\triangle EPC$ is isosceles, the point F is also the mid-point of PC.] Thus, $APFE$ is a rectangle, which implies that $AE = PF = FC$. Then, by similar triangles, we have $AD : DC = AE : PC = 1 : 2$.



That brings us to the end of another issue of Skoliad. This issue's winner of a copy of **MATHEMATICAL MAYHEM VOLUME 7** is Zack Wolske. Congratulations, Zack! Please, continue sending in contests and solutions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to **Mathematical Mayhem, Cairine Wilson Secondary School, 975 Orleans Blvd., Gloucester, Ontario, Canada. K1C 2Z7**. The electronic address is

mayhem-editors@cms.math.ca

The Mayhem Editor is Shawn Godin (Cairine Wilson Secondary School). The Assistant Mayhem Editor is John Grant McLoughlin (University of New Brunswick). The other staff members are Paul Ottaway (Dalhousie University) and Larry Rice (University of Waterloo).

Mayhem Problems

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mayhem-editors@cms.math.ca

N'oubliez pas d'inclure à toute correspondance votre nom, votre année scolaire, le nom de votre école, ainsi que votre ville, province ou état et pays. Nous sommes surtout intéressés par les solutions d'étudiants du secondaire. Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le *premier novembre 2003*. Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier et Hidemitsu Saeki, de l'Université de Montréal, d'avoir traduit les problèmes.

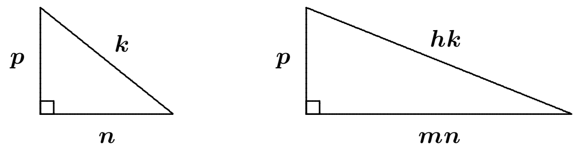
M94. *Proposé par J. Walter Lynch, Athens, GA, USA.*

On a douze balles qui, au premier coup d'oeil, semblent identiques. A la vérité, onze sont identiques et l'une d'entre elles diffère légèrement de poids d'avec les autres (sans qu'on sache en quel sens). Avec une balance à plateaux, identifier cette balle en trois pesées seulement.

You have twelve balls which are identical in appearance. Eleven of them are in fact identical, and the other one differs slightly in weight from each of these eleven. Using a balance scale, find the odd ball in only three weighings.

M95. *Proposé par l'équipe de Mayhem.*

Dans la figure ci-dessous, sachant que h, k, m, n, et p sont des entiers avec $h \neq 1$, déterminer la valeur de h.



In the figure above, given that h, k, m, n, and p are integers with $h \neq 1$, determine the value of h.

M96. *Proposé par l'équipe de Mayhem*

Déterminer le plus grand reste qu'il soit possible d'engendrer en divisant un nombre de trois chiffres par la somme de ceux-ci.

. . . .

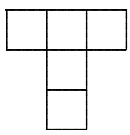
Determine the largest possible remainder that is attainable by dividing a three-digit number by the sum of its digits.

M97. *Proposé par l'équipe de Mayhem*

On sait qu'un échiquier normal de 8×8 comporte 64 cases. De plus, on dispose de plaques en forme de T de la dimension de cinq cases, comme dans la figure ci-dessous. On va en placer sur l'échiquier, en les faisant coïncider exactement avec le dessin des cases de ce dernier.

(a) Quel est le nombre maximal de plaques qu'on peut placer sur l'échiquier sans qu'il y ait recouvrement de plaques ?

(b) Si l'on tolère le recouvrement, quel est le nombre minimal de plaques requises pour couvrir l'échiquier ?



A standard 8×8 checkerboard consists of 64 unit squares. A T–shaped tile consists of five unit squares, as shown above. The tile must be placed on the checkerboard to cover exactly five unit squares on the board.

(a) What is the maximum number of non-overlapping tiles that can be placed on the board in this manner ?

(b) Assuming that overlapping is permitted, what is the minimum number of tiles required to cover the board ?

M98. *Proposé par l'équipe de Mayhem.*

On dit que N est un nombre *automorphe* si son carré se termine par une réplique des chiffres qui forment N. Par exemple, 6 est automorphe puisque 6^2 finit avec un 6.

(a) Trouver tous les nombres de deux chiffres en base 10 qui sont automorphes.

(b) Trouver tous les nombres de deux chiffres en base 6 qui sont automorphes.

. . . .

We say that N is an *automorphic* number if the value of N^2 ends with the string of digits forming N. For example, 6 is automorphic since 6^2 ends in 6.

(a) Find all two digit automorphic numbers in base 10.

(b) Find all two digit automorphic numbers in base 6.

M99. *Proposé par l'équipe de Mayhem.*

Montrer que pour tous les entiers positifs n,

$$1 \cdot \binom{n}{1} + 2 \cdot \binom{n}{2} + \ldots + n \cdot \binom{n}{n} \;=\; n \cdot 2^{n-1}\,.$$

. . . .

Prove that for all positive integers n,

$$1 \cdot \binom{n}{1} + 2 \cdot \binom{n}{2} + \ldots + n \cdot \binom{n}{n} \;=\; n \cdot 2^{n-1}\,.$$

M100. *Proposé par l'équipe de Mayhem.*

M. et Mme. Dupont partagent une invitation avec trois autres couples mariés qui ne se connaissent pas nécessairement. Il y a donc des présentations et des échanges de poignées de mains. Une fois les présentations terminées, Mme. Dupont demande aux sept autres personnes combien de poignées de mains elles ont donné. Chose surprenante, toutes donnent une réponse différente. Combien de poignées de mains M. Dupont a-t-il donné ?

. . . .

Mr. and Mrs. Smith are at a party with three other married couples. Since some of the guests are not acquainted with one another, various handshakes take place. No one shakes hands with his or her spouse, and of course, no one shakes hands with himself or herself. After all of the introductions have been made, Mrs. Smith asks the other seven people how many hands they shook. Surprisingly, they all give different answers. How many hands did Mr. Smith shake?

Mayhem Solutions

M44. Proposed by K.R.S. Sastry, Bangalore, India.

$ABCD$ is a Heron parallelogram (in which the sides, the diagonals and the area are natural numbers). The diagonals AC and BD have measures 85 and 41, respectively. Determine the measures of the sides AB and BC.

Solution by Kevin Chung, OAC student, Earl Haig S.S., North York, ON.

Let AB and BC be x and y, with x, $y \in \mathbb{N}$. By the Law of Cosines

$$\begin{aligned} \frac{x^2+y^2-BD^2}{2xy} &= \cos(\angle BAD) \\ &= -\cos(\angle ADC) \;=\; -\left(\frac{x^2+y^2-AC^2}{2xy}\right), \end{aligned}$$

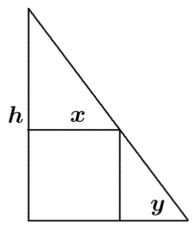
and so $x^2+y^2 = \dfrac{AC^2+BD^2}{2} = 4453$. Since $n^2 \equiv 0, 1, 4, 5, 6, 9 \pmod{10}$ for any $n \in \mathbb{Z}$, one of x^2, y^2 must be congruent to 4 (mod 10) and the other congruent to 9 (mod 10). A check of the possibilities shows that the equation $x^2+y^2 = 4453$ is satisfied only when $\{x, y\} = \{63, 22\}$ or $\{58, 33\}$. If $\{x, y\} = \{63, 22\}$, then $\cos(\angle BAD) = 1$. Thus, we reject $\{63, 22\}$. On the other hand, if $\{x, y\} = \{58, 33\}$, then $\cos(\angle BAD) = \dfrac{21}{29}$ and $\sin(\angle BAD) = \dfrac{20}{29}$, making the area $\dfrac{20}{29}\cdot 58\cdot 33 \in \mathbb{N}$. Hence, the only solutions are $(AB, BC) = (58, 33)$ or $(33, 58)$.

M45. *Proposed by a Canadian Customs officer, Pearson International Airport, Toronto, ON.*

A 10 metre long ladder is leaning upright against a wall, touching the edge of a cubic box. The box itself is put against the wall and measures 2 cubic metres. What is the height of the top of the ladder from the ground?

Solution by Kevin Chung, OAC student, Earl Haig S.S., North York, ON.

Let $x = \sqrt[3]{2}$, the length of the box. Let h and y be as shown in the diagram.



Then, by similar triangles, $\frac{h-x}{x} = \frac{x}{y}$. Then y, x, $h-x$ form a geometric progression. Let $x = yr$ and $h - x = xr$. Then, since $h^2 + (x+y)^2 = 100$, we have

$$\begin{aligned} (x+xr)^2 + \left(x + \frac{x}{r}\right)^2 &= 100 \\ r^2 + 2r + 1 + 1 + \frac{2}{r} + \frac{1}{r^2} &= \frac{100}{x^2} \\ \left(r + 1 + \frac{1}{r}\right)^2 &= \frac{100}{x^2} + 1\,. \end{aligned}$$

Thus, since $r > 0$, we have $r + 1 + \frac{1}{r} = \sqrt{\frac{100}{x^2} + 1}$; that is,

$$r^2 + r\left(1 - \sqrt{\frac{100}{x^2} + 1}\right) + 1 \;=\; 0\,.$$

From here and from $h = x(r+1)$, we find the height to be

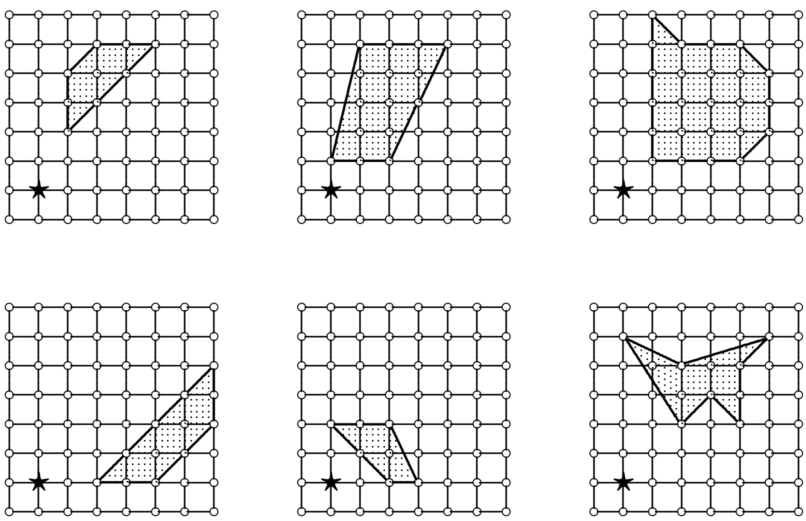
$$h \;=\; \frac{\sqrt{100+x^2} + x \pm \sqrt{(\sqrt{100+x^2}+x)(\sqrt{100+x^2}-3x)}}{2}\,;$$

that is, about 9.895 m or 1.444 m.

Twelve incorrect solutions were received.

M46. *Proposed by Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.*

The lattice polygons in the upper row of the figure are characterized by a common property, the lower ones by the reverse. Which property is it?



Solution by the proposer.

Let the lattice point denoted by the penta-star be the origin. Then all *vertices* of the upper polygons are visible points from the origin; that is, their coordinates are coprime. The vertices of the lower polygons are all invisible.

M47. *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

(a) Find all monic quadratic polynomials x^2+ax+b with integer roots, where 1, a, b is an arithmetic progression.

(b) Prove that there are no real numbers a, b, c such that 1, a, b, c is an arithmetic progression and $x^3 + ax^2 + bx + c$ has all real roots.

(a) *Solution by Kevin Chung, OAC student, Earl Haig S.S., North York, ON.*

Since 1, a, b is an arithmetic progression, we have $1 + b = 2a$, and $x^2 + ax + b = x^2 + ax + 2a - 1$. If we let $\alpha \geq \beta$ be the integer roots of this polynomial, then $\alpha+\beta = -a$ and $\alpha\beta = 2a-1$. Then $2\left[-(\alpha + \beta)\right]-1 = \alpha\beta$, from which $3 = \alpha\beta + 2\alpha + 2\beta + 4 = (\alpha + 2)(\beta + 2)$. Therefore, either

(1) $\alpha + 2 = 3$ and $\beta + 2 = 1$, or

(2) $\alpha + 2 = -1$ and $\beta + 2 = -3$.

In the first case, $(\alpha, \beta) = (1, -1)$, and hence $a = 0$; in the second case, $(\alpha, \beta) = (-3, -5)$, and hence $a = 8$. Thus, the monic quadratic polynomials are $x^2 - 1$ and $x^2 + 8x + 15$.

(b) *Solution by the proposer.*

If the roots of $x^3 + ax^2 + bx + c$ are r, s, and t, then

$$a = -r - s - t\,, \qquad b = rs + rt + st\,, \qquad c = -rst\,.$$

Hence, if 1, a, b, c is an arithmetic progression, then

$$-2r-2s-2t = rs+rt+st+1 \quad \text{and} \quad 2rs+2rt+2st = -rst-r-s-t\,.$$

Solving the first equation for t, we get

$$t = -\frac{2r + 2s + rs + 1}{r + s + 2}\,.$$

Plugging this into the second equation gives

$$4rs + 3r^2 + 3s^2 + r^2s^2 + 2r^2s + 2rs^2 + 2r + 2s + 1 = 0\,,$$

which we can write as

$$2(r^2 + s^2) + (r + 1)^2(s + 1)^2 = 0\,.$$

This obviously has no real solution.

Part (b) *was also solved by Kevin Chung, OAC student, Earl Haig S.S., North York, ON.*

M48. *Proposed by J. Walter Lynch, Athens, GA, USA.*

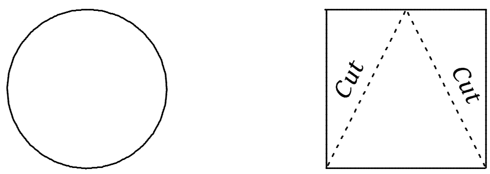
Tell how to make a single stopper that will stop a square hole, a round hole, and a triangular hole, and will pass through each.

If one wanted to give a hint, he might point out that a pyramid will stop a square hole and a triangular hole, and a cylinder will stop a square hole and a round hole.

Solution by the proposer.

Start with a right circular cylinder with equal height and diameter. This will already stop the round hole and the square hole.

Now set the cylinder on a table with one of the circular surfaces on top and cut a wedge off of each of two opposite sides. Do this by placing a knife along a diameter on the top of the cylinder and cutting down to the outer edge at the bottom of the cylinder. Replace the knife on the top of the cylinder and cut off the symmetric wedge.



Top View Side View

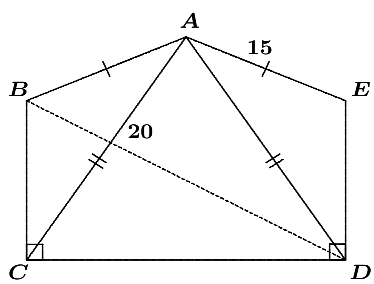
Now, in addition to the round hole and the square hole, the stopper will stop the triangular hole.

M49. *K.R.S. Sastry, Bangalore, India.*

The figure shows a Heron pentagon in which the sides, the diagonals and the area are natural numbers.

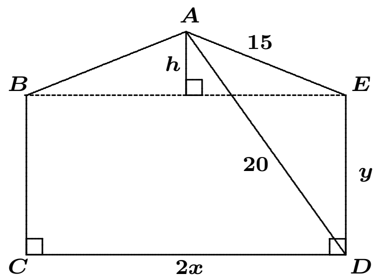
(a) $AB = AE = 15$, $AC = AD = 20$ and $BCDE$ is a rectangle. Find the length of BD.

(b) Give a set of general expressions for the sides, the diagonals and the area to generate an infinite family of such Heron pentagons $ABCDE$ as in the figure.



(a) *Solution by Kevin Chung, OAC student, Earl Haig S.S., North York, ON.*

Introduce variables h, x, and y, as shown in the diagram below. Then we have $h^2 + x^2 = 15^2$ and $(h+y)^2 + x^2 = 20^2$, which implies that $y(2h+y) = 175 = 5^2 \cdot 7 > y^2$. Thus, since $x > 0$, we must have $y = 7$, $h = 9$, and $x = 12$. This yields $BD^2 = 4x^2 + y^2 = 625$, giving $BD = 25$. This solution produces an area of 276 satisfying the condition that the area is an integer. Therefore, the solution is $BD = 25$.



(b) *Solution by the proposer.*

Set

$$\begin{aligned} AB &= AE = (m^2 - n^2)(m^2 + n^2)\,, \\ BC &= DE = 6m^2n^2 - m^4 - n^4\,, \\ CD &= BE = 4mn(m^2 - n^2)\,, \\ AC &= AD = 2mn(m^2 + n^2)\,, \\ BD &= CE = (m^2 + n^2)^2\,. \end{aligned}$$

Then the area is $[ABCDE] = 2mn(m^2 - n^2)(10m^2n^2 - m^4 - n^4)$, where $\gcd(m,n) = 1$ and $n < m < (\sqrt{2}+1)n$.

Part (b) *was also solved by Kevin Chung, OAC student, Earl Haig S.S., North York, ON, whose infinite family of solutions is a subset of the proposer's set.*

M50. *Proposed by the Mayhem Staff.*

This question is a bit of a variation of a well known and used problem. There are forms of the question where you want to use four 4's and some operations to make as long a list of values as possible. Thus

$$\frac{4+4}{4+4} = 1\,, \qquad 4 + 4 - \sqrt{4} - 4 = 2\,,$$

and so on. It is popular to use the digits of the year in such a problem (although, we will have to deal with a couple of zeros for a while).

The problem is to make as many numbers as possible using **up to** five π's. Thus, some acceptable results would be:

$$\frac{\pi + \pi + \pi}{\pi} = 3\,, \qquad \left\lfloor \sqrt{\pi^\pi} - \pi + \frac{\pi}{\pi} \right\rfloor ! = 6\,.$$

Solutions by Sabrina Liao, student, York Mills C.I., North York, ON; Adrian Florea, student, École secondaire St-Luc, Montréal, QC; Peng Liu, student, Glebe C.I., Ottawa, ON; James Meredith, Hudson H.S., Hudson Heights, QC; Rébecca Millette, student, École secondaire Dorval-Jean XXIII, Dorval, QC; Maxime Pelletier, student, Collège Sainte-Anne de Lachine, Lachine, QC; Jing Qin, student, École Émile-Legault, Saint-Laurent, QC; Danny Quan, student, Collège Jean de Brébeuf, Montréal, QC; Diana Rapeanu, student, Collège Notre-Dame Sacré-Coeur, Montréal, QC; Sarah Shaker, student, École secondaire Félix-Leclerc, Pointe-Claire, QC; and Bob Wang, student, Merivale H.S., Nepean, ON.

$$1 = \frac{\pi}{\pi} \qquad 2 = \frac{\pi+\pi}{\pi} \qquad 3 = \frac{\pi+\pi+\pi}{\pi}$$

$$4 = \lfloor\pi\rfloor! - \frac{\pi+\pi}{\pi} \qquad 5 = \frac{\pi+\pi}{\pi} + \lfloor\pi\rfloor \qquad 6 = \left(\frac{\pi+\pi+\pi}{\pi}\right)!$$

$$7 = \lfloor\pi\times\pi\rfloor - \frac{\pi+\pi}{\pi} \qquad 8 = \left\lfloor\sqrt{\pi^\pi}\right\rfloor + \frac{\pi+\pi}{\pi} \qquad 9 = \lfloor\pi+\pi+\pi\rfloor$$

$$10 = \lfloor\pi\times\pi\rfloor + \frac{\pi}{\pi} \qquad 11 = \left\lfloor\frac{\pi^\pi}{\pi}\right\rfloor$$

$$12 = \lfloor\pi+\pi+\pi+\pi\rfloor \qquad 13 = \left\lfloor\left(\pi+\frac{\pi}{\pi}\right)\times\pi\right\rfloor$$

$$14 = \lfloor(\sqrt[\pi]{\pi}+\pi)\times\pi\rfloor \qquad 15 = \lfloor\pi\rfloor! + \lfloor\pi+\pi+\pi\rfloor$$

Above is a taste of some of the wonderful solutions sent in. Jing Qin sent in a list from 1 to 100 with only a couple of omissions, while Bob Wang sent in 51 solutions. Thus, for this issue's prizes, we award a subscription of ***CRUX with MAYHEM*** to Jing Qin, and a copy of **ATOM** (A Taste Of Mathematics), volume 2, to Bob Wang. Continue sending us your solutions and problem proposals.

Pólya's Paragon

Paul Ottaway

For this month's installment we are going to take a little detour from the standard realm of mathematics and examine a very interesting problem in logic. This problem involves a group of 10 pirates who have come across a treasure consisting of 10 equally sized bars of gold. They struggle to come up with some equitable way to divide the bounty and eventually conclude that a democratic method would be best. So, the pirates number themselves by rank, with the captain being 1 and the lackey being 10. Pirate 10 begins by suggesting a way to divide the gold. Then, a vote is taken among all the pirates (including the one who gave the proposal) on whether they like this particular suggestion. If at least 50% are in favour, the gold is divided up according to the proposal and they go on their way. If more than 50% refuse, the pirate giving the proposal is killed and they repeat the procedure with the pirate with the next lower number.

There are a couple of assumptions that we need to make about the pirates in question. First of all, they are perfectly logical and they know that every other pirate is as well. Secondly, they are infinitely greedy — that is to say, they will always reject a proposal if they know they can get more gold later by doing so. Thirdly, they are bloodthirsty. That is to say, if they can get the same amount of gold from a later proposal, they will refuse the current proposal just because they would rather kill another pirate than not. (Hey, they are pirates after all!)

The problem is as follows: If you are the 10^{th} pirate, what distribution do you propose to get the most gold bars?

A slight variation of this problem appeared in the May, 1999 issue of Scientific American. (See also [2002 : 105].) Some of the assumptions I have used are a little different and lead to some very interesting results.

STOP READING NOW until you have tried the problem yourself.

First of all, it may not be entirely clear that there is anything you can suggest that will not end in your death! In problems like this one, we can only decide what you (the 10^{th}) should do if we know with certainty what the 9^{th} pirate will do if you die. Therefore, it seems like we should work backwards towards a solution.

If there happened to be only 1 pirate, he would simply suggest taking all the gold and get a unanimous vote! Unfortunately, this situation will never occur. If there were two pirates, number 2 would simply suggest that he would take all the gold and give the captain nothing. He will clearly get at least 50% of the votes since he will vote for it himself!

We now run into something a little more unexpected. If there were 3 pirates left, what does number 3 do? He only needs one vote besides his own to make his proposal pass. He can suggest 9 bars of gold for himself and 1 for the captain. Clearly, this is better for both of them, since the captain gets nothing if pirate number 3 dies. Therefore, the captain will agree and pirate number 2 will go away empty handed.

Continuing along these lines, number 4's proposal should be 9 bars for himself and 1 for pirate number 2 (who would otherwise get nothing if number 4 is killed). Number 5 should then "bribe" both 1 and 3 with 1 gold bar each and keep 8 for himself to get a majority vote of 3/5. Clearly, a pattern is emerging. We can easily verify that pirate number 10 can suggest keeping 6 for himself and giving 1 bar to each of 2, 4, 6, and 8. He gets exactly five of the ten votes and comes away quite well off!

The following table lists the proposals each pirate would make if it becames his turn. The entry in the n^{th} row and m^{th} column represents how much pirate m would receive with pirate n's proposal.

-	1	2	3	4	5	6	7	8	9	10
1	10	-	-	-	-	-	-	-	-	-
2	0	10	-	-	-	-	-	-	-	-
3	1	0	9	-	-	-	-	-	-	-
4	0	1	0	9	-	-	-	-	-	-
5	1	0	1	0	8	-	-	-	-	-
6	0	1	0	1	0	8	-	-	-	-
7	1	0	1	0	1	0	7	-	-	-
8	0	1	0	1	0	1	0	7	-	-
9	1	0	1	0	1	0	1	0	6	-
10	0	1	0	1	0	1	0	1	0	6

This leads to other interesting questions about pirates. Consider a variant to the original problem where a proposal requires more than 50% of the votes to be accepted. Again, I suggest that you try working on this new problem for a while before continuing.

When there are only 2 pirates left, the captain will never accept the proposal because he would always rather kill number 2 and take all the gold. Since 50% is no longer good enough, pirate 2 cannot save his own life. So given this, what does pirate 3 propose if it becomes his turn? Clearly we can use the logic from above and say that he can keep 9 bars and give 1 bar to number 2. What would happen if he proposes to keep all the gold himself? In particular, how does pirate 2 vote? If he votes for the proposal he gets no gold, but if he votes against it he dies because he cannot make any good proposal next! We did specify, however, that pirates are bloodthirsty. If pirate 2 is going to get no gold, he would rather vote no and see pirate 3 die, despite the fact that it would mean he himself dies next.

Now we get to number 4. He needs 2 of the other 3 to vote yes to get over 50% approval. Thus, he must "bribe" the first two pirates with 1 and 2 bars, respectively — keeping 7 for himself. We can continue with this line of reasoning to get the following table:

-	1	2	3	4	5	6	7
1	10	-	-	-	-	-	-
2	-	-	-	-	-	-	-
3	0	1	9	-	-	-	-
4	1	2	0	7	-	-	-
5	2	0	1	0	7	-	-
6	0	1	2	1	0	6	-
7	1	2	0	0	1	0	6

At this point I should mention that this is not the only proposal that pirate 7 can make and still be accepted. He could just as well propose giving 2 bars of gold to pirate 4 rather than pirate 2. This gets him the same amount of gold, so there is no reason for him to prefer one of these proposals over the other.

Now the really interesting consequence of this is that pirate 8 must account for the fact that it is unknown which proposal pirate 7 will make. If pirate 8 tries to get pirate 2's vote by giving him 1 gold bar, he will not succeed, because pirate 2 may get more by refusing. To avoid the use of probabilities, we assume that pirate 8 wants to ensure that his proposal will always pass the vote. Thus, to get a particular pirate's vote, he must offer that pirate more than the maximum the pirate could get in the next round. I will use *'s to indicate values that can be swapped within a particular proposal. Now, the 6^{th} and later rows look like this:

-	1	2	3	4	5	6	7	8	9	10
6	0	1	2	1	0	6	-	-	-	-
7	1	0*	0	2*	1	0	6	-	-	-
8	2	0	1	0	2	1	0	4	-	-
9	0	1	2*	1	0	0*	1	0	5	-
10	1	2*	0	2*	1	0	0*	1	0	3

Notice that the 9^{th} pirate can actually do better than the 8^{th} if he ever gets to make a proposal. Also, the 10^{th} pirate has 3 different ways to make a proposal that will certainly pass the vote. It is a worthwhile exercise to see how far this table can be extended. Eventually we will reach a pirate who cannot make a proposal that is guaranteed to pass. The first such pirate is number 14. I found it truly amazing, however, to discover that the 15^{th} pirate actually has a proposal that will be accepted!

Naturally, there are many more questions we can ask about such interesting pirates:

1. For both versions, given n pirates and m bars of gold, how many bars can the n^{th} pirate get?
2. For both versions, which values of n and m allow no good proposal for the n^{th} pirate?
3. In the second version, what happens if we introduce probabilities and say that pirates will vote yes if their "expected" number of gold bars is greater than if they vote no?

4. For both versions, what if the pirates decide that after a refused proposal ALL pirates who voted yes are to be killed? (This problem begins to infringe on classical game theory and is similar in some respects to a famous problem called the Prisoner's Dilemma).

THE OLYMPIAD CORNER

No. 230

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

We start this number of the *Corner* with ten problems shortlisted for the 2000 International Mathematical Olympiad in Korea. Thanks go to Andy Liu, Canadian Team Leader to IMO 2000, for collecting them for our use.

2000 INTERNATIONAL MATHEMATICAL OLYMPIAD
Shortlisted Problems

1. (*Brazil*) Determine all triples of positive integers (a, m, n) such that $a^m + 1$ divides $(a+1)^n$.

2. (*Bulgaria*) Prove that there exist infinitely many positive integers n such that $p = nr$, where p and r are respectively the semiperimeter and the inradius of a triangle with integer side lengths.

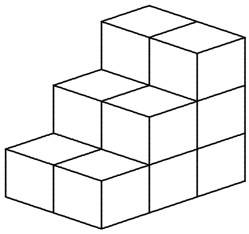
3. (*Colombia*) Let $n \geq 4$ be a fixed integer. A set $S = \{P_1, \dots, P_n\}$ of n points is given in the plane such that no three are collinear and no four concyclic. For $1 \leq t \leq n$, let a_t be the number of circles $P_iP_jP_k$ that contain P_t in their interior, and let $m(S) = a_1 + a_2 + \cdots + a_n$. Prove that there exists a positive integer $f(n)$, depending only on n, such that the points of S are the vertices of a convex polygon if and only if $m(S) = f(n)$.

4. (*Czech Republic*) Let n and k be positive integers such that $\frac{n}{2} < k \leq \frac{2n}{3}$. Determine the least positive integer m for which it is possible to place each of m pawns on a square of an $n \times n$ chessboard so that no column or row contains a block of k adjacent unoccupied squares.

5. (*France*) Let p and q be relatively prime positive integers. Determine the number of subsets S of $\{0, 1, 2, \dots\}$ such that $0 \in S$ and, for each element $n \in S$, the integers $n+p$ and $n+q$ belong to S.

6. (*France*) For a positive integer n, let $d(n)$ be the number of positive divisors of n. Find all positive integers such that $d(n)^3 = 4n$.

7. (*Italy*) The diagram shows a staircase-brick with 3 steps of width 2, made of 12 unit cubes. Determine all positive integers n for which it is possible to build an $n \times n \times n$ cube using such bricks.



8. (*Japan*) Determine all integers $n \geq 2$ such that for all integers a and b relatively prime to n, $a \equiv b \pmod{n}$ if and only if $ab \equiv -1 \pmod{n}$.

9. (*Romania*) Prove that the set of positive integers which cannot be represented as a sum of distinct perfect squares is finite.

10. (*Russia*) In the plane we have n rectangles with parallel sides. The sides of distinct rectangles lie on distinct lines. The boundaries of the rectangles cut the plane into connected regions. A region is said to be *nice* if it has at least one of the vertices of the n rectangles on its boundary. There can be non-convex regions, as well as regions with more than one boundary curve. Prove that the sum of the numbers of the vertices of all nice regions is less than $40n$.

While sorting materials for this number I discovered some solutions to the 1997 Selection Test for the Vietnamese Team filed with solutions for the April 2001 *Corner*. The problems were given [2000 : 264], and one solution was discussed in the *Corner* last November [2002 : 423–424]. My apologies. Here are the solutions.

3. Find the greatest real number α such that there exists an infinite sequence of whole numbers (a_n) $(n = 1, 2, 3, \dots)$ satisfying simultaneously the following conditions:

(i) $a_n > 1997^n$ for every $n \in \mathbb{N}^*$,

(ii) $a_n^\alpha \leq U_n$ for every $n \geq 2$, where U_n is the greatest common divisor of the set of numbers $\{a_i + a_j \mid i + j = n\}$.

Solution by Mohammed Aassila, Strasbourg, France, adapted by the editors.

The greatest number α is $\frac{1}{2}$.

Let F_n be the n^{th} Fibonacci number. Let m be an even positive integer such that $F_{2mn} > 1997^n$ for all $n \in \mathbb{N}^*$, and let $a_n = 3F_{2mn}$. The sequence $(a_n)_{n=1}^{\infty}$ satisfies condition (i). We will prove that it satisfies condition (ii) with $\alpha = \frac{1}{2}$.

[*Editor's note*: The following argument invokes various properties of the Fibonacci numbers, F_n, some of which may be unfamiliar to the reader.

The well-known recurrence relation $F_{n+2} = F_{n+1} + F_n$ serves to define F_n for all integers $n \geq 0$, given that $F_0 = 0$ and $F_1 = 1$. We can extend this definition by letting $F_{-n} = (-1)^{n+1}F_n$. Alternatively, we can define F_n for any integer n by the explicit formula

$$F_n \;=\; \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] .$$

For any integers k and ℓ, we have $F_{k+\ell} = F_kF_{\ell+1} + F_{k-1}F_\ell$. This identity may be taken as the basis for several of the assertions below.]

For all positive integers i and j, we have

$$F_{2mi} \;=\; F_{m(i+j)}F_{m(i-j)+1} + F_{m(i+j)-1}F_{m(i-j)} .$$

Interchanging i and j and recalling that m is even, we obtain

$$F_{2mj} \;=\; F_{m(i+j)}F_{m(i-j)-1} - F_{m(i+j)-1}F_{m(i-j)} .$$

Hence,

$$F_{2mi} + F_{2mj} \;=\; F_{m(i+j)}\left(F_{m(i-j)+1} + F_{m(i-j)-1}\right) .$$

If $i + j = n$, then $F_{mn} \mid (F_{2mi} + F_{2mj})$, and hence, $3F_{mn} \mid (a_i + a_j)$, which implies that $3F_{mn} \leq U_n$ (where U_n is defined as in the problem statement above). Furthermore,

$$F_{2mn} \;=\; F_{mn}\left(F_{mn+1} + F_{mn-1}\right) \;=\; F_{mn}\left(2F_{mn} + F_{mn-1}\right) \;\leq\; 3F_{mn}^2 .$$

Thus, $a_n = 3F_{2mn} \leq 9F_{mn}^2 \;\leq\; U_n^2$, and finally, $a_n^{1/2} \leq U_n$, as required in condition (ii).

Now, consider any $\alpha > 0$, and suppose that some sequence (a_n) satisfies the conditions (i) and (ii) for this α. We will prove that $\alpha \leq \frac{1}{2}$.

Consider any ε such that $0 < \varepsilon < 2$. Suppose that there exists $N \in \mathbb{N}^*$ such that $a_{2n} < a_n^{2-\varepsilon}$ for all $n > N$. Then, for any $n > N$, we have $\frac{\log a_{2n}}{2n} < \left(\frac{2-\varepsilon}{2}\right)\left(\frac{\log a_n}{n}\right)$, and therefore, $\frac{\log a_{2^kn}}{2^kn} < \left(\frac{2-\varepsilon}{2}\right)^k\left(\frac{\log a_n}{n}\right)$ for all $k \in \mathbb{N}^*$. Then $\lim\limits_{k\to\infty} \frac{\log a_{2^kn}}{2^kn} = 0$. But this is impossible, because condition (i) implies that $\frac{\log a_n}{n} \geq \log 1997$ for all n. Consequently, there must be infinitely many values of n for which $a_{2n} \geq a_n^{2-\varepsilon}$. For any such n,

$$a_n^{(2-\varepsilon)\alpha} \;\leq\; a_{2n}^{\alpha} \;\leq\; U_{2n} \;\leq\; 2a_n ,$$

and hence, $2 \geq a_n^{(2-\varepsilon)\alpha-1} \geq 1997^{n((2-\varepsilon)\alpha-1)}$. Therefore, $(2-\varepsilon)\alpha - 1 \leq 0$. Letting $\varepsilon \to 0$, we have $\alpha \leq \frac{1}{2}$.

4. Let $f : \mathbb{N} \to \mathbb{Z}$ be the function defined by:

$$\begin{aligned} f(0) &= 2\,, \qquad f(1) \;=\; 503\,, \\ f(n+2) &= 503\, f(n+1) - 1996\, f(n) \quad \text{for all } n \in \mathbb{N}\,. \end{aligned}$$

For every $k \in \mathbb{N}^*$, take k arbitrary integers $s_1, s_2, \ldots, s_k$ such that $s_i \geq k$ for all $i = 1, 2, \ldots, k$, and for every s_i $(i = 1, 2, \ldots, k)$, take an arbitrary prime divisor $p(s_i)$ of $f(2^{s_i})$.

Prove that for positive integers $t \leq k$, we have:

$$\sum_{i=1}^{k} p(s_i) \mid 2^t \quad \text{if and only if} \quad k \mid 2^t\,.$$

Solution by Mohammed Aassila, Strasbourg, France.

The problem should be: prove that

$$2^t \,\Big|\, \sum_{i=1}^{k} p(s_i) \iff 2^t \mid k\,.$$

First, note that $f(n) = 4^n + 499^n$ for all $n \in \mathbb{N}$. (Use induction for example). Also note that if p is an odd prime, if $m, n \in \mathbb{N}$ are not divisible by p, and if $p \mid m^{2^s} + n^{2^s}$ $(s \geq 0)$, then $p \equiv 1 \pmod{2^{s+1}}$. [*Editor's note*: Since p does not divide m, there is an integer a such that $am \equiv 1 \pmod{p}$. Since $m^{2^s} + n^{2^s} \equiv 0 \pmod{p}$, we have

$$\begin{aligned} (am)^{2^s} + (an)^{2^s} &\equiv 0 \pmod{p} \\ (an)^{2^s} &\equiv -1 \pmod{p}\,. \end{aligned}$$

Therefore, $(an)^{2^{s+1}} \equiv 1 \pmod{p}$, which implies that the order of $an \pmod{p}$ is 2^{s+1}. By Fermat's Theorem, 2^{s+1} divides $p-1$.]

Let $s_1, s_2, \ldots, s_k$ be k arbitrary integers such that $s_i \geq k$ for all $i = 1, 2, \ldots, k$, and let $p_i = p(s_i)$ be a prime divisor of $f(2^{s_i})$; that is, $p_i \mid 4^{2^{s_i}} + 499^{2^{s_i}}$ for each i. Therefore, $2^{s_i+1} \mid p_i - 1$, and in particular, $p_i \equiv 1 \pmod{2^k}$. Thus, $\sum_{i=1}^{k} p_i \equiv k \pmod{2^k}$. Consequently,

$$2^t \,\Big|\, \sum_{i=1}^{k} p(s_i) \iff 2^t \mid k\,.$$

5. Determine all pairs of positive real numbers a, b such that for every $n \in \mathbb{N}^*$ and for every real root x_n of the equation

$$4n^2 x = \log_2(2n^2 x + 1)$$

we have

$$a^{x_n} + b^{x_n} \geq 2 + 3x_n\,.$$

Solution by Mohammed Aassila, Strasbourg, France, adapted by the editors.

We have

$$\begin{aligned} 4n^2x \;=\; \log_2(2n^2x+1) &\iff 4^{2n^2x} \;=\; 2n^2x+1 \\ &\iff 2n^2x \;=\; -\tfrac{1}{2} \text{ or } 0 \\ &\iff x \;=\; -\tfrac{1}{4n^2} \text{ or } 0\,. \end{aligned}$$

Thus, the set of roots x_n is $E = \{0\} \cup \{-\frac{1}{4n^2} \mid n \in \mathbb{N}^*\}$. We claim that $a^x + b^x \geq 2 + 3x$ for all $x \in E$ if and only if $ab \leq e^3$.

Note first that

$$\lim_{x\to 0} \frac{a^x - 1}{x} \;=\; \log a\,, \qquad \lim_{x\to 0} \frac{b^x - 1}{x} \;=\; \log b\,,$$

and hence,

$$\lim_{x\to 0} \frac{a^x + b^x - 2}{x} \;=\; \log(ab)\,.$$

If $a^x + b^x \geq 2 + 3x$ for all $x \in E$, then for all n,

$$\frac{a^{-\frac{1}{4n^2}} + b^{-\frac{1}{4n^2}} - 2}{-\frac{1}{4n^2}} \;\leq\; 3\,.$$

Letting $n \to \infty$, we find that $\log(ab) \leq 3$, and therefore, $ab \leq e^3$.

Conversely, let us suppose that $ab \leq e^3$. Using the Power–Mean Inequality, we have, for $x < 0$,

$$\left(\frac{a^x + b^x}{2}\right)^{1/x} \;\leq\; \sqrt{ab} \;\leq\; e^{3/2}\,.$$

Then

$$\frac{a^x + b^x}{2} \;\geq\; e^{3x/2} \;\geq\; 1 + \frac{3x}{2}\,,$$

and thus, $a^x + b^x \geq 2 + 3x$. This inequality is also satisfied when $x = 0$, trivially. Therefore, it is satisfied for all $x \in E$.

Next we turn to some solutions of the Second Round problems of the 33$^{\text{rd}}$ Spanish Mathematical Olympiad [2001 : 92–93].

1. Calculate the sum of the squares of the first 100 terms of an arithmetic progression, given that the sum of the first 100 terms of the progression equals -1, and that the sum of the even numbered terms equals $+1$.

Solved by Robert Bilinski, Outremont, QC; and Pierre Bornsztein, Pontoise, France. We give Bilinski's write-up.

Let $a_n = a_0 + nb$ be the general term of our arithmetic progression. The first 100 terms are a_0 to a_{99}, and their sum is

$$\sum_{n=0}^{99}(a_0+nb) \;=\; 100a_0 \;+\; \frac{99\times 100}{2}b \;=\; 100a_0 \;+\; 4950b\,.$$

The sum of the even-numbered terms is

$$\sum_{n=0}^{49}(a_0+2nb) \;=\; 50a_0+\frac{49\times 50}{2}(2b) \;=\; 50a_0+2450b\,.$$

Solving the equations $100a_0 + 4950b = -1$ and $50a_0 + 2450b = 1$, we get $b = -\frac{3}{50}$ and $a_0 = \frac{148}{50}$. Since $a_n^2 = (a_0 + nb)^2 = a_0^2 + 2na_0b + n^2b^2$, we have

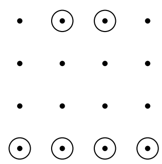
$$\sum_{n=0}^{99} a_n^2 \;=\; 100a_0^2 + 9900a_0b + 328350b^2 = \frac{14999}{50}\,.$$

[*Editor's Note*: It is interesting to observe that if we had numbered the terms of our arithmetic progression starting with a_1, instead of a_0, an analysis as above would yield the same progression as above, but taken in the opposite order, and the sum of the squares would still be the same.]

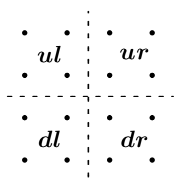
2. Let A be the set of the 16 lattice points forming a square of side 4. Find, with reasons, the largest number of points of A such that any THREE of them do NOT form an isosceles right triangle.

Solution by Pierre Bornsztein, Pontoise, France.

The answer is 6. An example of a subset of A with 6 points, no three of which form an isosceles right triangle, is shown in the figure below, where the points in the subset are circled.



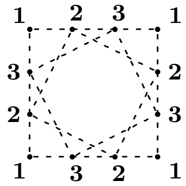
Let E be a subset of A such that any three points from E do not form an isosceles right triangle. Divide A into four 2×2 squares, say ul, ur, dl, dr ($u =$ up, $d =$ down, $l =$ left, $r =$ right). It is clear that we cannot have more than two points in E from any one of these four squares. In the same way we cannot have more than two points from the 2×2 square at the centre.



A point of E will be enclosed in a circle, and a point which is excluded from E will be crossed out. Points will be identified by coordinates (x, y) where x, $y \in \{1, 2, 3, 4\}$.

Suppose that E contains 7 points. Then, three of the four squares ul, ur, dl, dr each contain two of the points of E, and the fourth contains only one point of E. We consider the possible cases, showing that in each case a contradiction arises.

First case. No point of E belongs to the central 2×2 square. Divide the 12 points along the edges of A into three squares 1, 2, 3 (see the figure below). None of these squares can contain more than two points of E. Thus, $|E| \leq 6$, a contradiction.



Second case. The central 2×2 square contains a unique point of E. With no loss of generality, we may suppose that it is $(2, 2)$. Thus, the other points in this square are crossed out. One of the 2×2 squares ul and dr contains two points of E. With no loss of generality, we may suppose that it is ul.

(a) If $(1, 3)$ is circled, then $(1, 1)$, $(1, 2)$, $(2, 3)$, $(2, 4)$ must be crossed out. Thus, $(1, 4)$ is circled, which makes $(4, 3)$ crossed out.

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Considering the squares 1, 2, 3 of the first case, we see that no more than one point can still be circled for each of these squares. Thus, $|E| \leq 6$, a contradiction.

(b) If $(1, 3)$ is crossed out, then $(1, 4)$ and $(2, 4)$ are circled. Thus, $(4, 2)$, $(4, 3)$, $(4, 4)$ are crossed out, and $(3, 4)$ is circled (since one point of ur has to belong to E). Since there is only one circled point in ur, there must be two circled points in dr. That leads to $(3, 1)$ and $(4, 1)$ being circled.

⊙ ⊙ ⊙ ×
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The last three points have to be crossed out, and $|E| \leq 6$, a contradiction.

Third case. The central 2×2 square contains two points of E.

(a) If these two points of E are two horizontal points or two vertical points, then, with no loss of generality, we may suppose that they are horizontal, and that they are $(2,2)$ and $(2,3)$. Then $(2,1)$, $(3,1)$, $(2,3)$, $(3,3)$ are crossed out.

$$\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \times & \times & \cdot \\ \cdot & \odot & \odot & \cdot \\ \cdot & \times & \times & \cdot \end{array}$$

Either ul or ur contains two points of E. With no loss of generality, we may suppose that ul contains two points of E.

(i) If $(1,3)$ is circled, then $(1,1)$, $(1,2)$, $(2,4)$, $(4,4)$ are crossed out. It follows that $(1,4)$ is circled, and then $(3,4)$, $(4,3)$ are crossed out.

$$\begin{array}{cccc} \odot & \times & \times & \times \\ \odot & \times & \times & \times \\ \times & \odot & \odot & \cdot \\ \times & \times & \times & \cdot \end{array}$$

Thus, $|E| \leq 6$, a contradiction.

(ii) If $(1,3)$ is marked with a cross, then $(1,4)$ and $(2,4)$ are circled. Then, $(1,2)$, $(3,4)$, $(4,2)$, $(4,3)$, $(4,4)$ are crossed out.

$$\begin{array}{cccc} \odot & \odot & \times & \times \\ \times & \times & \times & \times \\ \times & \odot & \odot & \times \\ \cdot & \times & \times & \cdot \end{array}$$

Then $|E| \leq 6$, a contradiction.

(b) If the two points in the central square that are in E are diagonally opposite, then, with no loss of generality, we may suppose that they are $(2,2)$ and $(3,3)$. Then $(1,3)$, $(2,3)$, $(2,4)$, $(3,1)$, $(3,2)$, $(4,2)$ are crossed out.

$$\begin{array}{cccc} \cdot & \times & \cdot & \cdot \\ \times & \times & \odot & \cdot \\ \cdot & \odot & \times & \times \\ \cdot & \cdot & \times & \cdot \end{array}$$

It follows that ul and dr each contain at most one point of E, a contradiction.

3. Consider the parabolas

$$y \;=\; x^2 + px + q\,,$$

that intersect the coordinate axes at three distinct points. For these three points, a circle is drawn. Show that all the circles drawn when p and q vary over $\mathbb{R}$ pass through a fixed point, and determine the point.

Solved by Michel Bataille, Rouen, France; Robert Bilinski, Outremont, QC; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We give the solution by Díaz-Barrero.

Let $Q(0,q)$ be the intersection point of a generic parabola $y = x^2 + px + q$ (p, $q \in \mathbb{R}$) with the y–axis, and let $R(r,0)$ and $S(s,0)$ be the intersection points of this parabola with the x–axis. The centre C of the circle through Q, R, and S is the point common to the median line of RS, with equation $x + \frac{p}{2} = 0$, and the median line of QR, with equation $y = \frac{r}{q}x + \frac{q^2-r^2}{2q}$. That is,

$$C \;=\; \left(-\frac{p}{2}\,,\;\frac{q^2-r^2-rp}{2q}\right)\,. \tag{1}$$

Since $r + s = -p$ and $rs = q$ (Cardan–Viète formulas), we have $-rp = r^2 + rs = r^2 + q$. Substituting into (1), the centre of the circle becomes $C = \left(-\frac{p}{2}, \frac{q+1}{2}\right)$. Its radius is given by

$$d(C,Q) \;=\; \sqrt{\frac{p^2+(1-q)^2}{4}}\,.$$

Therefore, the circle equation is

$$\left(x+\frac{p}{2}\right)^2 \;+\; \left(y-\frac{q+1}{2}\right)^2 \;=\; \frac{p^2+(1-q)^2}{4}$$

or

$$x^2 + y^2 + px - (1+q)y + q \;=\; 0\,.$$

Finally, observe that for all p, $q \in \mathbb{R}$, the circles defined by the foregoing equation have the common point $(0,1)$, as can be checked by substitution.

4. Let p be a prime number. Find all integers $k \in \mathbb{Z}$ such that $\sqrt{k^2 - pk}$ is a positive integer.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; Murray S. Klamkin, University of Alberta, Edmonton, AB; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We use the solution by Díaz-Barrero, adapted by the editors.

Suppose that $\sqrt{k^2 - pk} = n$, where n is a positive integer. Then $k^2 - pk - n^2 = 0$, and

$$k = \frac{p \pm \sqrt{p^2 + 4n^2}}{2}\,. \tag{1}$$

It follows that $p^2 + 4n^2$ must be a square; that is, $p^2 + 4n^2 = m^2$ for some $m \in \mathbb{N}$. Then, $p^2 = (m + 2n)(m - 2n)$. Since $m + 2n > m - 2n$ and p is prime, it follows from the Fundamental Theorem of Arithmetic that $m + 2n = p^2$ and $m - 2n = 1$.

Solving, we obtain $m = \dfrac{p^2 + 1}{2}$ and $n = \dfrac{p^2 - 1}{4}$. These solutions are admissible provided that $p \neq 2$. (When $p = 2$, they are not integers.) Finally, substituting $n = (p^2 - 1)/4$ into (1), we obtain

$$k = \left(\frac{p+1}{2}\right)^2 \quad \text{or} \quad k = -\left(\frac{p-1}{2}\right)^2 .$$

These are the only valid solutions when $p \neq 2$. When $p = 2$ there are no solutions for k.

5. Show that for any convex quadrilateral with unit area, the sum of the sides and diagonals is not less than $2(2 + \sqrt{2})$.

Solved by Mohammed Aassila, Strasbourg, France; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We give the argument of Aassila.

Let θ be the angle between the diagonals. Then the area is equal to $\frac{1}{2} d_1 d_2 \sin\theta$. Then $d_1 d_2 \geq 2$, with equality when the diagonals are perpendicular. By the AM–GM Inequality, $d_1 + d_2 \geq 2\sqrt{2}$, with equality when the diagonals are equal and perpendicular.

If $s = \dfrac{a+b+c+d}{2}$, and B, D are opposite angles, then we have

$$(\text{area})^2 = (s-a)(s-b)(s-c)(s-d) - (abcd)\cos^2\left(\frac{B+D}{2}\right) .$$

Hence, $(s-a)(s-b)(s-c)(s-d) \geq 1$. By the AM–GM Inequality,

$$4 \leq (s-a) + (s-b) + (s-c) + (s-d) = a + b + c + d\,,$$

with equality when $a = b = c = d$.

We conclude that the sum of the sides and diagonals is at least $2(2 + \sqrt{2})$, with equality for a square.

As a third set of solutions this issue, we turn to reader submissions for the 7^{th} Japan Mathematical Olympiad, Final Round, 1997 [2001 : 93–94].

1. Prove that, whenever we put ten points in any way on a circle whose diameter is 5, we can find two points whose distance is less than 2.

Solved by Mohammed Aassila, Strasbourg, France; Marcus Emmanuel Barnes, student, York University; Robert Bilinski, Outremont, QC; George Evagelopoulos, Athens, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give the solution by Barnes.

A circle whose diameter is 5 has a circumference slightly less than 16 units. Divide the circumference into 8 arcs of equal length. Then the 8 arcs are of length slightly less than 2. Since there are 8 arcs, by the Pigeonhole Principle at least one of these arcs has 2 points on it. Since the arcs are less than 2 units, the distance between these two points is less than 2.

The solution by Evagelopoulos allows choices anywhere in the disc. We give it next.

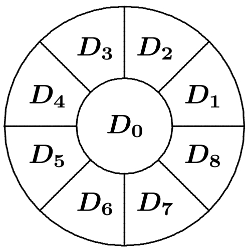


Figure 1

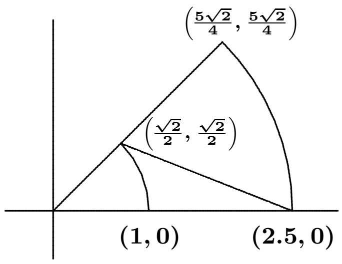


Figure 2

A disc whose diameter is 5 can be divided up as shown in Figure 1. Here the boundary of D_0 is a circle of radius 2 and is not included in D_0. It is easy to see that the distance between any two points in D_i is less than 2 (see Figure 2).

Comment by Pierre Bornsztein, Pontoise, France.

This problem appears in the 1983 British Olympiad. Solutions may be found in:

[1] A. Gardiner, *The Mathematical Olympiad Handbook*, Oxford Science Publication, pp. 179–180.

[2] R. Honsberger, *More Mathematical Morsels*, MAA, pp. 153–155.

[*Editor*: The problem appeared in [1983 : 109].]

2. Let a, b, c be positive integers. Prove that the inequality

$$\frac{(b+c-a)^2}{(b+c)^2+a^2} + \frac{(c+a-b)^2}{(c+a)^2+b^2} + \frac{(a+b-c)^2}{(a+b)^2+c^2} \geq \frac{3}{5}$$

holds. Determine also when the equality holds.

Solution by Mohammed Aassila, Strasbourg, France.
Let

$$\begin{aligned} A &= (b+c-a)^2\left((c+a)^2+b^2\right)\left((a+b)^2+c^2\right) \\ &\quad +(c+a-b)^2\left((b+c)^2+a^2\right)\left((a+b)^2+c^2\right) \\ &\quad +(a+b-c)^2\left((b+c)^2+a^2\right)\left((c+a)^2+b^2\right), \\ B &= \left((b+c)^2+a^2\right)\left((c+a)^2+b^2\right)\left((a+b)^2+c^2\right). \end{aligned}$$

We have to prove that $5A-3B \geq 0$.

We have

$$\begin{aligned} &5A-3B \\ &= 4\{3(a^6+b^6+c^6)+(a^5b+ab^5+b^5c+bc^5+c^5a+ca^5) \\ &\quad -(a^4b^2+a^2b^4+b^4c^2+b^2c^4+c^4a^2+c^2a^4) \\ &\quad +2(a^3b^3+b^3c^3+c^3a^3)+3abc(a^3+b^3+c^3) \\ &\quad -6abc(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2)+12a^2b^2c^2\} \\ &= 4\{bc(c-a)^2(a-b)^2+ca(a-b)^2(b-c)^2+ab(b-c)^2(c-a)^2\} \\ &\quad +4abc\{(a+b)(a-b)^2+(b+c)(b-c)^2+(c+a)(c-a)^2\} \\ &\quad +8\{a^3(b+c)(b-c)^2+b^3(c+a)(c-a)^2+c^3(a+b)(a-b)^2\} \\ &\quad +6\{(a^3-b^3)^2+(b^3-c^3)^2+(c^3-a^3)^2\} \\ &\quad +4\{ab(a^2+ab+b^2)(a-b)^2+bc(b^2+bc+c^2)(b-c)^2 \\ &\quad +ca(c^2+ca+a^2)(c-a)^2\} \\ &\geq 0, \end{aligned}$$

with equality when $a=b=c$.

4. Let A, B, C, D be points in space in a general position. Assume that $AX+BX+CX+DX$ is a minimum at $X=X_0$ ($X_0 \neq A, B, C, D$). Prove that $\angle AX_0B = \angle CX_0D$.

Solved by Mohammed Aassila, Strasbourg, France; and George Evagelopoulos, Athens, Greece. We give Aassila's presentation.

Let (x_1,y_1,z_1), $\ldots$, (x_4,y_4,z_4) be the coordinates of A, B, C, D, respectively, and let (x,y,z) be the coordinates of X. We set

$$\begin{aligned} f(X) &= AX+BX+CX+DX \\ &= \sum_{i=1}^{4}\sqrt{(x-x_i)^2+(y-y_i)^2+(z-z_i)^2}. \end{aligned}$$

Then

$$\frac{\partial f}{\partial x} = \sum_{i=1}^{4}\frac{(x-x_i)}{\sqrt{(x-x_i)^2+(y-y_i)^2+(z-z_i)^2}},$$

and similarly for $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$. Since f has a minimum at $X = X_0$, we have $\left.\frac{\partial f}{\partial x}\right|_{X_0} = \left.\frac{\partial f}{\partial y}\right|_{X_0} = \left.\frac{\partial f}{\partial z}\right|_{X_0} = 0$. Thus, we obtain

$$\frac{\overrightarrow{AX_0}}{AX_0} + \frac{\overrightarrow{BX_0}}{BX_0} + \frac{\overrightarrow{CX_0}}{CX_0} + \frac{\overrightarrow{DX_0}}{DX_0} = 0\,,$$

which implies

$$\frac{\overrightarrow{AX_0}}{AX_0} \cdot \frac{\overrightarrow{BX_0}}{BX_0} = \frac{\overrightarrow{CX_0}}{CX_0} \cdot \frac{\overrightarrow{DX_0}}{DX_0}\,.$$

Hence, $\angle AX_0D = \angle CX_0D$.

5. Let n be a positive integer. To each vertex of a regular 2^n–gon, we assign one of letters "A" or "B". Prove that we can do this in such a way that all possible sequences of n letters which appear in this 2^n–gon as an arc directed clockwise from some vertex are mutually distinct.

Solution by George Evagelopoulos, Athens, Greece, adapted by the editors.

We shall call a sequence of n characters "A" or "B", simply, a "word". The number of words is 2^n. Since this is equal to the number of vertices, every word has to appear just once. Therefore, it suffices to prove that we can arrange these 2^n words in a cycle

$$W_1 \longrightarrow W_2 \longrightarrow \cdots \longrightarrow W_{2^n-1} \longrightarrow W_{2^n} \longrightarrow W_1\,,$$

in such a way that any two adjacent words $W_i \longrightarrow W_j$ satisfy the following overlap condition:

$$\text{If } W_i = X_1X_2X_3\cdots X_n, \text{ then } W_j = X_2X_3\cdots X_nX_{n+1}\,, \qquad (1)$$

where each X_i is either A or B.

We prove this in three steps.

Step 1. We divide the 2^n words into short cycles.

We construct short cycles in the following way. Starting with a word $W_1 = X_1X_2X_3\cdots X_n$, we put

$$\begin{aligned} W_2 &= X_2X_3\cdots X_nX_1\,, \\ W_3 &= X_3\cdots X_nX_1X_2\,, \\ &\vdots \end{aligned}$$

Clearly, any two adjacent words $W_i \longrightarrow W_{i+1}$ satisfy (1). Letting d be the smallest integer such that $W_{d+1} = W_1$, we obtain a short cycle of length d:

$$W_1 \longrightarrow W_2 \longrightarrow \cdots \longrightarrow W_{d-1} \longrightarrow W_d \longrightarrow W_1\,.$$

Note that each word is contained in just one short cycle, and occurs just once in that cycle.

Step 2. We connect disjoint cycles.

Let $W_1 = X_1X_2\cdots X_{n-1}A$ and $W_1' = X_1X_2\cdots X_{n-1}B$ be two words that differ only in their last character, and belong to *disjoint* cycles

$$W_1 \longrightarrow W_2 \longrightarrow \cdots \longrightarrow W_{d-1} \longrightarrow W_d \longrightarrow W_1$$

and

$$W_1' \longrightarrow W_2' \longrightarrow \cdots \longrightarrow W_{e-1}' \longrightarrow W_e' \longrightarrow W_1' .$$

These need not be short cycles, but they must each have the following properties (all of which are possessed by short cycles):

(i) adjacent words in the cycle satisfy the overlap condition (1),

(ii) no word occurs more than once in the cycle, and

(iii) if the cycle contains a word W, then it contains every word that is in the same short cycle as W.

We combine the above cycles as follows:

$$W_1 \longrightarrow W_2 \longrightarrow \cdots \longrightarrow W_d \longrightarrow W_1' \longrightarrow W_2' \longrightarrow \cdots \longrightarrow W_e' \longrightarrow W_1 .$$

It is easy to see that this new cycle has the properties (i)—(iii). (Condition (1) is satisfied by the newly created pairs $W_d \longrightarrow W_1'$ and $W_e' \longrightarrow W_1$, because it was satisfied for the pairs $W_d \longrightarrow W_1$ and $W_e' \longrightarrow W_1'$.)

Step 3. We prove that there is a cycle in which any two adjacent words satisfy (1) and every word occurs exactly once.

Let $\mathcal{C}$ be a cycle which contains the word $AA\cdots A$, has the properties (i)—(iii) defined in Step 2, and is maximal with respect to the connecting process in Step 2. No word can occur more than once in $\mathcal{C}$ (property (ii)). We will prove that every word belongs to $\mathcal{C}$.

By induction on r, we shall prove that the word $AA\cdots ABX_1\cdots X_r$ belongs to $\mathcal{C}$, for each $r \in \{0, 1, \ldots, n-1\}$. In the case $r = 0$, the word $AA\cdots AB$ must belong to $\mathcal{C}$, because otherwise the short cycle containing this word could be connected to $\mathcal{C}$ following $AA\cdots A$, by the process in Step 2. This would contradict the maximality of $\mathcal{C}$.

Assuming the assertion is true for r, we shall prove it is true for $r+1$. By the induction hypothesis, $AA\cdots ABX_1\cdots X_r$ belongs to $\mathcal{C}$. The next word in the cycle $\mathcal{C}$ is either $AA\cdots ABX_1\cdots X_rA$ or $AA\cdots ABX_1\cdots X_rB$. Without loss of generality, we may suppose it is $AA\cdots ABX_1\cdots X_rA$. Then, $AA\cdots ABX_1\cdots X_rB$ must also be contained in $\mathcal{C}$; otherwise, the short cycle containing this word could be connected to $\mathcal{C}$ by the process in Step 2, contradicting the maximality of $\mathcal{C}$.

[*Editor's comment*: This problem appears in the book *Elements of Discrete Mathematics* by C.L. Liu, McGraw-Hill, 1985, as a worked example on pages 153–155, where the approach used is that of finding an eulerian circuit in a graph.]

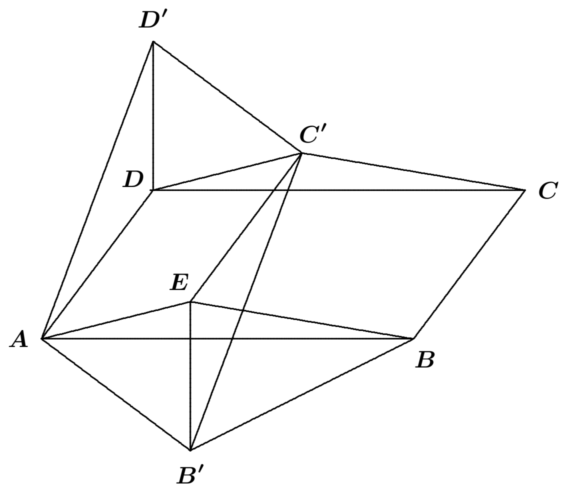
We conclude with a solution to Klamkin Quickie #5 from [2001 : 166], different from the solution given previously [2001 : 300].

5. $ABCD$ and $AB'C'D'$ are any two parallelograms in a plane with A opposite to C and C'. Prove that BB', CC' and DD' are possible sides of a triangle.

Solved by D.J. Smeenk, Zaltbommel, the Netherlands.

We are given that $ABCD$ and $AB'C'D'$ are parallelograms. Complete parallelogram $BCC'E$. Then $AEC'D$ is a parallelogram as well.

Since $C'D'$ is parallel to and equal in length to $B'A$ and since $C'D$ is parallel to and equal in length to EA, we have $\triangle AB'E \cong \triangle C'D'D$; whence, $EB' = DD'$. Observe that BB', BE, and $B'E$ are the sides of $\triangle B'BE$. Therefore, BB', CC', DD' are possible sides of a triangle.



That completes the *Corner* for this issue. Send me Olympiad contests and your nice solutions and generalizations.

BOOK REVIEW

John Grant McLoughlin

Doctor Ecco's Cyberpuzzles
by Dennis Shasha, published by W. W. Norton & Co., Inc. New York, 2002
ISBN 0-393-05120-X, hardcover, 231 + xv pages, US$24.95
Reviewed by ***Andy Liu****, University of Alberta, Edmonton, Alberta.*

This is the third book about Dr. Ecco by Dennis Shasha, a computing scientist at the Courant Institute and a worthy successor of Martin Gardner to the exalted position of the puzzle columnist in *Scientific American*. He has also contributed a column to *Dr. Dobb's Journal*, from which the material of this volume is drawn.

Dr. Ecco is a professional omniheurist, a solver of all kinds of problems. He is perhaps best described as a mathematical version of Sherlock Holmes. He has a consultant agency to which government, industry and individuals bring their problems. He even has a chronicler, *à la* Dr. John Watson, in the person of Prof. Justin Scarlet.

The first volume, *The Puzzling Adventures of Dr. Ecco*, is the text of a most successful course in discrete mathematics at the University of Alberta. The students are presented with a treasure trove of instructive, important, and interesting problems. The attempt to solve them draws upon their ingenuity as well as their mathematical background. Many students commented that this course makes them have a greater appreciation for other mathematics courses. (This can, of course, be taken in two ways, but we are now sufficiently confident that it is meant as a compliment.)

Dr. Ecco's clients are often satisfied with solutions to their problems, and are not particular about how Dr. Ecco comes up with them. In our course, we strive to understand Dr. Ecco's creative genius. Students also need to be challenged with supplementary problems. The University of Alberta has published a companion volume titled *Prof. Scarlet's Notebook* which contains additional examples and exercises as well as mathematics notes. The second volume, *Codes, Puzzles, and Conspiracy*, is also a good source.

One difficulty in using the books as texts is that Dr. Ecco's clients come in no particular order. It is unrealistic to expect that those whose problems are, for example, graph-theoretic in nature should come one after another. To present a coherent plan for study, we have to reorganize the puzzles into topic groups. This difficulty is overcome in the present volume in a most ingenious way.

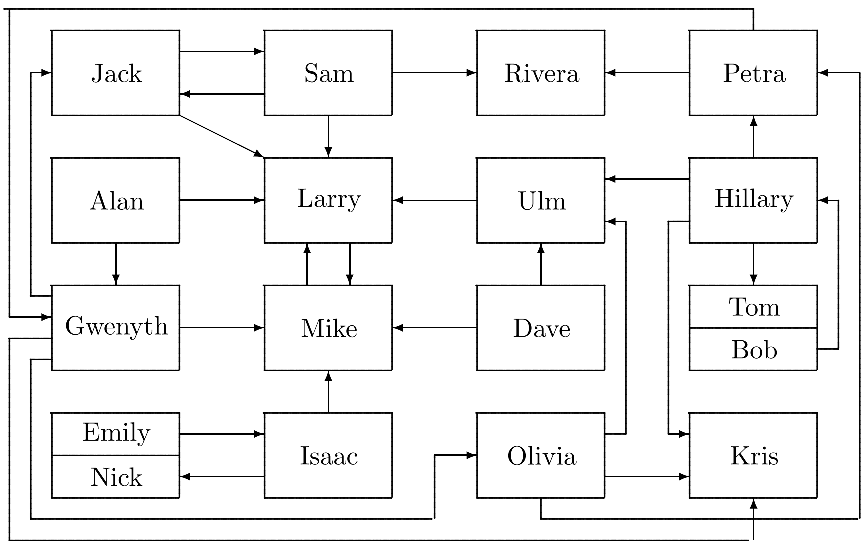
Dr. Evangeline Good, a regular in the first two volumes, makes only a cameo appearance here, but her intellectual contribution is more than adequately replaced by that of Dr. Ecco's precocious niece Liane. Her age varies from 10 to 13 in various puzzles, and not non-decreasingly! It turns out that the puzzles are not given in their correct order, so that problems of a similar nature can be grouped neatly into eight chapters. A bonus puzzle, which involves some decryption, is to recover the chronology of Dr. Ecco's latest cyber-adventures!

Almost all of the problems in the first two volumes arise from theoretical computing science, and while the level of mathematical sophistication varies, it is generally high. This third volume represents a departure in style, primarily because of the intended audience of the source journal.

The book introduces a new type of puzzle, in which the input space is relatively large, so that some form of computation aid is needed. At the same time, a mindless frontal assault is not likely to yield positive results. Such puzzles should appeal to people who are interested in creative computing, and are designated for *Cyberexperts*.

An example is Puzzle 25, in which twelve objects are to be coded with positive integers so that, when the codes of all objects in any subset of size at most four are summed, the resulting sums are distinct. The object is to minimize the largest such sum.

Other puzzles are designated for *Cybernovices*. This means that the reader can realistically expect to solve them with pencil and paper. One example is Puzzle 32, where eighteen people accuse one another of being liars. The data are summarized in the following chart. An arrow pointing from X to Y indicates that X is accusing Y of being a liar. What this means is that at least one of them is a liar. The problem is to determine the minimum number of liars in this group.



In an exclusive interview, the author shared some of his philosophy about omniheurists and the nature of problem-solving:

What is an omniheurist? He or she is a person who can solve all kinds of problems using all techniques available. Much like a shipwreck on an unknown island, an omniheurist must scrounge and invent new functionality from old objects. It is Dr. Ecco's belief (and I agree with him) that omniheurists must be generalists. Their experience must be wide enough to draw upon analogy and common sense as well as mathematics and computation. Further, omniheurists should view the computer as a prosthetic device rather than as a substitute for thinking. Whereas many of the puzzles in this book involve searching through a large space, most require far too much computation time for a simple brute force technique. You must use imagination and creativity first and then ask the computer to do the grunt work. A deep question about puzzles is whether it is a uniquely human ability, kind of like laughing and the ability to speak. I frankly have no idea how to design an experiment to determine whether, say, a bird is solving a puzzle when it builds a nest out of unfamiliar materials or doing something else. Psychologists have rats run mazes, but is that puzzle-solving or simply search with memory? Whatever it is, I'm glad that I'm not a rat.

In Memorium

H.S.M. Coxeter, 1907–2003

On March 31 Donald Coxeter died in his home at age 96, producing beautiful mathematics until his final day. The world has lost a great mathematician, and ***CRUX with MAYHEM*** has lost a great friend. Although his tangible contributions to our journal amounted to only a half-dozen problems and a few letters to the editor, the support and encouragement he gave to Léo Sauvé, the founding editor, proved to be of lasting value. After Léo was replaced by an editorial board, we could still turn to Coxeter for advice on any geometrical matter. Indeed, it was remarkable how he was able to find time for his vast correspondence. Until health problems slowed him down a couple of years ago, he dedicated each morning to answering letters. These letters came from everybody—not just from editors, but from school kids, mathematicians, scientists, artists, amateurs, and professionals. For him, answering letters was not a chore, but rather a way of discovering new ideas. We will all miss him.

An Inequality for a Product of Logarithms, Part II

Walther Janous and Li Zhou

In his recent article [1] Erhard Braune proved an interesting inequality for a certain product of logarithms. Rephrased slightly, the inequality is equivalent to

$$\log\frac{x+1}{x-1}\cdot\log\frac{y+1}{y-1} \;\geq\; \log^2\frac{\frac{x+y}{2}+1}{\frac{x+y}{2}-1} \tag{1}$$

for x, $y > 1$. In his proof, Braune employs the integral representation $\log\frac{x+1}{x-1} = \int_0^x \frac{e^{-xt}}{t}\sinh t\,dt$ and the Cauchy-Schwarz Inequality. In this note, we shall provide an alternative proof and some improvements and generalizations of Braune's Inequality (1). The key method we use is the log-convexity of certain functions, where a positive function $f(x)$ is called *log-convex* if $\log f(x)$ is a convex function of x. Since log-convexity incorporates the properties of logarithm and convexity, it is often useful in proving inequalities involving products, as we shall illustrate in this paper.

1. An Alternative Proof of Braune's Inequality

Consider the function $f(x) = \log\dfrac{x+1}{x-1}$ for $x > 1$. We claim that $f(x)$ is log-convex. To see this, we have to check that the second derivative of the function $\log f(x)$ is non-negative. Because $(\log f(x))'' = \dfrac{4(xf(x)-1)}{((x^2-1)f(x))^2}$ we have to show that the numerator of the last expression is non-negative; that is, $g(x) \geq 0$ for $x > 1$, where $g(x) = f(x) - 1/x$. This is proved by noting that $g'(x) = -\dfrac{x^2+1}{x^2(x^2-1)} < 0$ (implying that $g(x)$ is strictly decreasing for $x > 1$), and $\lim_{x\to\infty} g(x) = 0$.

The log-convexity of $f(x)$ implies that, for x, $y > 1$,

$$\begin{aligned}\log\frac{x+1}{x-1}\cdot\log\frac{y+1}{y-1} &= e^{\log f(x)+\log f(y)}\\ &\geq e^{2\log f(\frac{x+y}{2})} = \log^2\frac{\frac{x+y}{2}+1}{\frac{x+y}{2}-1}\,.\end{aligned}$$

2. Generalizations of Braune's Inequality

The log-convexity of $f(x)$ also yields the following standard generalizations.

a) If λ, $\mu \geq 0$ with $\lambda + \mu = 1$ and x, $y > 1$, then

$$\lambda \log f(x) + \mu \log f(y) \ \geq \ \log f(\lambda x + \mu y) \, .$$

Thus,

$$\log^{\lambda} \frac{x+1}{x-1} \cdot \log^{\mu} \frac{y+1}{y-1} \ \geq \ \log \frac{\lambda x + \mu y + 1}{\lambda x + \mu y - 1} \, .$$

b) We also have the n–variable inequality

$$\sum_{i=1}^{n} \lambda_i \log f(x_i) \ \geq \ \log f(\sum_{i=1}^{n} \lambda_i x_i) \, ;$$

that is,

$$\prod_{i=1}^{n} \log^{\lambda_i} \frac{x_i+1}{x_i-1} \ \geq \ \log \frac{\left(\sum\limits_{i=1}^{n} \lambda_i x_i\right) + 1}{\left(\sum\limits_{i=1}^{n} \lambda_i x_i\right) - 1} \, ,$$

where $n \geq 2$, and $\lambda_1, \ldots, \lambda_n$ are non-negative real numbers satisfying $\sum\limits_{i=1}^{n} \lambda_i = 1$ and $x_1, \ldots, x_n > 1$.

3. Immediate Improvements to Braune's Inequality

In fact, Braune's Inequality belongs to an infinite family of stronger inequalities, namely

$$\log \frac{x+1}{x-1} \cdot \log \frac{y+1}{y-1} \ \geq \ \log^2 \frac{\left(\frac{x^{1/\alpha} + y^{1/\alpha}}{2}\right)^{\alpha} + 1}{\left(\frac{x^{1/\alpha} + y^{1/\alpha}}{2}\right)^{\alpha} - 1} \, , \tag{2}$$

for x, $y > 1$ and $\alpha \geq 1$.

We first observe that (2) becomes increasingly stronger as $\alpha \to \infty$. This is clear, since

$$\left(\frac{x^{1/\alpha} + y^{1/\alpha}}{2}\right)^{\alpha} \ \geq \ \left(\frac{x^{1/\beta} + y^{1/\beta}}{2}\right)^{\beta}$$

for $1 \leq \alpha \leq \beta$, by the Power-Mean Inequality, and $\log \frac{x+1}{x-1}$ is a decreasing function for $x > 1$.

In order to prove (2), we exploit the log-convexity of $f(x) = \log \dfrac{x^\alpha + 1}{x^\alpha - 1}$ for $x > 1$ and $\alpha \geq 1$. This time we have

$$\big(\log f(x)\big)'' \;=\; \frac{2\alpha x^{\alpha-2}\big((\alpha x^{2\alpha} + x^{2\alpha} + \alpha - 1)f(x) - 2\alpha x^\alpha\big)}{\big((x^{2\alpha}-1)f(x)\big)^2}\,.$$

Therefore, we need to show that $g(x) = f(x) - \dfrac{2\alpha x^\alpha}{(\alpha+1)x^{2\alpha} + \alpha - 1} \geq 0$ for $x > 1$ and $\alpha \geq 1$. Indeed,

$$g'(x) \;=\; -\frac{2\alpha x^{\alpha-1}\big((\alpha+1)x^{4\alpha} + (4\alpha^2-2)x^{2\alpha} - (\alpha-1)\big)}{(x^{2\alpha}-1)\big((\alpha+1)x^{2\alpha} + \alpha - 1\big)^2}\,,$$

and $(\alpha+1)x^{4\alpha} + (4\alpha^2-2)x^{2\alpha} - (\alpha-1) > 4\alpha^2 > 0$. We also have $\lim_{x\to\infty} g(x) = 0$. We conclude that $f(x)$ is log-convex. Consequently, for x, $y > 1$ and $\alpha \geq 1$,

$$\begin{aligned}\log\frac{x+1}{x-1}\cdot\log\frac{y+1}{y-1} &= \log\frac{(x^{1/\alpha})^\alpha + 1}{(x^{1/\alpha})^\alpha - 1}\cdot\log\frac{(y^{1/\alpha})^\alpha + 1}{(y^{1/\alpha})^\alpha - 1}\\ &\geq \log^2\frac{\left(\dfrac{x^{1/\alpha}+y^{1/\alpha}}{2}\right)^\alpha + 1}{\left(\dfrac{x^{1/\alpha}+y^{1/\alpha}}{2}\right)^\alpha - 1}\,.\end{aligned}$$

4. Further Improvements to Braune's Inequality

The inequality (2) raises an interesting question. What is the limiting inequality as $\alpha \to \infty$? For fixed x, $y > 1$, $\left(\dfrac{x^{1/\alpha}+y^{1/\alpha}}{2}\right)^\alpha$ decreases to $\sqrt{xy}$ as $\alpha \to \infty$. Thus, we are led naturally to the inequality

$$\log\frac{x+1}{x-1}\cdot\log\frac{y+1}{y-1} \;\geq\; \log^2\frac{\sqrt{xy}+1}{\sqrt{xy}-1} \qquad (3)$$

for x, $y > 1$.

In order to prove this, we consider $f(x) = \log\dfrac{e^x+1}{e^x-1}$ for $x > 0$. Then

$$\big(\log f(x)\big)'' \;=\; \frac{2e^x\big((e^{2x}+1)f(x) - 2e^x\big)}{\big((e^{2x}-1)f(x)\big)^2}\,.$$

If $g(x) = f(x) - \dfrac{2e^x}{e^{2x}+1}$ for $x > 1$, then $g'(x) = -\dfrac{8e^{3x}}{(e^{2x}-1)(e^{2x}+1)^2} < 0$ and $\lim_{x\to\infty} g(x) = 0$. Hence, f is log-convex. Then, for x, $y > 1$,

$$\begin{aligned}\log\frac{x+1}{x-1}\cdot\log\frac{y+1}{y-1} &= \log\frac{e^{\log x}+1}{e^{\log x}-1}\cdot\log\frac{e^{\log y}+1}{e^{\log y}-1}\\ &\geq \log^2\frac{\sqrt{xy}+1}{\sqrt{xy}-1}\,.\end{aligned}$$

5. A Known Inequality

Notice that the inequalities (1), (2), and (3) can also be phrased for $0 < x,\, y < 1$. For example, (3) is equivalent to

$$\log\frac{1+x}{1-x}\cdot\log\frac{1+y}{1-y} \;\geq\; \log^2\frac{1+\sqrt{xy}}{1-\sqrt{xy}} \tag{3'}$$

for $0 < x,\, y < 1$. This reminds us of a known inequality. In [2] (Solution I), it was shown that $f(x) = \log\dfrac{1+\sin e^x}{1-\sin e^x}$ is log-convex for $x < \log(\pi/2)$ and hence,

$$\log\frac{1+\sin u}{1-\sin u}\cdot\log\frac{1+\sin v}{1-\sin v} \;\geq\; \log^2\frac{1+\sin\sqrt{uv}}{1-\sin\sqrt{uv}} \tag{4}$$

for $0 < u,\, v < \pi/2$. The fact that (4) is even stronger than (3′) follows from the claim that $\sin\sqrt{uv} \geq \sqrt{\sin u\cdot\sin v}$. Indeed, define $h(x) = \sin e^x$ for $x < \log(\pi/2)$. Then

$$\big(\log h(x)\big)'' \;=\; \frac{e^x\big(h(x)\cos e^x - e^x\big)}{\big(h(x)\big)^2} \;=\; \frac{e^x\big(\sin(2e^x) - 2e^x\big)}{2\big(h(x)\big)^2} \;<\; 0\,.$$

Thus, h is log-concave, establishing the claim. Interestingly, there is also an alternative proof of (4) ([2] Solution II) using the integral representation $\log\dfrac{1+\sin x}{1-\sin x} = 2\int_0^x \sec t\,dt$ and the Cauchy-Schwarz Inequality. So we have come full circle in our brief excursion.

6. Final Remarks

a) All the improved inequalities (2)–(4) also have, of course, the weighted generalizations of section 2. Interested readers can write them out easily.

b) Recall that $\log\dfrac{1+x}{1-x} = 2\tanh^{-1}x$ for $0 < x < 1$. Therefore, inequalities (1)–(4) are of the form

$$\tanh^{-1}x\cdot\tanh^{-1}y \;\geq\; \Big[\tanh^{-1}\big(S(x,y)\big)\Big]^2$$

for $0 < x,\, y < 1$, where $S(x,y)$ belongs to a certain family of functions symmetric in x and y. A close inspection reveals that in (1)–(4) we have taken $S(x,y)$ to be, respectively,

$$\frac{2xy}{x+y}\,,\quad \left(\frac{2(xy)^{1/\alpha}}{x^{1/\alpha}+y^{1/\alpha}}\right)^{\alpha},\quad \sqrt{xy}\,,\quad \text{and}\quad \sin\sqrt{\sin^{-1}x\cdot\sin^{-1}y}\,.$$

The interested reader might also explore other possibilities; for example,

$$\begin{aligned} S(x,y) &= \tan^{-1}\sqrt{\tan x\cdot\tan y}\,,\\ \text{or}\quad S(x,y) &= \sinh^{-1}\sqrt{\sinh x\cdot\sinh y}\,. \end{aligned}$$

c) Our technique can also be applied to prove the log-convexity of another function f that yields a more distant relative of (1). Consider the function $f : (0,\infty) \to (0,\infty)$ defined by $f(x) = \log\left(a + \frac{b}{x^\alpha}\right)$, where $a \geq 1$ and $b > 0$ are arbitrary real numbers and $\alpha > 0$. Then we have

$$\big(\log f(x)\big)'' = \frac{b\alpha\big((a(\alpha+1)x^\alpha + b)f(x) - b\alpha\big)}{\big(x(ax^\alpha + b)f(x)\big)^2}\,.$$

If $g(x) = f(x) - \dfrac{b\alpha}{a(\alpha+1)x^\alpha + b}$, then $\lim\limits_{x\to\infty} g(x) = \log a \geq 0$ and

$$g'(x) = -\frac{b\alpha\big(a^2(\alpha+1)x^{2\alpha} + ab(\alpha+1)(2-\alpha)x^\alpha + b^2\big)}{x(ax^\alpha + b)\big(a(\alpha+1)x^\alpha + b\big)^2}\,.$$

Therefore, if $a^2(\alpha+1)x^{2\alpha} + ab(\alpha+1)(2-\alpha)x^\alpha + b^2 \geq 0$ for all $x > 0$, then certainly $g(x)$ is a non-increasing function, implying the log-convexity of $f(x)$. This occurs if the discriminant Δ of the left-hand side is non-positive, or in other words, if $\Delta = a^2b^2\alpha^2(\alpha^2 - 2\alpha - 3) \leq 0$; that is, if $0 < \alpha \leq 3$.

7. References

[1] E. Braune, *An Inequality for a Product of Logarithms*, Crux Math. **28** (2002), 223–225.

[2] C.P. Niculescu, L. Zhou and D. Donini, *Problem 1597 and its solutions*, Math. Magazine **74** (2001), 158–159.

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PROBLEMS

Faire parvenir les propositions de problèmes et les solutions à Jim Totten, Département de mathématiques et de statistique, University College of the Cariboo, Kamloops, BC V2C 5N3. Les propositions de problèmes doivent être accompagnées d'une solution ainsi que de références et d'autres indications qui pourraient être utiles à la rédaction. Si vous envoyez une proposition sans solution, vous devez justifier une solution probable en fournissant suffisamment d'information. Un numéro suivi d'une astérisque (⋆) indique que le problème a été proposé sans solution.

Nous sollicitons en particulier des problèmes originaux. Cependant, d'autres problèmes intéressants pourraient être acceptables s'ils ne sont pas trop connus et si leur provenance est précisée. Normalement, si l'auteur d'un problème est connu, il faut demander sa permission avant de proposer un de ses problèmes.

Pour faciliter l'étude de vos propositions, veuillez taper ou écrire à la main (lisiblement) chaque problème sur une feuille distincte de format $8\frac{1}{2}''\times 11''$ *ou* A4, *la signer et la faire parvenir au rédacteur en chef. Les propositions devront lui parvenir au plus tard le* **1er décembre 2003**. *Vous pouvez aussi les faire parvenir par courriel à* **crux-editors@cms.math.ca**. (*Nous apprécierions de recevoir les problèmes et solutions envoyés par courriel au format* LaTeX). *Les fichiers graphiques doivent être de format « epic » ou « eps » (encapsulated postscript). Les solutions reçues après la date ci-dessus seront prises en compte s'il reste du temps avant la publication. Veuillez prendre note que nous n'acceptons pas les propositions par télécopieur.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5, et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6, et 8, le français précédera l'anglais.

Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier et Hidemitsu Saeki, de l'Université de Montréal, d'avoir traduit les problèmes.

2838★. *Proposé par Mohammed Aassila, Strasbourg, France.*

Soit P un polynôme réel à coefficients entiers tels qu'il existe une sous-suite infinie de la suite $\{P(k)\}_{k=1}^{\infty}$ jouissant de la propriété que la sous-suite ne possède qu'un nombre fini de facteurs premiers.

Montrer que P est de la forme $P(x) = (ax+b)^n$.

. . . .

Let P be a real polynomial with integer coefficients such that there is an infinite subsequence of the sequence $\{P(k)\}_{k=1}^{\infty}$ with the property that the subsequence has only finitely many prime divisors.

Prove that P is of the form $P(x) = (ax+b)^n$.

2839. *Proposé par Murray S. Klamkin, Université de l'Alberta, Edmonton, AB.*

Si x, y, et z sont des nombres réels, montrer que

$$\left(x^3+y^3+z^3\right)+3(xyz)^3 \;\geq\; 4\left(y^3z^3+z^3x^3+x^3y^3\right)\,.$$

Déterminer les cas où il y a égalité.

. . . .

Suppose that x, y, and z are real numbers. Prove that

$$\left(x^3+y^3+z^3\right)+3(xyz)^3 \;\geq\; 4\left(y^3z^3+z^3x^3+x^3y^3\right)\,.$$

Determine the cases of equality.

2840. *Proposé par Juan-Bosco Romero Márquez, Université de Valladolid, Valladolid, Espagne.*

Soit A' un point intérieur du côté BC d'un triangle ABC. Si les bissectrices des angles $BA'A$ et $CA'A$ coupent AB et CA en D et E, respectivement, montrer que AA', BE, et CD sont concourants.

. . . .

Let A' be an interior point of the line segment BC in $\triangle ABC$. The interior bisectors of $\angle BA'A$ and $\angle CA'A$ intersect AB and CA at D and E, respectively. Prove that AA', BE, and CD are concurrent.

2841. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Démontrer les inégalités suivantes :

$$\begin{aligned}\frac{\pi}{2}&\left(1-\frac{1}{4n}+\frac{3}{32n^2}-\frac{11}{128n^3}\right)\\ &\leq\;\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\frac{1}{2n+1}\\ &\leq\;\frac{\pi}{2}\left(1-\frac{1}{4n}+\frac{3}{32n^2}-\frac{11}{128n^3}+\frac{83}{2048n^4}\right)\,.\end{aligned}$$

. . . .

Prove the following inequalities:

$$\begin{aligned}\frac{\pi}{2}&\left(1-\frac{1}{4n}+\frac{3}{32n^2}-\frac{11}{128n^3}\right)\\ &\leq\;\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\frac{1}{2n+1}\\ &\leq\;\frac{\pi}{2}\left(1-\frac{1}{4n}+\frac{3}{32n^2}-\frac{11}{128n^3}+\frac{83}{2048n^4}\right)\,.\end{aligned}$$

2842. *Proposé par G. Tsintsifas, Thessalonique, Grèce.*
Soit x_1, x_2, ..., x_n des nombres réels positifs. Montrer que

(a) $$\frac{\sum_{k=1}^{n} x_k^n}{n \prod_{k=1}^{n} x_k} + \frac{n \left(\prod_{k=1}^{n} x_k \right)^{\frac{1}{n}}}{\sum_{k=1}^{n} x_k} \geq 2,$$

(b) $$\frac{\sum_{k=1}^{n} x_k^n}{\prod_{k=1}^{n} x_k} + \frac{\left(\prod_{k=1}^{n} x_k \right)^{\frac{1}{n}}}{\sum_{k=1}^{n} x_k} \geq 1.$$

. . . .

Let x_1, x_2, ..., x_n be positive real numbers. Prove that

(a) $$\frac{\sum_{k=1}^{n} x_k^n}{n \prod_{k=1}^{n} x_k} + \frac{n \left(\prod_{k=1}^{n} x_k \right)^{\frac{1}{n}}}{\sum_{k=1}^{n} x_k} \geq 2,$$

(b) $$\frac{\sum_{k=1}^{n} x_k^n}{\prod_{k=1}^{n} x_k} + \frac{\left(\prod_{k=1}^{n} x_k \right)^{\frac{1}{n}}}{\sum_{k=1}^{n} x_k} \geq 1.$$

2843. *Proposé par Bektemirov Baurjan, étudiant, Aktobe, Kazakstan.*
On suppose que $2\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 1 + \frac{x}{yz} + \frac{y}{zx} + \frac{z}{xy}$ pour des réels positifs x, y, z. Montrer que

$$(1 - x)(1 - y)(1 - z) \leq \frac{1}{64}.$$

. . . .

Suppose that $2\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 1 + \frac{x}{yz} + \frac{y}{zx} + \frac{z}{xy}$ for positive real x, y, z. Prove that

$$(1 - x)(1 - y)(1 - z) \leq \frac{1}{64}.$$

2844. *Proposé par Mihály Bencze, Brasov, Roumanie.*

On suppose que la suite $\{x_n\}$ satisfait $\sum\limits_{k=1}^{n} \frac{1}{k} - \ln(n + x_n) = \gamma$, où γ est la constante d'Euler.

(a) Montrer que $\{x_n\}$ est convergente et que $\lim\limits_{n\to\infty} x_n = \frac{1}{2}$.

(b) Trouver, pour le terme général x_n, une approximation asymptotique dont l'erreur est $O\left(\frac{1}{n^2}\right)$.

. . . .

Suppose that the sequence $\{x_n\}$ satisfies $\sum\limits_{k=1}^{n} \frac{1}{k} - \ln(n + x_n) = \gamma$, where γ is Euler's constant.

(a) Prove that $\{x_n\}$ is convergent and that $\lim\limits_{n\to\infty} x_n = \frac{1}{2}$.

(b) Determine an asymptotic approximation for the general term x_n, with an error that is $O\left(\frac{1}{n^2}\right)$.

2845. *Proposé par G. Tsintsifas, Thessalonique, Grèce.*

Soit Q un carré de côté 1 et soit S un ensemble fini de carrés dont la somme des aires est égale à $\frac{1}{2}$.

Montrer que l'ensemble S peut former une partie d'un carrelage du carré Q.

. . . .

Let Q be a square of side length 1, and let S be a set consisting of a finite number of squares such that the sum of their areas is $\frac{1}{2}$.

Prove that the set S can be packed inside the square Q.

2846. *Proposé par G. Tsintsifas, Thessalonique, Grèce.*

Un simplexe régulier $S_n = A_1A_2A_3\ldots A_{n+1}$ est inscrit dans la sphère unité Σ de $\mathbb{E}^n$. Soit O l'origine de $\mathbb{E}^n$, $M \in \Sigma$, $u_k = \overrightarrow{OA_k}$ et $v = \overrightarrow{OM}$.

Trouver la valeur maximale de $\sum\limits_{k=1}^{n+1} |u_k \cdot v|$.

. . . .

A regular simplex $S_n = A_1A_2A_3\ldots A_{n+1}$ is inscribed in the unit sphere Σ in $\mathbb{E}^n$. Let O be the origin in $\mathbb{E}^n$, $M \in \Sigma$, $u_k = \overrightarrow{OA_k}$ and $v = \overrightarrow{OM}$.

Find the maximum value of $\sum\limits_{k=1}^{n+1} |u_k \cdot v|$.

2847. *Proposé par G. Tsintsifas, Thessalonique, Grèce.*

Le *cercle-intérieur* inscrit dans un tétrahèdre est un cercle de rayon maximal inscrit dans ce tétrahèdre pour toutes les directions possibles dans $\mathbb{E}^3$.

Trouver le rayon du cercle-intérieur d'un tétrahèdre régulier.

....

The *inscircle* inscribed in a tetrahedron is a circle of maximum radius inscribed in the tetrahedron, considering every possible orientation in $\mathbb{E}^3$.

Find the radius of the inscircle of a regular tetrahedron.

2848. *Proposé par Murray S. Klamkin, Université de l'Alberta, Edmonton, AB et K.R.S. Sastry, Bangalore, Inde.*

On suppose que A, B, and C sont les angles d'un triangle ABC et que ω est son angle de Crelle-Brocard. Montrer que $A + \omega = \frac{\pi}{2}$ si et seulement si $\operatorname{tg} C$, $\operatorname{tg} A$, et $\operatorname{tg} B$, pris dans cet ordre, sont en progression géométrique.

....

Suppose that A, B, and C are the angles of $\triangle ABC$ and that ω is its Crelle-Brocard angle. Prove that $A + \omega = \frac{\pi}{2}$ if and only if $\tan\ C$, $\tan\ A$, $\tan\ B$ are in geometric progression in that order.

2849. *Proposé par Toshio Seimiya, Kawasaki, Japon.*

Dans un quadrilatère convexe $ABCD$, on a $\angle ABC = \angle BCD = 120°$. Montrer que si $AB^2 + BC^2 + CD^2 = AD^2$, alors $ABCD$ possède un cercle inscrit.

....

In a convex quadrilateral $ABCD$, we have $\angle ABC = \angle BCD = 120°$. Suppose that $AB^2 + BC^2 + CD^2 = AD^2$.

Prove that $ABCD$ has an inscribed circle.

2850. *Proposé par D.J. Smeenk, Zaltbommel, Pays-Bas.*

Trouver toutes les solutions entières de l'équation

$$x^2 - 4xy + 6y^2 - 2x - 20y \;=\; 29\,.$$

....

Find all integral solutions of

$$x^2 - 4xy + 6y^2 - 2x - 20y = 29\,.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2732. [2002 : 178] *Proposed by Mihály Bencze, Brasov, Romania.*

Let ABC be a triangle with sides a, b, c, medians m_a, m_b, m_c, altitudes h_a, h_b, h_c, and area Δ. Prove that

$$a^2+b^2+c^2 \;\geq\; 4\sqrt{3}\Delta \max\left\{\frac{m_a}{h_a},\frac{m_b}{h_b},\frac{m_c}{h_c}\right\}\,.$$

Solution by Michel Bataille, Rouen, France.

Without loss of generality, assume that $\frac{m_a}{h_a} = \max\left\{\frac{m_a}{h_a},\frac{m_b}{h_b},\frac{m_c}{h_c}\right\}$. Since $m_a=\frac{1}{2}\sqrt{2b^2+2c^2-a^2}$ and $\Delta=\frac{1}{2}ah_a$, the inequality

$$\frac{m_a}{h_a} \;\leq\; \frac{a^2+b^2+c^2}{4\sqrt{3}\Delta}$$

is successively equivalent to

$$\begin{aligned}
a\sqrt{2b^2+2c^2-a^2} &\;\leq\; \frac{a^2+b^2+c^2}{\sqrt{3}}\,,\\
3a^2(2b^2+2c^2-a^2) &\;\leq\; (a^2+b^2+c^2)^2\,,\\
4a^4+b^4+c^4-4a^2b^2-4a^2c^2+2b^2c^2 &\;\geq\; 0\,,\\
(b^2+c^2-2a^2)^2 &\;\geq\; 0\,.
\end{aligned}$$

Since the last inequality is clearly true, the result follows.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; PIERRE BORNSZTEIN, Pontoise, France; NIKOLAOS DERGIADES, Thessaloniki, Greece; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; VEDULA N. MURTY, Dover, PA, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2738. [2002 : 180] *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Let x, y and z be positive real numbers satisfying $x^2+y^2+z^2=1$. Prove that

$$\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2} \;\geq\; \frac{3\sqrt{3}}{2}\,.$$

Solution by Bogdan Ioniţă and Titu Zvonaru, Bucharest, Romania.
We will prove that if $x > 0$, then

$$x(1-x^2) \;\le\; \frac{2}{3\sqrt{3}}\,. \tag{1}$$

Setting $x = \frac{t}{\sqrt{3}}$, we have

$$\begin{aligned} x(1-x^2) \;\le\; \frac{2}{3\sqrt{3}} &\iff \frac{t}{\sqrt{3}}\left(1-\frac{t^2}{3}\right) \;\le\; \frac{2}{3\sqrt{3}} \\ &\iff 3t\left(1-\frac{t^2}{3}\right) \;\le\; 2 \\ &\iff t^3-3t+2 \;\ge\; 0 \\ &\iff (t-1)^2(t+2) \;\ge\; 0\,, \end{aligned}$$

which is true.

By (1) it follows that:

$$\frac{x}{1-x^2} \;\ge\; \frac{3\sqrt{3}}{2}x^2\,, \qquad \frac{y}{1-y^2} \;\ge\; \frac{3\sqrt{3}}{2}y^2\,, \qquad \frac{z}{1-z^2} \;\ge\; \frac{3\sqrt{3}}{2}z^2\,.$$

Thus,

$$\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2} \;\ge\; \frac{3\sqrt{3}}{2}(x^2+y^2+z^2) \;=\; \frac{3\sqrt{3}}{2}\,,$$

with equality if and only if $x^2 = y^2 = z^2 = 1/3$.

[*Editor's note*: Vasile Cartoaje, University of Ploiesti, Romania, presented the following generalization in his solution. Let $a_1, a_2, \ldots, a_n < 1$ be non-negative real numbers such that $a = \sqrt{\frac{a_1^2+\cdots+a_n^2}{n}} \ge \frac{\sqrt{3}}{3}$. Then

$$\frac{a_1}{1-a_1^2}+\frac{a_2}{1-a_2^2}+\cdots+\frac{a_n}{1-a_n^2} \;\ge\; \frac{na}{1-a^2}\,.]$$

Also solved by MICHEL BATAILLE, Rouen, France; PIERRE BORNSZTEIN, Pontoise, France; PAUL BRACKEN, CRM, Université de Montréal, Montréal, QC; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; M.R. MODAK, Pune, India; ZENOFON PAPANICOLAOU, Athens, Greece; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2739. [2002 : 244] *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Suppose that a, b and c are positive real numbers. Prove that

$$\frac{\sqrt{a+b+c}+\sqrt{a}}{b+c}+\frac{\sqrt{a+b+c}+\sqrt{b}}{c+a}+\frac{\sqrt{a+b+c}+\sqrt{c}}{a+b} \geq \frac{9+3\sqrt{3}}{2\sqrt{a+b+c}}\,.$$

Solution by M.R. Modak, S.P. College, Pune, India.

Let $s=a+b+c$ and $x=\frac{a}{s}$, $y=\frac{b}{s}$, $z=\frac{c}{s}$. Then $0<x,\,y,\,z<1$, $x+y+z=1$, and the given inequality can be written as

$$\frac{\sqrt{s}+\sqrt{a}}{b+c}+\frac{\sqrt{s}+\sqrt{b}}{c+a}+\frac{\sqrt{s}+\sqrt{c}}{a+b} \geq \frac{9+3\sqrt{3}}{2\sqrt{s}}\,, \qquad \text{or}$$

$$\frac{s+\sqrt{sa}}{b+c}+\frac{s+\sqrt{sb}}{c+a}+\frac{s+\sqrt{sc}}{a+b} \geq \frac{9+3\sqrt{3}}{2}\,. \qquad (1)$$

Dividing each numerator and denominator on the left by s, (1) becomes

$$\frac{1+\sqrt{x}}{y+z}+\frac{1+\sqrt{y}}{z+x}+\frac{1+\sqrt{z}}{x+y} \geq \frac{9+3\sqrt{3}}{2}\,, \qquad \text{or}$$

$$\frac{1+\sqrt{x}}{1-x}+\frac{1+\sqrt{y}}{1-y}+\frac{1+\sqrt{z}}{1-z} \geq \frac{9+3\sqrt{3}}{2}\,. \qquad (2)$$

By the Cauchy-Schwarz Inequality [*Ed*: Or simply the AM-HM Inequality], we have

$$[(1-x)+(1-y)+(1-z)]\left(\frac{1}{1-x}+\frac{1}{1-y}+\frac{1}{1-z}\right) \geq (1+1+1)^2$$

$$\text{or} \qquad \frac{1}{1-x}+\frac{1}{1-y}+\frac{1}{1-z} \geq \frac{9}{2}\,, \qquad (3)$$

since $(1-x)+(1-y)+(1-z)=3-(x+y+z)=2$. Also, replacing x, y, and z by $\sqrt{x}$, $\sqrt{y}$, and $\sqrt{z}$, respectively, in Problem 2738 above, we have

$$\frac{\sqrt{x}}{1-x}+\frac{\sqrt{y}}{1-y}+\frac{\sqrt{z}}{1-z} \geq \frac{3\sqrt{3}}{2} \qquad (4)$$

Adding (3) and (4), we get (2), and the proof is complete.

Also solved by MICHEL BATAILLE, Rouen, France; PIERRE BORNSZTEIN, Pontoise, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JIMMY CHUI, student, University of Toronto, Toronto, Ontario; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; JOE HOWARD, Portales, NM, USA; BOGDAN IONIŢĂ and TITU ZVONARU, Bucharest, Romania; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK (2 solutions); VEDULA N. MURTY, Dover, PA, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

As is always the case with inequalities, there are a variety of proof techniques used among the submitted solutions. These include calculus, convexity, Jensen's Inequality, weighted Jensen's Inequality, Cauchy-Schwarz Inequality, Lagrange Multipliers, and majorization!

2740. [2002 : 245] *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

In the plane are given three ellipses, E_1, E_2 and E_3. The points A, B and C satisfy the following conditions:

> A and B are the foci of E_1, B and C are the foci of E_2, C and A are the foci of E_3, C is on E_1, A is on E_2, B is on E_3.

With only an unmarked straightedge, construct the incentre of $\triangle ABC$.

Combination of solutions by Michel Bataille, Rouen, France and Peter Y. Woo, Biola University, La Mirada, CA, USA.

Pascal's Theorem allows one to construct the tangent to a conic at a point P given four other points Q, R, S, T of the conic — the tangent at P is the line joining P to the point where

$$RS \text{ meets the line that joins } TP \cdot QR \text{ to } ST \cdot PQ\,.$$

For our construction we are given all points on each conic in addition to the foci. Thus, we can construct the tangent to E_1 at C, to E_2 at A, and to E_3 at B (using the straightedge seven times for each tangent). These three lines form a triangle $A'B'C'$, labeled so that A' is opposite A, etc. Moreover, since A and B are the foci of E_1, and so on, the three tangents are the external angle bisectors of triangle ABC. As a consequence, the lines BB' and CC' intersect at the incentre of triangle ABC.

Also solved by VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; and the proposer.

2742. [2002 : 245] *Proposed by Manuel Murillo Tsijli, Instituto Tecnológico de Costa Rica, Cartago, Costa Rica.*

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) \;=\; \begin{cases} 5x & \text{if} \quad x \le \frac{1}{2}\,, \\ 5-5x & \text{if} \quad x > \frac{1}{2}\,. \end{cases}$$

Let $f^1(x) = f(x)$ and $f^n(x) = f\left(f^{n-1}(x)\right)$ for $n \ge 2$. Calculate the exact value of $f^{1998}\left(\dfrac{4}{5^{16}-1} + \dfrac{4}{5^{125}-1}\right)$.

Solution by Michel Bataille, Rouen, France.

For each non-negative integer k, we will let $a_k = 5^k \cdot \dfrac{4}{5^{16}-1}$ and $b_k = 5^k \cdot \dfrac{4}{5^{125}-1}$. We will prove that if $1 \le k \le 1999$ and k is not divisible by 16 nor by 125, then

$$f^k(a_0 + b_0) = a_r + b_s\,, \tag{R_k}$$

where $k \equiv r \pmod{16}$, $0 < r < 16$, and $k \equiv s \pmod{125}$, $0 < s < 125$. This result gives the answer:

$$f^{1998}\left(\frac{4}{5^{16}-1}+\frac{4}{5^{125}-1}\right) = 4\left(\frac{5^{14}}{5^{16}-1}+\frac{5^{123}}{5^{125}-1}\right).$$

First, we make the following two remarks:

(1) If $0 \le k \le 14$ and $0 \le \ell \le 123$, then

$$a_k = \frac{5^k}{1+5+\cdots+5^{15}} < \frac{5^k}{5^{15}} \le \frac{5^{14}}{5^{15}} = \frac{1}{5},$$

and likewise, $b_\ell < \frac{1}{5}$. Then $a_k + b_\ell < \frac{1}{2}$.

(2) $a_{15} > \frac{1}{2}$, since this inequality is equivalent to $8 > 5 - \frac{1}{5^{15}}$, which is clearly true. Furthermore,

$$5(1-a_{15}) = 5\cdot\frac{5^{15}-1}{5^{16}-1} = 1-\frac{4}{5^{16}-1} = 1-a_0.$$

Similarly, $b_{124} > \frac{1}{2}$ and $5(1-b_{124}) = 1-b_0$.

We will now show that if (R_k) is true for some positive integer $k \le 1998$ which is neither a multiple of 16 nor a multiple of 125, then $(R_{k'})$ is true, where k' is the next integer larger than k satisfying the same condition. The general result will follow, since (R_1) is obviously true.

Let k be such an integer as we have just described for which (R_k) is true. Observe that we cannot have both $r = 15$ and $s = 124$ simultaneously, since 2000 is the least common multiple of 16 and 125, and $k+1 < 2000$. There are five mutually exclusive possibilities, which we will now study in turn.

(a) $r \le 14$ and $s \le 123$. Then $k' = k+1$, and $a_r + b_s < \frac{1}{2}$, (remark (1)). Hence, $f^{k'}(a_0+b_0) = f(a_r+b_s) = 5(a_r+b_s) = a_{r+1}+b_{s+1}$. Thus, $(R_{k'})$ is true.

(b) $r = 15$ and $s \le 122$. Then $k' = k+2$, and $a_{15}+b_s > \frac{1}{2}$, (remark (2)). Hence,

$$\begin{aligned} f^{k+1}(a_0+b_0) &= 5(1-a_{15}-b_s) = 5(1-a_{15})-b_{s+1} \\ &= 1-(a_0+b_{s+1}). \end{aligned}$$

Since $a_0 + b_{s+1} < \frac{1}{2}$ (remark (1)), we have

$$f^{k'}(a_0+b_0) = 5[1-(1-(a_0+b_{s+1}))] = a_1+b_{s+2}.$$

This is $(R_{k'})$.

(c) $r \le 13$ and $s = 124$. This case is similar to (b).

(d) $r = 15$ and $s = 123$. This case occurs only for $k = 623$, as is easily checked. Then $k' = 626$. We calculate

$$\begin{aligned} f^{624}(a_0 + b_0) &= 5(1 - a_{15} - b_{123}) = 1 - (a_0 + b_{124}) < \tfrac{1}{2}, \\ f^{625}(a_0 + b_0) &= 5(1 - b_{124} - a_0) = 1 - (b_0 + a_1) > \tfrac{1}{2}, \\ f^{626}(a_0 + b_0) &= 5[1 - (1 - (b_0 + a_1))] = a_2 + b_1 . \end{aligned}$$

Thus, $(R_{k'})$ is true.

(e) $r = 14$ and $s = 124$. This case occurs only for $k = 1374$. It can be treated as in (d).

The proof is now complete.

Also solved by CHARLES DIMINNIE and ROGER ZARNOWSKI, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. Computer solutions were submitted by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and STAN WAGON, Macalester College, St. Paul, MN, USA. There were two incorrect solutions.

The behaviour of the sequence of functional iterates $f^k(a_0 + b_0)$ changes drastically when $k = 1999$. The sequence, hitherto confined to the interval $(0, 1)$, abruptly breaks out of this interval and diverges to $-\infty$. This was noted by Diminnie and Zarnowski, and by Zhou.

2743. [2002 : 246] *Proposed by Péter Ivády, Budapest, Hungary.*
Show that, for x, $y \in \left(0, \frac{\pi}{2}\right)$,

$$\left(\frac{x}{\sin x} + \frac{y}{\sin y}\right) \cos\left(\frac{x}{2}\right) \cos\left(\frac{y}{2}\right) < 2 .$$

Solution by Michel Bataille, Rouen, France.

Let $u = \dfrac{x}{2}$ and $v = \dfrac{y}{2}$. Then u, $v \in (0, \pi/4)$ and the stated inequality easily transforms into

$$4 \sin u \sin v - u \sin 2v - v \sin 2u > 0 . \qquad (1)$$

Assume for the moment the following two claims:

(2) if a, $b \in (0, \pi/4)$, then $2a \cos b - \sin a > 0$;

(3) if $t \in (0, \pi/4)$, then $4 \sin t - \sin 2t - 2t > 0$.

Consider the function $f(u) = 4 \sin u \sin v - u \sin 2v - v \sin 2u$, where $u \in [0, \pi/4)$ and v is an arbitrary fixed number in $(0, \pi/4)$. Differentiating twice, we get

$$\begin{aligned} f'(u) &= 4 \cos u \sin v - \sin 2v - 2v \cos 2u , \\ f''(u) &= 4 \sin u (2v \cos u - \sin v) . \end{aligned}$$

Using (2), we see that $f''(u) > 0$; whence, we have f' increasing. By (3), $f'(0) = 4\sin v - \sin 2v - 2v > 0$; hence, $f'(0) > 0$ for $u \in [0, \pi/4)$. Thus, f is increasing as well and (1) follows, since $f(0) = 0$.

It remains to prove (2) and (3):

(2) Consider $g(t) = 2t\cos b - \sin t$ for $t \in [0, \pi/4)$. Then

$$g'(t) \;=\; 2\cos b - \cos t \;>\; \sqrt{2} - 1 \;>\; 0\,,$$

so that g is increasing. Then (2) follows since $g(0) = 0$.

(3) The method is similar with $h(t) = 4\sin t - \sin 2t - 2t$, whose derivative $h'(t) = 4\cos t(1 - \cos t)$ is positive.

[*Editor's note*: Li Zhou notes that this problem (also proposed by Péter Ivády) appears with a solution by John Spellmann in *Math. Magazine*, **76**, 1 (Feb. 2003), 70–71.]

Also solved by CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OVIDIU FURDUI, Western Michigan University, Kalamazoo, MI, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There were 3 incorrect solutions.

2744. [2002 :246] *Proposed by K.R.S. Sastry, Bangalore, India.*

The cevians AD and BE of $\triangle ABC$ intersect at an interior point K. Assume that $\frac{AK}{KD} = \frac{BK}{KE} = \lambda \neq \pm 1$.

Show that $AD^2 + BE^2 = \left(\frac{2}{\lambda}\right) AB^2 + \left(\frac{\lambda-1}{\lambda}\right)^2 (BC^2 + CA^2)$.

Composite of solutions by David Loeffler, student, Trinity College, Cambridge, UK; D.J. Smeenk, Zaltbommel, the Netherlands; and Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.

Let a, b, and c denote the side lengths of the triangle. From $\dfrac{AK}{KD} = \dfrac{BK}{KE}$ it follows that the triangles ABK and DEK are similar. This implies that $\angle ABK = \angle DEK$ and thus $DE \parallel AB$. Hence, we have $DE = \dfrac{AB}{\lambda} = \dfrac{c}{\lambda}$, $CE = \dfrac{CA}{\lambda} = \dfrac{b}{\lambda}$ and $CD = \dfrac{CB}{\lambda} = \dfrac{a}{\lambda}$. Applying Stewart's Theorem to the cevian AD, we get

$$a\left(AD^2 + \frac{a}{\lambda}\left(a - \frac{a}{\lambda}\right)\right) \;=\; c^2\frac{a}{\lambda} + b^2\left(a - \frac{a}{\lambda}\right)$$

$$\text{or} \qquad AD^2 \;=\; \frac{c^2}{\lambda} + b^2\left(1 - \frac{1}{\lambda}\right) - \frac{a}{\lambda}\left(a - \frac{a}{\lambda}\right)\,. \tag{1}$$

Similarly, by applying Stewart's Theorem to the cevian BE, we get

$$BE^2 \;=\; \frac{c^2}{\lambda} + a^2\left(1 - \frac{1}{\lambda}\right) - \frac{b}{\lambda}\left(b - \frac{b}{\lambda}\right) \tag{2}$$

Adding (1) and (2) leads to

$$\begin{aligned} AD^2 + BE^2 &= \frac{2c^2}{\lambda} + \left(1 - \frac{1}{\lambda}\right)\left(a^2 + b^2 - \frac{a^2+b^2}{\lambda}\right) \\ &= \frac{2c^2}{\lambda} + \left(1 - \frac{1}{\lambda}\right)^2 (a^2 + b^2) \\ &= \frac{2}{\lambda} AB^2 + \left(\frac{\lambda - 1}{\lambda}\right)^2 (BC^2 + CA^2)\,, \end{aligned}$$

as desired.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (two solutions); MICHEL BATAILLE, Rouen, France; PIERRE BORNSZTEIN, Pontoise, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU and BOGDAN IONIŢĂ, Bucharest, Romania; and the proposer.

2746. [2002 : 246] *Proposed by K.R.S. Sastry, Bangalore, India.*

In triangle ABC, the sides are in arithmetic progression with $AB + BC = 2AC$. The median AD intersects the Gergonne cevian BE (that is, the line segment from B to the contact point E of the incircle with AC) at the point S.

Prove that $\triangle ABC$ is similar to a rational Heron triangle (that is, one with rational sides and area) if and only if $\frac{AS}{SD}$ is one-sixth of a rational square.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Let $x = \dfrac{AS}{SD}$ and $y = \dfrac{BS}{SE}$. As is customary, we will denote BC by a, CA by b, and AB by c. Let $s = \frac{1}{2}(a+b+c)$ be the semiperimeter. We will let $[XYZ]$ represent the area of a triangle XYZ. Set $[DSE] = t$. Then $[ASE] = xt$, $[BSD] = yt$, and $[ASB] = xyt$. Since D is the mid-point of BC, we have $[CDE] = [BDE] = (y+1)t$. Consequently,

$$(x+y+2)t \;=\; [ACD] \;=\; [ABD] \;=\; (xy+y)t\,.$$

Thus, $y = 1 + \dfrac{2}{x}$. Now

$$\frac{x+1}{2\left(1+\frac{1}{x}\right)} \;=\; \frac{(x+1)t}{(y+1)t} \;=\; \frac{AE}{EC} \;=\; \frac{s-a}{s-c} \;=\; \frac{3c-a}{3a-c}\,,$$

which yields $x = \dfrac{2(3c-a)}{3a-c}$.

In [1], Heron triangles in which $a + c = 2b$ are characterized by

$$a : b : c \;=\; (m^2 + 9n^2) : 2(m^2 + 3n^2) : 3(m^2 + n^2)\,.$$

Hence, if ABC is similar to a Heron triangle, then

$$x = \frac{2[9(m^2+n^2)-(m^2+9n^2)]}{3(m^2+9n^2)-3(m^2+n^2)} = \frac{1}{6}\left(\frac{2m}{n}\right)^2 .$$

Conversely, if $x = \frac{1}{6}\left(\frac{p}{q}\right)^2$, then $\frac{2(3c-a)}{3a-c} = \frac{p^2}{6q^2}$, which yields $\frac{a}{c} = \frac{p^2+9(2q)^2}{3[p^2+(2q)^2]}$, and thus,

$$a : b : c = [p^2+9(2q)^2] : 2[p^2+3(2q)^2] : 3[p^2+(2q)^2] .$$

Reference

[1] K.R.S. Sastry, Heron triangles: an incenter perspective, *Math. Magazine*, Vol 73, No 5, Dec 2000, 388–392.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TITU ZVONARU and BOGDAN IONIŢĂ, Bucharest, Romania; and the proposer. There was one incomplete solution.

Titu Zvonaru and Bogdan Ioniţă claim the result can be generalized as follows:
If $AB + BC = nAC$, where n is a positive rational number greater than 1, then AS/SD must be the square of a rational number divided by $2(n+1)(n-1)$.

2747. [2002 : 247] *Proposed by K.R.S. Sastry, Bangalore, India.*

[Corrected] Prove that the orthocentre of an acute-angled triangle lies inside or on the incircle according as the inradius is less than or equal to the mean proportional between the two segments of an altitude formed by the orthocentre.

Combination of solutions by Murray S. Klamkin, University of Alberta, Edmonton, AB and D.J. Smeenk, Zaltbommel, the Netherlands.

As usual, we denote the orthocentre by H, the incentre by I, and the inradius by r. We let AD be the altitude from A to BC. The result follows quickly from three basic formulas:

$IH^2 = 2r^2 - 4R^2\cos A\cos B\cos C$	([1], 5.8)
$AH = 2R\cos A$	([2] Theorem 252e, p. 162)
$HD = 2R\cos B\cos C$	([2] Theorem 252g, p. 162)

By definition, the orthocentre of a triangle lies inside or on the incircle according as $IH^2 < r^2$ or $IH^2 = r^2$. When the triangle is acute (so that the cosines of all three angles are positive), these conditions are equivalent to

$$r^2 \le 4R^2\cos A\cos B\cos C .$$

Since the mean proportional between the segments of an altitude is

$$\sqrt{AH \times HD} = \sqrt{4R^2\cos A\cos B\cos C} ,$$

the result now follows.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

The proposer's orginal statement of the problem was correct, but the editors misinterpreted his intention while trying to clarify the problem. In addition to the interpretation used in the featured solution, solvers dealt with the misinterpretation in two other ways:

1. *Bradley, Janous, and Zhou showed that the original statement of the problem could not be correct, and*
2. *the others correctly guessed the proposer's intention and proved the case of equality.*

It was Bataille alone who noticed that the triangle must be acute-angled.

Johnson, in [2], noted that if AH is extended to meet the circumcircle of triangle ABC at H', then $HD = DH'$, so that our quantity $AH \times HD$ equals $(1/2)AH \times HH'$, which is half the power of H with respect to the circumcircle (and once again we see that the products of the segments of the respective altitudes are equal).

References

[1] O. Bottema, R.Ž. Djordjević, R.R. Janić, D.S. Mitrinović & P.M. Vasić, *Geometric Inequalities*, Groningen, 1969

[2] Roger A. Johnson, *Advanced Euclidean Geometry*, Dover, 1960.

2748★. [2002 : 247] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let a_1, a_2, ..., a_n $(n \geq 1)$ be non-negative real numbers such that $a_1 \leq a_2 \leq \ldots \leq a_n$ and $\sum\limits_{k=1}^{n} a_k = 1$.

Determine the least upper bound of $a_n \sum\limits_{k=1}^{n} (n+1-k)a_k$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Let $S_n = a_n \sum_{k=1}^{n}(n+1-k)a_k$ and let L_n be the least upper bound of S_n. Clearly, $L_1 = 1$. Suppose that $n \geq 2$. If $a_j - a_i = d > 0$ for some i, j with $1 \leq i < j \leq n-1$, then we can replace a_i by $a_i + \frac{d}{2}$ and a_j by $a_j - \frac{d}{2}$ to achieve a larger value of S_n. Thus it suffices to consider the case $a_1 = a_2 = \ldots = a_{n-1}$. Then

$$\begin{aligned} S_n &= a_n \left[\frac{n+2}{2}(n-1)a_1 + a_n\right] = a_n \left[\frac{n+2}{2}(1-a_n) + a_n\right] \\ &= \frac{n}{2} a_n \left(\frac{n+2}{n} - a_n\right), \end{aligned}$$

which achieves a maximum $L_n = \dfrac{(n+2)^2}{8n}$ when $a_n = \dfrac{n+2}{2n}$ (while $a_1 = a_2 = \ldots = a_{n-1} = \dfrac{n-2}{2n(n-1)}$).

Also solved by MICHEL BATAILLE, Rouen, France; PIERRE BORNSZTEIN, Pontoise, France; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; and the proposer. There were two incorrect solutions submitted.

Loeffler noticed that there appears to be some confusion as to whether the question asks for least upper bound (as in the English version), or greatest lower bound (l'infimum, as in the French version). He then proceeded to find both lub *and* glb. *All of the other solvers have found the least upper bound only.*

2749. [2002 : 247] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Suppose that P is an interior point of $\triangle ABC$. The line through P parallel to AB meets BC at L and CA at M'. The line through P parallel to BC meets CA at M and AB at N'. The line through P parallel to CA meets AB at N and BC at L'.

Prove that

(a) $\left(\frac{BL}{LC}\right)\left(\frac{CM}{MA}\right)\left(\frac{AN}{NB}\right)\left(\frac{BL'}{L'C}\right)\left(\frac{CM'}{M'A}\right)\left(\frac{AN'}{N'B}\right) = 1$;

(b) $\left(\frac{BL}{LC}\right)\left(\frac{CM}{MA}\right)\left(\frac{AN}{NB}\right) \le \frac{1}{8}$;

(c) $[LMN] = [L'M'N']$; [Note: $[XYZ]$ denotes the area of $\triangle XYZ$.]

(d) $[LMN] \le \frac{[ABC]}{3}$.

Locate the point P when equality holds in parts (b) and (d).

Solution by David Loeffler, student, Trinity College, Cambridge, UK.

Suppose that P has areal coordinates $\lambda : \mu : \nu$, where $\lambda + \mu + \nu = 1$.

Let D be the foot of the perpendicular from P to BC. Then, by definition, we have

$$\tfrac{1}{2}aPD = [BCP] = \lambda[ABC] = \tfrac{1}{2}ah_a\,,$$

so that $PD = \lambda h_a$. Thus, $\triangle PLL'$ (which is clearly similar to ABC since the corresponding sides are all parallel) has its altitude a factor of λ smaller than that of ABC, and hence, its sides are in the same proportion.

We thus have

$$\begin{array}{lllll} BN' = PL = \lambda c\,, & & CM = PL' = \lambda b\,,\\ CL' = PM = \mu a\,, & & AN = PM' = \mu c\,,\\ AM' = PN = \nu b\,, & & BL = PN' = \nu a\,. \end{array}$$

(a)

$$\left(\frac{BL}{LC}\right)\left(\frac{CM}{MA}\right)\left(\frac{AN}{NB}\right)\left(\frac{BL'}{L'C}\right)\left(\frac{CM'}{M'A}\right)\left(\frac{AN'}{N'B}\right)$$
$$= \frac{\nu a}{(1-\nu)a}\frac{\lambda b}{(1-\lambda)b}\frac{\mu c}{(1-\mu)c}\frac{(1-\mu)a}{\mu a}\frac{(1-\nu)b}{\nu b}\frac{(1-\lambda)c}{\lambda c} = 1\,.$$

(b) We have

$$\left(\frac{BL}{LC}\right)\left(\frac{CM}{MA}\right)\left(\frac{AN}{NB}\right) = \frac{\nu a}{(1-\nu)a}\frac{\lambda b}{(1-\lambda)b}\frac{\mu c}{(1-\mu)c}$$
$$= \frac{\nu}{(1-\nu)}\frac{\lambda}{(1-\lambda)}\frac{\mu}{(1-\mu)}\,.$$

Consider the expression

$$\frac{\nu}{(1-\nu)} + \frac{\lambda}{(1-\lambda)} + \frac{\mu}{(1-\mu)}$$
$$= \frac{1}{(1-\nu)} + \frac{1}{(1-\lambda)} + \frac{1}{(1-\mu)} - 3 \le -3 + \frac{9}{3-\lambda-\mu-\nu} = \frac{3}{2}$$

(by the AM–HM Inequality). Since the sum of these three positive quantities is at most $\frac{3}{2}$, their product must consequently be at most $\frac{1}{8}$ (by the AM–GM Inequality). Equality occurs if and only if $\lambda = \mu = \nu$; that is, if and only if P is the centroid.

(c) We see that

$$[PMN] = \tfrac{1}{2}PM \cdot PN \cdot \sin\angle MPN$$
$$= \tfrac{1}{2}\mu a \cdot \nu b \cdot \sin C \quad (\text{since } \angle MPN = \pi - \angle PMA = \pi - C)$$
$$= \frac{abc}{4R}\mu\nu\,,$$
$$[PM'N'] = \tfrac{1}{2}PM' \cdot PN' \cdot \sin\angle M'PN'$$
$$= \tfrac{1}{2}\mu c \cdot \nu a \cdot \sin B = \frac{abc}{4R}\mu\nu\,.$$

Thus, $[PMN] = [PM'N']$, and similarly $[PNL] = [PN'L']$ and $[PLM] = [PL'M']$, so that $[LMN] = [L'M'N']$ (using the obvious fact that point P is inside both triangles).

(d) Using the formulae above, we see that

$$[LMN] = \frac{abc}{4R}(\lambda\mu + \mu\nu + \nu\lambda)\,.$$

We note that $\frac{abc}{4R}$ is a standard formula for $[ABC]$, and that

$$(\lambda\mu + \mu\nu + \nu\lambda) \;=\; \frac{1}{2}(\lambda + \mu + \nu)^2 - \frac{1}{2}(\lambda^2 + \mu^2 + \nu^2)\,.$$

We have $(\lambda^2 + \mu^2 + \nu^2)(1^2 + 1^2 + 1^2) \geq (\lambda + \mu + \nu)^2 = 1$ by the Cauchy-Schwartz Inequality, so that this is at most

$$\frac{1}{2} - \frac{1}{2}\left(\frac{1}{3}\right) \;=\; \frac{1}{3}\,.$$

Thus,

$$[LMN] \;\leq\; \frac{1}{3}[ABC]\,,$$

as claimed. Again equality occurs if and only if $\lambda = \mu = \nu$, so that P is the centroid.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; MICHEL BATAILLE, Rouen, France; HERBERT GÜLICHER, Westfalische Wilhelms-Universität, Münster, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU and BOGDAN IONIŢĂ, Bucharest, Romania; and the proposer.

2750. [2002 : 248] *Proposed by Paul Bracken, CRM, Université de Montréal, Montréal, QC.*

A triangle ABC has a right angle at C, and the product of the lengths of the sides AB and BC is constant.

If $\lambda > 2\sqrt{2}$, show that the quantity $AC + \lambda BC$ has a minimum when $AC = \left(\frac{\lambda+\sqrt{\lambda^2-8}}{2}\right) BC$, and a maximum when $AC = \left(\frac{\lambda-\sqrt{\lambda^2-8}}{2}\right) BC$.

Solution by Joe Howard, Portales, NM, USA.

Let a, b, c be the lengths of the sides opposite the angles A, B, C, respectively. Let $ac = k$. Since $a^2 + b^2 = c^2 = \frac{k^2}{a^2}$, we get

$$b \;=\; \frac{\sqrt{k^2 - a^4}}{a}\,.$$

If $f(a) \;=\; \frac{\sqrt{k^2 - a^4}}{a} + \lambda a$, then $f'(a) \;=\; \frac{-k^2 - a^4}{a^2\sqrt{k^2 - a^4}} + \lambda$. Now, setting $f'(a) = 0$, we have $b^2 - \lambda ab + 2a^2 = 0$, which has roots

$$b \;=\; a\left(\frac{\lambda \pm \sqrt{\lambda^2 - 8}}{2}\right)\,.$$

Computing the second derivative, we find

$$f''(a) \;=\; 2\frac{(b^2-2a^2)c^2}{a^2b^3}\,.$$

At $b = a\left(\frac{\lambda+\sqrt{\lambda^2-8}}{2}\right)$ we have

$$b^2-2a^2 \;=\; a^2\left(\tfrac{1}{2}\lambda^2+\tfrac{1}{2}\lambda\sqrt{\lambda^2-8}-4\right) \;>\; 0\,.$$

Hence, we have a local minimum. At $b = a\left(\frac{\lambda-\sqrt{\lambda^2-8}}{2}\right)$ we have

$$b^2-2a^2 \;=\; a^2\left(\tfrac{1}{2}\lambda^2-\tfrac{1}{2}\lambda\sqrt{\lambda^2-8}-4\right) \;<\; 0\,.$$

Hence, we have a local maximum. For the last inequalities it suffices to observe

$$\begin{aligned}\lambda^2-8 \;<\; \lambda\sqrt{\lambda^2-8} \quad &\Longleftrightarrow \quad \left(\lambda^2-8\right)^2 \;<\; \lambda^2(\lambda^2-8)\\ &\Longleftrightarrow \quad \lambda^2-8 \;<\; \lambda^2\,.\end{aligned}$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; PANOS E. TSAOUSSOGLOU, Athens, Greece; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.