SKOLIAD No. 69

Shawn Godin

Solutions may be sent to Shawn Godin, 2191 Saturn Cres., Orleans, ON, K4A 3T6, or emailed to

mayhem-editors@cms.math.ca.

We are especially looking for solutions from high school students. Please include your name, school or other affiliation (if applicable), city, province or state, and country on any correspondence. High school students should also include their grade in school. Please send your solutions to the problems in this edition by *1 October 2003*. A copy of **MATHEMATICAL MAYHEM Vol. 3** will be presented to the pre-university reader(s) who send in the best solutions before the deadline. The decision of the editor is final.

We will only print solutions to problems marked with an asterisk ($*$) if we receive them from students in grade 10 or under (or equivalent), or if we receive a unique solution or a generalization.

The items in this issue come from the 2001 Invitational Mathematics Challenge, a Canadian Mathematics Competition run by the Centre for Education in Mathematics and Computing (CEMC) at the University of Waterloo. This competition is written by the top 200 students who write the Cayley (Grade 10) and Fermat (Grade 11) contests (by invitation). Thanks go to Peter Crippin at CEMC for allowing us to use the contest material.

2001 Invitational Mathematics Challenge
Défi invitation de mathématiques 2001

(Grade 10 / 10^{e} année - Sec. IV au Québec)

TIME ALLOWED: 2 hours.

Calculators are permitted. It is expected that all calculations and answers will be expressed as exact numbers such as 4π, $2+\sqrt{7}$, etc. Marks are awarded for completeness, clarity and style of presentation. A correct solution, poorly presented, will not earn full marks.

1. Thirty years ago, the ages of Xavier, Yolanda, and Zoë were in the ratio $1:2:5$. Today, the ratio of Xavier's age to Yolanda's age is $6:7$. What is Zoë's present age?

. . . .

Il y a trente ans, l'âge de Xavier, l'âge de Yolande, et l'âge de Zoë étaient dans un rapport de $1:2:5$. Aujourd'hui, l'âge de Xavier et celui de Yolande sont dans un rapport de $6:7$. Quel est l'âge actuel de Zoë ?

2. (a) Determine the number of integers between 100 and 999, inclusive, that contain exactly two digits that are the same.

(b) Determine the probability that a positive integer less than 1000 contains exactly two digits that are the same.

. . . .

(a) Déterminer le nombre d'entiers, de 100 à 999, qui ont exactement deux chiffres identiques.

(b) Déterminer la probabilité pour qu'un entier strictement positif, inférieur à 1000, ait exactement deux chiffres identiques.

3. Solve the system of equations:

$$\begin{aligned} x + y + z &= 2 \\ x^2 - y^2 - z^2 &= 2 \\ x - 3y^2 + z &= 0\,. \end{aligned}$$

. . . .

Résoudre le système d'équations :

$$\begin{aligned} x + y + z &= 2 \\ x^2 - y^2 - z^2 &= 2 \\ x - 3y^2 + z &= 0. \end{aligned}$$

4. A flat mirror is perpendicular to the xy–plane and stands on the line $y = x + 4$. A laser beam from the origin strikes the mirror at $P(-1, 3)$ and is reflected to the point Q on the x–axis. Determine the coordinates of the point Q.

. . . .

Un miroir plat est placé sur la droite d'équation $y = x + 4$, tout en étant perpendiculaire au plan formé par les axes des x et des y. Un rayon laser part de l'origine, atteint le miroir au point $P(-1, 3)$, puis est réfléchi de manière à atteindre le point Q sur l'axe des x. Déterminer les coordonnées du point Q.

5. Determine all pairs of non-negative integers (m, n) which are solutions to the equation $3\,(2^m) + 1 = n^2$.

. . . .

Déterminer toutes les paires (m, n) d'entiers non négatifs qui vérifient l'équation $3\,(2^m) + 1 = n^2$.

(11^e année / Grade 11 - Sec.V au Québec)

DURÉE : 2 heures.

L'usage de la calculatrice est permis. Les réponses et les calculs doivent être exprimés à l'aide de nombres exacts, tels que 4π, $2+\sqrt{7}$, etc. Dans l'évaluation, on tiendra compte de la qualité, de la clarté et de la précision de la présentation. Une solution correcte, mais mal présentée, ne recevra pas le nombre maximum de points.

1. Un cultivateur a six contenants pouvant contenir respectivement 15, 16, 18, 19, 20, et 31 litres. Un des contenants est rempli de créme, tandis que les cinq autres sont remplis de lait blanc ou de lait au chocolat. Il y a deux fois plus de lait blanc que de lait au chocolat.

(a) Quel est le volume du contenant qui est rempli de crème ?

(b) La crème se vend 3 \$ le litre, le lait au chocolat se vend 2 \$ le litre et le lait blanc se vent 1 \$ le litre. Quelle est la valeur totale de ce qui est dans les six contenants ?

. . . .

Farmer Haas has six containers with capacities of 15, 16, 18, 19, 20, and 31 litres. One of these containers is filled with cream and the other five are filled with either white milk or chocolate milk. Farmer Haas has twice as much white milk as chocolate milk.

(a) What is the volume of the container that is filled with cream?

(b) The price of the cream is \$$3$ per litre, the price of the chocolate milk is \$$2$ per litre, and the price of the white milk is \$$1$ per litre. What is the total value of the contents of the six containers?

2. Voir question # 3 du concours de 10^e.

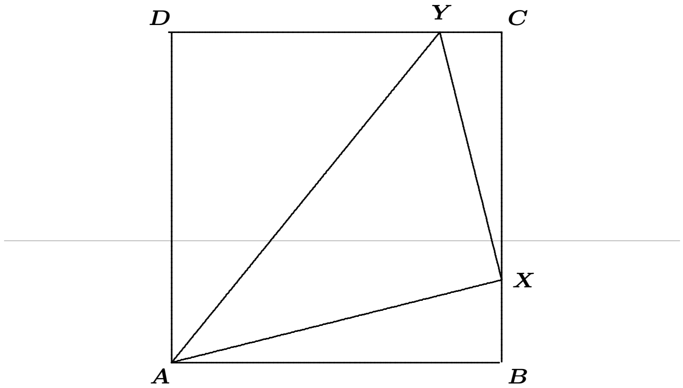
. . . .

See question # 3 from the grade 10 contest.

3. Les points X et Y sont situés sur les côtés respectifs BC et CD du carré $ABCD$. Les segments XY, AX, et AY ont une longueur respective de 3, 4, et 5 unités. Déterminer la longueur d'un côté du carré $ABCD$.

. . . .

Points X and Y are on sides BC and CD of square $ABCD$, as shown below. The lengths of XY, AX, and AY are 3, 4, and 5, respectively. Determine the side length of square $ABCD$.



4. Un miroir plat est placé le long d'une droite L, perpendiculairement au plan formé par les axes des x et des y. Un rayon laser part de l'origine, atteint le miroir au point $P(-1,5)$, puis est réfléchi pour atteindre le point $Q(24,0)$. Déterminer l'équation de la droite L.

. . . .

A flat mirror is perpendicular to the xy–plane and stands along a line L. A laser beam from the origin strikes the mirror at $P(-1,5)$ and is reflected to the point $Q(24,0)$. Determine the equation of the line L.

5. Soit $f(n) = n^4 + 2n^3 - n^2 + 2n + 1$.

(a) Démontrer que $f(n)$ peut être exprimé comme produit de deux polynômes du second degré ayant chacun des coefficients entiers.

(b) Déterminer tous les entiers n pour lesquels $|f(n)|$ est un nombre premier.

. . . .

Let $f(n) = n^4 + 2n^3 - n^2 + 2n + 1$.

(a) Show that $f(n)$ can be written as the product of two quadratic polynomials with integer coefficients.

(b) Determine all integers n for which $|f(n)|$ is a prime number.

Next we give solutions to the 1991 Canadian Mathematical Society Prize Exam given in the October 2002 issue [2002 : 391].

1. Show that $1 - \frac{2}{3} + \frac{3}{9} - \frac{4}{27} + \frac{5}{81} - \cdots \frac{100}{3^{99}} = \frac{9}{16}\left[1 - \frac{403}{3^{101}}\right]$.

Solution by Jefferson Lin, student, Brooklyn, NY, USA.

Let $S = 1 - \frac{2}{3} + \frac{3}{9} - \cdots \frac{100}{3^{99}}$. Then $\frac{S}{3} = \frac{1}{3} - \frac{2}{9} + \cdots + \frac{99}{3^{99}} - \frac{100}{3^{100}}$.

Adding these two we get

$$S + \frac{S}{3} \;=\; \left(1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} - \cdots - \frac{1}{3^{99}}\right) - \frac{100}{3^{100}}\,.$$

But the expression in the brackets is just a geometric series. Thus, we get

$$\frac{4}{3}S \;=\; \frac{1 - \left(-\frac{1}{3}\right)^{100}}{1 - \left(-\frac{1}{3}\right)} - \frac{100}{3^{100}}\,.$$

Multiplying both sides of this equation by $\frac{4}{3}$ yields

$$\frac{16}{9}S \;=\; 1 - \frac{1}{3^{100}} - \frac{400}{3^{101}} \;=\; \frac{3^{101} - 403}{3^{101}}\,.$$

Thus, $S = \frac{9}{16}\left[1 - \frac{403}{3^{101}}\right]$.

Also solved by Siwen Sun, grade 11 student, Collège Saint-Louis, Lachine, QC.

2. Solve for all real x: $\sqrt{x^2 - x + 2} + \sqrt{x^2 - x - 2} = 1$.

Solution by Geneviève Lalonde, Massey, ON.
Completing the square yields

$$x^2 - x + 2 \;=\; x^2 - x + \tfrac{1}{4} - \tfrac{1}{4} + 2 \;=\; \left(x - \tfrac{1}{2}\right)^2 + \tfrac{7}{4}\,.$$

We notice that $\sqrt{x^2 - x + 2} \geq \sqrt{\frac{7}{4}} > 1$ for all $x \in \mathbb{R}$. Therefore, the original equation has no solutions.

Three incorrect solutions were received. All incorrect solutions solved by squaring to eliminate the radicals, but failed to check for extraneous roots.

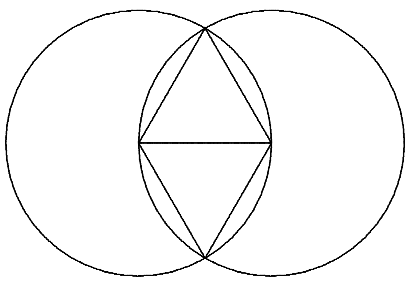
3. Two circles of equal radius pass through each other's centres. What are

(a) the perimeter and

(b) the area of the whole region enclosed by the circles?

Solution to part (a) *by Siwen Sun, grade* 11 *student, Collège Saint-Louis, Lachine, QC.*

Because both circles have the same radius and pass through each other's centres, they form 2 equilateral triangles with the 2 centres and the 2 points of intersection as vertices.



Therefore, the angle of the part inside the 2 circles is $2 \times 60^\circ = 120^\circ$. Thus,

$$P \;=\; 4\pi r - 2 \times \frac{2\pi r}{3} \;=\; \frac{8\pi r}{3}\,.$$

Also solved by Jefferson Lin, student, Brooklyn, NY, USA.

Solution to part (b) *by Geneviève Lalonde, Massey, ON.*

The area is made up of $\frac{2}{3}$ of each circle, plus two equilateral triangles. Thus,

$$\begin{aligned} A &= 2 \times \frac{2}{3}\pi r^2 + 2 \times \frac{\sqrt{3}r^2}{4} \\ &= \left(\frac{4}{3}\pi + \frac{\sqrt{3}}{2}\right) r^2\,. \end{aligned}$$

Two incorrect solutions were also received.

4. When the polynomial $x^4 + ax^3 - 7x^2 + bx - 49$ is divided by $(x - 3)$ the remainder is 53, and by $(x + 2)$ the remainder is -87. Find a and b.

Solution by Alfian, grade 11 *student, SMU Methodist* 1*, Palearbang, Indonesia.*

Let $f(x) = x^4 + ax^3 - 7x^2 + bx - 49$. The remainder theorem tells us that the remainder when $f(x)$ is divided by $(x - 3)$ is $f(3)$. Thus, $f(3) = 53$ which, after substitution, yields

$$27a + 3b \;=\; 84\,. \tag{1}$$

Similarly, we must have $f(-2) = -87$ which, after substitution, yields

$$4a + b \;=\; 13\,. \tag{2}$$

Multiplying (2) by 3 and subtracting from (1) gives $a = 3$. Then substitution gives $b = 1$.

Also solved by Jefferson Lin, student, Brooklyn, NY, USA.

5. From the letters of the word "antenna", we want to make all possible four-letter "words" (they may be nonsensical, for example, "aann"). How many can we make?

Solution by Geneviève Lalonde, Massey, ON.

We must examine a number of cases.

Case 1: The word has 3 n's. Then we must choose the 4^{th} letter from $\{a, e, t\}$. Thus, the total number of such words is $\binom{3}{1} \times 4 = 12$.

Case 2: The word has 2 n's and *not* 2 a's. Thus, we must choose two letters from $\{a, e, t\}$. The total number of such words is $\binom{3}{2} \times \frac{4!}{2!} = 36$.

Case 3: The word has 2 n's and 2 a's. The total number of such words is $\frac{4!}{2!2!} = 6$.

Case 4: The word has 2 a's and *not* 2 n's. Thus, we must choose two letters from $\{e, n, t\}$. The total number of such words is $\binom{3}{2} \times \frac{4!}{2!} = 36$.

Case 5: The word is made up of 4 distinct letters. The total number of such words is $4! = 24$.

The total number of four-letter words is $12 + 36 + 6 + 36 + 24 = 118$.

Two incorrect solutions were received.

6. If a and b are positive integers larger than 2, prove that $(2^a + 1)$ cannot be divisible by $(2^b - 1)$.

Solution by Jefferson Lin, student, Brooklyn, NY, USA.

Assume that we have $2^b - 1 \mid 2^a + 1$ with $a > b$, and assume that a is the smallest value for which this is true. Then there is an integer k such that $k(2^b - 1) = 2^a + 1$. Thus,

$$\begin{aligned} k \cdot 2^b - k &= 2^a + 1\,, \\ k \cdot 2^b - 2^a &= k + 1\,, \\ 2^b(k - 2^{a-b}) &= k + 1\,. \end{aligned}$$

Therefore, if we let $n = k - 2^{a-b}$, we get

$$\begin{aligned} k + 1 &= n \cdot 2^b\,, \\ k &= n \cdot 2^b - 1\,, \\ (n \cdot 2^b - 1)(2^b - 1) &= 2^a + 1\,. \end{aligned}$$

Simplifying yields $n(2^b - 1) = 2^{a-b} + 1$. Thus, $2^b - 1 \mid 2^{a-b} + 1$. But $2^{a-b} + 1 < 2^a + 1$, and $2^a + 1$ was assumed to be the smallest value with this property. Therefore, we have a contradiction, and $(2^a + 1)$ cannot be divisible by $(2^b - 1)$ for positive integers $a, b > 2$.

One incorrect solution was received.

That brings us to the end of another issue of Skoliad. This issue's winner of a copy of **MATHEMATICAL MAYHEM VOL. 6** is (drum roll ...) Jefferson Lin! Please continue sending in contest problems and solutions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to ~~**Mathematical Mayhem, Cairine Wilson Secondary School, 975 Orleans**~~ **Blvd., Gloucester, Ontario, Canada. K1C 2Z7**. The electronic address is

mayhem-editors@cms.math.ca

The Mayhem Editor is Shawn Godin (Cairine Wilson Secondary School). The Assistant Mayhem Editor is John Grant McLoughlin (University of New Brunswick). The other staff members are Paul Ottaway (Dalhousie University) and Larry Rice (University of Waterloo).

Mayhem Problems

Proposals and solutions may be sent to **Mathematical Mayhem, 2191 Saturn Crescent, Orleans, Ontario, K4A 3T6** or emailed to

mayhem-editors@cms.math.ca

Please include in all correspondence your name, school, grade, city, province or state and country. We are especially looking for solutions from high school students. Please send your solutions to the problems in this edition by *1 October 2003*. Solutions received after this time will be considered only if there is time before publication of the solutions.

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5 and 7, English will precede French, and in issues 2, 4, 6 and 8, French will precede English.

The editor thanks Jean-Marc Terrier and Hidemitsu Saeki of the University of Montreal for translations of the problems.

M88. *Proposed by the Mayhem Staff.*

A set S consists of six numbers. When we take all possible subsets of S containing 5 elements, the sums of the elements of these subsets are 87, 92, 98, 99, 104, and 110, respectively. Determine the six numbers in S.

. . . .

Un ensemble S contient six nombres. Si l'on prend tous les sous-ensembles de S ne contenant que 5 éléments, les sommes des éléments de ces sous-ensembles sont respectivement 87, 92, 98, 99, 104, et 110. Déterminer les six nombres dans S.

M89. *Proposed by the Mayhem Staff.*

Find all positive integers x for which $x(x + 60)$ is a perfect square.

. . . .

Trouver tous les entiers positifs x pour lesquels $x(x + 60)$ est un carré parfait.

M90. *Proposed by the Mayhem Staff.*

Determine the largest positive integer n for which 2002^n is a factor of $2002!$. What happens if 2002 is replaced by 2003 or 2004?

. . . .

Trouver le plus grand entier positif n tel que 2002^n soit un facteur de $2002!$. Qu'arrive-t-il si l'on remplace 2002 par 2003 ou 2004 ?

M91. *Proposed by Robert Morewood, Burnaby South Secondary School, Burnaby, BC.*

Let k be a four-digit integer. Determine all possible values of k for which k^{2003} ends in the four digits 2003. What happens if 2003 is replaced by 2002 or 2004?

. . . .

Soit k un nombre de quatre chiffres. Trouver toutes les valeurs possibles de k pour lesquelles le nombre k^{2003} se termine par 2003. Qu'arrive-t-il si l'on remplace 2003 par 2002 ou 2004 ?

M92. *Proposed by the Mayhem Staff.*

A 3×3 magic square consists of nine distinct values, such that each of the rows, columns, and diagonals have a constant sum. Below is an example of a 3×3 magic square.

Suppose that a 3×3 magic square has a constant sum of T. Let the middle entry of this square be E. Prove that $T = 3E$.

2	9	4
7	5	3
6	1	8

Un carré magique de 3×3 est formé de neuf valeurs distinctes, telles que la somme des éléments de chacune des lignes, des colonnes et des diagonales donne la même constante. Voir l'exemple ci-dessus.

Soit T la somme constante d'un carré magique 3×3. Si l'on désigne l'élément du centre par E, montrer que $T = 3E$.

M93. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

In triangle ABC, suppose that $\tan A$, $\tan B$, $\tan C$ are in harmonic progression. Show that a^2, b^2, c^2 form an arithmetic progression.

[*Note:* x, y, z are in harmonic progression if $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$ form an arithmetic progression.]

Dans un triangle ABC, on suppose que $\tan A$, $\tan B$, $\tan C$ sont en progression harmonique. Montrer que a^2, b^2, c^2 forment une progression arithmétique .

[*Note :* x, y, z sont en progression harmonique si $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$ forment une progression arithmétique.]

Mayhem Solutions

M38. *Proposed by the Mayhem staff.*

Find all values of n such that $1! + 2! + 3! + \cdots + n!$ is a perfect square.

Solution by Andrew Mao, grade 10 student, H.B. Beal S.S., London, ON.

We can check that

$$\begin{aligned} 1! &= 1 &\equiv& \ 1 \pmod 5 \\ 1! + 2! &= 3 &\equiv& \ 3 \pmod 5 \\ 1! + 2! + 3! &= 9 &\equiv& \ 4 \pmod 5 \\ 1! + 2! + 3! + 4! &= 33 &\equiv& \ 3 \pmod 5 \end{aligned}$$

If $n > 4$, then

$$1! + 2! + \cdots + n! = (1! + 2! + 3! + 4!) + (5! + 6! + \cdots + n!)\,.$$

But $k! \equiv 0 \pmod 5$ when $k \geq 5$. Thus, the sum in the second bracket is divisible by 5. Therefore,

$$1! + 2! + \cdots + n! \equiv 1! + 2! + 3! + 4! \equiv 3 \pmod 5\,.$$

Then $1! + 2! + \cdots + n!$ is not a square, since, for any natural number m, we have $m^2 \equiv 0$, $m^2 \equiv 1$, or $m^2 \equiv 4 \pmod 5$. Therefore, $n = 1$ and $n = 3$ are the only solutions.

Also solved by Kevin Chung, OAC student, Earl Haig S.S., North York, ON; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; and Antonio Lei, year 12 student, Colchester Royal Grammar School, Colchester, UK.

M39. *Proposed by the Mayhem staff.*

Given x is a positive real number and

$$x \;=\; 2002+\cfrac{1}{2002+\cfrac{1}{2002+\cfrac{1}{2002+\cfrac{1}{2002+\frac{1}{x}}}}}\,,$$

find x.

Solution by Alfian, grade 11 *student, SMU Methodist* 1*, Palearbang, Indonesia*.

Let us consider the more general problem:

$$x \;=\; A+\cfrac{1}{A+\cfrac{1}{A+\cfrac{1}{A+\cfrac{1}{A+\frac{1}{x}}}}}\,,$$

for some real number A.

Notice that we can rewrite

$$A+\frac{1}{x} \;=\; \frac{Ax+1}{x}\,.$$

Then the next level becomes

$$A+\frac{1}{A+\frac{1}{x}} \;=\; A+\frac{1}{\frac{Ax+1}{x}} \;=\; A+\frac{x}{Ax+1} \;=\; \frac{A^2x+x+A}{Ax+1}\,.$$

Continuing the process, we end up with

$$x \;=\; \frac{A^5x+4A^3x+3Ax+A^4+3A^2+1}{A^4x+3A^2x+x+A^3+2A}\,,$$

which leads to the quadratic equation

$$(A^4+3A^2+1)x^2-A(A^4+3A^2+1)x-(A^4+3A^2+1) \;=\; 0\,.$$

Thus, as long as $A^4+3A^2+1\neq 0$ (which is true for all real A), we have $x^2-Ax-1=0$. Hence,

$$x \;=\; \frac{A\pm\sqrt{A^2+4}}{2}\,.$$

Since x is given to be positive, we must choose

$$x \;=\; \frac{A+\sqrt{A^2+4}}{2}\,.$$

Now in our particular case, $A=2002$, and $x=1001+\sqrt{1001^2+1}$.

Also solved by Austrian 2002 *IMO team; Kevin Chung, OAC student, Earl Haig S.S., North York, ON; and Yuming Chen and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

M40. *Proposed by Louis-François Préville-Ratelle, student, Cégep Régional de Lanaudière à L'Assomption, Joliette, QC.*

Suppose a and b are two divisors of the integer n, with $a < b$. Prove:

$$\left\lfloor \frac{n}{a+1} \right\rfloor + \cdots + \left\lfloor \frac{n}{b} \right\rfloor \;=\; \left\lfloor \frac{n}{\frac{n}{b}+1} \right\rfloor + \cdots + \left\lfloor \frac{n}{\frac{n}{a}} \right\rfloor .$$

Here, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

For example, if $n = 24$, $a = 3$, and $b = 6$, this says:

$$\left\lfloor \frac{24}{4} \right\rfloor + \left\lfloor \frac{24}{5} \right\rfloor + \left\lfloor \frac{24}{6} \right\rfloor \;=\; \left\lfloor \frac{24}{5} \right\rfloor + \left\lfloor \frac{24}{6} \right\rfloor + \left\lfloor \frac{24}{7} \right\rfloor + \left\lfloor \frac{24}{8} \right\rfloor ,$$

which evaluates to the identity $6 + 4 + 4 = 4 + 4 + 3 + 3$.

Solution by the proposer.

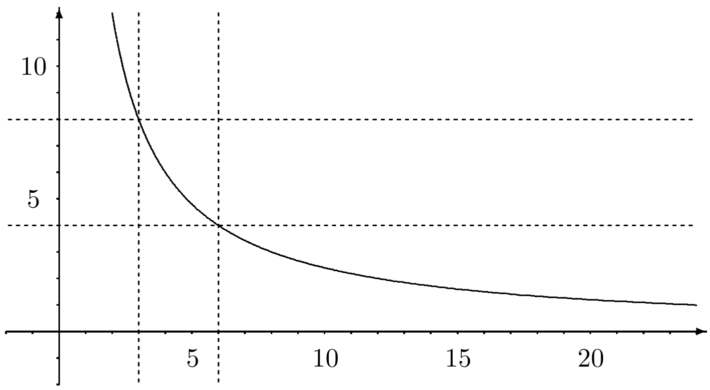
We want to show that

$$\left\lfloor \frac{n}{a+1} \right\rfloor + \cdots + \left\lfloor \frac{n}{b} \right\rfloor \qquad (1)$$

equals

$$\left\lfloor \frac{n}{\frac{n}{b}+1} \right\rfloor + \cdots + \left\lfloor \frac{n}{\frac{n}{a}} \right\rfloor . \qquad (2)$$

Consider the graph of the function $y = \frac{n}{x}$. Draw vertical lines at $x = a$ and $x = b$, and horizontal lines at $y = \frac{n}{b}$ and at $y = \frac{n}{a}$. (See diagram below for $n = 24$, $a = 3$, and $b = 6$.)



Then the number (1) is exactly the number of integer points lying below the graph, strictly to the right of the first vertical line and to the left of (or on) the second vertical line. Similarly, the number (2) is exactly the number of integer points lying to the left of the graph, below (or on) the upper vertical line and strictly above the lower vertical line.

To show that (1) and (2) are equal, we can forget the points which were counted in both (1) and (2). We need to show that the number of points counted in (1) but not in (2) is the same as the number of points counted in (2) but not in (1). This is relatively easy, because both of those regions are simply rectangular.

The points counted in (1) but not in (2) are on or below $y = \frac{n}{b}$, and between $x = a + 1$ and $x = b$. There are $\left(\frac{n}{b}\right) \times (b - (a + 1) + 1) = n\left(1 - \frac{a}{b}\right)$ of these. Similarly, the points counted in (2) but not in (1) are on or to the left of $x = a$, above $y = \frac{n}{b}$ and below or on $y = \frac{n}{a}$. There are $a \times \left(\frac{n}{a} - \left(\frac{n}{b} + 1\right) + 1\right) = n\left(1 - \frac{a}{b}\right)$ of these.

Thus, the two numbers are the same, and the equality is established.

M41. *Proposed by J. Walter Lynch, Athens, GA, USA.*

Find the number of orders of wins and losses that can occur in a World Series. For example if the series ends after five games there are eight possible orders: ANNNN NANNN NNANN NNNAN NAAAA ANAAA AANAA AAANA where A is for an American League win and N is for a National League win. Note that the series ends as soon as one team wins four games.

Solution by Sabrina Liao, student, York Mills C.I., North York, ON.

Since the series ends when a team wins 4 games, the series could end after 4, 5, 6, or 7 games.

- 4 game series: 2 possible orders: NNNN AAAA.
- 5 game series: 8 possible orders: NAAAA ANAAA AANAA AAANA and 4 more with the N and A reversed.
- 6 game series: If N wins, then the order must end with N, with 2 A's and 3 N's in the other 5 positions. Using a similar argument for when A wins, we find that the number of orders is $2 \times \binom{5}{2} = 20$.
- 7 game series: By the same reasoning, there will be $2 \times \binom{6}{3} = 40$ orders.

Thus, there altogether $2 + 8 + 20 + 40 = 70$ orders of wins and losses.

Also solved by George Adler, student, Gloucester H.S., Gloucester, ON; Steven Béliveau, student, Ecolé d'éducation internationale de Laval, Laval, QC; Robert Bilinski, Outremont, QC; Kevin Chung, OAC student, Earl Haig S.S., North York, ON; Jean-Philippe Lemieux, student, Ecolé secondaire Dorval-Jean XXIII, Dorval, QC; Peng Liu, student, Glebe C.I., Ottawa, ON; James Meredith, Hudson H.S., Hudson Heights, QC; Rébecca Millette, student, Ecolé secondaire Dorval-Jean XXIII, Dorval, QC; Alexandra Ortan, student, Ecolé Joseph-François-Perrault, Montréal, QC; Maxime Pelletier, student, Collège Sainte-Anne de Lachine, Lachine, QC; Jing Qin, student, Ecolé Emilé-Legault, Saint-Laurent, QC; Diana Rapeanu, student, Collège Notre-Dame Sacré-Coeur, Montréal, QC; Sarah Shaker, student, Ecolé secondaire Félix-Leclerc, Pointe-Claire, QC; Siwen Sun, student, Collège Sainte-Louis, Lachine, QC; Bob Wang, student, Merivale H.S., Nepean, ON; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Nicolas Wionzek, student, Almonte and District H.S., Almonte, ON. Four incorrect solutions were received.

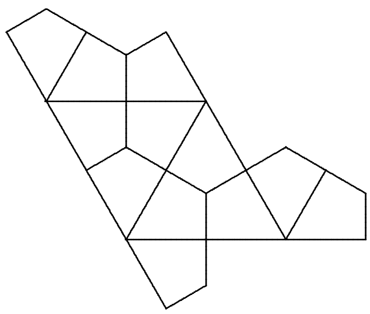
The proposer notes that of the 70 *possible orders,* 43 *have occurred. He provides the following list, of when the orders occurred, from most to least frequent, for your enjoyment.*

AAAA: *1914, 1927, 1928, 1932, 1938, 1939, 1950, 1966, 1989, 1998, 1999;*
ANAAA: *1913, 1941, 1943, 1949, 1961, 1974, 1984;*
NNNN: *1907, 1922, 1954, 1963, 1976, 1990;*
NAAANA: *1911, 1935, 1936, 1948, 1992;*
AANAA: *1916, 1929, 2000;*
NNANN: *1908, 1933, 1988;*
ANAANA: *1918, 1977, 1993;*
NAAAA: *1915, 1983;*
NNAAAA: *1978, 1996;*
AANNNAA: *1987, 1991;*
NNANAAA: *1958, 1985;*
ANANANN: *1940, 1946;*
ANNANAN: *1931, 1975;*
NANANAN: *1909, 1997;*
AAANA: *1910, 1937, 1970;*
AANNAA: *1917, 1930, 1953;*
AANNNAN: *1955, 1965, 1971;*
ANNNN: *1942, 1969;*
NANAAA: *1923, 1951;*
NANANAA: *1924, 1952;*
ANAANNN: *1925, 1979;*
ANNAANN: *1926, 1982;*
NAANNAN: *1960, 1964;*
NANNN: *1905;*
NNAANN: *1980;*
NNANAN: *1995;*
AANNANA: *1947;*
ANANNAA: *1973;*
AANNANN: *1986;*
NANNAAN: *1967;*
ANANAA: *1906;*
ANANNN: *1944;*
NNAAANA: *1956;*
ANAANNA: *1912;*
NANAANA: *1945;*
ANANNAN: *1957;*
NNAAANN: *2001.*
AANNNN: *1981;*
ANNNAN: *1959;*
AANANNA: *1972;*
ANANANA: *1962;*
NANNAAA: *1968;*
NANAANN: *1934;*

The proposer notes that the years 1903, 1919, 1920, and 1921 are not included, because in these years the winner won five games.

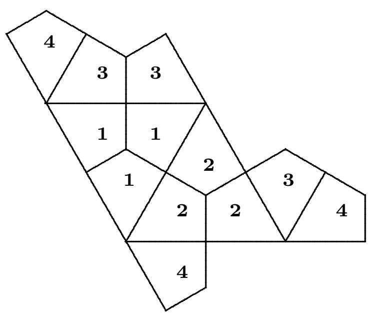
M42. *Proposed by Izidor Hafner, Tržaška 25, Ljubljana, Slovenia.*

The diagram below represents the net of a polyhedron. The faces of the solid are divided into smaller polygons. The task is to colour the polygons (or number them), so that each face of the original solid is a different colour.



Solution by Kevin Chung, OAC student, Earl Haig S.S., North York, ON.

The original polyhedron is a regular tetrahedron. The faces are numbered as below.



M43. *Proposed by the Mayhem staff.*

Prove that

$$\frac{29-5\sqrt{29}}{58}\left(\frac{7+\sqrt{29}}{2}\right)^{2002}+\frac{29+5\sqrt{29}}{58}\left(\frac{7-\sqrt{29}}{2}\right)^{2002}$$

is an integer.

Solution by Austrian 2002 *IMO team.*

Let a_n be the sequence with

$$a_n=\frac{29-5\sqrt{29}}{58}\left(\frac{7+\sqrt{29}}{2}\right)^{n}+\frac{29+5\sqrt{29}}{58}\left(\frac{7-\sqrt{29}}{2}\right)^{n}.$$

Thus, $\frac{7+\sqrt{29}}{2}$ and $\frac{7-\sqrt{29}}{2}$ are the two roots of the characteristic equation, $x^2+px+q=0$, of a_n. Therefore, we have

$$\begin{aligned} p &= -\left(\frac{7+\sqrt{29}}{2}+\frac{7-\sqrt{29}}{2}\right) = -7\,, \\ q &= \frac{7+\sqrt{29}}{2}\times\frac{7-\sqrt{29}}{2} = 5\,. \end{aligned}$$

Thus, the characteristic equation is $x^2=7x-5$. Therefore, the recurrence relation is $a_{n+2}=7a_{n+1}-5a_n$. Now we only need determine the first two values, which are $a_0=1$ and $a_1=1$.

Because a_0 and a_1 are integers and the coefficients of the recursion formula are integers, all values of the sequence are integers.

Also solved by Kevin Chung, OAC student, Earl Haig S.S., North York, ON. Seven incomplete or incorrect solutions were received.

The mail gremlins were at work last month; we received solutions from Andrew Mao, grade 10 student, H.B. Beal S.S., London, ON for M30, M32, M33, M34, M35, M36, and M37. Sorry about that Andrew. This issue's Mayhem Taunt winner is . . . Andrew Mao! Andrew will receive a subscription to ***Crux Mathematicorum*** for 2003.

High, Low, High, Low, It's Off To Work We Go

Zhe Li and Paul Belcher

In the British Mathematical Olympiad Round 1 in 1995 there was the following question:

5. The seven dwarfs walk to work each day in single file, with heights alternating up-down-up- or down-up-down-. For how long can they continue with a new order every day? What if Snow White always comes too?

We will generalize this problem. Let $w(n)$, $n \in \mathbb{N}^+$, represent the number of ways that n dwarves of different heights can walk to work with their heights alternating up, down, up, ..., or down, up, down, ... Directly listing and counting gives $w(1) = 1$, $w(2) = 2$, $w(3) = 4$, $w(4) = 10$, and $w(5) = 32$.

The heights of the dwarves can be represented, from smallest to largest simply as $1, 2, 3, \ldots, n$. (Since we are interested only in the relative heights, any increasing sequence will do here.) If we let $u(n)$ be the number of arrangements of the heights that go up, down, up, ..., and $d(n)$ be the number of arrangements that go down, up, down, ..., then we notice that $u(n) = d(n)$, for $n \geq 2$. We can prove this as follows.

Let $a_1, a_2, \ldots, a_n$ be an arrangement that goes up, down, up, ... To this we apply the transformation $a_i \to (n+1) - a_i$. This will give a unique new arrangement that goes down, up, down, ... To visualize this, let the height of the tunnel in which the dwarves are walking to work, singing, be $n+1$. Then consider another set of dwarves upside down with their feet on the roof of the tunnel, directly above the first set, as shown in Figure 1.

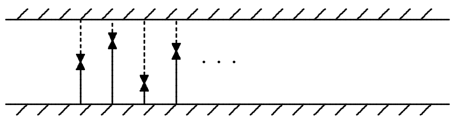


Figure 1

We now attempt to find a recurrence relation for $w(n)$. Let the tallest dwarf, of height n, who will be called Max, be in the i^{th} position. There are $i-1$ dwarves before Max, and they could be selected in $\binom{n-1}{i-1}$ ways. Their order would have to go ..., up, down, and finally up to Max. Apart from the cases where $i = 1$ or $i = 2$, the number of ways of correctly arranging these $i-1$ dwarves is $d(i-1) = \frac{1}{2}w(i-1)$, as we can see by considering the dwarves in reverse order. Following Max, the $n-i$ dwarves must have an order going down from Max, then up, down, ... Apart from the cases where $i = n - 1$ or $i = n$, the number of ways of correctly arranging these $n - i$ dwarves is $d(n-i) = \frac{1}{2}w(n-i)$.

We have a problem with $n = 1$, since $w(1) = u(1) = d(1) = 1$. To deal with this, we shall define a new sequence $W(n)$ by $W(1) = W(0) = 2$ and $W(n) = w(n)$ if $n \geq 2$. We consider $u(0) = d(0) = 1$. Then we have the recurrence relation

$$W(n) \;=\; \sum_{i=1}^{n} \binom{n-1}{i-1} \frac{W(i-1)}{2} \frac{W(n-i)}{2}\,, \quad n \;\geq\; 2\,, \tag{1}$$

with $W(0) = 2$, $W(1) = 2$.

From this recurrence relation we can build up our sequence as follows: $W(2) = 2$, $W(3) = 4$, $W(4) = 10$, $W(5) = 32$, $W(6) = 122$, $W(7) = 544$. This agrees with our directly computed list and answers the question for 7 dwarves: they can walk to work in 544 ways with an alternating height pattern. If Snow White wants to take part as well, then calculating one step further gives $W(8) = 2770$.

If we let $t(n) = [W(n)]/n!$, then $t(n) \to 0$ as $n \to \infty$, as we will now show. Suppose we take an arrangement of dwarves having the required property and exchange two successive dwarves. The new arrangement will not have the required property if $n \geq 3$. There are $n - 1$ ways of exchanging two dwarves, producing $(n-1)W(n)$ arrangements which are all different. Thus, the number of arrangements not having the desired property is at least $(n-1)W(n)$. Hence, $n! \geq (n-1)W(n) + W(n) = nW(n)$. Therefore,

$$\frac{W(n)}{n!} \;\leq\; \frac{1}{n}\,, \quad n \;\geq\; 3\,,$$

giving the result. Therefore, if the dwarves just arrange themselves randomly, then the probability that the arrangement has the desired property tends to zero as n tends to infinity.

We now attempt to find a non-recursive formula for $W(n)$ by means of a generating function for $t(n)$. From (1), replacing $i - 1$ by i, we have

$$W(n) \;=\; \frac{1}{4} \sum_{i=0}^{n-1} \frac{(n-1)!}{i!(n-1-i)!} W(i)W(n-1-i)\,, \quad n \;\geq\; 2\,.$$

Hence,

$$nt(n) \;=\; \frac{1}{4} \sum_{i=0}^{n-1} t(i)t(n-1-i)\,, \quad n \geq 2\,, \tag{2}$$

with $t(0) = 2$ and $t(1) = 2$. Consider the generating function $h(x)$:

$$h(x) \;=\; t(0) + t(1)x + t(2)x^2 + \cdots + t(n)x^n + \cdots\,.$$

In $\frac{1}{4}[h(x)]^2$, the first term is 1 and the coefficient of x^{n-1}, for $n \geq 2$, is $\frac{1}{4}\sum\limits_{i=0}^{n-1} t(i)t(n-1-i) = nt(n)$, using (2). Also,

$$h'(x) = t(1) + 2t(2)x + \cdots + nt(n)x^{n-1} + \cdots .$$

Therefore, $h'(x) = \frac{1}{4}[h(x)]^2 + 1$. Letting $h(x) = y$, we have the differential equation $\frac{dy}{dx} = \frac{1}{4}y^2 + 1$, which has solution $\arctan\left(\frac{y}{2}\right) = \frac{x}{2} + c$. When $x = 0$, we get $y = h(0) = t(0) = 2$, and also $c = \arctan(1) = \frac{\pi}{4}$. Thus, $y = h(x) = 2\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)$.

By considering the Taylor expansion of $h(x)$, we find that $t(n)$, the coefficient of x^n, is given by $t(n) = \frac{2}{n!}D^n \tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\Big|_{x=0}$. Thus,

$$W(n) \;=\; 2D^n \tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\Bigg|_{x=0} \;=\; \frac{1}{2^{n-1}} D^n \tan x\Bigg|_{x=\frac{\pi}{4}} .$$

The table below verifies this result for small values of n.

n	$D^n \tan(x)$	$D^n \tan(x)$ at $x = \frac{\pi}{4}$	$\frac{1}{2^{n-1}} D^n \tan(x)$ at $x = \frac{\pi}{4}$
0	$\tan x$	1	2
1	$1 + \tan^2 x$	2	2
2	$2\tan x(1 + \tan^2 x)$	4	2
3	$2(3\tan^2 x + 1)(1 + \tan^2 x)$	16	4
4	$8\tan x(3\tan^2 x + 2)(1 + \tan^2 x)$	80	10
5	$8(1 + \tan^2 x)$ $\times(15\tan^4 x + 15\tan^2 x + 2)$	512	32
6	$16\tan x(1 + \tan^2 x)$ $\times(45\tan^4 x + 60\tan^2 x + 17)$	3904	122

Our conclusion is that the number of ways, $w(n)$, of arranging n objects of different magnitudes in order so that the differences in magnitude alternate between positive and negative is given by

$$w(n) \;=\; \frac{1}{2^{n-1}} D^n \tan x\Bigg|_{x=\frac{\pi}{4}} ,$$

$n \geq 2$, with $w(1) = 1$.

Zhe Li is a 20-year-old student from China at Atlantic College who investigated this problem for his extended essay in Maths in the International Baccalaureate. Paul Belcher is Head of Maths at Atlantic College and was supervisor for the extended essay.

United World College of the Atlantic
St. Donat's Castle
Llantwit Major
Vale of Glamorgan
CF61 1WF, UK

Pólya's Paragon

Paul Ottaway

This month I would like to spend some time examining some difficult easy problems. That may seem rather contradictory. The three problems I have selected are taken from past grade 7 and 8 mathematics contests. This is what will make them easier than some of the problems I have posed in the past. On the other hand, they are the last questions from their respective contests and will therefore require a little more insight and creativity on the part of the reader. I should also note that, when attempting these problems, we must put on our "grade 8 hats" and forget all the mathematics we have come to learn since that time. I find it remarkable how much mathematics I take for granted. It is a worthwhile exercise to return to the basics.

One of my favourite aspects of grade 7 and 8 problems is that they can be used as a teaching tool. After years of doing mathematics contests, the tricks and techniques seem commonplace. I have to remember, however, that at some point even I saw them for the first time.

The following problem has a solution that is related to a particular set of numbers that a grade 7 student may not be familiar with. Naturally, this pattern is rather intriguing and can be used as a platform for many more problems in the future.

Gauss, Grade 7, $\#25$, 2001

A triangle can be formed having side lengths 4, 5, and 8. It is impossible, however, to construct a triangle having side lengths 4, 5, and 9. Ron has 8 sticks, each having an integer length. He observes that he cannot form a triangle using any three of these sticks as side lengths. The shortest possible length of the longest of the eight sticks is

(A) 20 (B) 21 (C) 22 (D) 23 (E) 24

Solution: Since we want the shortest possible length for the longest stick, it makes sense to start with the smallest sticks, each having length 1. Although a grade 8 student may not recognize it as such, the triangle inequality is involved in the question. It is clear that if sticks of length a and b are in our set, there cannot be any other stick of length less than $a+b$, since these three would then form a triangle. Using this information, we systematically add more sticks, taking the smallest possible length at each step. This gives us sticks with lengths 1, 1, 2, 3, 5, 8, 13, and 21. Is it much of a coincidence that these numbers form the Fibonacci sequence? Although this is not a formal proof that 21 is the shortest possible, a simple proof by contradiction could be used to show it is indeed a lower bound. This is left as an exercise.

I have found that the most prevalent formulas when solving problems in a contest setting are the formulas for the sums of arithmetic and geometric sequences. These cropped up in at least one question, it seems to me, from every contest that I can remember writing. Even though Gauss is said to have found one of these formulas at a ridiculously young age, most of our grade 7 and 8 students do not know them. How would we then approach the following problem?

Gauss, Grade 8, $\#21$, 1999:

The sum of seven consecutive integers is always:

(A) odd (B) a multiple of 7 (C) even
(D) a multiple of 4 (E) a multiple of 3

Solution 1: If we jump right to the formula for the sum of an arithmetic sequence, we can see that the sum starting at a is $7(a-1)+28$ which is always a multiple of 7. Easy enough, right? What if we did not know the formula for the sum of an arithmetic sequence?

Solution 2: It is easy for a grade 8 student to check the first couple of values obtained by computing these sums. For instance, $1+2+3+4+5+6+7=28$ and $2+3+4+5+6+7+8=35$. Now, since we know that one of the 5 possible answers must be correct we can now eliminate ones that are not satisfied by the values we have computed. Looking at 28, we can eliminate (A) and (E). Then, looking at 35 we can eliminate (C) and (D). That leaves only one possible answer, namely (B).

To quote Sir Arthur Conan Doyle, "*When you have eliminated the impossible, that which remains, however improbable, must be the truth.*"

This technique, although not appropriate when trying to solve problems in general, can be very useful and time-saving when attacking multiple choice questions. In my experience, this particular problem takes an average high school student longer than a grade 7 student. First, they remember that a formula exists; then, they spend quite a bit of time trying to remember or re-derive it before they can solve the problem!

For my last problem this month, I have chosen one of my favourites. It exemplifies the idea of using a problem as a teaching tool, as well as appealing to the more limited mathematical knowledge of a grade 7 student.

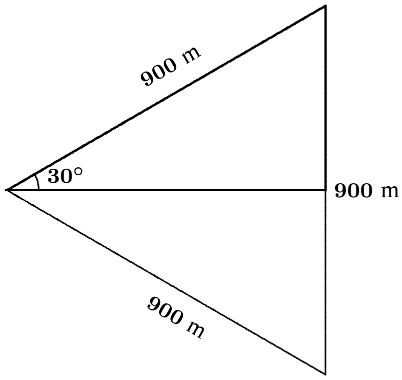
Variant on a Gauss, Grade 7**, problem**:

A particular ski lift takes passengers from the base of a mountain to its peak at a rate of 1.5 m/s. The lift travels at a 30 degree incline and it takes 10 minutes to reach the top. What is the height of the mountain?

Solution 1: We begin with a simple calculation: Ten minutes is the same as 600 seconds; hence, we have travelled a total of $(1.5)(600)=900$ metres. Fitting this to a diagram, we get a $30°$-$60°$-$90°$ triangle with 900 m along its hypotenuse. Some simple trigonometry will tell us that the height of the mountain is 450 m.

But wait! How many grade 7 students do you know who have a working understanding of trigonometry? It certainly is not common. How do you think a grade 7 student can solve this problem? The following solution is my best guess.

Solution 2: First, we recognize that a $30°$-$60°$-$90°$ triangle is really half of an equilateral triangle (we reflect along the side opposite the 60-degree angle).



Each side of this triangle is exactly 900 m. Now we see that the two $30°$-$60°$-$90°$ triangles are identical (congruent). Therefore, 900 m is twice the length that we are looking for. Thus, the height of the mountain is 450 m.

This, in my opinion, is one of the best forms of problem solving. It is not simply that we can calculate what the answer should be, but that we can calculate it without some of the "higher" mathematics that many of us take for granted. In some ways, knowing too much can slow us down and prevent us from being creative when the need arises.

I am always impressed when a student can solve the problem I have just shown here. If you remember back to when you first learned the $30°$-$60°$-$90°$ triangle and the 1-$\sqrt{3}$-2 side lengths, it was probably proven to you by taking an equilateral triangle, splitting it in two, and applying the Pythagorean theorem. By solving this problem, a grade 7 student has demonstrated that she can independently develop the same result years before it is taught in her curriculum. In my mind, this is problem-solving at its best.

THE OLYMPIAD CORNER

No. 229

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, AB, Canada, T2N 1N4.

We begin this issue with the problems of the 2000 Hungarian Mathematical Olympiad. Thanks go to Andy Liu, Canadian Team Leader to IMO 2000 in Korea, for collecting them.

2000 HUNGARIAN MATHEMATICAL OLYMPIAD

1. Consider the number of positive even divisors for each of the first n positive integers, and form the sum of these numbers. Form a similar sum of the numbers of positive odd divisors of the first n positive integers. Prove that the two sums differ by at most n.

2. Construct the point P inside a given triangle such that the feet of the perpendiculars from P to the sides of the triangle determine a triangle whose centroid is P.

3. Let k be a positive integer, and suppose that more than 2^k distinct integers are given. Prove that $k+2$ of these numbers can be chosen so that, for some positive integer m, the sums of the chosen numbers taken m at a time are all distinct.

Next we give the problems of the 2000 Iranian Mathematical Olympiad. Yet again we thank Andy Liu, Canadian Team Leader to IMO 2000 in Korea, for obtaining them for our use.

2000 IRANIAN MATHEMATICAL OLYMPIAD

1. In a tennis tournament, there are n participants A_1, A_2, $\dots$, A_n. Any two of them play at most once against each other, and the winner of each match receives 1 point. The number of matches that have been played is $k \leq n(n-1)/2$. Prove that the non-negative integers d_1, d_2, $\dots$, d_n are the scores obtained by A_1, A_2, $\dots$, A_n, respectively, if and only if $\sum_{i=1}^{n} d_i = k$ and, for every subset $X \subseteq \{1, 2, \dots, n\}$, the number of matches played among A_j with $j \in X$ is not greater than $\sum_{j \in X} d_j$.

2. Triangles $A_3A_1O_2$ and $A_1A_2O_3$ are constructed outside triangle $A_1A_2A_3$, with $O_2A_3 = O_2A_1$ and $O_3A_1 = O_3A_2$. A point O_1 is outside $A_1A_2A_3$ such that $\angle O_1A_3A_2 = \frac{1}{2}\angle A_1O_3A_2$ and $\angle O_1A_2A_3 = \frac{1}{2}\angle A_1O_2A_3$, and T is the foot of the perpendicular from O_1 to A_2A_3. Prove that

(a) A_1O_1 is perpendicular to O_2O_3, and

(b) $\dfrac{A_1O_1}{O_2O_3} = 2\dfrac{O_1T}{A_2A_3}$.

3. A circle Γ with radius R and centre W, and a line d are drawn in a plane, such that the distance of W from d is greater than R. Let M and N be two variable points on the line d such that the circle with diameter MN is tangent to the circle Γ. Prove that there exists a point P in the plane such that $\angle MPN$ is constant.

4. Let n be a positive integer, and let S be a set containing ordered n-tuples of non-negative integers such that if $(a_1, a_2, \ldots, a_n) \in S$, then every $(b_1, b_2, \ldots, b_n)$ for which $b_i \le a_i$, $1 \le i \le n$, is also in S. If $h_m(S)$ is the number of elements of S the sum of whose components is equal to m, prove that h_m is a polynomial in m for all sufficiently large m.

5. Suppose a, b, and c are real numbers such that for any positive real numbers $x_1, x_2, \ldots, x_n$,

$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^a \cdot \left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right)^b \cdot \left(\frac{1}{n}\sum_{i=1}^{n} x_i^3\right)^c \ \ge\ 1\,.$$

Prove that the vector (a, b, c) can be represented as a non-negative linear combination of the vectors $(-2, 1, 0)$ and $(1, -2, 1)$.

6. Prove that for every positive integer n, there exists a polynomial $p(x)$ with integer coefficients such that $p(1)$, $p(2)$, $\ldots$, $p(n)$ are distinct powers of 2.

Before turning to solutions from our readers to problems from the 2001 numbers of the *Corner*, we revisit problem 1 of the XXXIII Spanish Mathematical Olympiad 1996–97, First Round [2000 : 196–197]. Jean-Claude Andrieux offers the following geometric companion to the algebraic solution in [2002 : 294–295].

1. Show that any complex number $z \ne 0$ can be expressed as a sum of two complex numbers such that their difference and their quotient are purely imaginary (that is, with real part zero).

Additional observations by Jean-Claude Andrieux, Beaune, France.

Il me semble qu'une solution totalement géométrique pourrait éclairer la résolution algébrique.

On cherche deux complexes α et β tels que:

$$\begin{cases} \text{(i)} & z \;=\; \alpha+\beta\,, \\ \text{(ii)} & \beta-\alpha \in i\mathbb{R}\,, \\ \text{(iii)} & \dfrac{\beta}{\alpha} \in i\mathbb{R}\,. \end{cases}$$

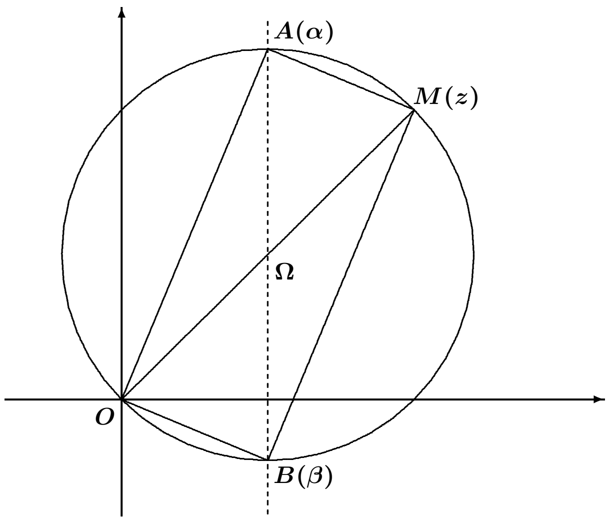
Le plan complexe étant rapporté à un repère orthonormal $(0, \vec{u}, \vec{v})$, notons M, A et B les points d'affixes respectives z, α et β.

(i) Traduit le fait que $OAMB$ est un parallélogramme.

(ii) Traduit le fait que $(\overrightarrow{OA}, \overrightarrow{OB}) = \frac{\pi}{2} \bmod \pi$. Donc $OAMB$ est un rectangle.

(iii) Traduit le fait que $\overrightarrow{AB}$ est colinéaire à $\vec{v}$. Donc $[AB]$ est le diamètre de direction $(0, \vec{v})$ du cercle circonscrit à $OAMB$.

On en déduit alors la construction des points A et B, uniques à l'ordre près.



Now we turn to solutions to the problems of the Hungary-Israel Bi-National Mathematical Competition 1997 [2001 : 8–9].

1. Is there an integer N such that

$$(\sqrt{1997} - \sqrt{1996})^{1998} \;=\; \sqrt{N} - \sqrt{N-1}\,?$$

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Robert Bilinski, Outremont, QC; Pierre Bornsztein, *Pontoise, France; Murray S. Klamkin, University of Alberta, Edmonton, AB; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. Here is Wang's solution, with historical comments.*

One such integer is

$$N = \left(\frac{(\sqrt{1997}+\sqrt{1996})^{1998}+(\sqrt{1997}-\sqrt{1996})^{1998}}{2}\right)^2 .$$

In the following, we shall establish a more general result.

Theorem. Let a, $b \in \mathbb{R}$, with $0 \le b \le a$. Then for all $n \in \mathbb{N}$, we have $(a-b)^n = \sqrt{k^2} - \sqrt{k^2-(a^2-b^2)^n}$, where $k = \frac{1}{2}\big((a+b)^n+(a-b)^n\big)$.

Proof. Since

$$k^2 - \left(\frac{(a+b)^n-(a-b)^n}{2}\right)^2 = (a^2-b^2)^n ,$$

we get

$$\sqrt{k^2} - \sqrt{k^2-(a^2-b^2)^n} = k - \frac{(a+b)^n-(a-b)^n}{2} = (a-b)^n .$$

Corollary 1. Let d, m, $n \in \mathbb{N}$, with $d \le m$. Then

$$(\sqrt{m}-\sqrt{m-d})^n = \sqrt{k^2} - \sqrt{k^2-d^n} ,$$

where $k = \frac{1}{2}\big((\sqrt{m}+\sqrt{m-d})^n + (\sqrt{m}-\sqrt{m-d})^n\big)$.

Proof. In the theorem, let $a = \sqrt{m}$ and $b = \sqrt{m-d}$.

Corollary 2. Let m, $n \in \mathbb{N}$. Then

$$(\sqrt{m}-\sqrt{m-1})^n = \sqrt{k^2} - \sqrt{k^2-1} ,$$

where $k = \frac{1}{2}\big((\sqrt{m}+\sqrt{m-1})^n + (\sqrt{m}-\sqrt{m-1})^n\big)$.

The given problem is the special case of Corollary 2 when $m = 1997$, $n = 1998$, and $N = k^2$. Note that $k \in \mathbb{N}$, since n is even and

$$k = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} \left(\sqrt{m}\right)^{n-2i} \left(\sqrt{m-1}\right)^{2i} .$$

Comments. Here is a brief history of this problem, as I know it. Over fifty years ago, the following problem appeared in the *American Mathematical Monthly* E950 [1951, 566]: Show that every positive integral power of $\sqrt{2}$ is of the form $\sqrt{m}-\sqrt{m-1}$. According to C.W. Trigg in the article "The Monthly Problems Departments, 1894–1954" (*Monthly*, Vol 64, No 7, 1957, Part II, *The Otto Dunkel Memorial Problem Book*, pp. 3–8), this was the second most popular problem proposed during that 50-year span, in terms of the number of people who submitted solutions. Browsing through the list of 65 solvers' names, I recognize such well-known mathematicians as Frank Harary, Leo Moser, and Albert Wilansky, to name just a few.

The published solution by S.T. Thompson dealt with the more general situation discussed in the solution above. The solution given here is basically the same as his, with minor changes. The original *Monthly* problem was used as the second question on the 1994 Canadian Mathematical Olympiad.

2. Find all real numbers α with the following property: for any positive integer n there exists an integer m such that

$$\left|\alpha - \frac{m}{n}\right| < \frac{1}{3n}.$$

Solved by Mohammed Aassila, Strasbourg, France; and Pierre Bornsztein, Pontoise, France. We present the solution by Bornsztein.

We will prove that the real numbers α with the specified property are the integers.

Note that if m, n are integers with $n > 0$, the condition $\left|\alpha - \frac{m}{n}\right| < \frac{1}{3n}$ may be rewritten as $|n\alpha - m| < \frac{1}{3}$, which is the same as $d(n\alpha, m) < \frac{1}{3}$, where d denotes the usual distance. Thus, the property described in the problem is equivalent to the following property P:

$$d(n\alpha, \mathbb{Z}) < \frac{1}{3}, \qquad \text{for any positive integer } n.$$

Case 1. Suppose α is irrational. Then, from a well-known theorem of Kronecker (see [1]), the set $\{n\alpha - \lfloor n\alpha\rfloor \mid n \in \mathbb{N}^*\}$ is dense in $[0,1]$. Hence, there exists $n \in \mathbb{N}^*$ such that $d\left(n\alpha - \lfloor n\alpha\rfloor, \frac{1}{2}\right) < 0.1$. Let $a = \lfloor n\alpha\rfloor$. Then $a + 0.4 < n\alpha < a + 0.6$ which leads to

$$d(n\alpha, \mathbb{Z}) = \min\{d(n\alpha, a), d(n\alpha, a+1)) > 0.4 > \frac{1}{3}.$$

Thus, α does not have the property P.

Case 2. Suppose α is rational. Let $\alpha = \frac{a}{b}$ where a, b are relatively prime integers and $b > 1$.

- If $b = 2k$ is even, then a is odd, and we have $k\alpha = a/2$. It follows that $d(k\alpha, \mathbb{Z}) = \frac{1}{2} > \frac{1}{3}$. Thus, α does not have the property P.

- If $b = 2k+1$ is odd and $k > 1$, we note that since a, b are relatively prime, there exist integers u, v with $u > 0$ such that $au + bv = 1$. Then

$$u\alpha = \frac{ua}{b} = \frac{1 - bv}{b} = -v + \frac{1}{b} = -v + \frac{1}{2k+1},$$

and hence,

$$ku\alpha \;=\; -kv + \frac{k}{2k+1}\,.$$

Then $d(ku\alpha, \mathbb{Z}) = \frac{k}{2k+1}$ or $d(ku\alpha, \mathbb{Z}) = 1 - \frac{k}{2k+1} = \frac{k+1}{2k+1}$. In either case $d(ku\alpha, \mathbb{Z}) > \frac{1}{3}$ (noting that $k > 1$). Thus, α does not have the property P.

- If $b = 3$, then $\alpha = p + \frac{c}{3}$ where p is an integer and $c \in \{1, 2\}$. Then $d(\alpha, \mathbb{Z}) = \frac{1}{3}$, and α does not have the property P.

We have shown that if $\alpha \notin \mathbb{Z}$, then α does not have the property P. Moreover, if $\alpha \in \mathbb{Z}$, then obviously α has the property P (choose $m = n\alpha$). Thus, α has the property P if and only if α is an integer, as claimed.

Reference:

[1] G.H. Hardy, E.M. Wright, *An introduction to the theory of numbers*, Oxford.

3. ABC is an acute-angled triangle whose circumcentre is O. The intersection points of the diameters of the circumcircle, passing through A, B, C, with the opposite sides are A_1, B_1, C_1, respectively. The circumradius of the triangle ABC is of length $2p$, where p is a prime. The lengths OA_1, OB_1, OC_1 are integers. What are the lengths of the sides of the triangle?

Solved by Mohammed Aassila, Strasbourg, France; Geoffrey A. Kandall, Hamden, CT, USA; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We give Kandall's solution.

The circumcentre O lies in the interior of ABC, and $OA = OB = OC = 2p$. Let $OA_1 = r$, $OB_1 = s$, $OC_1 = t$. Thus, r, s, t are positive integers.

Assertion I. $r = s = t = p$.
Proof. We have

$$\frac{OA_1}{AA_1} + \frac{OB_1}{BB_1} + \frac{OC_1}{CC_1} = \frac{[OBC]}{[ABC]} + \frac{[OAC]}{[ABC]} + \frac{[OAB]}{[ABC]} = \frac{[ABC]}{[ABC]}\,;$$

that is,

$$\frac{r}{r+2p} + \frac{s}{s+2p} + \frac{t}{t+2p} \;=\; 1\,.$$

This is equivalent to

$$r(s+2p)(t+2p) + s(r+2p)(t+2p) + t(r+2p)(s+2p) \\ = (r+2p)(s+2p)(t+2p)\,. \qquad (1)$$

It follows that $2rst \equiv 0 \pmod{2p}$; hence, $p \mid rst$. Witout loss of generality, suppose $p \mid r$. Since $0 < r < 2p$, we have $r = p$. Replacing r by p in (1), we obtain

$$3s(t+2p) + 3t(s+2p) \;=\; 2(s+2p)(t+2p)\,, \qquad (2)$$

which implies $4st \equiv 0 \pmod{2p}$; that is, $p \mid 2st$. Thus, either $p = 2$ or $p \mid st$.

(i) Suppose $p = 2$. Equation (2) reduces to $st + s + t = 8$; that is, $(s+1)(t+1) = 9$. Therefore, $s = t = 2$. Also $r = p = 2$.

(ii) Suppose $p \mid st$. Without loss of generality, $p \mid s$. Since $0 < s < 2p$, we have $s = p$. It then follows easily from equation (2) that $t = p$.

Assertion I has now been proved.

Assertion II. $AB = AC = BC = 2p\sqrt{3}$.

Proof. We have three pairs of congruent triangles: $\triangle OAB_1 \cong \triangle OBA_1$, $\triangle OBC_1 \cong \triangle OCB_1$, $\triangle OCA_1 \cong \triangle OAC_1$. Let

$$\begin{aligned} X &= [OAB_1] = [OBA_1]\,, \\ Y &= [OBC_1] = [OCB_1]\,, \\ Z &= [OAC_1] = [OCA_1]\,. \end{aligned}$$

Since $AO : OA_1 = 2 : 1$, we have $Y + Z = 2X$ and $X + Y = 2Z$, from which it follows easily that $X = Z$. Consequently, $BA_1 = CA_1$; hence, $OA_1 \perp BC$ and $BA_1 = CA_1 = p\sqrt{3}$. Therefore, $BC = 2p\sqrt{3}$. Similarly, $AB = AC = 2p\sqrt{3}$.

4. What is the number of distinct sequences of length 1997 that can be formed by using the letters A, B, C, where each letter appears an odd number of times?

Solved by Pierre Bornsztein, Pontoise, France; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We use Klamkin's write-up.

We will find the number of distinct sequences of length $2n+1$. We need only set $n = 998$ to solve the given problem.

The number we seek is

$$S = \sum \frac{(2n+1)!}{r!s!t!}\,,$$

where the summation is over all odd non-negative integers r, s, t such that $r+s+t = 2n+1$. To find this sum explicitly, we start with the multinomial expansion

$$(x+y+z)^{2n+1} = \sum \frac{x^r y^s z^t (2n+1)!}{r!s!t!}\,,$$

where the summation is over *all* non-negative integers r, s, t such that $r+s+t = 2n+1$. It then follows that the latter sum where r, s, t are all odd is given by

$$\begin{aligned} &\tfrac{1}{8}\big[(x+y+z)^{2n+1} \\ &\quad -(-x+y+z)^{2n+1} - (x-y+z)^{2n+1} - (x+y-z)^{2n+1} \\ &\quad +(-x-y+z)^{2n+1} + (-x+y-z)^{2n+1} + (x-y-z)^{2n+1} \\ &\quad -(-x-y-z)^{2n+1}\big]\,. \end{aligned}$$

Now, setting $x = y = z = 1$, it follows that

$$S = \frac{2(3)^{2n+1} - 6}{8} = \frac{3(3^{2n} - 1)}{4}.$$

In a similar fashion, by *adding* all the latter eight trinomials but this time each to the power $2n$, we find that the number of distinct sequences of length $2n$ that can be formed by using the letters A, B, C, where each letter appears an even number of times, is $[2(3)^{2n} + 6]/8$.

5. The three squares ACC_1A'', ABB_1A', $BCDE$ are constructed on the sides of a given triangle ABC, outwards. The center of the square $BCDE$ is P. Prove that the three lines $A'C$, $A''B$ and PA pass through one point.

Solved by Michel Bataille, Rouen, France; and Geoffrey A. Kandall, Hamden, CT, USA. We use Kandall's solution.

We may assume $\angle A (= \angle BAC) < 90^\circ$. Say $A'C$ meets AB at R, AP meets BC at S, and $A''B$ meets AC at T. Draw $A'B$, $A''C$, BP, and CP.

$$\begin{aligned}
\frac{AR}{RB} &= \frac{[A'AC]}{[A'BC]} = \frac{A'A \cdot AC \cdot \sin(A + 90^\circ)}{A'B \cdot BC \cdot \sin(B + 45^\circ)} \\
&= \frac{1}{\sqrt{2}} \cdot \frac{AC}{BC} \cdot \frac{\sin(A + 90^\circ)}{\sin(B + 45^\circ)}, \\
\frac{BS}{SC} &= \frac{[ABP]}{[ACP]} = \frac{AB \cdot BP \cdot \sin(B + 45^\circ)}{AC \cdot CP \cdot \sin(C + 45^\circ)} = \frac{AB}{AC} \cdot \frac{\sin(B + 45^\circ)}{\sin(C + 45^\circ)}, \\
\frac{CT}{TA} &= \frac{[A''CB]}{[A''AB]} = \frac{A''C \cdot BC \cdot \sin(C + 45^\circ)}{A''A \cdot AB \cdot \sin(A + 90^\circ)} \\
&= \frac{\sqrt{2}}{1} \cdot \frac{BC}{AB} \cdot \frac{\sin(C + 45^\circ)}{\sin(A + 90^\circ)}.
\end{aligned}$$

It follows easily that $\frac{AR}{RB} \cdot \frac{BS}{SC} \cdot \frac{CT}{TA} = 1$. Hence, $A'C$, $A''B$, and PA are concurrent.

6. Can a closed disk be decomposed into a union of two congruent parts having no common points?

Solved by Pierre Bornsztein, Pontoise, France; and Robert Bilinski, Outremont, QC. We give the comment by Murray S. Klamkin, University of Alberta, Edmonton, AB.

This problem was given as B-6 in the 25^{th} William Lowell Putnam Mathematical Competition, December 1964, and is given with solution in [1].

Reference:

[1] A.M. Gleason, R.E. Greenwood, L.M. Kelly, *The William Lowell Putnam Mathematical Competition, Problems and Solutions:* 1938–1964, Math. Assoc. of America, 1980, p. 599.

We turn next to solutions to some of the problems of the 36^{th} Armenian National Olympiad in Mathematics [2001 : 9–10].

1. Let
$$p(x) \;=\; (x-a_1)^{n_1}(x-a_2)^{n_2}(x-a_3)^{n_3}$$
be a polynomial, such that
$$p(x)-1 \;=\; (x-b_1)^{k_1}(x-b_2)^{k_2}(x-b_3)^{k_3}\,,$$
where the numbers a_1, a_2, a_3, as well as b_1, b_2, b_3, are distinct, and n_1, n_2, n_3, k_1, k_2, k_3 are natural numbers. Prove that the degree of the polynomial $p(x)$ does not exceed 5.

Solution by Michel Bataille, Rouen, France.

First, note that since $p(a_i)=0$ and $p(b_j)=1$, we cannot have $a_i=b_j$ $(i,\,j=1,\,2,\,3)$. Thus, a_1, a_2, a_3, b_1, b_2, b_3 are six distinct numbers.

Now, if $(x-c)^m$ divides the polynomial $q(x)$, then $(x-c)^{m-1}$ divides its derivative $q'(x)$. Noticing that $\big(p(x)-1\big)'=p'(x)$, we see that the polynomials $(x-a_i)^{n_i-1}$, $(x-b_j)^{k_j-1}$ divide $p'(x)$ (for i, $j=1,\,2,\,3$). Since a_1, a_2, a_3, b_1, b_2, b_3 are six distinct numbers, the product
$$(x-a_1)^{n_1-1}(x-a_2)^{n_2-1}(x-a_3)^{n_3-1}(x-b_1)^{k_1-1}(x-b_2)^{k_2-1}(x-b_3)^{k_3-1}$$
divides $p'(x)$ as well. We deduce that
$$\begin{aligned}\deg p'(x) &\geq (n_1-1)+(n_2-1)+(n_3-1)\\ &\qquad +(k_1-1)+(k_2-1)+(k_3-1)\\ &= 2\deg p(x)-6\,,\end{aligned}$$
since $n_1+n_2+n_3=k_1+k_2+k_3=\deg p(x)$. Also, $\deg p'(x)=\deg p(x)-1$. The desired result $\deg p(x)\leq 5$ follows at once.

2. Suppose a and b are natural numbers, such that $(a+b)$ is an odd number. Prove that for any division of the set of natural numbers into two groups, there will be two numbers from the same group, the difference of which is either a or b.

Solved by Pierre Bornsztein, Pontoise, France; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We give Bornsztein's solution.

Since $a+b$ is odd, we deduce that a and b have opposite parity. With no loss of generality, we may suppose that a is even and b is odd. For a contradiction, suppose that $\mathbb{N}=A\cup B$ with $A\cap B=\emptyset$, and for any x, $y\in A$ (respectively, B), $x-y\neq a$ and $x-y\neq b$.

With no loss of generality, we may suppose that $1\in A$. Then, $1+a$ and $1+b$ are in B. Thus, $1+2a$ and $1+2b$ are in A. By induction, we easily prove that, for each non-negative integer n, both $1+2na$ and $1+2nb$ are in A, while $1+(2n+1)a$ and $1+(2n+1)b$ are in B. In particular,

$1 + ab \in A$, since a is even, and $1 + ab \in B$, since b is odd. This contradicts the hypothesis that $A \cap B = \emptyset$.

The conclusion follows.

3. Prove that, for any points A, B, C, D, E, F, the following inequality holds:

$$AD^2 + BE^2 + CF^2 \;\leq\; 2(AB^2 + BC^2 + CD^2 + DE^2 + EF^2 + FA^2)\,.$$

Solved by Michel Bataille, Rouen, France; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We give Klamkin's write-up.

Let $\overrightarrow{A}$, $\overrightarrow{B}$, $\overrightarrow{C}$, $\overrightarrow{D}$, $\overrightarrow{E}$, $\overrightarrow{F}$ denote the respective vectors AB, BC, CD, DE, EF, FA. Then the inequality can be rewritten as

$$\left(\overrightarrow{A} + \overrightarrow{B} + \overrightarrow{C}\right)^2 + \left(\overrightarrow{B} + \overrightarrow{C} + \overrightarrow{D}\right)^2 + \left(\overrightarrow{C} + \overrightarrow{D} + \overrightarrow{E}\right)^2$$
$$\leq\; 2\left(\overrightarrow{A}^2 + \overrightarrow{B}^2 + \overrightarrow{C}^2 + \overrightarrow{D}^2 + \overrightarrow{E}^2 + \overrightarrow{F}^2\right).$$

Replacing $\overrightarrow{F}$ by $-\left(\overrightarrow{A} + \overrightarrow{B} + \overrightarrow{C} + \overrightarrow{D} + \overrightarrow{E}\right)$, the inequality can be rewritten in terms of a sum of squares:

$$\left(\overrightarrow{A} + \overrightarrow{B} + \overrightarrow{D} + \overrightarrow{E}\right)^2 + \left(\overrightarrow{A} + \overrightarrow{C} + \overrightarrow{E}\right)^2 + \left(\overrightarrow{A} + \overrightarrow{D}\right)^2 + \left(\overrightarrow{B} + \overrightarrow{E}\right)^2 \;\geq\; 0\,.$$

Thus, the inequality holds.

There is equality if and only if $\overrightarrow{A} + \overrightarrow{C} + \overrightarrow{E} = \overrightarrow{A} + \overrightarrow{D} = \overrightarrow{B} + \overrightarrow{E} = \overrightarrow{0}$. This requires that $ABCDEF$ be a planar centro-symmetric hexagon whose sides are parallel to the three main diagonals. Here, any side length is half the length of its parallel main diagonal.

4. It is known that the function $f(x)$ is defined on the set of natural numbers, taking values from the natural numbers, and that it satisfies the following conditions:

(a) $f(xy) = f(x) + f(y) - 1$ for any x, $y \in \mathbb{N}$,

(b) the equality $f(x) = 1$ is true for finitely many numbers,

(c) $f(30) = 4$.

Find $f(14400)$.

Solved by Jean-Claude Andrieux, Beaune, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We present the solution by Andrieux.

Prenons $x = y = 1$, on obtient $f(1) = f(1) + f(1) - 1$, donc $f(1) = 1$. Soit n un entier différent de 1. Une récurrence immédiate montre que, pour tout entier k :

$$f(n^k) \;=\; kf(n) - (k-1)\,.$$

Si $f(n) = 1$, alors pour tout entier k on a $f(n^k) = 1$. Les entiers n^k avec $k \in \mathbb{N}$ étant tous distincts, l'équation $f(x) = 1$ possède alors une infinité de solutions ce qui contredit (b). Donc, pour $n \geq 2$, on a $f(n) \geq 2$.

Soient alors 3 entiers n, p, et q, on a

$$\begin{aligned} f(npq) &= f((np)q) = f(np) + f(q) - 1 \\ &= f(n) + f(p) - 1 + f(q) - 1 = f(n) + f(p) + f(q) - 2\,. \end{aligned}$$

Puisque $f(30) = 4$, on a

$$\begin{aligned} f(2 \times 3 \times 5) &= 4\,, \\ f(2) + f(3) + f(5) - 2 &= 4\,, \\ f(2) + f(3) + f(5) &= 6\,. \end{aligned}$$

De $f(2) \geq 2$, $f(3) \geq 2$ et $f(5) \geq 2$, on en déduit

$$f(2) = f(3) = f(5) = 2\,.$$

Décomposons alors 14400 en produit de facteurs premiers,

$$14400 = 2^6 \times 3^2 \times 5^2\,,$$

d'où

$$\begin{aligned} f(14400) &= f(2^6 \times 3^2 \times 5^2) \\ &= f(2^6) + f(3^2) + f(5^2) - 2 \\ &= 6f(2) - 5 + 2f(3) - 1 + 2f(5) - 1 - 2 \\ &= 12 - 6 + 4 - 1 + 4 - 1 - 2 \\ &= 11\,. \end{aligned}$$

Finalement, $f(14400) = 11$.

Next we look at solutions to some of the problems of the Croatian National Mathematical Competition, Novi Vinodolski, IV$^{\text{th}}$ Class, 1997 [2001 : 89].

1. Find the last four digits of the number 3^{1000} and the number 3^{1997}.

Solved by Pierre Bornsztein, Pontoise, France; Murray S. Klamkin, University of Alberta, Edmonton, AB; Pavlos Maragoudakis, Pireas, Greece; Panos E. Tsaoussoglou, Athens, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give the solution by Tsaoussoglou.

(a) $3^{1000} = (3^4)^{250} = 81^{250}$, showing that the last digit is 1. Now

$$\begin{aligned} 3^{1000} &= \left(3^2\right)^{500} = (10-1)^{500} \\ &= 10^{500} - \frac{500}{1}10^{499} + \frac{499 \cdot 500}{1 \cdot 2}10^{498} - \cdots \\ &\quad - \frac{498 \cdot 499 \cdot 500}{1 \cdot 2 \cdot 3}10^3 + \frac{499 \cdot 500}{2}10^2 - \frac{500}{1}10 + 1\,. \end{aligned}$$

For the last 4 digits, we need only the last 3 terms, which simplify to

$$\begin{aligned} 499 \cdot 25 \cdot 10^3 - 5 \cdot 10^3 + 1 &= 5(499 \cdot 5 - 1) \cdot 10^3 + 1 \\ &= 1247 \cdot 10^4 + 1 \\ &= 12470001\,. \end{aligned}$$

Therefore, the last four digits are 0001.

(b) $3^{1997} = 3 \cdot 3^{1996} = 3(3^4)^{499}$, showing that the last digit is 3. Now

$$\begin{aligned} 3^{1996} &= \left(3^2\right)^{998} = (10-1)^{998} \\ &= 10^{998} - \frac{998}{1}10^{997} + \frac{997 \cdot 998}{1 \cdot 2}10^{996} - \cdots \\ &\qquad - \frac{996 \cdot 997 \cdot 998}{1 \cdot 2 \cdot 3}10^3 + \frac{997 \cdot 998}{1 \cdot 2}10^2 - \frac{998}{1}10 + 1 \\ &= \ldots 4321\,. \end{aligned}$$

Then

$$3^{1997} = 3(\ldots 4321) = \ldots 2963\,.$$

Thus, the last four digits of 3^{1997} are 2963.

We also give Klamkin's solution.

By Fermat's general theorem, $3^{\varphi(10000)} \equiv 1 \pmod{10000}$. Here, $\varphi(10000) = 4000$. Now $3^{4000} - 1 = (3^{1000} - 1)(3^{1000} + 1)(3^{2000} + 1)$. Since $3^{4n} \equiv 1 \pmod{10}$, it follows that $3^{1000} \equiv 1 \pmod{10000}$. Hence, the last four digits of the number 3^{1000} are 0001.

Let $3^{1997} \equiv a + 10b + 100c + 1000d \pmod{10000}$. Then

$$3^{2000} \equiv 1 \equiv 27a + 270b + 2700c + 27000d \pmod{10000}\,.$$

Hence, $a = 3$ and then $-8 \equiv 27b + 270c + 2700 \pmod{1000}$; hence, $b = 6$ and then $-17 \equiv 27c + 270d \pmod{100}$; hence, $c = 9$ and then $-26 \equiv 27d \pmod{10}$. Finally, $d = 2$. Thus, the last four digits of 3^{1997} must be 2963.

2. A circle k and the point K are on the same plane. For every two distinct points P and Q on k, the circle k' contains the points P, Q, and K. Let M be the intersection of the tangent to the circle k' at the point K and the line PQ. Find the locus of the points M when P and Q move over all points on k.

Solution by Michel Bataille, Rouen, France.

First two easy particular cases:

If K lies on k, the locus of M is clearly the tangent to k at K.

If K is the centre O of k, the tangent to k' at K is always parallel to PQ. Thus, the locus of M is empty.

Now to the general case where $K \notin k$ and $K \neq 0$ (Figure 1). Since the line PQ is the radical axis of k and k', the point M has the same power with respect to k and k'. The relation $MO^2 - R^2 = MK^2$ (in which R denotes the radius of k) follows immediately and shows that M belongs to the line L whose points N are characterized by $NO^2 - NK^2 = R^2$.

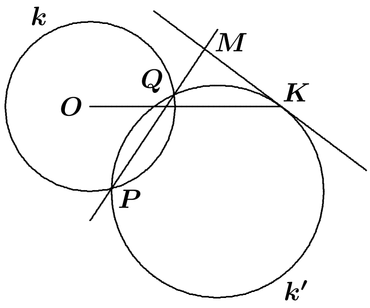


Figure 1

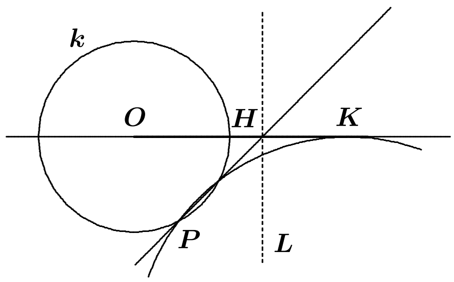


Figure 2

Conversely, let N be any point on L. Then

$$NO^2 - R^2 = NK^2 . \qquad (1)$$

Choose any point P on k, not on NK and not a point of tangency of k with a circle tangent to NK at K [this leaves infinitely many choices!]. Then there exists a unique circle Γ through P and tangent to NK at K: its centre is the point of intersection of the perpendicular bisector of KP and the perpendicular to NK at K. The circle Γ cuts k again at Q distinct from P (by the choice of P), and the line PQ is the radical axis of Γ ($= k'$) and k. Thus, PQ passes through any point that has the same power with respect to the two circles. In particular, PQ passes through N, in view of (1). Thus, $N = M$ is a point of the locus. In conclusion, the locus we seek is the line L.

Note. This line L is the perpendicular to OK at the point H defined by $IH = \frac{1}{2}R^2/OK$, where I is the mid-point of OK. Observe that K is not on L (since $K \notin k$) and that L is exterior to k (since $NO^2 > R^2$ for all N on L). The line L can easily be constructed by remarking that H is also the intersection of OK with the line through the common points of k and a circle tangent to OK at K (see Figure 2).

3. A function f is defined on the set of positive numbers, which has the following properties:

$$f(1) = 1 , \quad f(2) = 2 ,$$

$$f(n+2) = f\big(n+2-f(n+1)\big) + f\big(n+1-f(n)\big) , \quad (n \geq 1) .$$

(a) Show that $f(n+1) - f(n) \in \{0, 1\}$ for every $n \geq 1$.

(b) If $f(n)$ is odd, show that $f(n+1) = f(n) + 1$.

(c) For every natural number k determine all numbers n for which

$$f(n) = 2^{k-1} + 1 .$$

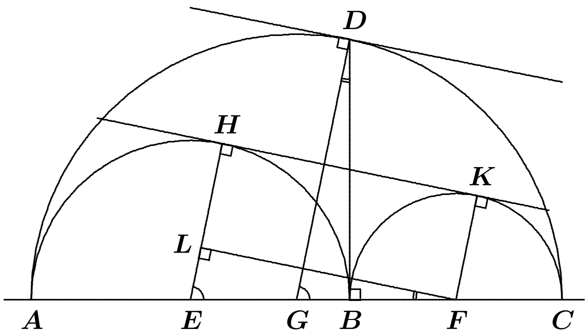
Comment by Michel Bataille, Rouen, France.

This problem is very similar to problem No 5, 1990 Canadian Mathematical Olympiad. A reference is *Mathematics Magazine* Vol 63, No 3, June 1990, p. 204.

Next we turn to solutions to some of the problems of the Croation National Mathematical Competition, Additional Competition for Selection of the 38^{th} IMO Team, 1997 [2001 : 90].

1. Three points A, B, C, are given on the same line, such that B is between A and C. Over the segments $\overline{AB}$, $\overline{BC}$, $\overline{AC}$, as diameters, semicircles are constructed on the same side of the line. The perpendicular from B to AC intersects the largest circle at point D. Prove that the common tangent of the two smaller semicircles, different from BD, is parallel to the tangent on the largest semicircle through the point D.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; and D.J. Smeenk, Zaltbommel, the Netherlands. We give Smeenk's solution.



Let the semicircles with diameters AB, BC, and AC be $\Gamma_1(E, R_1)$, $\Gamma_2(F, R_2)$, and $\Gamma_3(G, R_1 + R_2)$, respectively. Assume $R_1 > R_2$. The common tangent to Γ_1 and Γ_2 touches Γ_1 at H and Γ_2 at K. Let $L \in EH$ such that $FL \| KH$. Thus, $GD = EF = R_1 + R_2$, $GB = EL = R_1 - R_2$, and $\angle ELF = \angle GBD = 90^\circ$. We see that $\triangle ELF \cong \triangle GBD$, and therefore, $\angle FEL = \angle BGD$. This implies that EH is parallel to GD. We conclude that the tangent to Γ_3 at D is parallel to KH.

2. Let a, b, c, d be real numbers such that at least one is different from zero. Prove that all roots of the polynomial

$$P(x) = x^6 + ax^3 + bx^2 + cx + d$$

cannot be real.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; Murray S. Klamkin, University of Alberta, Edmonton, AB; Pavlos Maragoudakis, Pireas, Greece; and Heinz-Jürgen Seiffert, Berlin, Germany. We give Seiffert's write-up.

More generally, let $a_0, a_1, \ldots, a_{n-3}$, $n \geq 3$, be real numbers such that at least one is different from zero. Then, the polynomial

$$P(x) \;=\; x^n + \sum_{j=0}^{n-3} a_j x^j \qquad (1)$$

cannot have only real roots. (The proposal's result is the particular case $n = 6$.)

Proof. Assume, by way of contradiction, that $P(x)$ has the (not necessarily distinct) real roots x_1, x_2, ..., x_n. Since $P(x)$ has leading coefficient 1,

$$P(x) \;=\; \prod_{k=1}^{n} (x - x_k)\,. \qquad (2)$$

Comparing the coefficients of x^{n-1} and x^{n-2} gives

$$\sum_{k=1}^{n} x_k \;=\; 0 \quad \text{and} \quad \sum_{1 \leq j < k \leq n} x_j x_k \;=\; 0\,.$$

Hence,

$$\sum_{k=1}^{n} x_k^2 \;=\; \left(\sum_{k=1}^{n} x_k\right)^2 - \; 2 \sum_{1 \leq j < k \leq n} x_j x_k \;=\; 0\,,$$

which implies $x_1 = x_2 = \cdots = x_n = 0$, because $x_1, x_2, \ldots, x_n$ are all real. Thus, by (2), $P(x) = x^n$, and then, by (1), $a_0 = a_1 = \cdots = a_{n-3} = 0$, a contradiction.

Next we look at reader solutions to problems of the 1997 St. Petersburg City Mathematical Olympiad, Selection Round – 10^{th} Grade [2001 : 91].

1. Positive integers x, y, z satisfy the equation $2x^x + y^y = 3z^z$. Prove that they are equal.

Solution by Pierre Bornsztein, Pontoise, France.

Let x, y, z be positive integers, such that $2x^x + y^y = 3z^z$.

Suppose that $y > z$. Then, since they are integers, we have $y \geq z + 1$. Using the Binomial Theorem,

$$3z^z \;=\; 2x^x + y^y \;>\; (z+1)^{z+1} \;\geq\; z^{z+1} + (z+1)z^z \;\geq\; 3z^z\,.$$

This is a contradiction. Thus, $y \leq z$.

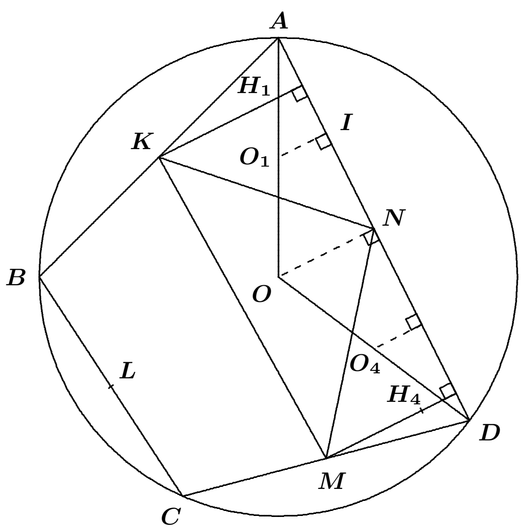
If $y < z$, then $2x^x = 3z^z - y^y > 2z^z$, and so $x > z$. As above, we then have $x \geq z + 1$ and

$$3z^z \;=\; 2x^x + y^y \;>\; 2(z+1)^{z+1} \;\geq\; 2z^{z+1} + (z+1)z^z \;>\; 3z^z\,.$$

This is a contradiction. Thus, $y = z$. Then $x^x = z^z$, which gives $x = z$, and we are done.

3. K, L, M, N are the mid-points of sides AB, BC, CD, DA, respectively, of an inscribed quadrangle $ABCD$. Prove that the orthocentres of triangles AKN, BKL, CLM, DMN are vertices of a parallelogram.

Solved by Michel Bataille, Rouen, France; and D.J. Smeenk, Zaltbommel, the Netherlands. We give Bataille's solution.



Let O be the centre of the circle $(ABCD)$, and let H_1, H_2, H_3, H_4 be the orthocentres of triangles AKN, BKL, CLM, DMN, respectively. Note that $\triangle AKN$ is the homothetic image of $\triangle ABD$ under the homothety with centre A and factor $\frac{1}{2}$. It follows that the circumcentre O_1 of $\triangle AKN$ is the mid-point of AO. Denoting by I the mid-point of AN, we thus have $\overrightarrow{O_1I} = \frac{1}{2}\overrightarrow{ON}$. Using the well-known relation $\overrightarrow{KH_1} = 2\overrightarrow{O_1I}$, we deduce that $\overrightarrow{KH_1} = \overrightarrow{ON}$.

Similarly, $\overrightarrow{MH_4} = \overrightarrow{ON}$. It then follows that $\overrightarrow{KH_1} = \overrightarrow{MH_4}$. Hence, $\overrightarrow{H_1H_4} = \overrightarrow{KM}$. In the same way, $\overrightarrow{H_2H_3} = \overrightarrow{KM}$. Thus, $\overrightarrow{H_1H_4} = \overrightarrow{H_2H_3}$, which means that $H_1H_2H_3H_4$ is a parallelogram.

Now we look at solutions from our readers to problems of the 1997 St. Petersburg City Mathematical Olympiad, Selection Round – 11^{th} Grade [2001 : 91–92].

1. Can a 75×75 table be partitioned into dominoes (that is, 1×2 rectangles) and crosses (that is, five-square figures consisting of a square and its four horizontal and vertical neighbours)?

Solved by Robert Bilinski, Outremont, QC; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We give the solution by Klamkin.

Our solution is indirect in that we assume it can be done and obtain a contradiction. Let the table be coloured black and white in the manner of a chess board, with the corner squares being black. Then the total number of black squares is one greater than the total number of white squares. Let x and y denote the number of dominoes and crosses, respectively, that cover the table. Then, $2x + 5y = 75^2$, and therefore, $x = 5n$. It follows that $2n + y = 75 \times 15 = 1125$. Now y must be odd, say $y = 2m + 1$.

While each domino covers one black and one white square, a cross can cover one white and four black squares, or one black and four white. Let p denote the number of crosses used each of which cover one white and four black. The remaining $2m+1-p$ crosses each cover one black and four white. The total number of black squares covered by the crosses is $4p+(2m+1-p)$, while the total number of white squares covered is $p+4(2m+1-p)$. Their difference is $6p-6m-3$. This difference must equal 1. Thus, $3p-3m = 2$, which is impossible. Hence, the table cannot be partitioned in the given manner.

2. Prove that for $x \geq 2$, $y \geq 2$, $z \geq 2$

$$(y^3+x)(z^3+y)(x^3+z) \;\geq\; 125xyz\,.$$

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; Murray S. Klamkin, University of Alberta, Edmonton, AB; Pavlos Maragoudakis, Pireas, Greece; Panos E. Tsaoussoglou, Athens, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Klamkin's generalization, adapted by the editors.

We show that

$$(x_1^{nr}+ax_2^{ns})(x_2^{nr}+ax_3^{ns})\cdots(x_n^{nr}+ax_1^{ns}) \;\geq\; (k^{n(r-s)}+a)^n P^{ns}\,, \quad (1)$$

where $P = x_1x_2\cdots x_n$, $r \geq s \geq 0$, $a \geq 0$, and $x_i \geq k \geq 0$.

Since $k^n \leq P$, we have $k^{n(r-s)} \leq P^{r-s}$, and hence,

$$\begin{aligned}(k^{n(r-s)}+a)P^s &\leq P^r + aP^s \\ &\leq (x_1^{nr}+ax_2^{ns})^{\frac{1}{n}}(x_2^{nr}+ax_3^{ns})^{\frac{1}{n}}\cdots(x_n^{nr}+ax_1^{ns})^{\frac{1}{n}}\,,\end{aligned}$$

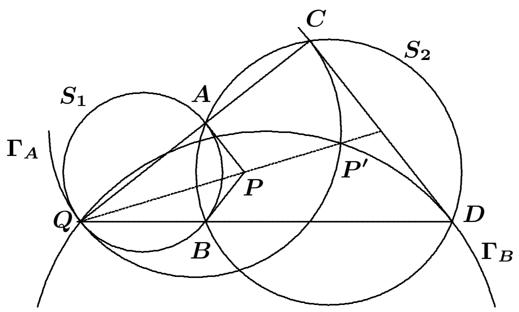
where the last step follows by Hölder's Inequality. Now raise both sides to the power n to obtain (1). There is equality if and only if $x_i = k$ for all i.

The given inequality corresponds to the special case $n = 3$, $r = 1$, $s = \frac{1}{3}$, $a = 1$ and $k = 2$.

Other extensions can be obtained by replacing each of the factors on the left side of (1) by a sum of more terms, and using Hölder's Inequality. For example, the first factor can be replaced by $x_1^{nr} + ax_2^{ns} + bx_3^{nt}$, with the other $n - 1$ factors given cyclically.

3. Circles S_1 and S_2 intersect at points A and B. A point Q is chosen on S_1. The lines QA and QB meet S_2 at points C and D; the tangents to S_1 at A and B meet at point P. The point Q lies outside S_2, the points C and D lie outside S_1. Prove that the line QP goes through the mid-point of CD.

Solved by Michel Bataille, Rouen, France; and Babis Stergiou, Lycio Psachnon Evias, Greece. We give Bataille's solution.



Since $QA \cdot QC = QB \cdot QD$, points A, C as well as B, D are inverse points through an inversion with centre Q. This inversion transforms the line CD into S_1 and the lines AP, BP into circles Γ_A, Γ_B passing through Q and tangent to CD at C, D, respectively. Let P' be the inverse of P. We see that Γ_A, Γ_B intersect at Q and P' and that CD is a common exterior tangent (at C, D) to Γ_A, Γ_B. As is well-known, QP' meets CD at its mid-point. Since Q, P, P' are collinear, the desired result is obtained.

That completes the *Corner* for this issue. Send me your nice solutions, generalizations, and also Olympiad contests!

BOOK REVIEW

John Grant McLoughlin

The Inquisitive Problem Solver
by Paul Vaderlind, Richard Guy, and Loren Larson, published by the Mathematical Association of America (MAA Problem Books Series), 2002
ISBN 0-88385-806-1, softcover, 327 + xv pages, US$34.95
Reviewed by ***Stan Wagon****, Macalester College, St. Paul, MN, USA.*

This welcome addition to the problem literature consists of 256 main problems (and much more), most of which come from a Swedish collection published in 1996 by Paul Vaderlind. Loren Larson came across the volume on a trip to Sweden and decided (correctly) that translation would be a worthwhile project. He and Richard Guy then turned it into a distinctive problem book by carefully working through the material, adding some problems of their own, and, most notably, adding much more material in the way of additional questions and remarks (and solutions to the additional questions).

The collection is noteworthy for the consistency and accessibility of the main problems. They are meant to appeal to a wide audience, with no mathematical prerequisites. The authors use the word "miniatures" to describe many problems in the collection.

Here is an example of a simple, but nonetheless inviting, problem (#202). Who has a winning strategy when the usual tic-tac-toe game is changed by adding a place at the right end of the first row? An extension to the problem asks for the smallest board for which the first player has a winning strategy.

Here is another nice one (#169), by no means difficult, yet with a compelling and surprising statement. Let W_k and L_k be the numbers of wins and losses for each of n players in a round-robin tournament. True or false: the sum of the squares of the W_k equals the sum of the squares of the L_k? There are many more such as these. Even though I am quite familiar with the problem literature, many were new to me.

An exceptional feature of the book is the variety of extensions and generalizations of the problems. Here is an example. At first we are presented with a not-too-difficult problem (#178): Start with $\{1, 2, \ldots, 2001\}$. Select a and b from the list, with $a \geq b$, remove them, and insert $a^2 - b^2$. Repeat until one number is left. Can it be 0? This problem has an easy parity-based solution (the answer is NO), but it has a surprising generalization. Define $S(n)$ to be the smallest possible number one can get by reducing the set $\{1, 2, 3, \ldots, n\}$ to a single number as before. For $n \geq 8$, the S-sequence is periodic, with period 12. For example, $S(10^{100})$ is 4. And the globally largest value of S is $S(6) = 63$.

These surprising results sent me to the website

http://www.math.uwaterloo.ca/JIS/HICKERSON/hickerson.html

to look up the paper in question. It is noteworthy that the discoverers of this result were motivated by the question by Guy, who posed it while working on the book under review.

There is one general point that applies to a few problems in this book, as well as occasional problems printed in ***Crux Mathematicorum*** or given on math contests. Students are usually as computer literate as their teachers, or more so. I find a problem such as #168 (If you add up the numbers 9, 99, 999, $\ldots$, $9999\ldots9$ with 99 digits, will the answer contain 99 ones?) a little silly as stated, since it is so easy to just add up the numbers in question and see that, yes, there are 99 ones. The same philosophy we have applied to calculus instruction for the past ten or more years, that one should not spend a lot of time on algebraic differentiation and integration when machines can do it far better than any human, should apply to problemists as well. If the method of solution is inherently interesting, find a way to pose it that makes solution by machine irrelevant.

There are a few errors. The solution given to problem #98 is incorrect, the solution to problem #209 does not match the given problem, and I found a few minor typos. Back to the plus side. The graphics are plentiful and superbly rendered, there is a 26-page section of hints, and there is a very valuable 48-page "Treasury" containing miscellaneous definitions, results, and techniques. The Treasury will be very valuable for students.

Overall the comments by Ron Graham on the back cover are right on the mark: "I don't know of a better book for introducing students (of all ages!) to the basic principles for attacking such problems". And the translators deserve extra credit for taking an inquisitive view of these elementary problems and adding much valuable material in the solutions and generalizations.

Acknowledgment:

As the Book Review Editor, I would like to thank Shauna Gammon for her assistance. Shauna, an undergraduate mathematics student at Memorial University of Newfoundland, has provided valuable technical support throughout the transition into the editorial role and my subsequent move to the University of New Brunswick.

Divisibility by Numbers Ending in Nine

H. Havens

There is a popular divisibility method used to find whether numbers are divisible by nine. You simply add the digits of the number together to get a new number (which is smaller than the number you started with). Then you add the digits of the new number. You repeat this process until you are left with a single-digit number. If the final number is 9, then the initial number is divisible by 9. If the final number is 8, then the remainder is 8, and so on.

Take the number 108, for example. Note that $1 + 0 + 8 = 9$, which means that 108 is divisible by 9; indeed, $108 = 9 \cdot 12$. Instead of decomposing 108 into its separate digits, suppose we decompose it into 10 and 8. Adding as before, we get $10+8 = 18$ and then $1+8 = 9$. Thus, we decomposed the number in a different manner, yet we still ended up with the same result. For reasons developed below, we will express the calculations as follows: $10 + (1) \cdot 8 = 18$, $1 + (1) \cdot 8 = 9$. Applying the same process to 109, we get $10 + (1) \cdot 9 = 19$, $1 + (1) \cdot 9 = 10$, and finally $1 + (1) \cdot 0 = 1$. Therefore, the remainder of $109/9$ is 1, or in terms of congruence, $109 \equiv 1 \pmod 9$.

Now we will try to find a divisibility theorem that works for division by any number ending in 9, such as 19. Note that $38/19 = 2$. If the algorithm is to correspond to the one we used for division by 9, then there is a number y that we can use to compute $3 + y \cdot 8$. We need to determine y so that $3+y \cdot 8 = 19$. Solving, we find $y = 2$. Note that $y = 2 = (19+1)/10$ here, and $y = 1 = (9 + 1)/10$ above.

For any number x ending in 9, the divisibility of another number v by x can be found by the following recursive process: take the units digit from v and multiply it by $y = (x + 1)/10$; add the result to v with its units digit removed; repeat this until you get a number less than or equal to x. If this final number turns out to be x, then the original number v is divisible by x; otherwise, v is not divisible by x.

Let us now analyze this algorithm and verify that it works as claimed. Given two numbers v and x, where x has 9 as its last digit, we want to divide v by x. We write $v = 10a + b$ and $x = 10c + 9$. Then $a = (v - b)/10$ and $c = (x - 9)/10$. We also let $y = c + 1 = (x + 1)/10$. The algorithm states that the next number, v_1, which will be used in place of v, is computed as:

$$v_1 \;=\; a + y \cdot b \;=\; \frac{v-b}{10} + \frac{x+1}{10} \cdot b \;=\; \frac{v + x \cdot b}{10}\,. \qquad (1)$$

Observe that $10 \cdot y = x + 1 \equiv 1 \pmod{x}$ and that $x \cdot b \equiv 0 \pmod{x}$. Thus, $v \equiv v + x \cdot b \pmod{x}$, and hence,

$$v_1 \;=\; \frac{v + x \cdot b}{10} \;\equiv\; y \cdot (v + x \cdot b) \;\equiv\; y \cdot v \pmod{x}\,.$$

As a consequence, $v_1 \equiv 0 \pmod{x}$ if and only if $y \cdot v \equiv 0 \pmod{x}$. Since x and y are relatively prime, $y \cdot v \equiv 0 \pmod{x}$ if and only if $v \equiv 0 \pmod{x}$. Therefore, $v_1 \equiv 0 \pmod{x}$ if and only if $v \equiv 0 \pmod{x}$.

To continue the process, we write $v_1 = 10a_1 + b_1$, and set

$$v_2 \;=\; a_1 + y \cdot b_1\,.$$

It is easy to check that $10v_2 = v_1 + x \cdot b_1$. Reasoning similar to the above tells us that $v_1 \equiv 0 \pmod{x}$ if and only if $v_2 \equiv 0 \pmod{x}$. We continue until we get $v_n \leq x$, and we conclude that $v_n = x$ if and only if x divides v.

For $x = 59$ and $v = 4779$, we have $y = (x+1)/10 = 6$. Also, $v = 10 \cdot 477 + 9$, implying that $a = 477$ and $b = 9$. Thus,

$$v_1 \;=\; a + y \cdot b \;=\; 477 \;+\; 6 \cdot 9 \;=\; 531\,,$$

Now $a_1 = 53$ and $b_1 = 1$; whence,

$$v_2 \;=\; a_1 + y \cdot b_1 \;=\; 53 \;+\; 6 \cdot 1 \;=\; 59\,.$$

The process stops here because 59 is less than or equal to 59. Since we get 59 itself, 4779 is divisible by 59.

For $x = 79$ and $v = 4431$, we have $y = (x+1)/10 = 8$. Also, $v = 10 \cdot 443 + 1$, which implies that $a = 443$ and $b = 1$. Therefore,

$$v_1 \;=\; a + y \cdot b \;=\; 443 + 8 \cdot 1 \;=\; 451\,.$$

Now $a_1 = 45$ and $b_1 = 1$; whence,

$$v_2 \;=\; a_1 + y \cdot b_1 \;=\; 45 + 8 \cdot 1 \;=\; 53\,.$$

Since $53 \not\equiv 0 \pmod{79}$, we conclude that $4431 \not\equiv 0 \pmod{79}$.

How can we find the remainder? First, how do we find the remainder of $16/9$? We have $1+6 = 7$. Since only one step was taken, we compute $10^1 \cdot 7$ and divide by 9 to get our remainder. This method can be applied when we divide by 19 or any other number ending with 9. Consider $39/19$. We have $3 + 2 \cdot 9 = 21$, $2 + 2 \cdot 1 = 4$. The process took two steps. To determine the remainder, we could compute $10^2 \cdot 4 \pmod{19}$. However, it is easier to note that $10 \cdot 4 = 40 \equiv 2 \pmod{19}$ and that $10 \cdot 2 = 20 \equiv 1 \pmod{19}$. This gives the correct remainder of 1.

We now apply this method to find the remainder in the example presented above, where $x = 79$ and $v = 4431$. It took two steps to get to the number $v_2 = 53$ that is less than 79. We could multiply v_2 by 10^2, and reduce the result modulo 79 to get the required remainder. But it is easier computationally to proceed as follows: $10 \cdot 53 \equiv 56 \pmod{79}$, $10 \cdot 56 \equiv 7 \pmod{79}$. Therefore, $4431 \equiv 7 \pmod{79}$.

Now we describe in general how to find the remainder $r \equiv v \pmod{x}$. Let n be the number of steps in the process. We claim

$$10^n \cdot v_n \equiv v \pmod{x}.$$

It follows from (1) that $10 \cdot v_1 = v + x \cdot b \equiv v \pmod{x}$. Since $10 \cdot v_2 = v_1 + x \cdot b_1$, we have $10^2 \cdot v_2 = 10 \cdot v_1 + 10x \cdot b_1 \equiv v \pmod{x}$. Repeating this procedure n times, we have proved the claim.

There is a well-known divisibility method to see if numbers are divisible by 3. There is also an algorithm similar to the one presented above for divisibility by numbers ending in 3. We will now describe this algorithm.

We start with some observations. Take the number $360 = 3 \cdot 120$. Because $3 + 6 + 0 = 9$ and $9 = 3 \cdot 3$, we find that 360 is divisible by 3. The computations can be written as

$$\begin{aligned} 36 + 0 \cdot 1 &= 36\,, \\ 3 + 6 \cdot 1 &= 9\,. \end{aligned}$$

The number 1 can be obtained as $1 + 3 \cdot 0 = 1$.

For 52 and 13 one has $52 = 13 \cdot 4$. Define v so that $5 + v \cdot 2 = 13$; that is, $v = 4$.

$$\begin{aligned} 13 &= 10 \cdot 1 + 3 \quad \Longrightarrow \quad 1 + 3 \cdot 1 = 4 = v\,, \\ 52 &= 10 \cdot 5 + 2 \quad \Longrightarrow \quad 5 + 2 \cdot 4 = 13\,. \end{aligned}$$

The last 13 is divisible by 13, so 52 is also. The number 4 can be obtained as $1 + 3 \cdot 1 = 4$.

Based on these examples we state the following algorithm for checking the divisibility of a number v by $10x + 3$:

a. If $v \leq 10x + 3$, then v is divisible by $10x + 3$ if and only if $v = 10x + 3$.

b. If v is greater than $10x + 3$, compute the number $1 + 3 \cdot x$.

c. Truncate the last digit of v, multiply it by $1 + 3 \cdot x$, and add the result to the remaining part of v, obtaining v'. If $v' > v$, repeat the process.

d. If $v \geq v' > 4(10x + 3)$, repeat the process until a number less than or equal to $4(10x + 3)$ is obtained.

e. If $v' \leq 4(10x + 3)$, then the original number v is divisible by $10x + 3$ if and only if $v' = 10x + 3$, $2(10x + 3)$, $3(10x + 3)$, or $4(10x + 3)$.

Why does this algorithm work? Let $n = 10a + b$ and $x = 10c + 3$. Set $y = 1 + 3c = (x - c)/3$. Since 10 is relatively prime to x, the number $v = a + by$ is divisible by x if and only if $10v$ is. Now,

$$\begin{aligned} 10v &= 10(a + by) = 10a + 10b(1 + 3c) \\ &= 10a + b + 3b(10c + 3) = n + 3bx\,. \end{aligned}$$

Therefore, $10v = n + 3bx$ is divisible by x if and only if n is. If we now let $n_1 = a + by = 10a_1 + b_1$ and set $n_2 = a_1 + b_1 y$, then the process continues.

We can also find the remainder. The number $x = 23$ has the form $10c + 3$ and $10557 = 459 \cdot 23$. In this case, $1 + 3c = 7$. Hence,

$$\begin{aligned} 10557 &= 1055 \cdot 10 + 7 & \Longrightarrow && 1055 + 7 \cdot 7 &= 1104\,, \\ 1104 &= 110 \cdot 10 + 4 & \Longrightarrow && 110 + 4 \cdot 7 &= 138\,, \\ 138 &= 13 \cdot 10 + 8 & \Longrightarrow && 13 + 8 \cdot 7 &= 69\,. \end{aligned}$$

Since $69 = 3 \cdot 23$, we conclude that 23 divides 10557.

This implies that $10558/23$ will have remainder 1. We will find this using the algorithm. Note

$$\begin{aligned} 10558 &= 1055 \cdot 10 + 8 & \Longrightarrow && 1055 + 8 \cdot 7 &= 1111\,, \\ 1111 &= 111 \cdot 10 + 1 & \Longrightarrow && 111 + 1 \cdot 7 &= 118\,, \\ 118 &= 11 \cdot 10 + 8 & \Longrightarrow && 11 + 8 \cdot 7 &= 67\,. \end{aligned}$$

Notice that 67 is less than $3 \cdot 23$ and $69 - 67 = 2 = c \cdot$ (remainder).

The remainder r can be expressed as

$$r = (a(10c + 3)) - v')/c\,,$$

when a is 1, 2, 3 such that the difference $a(10c + 3) - v'$ is a minimum. The number v' is obtained from $v/(10c + 3)$ and $v' \leq 3(10c + 3)$. Here c must not equal zero, so that this process does not work if $10c + 3 = 3$.

Concluding Remarks. Clearly, the numbers 9 and 3 play an important role in the algorithms above, and no such simple analogues can be seen for other numbers. Are there any other similar algorithms? From another point of view, it might be interesting to try to find divisibility algorithms in number systems with bases different from 10.

Acknowledgement. I would like to thank Professor F. Bogomolov of NYU, and Professors L. Katzarkov and M. Yotov of UCI for useful remarks. My work on this was done with some support from UCI Partnership and COSMOS programs.

H. Havens
1723 Miramar Drive
Newport Beach, CA 92661
USA

PROBLEMS

Problem proposals and solutions should be sent to Jim Totten, Department of Mathematics and Statistics, University College of the Cariboo, Kamloops, BC, Canada, V2C 4Z9. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a proposal is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk (⋆) after a number indicates that a problem was proposed without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ *or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than* **1 November 2003**. *They may also be sent by email to* **crux-editors@cms.math.ca**. (*It would be appreciated if email proposals and solutions were written in* LaTeX.) *Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

In the solutions section, the problem will be given in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier and Hidemitsu Saeki of the University of Montreal for translations of the problems.

2826. *Proposed by Bernardo Recamán Santos, Bogota, Colombia.*

Show that, for every sufficiently large integer n, it is possible to split the integers 1, 2, $\dots$, n into two disjoint subsets such that the sum of the elements in one set equals the product of the elements in the other.

. . . .

Montrer que pour tout entier n suffisamment grand, il est possible de séparer les entiers 1, 2, $\dots$, n en deux sous-ensembles disjoints de telle sorte que la somme des éléments du premier soit égale au produit des éléments du second.

2827. *Proposed by José Luis Díaz-Barrero and Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let n be a non-negative integer. Determine

$$\sum_{k=0}^{\infty} \frac{\tanh\left(2^k\right)}{2+2\sinh^2\left(2^k\right)}\,.$$

. . . .

Soit n un entier non négatif. Calculer

$$\sum_{k=0}^{\infty} \frac{\tanh\left(2^k\right)}{2+2\sinh^2\left(2^k\right)}\,.$$

2828. *Proposed by Achilleas Pavlos Porfyriadis, Student, American College of Thessaloniki "Anatolia", Thessaloniki, Greece* (*adapted by the Editors*).

Suppose that f satisfies the functional equation

$$f(x)+2f\left(\frac{x+2000}{x-1}\right) \;=\; 4011-x\,.$$

Find the value of $f(2002)$.

. . . .

On suppose que f satisfait l'équation fonctionnelle

$$f(x)+2f\left(\frac{x+2000}{x-1}\right) \;=\; 4011-x\,.$$

Trouver la valeur de $f(2002)$.

2829. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Given $\triangle ABC$ with sides a, b, c, prove that

$$\frac{2\left(a^4+b^4+c^4\right)}{\left(a^2+b^2+c^2\right)^2} + \frac{ab+bc+ca}{a^2+b^2+c^2} \;\geq\; 2\,.$$

. . . .

Montrer que, dans un triangle ABC de côtés a, b, c,

$$\frac{2\left(a^4+b^4+c^4\right)}{\left(a^2+b^2+c^2\right)^2} + \frac{ab+bc+ca}{a^2+b^2+c^2} \;\geq\; 2\,.$$

2830. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*
Suppose that $\Gamma(O,R)$ is the circumcircle of $\triangle ABC$. Suppose that side AB is fixed and that C varies on Γ (always on the same side of AB).

Suppose that I_a, I_b, I_c, are the centres of the excircles of $\triangle ABC$ opposite A, B, C, respectively. If Ω is the centre of the circumcircle of $\triangle I_aI_bI_c$, determine the locus of Ω as C varies.

. . . .

Soit $\Gamma(O,R)$ le cercle circonscrit du triangle ABC. Le côté AB étant fixé, on fait varier C sur Γ (mais sans le faire passer de l'autre côté de AB).

Soit I_a, I_b, et I_c les centres des cercles exinscrits du triangle ABC opposés aux sommets A, B, et C, respectivement. Ω désignant le centre du cercle circonscrit au triangle $I_aI_bI_c$, déterminer le lieu de Ω lorsque C varie.

2831. *Proposed by Achilleas Pavlos Porfyriadis, Student, American College of Thessaloniki "Anatolia", Thessaloniki, Greece.*
For a convex polygon, prove that it is impossible for two sides without a common vertex to be longer than the longest diagonal.

. . . .

Montrer que, dans un polygone convexe, il est impossible que deux côtés non adjacents soient plus longs que la plus longue diagonale.

2832★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let n be a positive integer, and let

$$a(n) = \left|\sum_{j=0}^{3n}(-2)^j\left(\binom{6n+2-j}{j+1}+\binom{6n+1-j}{j}\right)\right|.$$

Prove that

(a) $a(n)=3$ if and only if $n=1$, and

(b) the sequence $\{a(n)\}_{n=1}^{\infty}$ is strictly increasing.

. . . .

Soit n un entier positif, et soit

$$a(n) = \left|\sum_{j=0}^{3n}(-2)^j\left(\binom{6n+2-j}{j+1}+\binom{6n+1-j}{j}\right)\right|.$$

Montrer que

(a) $a(n)=3$ si et seulement si $n=1$, et

(b) la suite $\{a(n)\}_{n=1}^{\infty}$ est strictement croissante.

2833★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let a be a positive real number, and let $n \geq 2$ be an integer. For each $k = 1, 2, \ldots, n$, let x_k be a non-negative real number, λ_k be a positive real number, and let $y_k = \lambda_k x_k + \frac{x_{k+1}}{\lambda_{k+1}}$. Here and elsewhere, indices greater than n are to be reduced modulo n.

(a) If $a > 1$, prove that

$$n + \sum_{k=1}^{n} a^{y_k} \geq 2 \sum_{k=1}^{n} a^{x_k} \qquad \text{and} \qquad 3n + \sum_{k=1}^{n} a^{y_k + y_{k+1}} \geq \sum_{k=1}^{n} (1 + a^{x_k})^2 .$$

(b) If $0 < a < 1$, prove that the opposite inequalities hold.

[The proposer has proofs for the cases $n = 3$ and $n = 4$.]

. . . .

Soit a un nombre réel positif, et soit $n \geq 2$ un entier. Pour chaque $k = 1, 2, \ldots, n$, soit x_k un nombre réel non négatif et λ_k un nombre réel positif, et soit $y_k = \lambda_k x_k + \frac{x_{k+1}}{\lambda_{k+1}}$.Ici et dans ce qui suit, on convient que les indices plus grands que n sont réduits modulo n.

(a) Si $a > 1$, montrer que

$$n + \sum_{k=1}^{n} a^{y_k} \geq 2 \sum_{k=1}^{n} a^{x_k} \qquad \text{et} \qquad 3n + \sum_{k=1}^{n} a^{y_k + y_{k+1}} \geq \sum_{k=1}^{n} (1 + a^{x_k})^2 .$$

(b) Si $0 < a < 1$, montrer que les inégalités sont inversées.

[Le proposeur a une preuve pour les cas $n = 3$ et $n = 4$.]

2834. *Proposed by Michel Bataille, Rouen, France.*

Let $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for integers $n > 2$. Then define

$$g_n = f_{n+6} + 3f_{n+2} + 3f_{n-2} + f_{n-6}$$

for integers $n > 6$. Find $\gcd\{g_{f_{6666}}, g_{f_{666}}\}$.

. . . .

Soit $f_1 = f_2 = 1$ et $f_n = f_{n-1} + f_{n-2}$ où n est un entier, $n > 2$. On définit

$$g_n = f_{n+6} + 3f_{n+2} + 3f_{n-2} + f_{n-6}$$

avec n entier et $n > 6$. Trouver le plus grand commun diviseur de $g_{f_{6666}}$ et $g_{f_{666}}$.

2835. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

For non-negative real numbers x and y, not both equal to 0, prove that

$$\frac{x^4+y^4}{(x+y)^4}+\frac{\sqrt{xy}}{x+y}\geq\frac{5}{8}.$$

. . . .

Si x et y sont deux nombres réels non négatifs non tous nuls, montrer que

$$\frac{x^4+y^4}{(x+y)^4}+\frac{\sqrt{xy}}{x+y}\geq\frac{5}{8}.$$

2836. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Suppose that $\triangle ABC$ is equilateral and that P is an interior point. The lines AP, BP, CP intersect the opposite sides at D, E, F, respectively. Suppose that $PD = PE = PF$. Determine the locus of P.

. . . .

Soit ABC un triangle équilatéral et P un point intérieur. Les droites AP, BP, CP coupent respectivement les côtés opposés en D, E, F. Si l'on suppose que $PD = PE = PF$, déterminer le lieu de P.

2837. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Suppose that Γ is a circle and that I, J, and K are three distinct points in the plane of Γ, but not on Γ. Let A be any point on Γ. Points B, C, D, E, F, and G on Γ are defined by the conditions that chords AB and DE intersect at I, chords BC and EF intersect at J, and chords CD and FG intersect at K. (A tangent is to be regarded as a chord with its point of contact defined to be a pair of coincident points.)

Is it possible to select the positions of I, J, and K so that G coincides with A for all points A lying on Γ? (Justification required!)

. . . .

Soit I, J, et K trois points dans le plan d'un cercle Γ et A un point quelconque sur Γ. On désigne par B, C, D, E, F, et G les points sur Γ tels que les cordes AB et DE, BC et EF, CD et FG se coupent respectivment en I, J, et K. (Une tangente est considérée comme une corde engendrée par deux points qui coïncident.)

Est-il possible de choisir les points I, J, et K de telle sorte que G coïncide avec A pour tous les points A situés sur Γ ? (On demande une justification !)

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2718. [2002 : 112] *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit $A_k \in M_m(\mathbb{R})$ tels que $A_iA_j = O_m$, $i, j \in \{1, 2, \ldots, n\}$, avec $i < j$ et $x_k \in \mathbb{R}^*$, $(k = 1, 2, \ldots, n)$. Montrer que

$$\det\left(I_m + \sum_{k=1}^{n}\left(x_kA_k + x_k^2A_k^2\right)\right) \geq 0\,.$$

Solution par Michel Bataille, Rouen, France.

Pour chaque $p \in \{1, 2, \ldots, n\}$ posons

$$B_p = I_m + \sum_{k=1}^{p}(x_kA_k + x_k^2A_k^2) \qquad \text{et} \qquad C_p = I_m + x_pA_p + x_p^2A_p^2\,.$$

On désire prouver : $\det(B_n) \geq 0$.

En utilisant $A_1A_2 = 0_m$, on voit aussitôt que $C_1C_2 = B_2$. Puis, en utilisant $A_1A_3 = A_1A_2 = 0_m$, il suit que $C_1C_2C_3 = B_2C_3 = B_3$. Par une récurrence immédiate, on arrive à $B_n = C_1C_2\cdots C_n$, et donc

$$\det(B_n) = \det(C_1)\det(C_2)\cdots\det(C_n)\,.$$

Il suffit donc de prouver $\det(C_p) \geq 0$, $(p = 1, 2, \ldots, n)$. Maintenant,

$$C_p = (I_m - \omega x_pA_p)(I_m - \overline{\omega}x_pA_p) = (I_m - \omega x_pA_p)\overline{(I_m - \omega x_pA_p)}\,,$$

où $\omega = \exp(2\pi i/3)$. (La barre indiquant la conjugaison complexe. Il s'ensuit que

$$\begin{aligned}\det(C_p) &= \det(I_m - \omega x_pA_p)\det\overline{(I_m - \omega x_pA_p)}\\ &= \det(I_m - \omega x_pA_p)\overline{\det(I_m - \omega x_pA_p)}\\ &= |\det(I_m - \omega x_pA_p)|^2 \geq 0\,.\end{aligned}$$

Also solved by TIM D. AUSTIN, student, Colchester Royal Grammar School, Colchester, UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There was one incorrect solution.

2724★. [2002 : 174] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let a, b, c be the sides of a triangle and h_a, h_b, h_c, respectively, the corresponding altitudes. Prove that the maximum range of validity of the inequality

$$\left(\frac{h_a^t+h_b^t+h_c^t}{3}\right)^{1/t} \;\leq\; \frac{\sqrt{3}}{2}\left(\frac{a^t+b^t+c^t}{3}\right)^{1/t},$$

where $t \neq 0$, is $\dfrac{-\ln 4}{\ln 4-\ln 3} \;<\; t \;<\; \dfrac{\ln 4}{\ln 4-\ln 3}$.

Partial solution by Murray S. Klamkin, University of Alberta, Edmonton, AB.

We show that the inequality is valid for a narrower range of values, namely $0 < t \leq \dfrac{\ln 9-\ln 4}{\ln 4-\ln 3}$.

If we denote the area of the triangle by F and the circumradius by R, then letting $h_a = 2F/a$, $a = 2R^2 \sin A \sin B \sin C$, etc., the given inequality can be rewritten as

$$\sum_{\text{cyclic}} \sin^t A \;\geq\; \left(\frac{2}{\sqrt{3}}\right)^t \sum_{\text{cyclic}} \sin^t B \sin^t C\,. \tag{1}$$

The inequality

$$\sum_{\text{cyclic}} \sin^t A \;\leq\; 3\left(\frac{\sqrt{3}}{2}\right)^t \tag{2}$$

is a known inequality ([1]) having maximum range $0 \leq t < \dfrac{\ln 9-\ln 4}{\ln 4-\ln 3}$. Since $3(yz+zx+xy) \leq (x+y+z)^2$, we have

$$3\left(\sum_{\text{cyclic}} \sin^t B \sin^t C\right) \;\leq\; \left(\sum_{\text{cyclic}} \sin^t A\right)^2$$

$$\frac{3\left(\sum_{\text{cyclic}} \sin^t B \sin^t C\right)}{\sum_{\text{cyclic}} \sin^t A} \;\leq\; \sum_{\text{cyclic}} \sin^t A \;\leq\; 3\left(\frac{\sqrt{3}}{2}\right)^t,$$

from which (1) follows. This proves (1) for $0 < t < \dfrac{\ln 9-\ln 4}{\ln 4-\ln 3}$, which proves the proposed inequality for this range of t.

Incidentally, the value $\dfrac{\ln 9-\ln 4}{\ln 4-\ln 3}$ is obtained from (2) by equality for the degenerate triangle with angles of 90°, 90°, and 0°. The proposed upper bound on t is obtained from (1) by equality for the same degenerate triangle.

References.

[1] O. Bottema, R.Ž. Djordjević, R.R. Janić, D.S. Mitrinović & P.M. Vasić, *Geometric Inequalities*, Groningen, 1969.

[*Editor's Note:* For negative values of t, when we replace t by $-t$ in the inequality (1), we obtain

$$\left(\frac{2}{\sqrt{3}}\right)^{-t}\left(\sin^{-t}B\sin^{-t}C+\sin^{-t}C\sin^{-t}A+\sin^{-t}A\sin^{-t}B\right) \leq \sin^{-t}A+\sin^{-t}B+\sin^{-t}C\,.$$

Then, multiplying through by $\left(\frac{2}{\sqrt{3}}\right)^{t}\sin^t A\sin^t B\sin^t C$, we get the opposite inequality to (1). Thus, the proposed range of validity for the inequality in this problem must be amended to

$$0 < t < \frac{\ln 4}{\ln 4-\ln 3}\,.$$

We invite readers to find a solution that considers this entire range.]

2725. [2002 : 175] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For $k \geq 1$, let $S_k(n) = \sum\limits_{j=1}^{n}(2j-1)^k$ be the sum of the k^{th} powers of the first n odd numbers.

1. Show that the sequence $\{S_3(n), n \geq 1\}$ contains *infinitely many squares*.
2. ⋆ Prove that this sequence contains only *finitely many squares* of other exponents k.

[*Editor's comments:* Clearly, for Part 2 of the problem to be non-trivial, we must state that $k > 1$ (since $S_1(n) = n^2$, which is a square for all n). This was actually stipulated by the proposer in his original submission, but was inadvertently omitted. Apparently, all the solvers realized this omission and interpreted the question correctly. Also, ".. OF other exponents" was clearly a typo for ".. FOR other exponents".]

Solution to Part 1 *by the Austrian IMO Team 2002* (*slightly modified by the editor*).

Note first that

$$\begin{aligned} S_3(n) &= \sum_{j=1}^{2n} j^3 - \sum_{j=1}^{n}(2j)^3 \\ &= \left(\frac{2n(2n+1)}{2}\right)^2 - 8\left(\frac{n(n+1)}{2}\right)^2 \;=\; n^2(2n^2-1)\,. \end{aligned}$$

Hence, $S_3(n)$ is a square if and only if $2n^2 - 1 = m^2$ for some integer m. The Diophantine equation $m^2 - 2n^2 = -1$ is a Pell-type equation which is well known to have infinitely many solutions, since 2 is not a perfect square. Indeed, $m = n = 1$ is clearly a solution. If we define m_k and n_k for $k \in \mathbb{N}$ by

$$m_1 = n_1 = 1\,, \qquad m_k + \sqrt{2}n_k \;=\; (m_1 + \sqrt{2}n_1)^{2k-1}\,,$$

then $m_k - \sqrt{2}n_k = (m_1 - \sqrt{2}n_1)^{2k-1}$, and hence,

$$\begin{aligned} m_k^2 - 2n_k^2 &= \Big((m_1 + \sqrt{2}n_1)(m_1 - \sqrt{2}n_1)\Big)^{2k-1} \\ &= (m_1^2 - 2n_1^2)^{2k-1} \;=\; -1\,. \end{aligned}$$

Therefore, every pair (m_k, n_k) is a solution. It follows that the sequence $\{S_3(n)\}_{n=1}^{\infty}$ contains infinitely many squares.

Also solved (Part 1 only) by NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There was one incorrect solution.

All the submitted solutions are similar to the one featured above. Both Klamkin and Zhou simply quoted known results about the existence of infinitely many solutions to the Pell equation involved. In fact, from known theory, the solutions obtained above actually yield all the n values for which $S_3(n)$ is a square. This was explicitly pointed out only by Guersenzvaig. No solution to Part 2 was received, so it remains open. Klamkin commented that this may be a very difficult problem in general, but the case $k = 2$ might be solvable, since the corresponding Diophantine equation is $n(2n-1)(2n+1) = 3m^2$.

2726. [2002 : 175] *Proposed by Armend Shabini, University of Prishtina, Prishtina, Kosovo, Serbia.*

Given the finite sequence of real numbers, $\{a_k\}$, $1 \le k \le 2n$, where the terms satisfy

$$a_{2k} - a_{2k-1} \;=\; d\,,\; 1 \le k \le n\,, \qquad \text{and} \qquad \frac{a_{2k+1}}{a_{2k}} \;=\; q\,,\; 1 \le k \le n-1\,,$$

prove that, when $q \neq 1$,

(a) $\displaystyle\sum_{k=1}^{2n} a_k \;=\; \frac{2qa_{2n} - 2a_1 - nd(1+q)}{q-1}$, and

(b) $\displaystyle a_{2n} \;=\; a_1\,q^{n-1} + d\left(\frac{1-q^n}{1-q}\right)$.

[*Editor's Note:* The formula in (b) was misprinted originally but has been corrected above. The editors were at fault here, not the proposer. All solvers corrected the error.]

Solution by Joe Howard, Portales, NM, USA.

$$\begin{aligned}\sum_{k=1}^{2n} a_k &= a_1 + (1+q)a_2 + \cdots + (1+q)a_{2n} - qa_{2n} \\ &= (1+q)(a_2 + a_4 + \cdots + a_{2n}) - qa_{2n} + a_1 \\ &= (1+q)(a_1 + a_3 + \cdots + a_{2n-1} + nd) - qa_{2n} + a_1 \\ &= (a_3 + a_5 + \cdots + a_{2n-1}) + q(a_1 + a_3 + \cdots + a_{2n-1}) \\ &\quad -qa_{2n} + 2a_1 + nd(1+q) \\ &= q(a_2 + a_4 + \cdots + a_{2n}) + q(a_1 + a_3 + \cdots + a_{2n-1}) \\ &\quad -2qa_{2n} + 2a_1 + nd(1+q) \\ &= q\sum_{k=1}^{2n} a_k - 2qa_{2n} + 2a_1 + nd(1+q)\,,\end{aligned}$$

from which (a) follows.

An easy induction argument shows that, for $k = 1, 2, \ldots, n$,

$$a_{2k} = a_1 q^{k-1} + d(1 + q + q^2 + \cdots + q^{k-1})\,.$$

By setting $k = n$ and summing the geometric progression, we obtain (b).

Also solved by MICHEL BATAILLE, Rouen, France; PIERRE BORNSZTEIN, Pontoise, France; PAUL BRACKEN, CRM, Université de Montréal, Montréal, QC; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; VEDULA N. MURTY, Dover, PA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU and BOGDAN IONIŢĂ, Bucharest, Romania; and the proposer.

2727. [2002 : 176] *Proposed by Armend Shabini, University of Prishtina, Prishtina, Kosovo, Serbia.*

Given the finite sequence of real numbers, $\{a_k\}$, $1 \le k \le n$, where the terms satisfy

$$a_k - a_{k-1} \;=\; a_{k-1} - a_{k-2} + d\,, \quad k > 2\,,\;\; d \in \mathbb{R}\,,$$

find a closed form expression for $\sum\limits_{k=1}^{n} a_k$.

Use this to find the value of $\sum\limits_{k=0}^{n-1} \binom{2k+2}{2k}$.

Solution by David Loeffler, student, Trinity College, Cambridge, UK.

Solving the non-homogeneous recurrence relation, we see that $a_k = \frac{1}{2}dk^2 + Bk + C$, for constants B and C depending on a_1 and a_2. To verify that this is a solution, observe:

$$a_k - 2a_{k-1} + a_{k-2} \;=\; \frac{1}{2}d\left(k^2 - 2(k-1)^2 + (k-2)^2\right) \;=\; d$$

(the linear terms clearly cancel). Solving for B and C gives

$$B = a_2 - a_1 - \frac{3}{2}d \quad \text{and} \quad C = -a_2 + 2a_1 + d\,.$$

Hence,

$$\sum_{k=1}^{n} a_k = \frac{1}{2}d\left[\frac{1}{6}n(n+1)(2n+1)\right] + B\left[\frac{1}{2}n(n+1)\right] + Cn\,,$$

using standard summation results.

We see that if $a_k = \binom{2k}{2}$, then $a_k - a_{k-1} = 4k - 3$. Thus, $d = 4$. Checking the first two terms, we have $B = -1$, $C = 0$. Therefore,

$$\begin{aligned} \sum_{k=0}^{n-1} \binom{2k+2}{2k} &= \sum_{k=1}^{n} a_k = 2\left[\frac{1}{6}n(n+1)(2n+1)\right] - \left[\frac{1}{2}n(n+1)\right] \\ &= \frac{1}{6}n(n+1)(4n-1)\,. \end{aligned}$$

Also solved by MICHEL BATAILLE, Rouen, France; PAUL BRACKEN, CRM, Université de Montréal, Montréal, QC; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; NORVALD MIDTTUN, Royal Norwegian Naval Academy, Bergen, Norway; VEDULA N. MURTY, Dover, PA, USA; ROBERT P. SEALY, Mount Allison University, Sackville, NB; SKIDMORE COLLEGE PROBLEM GROUP, Saratoga Springs, New York; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Editor's note: Some solutions used the fact that the third differences of a_k are zero; therefore, an expression for a_k can be obtained using Newton-Gregory forward differences.

2728. [2002 : 177] *Proposed by Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.*

The distance between two well-known points in $\triangle ABC$ is

$$\frac{bc}{a+b+c}\sqrt{2(\cos A + 1)}\,.$$

What are the points?

A combination of almost identical solutions by Nikolaos Dergiades, Thessaloniki, Greece and Li Zhou, Polk Community College, Winter Haven, FL, USA.

Let I be the incentre of $\triangle ABC$, and let r be the inradius. Let $[ABC]$ denote the area of $\triangle ABC$, and let s be its semiperimeter. Then

$$\begin{aligned} AI &= \frac{r}{\sin(A/2)} = \frac{[ABC]}{s\,\sin(A/2)} = \frac{bc\,\sin(A)}{(a+b+c)\sin(A/2)} \\ &= \frac{bc}{a+b+c}\left(2\cos(A/2)\right) = \frac{bc}{a+b+c}\sqrt{2(\cos A + 1)}\,. \end{aligned}$$

The answer is: the points are the vertex A and the incentre I.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; VEDULA N. MURTY, Dover, PA, USA; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; TITU ZVONARU and BOGDAN IONIŢĂ, Bucharest, Romania; and the proposer.

2729. [2002 : 177] *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

Let $Z(n)$ denote the number of trailing zeroes of $n!$, where $n \in \mathbb{N}$.

(a) Prove that $\dfrac{Z(n)}{n} < \dfrac{1}{4}$.

(b)⋆ Prove or disprove that $\lim\limits_{n\to\infty} \dfrac{Z(n)}{n} = \dfrac{1}{4}$.

Solution by Natalio H. Guersenzvaig, Universidad CAECE, Buenos Aires, Argentina.

(a) Clearly, $\mathbb{Z}(n) = E_5(n)$, where $E_5(n)$ denotes the largest integer d such that $5^d \mid n!$. By the well-known Legendre formulas [Ed: see, for example, Calvin T. Long, *Elementary Introduction to Number Theory*, 3rd edition, pp 64–67, Theorems 2.29 and 2.30], we have

$$E_5(n) = \sum_{k\geq 1} \left\lfloor \frac{n}{5^k} \right\rfloor = \frac{n - S_5(n)}{5-1} = \frac{n - S_5(n)}{4},$$

where $S_5(n)$ is the sum of the digits in the base-5 representation of n. Since $S_5(n) \geq 1$, it follows immediately that $\dfrac{\mathbb{Z}(n)}{n} < \dfrac{1}{4}$.

(b) Let $n = (n_r \cdots n_1 n_0)_5$ be the base-5 representation of n, where $n_i \in \{0, 1, 2, 3, 4\}$ for all $i = 0, 1, 2, \ldots, r$. From $5^r \leq n < 5^{r+1}$ we get $r = \lfloor \log_5 n \rfloor$. Hence, $S_5(n) \leq 4(r+1) = 4\big(\lfloor \log_5 n \rfloor + 1\big)$, and therefore, $\dfrac{S_5(n)}{4n} \leq \dfrac{\lfloor \log_5 n \rfloor + 1}{n}$. Since $0 \leq \dfrac{S_5(n)}{4n}$ and $\lim\limits_{n\to\infty} \dfrac{\lfloor \log_5 n \rfloor + 1}{n} = 0$, we have $\lim\limits_{n\to\infty} \dfrac{S_5(n)}{4n} = 0$ by the Squeeze Theorem. It follows that

$$\lim_{n\to\infty} \frac{\mathbb{Z}(n)}{n} = \lim_{n\to\infty} \left(\frac{1}{4} - \frac{S_5(n)}{4n} \right) = \frac{1}{4}.$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; TIM D. AUSTIN, Student, Colchester Royal Grammar School, Colchester, UK; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; PIERRE BORNSZTEIN, Pontoise, France; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; KEITH EKBLAW, Walla Walla, WA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, *Austria; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; OLOV WILANDER, student, Christ's College, Cambridge, UK; LI ZHOU, Polk Community College, Winter Haven, FL, USA. Part* (a) *only was also solved by CHARLES ASHBACHER, Hiawatha, IA, USA; The AUSTRIAN IMO TEAM, 2002; NIKOLAOS DERGIADES, Thessaloniki, Greece; and the proposer. There were two incorrect solutions to part* (b).

Both Loeffler and Wilander studied the general problem of expanding $n!$ in base b. Both of them remarked that if $b = \prod_i p_i^{\alpha_i}$, then

$$\lim_{n\to\infty} \frac{\mathbb{Z}_b(n)}{n} = \min_i \left(\frac{1}{\alpha_i(p_i-1)}\right),$$

where $\mathbb{Z}_b(n)$ denotes the number of trailing zeroes of $n!$ in base b.

2730. [2002 : 177] *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Let $\mathrm{AM}(x_1, x_2, \ldots, x_n)$ and $\mathrm{GM}(x_1, x_2, \ldots, x_n)$ denote the arithmetic mean and the geometric mean of the real numbers x_1, x_2, $\ldots$, x_n, respectively.

Given positive real numbers a_1, a_2, $\ldots$, a_n, b_1, b_2, $\ldots$, b_n, prove that

(a) $\mathrm{GM}(a_1+b_1, a_2+b_2, \ldots, a_n+b_n)$
$\geq \mathrm{GM}(a_1, a_2, \ldots, a_n) + \mathrm{GM}(b_1, b_2, \ldots, b_n)$.

For each real number $t \geq 0$, define $f(t) = \mathrm{GM}(t+b_1, t+b_2, \ldots, t+b_n) - t$.

(b) Prove that $f(t)$ is a monotonic increasing function of t, and that

$$\lim_{t\to\infty} f(t) = \mathrm{AM}(b_1, b_2, \ldots, b_n).$$

[*Editor's note:* Several solvers pointed out that part (a) is ***CRUX with MAYHEM*** problem **2176** [1996 : 275; 1997 : 444].]

Solution to Part (b) *by Murray S. Klamkin, University of Alberta, Edmonton, AB.*

Let $G(x) = \mathrm{GM}(1+b_1x, 1+b_2x, \ldots, 1+b_nx)$. Then $f(t) = \dfrac{G(x)-1}{x}$ where $x = \dfrac{1}{t}$. Thus, $\lim\limits_{t\to\infty} f(t) = \lim\limits_{x\to 0} \dfrac{G(x)-1}{x}$. By l'Hôpital's Rule,

$$\lim_{x\to 0} \frac{G(x)-1}{x} = \lim_{x\to 0} G'(x) = \lim_{x\to 0} \frac{G(x)}{n} \sum_{k=1}^{n} \frac{b_k}{1+b_kx} = \mathrm{AM}(b_1, b_2, \ldots, b_n).$$

Also solved by TIM AUSTIN, Colchester Royal Grammar School, Colchester, UK; MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; JOSÉ LUIS DÍAZ-BARRERO and JUAN JOSÉ EGOZCUE, Universitat Politècnica de Catalunya, Barcelona, Spain; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; M. PERISASTRY, Maharaja's College, Vizianagaram, India (part (a))*; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de*

Valladolid, Valladolid, Spain; VEDULA N. MURTY, Dover, PA, USA; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

According to Howard, Part (a) *also appears in* Equations and Inequalities, *by Herman, Kucera, and Simsa, Springer-Verlag, p. 158, and in* Aspects of Calculus, *by Klambauer, p. 203.*

2731. [2002 : 178] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let C be a conic with foci F_1, F_2, and directrices D_1, D_2, respectively.

Given any point M on the conic, draw the line passing through M, perpendicular to the directrices, intersecting D_1, D_2, at M_1, M_2, respectively. Let R be the point of intersection of the lines M_1F_1 and M_2F_2. Prove that

(a) $\dfrac{\overline{F_1R}}{\overline{M_1R}}$ is independent of the choice of M;

(b) the normal to the conic at M passes through R.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

(a) Since M_1M_2 is parallel to F_1F_2 it represents the (fixed) distance between the directrices. Moreover, by similar triangles, $\dfrac{F_1R}{M_1R} = \dfrac{F_1F_2}{M_1M_2}$. Thus, the ratio is independent of the choice of M.

(b) Suppose that MR intersects F_1F_2 at S. Then

$$\frac{F_1S}{M_1M} = \frac{RS}{RM} = \frac{F_2S}{M_2M}.$$

By the definition of focus and directrix, we also have $\dfrac{F_1M}{M_1M} = e = \dfrac{F_2M}{M_2M}$, where e is the eccentricity. Hence, $\dfrac{F_1S}{F_1M} = \dfrac{F_2S}{F_2M}$. In the case of an ellipse, this condition means that RM is the angle bisector of $\angle F_1MF_2$, and is thus normal to the conic at M, as desired. For the hyperbola case, we draw through F_1 the line parallel to F_2M, intersecting RS at T. Then $\dfrac{F_1S}{F_1T} = \dfrac{F_2S}{F_2M}$; hence, $F_1T = F_1M$. Consequently, $\angle F_1MS = \angle F_2MR$, and we conclude again that RM is the normal at M.

Also solved by MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

Loeffler notes that M does not need to lie on the conic for Part (a) *of the problem. Furthermore, Bataille shows that the ratio, F_1R/M_1R, in Part* (a) *is equal to e^2 as follows: In standard notation, $c = F_1F_2/2$, a is half the distance between the intersection points of F_1F_2 with the conic, a/e is the distance from the centre of the conic to a directrix, and $c/a = e$. Then*

$$\frac{F_1R}{M_1R} = \frac{F_1F_2}{M_1M_2} = \frac{c}{(a/e)} = e^2.$$

2733★. [2002 : 179] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, AB.*

It is a known result that if O is the circumcentre of $\triangle A_1A_2A_3$, and if O_1, O_2, O_3, are the circumcentres of $\triangle OA_2A_3$, $\triangle OA_3A_1$, $\triangle OA_1A_2$, respectively, then the lines A_1O_1, A_2O_2 and A_3O_3 are concurrent.

Does the corresponding result hold for simplexes? That is, if O is the circumcentre of a simplex $A_0A_1\ldots A_n$ and O_k is the circumcentre of the simplex determined by O and the face opposite A_k, are the lines O_kA_k, $k = 0, 1, \ldots, n$, concurrent?

Solution by David Loeffler, student, Trinity College, Cambridge, UK.

Professor Klamkin's result does not hold in 3 dimensions. If we consider a Cartesian coordinate system and consider the tetrahedron $ABCD$ where

$$A_1 = (0,0,0)\,, \quad A_2 = (1,0,0)\,, \quad A_3 = (0,1,0)\,, \quad A_4 = (0,0,2)\,,$$

then it is readily verified that

$$O = \left(\frac{1}{2},\frac{1}{2},1\right)\,, \quad O_1 = \left(-1,-1,\frac{1}{4}\right)\,, \quad O_2 = \left(-1,\frac{1}{2},1\right)\,,$$
$$O_3 = \left(\frac{1}{2},-1,1\right)\,, \quad O_4 = \left(\frac{1}{2},\frac{1}{2},\frac{1}{4}\right)\,.$$

Then A_1, O_1, A_2, and O_2 are not coplanar, since

$$\begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ -1 & -1 & \frac{1}{4} & 1 \\ -1 & \frac{1}{2} & 1 & 1 \end{vmatrix} = \frac{9}{8} \neq 0\,.$$

Therefore, the lines A_1O_1 and A_2O_2 are not concurrent.

No other solutions were submitted. Is there some other way to generalize to 3 dimensions?

2734. [2002 : 179] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, AB.*

Prove that

$$(bc)^{2n+3} + (ca)^{2n+3} + (ab)^{2n+3} \geq (abc)^{n+2}\left(a^n + b^n + c^n\right)\,,$$

where a, b, c, are non-negative reals, and n is a non-negative integer.

I. *Solution by Michel Bataille, Rouen, France.*

The inequality obviously holds if a, b, or c is 0. Thus, we may assume that a, b, $c > 0$. Dividing by $(abc)^{2n+3}$ and setting $x = \frac{1}{a}$, $y = \frac{1}{b}$, $z = \frac{1}{c}$, the given inequality becomes

$$x^{2n+3} + y^{2n+3} + z^{2n+3} \geq xy^{n+1}z^{n+1} + yz^{n+1}x^{n+1} + zx^{n+1}y^{n+1}\,, \quad (1)$$

for x, y, $z > 0$. Without loss of generality, we suppose that $x \geq y \geq z$. Now,

$$x^{2n+3} + y^{2n+3} + z^{2n+3} \;\geq\; x^{n+1}y^{n+2} + y^{n+1}z^{n+2} + z^{n+1}x^{n+2}\,,$$

since $(x^{n+2} - y^{n+2})(x^{n+1} - z^{n+1}) + (y^{n+2} - z^{n+2})(y^{n+1} - z^{n+1}) \geq 0$, and

$$\begin{aligned} & x^{n+1}y^{n+2} + y^{n+1}z^{n+2} + z^{n+1}x^{n+2} \\ & \qquad \geq\; xy^{n+1}z^{n+1} + yz^{n+1}x^{n+1} + zx^{n+1}y^{n+1}\,, \end{aligned}$$

since $z^{n+1}(x-y)(x^{n+1} - y^{n+1}) + y^{n+1}(y-z)(x^{n+1} - z^{n+1}) \geq 0$. Then (1) readily follows.

II. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA*.

The inequality is clearly true if $abc = 0$. Therefore, we may suppose $0 < a \leq b \leq c$. Then $\frac{1}{a} \geq \frac{1}{b} \geq \frac{1}{c}$ and $\frac{bc}{a} \geq \frac{ca}{b} \geq \frac{ab}{c}$. Hence, by the Rearrangement Inequality [*Ed*. See, for example, Dragos Hrimiuc, π *in the Sky*, Pacific Institute for the Mathematical Sciences, Dec 2000, pp. 21–23], we have

$$\begin{aligned} & \frac{1}{a}\left(\frac{bc}{a}\right)^{n+1} + \frac{1}{b}\left(\frac{ca}{b}\right)^{n+1} + \frac{1}{c}\left(\frac{ab}{c}\right)^{n+1} \\ & \geq\; \frac{1}{2}\left[\frac{1}{c}\left(\frac{bc}{a}\right)^{n+1} + \frac{1}{a}\left(\frac{ca}{b}\right)^{n+1} + \frac{1}{b}\left(\frac{ab}{c}\right)^{n+1}\right. \\ & \qquad\qquad \left. + \frac{1}{b}\left(\frac{bc}{a}\right)^{n+1} + \frac{1}{c}\left(\frac{ca}{b}\right)^{n+1} + \frac{1}{a}\left(\frac{ab}{c}\right)^{n+1}\right] \\ & =\; \frac{1}{2}\left[\frac{1}{c}\left(\left(\frac{bc}{a}\right)^{n+1} + \left(\frac{ca}{b}\right)^{n+1}\right) + \frac{1}{a}\left(\left(\frac{ca}{b}\right)^{n+1} + \left(\frac{ab}{c}\right)^{n+1}\right)\right. \\ & \qquad\qquad \left. + \frac{1}{b}\left(\left(\frac{ab}{c}\right)^{n+1} + \left(\frac{bc}{a}\right)^{n+1}\right)\right] \\ & =\; \frac{1}{2}\left[c^n\left(\left(\frac{b}{a}\right)^{n+1} + \left(\frac{a}{b}\right)^{n+1}\right) + a^n\left(\left(\frac{c}{b}\right)^{n+1} + \left(\frac{b}{c}\right)^{n+1}\right)\right. \\ & \qquad\qquad \left. + b^n\left(\left(\frac{a}{c}\right)^{n+1} + \left(\frac{c}{a}\right)^{n+1}\right)\right] \\ & \geq\; c^n + a^n + b^n\,, \end{aligned}$$

since $x + \frac{1}{x} \geq 2$ for all $x > 0$. Multiplying by $(abc)^{n+2}$ yields the desired inequality.

III. *Essentially the same solution by Pierre Bornsztein, Pontoise, France; Zeljko Hanjš, University of Zagreb, Zagreb, Croatia; Joe Howard, Portales, NM, USA; and Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.*

The desired inequality may be rewritten as:

$$\sum_{\text{cyclic}} a^{2n+3}b^{2n+3} \;\geq\; \sum_{\text{cyclic}} a^{2n+2}b^{2n+2}c^{n+2}\,.$$

Since the vector $(2n+3, 2n+3, 0)$ majorizes the vector $(2n+2, n+2, n+2)$, this inequality is a direct consequence of the Majorization Inequality, also known as Muirhead's Theorem. (See, for example, G. Hardy, J.E. Littlewood, and G. Pólya, *Inequalities*, 2nd ed. Cambridge University Press.) Note that equality occurs if and only if $a = b = c$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; THE AUSTRIAN IMO TEAM 2002; PAUL BRACKEN, CRM, Université de Montréal, Montréal, QC; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; VEDULA N. MURTY, Dover, PA, USA; ZENOFON PAPANICOLAOU, Athens, Greece; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

Using the Rearrangement Inequality, Janous obtained the more general result that if real numbers α and β satisfy $\alpha \geq \beta > 0$ and $\alpha \leq 2\beta$, then

$$(bc)^\alpha + (ca)^\alpha + (ab)^\alpha \geq (abc)^\beta (a^{2\alpha-3\beta} + b^{2\alpha-3\beta} + c^{2\alpha-3\beta})$$

for all positive reals a, b, and c. Of course, this also follows immediately from the Majorization Inequality, since clearly the vector $(2\alpha - 2\beta, \beta, \beta)$ is majorized by the vector $(\alpha, \alpha, 0)$.

2735★. [2002 : 179] *Proposed by Richard I. Hess, Rancho Palos Verdes, CA, USA.*

Given three Pythagorean triangles with the same hypotenuse, is it possible that the area of one triangle is equal to the sum of the areas of the other two triangles?

Editor's Note: No solutions have been submitted for this problem.

2736. [2002 : 180] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let $ABCD$ be a convex quadrilateral. From points A and B, draw lines parallel to sides BC and AD, respectively, giving points G and F on CD, respectively.

Let P and Q be the points of the intersection of the diagonals of the trapezoids $ABFD$ and $ABCG$, respectively.

Prove that $PQ \parallel CD$.

Solution by David Loeffler, student, Trinity College, Cambridge, UK.

Consider the following locus problem: points R and S are fixed, and T varies on a fixed line l, where R and S are on the same side of l. Point U lies on l such that $RSTU$ is a convex trapezoid. Point X is the intersection of the diagonals of this trapezoid. Show that the locus of X is a line parallel to l.

This is not difficult.Let R', S', and X' be the feet of the perpendiculars from R, S, and X, respectively, to l. Then

$$XX' = \frac{UX}{US} \cdot SS' = \frac{XT}{RT} \cdot RR' .$$

However, $\triangle RUX$ is similar to $\triangle TSX$, so that

$$\frac{UX}{XS} = \frac{RX}{XT} ,$$

and therefore,

$$\frac{UX}{UX+XS} = \frac{RX}{RX+XT} \qquad \text{or} \qquad \frac{UX}{US} = \frac{RX}{RT} .$$

Hence,

$$\frac{XX'}{RR'} + \frac{XX'}{SS'} = \frac{UX}{US} + \frac{XT}{RT} = \frac{RX}{RT} + \frac{XT}{RT} = 1 ,$$

implying that XX' is constant as T varies (since RR' and SS' are clearly constant). Hence, the locus of X is a line parallel to l, as claimed.

This immediately implies the result in the problem. Let $R = A$, $S = B$, and let l be the line CD. Let C and F be two possible positions of the variable point T. Then U is successively G and D, and X is first Q, then P. Hence, both P and Q lie on the locus defined above, which we showed to be a line parallel to l. Thus, PQ is parallel to CD, as desired.

Also solved by MICHEL BATAILLE, Rouen, France; NIKOLAOS DERGIADES, Thessaloniki, Greece; DAVID DOSTER, Choate Rosemary Hall, Wallingford, CT, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU and BOGDAN IONIŢĂ, Bucharest, Romania; and the proposer. Zhou, Zvonaru and Ioniţă have noted that, if AD is parallel to BC, then $P = Q$ and PQ is not well-defined.

2737. [2002 : 180] *Proposed by Lyubomir Lyubenov, teacher, and Ivan Slavov, student, Foreign Language High School "Romain Rolland", Stara Zagora, Bulgaria.*

Find all solutions of the equation

$$x^n - 2nx^{n-1} + 2n(n-1)x^{n-2} + ax^{n-3} + bx^{n-4} + \cdots + c = 0 ,$$

given that there are n real roots.

Solution by Mihály Bencze, Brasov, Romania and Li Zhou, Polk Community College, Winter Haven, FL, USA.

Let $x_1, x_2, \ldots, x_n$ be the real roots of the equation. Then

$$\sum_{i=1}^{n} x_i \;=\; 2n \qquad \text{and} \qquad \sum_{1\le i<j\le n} x_i x_j \;=\; 2n(n-1)\,,$$

so that

$$\sum_{i=1}^{n} x_i^2 \;=\; \left(\sum_{i=1}^{n} x_i\right)^2 - \;2\sum_{1\le i<j\le n} x_i x_j \;=\; 4n^2 - 4n(n-1) \;=\; 4n\,.$$

Therefore,

$$\sum_{i=1}^{n}(x_i-2)^2 \;=\; \sum_{i=1}^{n} x_i^2 - 4\sum_{i=1}^{n} x_i + 4n \;=\; 4n - 8n + 4n \;=\; 0\,,$$

whence $x_1 = x_2 = \cdots = x_n = 2$.

Also solved by TIM AUSTIN, student, Colchester Royal Grammar School, Colchester, UK; MICHEL BATAILLE, Rouen, France; PIERRE BORNSZTEIN, Pontoise, France; NIKOLAOS DERGIADES, Thessaloniki, Greece; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; TITU ZVONARU and BOGDAN IONIŢĂ, Bucharest, Romania; and the proposers. There was one incomplete solution submitted.