THE ACADEMY CORNER

No. 49

Bruce Shawyer

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APICS Mathematics Competition 2002

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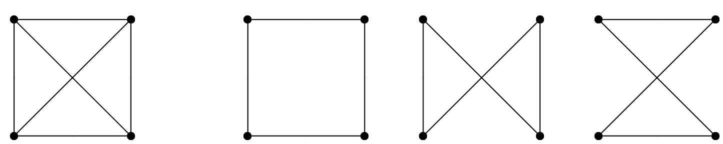
1. For each positive integer n, define $s_n = (n+13)(n+77)$ and $t_n = n(n+91)$. Let $S = s_1 + s_2 + \cdots + s_{2002}$ and $T = t_1 + t_2 + \cdots + t_{2002}$. Which is larger, S or T? Prove your answer.

2. K_n is a graph on n vertices, where every pair of vertices is joined by an edge. There are three different cycles of length 4 in the graph of K_4.

K_4



How many different cycles of length t are there in K_n?

3. Let $f(a,b)$ be the sum of all the positive integers between a and b inclusive. For examples, $f(1,5) = 1+2+3+4+5 = 15$.

(a) Determine the value of $f(13,53)$.

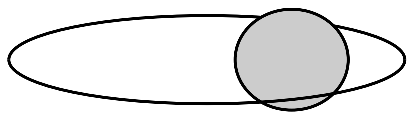
(b) Determine the value of $f(13333\ldots33, 533\ldots3333)$, where there are n 3's in each expression.

4. Let P be a point in the plane with positive integer coordinates, and let O be the origin. Consider the circle with centre O that passes through P. Let T be the tangent to the circle at P, and let T meet the x– and y–axes at X and Y, respectively.

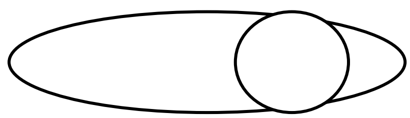
Prove that the area of $\triangle OXY$ cannot equal 2002.

5. For each positive integer n, let M_n be the square matrix where each diagonal entry is 2002, and every other entry is 1. Determine the smallest positive integer n for which $\det(M_n)$ is a perfect square.

6. An ellipse with formula $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a>b)$ lies in the plane $z=0$. What is the locus of the centre of a sphere with radius b that moves so as to make contact with the ellipse at two points. (The sphere may be visualized as rolling on an elliptical ring so that the sphere is just small enough to pass through the centre of the ring.)



Oblique view



Top view

7. At the Sackville Dim Sum restaurant, all dishes come in three sizes: small, medium and large. Small dishes cost x dollars, medium dishes cost y dollars, and large dishes cost z dollars, where x, y and z are positive integers with $x<y<z$. At this restaurant, there is no tax on any dish and the prices have not changed for a long time.

Margaret, Art and Edgar had dinner there last night, and together, they ordered 9 small dishes, 6 medium dishes and 8 large dishes. When the bill came, the following conversation ensued:

Margaret: "This bill is exactly twice as much as when I last came here."

Art: "This is exactly three times as much as when I last came here."

Edgar: "Oh, that was a delicious meal, and very reasonably priced too. Even if we give the waiter a 10% tip, the total is still less than $\$100$."

Determine the values of x, y and z, and prove that your answer is unique.

This is the last Academy Corner. When I started it, I thought that there would be lots of competition papers from many places. Apart from the Putnam and the International Competition for University Students, it appears that these occur only in Atlantic Canada. Sadly, I bid you farewell.

THE OLYMPIAD CORNER

No. 226

R.E. Woodrow

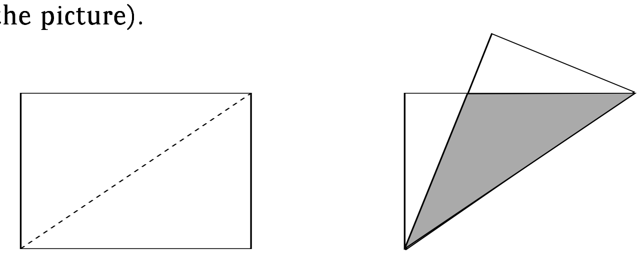
All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

We begin this number with the problems of the XV Gara Nazionale di Matematica 1999. Thanks go to Ed Barbeau, Canadian team Leader to the IMO in Romania for collecting them.

XV GARA NAZIONALE DI MATEMATICA

Cesenatico, 7 maggio 1999

1. Given a rectangular sheet with sides a and b, with $a > b$, fold it along a diagonal. Determine the area of the overlapped triangle (the shaded triangle in the picture).



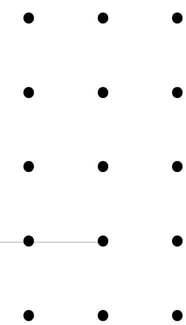
2. A natural number is said to be *balanced* if the number of its decimal digits equals the number of its distinct prime factors (for instance 15 is balanced, whereas 49 is not balanced). Prove that there are only finitely many balanced numbers.

3. Let Γ, Γ_1, Γ_2 be three circles with radii r, r_1, r_2, respectively, with $0 < r_1 < r_2 < r$. The circles Γ_1 and Γ_2 are internally tangent to Γ at two distinct points A and B and meet in two distinct points. Prove that the segment AB contains an intersection point of Γ_1 and Γ_2 if and only if $r_1 + r_2 = r$.

4. Albert and Barbara play the following game. On a table there are 1999 sticks. Each player in turn must remove from the table some sticks, provided that he removes at least one stick and at most one half of the sticks remaining on the table at the moment of his move. The player who leaves just one stick on the table loses the game. Barbara moves first.

Determine for which of the players there exists a winning strategy, and describe this strategy.

5. On a lake there is a village of pile-built dwellings, set on the nodes of a rectangular array $m \times n$ (see the example in the picture). Each dwelling is an end-point of exactly p bridges which connect the dwelling with one or more of the adjacent dwellings (here adjacent means with respect to the array; hence diagonal connection is not allowed). Determine for which values of m, n and p it is possible to place the bridges so that from any dwelling one can reach any other dwelling. (Clearly two adjacent dwellings can be connected by more than one bridge).



6. (a) Determine all pairs (x, k) of positive integers which satisfy the equation

$$3^k - 1 = x^3 .$$

(b) Prove that if n is an integer greater than 1 and different from 3 there are no pairs (x, k) of positive integers satisfying the equation

$$3^k - 1 = x^n .$$

Next we give the 2000 Chinese Mathematical Olympiad. Thanks go to Andy Liu, Canadian Team Leader to the IMO 2000 in Korea, who collected them.

2000 CHINESE MATHEMATICAL OLYMPIAD

1. In triangle ABC, $a \leq b \leq c$, where $a = BC$, $b = CA$ and $c = AB$, the circumradius is R and the inradius is r. What can be said about $\angle C$ if $a + b - 2R - 2r$ is

(a) positive; (b) zero; (c) negative?

2. The sequence $\{a_n\}$ is defined by $a_1 = 0$, $a_2 = 1$ and $a_n = \frac{n}{2}a_{n-1} + \frac{n(n-1)}{2}a_{n-2} + (-1)^n(1 - \frac{n}{2})$ for $n \geq 3$. Simplify

$$a_n + 2\binom{n}{1}a_{n-1} + 3\binom{n}{2}a_{n-2} + \cdots + (n-1)\binom{n}{n-2}a_2 + n\binom{n}{n-1}a_1 .$$

3. In a table-tennis tournament, all games are between pairs of participants. Each participant is a member of at most two pairs. No participant ever plays against another if the two form a pair. Two pairs play exactly once against each other as long as the preceding rule is not violated.

A set $\{a_1, a_2, \ldots, a_k\}$ is given, where k is a positive integer and $0 < a_1 < a_2 < \cdots < a_k$ are multiples of 6. What is the minimum number of participants so that at the end of the tournament, the number of games played by each participant is a_i for some i, and for each i, at least one participant has played exactly a_i games?

4. Let $\langle a_1, a_2, \ldots, a_n \rangle$ be any permutation of 1, 2, $\ldots$, n. Define $b_k = \max\{a_i : 1 \le i \le k\}$ for $k = 1$, 2, $\ldots$, n. Determine the average value of the first term, a_1, of all permutations for which the sequence $\{b_1, b_2, \ldots, b_n\}$ takes on exactly two distinct values.

5. Find all positive integers n for which there exist k integers n_1, n_2, $\ldots$, n_k, each greater than 3, such that

$$n = n_1 n_2 \cdots n_k = \sqrt[2^k]{2^{(n_1-1)(n_2-1)\cdots(n_k-1)}} - 1 .$$

6. A multiple-choice examination has 5 questions, each with 4 choices. Each of 2000 students picks exactly 1 choice for each question. Among any n students for some positive integer n, there exist 4 such that any 2 of them give the same answers to at most 3 questions. Determine the minimum value of n.

As a third set we give the 2000 Russian Mathematical Olympiad. Thanks again to Andy Liu.

2000 RUSSIAN MATHEMATICAL OLYMPIAD

1. Prove that there exist ten different real numbers a_1, a_2, $\ldots$, a_{10} such that the equation

$$(x - a_1)(x - a_2)\cdots(x - a_{10}) = (x + a_1)(x + a_2)\cdots(x + a_{10})$$

has exactly 5 different real roots.

2. The altitude and the base radius of a cylinder are equal to 1. Determine the minimal number of balls of radius 1 that cover this cylinder.

3. Let a_1, a_2, $\ldots$, a_{2000} be real numbers such that $a_1^3 + a_2^3 + \cdots + a_n^3 = (a_1 + a_2 + \cdots + a_n)^2$ for all n, $1 \le n \le 2000$. Prove that every element of the sequence is an integer.

4. Determine the smallest positive integer n such that an $n \times n$ square can be cut into 40×40 and 49×49 squares, with at least one square of each type.

5. Prove that

$$\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} < x, y \le 1 .$$

6. The incircle of triangle ABC with centre O touches the side AC at K. Another circle with the same centre intersects each side at two points.

The points of intersection on AC are B_1 and B_2, with B_1 closer to A. E is the point of intersection on AB closer to B, and F is the point of intersection on BC closer to B. Let P be the point of intersection of B_2E and B_1F. Prove that B, K and P are collinear.

7. Each of the numbers 1, 2, $\ldots$, N is black or white. In a move, we may change simultaneously the colours of any three of the numbers if one of them is the arithmetic mean of the other two. For which N is it possible to make all numbers white after a finite number of moves?

8. There are 2000 cities in a country and some pairs of them are connected by roads. From each city, the number of tours by roads which visit an even number of different cities before returning to the starting city is at most some positive integer N. Prove that the country can be separated into $N+2$ republics such that any two cities in the same republic are not connected by a road.

We begin our solutions set this issue with an even quicker Quickie Solution that Murray S. Klamkin supplied us than the one published in the last issue [2002 : 418].

2. Determine the maximum and minimum values of

$$\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}\;+\;\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$$

where a and b are given constants.

Solution.

Let $S = \sqrt{a^2\cos^2\theta+b^2\sin^2\theta}+\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$. Squaring we get,

$$S^2 \;=\; a^2+b^2+2\sqrt{(a^4+b^4)\sin^2\theta\cos^2\theta+a^2b^2(\cos^4\theta+\sin^4\theta)}\,.$$

Since $\cos^4\theta+\sin^4\theta=1-2\sin^2\theta\cos^2\theta$, the expression inside the radical can be written as

$$\frac{(a^2-b^2)^2(\sin^2 2\theta)}{4}+a^2b^2\,.$$

Hence, the maximum and minimum are taken on for $=\frac{\pi}{4}$ and 0, respectively, giving $S_{\text{max}}=\sqrt{2(a^2+b^2)}$ and $S_{\text{min}}=a+b$.

We continue our solutions set with readers' solutions to problems of the Mathematical Olympiad in Bosnia and Herzegovina, 1997, First Day [2000 : 325–326].

1. Solve the system of equations in $\mathbb{R}^3$:

$$\begin{aligned} 8(x^3+y^3+z^3) &= 73\,, \\ 2(x^2+y^2+z^2) &= 3(xy+yz+zx)\,, \\ xyz &= 1\,. \end{aligned}$$

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; by Pavlos Maragoudakis, Pireas, Greece; by Panos E. Tsaoussoglou, Athens, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution of Amengual Covas.

Let $u = x+y+z$, $v = xy+yz+zx$, $w = xyz$ and transform the given system into an equivalent system in u, v and w.

We have

$$x^2+y^2+z^2 \;=\; (x+y+z)^2 - 2(xy+yz+zx) \;=\; u^2-2v$$

and

$$\begin{aligned} x^3+y^3+z^3 &= (x+y+z)^3 - 3(x+y+z)(xy+yz+zx) + 3xyz \\ &= u^3 - 3uv + 3w\,. \end{aligned}$$

Hence, the given system is equivalent to

$$\begin{aligned} 8(u^3-3uv+3w) &= 73\,, \\ 2(u^2-2v) &= 3v\,, \\ w &= 1\,, \end{aligned}$$

whose solution is $u = \frac{7}{2}$, $v = \frac{7}{2}$, $w = 1$.

Since

$$(\zeta - x)(\zeta - y)(\zeta - z) \;=\; \zeta^3 - u\zeta^2 + v\zeta - w\,,$$

we see that the roots of

$$\zeta^3 - \frac{7}{2}\zeta^2 + \frac{7}{2}\zeta - 1 \;=\; 0 \qquad (1)$$

constitute a solution of the original system, and, since the equations are symmetrical, any one of the six permutations of these roots is also a solution.

The cubic (1) evidently has the root $\zeta = 1$; and the other two are easily found to be $\frac{1}{2}$ and 2.

Hence, we have the following six solutions (x,y,z):

$$\left(\frac{1}{2},1,2\right)\,,\;\left(\frac{1}{2},2,1\right)\,,\;\left(1,\frac{1}{2},2\right)\,,\;\left(1,2,\frac{1}{2}\right)\,,\;\left(2,\frac{1}{2},1\right)\,,\;\left(2,1,\frac{1}{2}\right)\,.$$

2. In an isosceles triangle ABC with the base $\overline{AB}$, point M lies on the side $\overline{BC}$. Let O be the centre of its circumscribed circle, and S be the centre of the inscribed circle in the triangle ABC. Prove that:

$$SM \parallel AC \iff OM \perp BS\,.$$

Solutions by Michel Bataille, Rouen, France; by D.J. Smeenk, Zaltbommel, the Netherlands; by Achilleas Sinefakopoulos, student, University of Athens, Greece; and by Toshio Seimiya, Kawasaki, Japan. We give the solution of Sinefakopoulos.

First, let the line BS meet the lines MO and CA at K and N, respectively. Also, let the points of tangency of the incircle with the sides BA and CA be L and L', respectively. Notice that BS bisects $\angle B$ and that the line COS bisects $\angle C$ and is perpendicular to AB at L. Moreover, it is easy to see that

$$\angle OBS \;=\; \frac{\angle B - \angle C}{2} \;=\; \angle NSL'\,. \tag{1}$$

Now if SM is parallel to AC, then $\angle SMB = \angle C = \angle SOB$; that is, $SOMB$ is cyclic. Hence,

$$\begin{aligned}\angle OMB &= \angle OMS + \angle SMB \;=\; \angle OBS + \angle SMB \\ &= \frac{\angle B - \angle C}{2} + \angle C \;=\; \frac{\angle B + \angle C}{2}\,,\end{aligned}$$

and thus,

$$\angle KOS + \angle SKO \;=\; \angle SBM + \angle OMB \;=\; \frac{\angle B}{2} + \frac{\angle B + \angle C}{2} \;=\; 90^\circ\,,$$

which means that OM is perpendicular to BS at K.

Conversely, if OM is perpendicular to BS at K, then the right triangles OKB and OKS are similar to the right triangles $NL'S$ by (1) and BLS, respectively. Thus,

$$\frac{NS}{SL} \;=\; \frac{NS}{SL'} \;=\; \frac{OB}{KB} \;=\; \frac{CO}{KB} \quad\text{and}\quad \frac{SL}{SB} \;=\; \frac{SK}{OS}\,.$$

By multiplying, we get

$$\frac{NS}{SB} \;=\; \frac{CO}{KB}\cdot\frac{SK}{OS} \;=\; \frac{CM}{MB}\,,$$

where the last equality follows from the Theorem of Menelaus in the triangle CSB for the line MOK. Therefore, SM is parallel to AC and the proof is complete.

3. Let $f : A \to \mathbb{R}$, $A \subseteq \mathbb{R}$, be a function with the following characteristic:

$$f(x+y) \;=\; f(x)\cdot f(y) - f(xy) + 1\,, \quad (\forall x,\, y \in A)\,.$$

(a) If $f : A \to \mathbb{R}$, $\mathbb{N} \subset A \subseteq \mathbb{R}$, is such a function, prove that the following is true:

$$f(n) \;=\; \begin{cases} \frac{c^{n+1}-1}{c-1}\,, & \forall n \in \mathbb{N}\,,\; c \neq 1\,, \\ n+1\,, & \forall n \in \mathbb{N}\,,\; c = 1\,, \end{cases}$$

$(c = f(1) - 1)$.

(b) Solve the given functional equation for $A = \mathbb{N}$.

(c) If $A = \mathbb{Q}$, find all the functions f which satisfy the given equation and the condition $f(1997) \neq f(1998)$.

Solution by Pierre Bornsztein, Pontoise, France.

(a) Let $A \subset \mathbb{R}$ such that $\mathbb{N} \subset A$ and $f : A \longrightarrow \mathbb{R}$ such that, for all x, $y \in A$,

$$f(x+y) \;=\; f(x)f(y) - f(xy) + 1\,. \tag{1}$$

Let $c = f(1) - 1$. Thus, $f(1) = c + 1$.

For $x = y = 0$, (1) leads to $f(0) = f^2(0) - f(0) + 1$ and then $f(0) = 1$.

For $y = 1$, (1) gives

$$f(x+1) \;=\; cf(x) + 1 \quad \text{for all } x \in A\,. \tag{2}$$

• If $c = 1$: From (2) we deduce $f(x+1) = f(x) + 1$ for all $x \in A$. And, from $f(0) = 1$, an easy induction leads to $f(n) = n + 1$ for all $n \in \mathbb{N}$.

• If $c \neq 1$: Since $\dfrac{c^{n+2}-1}{c-1} = c\dfrac{c^{n+1}-1}{c-1} + 1$ and $f(0) = 1$, another easy induction leads to $f(x) = \dfrac{c^{n+1}-1}{c-1}$ for all $n \in \mathbb{N}$. Thus (a) is proved.

(b) We suppose now that $A = \mathbb{N}$. Let $f : A \longrightarrow \mathbb{R}$ which satisfies (1).

• If $c \neq 1$, using (a) and that $f(4) = f(2+2) = f^2(2) - f(4) + 1$, we must have

$$\frac{c^5-1}{c-1} \;=\; \left(\frac{c^3-1}{c-1}\right)^2 - \frac{c^5-1}{c-1} + 1$$

which is equivalent to $c^6 - 2c^5 + 2c^3 - c^2 = 0$. That is,

$$c^2(c-1)^3(c+1) \;=\; 0$$

and then $c \in \{0,\, -1\}$.

For $c = 0$ we have $f(n) = 1$ for all $n \in \mathbb{N}$.

For $c = -1$ we have $f(n) = 1$ if n is even, and $f(n) = 0$ if n is odd.

Conversely, it is easy to see that the three possibilities above give indeed three solutions. Then for $A = \mathbb{N}$ the functions which satisfy (1) are

(i) $f \equiv 1$

(ii) $f : (n \mapsto n + 1)$

(iii) $f : (n \mapsto \frac{(-1)^n + 1}{2})$

(c) We now suppose that $A = \mathbb{Q}$. Let $f : A \longrightarrow \mathbb{R}$ which satisfies (1) and such that $f(1997) \neq f(1998)$.

Then the restriction of f to $\mathbb{N}$ satisfies (1) and from (b), must be of the form (i), (ii) or (iii). The condition $f(1997) \neq f(1998)$ eliminates (i).

In the two remaining cases we have $f(1) \in \{0, 2\}$ and $f(0) = 1$.

For $y = -x$, (1) leads to $f(0) = 1 = f(x)f(-x) - f(-x^2) + 1$. Thus,

$$f(x)f(-x) = f(-x^2) \quad \text{for all } x \in A. \tag{3}$$

In particular $f(1)f(-1) = f(-1)$. Since $f(1) \in \{0,2\}$, we have $f(-1) = 0$.

- If $f(1) = 0$ (case (iii)): Then, for all $x \in \mathbb{Q}$ we have:

$$f(1 - x) = f(1)f(-x) - f(-x) + 1 = -f(-x) + 1$$

and

$$f(x - 1) = f(-1)f(x) - f(-x) + 1 = -f(-x) + 1\,.$$

Thus, $f(x - 1) = f(1 - x)$.

Setting $X = x - 1$, we easily deduce that f is even. Then, from (3), we have $f^2(x) = f(x^2)$ for all $x \in \mathbb{Q}$. For $x = y$, (1) leads to

$$f(2x) = f^2(x) - f(x^2) + 1 = 1 \quad \text{for all } x \in \mathbb{Q}.$$

Then $f \equiv 1$. In particular $f(1997) = f(1998)$, which is a contradiction.

- If $f(1) = 2$ (case (ii)): From (2) we have $f(x+1) = f(x) + 1$ for all $x \in \mathbb{Q}$. By an easy induction, we deduce that:

$$f(x + p) = f(x) + p \quad \text{for all } x \in \mathbb{Q} \quad \text{and} \quad p \in \mathbb{Z}\,.$$

But

$$\begin{aligned} f(x + p) &= f(x)f(p) - f(px) + 1 \\ &= (p + 1)f(x) - f(px) + 1 \quad \text{(from (b) (ii))}\,. \end{aligned}$$

Thus,

$$\begin{aligned} f(px) &= (p + 1)f(x) + 1 - f(x) - p \\ &= pf(x) + 1 - p \quad \text{for all } x \in \mathbb{Q} \quad \text{and} \quad p \in \mathbb{Z}\,. \end{aligned}$$

Let a, b be integers with $b \neq 0$. From the above we then have $f(\frac{a}{b}\cdot b) = f(a) = a+1$ and $f(\frac{a}{b}\cdot b) = bf(\frac{a}{b}) + 1 - b$. Then

$$f\left(\frac{a}{b}\right) \;=\; \frac{a}{b}+1\,.$$

Thus, $f(x) = x+1$ for all $x \in \mathbb{Q}$.

It is not difficult to verify that $f:(x \mapsto x+1)$ is indeed a solution. Then, for $A = \mathbb{Q}$ the only function $f : A \longrightarrow \mathbb{R}$ which satisfies (1) and such that $f(1997) \neq f(1998)$ is $f:(x \mapsto x+1)$.

Next we look at solutions on file for the Second Day of the Mathematical Olympiad in Bosnia and Herzegovina [2000 : 326–327].

1. (a) Let A_1, B_1, C_1 be the points of contact of the circle inscribed in the triangle ABC and the sides BC, CA, AB, respectively. Let B_1C_1, A_1C_1, B_1A_1 be the arcs which do not contain points A_1, B_1, C_1 respectively. Let I_1, I_2, I_3 be their respective arc lengths. Prove the following inequality:

$$\frac{a}{I_1}+\frac{b}{I_2}+\frac{c}{I_3} \;\geq\; 9\frac{\sqrt{3}}{\pi}$$

(where a, b, c denote the lengths of sides of the given triangle).

(b) Let $ABCD$ be a tetrahedron with:

$$AB \;=\; CD \;=\; a\,,$$
$$BC \;=\; AD \;=\; b\,,$$
$$AC \;=\; BD \;=\; c\,.$$

Express the height of the tetrahedron in terms of the lengths a, b and c.

Solution to (a) by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. Editor's Note: Pavlos Maragoudakis points out that part (a) is ***Crux Mathematicorum*** *1651.*

(a) Denote the incentre and inradius of triangle ABC by I and r, respectively.

The perpendiculars IA_1, IB_1, IC_1 partition the triangle into three quadrilaterals, each of which is clearly cyclic. If A, B, C denote the angles in radians of $\triangle ABC$, then

$$I_1 \;=\; r(\pi - A)\,,\quad I_2 \;=\; r(\pi - B)\,,\quad I_3 \;=\; r(\pi - C)\,.$$

We employ the AM–GM inequality and obtain

$$\begin{aligned}\frac{a}{I_1}+\frac{b}{I_2}+\frac{c}{I_3} \;&=\; \frac{1}{r}\left(\frac{a}{\pi-A}+\frac{b}{\pi-B}+\frac{c}{\pi-C}\right)\\ &\geq\; \frac{1}{r}\cdot 3\sqrt[3]{\frac{abc}{(\pi-A)(\pi-B)(\pi-C)}}\,.\end{aligned} \tag{1}$$

Now,

$$\sqrt[3]{(\pi - A)(\pi - B)(\pi - C)} \;\le\; \frac{(\pi - A) + (\pi - B) + (\pi - C)}{3} \;=\; \frac{2\pi}{3},$$

whence by (1)

$$\begin{aligned} \frac{a}{I_1} + \frac{b}{I_2} + \frac{c}{I_3} \;&\ge\; \frac{1}{r} \cdot \frac{9}{2\pi}\sqrt[3]{abc} \;=\; \frac{9}{2\pi r}\sqrt[3]{4Rrs} \\ &\ge\; \frac{9}{\pi r}\sqrt[3]{r^2 s} \;=\; \frac{9}{\pi}\sqrt[3]{\frac{s}{r}} \qquad (2) \end{aligned}$$

where R and s are the circumradius and semiperimeter of $\triangle ABC$ and we have used Euler's Inequality $R \ge 2r$.

Finally, using the inequality $\frac{s}{r} \ge 3\sqrt{3}$ (item 5.3 of Bottema et al. *Geometric inequalities*) we obtain from (2)

$$\frac{a}{I_1} + \frac{b}{I_2} + \frac{c}{I_3} \;\ge\; \frac{9}{\pi}\sqrt[3]{3\sqrt{3}} \;=\; \frac{9\sqrt{3}}{\pi}.$$

Equality occurs only if $\triangle ABC$ is equilateral.

2. (a) Prove that for every positive integer n there exists a set M_n of positive integers which has n elements and possesses the property:

(i) the arithmetic mean of elements of an arbitrary non-empty subset of M_n is an integer

(ii) the geometric mean of elements of an arbitrary non-empty subset of M_n is an integer

(iii) both arithmetic and geometric mean of elements of an arbitrary non-empty subset of M_n are integral.

(b) Is there an infinite set M of natural numbers which has the property that the arithmetic mean of an arbitrary non-empty subset of M is an integer?

Solution by Pierre Bornsztein, Pontoise, France.

(a) We directly prove (iii).

Let $n \in \mathbb{N}^*$. Let $M_n = \{a_1, a_2, \ldots, a_n\}$ where

$$a_k \;=\; (k \cdot n!)^{n!} \quad \text{for} \quad k = 1, \ldots, n.$$

Then the a_k's are positive integers, pairwise distinct. Let $r \in \{1, \ldots, n\}$. Each of the a_k's is a multiple of r and is a perfect r–power. Then the sum of r of the a_k's is always a multiple of r, and the product of r of the a_k's is always a perfect r–power. The conclusion follows.

(b) No, such a set does not exist.

Suppose, for a contradiction, that such a set does exist.

Let a, b be two distinct elements of M. Since $a - b \neq 0$, there exists a prime number p which does not divide $a - b$.

Let $x_1, \ldots, x_{p-1}$ be pairwise distinct elements of $M - \{a, b\}$ (since M is infinite). Using the non-empty subsets $\{x_1, \ldots, x_{p-1}, a\}$ and $\{x_1, \ldots, x_{p-1}, b\}$ of M, we must have

$$x_1 + \cdots + x_{p-1} + a \equiv 0 \pmod{p} \quad \text{and} \quad x_1 + \cdots + x_{p-1} + b \equiv 0 \pmod{p} .$$

Thus, $a - b \equiv 0 \pmod{p}$, a contradiction.

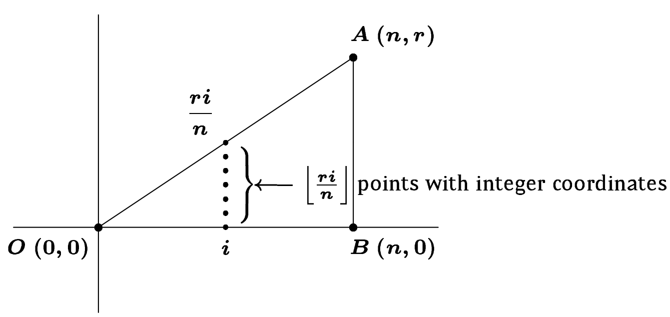
Next we turn to the solutions we have from our readers to problems of the 5th Japan Mathematical Olympiad 1995, Final Round, given [2000 : 327–328].

1. Let n and r be positive integers such that $n \geq 2$ and $r \not\equiv 0 \pmod{n}$, and let g be the greatest common divisor of n and r. Prove that

$$\sum_{i=1}^{n-1} \left\langle \frac{ri}{n} \right\rangle = \frac{1}{2}(n - g) ,$$

where $\langle x \rangle = x - \lfloor x \rfloor$ is the fractional part of x.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; by Pavlos Maragoudakis, Pireas, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the novel solution of Maragoudakis.



By Pick's Theorem, if P is a polygon and its vertices have integer coordinates, then we can find its area E by the formula $E = k - 1 + \frac{\ell}{2}$ where k is the number of points inside P with integer coordinates and ℓ is the number of points with integer coordinates on the sides of P.

Let $O(0,0)$, $A(n,r)$, $B(n,0)$. Then $OA : y = \frac{r}{n}x$. Since $g = (n,r)$, there are $n', r' \in \mathbb{N}$ such that $n = n'g$, $r = r'g$ and $(n', r') = 1$. If $i \in \{0, 1, \ldots, n\}$ then

$$\frac{ri}{n} \in \mathbb{N} \iff \frac{r'i}{n'} \in \mathbb{N} \iff n' \mid r'i \overset{(n',r')=1}{\iff} n' \mid i \iff i = n't\,, \quad t \in \mathbb{N}\,.$$

We have $0 \le i \le n \iff 0 \le n't \le n'g \iff 0 \le t \le g$.

Finally, the point $(x, \frac{rx}{n})$ on segment OA has integer coordinates if and only if $x \in \{0, 1, \ldots, n\}$ and $x = n't$, $0 \le t \le g$, $t \in \mathbb{N}$.

Thus, on OA there are $g+1$ points with integer coordinates, on AB there are r more, and on OB $n-1$ more. Now $\ell = g+1+r+n-1 = g+r+n$.

Also, $k = \sum_{i=1}^{n-1} \lfloor \frac{ri}{n} \rfloor - g + 1$ since on line $x = i$ there are $\lfloor \frac{ri}{n} \rfloor$ points with integer coordinates inside triangle OAB. We exclude the $g-1$ points which are on OA between O, A and have integer coordinates.

Now Pick's formula gives:

$$\frac{1}{2}nr \;=\; \sum_{i=1}^{n-1} \left\lfloor \frac{ri}{n} \right\rfloor - g + 1 - 1 + \frac{g+n+r}{2} \Longrightarrow \sum_{i=1}^{n-1} \left\lfloor \frac{ri}{n} \right\rfloor \;=\; \frac{1}{2}nr + \frac{g-n-r}{2}\,.$$

Thus,

$$\begin{aligned}
\sum_{i=1}^{n-1} \left\langle \frac{ri}{n} \right\rangle &= \sum_{i=1}^{n-1} \left(\frac{ri}{n} - \left\lfloor \frac{ri}{n} \right\rfloor \right) \\
&= \frac{r}{n} \sum_{i=1}^{n-1} i - \sum_{i=1}^{n-1} \left\lfloor \frac{ri}{n} \right\rfloor \;=\; \frac{r}{n}\frac{(n-1)n}{2} - \frac{1}{2}nr + \frac{r+n-g}{2} \\
&= \frac{1}{2}nr - \frac{1}{2}r - \frac{1}{2}nr + \frac{r+n-g}{2} \;=\; \frac{1}{2}(n-g)\,.
\end{aligned}$$

Comment. This problem was suggested to my class when I was a student in Athens University by Mr. P. Tsagaris during his lessons on Number Theory.

5. Let k and n be integers such that $1 \le k \le n$, and assume that a_1, a_2, ..., a_k satisfy

$$\begin{aligned}
a_1 + a_2 + \cdots + a_k &= n\,, \\
a_1^2 + a_2^2 + \cdots + a_k^2 &= n\,, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
a_1^k + a_2^k + \cdots + a_k^k &= n\,.
\end{aligned}$$

Prove that

$$(x+a_1)(x+a_2)\cdots(x+a_k) \;=\; x^k + \binom{n}{1}x^{k-1} + \binom{n}{2}x^{k-2} + \cdots + \binom{n}{k}\,.$$

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give Klamkin's solution.

Let T_r denote the elementary symmetric sum of the products of the a_i's taken r at a time and let $S_r = a_1^r + a_2^r + \cdots + a_k^r$ for $r = 1, 2, \ldots, k$. It is known [1] that $T_r = \left(\frac{1}{r!}\right)$ times an $r \times r$ determinant whose successive rows are

$$\begin{array}{cccccc} S_1, & 1, & 0, & & \ldots, & 0, \\ S_2, & S_1, & 2, & 0, & \ldots, & 0, \\ \vdots & & & & & \vdots \\ S_r, & S_{r-1}, & S_{r-2}, & & \ldots, & S_1. \end{array}$$

Our proof is now by induction. Assume that $T_r = \binom{n}{r}$ for $r = 1, 2, \ldots, m$. In the determinant for T_{m+1}, subtract the m^{th} row from the $(m+1)^{\text{st}}$ row giving the new $(m+1)^{\text{st}}$ row as $0, 0, \ldots, 0, n-m$, since all the S_j's equal n. Hence,

$$T_{m+1} \;=\; \frac{m!}{(m+1)!\,(n-m)}\binom{n}{m} \;=\; \binom{n}{m+1}$$

and hence this is valid for all m.

Reference:

[1] A. Mostowski, M. Stark, "Introduction to Higher Algebra," Pergamon Press, London, 1964.

We continue with solutions to problems of the 23$^{\text{rd}}$ All Russian Olympiad of the Secondary Schools, 11$^{\text{th}}$ Grade, First Day given [2000 : 388–389].

1. Solve, in integers, the equation

$$(x^2 - y^2)^2 \;=\; 1 + 16y\,.$$

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; and by Panos E. Tsaoussoglou, Athens, Greece. We give the write-up by Bornsztein.

We will prove that the solutions of

$$(x^2 - y^2)^2 \;=\; 1 + 16y \qquad (1)$$

are $(-4,5)$, $(4,5)$, $(-1,0)$, $(1,0)$, $(-4,3)$, $(4,3)$.

Let x, y be two integers satisfying (1). We note that:

- $y \geq 0$ unless $1 + 16y < 0$, for then $1 + 16y$ would not be a square.
- (x, y) is solution of (1). Then, with no loss of generality, we may suppose that $x \geq 0$.

Case 1: if $x \geq y$.

Then $x = y + a$ with $a \in \mathbb{N}$. Equation (1) can be written as:

$$4a^2y^2 + 4(a^3 - 4)y + a^4 - 1 \;=\; 0\,.$$

Thus, y is a solution of the quadratic

$$4a^2X^2 + 4(a^3 - 4)X + a^4 - 1 \;=\; 0\,.$$

Since $\Delta = 16(-8a^3 + a^2 + 16)$, we must have $f(a) = -8a^3 + a^2 + 16 \geq 0$. It is easy to see that f is decreasing for $a \in [1, +\infty)$, and that $f(2) = -44 < 0$. Then $f(a) < 0$ for $a \geq 2$.

It follows that $a \in \{0,\, 1\}$.

For $a = 0$, we have $x = y$, and (1) leads to $1 + 16y = 0$, which is impossible since y is an integer.

For $a = 1$, then $x = y + 1$ and (1) leads to $4y^2 - 12y = 0$. Thus, $y = 0$ or $y = 3$.

Conversely, it is easy to verify that $(1, 0)$ and $(4, 3)$ are solutions of (1) (then, so are $(-1, 0)$ and $(-4, 3)$).

Case 2: if $0 \leq x < y$.

Then $0 \leq x \leq y - 1$ and $y \geq 1$.

It follows that

$$x^2 - y^2 \;\leq\; 1 - 2y \;<\; 0$$

and then

$$1 + 16y \;=\; (x^2 + y^2)^2 \;\geq\; (1 - 2y)^2 \;=\; 1 + 4y^2 - 4y\,.$$

This leads to:

$$4y^2 - 20y \;\leq\; 0\,.$$

That is, $y \in \{1,\, 2,\, 3,\, 4,\, 5\}$.

For $y \in \{1,\, 2,\, 4\}$, note that $1 + 16y$ is not a square and then (1) does not hold.

For $y = 3$, (1) leads to $(x^2 - 9)^2 = 49$ with $x^2 - 9 < 0$. Then $x^2 - 9 = -7$, which does not have any integer solution.

For $y = 5$, (1) leads to $(x^2 - 25)^2 = 81$ with $x^2 - 25 < 0$. Then $x^2 - 25 = -9$ and $x = 4$.

Conversely, it is easy to verify that $(4, 5)$ is a solution of (1) (and then so is $(-4, 5)$), which completes the proof.

2. The Council of Wizards is tested in the following way: The King lines the wizards up in a row and places on the head of each of them either a white hat or a blue hat or a red hat. Each wizard sees the colours of hats of the people standing in front of him but he sees neither the colour of his hat nor the colours of hats of the people standing behind. Every minute some of the wizards must announce one of the three colours (it is allowed to speak out just once). After completion of this procedure the King executes all the wizards who failed to guess the right colour of their hats. Prior to this ceremony all 100 members have agreed to minimize the number of executions. How many of them are definitely secure against the punishment?

Solution by Peter Du, student, Calgary, Alberta.

The first wizard to guess has no information given to him about his own hat colour. Therefore, whatever colour he announces will end up being just a random guess for his own hat colour. Hence, it is impossible to secure all 100 members against punishment.

It is possible, however, to secure 99 members, with the following method:

The last wizard in line will announce a colour first, followed by the next wizard in front of him, then the next, and so on, until the wizard standing at the front of the line announces a colour last.

Assign the three colours three numbers: 0 for white, 1 for blue, and 2 for red. The last wizard will add up all the hat numbers that he sees in front of him, and then say the corresponding hat colour of the remainder when that sum is divided by 3. Let this remainder be s_1. The second-last wizard now can add up all the hat numbers in front of him, and find the remainder when that sum is divided by 3. Call this remainder r_2. The hat number of the second wizard is then $s_1 - r_2$. He can now call out the corresponding hat colour and save his own life. Let s_n denote the hat number of the hat colour called out by the n^{th} last wizard in line, and let r_n denote the remainder when the sum of the hat numbers in front of the n^{th} last wizard in line is divided by 3. Then, we only need to have the recursive sequence

$$s_n \equiv \left(s_1 - \sum_{k=2}^{n-1} s_k - r_n \right) \pmod{3} , \quad 0 \le s_n \le 2 .$$

And the n^{th} wizard can get his hat colour correct, guaranteed when $n > 1$. Therefore, we have 99 wizards definitely secure against punishment.

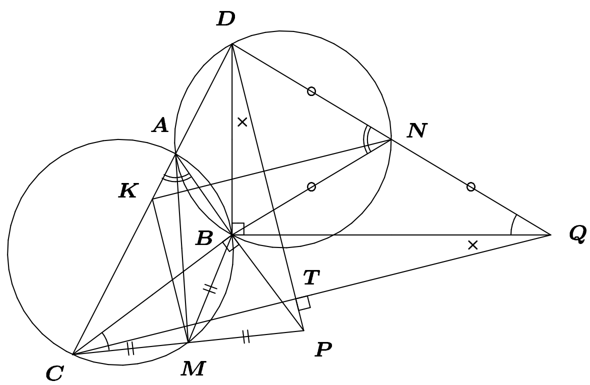
3. Two circles intersect at the points A and B. A line is drawn through the point A. This line crosses the first circle again at the point C and it crosses the second circle again at the point D. Let M and N be the mid-points of the arcs BC and BD respectively (these arcs do not contain A). Let K be the mid-point of the segment CD. Prove that the angle MKN is a right angle. (It may be assumed that A lies between C and D).

Solutions by Michel Bataille, Rouen, France; by Toshio Seimiya, Kawasaki, Japan; and by Babis Stergiou Chalkida, Greece. We first give the solution by Bataille.

From $\angle CMB = 180^\circ - \angle CAB = \angle BAD = 180^\circ - \angle BND$, we get $\angle CMB + \angle BND = 180^\circ$. Thus, the composition $R_N \circ R_M$ of the rotations R_M (centre M, transforming C into B) and R_N (centre N, transforming B into D) is a symmetry about a point. But, since $MC = MB$ and $NB = ND$, we have $R_N \circ R_M(C) = D$ so that $R_N \circ R_M$ is the symmetry about K.

Now, let $M' = R_N \circ R_M(M) = R_N(M)$. The triangle MNM' is isosceles ($NM = NM'$) and K is the mid-point of MM'. It follows that $NK \perp MM'$ and $\angle MKN = 90^\circ$.

Next we give Seimiya's solution.



Let P be a point on CM produced beyond M such that $CM = MP$, and let Q be a point on DN produced beyond N such that $DN = NQ$.

Since M is the mid-point of arc BC, we get $BM = CM$, so that $BM = CM = MP$. Hence, $\angle CBP = 90^\circ$.

Similarly, we get $BN = DN = NQ$ and $\angle DBQ = 90^\circ$. Thus, we have

$$\angle BQD = \frac{1}{2}\angle BND = \frac{1}{2}\angle BAC = \angle BAM = \angle BCM = \angle BCP .$$

Since $\angle CBP = 90^\circ = \angle DBQ$, we have $\triangle BCP \sim \triangle BQD$.

Since $\triangle BCP$ and $\triangle BQD$ are directly similar, we get $\triangle BCQ \sim \triangle BPD$, so that $\angle BQC = \angle BDP$. Let T be the intersection of CQ and DP. Since $\angle BQC = \angle BDP$, we get $\angle BQT = \angle BDT$, so that $\angle DTQ = \angle DBQ = 90^\circ$; that is, $DP \perp CQ$. Since K, M and N are mid-points of CD, CP and DQ, respectively, we have

$$KM \parallel DP \quad \text{and} \quad KN \parallel CQ .$$

Because $DP \perp CQ$, we get $KM \perp KN$. Therefore, $\angle MKN = 90^\circ$.

5. Given all possible quadratic trinomials of the type x^2+px+q, with integer coefficients p and q, $1 \leq p \leq 1997$, $1 \leq q \leq 1997$. Consider the sets of the trinomials:

(a) having integer zeros,

(b) not having real zeros.

Which of those sets is larger?

Solution by Pierre Bornsztein, Pontoise, France (*adapted by the editor*).

We will prove that there are more trinomials not having real zeros than trinomials having integer zeros.

Let $G = \{1, \ldots, 1997\}$. Note that such a trinomial has integer zeros if and only if (p,q) belongs to

$$F_1 \;=\; \{(p,q) \in G^2 : p^2 - 4q \quad \text{is a perfect square,}\}$$

and that a trinomial does not have real zeros if and only if (p,q) belongs to

$$F_2 \;=\; \{(p,q) \in G^2 : p^2 - 4q < 0\}$$

Thus, we will prove that $\mathrm{Card}(F_2) > \mathrm{Card}(F_1)$.

Let $(p,q) \in F_1$. Then there exists a non-negative integer a such that

$$p^2 - 4q \;=\; a^2\,.$$

Thus, $a = \sqrt{p^2-4q}$, a and p have the same parity, and (since $q \geq 1$) we have $a < p$. It follows that $a \leq p-2$. Thus, $p-a-1 \in \mathbb{N}^*$. Moreover,

$$\begin{aligned}(p-a-1)^2 - 4q &= p^2 - 4q + a^2 + 1 - 2ap - 2p + 2a \\ &= 2a^2 + 1 - 2ap + 2a - 2p \\ &= 2a(a-p) + 1 - 2(p-a)\,,\end{aligned}$$

where $2a(a-p) \leq 0$ and $1 - 2(p-a) < 0$. Thus, $(p-a-1)^2 - 4q < 0$.

Since $p-a-1 \in G$, we deduce that $(p-1-\sqrt{p^2-4q}, q)$ belongs to F_2.

Let

$$\begin{aligned} f : F_1 &\longrightarrow F_2 \\ (p,q) &\mapsto (p-1-\sqrt{p^2-4q}, q)\,. \end{aligned}$$

If (p,q), $(p',q') \in F_1$ such that $f(p,q) = f(p',q')$, then $q = q'$ and

$$p - 1 - \sqrt{p^2-4q} \;=\; p' - 1 - \sqrt{(p')^2 - 4q}\,.$$

Thus,

$$p - p' \;=\; \sqrt{p^2-4q} - \sqrt{(p')^2-4q}\,.$$

Suppose that $p \neq p'$. Then $\sqrt{p^2-4q}+\sqrt{(p')^2-4q} > 0$ and we have

$$p - p' \;=\; \frac{(p-p')(p+p')}{\sqrt{p^2-4q}+\sqrt{(p')^2-4q}}\,.$$

Thus,

$$p + p' \;=\; \sqrt{p^2-4q}+\sqrt{(p')^2-4q}\,. \qquad (1)$$

Since $q \geq 1$, we have

$$\sqrt{p^2-4q} \;<\; \sqrt{p^2} \;=\; p$$

and

$$\sqrt{p'^2-4q} \;<\; \sqrt{(p')^2} \;=\; p'\,.$$

Thus, (1) is not satisfied. A contradiction.

Therefore, $p = p'$, and f is injective.

It follows that (since F_1 and F_2 are finite):

$$\mathrm{Card}(F_2) \;\geq\; \mathrm{Card}(F_1)\,,$$

and it now suffices to prove that f is not surjective.

Suppose, for a contradiction, that there exists $(p,q) \in F_1$ such that $f(p,q) = (2,3)$. Then $q = 3$ and

$$p - 1 - \sqrt{p^2-12} \;=\; 2\,.$$

Therefore,

$$(p-3)^2 \;=\; p^2 - 12\,,$$

and thus, $p^2 - 6p + 9 \;=\; p^2 - 12$ giving $6p \;=\; 21$, which has no integral solution for p. It follows that there is no $(p,q) \in F_1$ such that $f(p,q) = (2,3)$. Since $(2,3) \in F_2$, we deduce that f is not surjective. Then $\mathrm{Card}(F_2) > \mathrm{Card}(F_1)$ and the proof is complete.

Next, we look at solutions on file for problems of contests given in the November 2000 number of ***Crux Mathematicorum***. We first turn to the Fourth National Mathematical Olympiad of Turkey given [2000 : 389–390].

1. Let $\{A_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ be sequences of positive integers. Assume that, for each positive integer x there exist a unique positive integer N and a unique N–tuple $(x_1, x_2, \ldots, x_N)$ of integers such that

$$x \;=\; \sum_{n=1}^{N} x_n A_n\,, \quad 0 \leq x_n \leq \alpha_n \quad (n = 1, 2, \ldots, N) \quad \text{and} \quad x_N \;\neq\; 0\,.$$

Prove that

(i) $A_{n_0} = 1$ for some n_0;

(ii) if $k \neq j$, then $A_k \neq A_j$,

(iii) if $A_k \leq A_j$, then A_k divides A_j.

Solution by Pierre Bornsztein, Pontoise, France.

(i) Assume that $A_i \geq 2$ for all i. Then for each $N \in \mathbb{N}^*$, $x_1, \ldots, x_N$ such that $0 \leq x_i \leq \alpha_i$ for $i = 1, \ldots, N$ with $x_N \neq 0$, we have

$$\sum_{i=1}^{N} x_i A_i \;\geq\; x_N A_N \;\geq\; A_N \;\geq\; 2\,.$$

It follows that 1 cannot be written in the desired form. A contradiction. Thus, $A_{n_0} = 1$ for some n_0.

(ii) Assume that there exist $j < k$ such that $A_j = A_k$. Then

$$A_j \;=\; \sum_{i=1}^{j} x_i A_i \;=\; \sum_{i=1}^{k} x_i' A_i\,,$$

where $x_i = 0$ for $i < j$, $x_i' = 0$ for $i < k$, $x_j = x_k' = 1$. This contradicts the assumption that the number A_j has a unique representation of the desired form. Then, if $j \neq k$, we have $A_j \neq A_k$.

(iii) The statement may be rewritten as: for each positive integer x, there exists a unique representation of x of the form

$$x \;=\; \sum_{i=1}^{+\infty} x_i A_i\,,$$

where only a finite number of x_i are non-zero (at least one), and $x_i \in \{0, 1, \ldots, \alpha_i\}$ for all i.

From (i) and (ii), we may consider the sequence $\{A_{n_k}\}_{k=0}^{+\infty}$ defined by $A_{n_0} = 1$ and, for $k \geq 0$

$$A_{n_{k+1}} \;=\; \min\{A_n : n \in \mathbb{N}^*\,,\quad n \notin \{n_0, n_1, \ldots, n_k\}\}\,.$$

Then $\{A_n : n \in \mathbb{N}^*\} = \{A_{n_k} : k \in \mathbb{N}\}$ and $\{A_{n_k}\}_{k=0}^{+\infty}$ is increasing.

Claim. For $k \in \mathbb{N}$, we have

(a) $A_{n_k} = 1 + \sum_{i=0}^{k-1} \alpha_{n_i} A_{n_i}$;

(b) $A_{n_k} = \prod_{i=0}^{k-1} (\alpha_{n_i} + 1)$;

(c) when x_{nj} takes each value in $\{0, \ldots, \alpha_{nj}\}$ for all $j \leq k$, the number

$$x \;=\; \sum_{j=0}^{k} x_{nj} A_{nj}$$

takes each value in the set $\{0, 1, \ldots, \sum_{i=0}^{k} \alpha_{n_i} A_{n_i}\}$ (with the usual convention that $\sum_\emptyset = 0$ and $\prod_\emptyset = 1$).

Proof. By induction on k. We have seen that $A_{n_0} = 1$. Thus, the claim is obviously true for $k = 0$.

Let k be a fixed non-negative integer. Suppose that the claim holds for k.

From the uniqueness of the representation and from (c), we deduce that

$$A_{n_{k+1}} \geq x \quad \text{where} \quad x = 1 + \sum_{j=0}^{k} \alpha_{n_j} A_{n_j} .$$

Suppose that $A_{n_{k+1}} > x$.

Then, since $\{A_{n_k}\}$ is increasing, we have $A_{n_j} > x$ for $j \geq k$. Moreover, since x has a representation of the desired form, we must have

$$x = \sum_{j=1}^{N} x_j A_j \quad \text{for some} \quad N \in \mathbb{N}^* ,$$

where $x_j \neq 0$ for some $j \notin \{n_0, \ldots, n_k\}$. Thus, $A_j \geq A_{n_{k+1}} > x$.

Therefore, $x \geq x_j A_j \geq A_j > x$, a contradiction. Thus, $A_{n_{k+1}} = x$ and (a) is proved for the value $k+1$.

From (a) and (b), we then have

$$\begin{aligned} A_{n_{k+1}} &= 1 + \sum_{j=0}^{k} \alpha_{n_j} A_{n_j} \\ &= \alpha_{n_k} A_{n_k} + 1 + \sum_{j=0}^{k-1} \alpha_{n_j} A_{n_j} \\ &= \alpha_{n_k} A_{n_k} + A_{n_k} \\ &= (\alpha_{n_k} + 1) A_{n_k} \\ &= \prod_{j=0}^{k} (\alpha_{n_j} + 1) , \end{aligned}$$

and (b) is proved for the value $k+1$.

Moreover, if $x = \sum_{j=0}^{k+1} x_{n_j} A_{n_j}$ with $x_{n_j} \in \{0, 1, \ldots, \alpha_{n_j}\}$ for $j = 0, \ldots, k+1$, we have

$$x = x_{n_{k+1}} A_{n_{k+1}} + y ,$$

where y takes all possible values in the set $\{0, 1, \ldots, \sum_{j=0}^{k} \alpha_{n_j} A_{n_j}\} = \{0, 1, \ldots, A_{n_{k+1}} - 1\}$.

It follows that x takes all possible values in the set $\{0, 1, \ldots, \sum_{j=0}^{k+1} \alpha_{n_j} A_{n_j}\}$, and the proof of the claim is complete.

We deduce that, for all $k \in \mathbb{N}$,

$$A_{n_k} = \prod_{j=0}^{k-1} (\alpha_{n_j} + 1) .$$

Let j, k be two positive integers such that

$$A_j \leq A_k .$$

Then, there exist p, $q \in \mathbb{N}$ such that $A_j = A_{n_p}$ and $A_k = A_{n_q}$. From $A_{n_p} \leq A_{n_q}$ and since $\{A_{n_k}\}$ is increasing, we deduce that $p \leq q$. Thus,

$$A_k = A_{n_q} = \prod_{j=0}^{q-1} (\alpha_{n_j} + 1) = A_{n_p} \prod_{j=p}^{q-1} (\alpha_{n_j} + 1) .$$

And thus, A_j divides A_k, as desired.

2. Given a square $ABCD$ of side length 2, let M and N be points on the edges $[AB]$ and $[CD]$, respectively. The lines CM and BN meet at P, while the lines AN and MD meet at Q. Show that $|PQ| \geq 1$.

Solutions by Pierre Bornsztein, Pontoise, France; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and Toshio Seimiya, Kawasaki, Japan. We give Klamkin's generalization.

More generally we replace the square by a rectangle with $AB = 2\ell$ and $CB = 2w$, and show that $|PQ| \geq \ell$.

Let the rectangular coordinates be $A(-\ell, w)$, $M(a, w)$, $B(\ell, w)$, $C(\ell, -w)$, $N(b, -w)$, and $D(-\ell, -w)$. The equations of lines AN and MD are

$$\frac{y - w}{2w} = -\frac{-(x + \ell)}{\ell + b} \quad \text{and} \quad \frac{y - w}{2w} = \frac{x - a}{a + \ell} .$$

Solving them gives point Q:

$$x_Q = \frac{ab - \ell^2}{a + b + 2\ell} , \quad y_Q = \frac{w(b - a)}{a + b + 2\ell} .$$

The equations of lines BN and MC are

$$\frac{y - w}{2w} = \frac{x - \ell}{\ell - b} \quad \text{and} \quad \frac{y - w}{2w} = \frac{x - a}{a - \ell} ,$$

and their point of intersection P is given by

$$x_P = \frac{ab - \ell^2}{a + b - 2\ell} , \quad y_P = \frac{w(b - a)}{a + b - 2\ell} .$$

We now need to show that $|PQ|^2 \geq \ell^2$ or

$$16[(ab-\ell^2)^2 + w^2(b-a)^2] \;\geq\; [4\ell^2 - (a+b)^2]^2 \,.$$

Since $-\ell \leq a$, $b \leq \ell$, it suffices to have

$$4(\ell^2 - ab) \;\geq\; 4\ell^2 - (a+b)^2$$

or $(a-b)^2 \geq 0$. There is equality if and only if $a = b$.

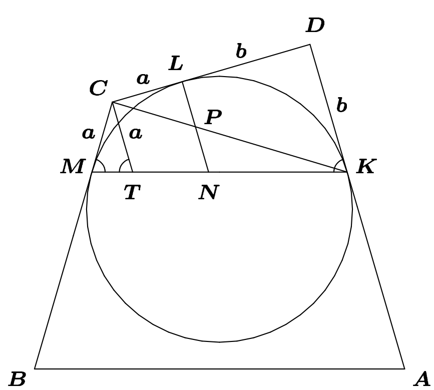
Comment. It follows that $\max|PQ|$ = the length of a diagonal of $ABCD$, this being the greatest chord of $ABCD$. It is achieved by letting N coincide with D and letting M approach B.

It also follows that points P and Q are always collinear with the centre of $ABCD$. This will also be true if $ABCD$ were a parallelogram.

4. Given a quadrangle $ABCD$, the circle which is tangential to $[AD]$, $[DC]$ and $[CB]$ touches these edges at K, L and M, respectively. Denote the point at which the line which passes through L and is parallel to AD meets $[KM]$ by N, and the point at which $[LN]$ and $[KC]$ meet by P. Prove that

$$|PL| \;=\; |PN| \,.$$

Solutions by Toshio Seimiya, Kawasaki, Japan; and by Achilleas Sinefakopoulos, student, University of Athens, Greece. We give the solution by Seimiya.



Since CM, CL, DL and DK are tangent to the circle, we have $CM = CL$ and $DL = DK$. We put $CM = CL = a$ and $DL = D = b$.

Let T be the point on MK such that $CT \parallel DK$. Then $\angle CTM = \angle DKM = \angle CMK = \angle CMT$. Thus, $CT = CM = a$. Since $LP \parallel DK$, we get $\frac{LP}{DK} = \frac{CL}{CD}$; that is, $\frac{LP}{b} = \frac{a}{a+b}$. Thus,

$$LP \;=\; \frac{ab}{a+b} \,. \tag{1}$$

Since $CT \parallel LN \parallel DK$, it follows that

$$\frac{PN}{CT} = \frac{KP}{KC} = \frac{DL}{DC} = \frac{b}{a+b}.$$

Since $CT = a$, we obtain

$$PN = \frac{ab}{a+b}. \tag{2}$$

From (1) and (2), we obtain $LP = PN$.

5. Show that $\prod_{k=0}^{n-1}(2^n - 2^k)$ is divisible by $n!$ for each positive integer n.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give Bataille's write-up.

Let $\mathbb{F}_2 = \{0, 1\}$ be the two-element field and $\mathbb{E} = (\mathbb{F}_2)^n$ be the vector space of all n–tuples $\vec{u} = (a_1, a_2, \ldots, a_n)$ with $a_i \in \mathbb{F}_2$ for $i = 1, 2, \ldots, n$. Then, $\prod_{k=0}^{n-1}(2^n - 2^k)$ is the number of ordered bases $(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n)$ of $\mathbb{E}$. Note that in such an ordered basis, the vectors $\vec{u}_1$, $\vec{u}_2$, …, $\vec{u}_n$ are distinct. It follows that each unordered basis $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ is a set of n vectors enabling one to make up exactly $n!$ ordered bases. Thus, if A denotes the number of unordered bases, we have

$$\sum_{k=0}^{n-1}(2^n - 2^k) = n! \times A.$$

The result follows.

6. Let $\mathbb{R}$ stand for the set of all real numbers. Show that there is no function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) > f(x)(1 + yf(x))$$

for all positive real x, y.

Solution by Pierre Bornsztein, Pontoise, France.

Suppose, for a contradiction, that such a function does exist. Then, for some positive x, y, we have

$$f(x+y) - f(x) > yf^2(x) \geq 0.$$

It follows that f is increasing on $\mathbb{R}^+$. Thus, f cannot be identically zero and there exists $\alpha > 0$ such that $f(\alpha) \neq 0$.

Since, for all $y > 0$,

$$f(x+y) > f(x) + yf^2(\alpha)$$

with $\lim_{y\to+\infty}(f(\alpha)+yf^2(\alpha)) = +\infty$, we deduce that $\lim_{x\to+\infty} f(x) = +\infty$. Then, we may consider $a > 0$ such that $f(a) > 0$. Thus, for $x \geq a$, we have $f(x) \geq f(a) > 0$. For such a real x, if $y = \frac{1}{f(x)}$, we have

$$f\left(x + \frac{1}{f(x)}\right) \;>\; 2f(x)\,. \tag{1}$$

Let (x_n) be the sequence defined by $x_0 = a$ and $x_{n+1} = x_n + \frac{1}{f(x_n)}$ for $n \in \mathbf{N}$.

An easy induction shows that (x_n) is well defined and increasing. Thus, $x_n \geq a$ for all $n \geq 0$. For all $n \geq 0$, define $U_n = f(x_n)$. Therefore, $U_n > 0$ and, from (1), we have

$$\begin{aligned} U_{n+1} &= f(x_{n+1}) \;=\; f\left(x_n + \frac{1}{f(x_n)}\right) \\ &> 2f(x_n) \;=\; 2U_n\,. \end{aligned}$$

This leads to $U_n \geq 2^n U_0$ for all $n \geq 0$ and therefore,

$$\lim_{n\to+\infty} U_n \;=\; +\infty\,. \tag{2}$$

But, for all $n \geq 0$,

$$x_{n+1} - x_n \;=\; \frac{1}{f(x_n)} \;=\; \frac{1}{U_n} \;\leq\; \frac{1}{2^n U_0}\,.$$

Summing, we obtain:

$$x_{n+1} \;\leq\; a + \frac{1}{U_0}\sum_{k=0}^{n}\frac{1}{2^k} \;<\; a + \frac{1}{U_0}\sum_{k=0}^{+\infty}\frac{1}{2^k}\,.$$

Thus, $x_{n+1} < a + \frac{2}{U_0}$.

It follows that, for all $n \geq 0$,

$$U_{n+1} \;=\; f(x_{n+1}) \;<\; f\left(a + \frac{2}{U_0}\right)\,,$$

where $f\left(a + \frac{2}{U_0}\right)$ is independent of n.

Thus, the sequence (U_n) is bounded from above, which contradicts (2). The conclusion now follows.

That completes the corner for this year. Send me your nice solutions, generalizations, and comments, as well as Olympiad Contests!

BOOK REVIEWS

JOHN GRANT McLOUGHLIN

Geometry for College Students
by I. Martin Isaacs, published by Brooks/Coles Publishing, 2001,
ISBN 0-5343-5179-4, hardcover, 217 pages, $81.95 (Cdn).
Reviewed by ***J. Chris Fisher****, University of Regina, Regina, SK.*

I have been looking for this geometry text for a long time. Many universities have a "college geometry" course whose principal audience consists of mathematics students specializing in secondary education. For them the course is required, but there are also students who take it as an elective. Isaacs' book is pitched at just the right level — to second- and third-year undergraduates who have at best a fuzzy knowledge of high-school geometry. The topics have been chosen both for their intrinsic interest and because they can be used to prove yet other interesting results. The emphasis is on attractive theorems and informative proofs, which means avoiding unsurprising theorems, formalism, and axiomatics. The author correctly points out that in many instances the students themselves will demand proofs because the assertions are otherwise so incredible.

The book is extremely well written — clear, self-contained, and authoritative. There are six chapters.

1. *The Basics* provides a review of the fundamental theorems that all students are supposed to have seen in high school. These theorems are used repeatedly throughout the text. The best way to become familiar with such results is to use them, so I limited class time to discussing only the less familiar (such as angles inscribed in the same arc are equal) and the problematic (similarity in general). The chapter is so carefully written that most students can (and do) pick up what they require without any help from the instructor.

2. *Triangles* discusses the Euler line, nine-point circle, Morley's Theorem, and lots more.

3. *Circles and Lines.*

4. *Ceva's Theorem and Its Relatives.*

5. *Vector Methods of Proof.*

6. *Geometric Constructions.*

There is more material here than will fit in a one-semester course — in the course I taught last semester (39 hours of class time) we spent most of our time in chapters 2, 4, and 6.

What makes a college geometry course particularly difficult for teacher and students is that there is so much that must be done in so short a time. Our students no longer have had a systematic introduction to geometry before their arrival at university. Few have any notion of what constitutes a convincing argument. The principal goal is therefore to learn the nature of mathematical proof — how to draw valid conclusions from hypotheses and how to detect and avoid invalid reasoning. Geometry clearly offers the right mix of depth and concreteness for this purpose. Thus, in addition to learning geometric facts, a student must learn how to read arguments, how to present arguments, and (most importantly!) how to allocate time to learn the material. There is a widespread belief among today's students that homework is something to be looked at an hour or two before it is to be submitted. For many, this is the first course where they either must change that attitude (sometimes after having received a grade of zero on their first assignment) or drop the course.

The author begins his text with an apology for not knowing any simple way to explain what a proof is. The only method seems to be providing ample examples of good proofs. He then provides a nice discussion of what constitutes a proof, illustrating his remarks by means of a carefully worked example that shows what can and what cannot be read off from a diagram. What distinguishes this text from its competitors is that all the proofs are complete, correct, and very easy to follow. There are no instances where the author proves the easy parts and dismisses the rest as an exercise or as something beyond the text's scope. What distinguishes this text from other excellent books, in particular from Coxeter and Greitzer's *Geometry Revisited* (which Isaacs acknowledges as both source and model for his text), is the detailed explanations that are included with each proof. Fewer demands are made upon the skills and the tenacity of the reader. The proofs are just right for the intended audience. As a consequence, the teacher can use class time to motivate each result and provide the basic ideas behind the proofs, allowing the students to consult the text for the detailed proof.

Part of the reason for his success here is that Isaacs is both a fine mathematician and an excellent writer — he has written two successful algebra texts and numerous expository articles. The book is not without shortcomings. There are not nearly enough figures, averaging around one per page; on the other hand, the omission of figures has kept the book short (217 pages), and it could be argued that the readers need practice at making diagrams for themselves as they read. Also, although the exercises contain good problems for further practice, they generally follow quickly from what has come just before in the text; in addition, there is often a hint accompanying the problem. It would be nice to have supplementary exercises that provide a genuine challenge. Of course, one can find plenty of such problems in *Crux with Mayhem* to supplement the text. Finally, why is it an expensive, hardcover text? Why is it not a paperback selling for $20?

A Revival of an Old Construction Problem

Barukh Ziv

The purpose of this note is to revive an old triangle construction problem and to provide an apparently missing elementary justification for part of it.

In 1988, I. Sakmar proposed the following problem [1]:

Inverting a Transformation by Equilateral Triangles

Let P, Q, R be the new vertices of equilateral triangles constructed outwardly on the edges of a given triangle ABC. (compare H.S.M. Coxeter, Introduction to Geometry, New York, 1961, p. 22).

(a) Show that any triangle PQR which can be obtained in this way arises from a unique triangle ABC, and give a construction for recovering triangle ABC from triangle PQR.

(b) Show that not every triangle PQR can be so obtained.

The solution followed three years later [2]. The first part of the solution is really delightful in its simplicity and beauty. It goes as follows: Let U, V, W be the new vertices of equilateral triangles constructed inwardly on the edges of the triangle PQR. If P, Q, R are vertices of equilateral triangles constructed outwardly on the sides of a given triangle ABC, then A, B, C are necessarily the mid-points of the sides of the triangle UVW (Figure 1).

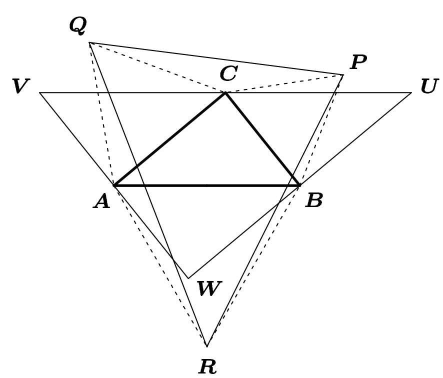


Figure 1

Later, from other sources [3, 4], the author has learned that this problem has had a long history. It is attributed to E. Lemoine who proposed it as early as 1868 in Nouvelles Annales de Mathématique (Question 864), and the first solution by L. Kiepert followed a year later (his solution was similar to that of Mauldon but proceeded by constructing outward equilateral triangles). The problem reappeared thereafter in various sources ([4] and a list of references cited therein). When justifying the construction, various methods were used, from analytical to purely elementary. For instance, Kiepert exploits the properties of the Fermat point of the triangle. The most elegant argument, as pointed out by J.E. Wetzel [3], was given in 1956 by H.G. Steiner, who used rotations: If X^{α} designates the rotation about a point X through an angle α (a positive angle is measured counterclockwise), then the sequence of rotations $Q^{60}P^{60}R^{60}$ is a half turn that fixes point A, and takes W into V. Therefore, A is a mid-point of VW. The same argument holds for vertices B and C.

And what about the second part of the problem? In fact, it is the discussion of the first part, and without it no solution to a construction problem may be considered complete. In [2], the existence condition was stated and proved analytically. In a much earlier note [5] a geometric condition was stated, without proof (see Figure 2):

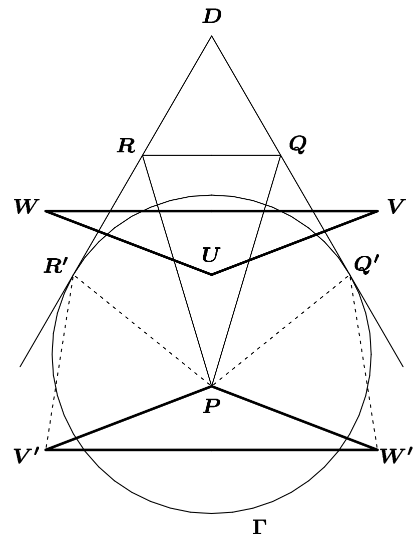


Figure 2

Consider any side of the given triangle PQR, say QR, and the corresponding outward equilateral triangle QRD. Extend the sides DQ and DR beyond Q and R to points Q' and R' so that $\overline{DQ} = \overline{QQ'}$ and $\overline{DR} = \overline{RR'}$. Let Γ be the circle tangent to DQ at Q' and DR at R'. The solution exists if vertex P lies inside Γ.

J. Wetzel in his elaboration of this and similar constructions [3] gave the same condition and provided proof using complex coordinates. In this note, the purely elementary, geometric argument is presented.

We begin with the claim that when triangle PQR is constructed from triangle ABC, then PQR and ABC have the same orientation. To prove this, assume that C is the largest angle in ABC. Because angles RAB and RBA are 60° (Figure 1), both angles RCA and RCB are less than 120°, so that P and Q are separated by the line CR. Also, because angles CBP and QAC are 60°, angles ABP and BAQ are less than 180°, so that pairs P, R and Q, R are separated by the line AB, and the claim follows. It then follows immediately that if after constructing ABC as the mid-point triangle of UVW, its orientation is opposite to PQR, then the vertices P,

Q, R cannot be obtained in the prescribed way from any triangle. Thus, we need to show that PQR has the same orientation as the mid-point triangle of UVW exactly when P lies inside the circle Γ.

No generality is lost in assuming triangle PQR counterclockwise oriented. Construct equilateral triangles $PR'V'$ and $Q'PW'$ oriented the same as PQR (Figure 2). Since $\overline{DQ} = \overline{QQ'}$ and $\overline{DR} = \overline{RR'}$, each of Q, Q', U and R, R', U form equilateral triangles. Also, because chord $Q'R'$ subtends an arc of $120°$ on Γ, one sees that, for P inside Γ, angle $Q'PR'$ varies from $60°$ to $240°$. Then, angle $V'PW'$ is always between $0°$ and $180°$, meaning the triangle $PV'W'$ is counterclockwise oriented. Similarly, when P is outside Γ, the angle $V'PW'$ is between $180°$ and $360°$, which corresponds to the clockwise oriented triangle $PV'W'$. Consider then the transformation $R'^{60}P^{-60}$; that is, the translation carrying point U into V and V' into P. Thus, the segment PV' is carried into segment UV by a half turn. Analogously, the translation $Q'^{-60}P^{60}$ carries U into W and W' into P, so that the segment PW' is carried into UW by a half turn. Therefore, triangles UVW and $PV'W'$ are interchanged by a half turn and hence have the same orientation. Of course, the mid-point triangle of UVW will have the same orientation as UVW. This concludes the proof.

It is interesting to note that P lying on the circle Γ corresponds to angle $V'PW' = 0°$ or $180°$; that is, points A, B, C are collinear. Specifically, when P lies on a small arc $Q'R'$, point A lies outside segment BC; when P lies on a big arc, A lies on the segment BC.

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5. Editorial Note to the Solution to Problem E170, *Amer. Math. Monthly* 43 (1936) 110.

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MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to **Mathematical Mayhem, Cairine Wilson Secondary School, 977 Orleans Blvd., Gloucester, Ontario, Canada. K1C 2Z7**. The electronic address is
`mayhem-editors@cms.math.ca`

The Mayhem Editor is Shawn Godin (Caairine Wilson Secondary School). The Assistant Mayhem Editor is Chris Cappadocia (University of Waterloo). The other staff member is Paul Ottaway (Dalhousie University).

Mayhem Problems

Envoyez vos propositions et solutions à MATHEMATICAL MAYHEM, Faculté de mathématiques, Université de Waterloo, 200 University Avenue West, Waterloo, ON, N2L 3G1, ou par courriel à
`mayhem-editors@cms.math.ca`
N'oubliez pas d'inclure à toute correspondance votre nom, votre année scolaire, le nom de votre école, ainsi que votre ville, province ou état et pays. Nous sommes surtout intéressés par les solutions d'étudiants du secondaire. Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le *1er avril 2003*. Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

Pour être admissibles au DÉFI MAYHEM de ce mois-ci, les solutions doivent avoir été postées *avant le* 1er avril 2003, cachet de la poste faisant foi.

M69. *Proposé par l'Equipe de Mayhem.*

Une suite numérique est formée en écrivant les chiffres des entiers naturels dans le même ordre qu'ils apparaissent dans les entiers. Les premiers termes de cette suite sont alors :

$$\mathbf{1, 2, 3, 4, 5, 6, 7, 8, 9, 1, 0, 1, 1, 1, 2, \ldots .}$$

Quel est le $\mathbf{2002}^{\text{ième}}$ terme dans la suite ?

A sequence of digits is formed by writing the digits from the natural numbers in the order that they appear. The sequence starts:

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 1,\ 0,\ 1,\ 1,\ 1,\ 2,\ \ldots.$$

What is the 2002^{nd} digit in the sequence?

M70. *Proposé par l'Equipe de Mayhem.*

Quel est le plus petit multiple positif de 15 qui est formé uniquement par les chiffres 0, 4 et 7, chacun apparaissant le même nombre de fois ?

. . . .

What is the smallest positive multiple of 15 that is made up of only the digits 0, 4 and 7, each appearing the same number of times?

M71. *Proposé par Richard Hoshino, Université Dalhousie, Halifax, Nouvelle Ecosse.*

Soit a, b et c trois nombres premiers et posons $x = a + b - c$, $y = a + c - b$ et $z = b + c - a$. Sachant que $x^2 = y$ et que $\sqrt{z} - \sqrt{y}$ est le carré d'un nombre premier, déterminer toutes les valeurs possibles pour le produit abc.

. . . .

Let $x = a + b - c$, $y = a + c - b$ and $z = b + c - a$, where a, b and c are prime numbers. Given that $x^2 = y$ and $\sqrt{z} - \sqrt{y}$ is the square of a prime number, determine all possible values for the product abc.

M72. *Proposé par J. Walter Lynch, Athens, GA, USA.*

Vous avez une tasse de café et une tasse de thé. Les tasses sont identiques et chacune contient la même quantité de liquide que l'autre. Vous prenez une cuillère de café de la tasse de café et la mettez dans la tasse de thé. Vous prenez ensuite une cuillère du mélange de la tasse de thé et vous la mettez dans la tasse de café. Comparer les quantités suivantes :

- la quantité de café dans la tasse de thé
- la quantité de thé dans la tasse de café

. . . .

You have a cup of coffee and a cup of tea. The cups are identical and each contains the same amount of liquid as the other. You take a teaspoon full of coffee out of the coffee cup and put it into the teacup. You then take a teaspoon full of the mixture out of the teacup and put it into the coffee cup. Which is greater, the amount of coffee in the teacup, the amount of tea in the coffee cup, or are they the same?

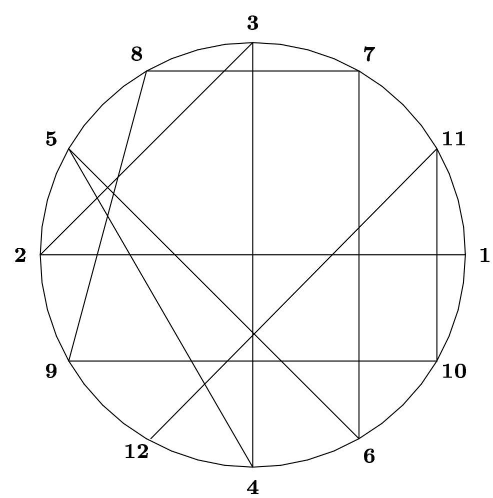
M73. *Proposé par J. Walter Lynch, Athens, GA, USA.*

Un cylindre droit de rayon r et de hauteur h contient du liquide jusqu'à x du haut du cylindre. Trouver l'angle d'inclinaison du cylindre afin que le liquide commence à être versé. (On suppose qu'il y a suffisament de liquide afin que la surface du liquide n'intersecte pas le fond du cylindre avant que le liquide commence à être versé.)

A right circular cylinder with radius r and height h contains a liquid to within x of the top of the cylinder. Find the angle through which the cylinder must be tilted in order for the liquid to start to pour out. (Assume that there is enough liquid in the cylinder so that the surface of the liquid does not intersect the bottom of the cylinder before the liquid starts to pour out.)

M74. *Proposé par l'Equipe de Mayhem.*

12 points sont également espacés sur la circonférence d'un cercle. De combien de façons peut-on associer les entiers de 1 à 12 à ces 12 points de telle manière que lorsqu'on rejoint ces points par les lignes droites, en ordre, ces droites ne sont pas concourantes ? Un exemple d'un **mauvais** arrangement est illustré ci-dessous.



A circle has 12 equally spaced points placed on its circumference. How many ways can the numbers 1 through 12 be assigned to the points so that if the points 1 through 12 are connected with line segments, in order, the segments do not cross? An example of a **bad** arrangement is illustrated above.

M75. *Proposé par l'Equipe de Mayhem.*

Les termes de la suite croissante 1, 5, 6, 25, 26, 30, 31, 125, 126, ... sont obtenus en ajoutant des puissances distinctes de 5. Quel est le $75^{\text{ième}}$ terme de cette suite ?

....

The increasing sequence 1, 5, 6, 25, 26, 30, 31, 125, 126, ... consists of positive integers that can be formed by adding distinct powers of 5. What is the 75^{th} integer in the sequence?

Mayhem Solutions

M19. *Proposed by the Mayhem staff.*

On the magical island of Xurc, there lives a giant Ecurb. Ecurb has an unlimited supply of special coins that are worth one million dollars each. Ecurb allows people to go into his castle and take as many of these coins as they like, but, they must give some up in order to cross the bridges to leave his island. At each of the five bridges Ecurb demands that you give $\frac{99}{100}$ of a coin more than $\frac{99}{100}$ of the coins in your possession. Coins cannot be cut or broken in any way. If the demand cannot be met Ecurb takes all of your coins and eats one of your feet. How many coins do you have to start with in order to make it off the island with exactly one coin (and both feet)?

Solution.

Suppose you have Z coins before you come to a bridge, and you have X coins after you cross the bridge. The relationship between Z and X is given by

$$\begin{aligned} Z - (0.99 + 0.99Z) &= X\,, \\ Z &= 100X + 99\,. \end{aligned}$$

You have one coin after you cross the fifth bridge. Using the relationship above, you will have had 199 coins just before crossing the fifth bridge. Apply this relationship repeatedly; then you will have had 19999 coins just before crossing the fourth bridge, 1999999 coins before crossing the third, 199999999 before crossing the second, and 19999999999 before crossing the first.

Therefore, you have to start with exactly 19999999999 coins to make it off the island with exactly one coin (and both feet).

M20. *Proposed by Richard Hoshino, Dalhousie University, Halifax, Nova Scotia.*

Suppose that A, B and C are positive integers in arithmetic progression with $A^\circ < B^\circ < C^\circ < 180^\circ$.

If $\sin A^\circ + \sin B^\circ = \sin C^\circ$ and $\cos A^\circ - \cos B^\circ = \cos^\circ C$, determine the triplet (A, B, C).

Solution.

Let $A^\circ = B^\circ - d^\circ$ and $C = B^\circ + d^\circ$ for some integer $0^\circ \leq d^\circ < 90^\circ$. Then

$$\begin{aligned} \sin(B^\circ - d^\circ) + \sin B^\circ &= \sin(B^\circ + d^\circ)\,, \\ \sin B^\circ \cos d^\circ - \sin d^\circ \cos B^\circ + \sin B^\circ &= \sin B^\circ \cos d^\circ + \sin d^\circ \cos B^\circ\,, \\ \sin B^\circ &= 2\sin d^\circ \cos B^\circ\,, \qquad (1) \end{aligned}$$

and

$$\begin{aligned} \cos(B^\circ - d^\circ) - \cos B^\circ &= \cos(B^\circ + d^\circ)\,, \\ \cos B^\circ \cos d^\circ + \sin B^\circ \sin d^\circ - \cos B^\circ &= \cos B^\circ \cos d^\circ - \sin B^\circ \sin d^\circ\,, \\ \cos B^\circ &= 2\sin B^\circ \sin d^\circ\,. \end{aligned} \qquad (2)$$

Substituting (2) into (1) gives

$$\begin{aligned} \sin B^\circ &= 2\sin d^\circ(2\sin B^\circ \sin d^\circ)\,, \\ \sin B^\circ &= 4\sin^2 d^\circ \sin B^\circ\,. \end{aligned}$$

And $\sin B^\circ \neq 0$, since $0^\circ < B^\circ < 180^\circ$. Hence, $\sin^2 d^\circ = \frac{1}{4}$, whereupon $\sin d^\circ = \frac{1}{2}$, since $0^\circ \leq d^\circ < 90^\circ$. Therefore, $d^\circ = 30^\circ$.

Substituting $\sin d^\circ = \frac{1}{2}$ into (1) gives $\sin B^\circ = \cos B^\circ$, from which it follows that $B^\circ = 45^\circ$. Therefore, $A = 15$, $B = 45$, and $C = 75$.

M21. *Proposed by the Mayhem staff.*

Find all positive integers a, b, c, d, and e which satisfy

$$a! = b! + c! + d! + e!\,.$$

Solution.

Suppose that $a > b \geq c \geq d \geq e \geq 1$. Then, $a! \geq 4e! \geq 4$, implying $a \geq 3$.

Since $a > b$, we have $a! \geq a(b!)$. But then $a! = b! + c! + d! + e!$ gives $a! \leq 4b!$, which forces $a \leq 4$. Thus, we have $a = 3$ or $a = 4$.

When $a = 3$, we have $b! + c! + d! + e! = 6$. We must have b, c, d, and e all strictly less than 3, and at least 1. A quick inspection shows that b, c, d, and e must be some arrangement of 2, 2, 1, and 1, after we lift the restriction on the order of b, c, d, and e.

When $a = 4$, we have $b! + c! + d! + e! = 24$. We must have b, c, d, and e all strictly less than 4. Thus, $b! + c! + d! + e! \leq 3! + 3! + 3! + 3! = 24$ and, in fact, we must take $b = c = d = e = 3$ so that the equality holds.

M22. *Proposed by the Mayhem staff.*

George is walking across a bridge on the train track. When he is $\frac{5}{12}$ of the way across the bridge he notices a train bearing down on him at 90 km/h. If he can just escape death by running in either direction, how fast can George run?

Solution.

Let the length of the bridge be b metres. George is $\frac{5}{12}b$ metres away from one end of the bridge and $\frac{7}{12}b$ metres away from the other end of the bridge.

The train is heading towards George at 90 km/h, or 25 m/s, and the end of the bridge that is closest to George is between George and the train (since the train is "bearing down on him"). Let the train be d metres away from the closest end of the bridge.

Let George's top speed be x m/s.

Now, it takes both George and the train the same amount of time to reach the end of the bridge that is closest to George. George takes $\frac{5b}{12x}$ seconds to reach the close end of the bridge. The train takes $\frac{d}{25}$ seconds to reach the close end of the bridge.

Thus, $\frac{5b}{12x} = \frac{d}{25}$ or

$$12x = 125\frac{b}{d} \qquad (1)$$

It also takes both George and the train the same amount of time to reach the far end of the bridge. From this we obtain the equality $\frac{7b}{12x} = \frac{b+d}{25}$ or

$$12x = 175\frac{b}{b+d} \qquad (2)$$

Comparing (1) and (2) we get $125\frac{b}{d} = 175\frac{b}{b+d}$ and solving for $\frac{b}{d}$ gives $\frac{b}{d} = \frac{2}{5}$.

Substituting this value into (1) gives $12x = 125\frac{2}{5}$ or $x = \frac{25}{6}$. Therefore, George can run at a speed of $\frac{25}{6}$ m/s $= 15$ km/h.

M23. *Proposed by José Luis Díaz-Barrero and Juan José Egozcue, Barcelona, Spain.*

Find all complex solutions of the following system of equations

$$\begin{aligned} x^3 + y^3 + z^3 + t^3 &= 12\,, \\ x^2 + y^2 + z^2 + t^2 &= 0\,, \\ xy + zt + (x+y)(z+t) &= 0\,, \\ xyzt &= 3\,. \end{aligned}$$

Solution by Mihály Bencze, Brasov, Romania.

$$\begin{aligned} &(x+y+z+t)^2 \\ &\quad = \; x^2 + y^2 + z^2 + t^2 + 2xy + 2zt + 2xz + 2xt + 2yz + 2yt \\ &\quad = \; (x^2 + y^2 + z^2 + t^2) + 2(xy + zt + (x+y)(z+t)) \\ &\quad = \; 0 + 2(0) \; = \; 0\,. \end{aligned}$$

Thus, $x + y + z + t = 0$.

Now notice that

$$\begin{aligned} 0 &= (x+y+z+t)(x^2+y^2+z^2+t^2-xy-zt-xz-xt-yz-yt) \\ &= x^3+y^3+z^3+t^3-3(xyz+yzt+ztx+txy)\,. \end{aligned}$$

And since $x^3+y^3+z^3+t^3=12$, we have $xyz+yzt+ztx+txy=4$. Thus, we have,

$$\begin{aligned} x+y+z+t &= 0\,, \\ xy+zt+xz+xt+yz+yt &= 0\,, \\ xyz+yzt+ztx+txy &= 4\,, \\ xyzt &= 3\,. \end{aligned}$$

Therefore, x, y, z, and r are the roots of the quartic equation

$$\begin{aligned} r^4-0r^3+0r^2-4r+3 &= 0\,, \\ r^4-4r+3 &= 0\,, \\ (r-1)^2(r^2+2r+3) &= 0\,, \end{aligned}$$

with the roots 1, 1, $-1+i\sqrt{2}$, and $-1-i\sqrt{2}$. Therefore, x, y, z, and r are some arrangement of these four roots.

M24. *Proposed by the Mayhem staff.*

A school math club is deciding on a name for its mascot, a stuffed rabbit. They have narrowed the choices down to three: Euler, Galois and Ramanujan. To pick the name they have each of the 100 club members rank the names in order of preference. When the polls were totalled it was found that 60 people preferred Galois over Ramanujan and 62 preferred Ramanujan over Euler. It was suggested, by a Galois supporter, that Euler should be dropped. A staunch Eulerist objected and demanded the counting continue. When the final totals came in, it was found that 68 preferred Euler over Galois! If each possible ranking was picked by at least one member, how many picked each name as their first choice?

Editor's Note: The problem as stated does not have a unique solution. The problem will be restated in a future issue in a form that is solvable. Sorry for the inconvenience.

M25. *Proposed by the Mayhem staff.*

What is the smallest number with the property that when the first digit (leftmost) is moved to the rightmost position, the new number is three times the original?

Solutions by Robert Bilinski, Outremont, PQ; and by Gustavo Krimker, Universidad CAECE, Argentina. We give Bilinski's solution.

$\sum\limits_{k=0}^{n} 10^k a_k = 10^n a_n + \sum\limits_{k=0}^{n-1} 10^k a_k$, with a_0, …, a_n single digits from 0 to 9, with $a_n \neq 0$ and $a_{n-1} \neq 0$.

We want to find the smallest n which allows a solution to

$$\left(\sum_{k=0}^{n-1} 10^{k+1} a_k\right) + a_n \;=\; 3(10^n)a_n + 3\sum_{k=0}^{n-1} 10^k a_k\,,$$

which simplifies to

$$7\sum_{k=0}^{n-1} 10^k a_k \;=\; ((3)10^n - 1)a_n\,.$$

Then $a_n \neq 7$ since the left side has $n-1$ digits while the right side has n digits. Hence, we must find the first n such that $(3)10^n - 1$ is divisible by 7. By inspection, $n = 5$. Now $(3)10^n - 1 = 299999 = 7(42857)$.

Therefore, $\sum\limits_{k=0}^{n-1} 10^k a_k = 42857 a_5$.

Of course, we want the smallest possible a_5. Thus, we get the solution $a_5 = 1$, $a_4 = 4$, $a_3 = 2$, $a_2 = 8$, $a_1 = 5$, and $a_0 = 7$. Therefore, 142857 is the smallest number satisfying the given conditions.

Pòlya's Paragon

Paul Ottaway

This month's article is inspired by a rather famous tiling problem involving dominoes and a chessboard. In case you are unfamiliar with it, I will pose it again here:

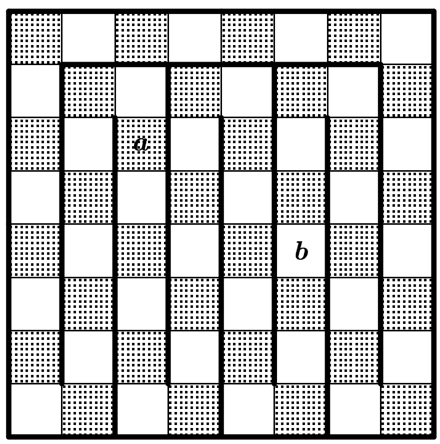
You are given an 8×8 chessboard and a supply of 2×1 dominoes that cover exactly 2 adjacent spaces on the chessboard. Is it possible to cover a board that is missing opposite corners with exactly 31 dominoes? None of the dominoes may overlap or hang off the edge of the board. I will for the rest of the article, call this a 'tiling'.

Clearly, it is easy to tile a complete board with exactly 32 dominoes by placing 4 along each row (or column). If we remove 2 spaces from the board, it would seem that shifting the dominoes in some way would allow us to tile the board with one less domino. In this particular case, it can never be done! The proof of this is one of my favourites since it is so simple and concise.

When a domino is placed on the board, it covers exactly 1 black square and 1 white square (assuming the standard alternating colouring of such a board). When we remove opposite corners, they have the same colour (say white). Now our board has 32 black squares and 30 white squares. Since we can cover at most 1 black space with each domino, it will be impossible to cover the entire board with exactly 31 dominoes. Of course, if we were to remove any two squares of the same colour, there would be no way to tile the board for exactly the same reason.

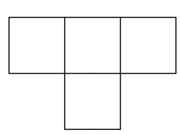
The natural problem to now ask is: If we remove one black square and one white square from the board, is there always a way to cover the board with 31 dominoes?

In some senses, this should be easier than the original problem. To prove that something never occurs, we must be sure to do it in complete generality. To show that something is possible, all we have to do is show how to do it. For this reason, our previous argument is no longer useful — if we remove opposite colour spaces we only know that a tiling may exist. We need to construct a tiling that uses 31 dominoes that will cover a chessboard that is missing one black and one white space. Consider the following diagram:



This snake-like pattern is what will give us the result. If we remove one space of each colour (a and b), we are left with 2 'paths' from a to b. Each path will consist of an even number of spaces since it must begin with one colour and end with the other. Thus, we can cover each path perfectly with some number of dominoes. Since the two paths cover all the spaces except a and b, we have tiled all 62 remaining spaces of the board with exactly 31 dominoes. I should be careful and note that if two adjacent spaces are removed, we can sometimes get a path of length 0 between them, but that will not hinder the proof any. This result is more commonly known as Gomory's Theorem.

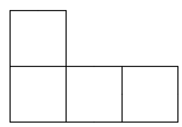
Let us now move on to some similar problems that we might tackle in much the same way: Is it possible to tile a 10×10 board with a supply of pieces like this?



Well, let us begin by assuming our board is coloured much like a chess-board with alternating black and white spaces. Each piece will cover 3 spaces of one colour and 1 space of the other. In either case, it is important to note that they cover an odd number of spaces of a given colour. Now we examine the board as a whole. If a tiling does exist, we must use exactly 25 pieces (since each covers 4 spaces, and the board has 100 spaces). Here is where we run into a problem. Let us look at how many black spaces we are covering. There are 25 tiles that cover 1 or 3 black spaces on the board. We also know that the board has 50 black spaces. That means that the sum of 25 odd numbers must be exactly 50! Of course this can never be true, so that a tiling does not exist.

That was certainly more work than the first problem, but still came down to a relatively simple argument about even and odd numbers — parity. Let us try this problem one more time:

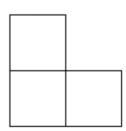
Is it possible to tile a 10×10 board with a supply of pieces like this?



It seems as though our previous method is no longer going to work. When we place this piece on the board it will cover 2 black squares and 2 white squares which will only end up telling us that a tiling may exist. I am going to leave this problem as an exercise, but I will give you a hint — try a different colouring of the board!

Here is another interesting tiling problem which I ran across last year:

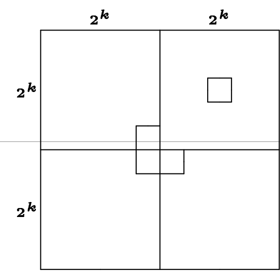
Is it possible (for all $n > 0$) to tile a $2^n \times 2^n$ board with exactly 1 (arbitrarily chosen) space removed if we are given a supply of pieces like this?



The answer to this problem is yes! The proof is by induction:

First note that for $n = 1$, we need only 1 piece to tile the board. Let us now assume that we can tile the board for some $n = k$ and construct a tiling of the board for $n = k + 1$.

First, we divide the board into 4 quadrants each of side length 2^k. The missing space is in one of the quadrants, so that, by induction, we can tile that quadrant perfectly. For each of the other quadrants, we can tile them perfectly by leaving exactly one space empty. If we tile them all so that space closest to the middle of the board is left empty, we are left with a board that is completely tiled except for 3 spaces in the centre that can be filled by one more piece.



Since we have now completely tiled the board for $n = k + 1$, by induction we conclude that it is possible to tile all such boards.

In my experience, these types of problems are the most fun to work on because they do not require too much mathematical background — and you get to play with little pieces. Here are some related tiling problems that I hope you find entertaining:

PROBLEMS:

1. We have already discussed the 10×10 board using 2 different 4–space tiles (tetris pieces). Try repeating this problem with the other 3 types of 4–space tiles (the line, box and zig-zag). Which one(s) work and what sort of colourings do you need to prove otherwise?

2. If we do not count reflections, there are actually 7 different 4–space tiles (tetris pieces). Is it possible to tile any rectangle with each piece used exactly once?

3. Is it possible to 'tile' an $8 \times 8 \times 8$ box with 'tiles' of size $2 \times 1 \times 1$, if opposite $1 \times 1 \times 1$ corners are removed from the box? Think about the colouring carefully!

4. Is it possible to 'tile' a box with side length 2^n with a single $1 \times 1 \times 1$ space missing with $2 \times 2 \times 2$ 'tiles' that are also missing a $1 \times 1 \times 1$ space. This is a little hard to visualize, but it is the 3–dimensional version of induction proof above.

Year End Wrap Up

Shawn Godin

Well another year has come to an end (funny how the year ends in October for an editor!). The new format of Mayhem seems to be taking form and we are slowly starting to hear from more students, which is what we want. To you new contributors to Crux with Mayhem, welcome, and please continue to be part of our little family. We are always open to suggestions on how to make your journal better for you.

This year saw the introduction of the Mayhem Taunt, thanks to the generous support of the Endowment Fund of the CMS. We are just sorting out prizes for the first four Mayhem Taunts, and the prizes will be announced, and solutions printed, in 2003. The money from the Endowment Fund will be used up on prizes from the 2002 Taunt, but we are hoping to offer some sort of continued prizes ... keep posted.

At this point I would like to thank some people who make Mayhem possible. First, and foremost, I would like to thank Mayhem assistant editor CHRIS CAPPADOCIA. Chris has taken on a larger, more regular, role this year and has really pulled through for me when I needed him.

GRAHAM WRIGHT at the CMS headquarters in Ottawa has continued to be a large asset. Graham has been a strong supporter of my ideas with MAYHEM. Along with the staff at the CMS office, they have been able to provide me with materials for Taunt prizes and the necessities of editing this section.

The rest of the Mayhem Staff: JIMMY CHUI and PAUL OTTAWAY have continued to dedicate their time to make MAYHEM all it is. We bid farewell to Jimmy as he pursues his studies, all the best. Your dedication and hard work over the years is greatly appreciated; we will miss you.

Some other people who have been helpful over the year need to be mentioned for their continued help "behind the scenes" : ARLENE ANGEL, DAVID BRIGGS, BILL CLARKE, ELIZABETH ELTON, RICHARD HOSHINO, and JOSEPH KHOURY.

Finally a big thank you and farewell to our outgoing Editor-in-Chief BRUCE SHAWYER. Bruce has been a big asset and very supportive over my first two years on the job. Since taking over at Crux, Bruce has continually improved an already great journal. All the best in your future endeavours, and if you get too restless with your new found freedom, we are always looking for material.

All the best of the season to our readers and contributors. **Crux with Mayhem** is a function of our staff and our readers. Continue helping us make our journal grow. Happy problem solving. We will see you in 2003.

SKOLIAD No. 66

Shawn Godin

Solutions may be sent to Shawn Godin, Cairine Wilson S.S., 975 Orleans Blvd., Orleans, ON, CANADA, K1C 2Z5, or emailed to

mayhem-editors@cms.math.ca.

Please include on any correspondence your name, school, grade, city, province or state and country. We are especially looking for solutions from high school students. Please send your solutions to the problems in this edition by *1 April 2003*. A copy of **MATHEMATICAL MAYHEM Vol. 8** will be presented to the pre-university reader(s) who send in the best set of solutions before the deadline. The decision of the editor is final.

Our first entry this issue comes from the Junior Balkan Mathematical Olympiad. My thanks go to the Competition Committee of the Greek Mathematical Society for forwarding the material to Robert Woodrow, and my thanks to Robert for forwarding the material to me.

2nd Junior Balkan Mathematical Olympiad

16 – 21 June, 1998, Athens, Greece

(for students up to 15.5 years old)

1. (Yugoslavia) Prove that the number

$$\underbrace{11\ldots111}_{1997}\underbrace{22\ldots222}_{1998}5$$

is a perfect square.

2. (Greece) Let us consider a convex pentagon $ABCDE$, with $AB = AE = CD = 1$, $\angle ABC = \angle DEA = 90^\circ$ and $BC + DE = 1$. Find the area of the pentagon.

3. (Albania) Find all pairs of positive integers (x, y) that satisfy the following equation: $x^y = y^{x-y}$.

4. (Bulgaria) Using only three digits can one write 16 three-digit numbers, such that no two of them are giving the same remainder divided by 16.

Our next contest comes from the BC Mathematics Competitions. My thanks go to Jim Totten of the University College of the Cariboo and Clint Lee of Okanagan University College for forwarding the material to me.

BRITISH COLUMBIA COLLEGES
Junior High School Mathematics Contest, 2002
Preliminary Round
March 6, 2002

1. The year 2002 is a palindromic number; that is, it reads the same forwards and backwards. The number of years in the third millennium; that is, between the years 2001 and 3000, that have this property is:

(a) 1 (b) 10 (c) 11 (d) 99 (e) 100

2. I have just won 50 of 75 games of "Free Cell" on the computer. The number of games out of the next 30 games that I must win in order to have won 60% overall is:

(a) 10 (b) 13 (c) 15 (d) 20 (e) 25

3. Mark sold two computers each for $\$198$. The first was sold for a profit of 10%, the other for a loss of 10%. Overall, Mark had:

(a) a loss of $8 (b) a loss of $4 (c) no profit or loss
(d) a gain of $4 (e) a gain of $8

4. Among the following numbers the one that is a multiple of 9 is:

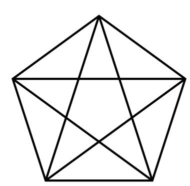
(a) 1233124 (b) 4623747 (c) 37438974 (d) 67346438 (e) 5955006

5. A boy and a girl are sitting on the steps outside of their school. "I'm a boy", said the one with black hair. "I'm a girl", said the one with red hair. If at least one of them is lying, then:

(a) the boy has red hair and the girl has black hair
(b) the boy has red hair and the girl has red hair
(c) the boy has black hair and the girl has black hair
(d) the boy has black hair and the girl has red hair
(e) the hair colours cannot be determined

6. The number of triangles in the figure is:

(a) 15 (b) 20 (c) 25
(d) 30 (e) more than 30



7. If $\frac{a}{b} = \frac{3}{4}$, $\frac{b}{c} = \frac{8}{9}$, and $\frac{c}{d} = \frac{2}{3}$, then the value of $\frac{ad}{b^2}$ is:

(a) $\frac{9}{16}$ (b) $\frac{81}{64}$ (c) $\frac{27}{32}$ (d) $\frac{4}{9}$ (e) $\frac{9}{64}$

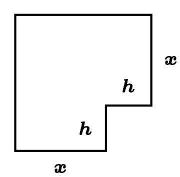
8. In the preliminary round of the British Columbia Colleges High School Mathematics Contest there are five points awarded for each correct answer, no points for each incorrect answer, and one point for each unanswered question to a maximum of five unanswered questions. The number of scores between 0 and 75 that are *not* achievable as a valid score for the fifteen questions on the contest is:

(a) 4 (b) 5 (c) 6 (d) 7 (e) more than 8

9. Old Mr. Jones celebrated his 86^{th} birthday with all of his descendants and their spouses, a total of 86 people including Mr. Jones. This included all of his children each with his or her spouse, all of his grandchildren each with his or her spouse, and all of his great-grandchildren, none of whom are married. He has three times as many grandchildren as children and three times as many great-grandchildren as grandchildren. The number of spouses of his children and grandchildren present is:

(a) 5 (b) 10 (c) 15 (d) 17 (e) 20

10. In the diagram all adjacent sides meet at right angles. If the area of the figure is 60 square units, and if $3 < x < 5$, then:

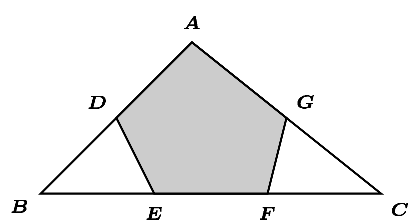


(a) $0.5 < h < 3.5$ (b) $1.5 < h < 5.5$

(c) $2.5 < h < 6.5$ (d) $3.5 < h < 8.5$

(e) $6 < h < 10$

11. In the triangle ABC in the diagram, point D bisects side AB, point G bisects side AC, and the points E and F trisect side BC. If the area of triangle ABC is 84, then the area of the shaded polygon $ADEFG$ is:



(a) $\frac{252}{5}$ (b) $\frac{336}{5}$ (c) 42

(d) 56 (e) 63

12. Antonino has two equal square pieces of cookie dough, each of the same uniform thickness. On one square Antonino makes the largest possible circular cookie and on the other he makes sixteen equal circular cookies. Assuming each circle is inscribed in identical non-overlapping square sections which comprise the whole of the original square piece of cookie dough, the ratio of the volume of the large cookie to the total volume of the sixteen smaller cookies is:

(a) $\sqrt{2} : 1$ (b) $1 : \sqrt{2}$ (c) $2\sqrt{2} : \pi$ (d) $\pi : 2\sqrt{2}$ (e) $1 : 1$

13. Samantha spent a total of $\$420$ for daily expenses during her vacation in Mexico. She noted that if she spent $\$7$ less per day, she could have stayed for another five days. The number of days she actually spent on her vacation is:

(a) 15 (b) 18 (c) 22 (d) 29 (e) 30

14. A solid rectangular block with a 5 cm by 5 cm square base has a height of x cm. If the *surface area* of the block is 120 cm^2, then x equals:

(a) 2 (b) 2.5 (c) 3.5 (d) 4 (e) 4.8

15. The number of real numbers for which the reciprocal of the number is exactly one third of the number is:

(a) 4 (b) infinite (c) 0 (d) 1 (e) 2

Next we turn to the solutions of the Eighteenth W.J. Blundon Contest presented in issue 3 of this year [2002 : 161].

1. (a) At a meeting of 100 people, every person shakes hands with every other person exactly once. How many handshakes are there in total?

Solution by Sang-oh Lee, student, grade 11, Rideau High School, Ottawa, Ontario.

The first person would shake hands with 99 people, the second person would shake hands with 98 people, and so on. Thus, the total number of handshakes is

$$99 + 98 + 97 + \cdots + 2 + 1 + 0 = \frac{99 * 100}{2} = 4950\,.$$

Also solved by Pieta Brown, student, year 10, Diolesan School For Girls, Auckland, New Zealand and Alfian Smu, Indonesia.

(b) How many four digit numbers are divisible by 5?

Solution by Pieta Brown, student, year 10, Diolesan School For Girls, Auckland, New Zealand.

Four digit numbers divisible by 5 have last digits of 0 or 5. There are 9 possible choices for the first digit (zero may not be included) then 10 for the second and third giving $9 \times 10 \times 10 \times 2 = 1800$ such numbers.

Also solved by Sang-oh Lee, student, grade 11, Rideau High School, Ottawa, Ontario and Alfian Smu, Indonesia.

2. Show that $n^2 + 2$ is divisible by 4 for no integer n.

I. *Solution by Sang-oh Lee, student, grade 11, Rideau High School, Ottawa, Ontario.*

If n is odd, then n^2 and $n^2 + 2$ are both odd; thus, there are no odd solutions.

If n is even, let $n = 2k$ for some integer k. Thus, $n^2 = (2k)^2 = 4k^2$ is divisible by 4, so that $n^2 + 2$ is not divisible by 4.

Thus, there are no integers n such that $n^2 + 2$ is divisible by 4.

II. *Solution by Pieta Brown, student, year 10, Diolesan School For Girls, Auckland, New Zealand.*

$n^2 \equiv 0, 1 \pmod 4$ for any integer n. Hence, $n^2 + 2 \equiv 2, 3 \pmod 4$ and can therefore not be divisible by 4.

Also solved by Siwen Sun, student, sec. 3, Collège Saint-Louis, Lachine, Quebec.

3. Prove that the difference of squares of two odd integers is always divisible by 8.

Solution by Pieta Brown, student, year 10, Diolesan School For Girls, Auckland, New Zealand.

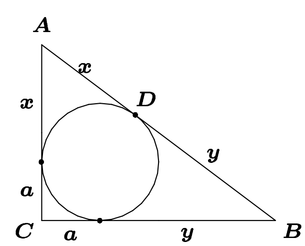
Odd integers are congruent to 1, 3, 5, or $7 \pmod 8$, the squares of all such numbers being $1 \pmod 8$. Therefore, if a and b are two odd numbers then $a^2 - b^2 \equiv 1 - 1 \equiv 0 \pmod 8$, as desired.

Also solved by Sang-oh Lee, student, grade 11, Rideau High School, Ottawa, Ontario, Alfian Smu, Indonesia and Siwen Sun, student, sec. 3, Collège Saint-Louis, Lachine, Quebec.

4. The inscribed circle of a right triangle ABC is tangent to the hypotenuse AB at D. If $AD = x$ and $DB = y$, find the area of the triangle in terms of x and y.

Solution by Siwen Sun, student, sec. 3, Collège Saint-Louis, Lachine, Quebec.

Since the circle touches the other sides at equal distance from the common vertices (see diagram), then the distance between C and the point where the circle is tangent to the other two sides is a. The Pythagorean theorem gives:



$$\begin{aligned}(x+a)^2+(y+a)^2 &= (x+y)^2\,,\\ x^2+2ax+a^2+y^2+2ay+a^2 &= x^2+2xy+y^2\,,\\ 2a(x+y)+2a^2 &= 2xy\,,\\ a(x+y+a) &= xy\,.\end{aligned}$$

Thus, the area of the triangle is

$$\begin{aligned}\frac{(x+a)(y+a)}{2} &= \frac{xy+ax+ay+a^2}{2}\\ &= \frac{xy+a(x+y+a)}{2}\\ &= \frac{xy+xy}{2} = xy\,.\end{aligned}$$

Therefore, the area of the triangle is equal to xy.

Also solved by Pieta Brown, student, year 10, Diolesan School For Girls, Auckland, New Zealand, Sang-oh Lee, student, grade 11, Rideau High School, Ottawa, Ontario, and Alfian Smu, Indonesia.

5. Find all integers x and y such that

$$2^x+3^y = 3^{y+2}-2^{x+1}\,.$$

Solution by Alfian Smu, Indonesia.

$$\begin{aligned}2^x+3^y &= 3^{y+2}-2^{x+1}\,,\\ 2^x+2^{x+1} &= 3^{y+2}-3^y\,,\\ 2^x+2\cdot 2^x &= 3^2\cdot 3^y-3^y\,,\\ 3\cdot 2^x &= 8\cdot 3^y\,,\\ 2^{x-3} &= 3^{y-1}\,.\end{aligned}$$

The only integral powers of 2 and 3 that are equal are $2^0=3^0=1$. Thus, $x-3=0 \Longrightarrow x=3$ and $y-1=0 \Longrightarrow y=1$.

Also solved by Pieta Brown, student, year 10, Diolesan School For Girls, Auckland, New Zealand and Siwen Sun, student, sec. 3, Collège Saint-Louis, Lachine, Quebec.

6. Find the number of points (x,y), with x and y integers, that satisfy the inequality $|x|+|y|<100$.

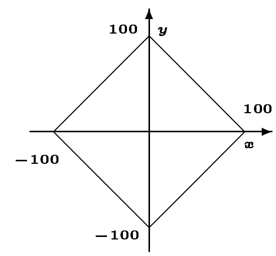
I. *Solution by Sang-oh Lee, student, grade 11, Rideau High School, Ottawa, Ontario.*

When $x=0$, $y\in\{-99,-98,\ldots,98,99\}$, so that there are 199 points. When $x=\pm 1$, $y\in\{-98,\,-97,\,\ldots,\,97,\,98\}$, so that there are 197 solutions, and so on. Thus, there are

$$(197+195+\cdots+3+1)\times 2+199 \;=\; 19801 \text{ points in total.}$$

II. *Solution by Geneviève Lalonde, Massey, Ontario.*

The points are the lattice points inside the square with sides $|x| + |y| = 100$ (see figure). Pick's Theorem states that the area A of a polygon with vertices at lattice points is given by $A = \frac{B}{2} + I - 1$ where B is the number of lattice points on the border and I is the number of lattice points in the interior. Clearly, the square has side length $100\sqrt{2}$ units and there are a total of 400 points on the boundary.

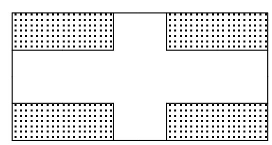


Putting this information into Pick's Theorem yields:

$$20000 = \frac{400}{2} + I - 1\,, \qquad 19801 = I\,.$$

Thus, there are $19\,801$ integer points that satisfy the inequality.

7. A flag consists of a white cross on a red field. The white stripes are of the same width, both vertical and horizontal. The flag measures $48\text{cm} \times 24\text{cm}$. If the area of the white cross equals the area of the red field, what is the width of the cross?



Solution by Alfian Smu, Indonesia.

Let the width of the cross be x cm. Then the area of the cross is $48x + 24x - x^2 = 72x - x^2$. The area of the red field is $48 \times 24 - (72x - x^2) = 1152 - (72x - x^2)$. Therefore, we must have

$$\begin{aligned} 72x - x^2 &= 1152 - (72x - x^2)\,, \\ 2(72x - x^2) &= 1152\,, \\ x^2 - 72x + 576 &= 0\,. \end{aligned}$$

Thus, $x = \frac{72 \pm \sqrt{72^2 - 4 \cdot 1 \cdot 576}}{2} = 36 \pm 12\sqrt{5}$. But since the flag is 24 cm wide, we must have $x < 24$, so that the width of the cross is $(36 - 12\sqrt{5})$ cm.

Also solved by Sang-oh Lee, student, grade 11, Rideau High School, Ottawa, Ontario and Siwen Sun, student, sec. 3, Collège Saint-Louis, Lachine, Quebec.

8. Solve $\frac{x+1}{2+\sqrt{x}} - \frac{1}{2-\sqrt{x}} = 3$.

Solution by Alfian Smu, Indonesia.

$$\begin{aligned} \frac{x+1}{2+\sqrt{x}} - \frac{1}{2-\sqrt{x}} &= 3\,, \\ \frac{(x+1)(2-\sqrt{x}) - (2+\sqrt{x}}{4-x} &= 3\,, \\ 2x - x\sqrt{x} - 2\sqrt{x} &= 12 - 3x\,, \\ -x\sqrt{x} - 2\sqrt{x} &= 12 - 5x\,. \end{aligned}$$

Squaring both sides yields

$$\begin{aligned} x^3 + 4x^2 + 4x &= 144 - 120x + 25x^2\,, \\ x^3 - 21x^2 + 124x - 144 &= 0\,, \\ (x-9)(x^2 - 12x + 16) &= 0\,, \\ x &= 9\,,\ 6 \pm 2\sqrt{5}\,. \end{aligned}$$

Thus, the solutions are $x = 9$, $x = 6 + 2\sqrt{5}$ and $x = 6 - 2\sqrt{5}$.

9. Let $P(x)$ and $Q(x)$ be polynomials with "reversed" coefficients

$$\begin{aligned} P(x) &= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0\,, \\ Q(x) &= a_0x^n + a_1x^{n-1} + \cdots + a_{n-2}x^2 + a_{n-1}x + a_n\,. \end{aligned}$$

where $a_n \neq 0$, $a_0 \neq 0$. Show that the roots of $Q(x)$ are the reciprocals of the roots of $P(x)$.

Solution by Pieta Brown, student, year 10, Diolesan School For Girls, Auckland, New Zealand.

Let q be a root of $Q(x) = 0$. Then

$$\begin{aligned} P\left(\frac{1}{q}\right) &= a_n\left(\frac{1}{q}\right)^n + \cdots + a_0 \\ &= \left(\frac{1}{q}\right)^n (a_n + a_{n-1}q + \cdots + a_0q^n) \;=\; \left(\frac{1}{q}\right)^n Q(q) \;=\; 0\,. \end{aligned}$$

Thus, if q is a root of $Q(x) = 0$, then $\frac{1}{q}$ is a root of $P(x) = 0$.

10. If 1997^{1998} is multiplied out, what is the units digit of the final product?

Solution by Siwen Sun, student, sec. 3, Collège Saint-Louis, Lachine, Quebec.

Note that $7^1 = 7$, $7^2 = 49$, $7^3 = 343$ and $7^4 = 2401$. Thus, the last digit goes 7, 9, 3, and 1. Then it restarts with the unit digit 7, 9, 3, and 1 repeating over and over.

Therefore, we can see that the cycle restarts every multiple of 4. Then $1998 = 499 \times 4 + 2$, so that the units digit of 7^{1998} is the same as the units digit of 7^2, which is 9.

Also solved by Pieta Brown, student, year 10, Diolesan School For Girls, Auckland, New Zealand, Sang-oh Lee, student, grade 11, Rideau High School, Ottawa, Ontario, and Alfian Smu, Indonesia.

This month's prize of a copy of **MATHEMATICAL MAYHEM Vol. 3** goes to Pieta Brown. Congratulations Pieta, and thanks to all of our solvers.

That ends another year of the Skoliad Corner (its first within the Mayhem area). Continue to send us your problems and solutions. It looks like for the next year, at least, we will be able to provide past volumes of Mayhem as prizes to some of our problem solvers. Happy solving!

CANADIAN MATHEMATICS COMPETITIONS

Every year a number of local and national contests are written by Canadian students. Here is a list of upcoming contest dates and information. In the tables below we have indicated two types: **written** requires written solutions, and **MC** is multiple choice. In all competitions, the grade level indicates the maximum grade level of competitors (lower grades are welcome).

1. **CANADIAN MATHEMATICAL SOCIETY**.

   The Canadian Mathematical Society co-sponsors a number of mathematics competitions, including all others mentioned on this page. The society also runs the Canadian Mathematical Olympiad (CMO), which is used to determine Canada's team at the International Mathematical Olympiad (IMO). Some students are also chosen to write the Asian Pacific Mathematics Olympiad (APMO).

Contest	**Grade**	**Type**	**Date**
CMO	12	Written	Wed. Mar. 26, 2003
APMO	12	Written	Mon. Mar. 10, 2003
IMO	12	Written	July 7–19, 2003 (Japan)

   For more information and sample contests you can visit the competition pages of the CMS website `www.cms.math.ca/Competitions`.

2. **The Canadian Mathematics Competitions**.

The Canadian Mathematics Competition is an activity of the Centre for Education in Mathematics and Computing, Faculty of Mathematics, University of Waterloo. It runs a number of national mathematics competitions, like the Canadian Open Mathematics Challenge (COMC) with the CMS, and the Canadian Computing Contest (CCC).

Contest	Grade	Type	Date
Gauss	8	MC	Wed. May 14, 2003
Pascal	9	MC	Wed. Feb. 19, 2003
Fryer	9	Written	Wed. Apr. 16, 2003
Cayley	10	MC	Wed. Feb. 19, 2003
Galois	10	Written	Wed. Apr. 16, 2003
Fermat	11	MC	Wed. Feb. 19, 2003
Hypatia	11	Written	Wed. Apr. 16, 2003
Euclid	12	Written	Tue. Apr. 15, 2003
CCC	13	Written	Tue. Feb. 25, 2003
COMC	13	Written	Wed. Nov. 27, 2002

For more information or copies of each of these contests, consult their website `www.cemc.uwaterloo.ca`.

3. **BRITISH COLUMBIA COLLEGES**.

British Columbia Colleges High School Mathematics Contest is an activity sponsored by nine colleges in British Columbia. It is written in two stages: the preliminary round is marked at the school by the student's teacher, and the top students in each school go to the local participating college to write the final round.

Contest	Grade	Type	Date
Junior Preliminary	10	MC	Wed. Mar. 5, 2003
Junior Final	10	MC and Written	Fri. May 2, 2003
Senior Preliminary	12	MC	Wed. Mar. 5, 2003
Senior Final	12	MC and Written	Fri. May 2, 2003

For more information and sample contests you can visit their website `www.cariboo.bc.ca/psd/math-sta/mathcompetition/mathcomp.html`.

4. **MANITOBA MATHEMATICAL CONTEST**.

The Manitoba Mathematical Contest (MMC) is an activity sponsored by the Winnipeg Actuaries Club, the Manitoba Association of Mathematics Teachers and the University of Manitoba.

Contest	Grade	Type	Date
MMC	12	Written	Wed. Feb. 19, 2003

For more information, contact Diane Dowling `ddowling8@shaw.ca`.

PROBLEMS

Faire parvenir les propositions de problèmes et les solutions à Jim Totten, Département de mathématiques et de statistique, University College of the Cariboo, Kamloops, C.-B. V2C 5N3. Les propositions de problèmes doivent être accompagnées d'une solution ainsi que de références et d'autres indications qui pourraient être utiles à la rédaction. Si vous envoyez une proposition sans solution, vous devez justifier une solution probable en fournissant suffisamment d'information. Un numéro suivi d'une astérisque (⋆) indique que le problème a été proposé sans solution.

Nous sollicitons en particulier des problèmes originaux. Cependant, d'autres problèmes intéressants pourraient être acceptables s'ils ne sont pas trop connus et si leur provenance est précisée. Normalement, si l'auteur d'un problème est connu, il faut demander sa permission avant de proposer un de ses problèmes.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

NOTE: Problems **2786** and **2787** in the last issue should be starred.

2769. Correction. *Proposé par Aram Tangboondouangjit, étudiant, University of Maryland, College Park, Maryland USA.*

Dans un triangle ABC, on suppose que $\cos B - \cos C = \cos A - \cos B \geq 0$. Montrer que

$$\left(b^2+c^2\right)\cos A - \left(a^2+b^2\right)\cos C \;\leq\; \left(c^2-a^2\right)\sec B\,.$$

. . . .

In $\triangle ABC$, suppose that $\cos B - \cos C = \cos A - \cos B \geq 0$. Prove that

$$\left(b^2+c^2\right)\cos A - \left(a^2+b^2\right)\cos C \;\leq\; \left(c^2-a^2\right)\sec B\,.$$

2789. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit A, $B \in M_n(\mathbb{C})$ tels que $A + B = I_n$ et que, pour un certain $k \in \mathbb{N}^*$, $A^{2k+1} = A^{2k}$. Montrer que $I_n + A^kB$ est inversible, et trouver son inverse.

. . . .

Let A, $B \in M_n(\mathbb{C})$ such that $A + B = I_n$ and, for some $k \in \mathbb{N}^*$, $A^{2k+1} = A^{2k}$. Prove that $I_n + A^kB$ is invertible, and find its inverse.

2790. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Montrer que $$\sum_{1\le i\le j\le n} \sin^2\left(\frac{(j-i)\pi}{n}\right) \;=\; \frac{n^2}{4}\,.$$

. . . .

Prove that $$\sum_{1\le i\le j\le n} \sin^2\left(\frac{(j-i)\pi}{n}\right) \;=\; \frac{n^2}{4}\,.$$

2791. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit $f:[0,1]\to(0,\infty)$ une fonction continue. Montrer que s'il existe $\alpha>0$ tel que, pour $n\in\mathbb{N}$,

$$\int_0^1 x^\alpha\,(f(x))^n\,dx \;\ge\; \frac{1}{(n+1)\alpha+1} \;\ge\; \int_0^1 (f(x))^{n+1}\,dx\,,$$

alors α est unique.

. . . .

Suppose that $f:[0,1]\to(0,\infty)$ is a continuous function. Prove that if there exists $\alpha>0$ such that, for $n\in\mathbb{N}$,

$$\int_0^1 x^\alpha\,(f(x))^n\,dx \;\ge\; \frac{1}{(n+1)\alpha+1} \;\ge\; \int_0^1 (f(x))^{n+1}\,dx\,,$$

then α is unique.

2792. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit $A_k\in M_n(\mathbb{R})$ $(k=1,2,\ldots,m\ge 2)$ tel que

$$\sum_{1\le i\le j\le m} (A_iA_j+A_jA_i) \;=\; 0_n\,.$$

Montrer que $$\det\left(\sum_{k=1}^m (I_n+A_k)^2-(m-2)I_n\right) \;\ge\; 0\,.$$

. . . .

Let $A_k\in M_n(\mathbb{R})$ $(k=1,2,\ldots,m\ge 2)$ for which

$$\sum_{1\le i\le j\le m} (A_iA_j+A_jA_i) \;=\; 0_n\,.$$

Prove that $$\det\left(\sum_{k=1}^m (I_n+A_k)^2-(m-2)I_n\right) \;\ge\; 0\,.$$

2793. *Proposé par Paul Deiermann, Southeast Missouri State University, Cape Girardeau, Missouri, USA.*

Montrer que l'aire de l'image de la portion du disque unité comprise dans le premier quadrant par l'application $\zeta = \cosh^{-1}(z)$ est une constante bien connue.

. . . .

Show that the area of the image of the portion of the unit disc which lies in the first quadrant under the mapping $\zeta = \cosh^{-1}(z)$ is a well-known constant.

2794. *Proposé par Mihály Bencze, Brasov, Roumanie.*

On suppose que $z_k \in \mathbb{C}^*$ $(k = 1, 2, \ldots, n)$ sont tels que

$$\begin{aligned}|z_1 + z_2 + \cdots + z_n| + |z_2 + z_3 + \cdots + z_n| + \cdots + |z_{n-1} + z_n| + |z_n| \\ = \quad |z_1 + 2z_2 + \cdots + nz_n| \;.\end{aligned}$$

Montrer que les z_k sont colinéaires.

. . . .

Suppose that $z_k \in \mathbb{C}^*$ $(k = 1, 2, \ldots, n)$ such that

$$\begin{aligned}|z_1 + z_2 + \cdots + z_n| + |z_2 + z_3 + \cdots + z_n| + \cdots + |z_{n-1} + z_n| + |z_n| \\ = \quad |z_1 + 2z_2 + \cdots + nz_n| \;.\end{aligned}$$

Prove that the z_k are collinear.

2795. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Un polygone convexe de côtés a_1, a_2, ..., a_n, est inscrit dans un cercle de rayon R. Montrer que

$$\sum_{k=1}^{n} \sqrt{4R^2 - a_k^2} \;\leq\; 2nR\sin\left(\frac{(n-2)\pi}{n}\right) \;.$$

. . . .

A convex polygon with sides a_1, a_2, ..., a_n, is inscribed in a circle of radius R. Prove that

$$\sum_{k=1}^{n} \sqrt{4R^2 - a_k^2} \;\leq\; 2nR\sin\left(\frac{(n-2)\pi}{n}\right) \;.$$

2796★. *Proposé par Fernando Castro G., Matirín Estado Monagas, Vénézuéla.*

Soit $\{p_n\}$ la suite des nombres premiers. Montrer que, pour $n \geq 2$, l'ensemble $I = \{1, 2, \ldots, n\}$ peut être partitionné en deux ensembles A et B, avec $A \cup B = I$, de sorte que,

$$1 \leq \frac{\prod_{i \in A} p_i}{\prod_{j \in B} p_j} \leq 2.$$

. . . .

Let $\{p_n\}$ be the sequence of prime numbers. Prove that, for each $n \geq 2$, the set $I = \{1, 2, \ldots, n\}$ can be partitioned into two sets A and B, where $A \cup B = I$, in such a way that,

$$1 \leq \frac{\prod_{i \in A} p_i}{\prod_{j \in B} p_j} \leq 2.$$

2797. *Proposé par Juan-Bosco Romero Márquez, Université de Valladolid, Valladolid, Espagne.*

Dans un triangle ABC, supposons que AD soit une hauteur. Supposons de plus que les perpendiculaires issues de D coupent les côtés AB et AC en E et F, respectivement. Supposons enfin que G et H sont des points de AB et AC, respectivement, tels que $DG \| AC$ et $DH \| AB$. Montrer que

(a) EF et GH se coupent en A^* sur BC.

En définissant B^* et C^* de manière analogue, montrer que

(b) A^*, B^* et C^* sont colinéaires.

. . . .

In $\triangle ABC$, suppose that AD is an altitude. Suppose that perpendiculars from D meet the sides AB and AC at E and F, respectively. Suppose that G and H are points of AB and AC, respectively, such that $DG \| AC$ and $DH \| AB$. Prove that

(a) EF and GH intersect at A^* on BC.

Defining B^* and C^* similarly, prove that

(b) A^*, B^* and C^* are collinear.

2798★. *Proposé par Šefket Arslanagić, Université de Sarajevo, Sarajevo, Bosnie et Herzégovine.*

Prouver ou réfuter l'inégalité $\displaystyle\sum_{j=1}^{n} \frac{1}{1 - \frac{P}{x_j}} \leq \frac{n}{1 - \left(\frac{1}{n}\right)^{n-1}}$, où $\displaystyle\sum_{j=1}^{n} x_j = 1$, $x_j \geq 0$ $(j = 1, 2, \ldots, n)$, et $P = \displaystyle\prod_{j=1}^{n} x_j$.

Prove or disprove the inequality $\sum\limits_{j=1}^{n} \frac{1}{1-\frac{P}{x_j}} \leq \frac{n}{1-\left(\frac{1}{n}\right)^{n-1}}$, where $\sum\limits_{j=1}^{n} x_j = 1$, $x_j \geq 0$ ($j = 1, 2, \ldots, n$), and $P = \prod\limits_{j=1}^{n} x_j$.

2799★. *Proposé par Šefket Arslanagić, Université de Sarajevo, Sarajevo, Bosnie et Herzégovine.*

Prouver ou réfuter l'inégalité

$$\sum_{\substack{i,\, j \,\in\, \{1,\, 2,\, \ldots,\, n\} \\ 1 \leq i \leq j \leq n}} \frac{1}{1-x_i x_j} \leq \binom{n}{2} \frac{1}{1-\frac{1}{n^2}},$$

où $\sum\limits_{j=1}^{n} x_j = 1$, $x_j \geq 0$.

. . . .

Prove or disprove the inequality

$$\sum_{\substack{i,\, j \,\in\, \{1,\, 2,\, \ldots,\, n\} \\ 1 \leq i \leq j \leq n}} \frac{1}{1-x_i x_j} \leq \binom{n}{2} \frac{1}{1-\frac{1}{n^2}},$$

where $\sum\limits_{j=1}^{n} x_j = 1$, $x_j \geq 0$.

2800. *Proposé par Mihály Bencze, Brasov, Roumanie.*

On suppose que les $z_k \in \mathbb{C}^*$ ($k = 1, 2, \ldots, n$) sont tels que $|z_k| = |M|$ (constant) et

$$\left|\sum_{k=1}^{n} z_k\right| = \left|\sum_{k=1}^{n} z_k - z_1\right| = \left|\sum_{k=1}^{n} z_k - z_2\right| = \cdots = \left|\sum_{k=1}^{n} z_k - z_n\right|$$

($k \in \{1, 2, \ldots, n\}$). Montrer que $\left(\sum\limits_{k=1}^{n} z_k\right)\left(\sum\limits_{k=1}^{n} \frac{1}{z_k}\right) = \frac{n}{2}$.

. . . .

Suppose that $z_k \in \mathbb{C}^*$ ($k = 1, 2, \ldots, n$) such that $|z_k| = |M|$ (constant) and

$$\left|\sum_{k=1}^{n} z_k\right| = \left|\sum_{k=1}^{n} z_k - z_1\right| = \left|\sum_{k=1}^{n} z_k - z_2\right| = \cdots = \left|\sum_{k=1}^{n} z_k - z_n\right|$$

($k \in \{1, 2, \ldots, n\}$). Prove that $\left(\sum\limits_{k=1}^{n} z_k\right)\left(\sum\limits_{k=1}^{n} \frac{1}{z_k}\right) = \frac{n}{2}$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

We apologise for omitting the names of MICHEL BATAILLE, Rouen, France and LI ZHOU, Polk Community College, Winter Haven, FL, USA, from the lists of solvers of 2665 and 2675, respectively.

2677. [2001 : 460] *Proposed by P. Ivady, Budapest, Hungary*

For $0 < x < \frac{\pi}{2}$, show that $\dfrac{\pi^2 - x^2}{\pi^2 + x^2} \;<\; \cos\left(\dfrac{x}{\sqrt{3}}\right)$.

Solution by Kee-Wai Lau, Hong Kong, China.

We prove, for $0 < x < \frac{\pi}{2}$, the stronger inequality

$$\frac{\pi^2 - x^2}{\pi^2 + x^2} \;<\; \cos(ax)\,, \tag{1}$$

where

$$a \;=\; \frac{4}{\pi}\tan^{-1}\left(\frac{1}{2}\right) \;>\; \frac{1}{\sqrt{3}}\,.$$

The substitution $x = \pi\tan\theta$ transforms the inequality (1) to

$$\frac{1 - \tan^2\theta}{1 + \tan^2\theta} \;<\; \cos(a\pi\tan\theta)$$

or,

$$\cos 2\theta \;<\; \cos(a\pi\tan\theta) \quad \text{for} \quad 0 < \theta < \tan^{-1}\left(\frac{1}{2}\right)\,.$$

Since $0 \;<\; 2\theta \;<\; \frac{\pi}{2}$ and $0 \;<\; a\pi\tan\theta \;<\; \frac{\pi}{2}$, it suffices to show that $2\theta > a\pi\tan\theta$ or

$$\frac{\tan\theta}{\theta} \;<\; \frac{2}{a\pi} \quad \text{for} \quad 0 < \theta < \tan^{-1}\left(\frac{1}{2}\right)\,.$$

This follows from the fact that the function $\dfrac{\tan\theta}{\theta}$ is strictly increasing for $0 < \theta < \tan^{-1}\left(\frac{1}{2}\right)$, which completes the proof.

Also solved by MICHEL BATAILLE, Rouen, France; PAUL BRACKEN, CRM, Université de Montréal, Montréal, Québec; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There were also three incorrect solutions submitted.

Several solvers have submitted similar improvements of Uvady's inequality. They have shown that

$$\frac{\pi^2 - x^2}{\pi^2 + x^2} \;<\; \cos(ax)\,,$$

for $\alpha > \frac{1}{\sqrt{3}} \approx 0.57735$*, as follows: Lau's constant is* $\alpha = \frac{4}{\pi}\tan^{-1}\left(\frac{1}{2}\right) \approx 0.59033$*; Seiffert has proved that the constant* $\alpha = \dfrac{8}{\pi\sqrt{10+4\sqrt{5}}} \approx 0.58506$ *works. Janous has proved that the inequality holds for* $\alpha = \frac{2}{\pi} \approx 0.63662$ *and posed the natural problem: Determine the largest real number* $\alpha > 0$ *such that the inequality*

$$\frac{\pi^2 - x^2}{\pi^2 + x^2} \;<\; \cos(\alpha x)$$

is valid for all x *in the interval* $\left(0, \frac{\pi}{2}\right)$.

2678. [2001 : 460] *Proposed by David Chow, student, Clifton College, Bristol, England.*

Prove that $\triangle ABC$ is isosceles if and only if

$$a\left(a^2 - b^2\right)\sin B + b\left(b^2 - c^2\right)\sin C + c\left(c^2 - a^2\right)\sin A \;=\; 0\,.$$

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Since $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$, the given condition is equivalent to

$$ab(a^2 - b^2) + bc(b^2 - c^2) + ca(c^2 - a^2) \;=\; 0\,,$$

or $(a+b+c)(a-b)(b-c)(c-a) \;=\; 0$, from which the claim follows immediately.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; RALF BANISCH, Landesschule Pforta, Schulpforte, Germany; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; MIHÁLY BENCZE, Brasov, Romania; PIERRE BORNSZTEIN, Pontoise, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; SCOTT H. BROWN, Auburn University at Montgomery, Montgomery, AL, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; ŽELJKO HANJŠ, University of Zagreb, Zagreb, Croatia; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PAUL JEFFERYS, student, Berkhamsted Collegiate School, UK; D. KIPP JOHNSON, Beaverton, OR, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; KEE-WAI LAU, Hong Kong; HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; ROBERT McGREGOR, Auburn, AL, USA; VEDULA N. MURTY, Dover, PA, USA; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; JOEL SCHLOSBERG, student, New York University, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; ANDREI SIMION, student, Cornell University, Ithaca, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Virtually all the submitted solutions are the same as, or are similar to the one featured above. Klamkin generalized the problem slightly by showing that $\triangle ABC$ *is isosceles if and only if*

$$a(a^n - b^n)\sin B + b(b^n - c^n)\sin C + c(c^n - a^n)\sin A \;=\; 0\,,$$

where n *is any positive integer.*

2679. [2001 : 460] *Proposed by Charles R. Diminnie, Angelo State University, San Angelo, TX, USA.*

Find all solutions of $\sin x + \sin 2x = \sin 4x$.

Editor's comment.

The previous Editor-in-Chief once challenged me to set a very simple problem once in a while. This was it. There were a few solvers who gave solutions only in a one period range and did not mention the periodicity. Rather than class them as incomplete, we are crediting them with a complete solution.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Since $\sin 4x - \sin 2x = 2\cos 3x \sin x$, the equation becomes

$$(2\cos 3x - 1)\sin x = 0.$$

Thus, $x = n\pi$ and $\frac{(6n \pm 1)\pi}{9}$, where n is an integer.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; BRIAN BEASLEY, Presbyterian College, Clinton, SC, USA; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; MIHÁLY BENCZE, Brasov, Romania; STEFFEN BIALLAS, Albert-Einstein-Gymnasium, Magdeburg, Germany; ROBERT BILINSKI, Outremont, Quebec; PIERRE BORNSZTEIN, Pontoise, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; JASON DAVIS, student, Angelo State University, San Angelo, TX, USA; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; DOUGLASS L. GRANT, University College of Cape Breton, Sydney, Nova Scotia; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; ZELJKO HANJS, University of Zagreb, Zagreb, Croatia; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PAUL JEFFERYS, student, Berkhamstead Collegiate School, UK; D. KIPP JOHNSON, Beaverton, OR, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; ROBERT McGREGOR, Auburn, AL, USA; NORVALD MIDTUN, Royal Norwegian Naval Academy, Bergen, Norway; VEDULA N. MURTY, Dover, PA, USA; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; HARRY SEDINGER, St. Bonaventure University, St. Bonaventure, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PATRICK STAHL, student, Walt Whitman High School, Bethesda, MD, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; JAMES VALLES Jr., Angelo State University, San Angelo, TX, USA; Mª JESÚS VILLAR RUBIO, Santander, Spain; STEFFEN WEBER, Georg-Cantor-Gymnasium, Halle, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

2683. [2001 : 462] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Find the value of $\lim\limits_{n\to\infty} \left(\frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n+1}{2k+1} \binom{n+1}{2k+1} \right)$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

First, note that

$$\begin{aligned} f(n) &:= \sum_{k=0}^{\lfloor n/2\rfloor} \frac{n+1}{2k+1}\binom{n+1}{2k+1} \\ &> \sum_{k=0}^{\lfloor n/2\rfloor} \frac{n+2}{2k+2}\binom{n+1}{2k+1} \;=\; \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n+2}{2k+2} \;=\; 2^{n+1}-1\,. \end{aligned}$$

On the other hand, for any $\epsilon > 0$, there exists K such that $\dfrac{2k+2}{2k+1} < 1+\epsilon$ for all $k > K$. Thus,

$$\begin{aligned} f(n) &= \frac{n+1}{n+2}\sum_{k=0}^{\lfloor n/2\rfloor} \frac{2k+2}{2k+1}\binom{n+2}{2k+2} \\ &< \sum_{k=0}^{\lfloor n/2\rfloor} \frac{2k+2}{2k+1}\binom{n+2}{2k+2} \\ &< \sum_{k=0}^{K} \frac{2k+2}{2k+1}\binom{n+2}{2k+2} + \sum_{k=K+1}^{\lfloor n/2\rfloor}(1+\epsilon)\binom{n+2}{2k+2} \\ &< 2\sum_{k=0}^{K}\binom{n+2}{2k+2} + (1+\epsilon)2^{n+1}\,. \end{aligned}$$

Note that, for fixed K, we have $\displaystyle\lim_{n\to\infty}\frac{1}{2^n}\sum_{k=0}^{K}\binom{n+2}{2k+2} \;=\; 0$. Hence, $\displaystyle\lim_{n\to\infty}\frac{1}{2^n}f(n) \;=\; 2$.

Also solved by CHARLES R. DIMINNIE and KARL HAVLAK, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; and the proposer. There were three incorrect submissions.

2684. [2001 : 461] *Proposed by Mohammed Aassila, Strasbourg, France.*

Does there exist an infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that for every rational number p, the n^{th} derivative $f^{(n)}(p)$ is a rational number whenever n is even, and is an irrational number whenever n is odd?

Editor's comment.

No solutions were submitted except that of the proposer. However, Michel Bataille, Rouen, France pointed out that the problem is known; it is number 22 of the problem section "A vos stylos" in the journal *L'Ouvert*. A solution appeared in *L'Ouvert*, 70 (1993), p. 48. This (anonymous) solution proves the existence of such a function by constructing it as the sum of a series of infinitely differentiable functions f_i which satisfies:

- for $n \leq j$, $\left|f_j^{(n)}\right| \leq 2^{-j}$,
- for $i < j$, f_j vanishes in a neighbourhood of r_i,
- $\left(\sum_{i<j} f_i^{(n)}\right)(r_j)$ is rational for n even and irrational for n odd.

[(r_j) denotes an enumeration of the rational numbers; the sequence (f_j) is constructed by induction.]

Solved by the proposer.

2685. [2001 : 461] *Proposed by Mohammed Aassila, Strasbourg, France.*

(a) Let $\mathcal{C}$ be a bounded, closed and convex domain in the plane. Construct a parallelogram $\mathcal{P}$ contained in $\mathcal{C}$ such that $\mathcal{A}(\mathcal{P}) \geq \frac{1}{2}\mathcal{A}(\mathcal{C})$, where $\mathcal{A}$ denotes area.

(b)★ Prove that if, further, $\mathcal{C}$ is centrally symmetric, then one can construct a parallelogram $\mathcal{P}$ such that $A(\mathcal{P}) \geq \frac{2}{\pi}A(\mathcal{C})$.

Editor's remark. Michel Bataille informs us that this problem is known, and occurs as number 19 of the problem section "A vos stylos" in the journal *L'Ouvert*, 67 (1992) pp. 38–40. In the solution of (a), Pierre Renfer even adds the constraint of a given direction for a side of P. Further, (b) is indicated as a conjecture of Gustave Choquet.

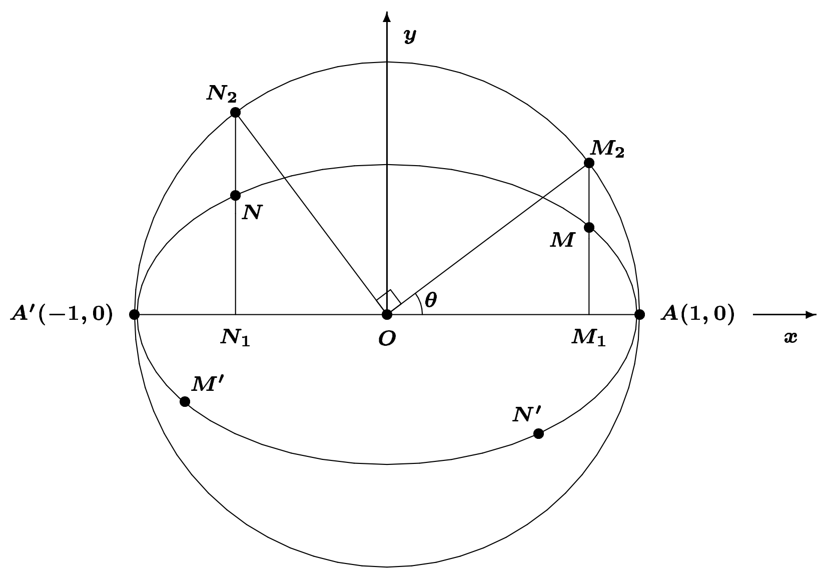
Judging by the small number of submissions, this problem is not well known.

Solution by G. Tsintsifas, Thessaloniki, Greece.

(a) It is well known that there exists a quadrilateral $ABCD$ of minimum area circumscribed to $\mathcal{C}$. The mid-points M, N, L, T, of the sides AB, BC, CD, DA, respectively, are on $\mathcal{C}$. See [1] or [2].

It is elementary that $\mathcal{A}(ABCD) = 2\mathcal{A}(MNLT)$, and that $MNLT$ is a parallelogram. But $\mathcal{A}(ABCD) \geq \mathcal{A}(\mathcal{C})$, giving $\mathcal{A}(MNLT) \geq \frac{1}{2}\mathcal{A}(\mathcal{C})$.

(b) Let AA' be a diameter of $\mathcal{C}$. We choose a Cartesian orthogonal system of coordinates such that A is $(1,0)$ and A' is $(-1,0)$. We consider the circle $(0,1)$ (centre 0, radius 1). Let $M \in \mathcal{C}$ and the perpendicular MM_1 to AA' intersects the positive semicircle $(0,1)$ at the point M_2, etc. (see figure).



We set $f(\theta) = \dfrac{MM_1}{M_1M_2}$ and we parameterize:

$$\left.\begin{array}{lcl} x_M &=& x(\theta) = \cos\theta \\ y_M &=& y(\theta) = f(\theta)\cdot\sin\theta \end{array}\right\} \tag{1}$$

Also, we have

$$\left.\begin{array}{lcl} x_N &=& x\left(\theta+\frac{\pi}{2}\right) = \cos\left(\theta+\frac{\pi}{2}\right) \\ y_N &=& y\left(\theta+\frac{\pi}{2}\right) = f\left(\theta+\frac{\pi}{2}\right)\cdot\sin\left(\theta+\frac{\pi}{2}\right) \end{array}\right\} \tag{2}$$

The function $f(\theta)$ is continuous, positive and periodic so that $f(0) = 0$, $f(\theta+\pi) = f(\theta)$ because of the central symmetry of $\mathcal{C}$.

We consider the points M' and N' symmetrically opposite to M and N with respect to O. We have

$$\mathcal{A}(MNM'N') = 4\mathcal{A}(MON)\,, \qquad 2\mathcal{A}(MON) = |x_My_N - x_Ny_M|\,.$$

Substituting from (1) and (2) gives

$$2\mathcal{A}(MON) = f\left(\theta+\frac{\pi}{2}\right)\sin^2\left(\theta+\frac{\pi}{2}\right) + f(\theta)\sin^2\theta\,, \tag{3}$$

or

$$\mathcal{A}(MNM'N') = -2\left(y(\theta)\dot{x}(\theta) + y\left(\theta+\frac{\pi}{2}\right)\dot{x}\left(\theta+\frac{\pi}{2}\right)\right)\,, \tag{4}$$

where $\dot{x}(\phi) = \dfrac{dx(\phi)}{d\phi}$.

Using these, we have

$$\mathcal{A}(\mathcal{C}) \;=\; \left|\int_0^{\pi} y\dot{x}\,d\theta\right| + \left|\int_{\pi}^{2\pi} y\dot{x}\,d\theta\right| \,.$$

Denoting by $\mathcal{P}$ the parallelogram of maximum area inscribed in $\mathcal{C}$, we have, from (4),

$$\mathcal{A}(\mathcal{P})\int_0^{2\pi} d\theta \;\ge\; 4\left|\int_0^{\pi} y\dot{x}\,d\theta\right| + 4\left|\int_{\pi}^{2\pi} y\dot{x}\,d\theta\right| \,,$$

or

$$\mathcal{A}(\mathcal{P}) \;\ge\; \frac{4}{2\pi}\mathcal{A}(\mathcal{C}) \,.$$

The equality follows from (3). The derivative of the second part must be zero, and we easily find that $f(\theta) = \text{constant} = k$.

Thus, the parametric equations of $\mathcal{C}$ are: $x = \cos\theta$, $y = k\sin\theta$; that is, $\mathcal{C}$ must be an ellipse.

References

1. Mahlon M. Day, *Polygons Circumscribed about Closed Convex Curves*, Trans. Amer. Math. Soc., 62 (1975) 315–319.

2. G.D. Chakevian and L.H. Lange, *Geometric Extremum Problems*, Mathematics Magazine, 44 (1971) 57.

Part (a) *was also solved by DENIS LIEUTIER and PIERRE BORNSZTEIN, Pontoise, France; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. Woo also proved the existence of the parallelogram for part* (b).

Tsintsifas comments that this opens questions:

1. *What about the minimum area of the parallelogram circumscribed about* $\mathcal{C}$*?*
2. *The same problems with perimeters.*

He submitted solutions, but we will let our readers digest his proof, and see if they can come up with nice solutions as well.

2686★. [2001 : 461] *Proposed by Mohammed Aassila, Strasbourg, France.*

Let $\mathcal{C}$ be a bounded, closed and convex domain in space. Construct a parallelepiped $\mathcal{P}$ contained in $\mathcal{C}$ such that $\mathcal{V}(\mathcal{P}) \ge \frac{4}{9}\mathcal{V}(\mathcal{C})$, where $\mathcal{V}$ denotes volume.

Editor's remark.

We have received one attempt of a solution to this problem: One solver has shown that if $\mathcal{C}$ is a convex body in $\mathbb{E}^3$, then there exists an inscribed parallelepiped $\mathcal{P}$ in $\mathcal{C}$ such that

$$\mathcal{V}(\mathcal{P}) \;\ge\; \frac{2}{9}\mathcal{V}(\mathcal{C}) \,.$$

However, the solution does not address the question whether the statement of the original problem is true or false. The editors would be pleased to see some new insights on this question.

2687. [2001 : 461] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Determine the locus of points (x, y) (in the real plane) for which the equation in z, $x z^3 + y z^2 + 1 = 0$, has two complex roots of modulus twice the modulus of its real root.

Solution by Henry Liu, student, University of Memphis, Memphis, TN, USA.

Let α and $\bar{\alpha}$ be the complex roots, and β be the real root. We have $|\alpha| = |\bar{\alpha}| = 2|\beta|$. Also [since x must be non-zero],

$$(z-\alpha)(z-\bar{\alpha})(z-\beta) \;=\; z^3 + \frac{y}{x}z^2 + \frac{1}{x}\,.$$

This yields the simultaneous equations

$$\frac{y}{x} \;=\; -(\alpha+\bar{\alpha}+\beta) \qquad (1)$$

$$0 \;=\; \alpha\bar{\alpha} + \alpha\beta + \bar{\alpha}\beta \qquad (2)$$

$$\frac{1}{x} \;=\; -\alpha\bar{\alpha}\beta \qquad (3)$$

Equation (2) gives

$$\alpha + \bar{\alpha} \;=\; -\frac{|\alpha|^2}{\beta} \;=\; -\frac{4|\beta|^2}{\beta} \;=\; -4\beta\,.$$

[*Editor's comment*. Note that when α is written $\alpha = \pm 2\beta e^{i\theta}$, equation (2) further implies that $\cos\theta = \pm 1$, so that α and $\bar{\alpha}$ must be real. Most readers agreed that the wording of the problem implies that there should be two non-real roots, in which case the locus we seek is empty. However, if we interpret (in Zhou's words) "more generously to allow three real roots," we can continue.]

Substituting into (1) gives

$$\frac{y}{x} \;=\; 3\beta\,. \qquad (4)$$

Equation (3) gives

$$\frac{1}{x} \;=\; -|\alpha|^2\beta \;=\; -4\beta^3\,. \qquad (5)$$

Equations (4) and (5) together give parametric equations for the locus of (x, y) :

$$x \;=\; -\frac{1}{4\beta^3}\,, \quad y \;=\; -\frac{3}{4\beta^2}\,.$$

The locus is therefore the semicubical parabola

$$4y^3 + 27x^2 \;=\; 0$$

without the cusp point at the origin.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GERRY LEVERSHA, St. Paul's School, London, England; JOEL SCHLOSBERG, student, New York University, NY, USA; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; Mª JESÚS VILLAR RUBIO, Santander, Spain; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2688. [2001 : 461] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Suppose that P is an arbitrary point inside cyclic quadrilateral $ABCD$. Let K, L, M and N be the projections of P onto AB, BC, CD and DA respectively.

Show that $AB \cdot PM + CD \cdot PK = BC \cdot PN + DA \cdot PL$.

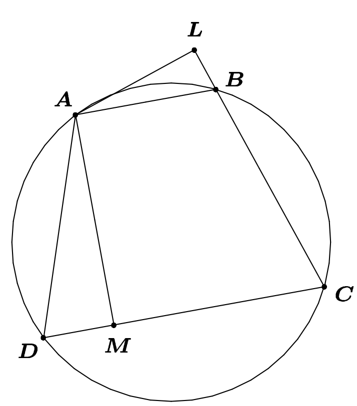
I. *Solution by David Loeffler, student, Trinity College, Cambridge, UK.*
For a fixed cyclic quadrilateral $ABCD$ and a variable point P, let

$$f(P) \;=\; AB \cdot PM - BC \cdot PN + CD \cdot PK - DA \cdot PL\,.$$

Then we see that $f(P)$ is a linear function of P, which may be extended to the entire plane if we regard the lengths PL, etc, as being signed.

Now, any linear function of two variables which is zero at three non-collinear points must be zero everywhere. Therefore, we shall show that $f(A) = 0$; it follows by cyclic permutations that $f(B)$, $f(C)$, and $f(D)$ are all zero, and hence, that $f(P) = 0$ for all points P.

If $P = A$, then PK and PN are both zero. We obtain this diagram:



We clearly have $\angle ADM = \angle ABL = \theta$, say, since $ABCD$ is cyclic; hence, $AM = AD\sin\theta$ and $AL = AB\sin\theta$. Thus,

$$f(A) \;=\; AB \cdot AD\sin\theta - AD \cdot AB\sin\theta \;=\; 0.$$

Therefore, $f(A)$, and similarly $f(B)$, $f(C)$, and $f(D)$ are all zero, whence, $f(P) = 0$ for all P as claimed.

II. *Solution by Henry Liu, student, University of Memphis, Memphis, TN, USA.*

There are four things that can happen regarding the location of the points K, L, M, and N:

1. K, L, M, and N lie on the sides AB, BC, CD, and DA, respectively.
2. Without loss of generality, K lies on the extension of AB, and L, M, and N lie on the sides BC, CD, and DA, respectively.
3. Without loss of generality, K and L lie on the extensions of AB and BC, and M and N lie on the sides CD and DA, respectively.
4. Without loss of generality, K and M lie on the extensions of AB and DC, and L and N lie on the sides BC and DA, respectively.

See Figure 1.

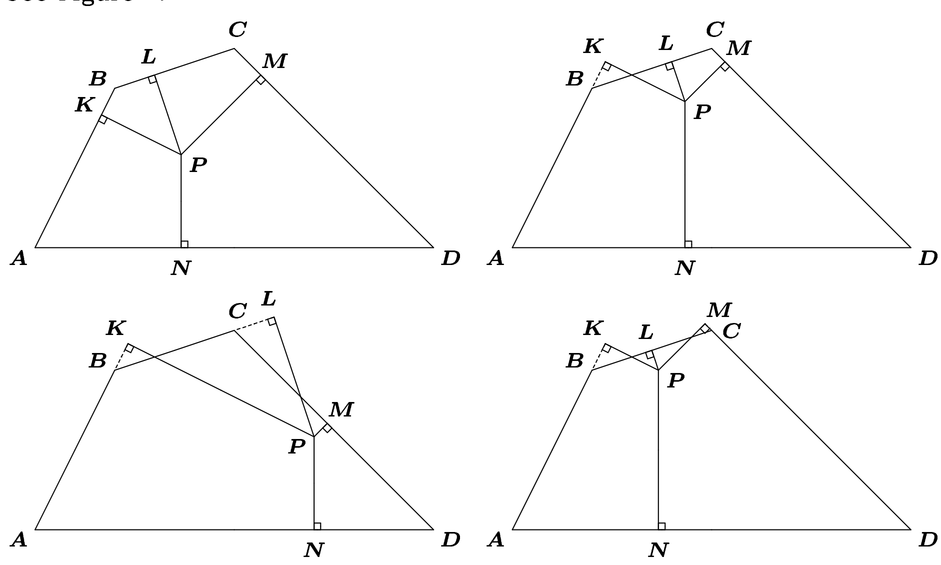


Figure 1

In each case, we can fit together two similar copies of each of $AKPN$, $BLPK$, $CMPL$, and $DNPM$, as shown in Figure 2. (The size of each quadrilateral in the figure below has been enlarged by the factor that is circled inside the quadrilateral. Also, the three vertices of each quadrilateral different from the point P are labelled on the interior of each quadrilateral.) Figure 2 actually corresponds only to case 1 in Figure 1. The remaining cases are somewhat similar, but more involved than this case. The resulting figure (without extended triangles, if these occur) is easily seen to be a parallelogram, and the result follows.

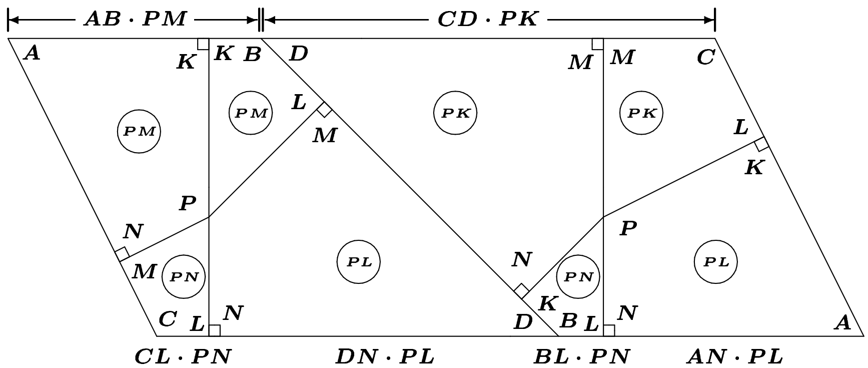


Figure 2

Also solved by MICHEL BATAILLE, Rouen, France; TOSHIO SEIMIYA, Kawasaki, Japan; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. A partial solution was submitted by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK.

2689. [2001 : 534] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Given $\triangle ABC$ and a point P not on it, draw $BD \| AC$ such that D lies on AP, draw $AE \| CB$ such that E lies on CP, and draw $CF \| AB$ such that F lies on BP. Let X be the point of intersection of the lines AB and CD, let Y be the point of intersection of the lines AC and BE, and let Z be the point of intersection of the lines BC and AF. Prove that X, Y and Z are collinear.

I. *Solution independently submitted by Toshio Seimiya, Kawasaki, Japan and by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let L, M and N be the intersections of PA, PB and PC with BC, CA and AB, respectively.

By Ceva's Theorem we get

$$\frac{BL}{LC} \cdot \frac{CM}{MA} \cdot \frac{AN}{NB} = 1 .$$

Since $AE \| BC$ we have

$$\frac{AY}{YC} = \frac{AE}{BC} = \frac{NA}{NB} = \frac{-AN}{NB} .$$

Since $CF||AB$,

$$\frac{CZ}{ZB} = \frac{CF}{AB} = \frac{CM}{AM} = \frac{CM}{-MA}.$$

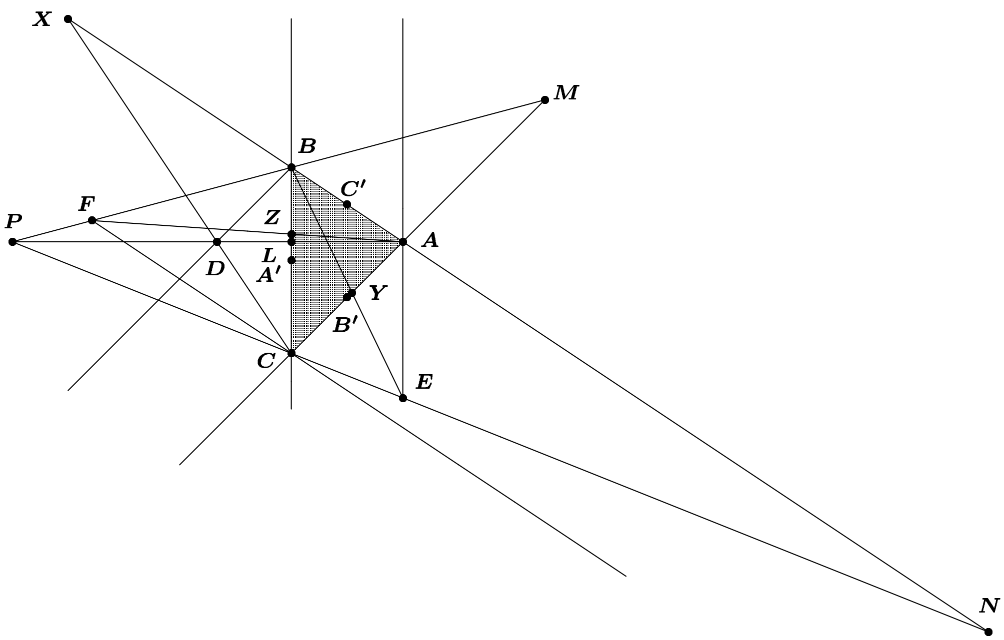
Since $BD||AC$,

$$\frac{BX}{XA} = \frac{BD}{CA} = \frac{BL}{CL} = \frac{BL}{-LC}.$$

From these four equations, we get

$$\frac{AY}{YC} \cdot \frac{CZ}{ZB} \cdot \frac{BX}{XA} = -\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = -1$$

Therefore, by Menelaus's Theorem X, Y and Z are collinear.



II. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

The claim follows immediately from three familiar results of projective geometry. Suppose [as in solution I] that AP, BP, CP intersect BC, CA, AB at L, M, N, respectively, and let A', B', C' be the mid-points of BC, CA, and AB, respectively. Then the hexagon $LB'MC'NA'$ is inscribed in a conic. (This *nine-point conic* is the affine image of a nine-point circle; see [1], p. 19.) Since $BD \parallel AC$, we have that L, B' and X are collinear.

[This is again a basic affine theorem; it is a simple consequence of Ceva's Theorem.] N, A', Y and M, C', Z are likewise collinear. Hence,

$$X = C'N \cdot LB', \quad Y = B'M \cdot NA', \quad Z = A'L \cdot MC'$$

are the points of intersection of opposite sides of the inscribed hexagon $LB'MC'NA'$, and thus, collinear by Pascal's Theorem ([1], p. 254).

Reference

[1] H.S.M. Coxeter, *Introduction to Geometry,* 2nd ed., Wiley, 1969.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

2690. [2001 : 534] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let $\triangle ABC$ be such that $\angle A$ is the largest angle. Let r be the inradius and R the circumradius. Prove that

$$A \gtrless 90^\circ \iff R+r \gtrless \frac{b+c}{2}\,.$$

Solution by Heinz-Jürgen Seiffert, Berlin, Germany.

The relation $A < 90^\circ$ if and only if $R+r < \frac{b+c}{2}$, under the same condition on A, was previously proposed by the same proposer as Problem 10713 in American Mathematical Monthly. In [1], it was shown that

$$R+r-\frac{b+c}{2} = R\left(\sin\frac{A}{2}-\cos\frac{A}{2}\right)\left(2\sin\frac{B}{2}\sin\frac{C}{2}+\cos\frac{B-C}{2}-\cos\frac{A}{2}\right),$$

and that the last factor on the right hand side is positive under the condition $A \geq |B-C|$. Since $A \gtrless 90^\circ$ if and only if $\sin\frac{A}{2} \gtrless \cos\frac{A}{2}$, the above equality obviously implies the desired relations under the weaker condition $A \geq |B-C|$.

[1] H.-J. Seiffert, A Cute Characterization of Acute Triangle (Solution to Problem 10713), American Mathematical Monthly, 5 (2000), p. 464.

Also solved by MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; PIERRE BORNSZTEIN, Pontoise, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; HENRY LIU, student, University of Memphis, Memphis, TN, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VEDULA N. MURTY, Dover, PA, USA; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2691. [2001 : 535] *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

The length of one base of an isosceles trapezoid, the equal sides, and the equal diagonals, are all odd integers. Show that if the remaining base also has integer length, then it is divisible by 8.

Solution by Michel Bataille, Rouen, France.
Let $ABCD$ be the given trapezoid with $AB \parallel CD$ and $AC = BD = a$, $AD = BC = b$, $AB = p$, $CD = q$. We suppose that $ABCD$ is convex so that $a > b$; if $ABCD$ is self-crossing, what follows remains valid by interchanging a and b. Note also that $p \neq q$, since otherwise $ABCD$ would be a rectangle and $a^2 = p^2 + b^2$ would contradict a, b, and p being odd.

Since $\angle BAC = \angle DCA$, the Law of Cosines gives

$$\frac{a^2+q^2-b^2}{2aq} = \frac{a^2+p^2-b^2}{2ap}$$

or $(a^2-b^2)(p-q) = pq(p-q)$. Hence, $a^2 - b^2 = pq$. But a and b are odd, whence $a^2 \equiv 1 \equiv b^2 \pmod 8$. Thus, $pq \equiv 0 \pmod 8$. If p, say, is an odd integer, then p and 8 are coprime integers, implying that $q \equiv 0 \pmod 8$. This is the desired result.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; ROBERT McGREGOR, Auburn, AL, USA; K.R.S. SASTRY, Bangalore, India; JOEL SCHLOSBERG, student, Bayside, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; Mª JESÚS VILLAR RUBIO, Santander, Spain; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There was one incorrect solution.

Many solvers used Ptolemy's Theorem to get the relation $a^2 - b^2 = pq$. Many others dropped perpendiculars from one base to the other and used the Theorem of Pythagoras to get this relation (or an equivalent one). Englehaupt constructs an infinite family of such trapezoids using the following Theorem: If $q > 4k$, $b = q - 2k$, and $a = q + 2k$, then we have $p = 8k$.

Proof: By the Law of Cosines we have

$$\cos(\angle ADC) = \frac{q^2+(q-2k)^2-(q+2k)^2}{2q(q-2k)} = \frac{q-8k}{2(q-2k)}.$$

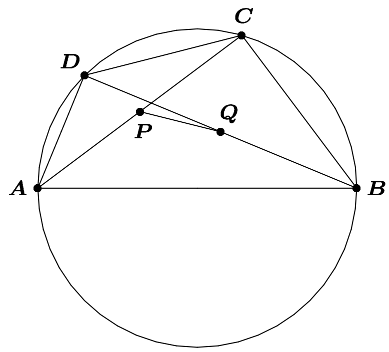
Since $p = q - 2b\cos(\angle ADC)$, it follows that $p = 8k$.

2692. [2001 : 535] *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

Let PQ be the distance between the mid-points of the diagonals of quadrilateral $ABCD$ with sides a, b, c, d and diagonals p and q. Give an example of such a quadrilateral where a, b, c, d, p, q and PQ are all positive integers.

I. *Solution by Michel Bataille, Rouen, France.*

Consider a circle Γ with diameter $AB = 65$. On the same side of AB, let $C \in \Gamma$ with $BC = 39$ and $D \in \Gamma$ with $AD = 25$. The Theorem of Pythagoras gives $AC = 52$ and $BD = 60$. By Ptolemy's Theorem, $AB \cdot CD = AC \cdot BD - BC \cdot AD$, so that $CD = 33$. Finally, the known relation $4PQ^2 = AB^2+BC^2+CD^2+DA^2-AC^2-BD^2$ (see Crux [1994 : 145]) gives $PQ = 17$.



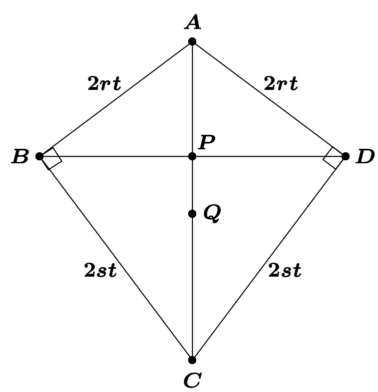
II. *Solution by Geoffrey A. Kandall, Hamden, CT, USA.*

Let (r, s, t) be any Pythagorean triple: $r^2+s^2 = t^2$. Then $(2rt, 2st, 2t^2)$ is also a Pythagorean triple.

Consider a quadrilateral $ABCD$ where $AB = DA = 2rt$, $BC = CD = 2st$ and $AC = 2t^2$. Quadrilateral $ABCD$ is cyclic, so that, by Ptolemy's Theorem, $BD = 4rs$. From the relation $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4PQ^2$ (Crux [1994 : 145]), we obtain $2t^2(r^2+s^2) = t^4+4r^2s^2+PQ^2$. Replacing t^2 by $r^2 + s^2$, we obtain

$$PQ^2 = (r^2+s^2)^2 - 4r^2s^2 = (r^2-s^2)^2,$$

whence, $PQ = |r^2 - s^2|$.



For example, if $r = 3$, $s = 4$ and $t = 5$, then $AB = DA = 30$, $BC = CD = 40$, $AC = 50$, $BD = 48$ and $PQ = 7$.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; HENRY LIU, student, University of Memphis, Memphis, TN, USA (two solutions); DAVID LOEFFLER, student, Trinity College, Cambridge, UK; K.R.S. SASTRY, Bangalore, India (three solutions); HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Sastry and Zhou have also found infinite families of solutions.

2693. [2001 : 535] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Given triangle ABC and a point P, the line through P parallel to BC, intersects AC, AB at Y_1, Z_1 respectively. Similarly, the parallel to CA intersects BC, AB at X_2, Z_2, and the parallel to AB intersects BC, AC at X_3, Y_3. Locate the point P for which the sum $Y_1P \cdot PZ_1 + Z_2P \cdot PX_2 + X_3P \cdot PY_3$ of products of *signed* lengths is maximal.

Solution by Nikolaos Dergiades, Thessaloniki, Greece.

If x, y and z are the signed distances from P to the sides of $\triangle ABC$, and $S = 2aR\sin B\sin C$ is twice the area of $\triangle ABC$, we then have

$$ax + by + cz = S\,, \qquad (1)$$

$$Y_1P \cdot PZ_1 = \frac{y}{\sin C} \cdot \frac{z}{\sin B} = \frac{4R^2}{abc} ayz\,,$$

and, hence,

$$Y_1P \cdot PZ_1 + Z_2P \cdot PX_2 + X_3P \cdot PY_3 = \frac{4R^2}{abc}\left(ayz + bzx + cxy\right)\,.$$

Substituting for z from (1), we get the trinomial in x:

$$\frac{4R^2}{abc^2} \cdot \Big(- abx^2 - \big(a^2y + b^2y - c^2y - bS\big)\, x + ayS - aby^2\Big)\,.$$

This trinomial attains its maximum if and only if

$$x = \frac{-(a^2y + b^2y - c^2y - bS)}{2ab} = -y\cos C + R\sin B\sin C\,,$$

or

$$x + y\cos C = R\cos A + R\cos B\cos C,$$

and, similarly,

$$y + z\cos A = R\cos B + R\cos C\cos A,$$

$$a + x\cos B = R\cos C + R\cos A\cos B.$$

The unique solution of this system is clearly $x = R\cos A$, $y = R\cos B$, $z = R\cos C$, and this is the property of only the circumcentre O of $\triangle\, ABC$.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

The proposer remarks that he found this interesting relation, that the sum of the signed products is equal to the power of P with respect to the circumcircle, in the solution of Problem 166 in The Analyst, *a Journal of Pure and Applied Mathematics, volume IV, 1877, pp. 96, 127. The problem was posed by W.P. Casey of San Francisco, CA, USA, and the printed solution was by the proposer.*

The same volume (pp. 159, 188) also contains the easier Problem 178, which asks for the proof of

$$\frac{Y_1Z_1}{BC} + \frac{Z_2X_2}{CA} + \frac{X_3Y_3}{AB} = 2\,.$$

2694. [2001 : 535] *Proposed by Aaron Lee and Jason Wilson, students, Biola University, La Mirada, CA, USA.*

Given a line segment AB, construct a square $ABCD$ using four or fewer circular arcs and a straightedge. The construction should use fewer arcs than those usually given in texts.

2695. [2001 : 535] *Proposed by Aaron Lee and Jason Wilson, students, Biola University, La Mirada, CA, USA.*

Given a line ℓ and a point P not on it, construct the line through P parallel to ℓ, using two or fewer circular arcs and a straightedge. The construction should use fewer arcs than those usually given in texts.

2696. [2001 : 535] *Proposed by Aaron Lee and Jason Wilson, students, Biola University, La Mirada, CA, USA.*

Using only a straightedge, construct the tangents from a point outside a given circle (and its centre).

Editor's comments.

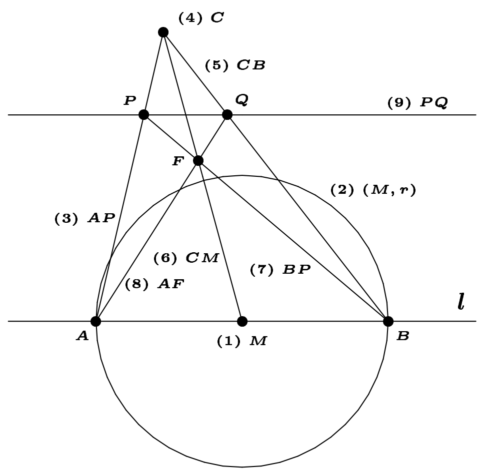
Several readers mentioned that every ruler-and-compass construction can be carried out with ruler and a *single fixed circle* that is drawn with its centre. Poncelet outlined the idea in 1822, and Steiner later (1833) provided a treatment that Eves calls "complete and systematic" ([2], section 4.6). The theorem can be found in books on geometric constructions such as [3], in geometry texts like [2], in general mathematics texts [1], and in books on projective geometry (a subject that is sometimes described as the study of figures that can be constructed with ruler alone). On the other hand, efficient constructions have also been a goal; Eves ([2], p. 181) describes Lemoine's system (from 1907) of evaluating efficiency, which will be simplified here by counting as a single step each inserted point, each line joining a pair of points, and each circle drawn with given centre and radius. The solutions below have been arranged into two parts. The first provides constructions that use only one circle in the spirit of the Poncelet-Steiner theorem; the second displays the most efficient constructions submitted in the spirit of the students who proposed these problems. All solvers included proofs that their constructions are correct, but we shall omit them on the grounds that most are straightforward, and the rest are easily found in references.

Notation. Denote the circle with centre X and radius of length y by the pair (X, y).

Part I: *Constructions using a single circle.*

2695. Given line l and point P not on it, construct the line through P that is parallel to l.

Solution by Hans Engelhaupt, Franz–Ludwig–Gymnasium, Bamberg, Germany, Geoffrey A. Kandall, Hamden, CT, USA, Václav Konečný, Ferris State University, Big Rapids, MI, USA (who cited reference [2] problem 4.6.1. on p. 175), *and Zhou* (who referred to [1], p. 166, and [4] Construction 3, p. 363).

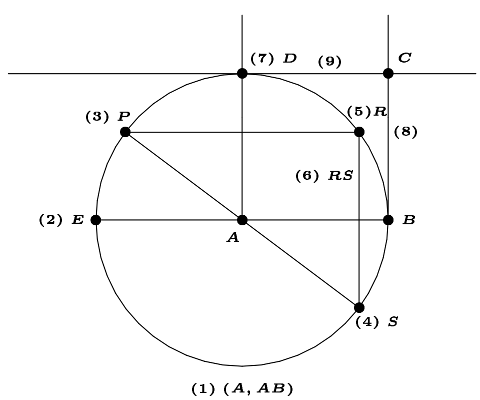


1. Choose a point M on l.
2. For a convenient radius r draw (M, r), calling A and B the points where it intersects l.
3. Draw AP.
4. Extend AP to a point C beyond P.
5. Draw CB.
6. Draw CM.
7. Draw BP calling F the point where it intersects CM.
8. Draw AF calling Q the point where it intersects CB.
9. PQ is the desired line parallel to $AB = l$.

Of course, the Poncelet-Steiner theorem does not require the given circle to have its centre on l. Loeffler showed how to construct a diameter of that circle which is parallel to an arbitrary line l. References [1] and [2] both describe Steiner's method for constructing PQ when l is not a diameter of the given circle.

2694. Given segment AB construct square $ABCD$.

Solution by Hans Engelhaupt, Franz–Ludwig–Gymnasium, Bamberg, Germany.



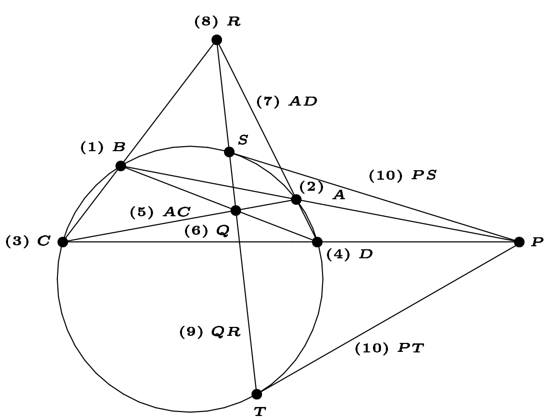
1. Construct (A, AB).
2. Extend the line BA to the point E where it meets the circle again.
3. Choose a point P on the circle.
4. Draw PA and call S the point where it meets the circle again.
5. Use problem 2695 above to construct PR parallel to AE, where R is the point where this line meets the circle again.

6. Draw RS. (Note that because PS is a diameter while $PR||AB$, we have $RS \perp AB$.)
7. Construct the line parallel to RS through A and call D the point where it hits the circle.
8. Construct the line through B parallel to RS.
9. Construct the line through D parallel to AE. Call C the point where it meets the line in 8.

$ABCD$ is the desired square.

2696. Construct the tangents to a given circle from an external point P. (The centre is not required.)

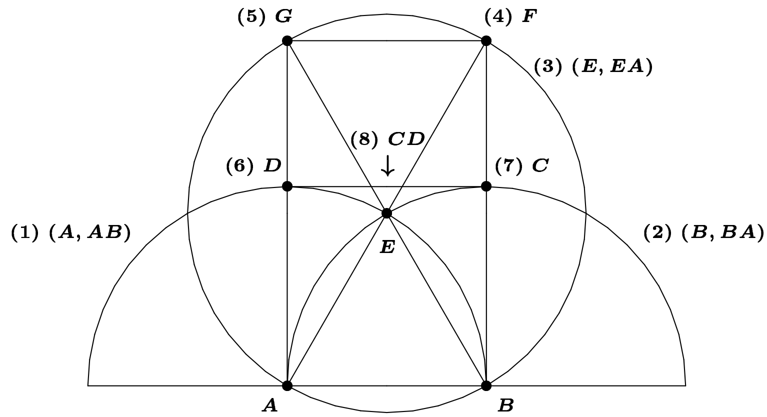
All solvers provided essentially the same construction. It can be found in geometry texts that have a chapter on projective geometry, as well as in all references listed below.



1. Choose a point B on the circle.
2. Draw PB and call A the second point where it meets the circle.
3. Choose a point C on the circle different from A and B.
4. Draw PC and call D the second point where it meets the circle.
5. Draw AC.
6. Draw BD and call Q the point where it meets AC.
7. Draw AD.
8. Draw BC and call R the point where it meets AD.
9. Draw QR and call S and T where it meets the circle.
10. Draw PS and PT. These are the required tangents.

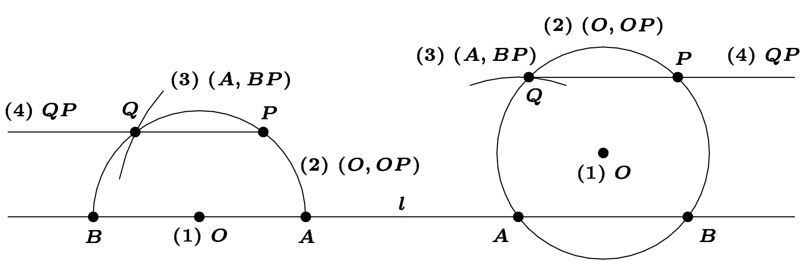
Part II: *Efficient constructions with few circles.*

2694. A square was constructed on edge AB using 3 circles in 8 steps *by Michel Bataille, Rouen, France, Charles R. Diminnie, Angelo State University, San Angelo, TX, USA, David Loeffler, student, Trinity College, Cambridge, UK, Robert McGregor, Auburn, AL, USA, and the proposers.*



1. Construct (A, AB).
2. Construct (B, BA), and call E one of the points where it meets the circle in (1).
3. Construct (E, EA).
4. Draw AE and call F the point where it again meets the circle in (3).
5. Draw BE and call G the point where it again meets the circle in (3).
6. Draw AG. D is one point where it meets (A, AB).
7. Draw BF. C is the point where it meets (B, BA) on the same side of AB as D.
8. Draw CD. $ABCD$ is the desired square.

2694. The parallel to l through P was constructed using 2 circles in 4 steps *by Christopher J. Bradley, Clifton College, Bristol, UK* (using O not on l) and by the proposers (using O on l).



1. Pick a point O either on l *(as in the figure on the left) or near* l *(as on the right)*.
2. Construct (O, OP) and call A, B the points where it meets l.

3. Construct (A, BP) and call Q the point where it meets the circle in (2) on same side of l as P.

4. Draw PQ. This is the desired parallel to l.

References

[1] Heinrich Dörrie, *100 Great Problems of Elementary Mathematics*, Dover, 1965.

[2] Howard Eves, *A Survey of Geometry*, Allyn and Bacon, 1972.

[3] George E. Martin, *Geometric Constructions*. Springer, 1998.

[4] Peter Woo, Straightedge constructions, given a parabola. *College Math. J.* **31**:5 (November, 2000), 364-372.

Solutions by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark (2696 only); CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; JOSHUA GREEN, BENJAMIN ARMBRUSTER, and ALICE TRIMBLE, Tucson, AZ, USA (2694 and 2695 only); JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta (2694 and 2696 only); GEOFFREY A. KANDALL, Hamden, CT, USA (2694 and 2695 only); VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; HENRY LIU, student, University of Memphis, Memphis, TN, USA (2694 and 2695 only); DAVID LOEFFLER, student, Trinity College, Cambridge, UK; ROBERT McGREGOR, Auburn, AL, USA (multiple solutions to 2694 and 2695); VICTOR PAMBUCCIAN, Phoenix, AZ, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposers.

2697. [2001 : 535] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Find a closed form for $\displaystyle\sum_{k=1}^{n} k\sin^2(kx)$.

Solution by Christopher J. Bradley, Clifton College, Bristol, UK (modified slightly by the editor).

Let S denote the given sum. Then

$$\begin{aligned} S &= \frac{1}{2}\sum_{k=1}^{n} k \;-\; \frac{1}{2}\sum_{k=1}^{n} k\cos(2kx) \\ &= \frac{1}{4}n(n+1) \;-\; \frac{1}{4}\frac{d}{dx}\left(\sum_{k=1}^{n}\sin(2kx)\right) . \qquad (1) \end{aligned}$$

Now,

$$\begin{aligned} (\sin x)\left(\sum_{k=1}^{n}\sin(2kx)\right) &= \frac{1}{2}\sum_{k=1}^{n}\big(\cos\left((2k-1)x\right) - \cos\left((2k+1)x\right)\big) \\ &= \frac{1}{2}\Big(\cos x - \cos\left((2n+1)x\right)\Big) . \qquad (2) \end{aligned}$$

From (1) and (2), we get

$$S = \frac{1}{4}n(n+1) - \frac{1}{8}\frac{d}{dx}\left(\frac{\cos x - \cos\left((2n+1)x\right)}{\sin x}\right). \tag{3}$$

By straightforward differentiation and some simplifications, we find that

$$\begin{aligned}\frac{d}{dx}&\left(\frac{\cos x - \cos\left((2n+1)x\right)}{\sin x}\right)\\ &= \frac{(2n+1)\sin x \sin\left((2n+1)x\right) + \cos x \cos\left((2n+1)x\right) - 1}{\sin^2 x}\\ &= \frac{2n \sin x \sin\left((2n+1)x\right) + \cos(2nx) - 1}{\sin^2 x}\\ &= \frac{2n\sin\left((2n+1)x\right)}{\sin x} - \frac{2\sin^2(nx)}{\sin^2 x}. \end{aligned} \tag{4}$$

Substituting (4) into (3), we get

$$S = \frac{1}{4}\left(n(n+1) - \frac{n\sin\left((2n+1)x\right)}{\sin x} + \frac{\sin^2(nx)}{\sin^2 x}\right).$$

[Ed.: Of course, this expression is valid only if $x \neq k\pi$, $k \in \mathbb{Z}$. Otherwise, S is clearly zero.]

Also solved by MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; PIERRE BORNSZTEIN, Pontoise, France; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; PAUL DEIERMANN, Southeast Missouri State University, Cape Girardeau, MO, USA; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; ŽELJKO HANJŜ, Uinversity of Zagreb, Zagreb, Croatia; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; URSEL LENK, Essen, Germany; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; ROBERT McGREGOR, Auburn, AL, USA; NORVALD MIDTTUN, Royal Norwegian Naval Academy, Laksevåg, Norway; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; STEFFEN WEBER, Georg-Cantor-Gymnasium Halle, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Many solvers considered functions like e^{2kxi} in their solutions. This approach, however, did not seem to make the computations any simpler. Some others used integration instead of differentiation. About one third of the solvers simply quoted the known formula:

$$\sum_{k=1}^{n} \sin(2kx) = \frac{\sin(nx)\sin\left((n+1)x\right)}{\sin x}.$$

Of course, the final expression for S can (and did) take many different forms.

2698. [2001 : 535] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands (adapted by the editor).*

The perimeter of a right triangle with integer sides is a perfect square. The area of the triangle is the cube of an integer. Find the smallest triangle satisfying these conditions.

[Ed. Smeenk asked the case when the hypotenuse has length 240.]

Solution by Henry Liu, student, University of Memphis, Memphis, TN, USA.

Throughout, all variables are positive integers.

A right triangle with sides a, b and c, where c is the hypotenuse, has the parametric equations

$$\begin{aligned}(a,b,c) &= \left(2kmn, k(m^2-n^2), k(m^2+n^2)\right) \quad \text{or} \\ (a,b,c) &= \left(k(m^2-n^2), 2kmn, k(m^2+n^2)\right),\end{aligned}$$

where $m > n$, $(m,n) = 1$, and m and n have different parity.

Assume, without loss of generality, that

$$(a,b,c) = \left(2kmn, k(m^2-n^2), k(m^2+n^2)\right).$$

For some x and y, we have

$$a+b+c = 2m(m+n)k = x^2, \quad \frac{1}{2}ab = mn(m-n)(m+n)k^2 = y^3.$$

Taking $(m,n) = (2,1)$ and $k = 48$ gives $x = y = 24$, whence $(a,b,c) = (192,144,240)$. We claim that this is the smallest triangle (in terms of area) with the given properties. We will show that the system of equations

$$\begin{aligned} 2m(m+n)k &= x^2, & (1)\\ mn(m-n)(m+n)k^2 &= y^3, & (2)\end{aligned}$$

has no solutions for $2 \le y \le 23$.

Now, m, $m+n > 1$. Either they have different parity, or they are both odd, whence n is even, so that $n > 1$. It follows that y must be even, and it must also have an odd prime factor. It must have exactly one odd prime factor; otherwise $y \ge 2\cdot 3\cdot 5 = 30$. Thus, $y = 2^\alpha p^\beta$, for some α, β and odd prime p.

Next, we show that m, n, $m+n$ and $m-n$ are pairwise coprime. Note that $m+n$ and $m-n$ are both odd. If $q|m$ and $q|m+n$ for some odd prime q, then $q|n$, a contradiction. Thus, $(m, m+n) = 1$. Similarly, $(n, m+n) = (m, m-n) = (n, m-n) = 1$. If $q|m+n$ and $q|m-n$, then $q|2m$ and $q|2n$, so that $q|m$ and $q|n$, a contradiction. Thus, $(m+n, m-n) = 1$.

Since m, $m+n > 1$, it follows that $p|m+n$ and $2|m$. Since $m-n$ is odd and $(m+n, m-n) = 1$, we have $m-n = 1$. Also, since n is odd and $(n, m+n) = 1$, we have $n = 1$. Thus, $(m,n) = (2,1)$. Equation (2) becomes $6k^2 = 2^{3\alpha}p^{3\beta}$, so that $p = 3$. Thus, $y = 6$, 12 or 18. Write $y = 6t$ for $t = 1$, 2 or 3. Equation (2) becomes $k^2 = 6^2t^3$, and hence $t = 1$, and $k = 6$. Now we have $x^2 = 72$, a contradiction.

Also solved with a mathematical proof by CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; ED KOCNER, student, Arizona State University, Glendale, AZ, USA; and the proposer. The result was obtained using a search technique by CHARLES ASHBACHER, Cedar Rapids, IA, USA; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; HENRY LIU, student, University of Memphis, Memphis, TN, USA; and DAVID LOEFFLER, student, Trinity College, Cambridge, UK.

Richard I. Hess found the next smallest set to be $(360, 150, 390)$.

2699. [2001 : 536] *Proposed by Maureen P. Cox and Albert White, St. Bonaventure University, St. Bonaventure, NY, USA.*

Evaluate $$\prod_{k=1}^{n} \left(\frac{4k+4n-3}{4n}\right)^{\frac{4n+4k-3}{4n^2}}.$$

Editor's comment. The intended problem was, in fact, to evaluate the **limit** of the given product. The limit sign was inadvertently left off when the problem was first printed. Fortunately, all of the solvers interpreted the problem in the way that was intended.

Solution by Karl Havlak and Charles R. Diminnie, Angelo State University, San Angelo, TX, USA.

Let $P_n = \prod_{k=1}^{n} \left(\frac{4k+4n-3}{4n}\right)^{\frac{4n+4k-3}{4n^2}}$. Then

$$\begin{aligned} \ln(P_n) &= \sum_{k=1}^{n} \left(\frac{4k+4n-3}{4n}\right) \ln\left(\frac{4k+4n-3}{4n}\right) \cdot \frac{1}{n} \\ &= \sum_{k=1}^{n} \left(1 + \frac{4k-3}{4n}\right) \ln\left(1 + \frac{4k-3}{4n}\right) \cdot \frac{1}{n}. \end{aligned}$$

Note that $\frac{k-1}{n} < \frac{4k-3}{4n} < \frac{k}{n}$, for $k = 1.\ 2, \ldots, n$. Hence, $\ln(P_n)$ is a Riemann sum for $f(x) = (1+x)\ln(1+x)$ on the interval $[0,1]$ (using the partition $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$). Since $f(x)$ is continuous on $[0,1]$, we get that

$$\lim_{n\to\infty} \ln(P_n) = \int_0^1 (1+x)\ln(1+x)\,dx = 2\ln 2 - \frac{3}{4}.$$

Therefore, $\lim_{n\to\infty} P_n = e^{2\ln 2 - \frac{3}{4}} = 4e^{-\frac{3}{4}}$.

Also solved (with virtually the same proof) by MICHEL BATAILLE, Rouen, France; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposers.

Seiffert considered the more general problem of finding the value of $\lim_{n\to\infty} Q_n$*, where* $Q_n = \prod_{k=1}^{n} \left(\frac{ak+bn-c}{bn}\right)^{\frac{ak+bn-c}{bn^2}}$*, where* a*,* b *and* c *are positive real numbers such that* $a > c$ *and* $b > c$*. Using Gamma functions and Stirling's Formula, he showed that*

$$\lim_{n\to\infty} Q_n = \left(\frac{a+b}{b}\right)^{\frac{(a+b)^2}{2ab}} \cdot \exp\left(-\frac{(a+2b)}{4b}\right).$$

The proposed problem is the special case when $a = b = 4$ *and* $c = 3$*.*

2700. [2001 : 536] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Let n be a positive integer. Show that

$$\sum_{k=1}^{n} \frac{k}{n+k}\binom{n}{k} < \sum_{k=1}^{n} \binom{n}{k} \log\left(\frac{n+k}{n}\right) < 2^{n-1}.$$

[Ed. "log" is, of course, the natural logarithm.]

Solution by Michel Bataille, Rouen, France.

Note first that $\frac{x-1}{x} < \log(x) < x-1$ for $x > 0$, $x \neq 1$.

[$\log(x) < x-1$ results from the strict concavity of $\log$, and substituting $\frac{1}{x}$ for x gives $\log(x) > \frac{x-1}{x}$.]

For each $k = 1, 2, \ldots, n$, letting $x = \frac{n+k}{n}$ then yields

$$\frac{k}{n+k} < \log\left(\frac{n+k}{n}\right) < \frac{k}{n}.$$

Hence,

$$\sum_{k=1}^{n} \frac{k}{n+k}\binom{n}{k} < \sum_{k=1}^{n} \binom{n}{k} \log\left(\frac{n+k}{n}\right) < \sum_{k=1}^{n} \binom{n}{k}\frac{k}{n}.$$

The result follows since

$$\sum_{k=1}^{n} \binom{n}{k}\frac{k}{n} = \sum_{k=1}^{n} \binom{n-1}{k-1} = \sum_{j=0}^{n-1} \binom{n-1}{j} = 2^{n-1}.$$

Also solved by MIHÁLY BENCZE, Brasov, Romania; PIERRE BORNSZTEIN, Pontoise, France; PAUL BRACKEN, CRM, Université de Montréal, Montréal, Québec; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; KEE-WAI LAU, Hong Kong, China; HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA;

VEDULA N. MURTY, Visakhapatnam, India; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; ROGER ZARNOWSKI, Angelo State University, San Angelo, TX, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Using some known inequalities in the literature and Jensen's Inequality, Seiffert established the following sharper results:

$$\sum_{k=1}^{n} \frac{2k}{2n+k}\binom{n}{k} \;<\; \sum_{k=1}^{n}\binom{n}{k}\log\left(\frac{n+k}{n}\right) \;<\; \sqrt{\frac{2}{3}}\,2^{n-1}\,.$$

Bencze listed the following generalization and eight similar inequalities:

$$\sum_{k=1}^{n} \frac{k^\alpha}{n^\alpha+k^\alpha}\binom{n}{k} \;<\; \sum_{k=1}^{n}\binom{n}{k}\log\left(\frac{n^\alpha+k^\alpha}{n^\alpha}\right) \;<\; \sum_{k=1}^{n}\frac{k^\alpha}{n^\alpha}\binom{n}{k}$$

for $\alpha = 1, 2, \ldots$. *He also referred to two of his papers in the* Octogon Mathematical Magazine (7(1), 1999 pp. 36–58 and 8(2), 2000 pp. 339–359, *for other generalizations.*

2700A. [2001 : 536] *Proposed by Paul Bracken, CRM, Université de Montréal, Montréal, Québec.*

Show that the function e^{-xn^2} can be written in the following form,

$$\begin{aligned} e^{-xn^2} &= \sum_{k=0}^{n-1}(-1)^k\frac{x^k n^{2k}}{k!} + (-1)^n\frac{x^n n^{2n}}{n!}\phi_x(n)\,, \quad \text{where} \\ \phi_x(n) &= 1 - \int_0^{xn^2} e^{-t}\left(1-\frac{t}{xn^2}\right)^n dt\,. \end{aligned}$$

Determine the leading large n behaviour of $\phi_x(n)$, and show that

$$\lim_{n\to\infty} n\phi_x(n) \;=\; 1/x\,.$$

Solution by Henry Liu, student, University of Memphis, Memphis, TN, USA.

By Taylor's Theorem, for an infinitely differentiable function $f(x)$ on an interval I and $a \in I$, we have

$$f(x) \;=\; \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k + R_n$$

for all $x \in I$, where

$$R_n \;=\; \frac{1}{(n-1)!}\int_a^x f^{(n)}(t)(x-t)^{n-1}\,dt\,.$$

With $f(x) = e^x$, $I = \mathbb{R}$ and $a = 0$, we have

$$e^x \;=\; \sum_{k=0}^{n-1}\frac{x^k}{k!} + \frac{1}{(n-1)!}\int_0^x e^t(x-t)^{n-1}\,dt\,.$$

Replacing x by $-xn^2$ gives

$$e^{-xn^2} = \sum_{k=0}^{n-1}(-1)^k\frac{x^k n^{2k}}{k!} + \frac{1}{(n-1)!}\int_0^{-xn^2} e^t(-xn^2-t)^{n-1}\,dt\,.$$

We have

$$\begin{aligned}
&\frac{1}{(n-1)!}\int_0^{-xn^2} e^t(-xn^2-t)^{n-1}\,dt \\
&= \frac{(-1)^n(xn^2)^{n-1}}{(n-1)!}\int_0^{xn^2} e^{-t}\left(1-\frac{t}{xn^2}\right)^{n-1}\,dt \\
&= \frac{(-1)^n(xn^2)^{n-1}}{(n-1)!}\left(\left[e^{-t}\left(-\frac{xn^2}{n}\right)\left(1-\frac{t}{xn^2}\right)^n\right]_0^{xn^2} \right.\\
&\qquad\left. + \int_0^{xn^2} e^{-t}\left(-\frac{xn^2}{n}\right)\left(1-\frac{t}{xn^2}\right)^n\,dt\right) \\
&= \frac{(-1)^n(xn^2)^{n-1}}{(n-1)!}\cdot\frac{xn^2}{n}\left(1-\int_0^{xn^2} e^{-t}\left(1-\frac{t}{xn^2}\right)^n\,dt\right) \\
&= (-1)^n\frac{x^n n^{2n}}{n!}\phi_x(n)\,,
\end{aligned}$$

as required.

Since

$$\left(1-\frac{t}{xn^2}\right)^n = 1-\frac{t}{xn}+\mathrm{O}\left(n^{-2}\right)\,,$$

to leading order, we have

$$\begin{aligned}
\phi_x(n) &\sim 1-\int_0^{xn^2} e^{-t}\left(1-\frac{t}{xn}\right)\,dt \\
&= 1-\left[\left(1-\frac{t}{xn}\right)(-e^{-t})\right]_0^{xn^2} + \int_0^{xn^2}(-e^{-t})\left(-\frac{1}{xn}\right)\,dt \\
&= 1+((1-n)e^{-xn^2}-1)+\frac{1}{xn}\int_0^{xn^2} e^{-t}\,dt \\
&= (1-n)e^{-xn^2}-\frac{1}{xn}(e^{-xn^2}-1)\,.
\end{aligned}$$

Since $\lim\limits_{n\to\infty} n(1-n)e^{-xn^2}=0$ and $\lim\limits_{n\to\infty} e^{-xn^2}=0$, we have

$$\lim_{n\to\infty} n\phi_x(n) = \lim_{n\to\infty}\left(n(1-n)e^{-xn^2}-\frac{1}{x}(e^{-xn^2}-1)\right)=\frac{1}{x}\,.$$

Also solved by MICHEL BATAILLE, Rouen, France; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; KEE-WAI LAU, Hong Kong, China; GERRY LEVERSHA, St. Paul's School, London, England; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; ROGER ZARNOWSKI, Angelo State University, San Angelo, TX, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

YEAR END FINALE

Well, this is it! This is my last chance to write to everyone, and to say a heartfelt "thank you" to you all for the last seven years. I have really enjoyed being the Editor, and I shall miss it. However, it is time to move on, and I am convinced that Jim Totten will be a superb Editor. I commend him to you.

There are many people that I wish to thank most sincerely for particular contributions. I thank most sincerely, ILIYA BLUSKOV, CHRIS FISHER, CLAYTON HALFYARD, GEORG GUNTHER, JIM TOTTEN, and EDWARD WANG, for their regular yeoman service in assessing the solutions; BRUCE GILLIGAN, for ensuring that we have quality articles; JOHN GRANT McLOUGHLIN, for ensuring that we have quality book reviews, ROBERT WOODROW and SHAWN GODIN for their interesting corners, and RICHARD GUY for sage advice whenever necessary.

You will have noticed the increased posing of problems in French. I thank JEAN-MARC TERRIER and HIDEMITSU SAYEKI for taking on this job at short notice, and for dealing with shorter than desirable deadlines.

The editors of the ***MATHEMATICAL MAYHEM*** section, SHAWN GODIN, CHRIS CAPPADOCIA, ADRIAN CHAN, DONNY CHEUNG, JIMMY CHUI, PAUL OTTAWAY and DAVID SAVITT, all do a sterling job.

I also thank all the editors who assist with proofreading. As well, I thank MOHAMMED AASSILA for helping out in this task. The quality of the work of all these people is a vital part of what makes ***CRUX with MAYHEM*** what it is. Thank you one and all.

As well, I would like to give special thanks to our Associate Editor, CLAYTON HALFYARD, for continuous sage advice, and for keeping me from printing too many typographical and mathematical errors; and to my colleagues, YURI BAHTURIN, HERMANN BRUNNER, ROLAND EDDY, ANDY FOSTER, EDGAR GOODAIRE, MAURICE OLESON, MIKE PARMENTER, DAVID PIKE, DONALD RIDEOUT, NABIL SHALABY, BRUCE WATSON, in the Department of Mathematics and Statistics at Memorial University for their occasional sage advice. I have also been helped by some Memorial University students, RENEÉ BARTER, ANDREA BURGESS, JONATHAN KAVANAGH and SHANNON SULLIVAN.

The staff of the Department of Mathematics and Statistics at Memorial University deserve special mention for their excellent work and support: ROS ENGLISH, MENIE KAVANAGH, WANDA HEATH, and LEONCE MORRISSEY; as well as the computer and networking expertise of DWAYNE HART, CRAIG SQUIRES and MARIAN WISSINK.

Also the LaTeX expertise of JOANNE LONGWORTH at the University of Calgary, JUNE ALEONG at Wilfrid Laurier University, the ***MAYHEM*** staff, and all others who produce material, is much appreciated.

Thanks to GRAHAM WRIGHT, the Managing Editor, who keeps me on the right track, and to the U of T Press, who continue to print a fine product.

The online version of ***CRUX with MAYHEM*** continues to attract attention. We recommend it highly to you. Thanks are due to LOKI JORGENSON, JUDI BORWEIN, and the rest of the team at SFU who are responsible for this.

Finally, I would like to express real and heartfelt thanks to the Head of my Department, HERBERT GASKILL, and to the Dean of Science, BOB LUCAS. Without their support and understanding, I would not be able to do the job of Editor-in-Chief.

Last but not least, I send my thanks to you, the readers of ***CRUX with MAYHEM***. Without you, ***CRUX with MAYHEM*** would not be what it is. We receive over 150 proposals each year, and, as you know, we publish 100 problem proposals in each volume. Of course, we receive hundreds of solutions, as you will see in the index that follows. Every year, we receive solutions from new readers. This is very pleasing. More and more proposals and solutions are arriving by email. The editor is able to process files sent in LaTeX, in WORD and in WordPerfect formats, as well (although less desirable) image files. One small reminder: please ensure that your name and address is on EVERY problem or proposal, and that each starts on a fresh sheet of paper. Otherwise, there may be filing errors, resulting in a submitted solution being lost.

This must be the quickest response ever to a problem — Natalio H. Guersenzvaig, Universidad CAECE, Buenos Aires, Argentina, emailed that the answer to question ?? [2002 : 460] is that they are the $\mathbf{21}^{\mathrm{st}}$, $\mathbf{35}^{\mathrm{th}}$ and $\mathbf{56}^{\mathrm{th}}$ convergents in the continued fraction expansion of π.

Keep those contributions and letters coming in. We need your ARTICLES, PROPOSALS and SOLUTIONS to keep ***CRUX with MAYHEM*** alive and well.

Wishing all our readers the compliments of the season, and a very happy, peaceful and prosperous year 2003.

I have enjoyed knowing you all. Fare thee well!

Bruce Shawyer

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Mitko Kunchev 2601, 2605, 2676, 2678, 2691
Kee-Wai Lau 2601, 2603, 2605, 2610, 2619, 2620, 2625, 2630, 2631, 2633, 2644, 2648, 2650, 2652, 2655, 2662, 2663, 2668, 2672, 2675, 2676, 2678, 2683, 2691, 2697, 2700, 2700A
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Robert P. Sealy 2610, 2687, 2697
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Heinz-Jürgen Seiffert 2601, 2604, 2605, 2606, 2607, 2610, 2619, 2621, 2622, 2625, 2627, 2631, 2634, 2637, 2642, 2644, 2645, 2646, 2649, 2650, 2653, 2654, 2655, 2658, 2658, 2664, 2672, 2675, 2676, 2677, 2678, 2679, 2691, 2692, 2697, 2699, 2700
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Austrian IMO Team 2001 2601, 2603, 2605, 2607, 2610
Con Amore Problem Group 2630, 2655, 2679, 2696, 2697, 2698
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Southwest Missouri State Problem Solving Group 2619