THE ACADEMY CORNER

No. 45

Bruce Shawyer

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In this issue, we present the hints and answers to the The Bernoulli Trials 2001, which we gave in [2001 : 353].

The Bernoulli Trials 2001
by
Ian VanderBurgh and Christopher G. Small

Hints:

1. To miss on the 501^{st} question, the first 500 questions must have been selected from the 1500 that were not too hard, and the 501^{st} from the 501 that were too hard.

2. Consider Tony and Maria as each running 2100 m at constant speeds (the first 1400 m being the "uphill" portion).

3. $$\frac{EF}{BC} = \frac{AE}{AB} = \frac{AE/AD}{AB/AD} = \frac{\cos 30^\circ}{\sec 30^\circ} .$$

4. Note that
$$H_n = \frac{1}{n}\sum_{i=0}^{n-1} f\left(\frac{i}{n}\right) ,$$
where $f(x) = 1/(1+3x)$. Show that $\lim H_n = (\ln 4)/3$. No calculators are allowed for the last step!

5. Since $2001 = 3 \times 23 \times 29$, it follows that $d(2001^n) = (n+1)^3$. Then, $(n+1)^3 = 2kn+1$ reduces to
$$n^2 + 3n + (3-2k) = 0 .$$
Thus,
$$n = \frac{-3+\sqrt{8k-3}}{2}$$
which will be a positive integer provided $8k-3$ is a perfect square.

6. Divide the numbers into pairs as

$$\{2n-1, 2n\}$$

for $n = 1, \ldots, 1000$. This leaves 2001 by itself. Suppose Alice chooses 2001 as her starting number. For each number that Barbara chooses, Alice can choose the other number of the pair.

7. Let the side length of each of the solids be s. Notice that the octahedron is the combination two square based pyramids with all side lengths s. Therefore, the volume of the octahedron is

$$2 \times \frac{1}{3} \times (\text{base of pyramid}) \times (\text{height of pyramid})\,.$$

Write this as a function of s. Show that the volume of the tetrahedron is

$$\frac{\sqrt{2}}{12}s^3\,,$$

which is 1, by assumption.

8. Consider

$$f(x) \;=\; \frac{2001}{x^2}\,.$$

9. Let $x =$ FOR and let $y =$ WAT. Then

$$6(1000x+y) \;=\; 7(1000y+x)\,.$$

This reduces to

$$5993x \;=\; 6994y\,.$$

But 5993 and 6994 are both divisible by 13. Thus, $461x = 538y$. However, 461 and 538 are relatively prime!

10. For the angles of a triangle,

$$\tan A + \tan B + \tan C \;=\; \tan A \tan B \tan C\,.$$

Thus, if x, $1+x$ and $1-x$ were the tangents, then we would have

$$\begin{aligned} x+1+x+1-x &= x(1-x)(1+x) \\ x+2 &= x-x^3 \\ x^3 &= -2 \\ x &= -\sqrt[3]{2}\,. \end{aligned}$$

Thus, $x < 0$ and $1+x < 0$. Is this possible?

11. This would be trivial with an illegal calculator. However, ten minutes is enough time to number crunch this, even without a calculator. To do so is to miss the fact that there is a more mathematically interesting argument.

Obviously there are no solutions for $0 < x \leq 1$. Since

$$f(x) = x^{x^{2001}}$$

is increasing for $x > 1$, there is at most one solution for $x > 1$. It is easy to check that

$$x = 2001^{1/2001}$$

is a solution. But

$$2001^{1/2001} = \left(\sqrt[3]{2001}\right)^{1/667} > 12.5^{1/667} > 12.5^{0.0014}.$$

12. We can write

$$\begin{aligned} &\lfloor (1+\sqrt{3})^{2001} \rfloor \\ &= (1+\sqrt{3})^{2001} + (1-\sqrt{3})^{2001} \\ &= (1+\sqrt{3})(4+2\sqrt{3})^{1000} + (1-\sqrt{3})(4-2\sqrt{3})^{1000} \\ &= 2^{1000}[(1+\sqrt{3})(2+\sqrt{3})^{1000} + (1-\sqrt{3})(2-\sqrt{3})^{1000}]. \end{aligned}$$

Can an additional factor of 2 be pulled out?

Answers:

1.	True.	2.	False.	3.	True.	4.	False.
5.	False.	6.	False.	7.	True.	8.	False.
9.	True.	10.	False.	11.	True.	12.	True.

THE OLYMPIAD CORNER

No. 218

R.E. Woodrow

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Oops! We have noticed a duplication — the problems of the 38$^{\text{th}}$ National Mathematical Olympiad of Slovenia 1994, which have just appeared in the previous issue [2001 : 359] have already appeared on [1998 : 132] with all the solutions published in [1999 : .208–211] and [1999 : 266–269]. Sorry.

We begin this number with a contest from France, from the "Concours Général des lyceés" and the Composition de Mathématiques (Classe terminale S) 1999. Thanks go to Michel Bataille, Rouen, France for forwarding the set to me.

COMPOSITION DE MATHÉMATIQUES 1999

Classe terminale S

Durée : 5 heures

1. Quel est le volume maximum d'un cylindre, ayant même axe de révolution qu'un cône donné et intérieur à ce cône ?

Quel est le volume maximum d'une boule, centrée sur cet axe et intérieure au cône ?

Comparer les deux maximums trouvés.

2. Résoudre dans $\mathbb{N}$ l'équation en n :

$$(n+3)^n = \sum_{k=3}^{n+2} k^n .$$

3. Pour quels triangles aux angles tous aigus le quotient du plus petit côté par le rayon du cercle inscrit est-il maximum ?

4. Sur une table trônent 1 999 bonbons rouges et 6 661 bonbons jaunes rendus indiscernables par des emballages uniformes. Un gourmand applique jusqu'à épuisement du stock l'algorithme ci-dessous :

(a) s'il reste des bonbons, il en tire un au hasard, note sa couleur, le mange et va en (b) ;

(b) s'il reste des bonbons, il en tire un au hasard et note sa couleur :

— si elle est la même que celle du dernier bonbon avalé, il le mange et retourne en (b),

— sinon, il le remmaillote, le pose et retourne en (a).

Montrer que tous les bonbons seront mangés et donner la probabilité pour que le dernier bonbon mangé soit rouge.

5. Montrer que les symétriques de chaque sommet d'un triangle par rapport au côté opposé sont alignés si, et seulement si, la distance de l'orthocentre au centre du cercle circonscrit est égale à son diamètre.

Next we give the problems of the three rounds of the Iranian Mathematical Olympiad 1998-1999. Thanks go to Ed Barbeau, Canadian Team Leader to the IMO at Bucharest for collecting them for our use.

16th IRANIAN MATHEMATICAL OLYMPIAD 1998-1999

First Round

1. Suppose that $a_1 < a_2 < \cdots < a_n$ are real numbers. Prove that:

$$a_1a_2^4 + a_2a_3^4 + \cdots + a_{n-1}a_n^4 + a_na_1^4 \;\geq\; a_2a_1^4 + a_3a_2^4 + \cdots + a_na_{n-1}^4 + a_1a_n^4 \,.$$

2. Suppose that n is a natural number. The n–tuple $(a_1, a_2, \ldots, a_n)$ is said to be *good*, if $a_1 + a_2 + \cdots + a_n = 2n$ and furthermore, no subset of $\{a_1, \ldots, a_n\}$ has a sum equal to n. Find all good n–tuples.

3. Let I be the incentre of the triangle ABC and AI meet the circumcircle of ABC at point D. Denote the foot of perpendiculars dropped from I on BD and CD by E and F respectively. If $IE + IF = \frac{1}{2}AD$, find the value of $\angle BAC$.

4. Let ABC be a triangle with $BC > CA > AB$. Select points D on BC and E on the extension of AB such that $BD = BE = AC$. The circumcircle of BED intersects AC at point P and BP meets the circumcircle of ABC at point Q. Show that $AQ + CQ = BP$.

5. Suppose that n is a positive integer and $d_1 < d_2 < d_3 < d_4$ are the four smallest positive integers, dividing n. Find all integers n satisfying $n = d_1^2 + d_2^2 + d_3^2 + d_4^2$.

6. Suppose that $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ are two 0/1 sequences. The distance of A from B is defined to be the number of i for which $a_i \neq b_i$ $(1 \leq i \leq n)$ and is denoted by $d(A, B)$.

Suppose that A, B, C are three $0/1$ sequences and $d(A,B) = d(A,C) = d(B,C) = \delta$.

(a) Prove that δ is an even number.

(b) Prove that there exists a $0/1$ sequence D such that

$$d(D,A) \;=\; d(D,B) \;=\; d(D,C) \;=\; \frac{1}{2}\delta\,.$$

Second Round

1. Define the sequence $\{x_i\}_{i=0}^{\infty}$ by $x_0 = 0$ and,

$$\begin{aligned} x_n &= x_{n-1} + \frac{3^r - 1}{2}\,, \quad \text{if } n = 3^{r-1}(3k+1)\,, \\ x_n &= x_{n-1} - \frac{3^r + 1}{2}\,, \quad \text{if } n = 3^{r-1}(3k+2)\,, \end{aligned}$$

where k and r are integers. Prove that every integer occurs exactly once in this sequence.

2. Suppose that $n(r)$ denotes the number of points with integer coordinates on a circle of radius $r > 1$. Prove that,

$$n(r) < 6\sqrt[3]{\pi r^2}\,.$$

3. Suppose that $ABCDEF$ is a convex hexagon with $AB = BC$, $CD = DE$, and $EF = FA$. Prove that

$$\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} \;\geq\; \frac{3}{2}\,.$$

4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying,

$$f(f(x) + y) \;=\; f(x^2 - y) + 4f(x)y\,,$$

for all real numbers x, $y \in \mathbb{R}$.

5. In triangle ABC, the angle bisector of $\angle BAC$ meets BC at point D. Suppose that Γ is the circle which is tangent to BC at D and passes through the point A. Let M be the second point of intersection of Γ and AC and BM meets the circle at P. Prove that AP is a median of triangle ABD.

6. Suppose that ABC is a triangle. If we paint the points of the plane in red and green, prove that there exist either two red points which are one unit apart or three green points forming a triangle equal to ABC.

Third Round

1. Suppose that $X = \{1, 2, \ldots, n\}$ and $A_1, A_2, \ldots, A_k$ are subsets of X such that for every $1 \le i_1, i_2, i_3, i_4 \le k$, we have,

$$|A_{i_1} \cup A_{i_2} \cup A_{i_3} \cup A_{i_4}| \;\le\; n - 2\,.$$

Prove that $k \le 2^{n-2}$.

2. Suppose that a circle passing through the points A and C of triangle ABC meets AB and BC at points D and E respectively. In the arcwise triangle EBD, inscribe a circle Γ with centre S. Suppose that Γ is tangent to arc DE at point M. Prove that the angle bisector of $\angle AMC$ passes through the incentre of triangle ABC.

3. Suppose that $C_1, \ldots, C_n$ are circles of radius one in the plane such that no two of them are tangent, and the subset of the plane, formed by the union of these circles, is connected. If $S = \{C_i \cap C_j \mid 1 \le i < j \le n\}$, prove that $|S| \ge n$.

4. Suppose that $x_1, \ldots, x_n$ are real numbers and $-1 \le x_i \le 1$ and $x_1 + \cdots + x_n = 0$. Prove that there exists a permutation σ such that for all $1 \le p \le q \le n$, we have,

$$|x_{\sigma(p)} + \cdots + x_{\sigma(q)}| \;\le\; 2 - \frac{1}{n}\,.$$

Prove that the right side cannot be replaced by $2 - \frac{4}{n}$.

5. Suppose that $ABCDEF$ is a convex hexagon with $\angle B + \angle D + \angle F = 360^\circ$ and

$$\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} \;=\; 1\,.$$

Prove that

$$\frac{BC}{CA} \cdot \frac{AE}{EF} \cdot \frac{FD}{DB} \;=\; 1\,.$$

6. Suppose that $r_1, \ldots, r_n$ are real numbers. Prove that there exists $I \subseteq \{1, 2, \ldots, n\}$ such that I meets $\{i, i+1, i+2\}$ in at least one and at most two elements, for $1 \le i \le n-2$ and

$$\left|\sum_{i \in I} r_i\right| \;\ge\; \frac{1}{6} \sum_{i=1}^{n} |r_i|\,.$$

As a final set of problems for the year we give the 1999 Chinese Mathematical Olympiad. Thanks again go to Ed Barbeau, Canadian Team Leader at the IMO at Bucharest, for collecting them for the *Corner*.

1999 CHINESE MATHEMATICAL OLYMPIAD

First Day

January 11, 1999 — Time: 4.5 hours

1. In acute triangle $\triangle ABC$, $\angle ACB > \angle ABC$. Point D is on BC such that $\angle ADB$ is obtuse. Let H be the orthocentre of $\triangle ABD$. Suppose point F is inside $\triangle ABC$ and on the circumcircle of $\triangle ABD$. Prove that point F is the orthocentre of $\triangle ABC$, if and only if HD is parallel to CF and H is on the circumcircle of $\triangle ABC$.

2. For a given real number a, suppose the sequence of real coefficient polynomials $\{f_n(x)\}$ satisfies

$$\begin{cases} f_0(x) &= 1 \\ f_{n+1}(x) &= xf_n(x) + f_n(ax)\,, \quad n = 0, 1, 2, \ldots. \end{cases}$$

(a) Prove that $f_n(x) = x^n f_n(\frac{1}{x})$, $n = 0, 1, 2, \ldots$;

(b) Give explicit formulas for $f_n(x)$.

3. Space city MO consists of 99 space stations. Each two stations are connected by a tube passage. Among all these tubes, 99 are two-way and the others are strictly one-way. For any 4 stations, we call them a strongly connected quadruple if, from each of the 4 stations, one can get to the other 3 through the tubes connecting these 4 stations.

Design a scheme for the space city MO such that it has the maximal number of strongly connected quadruples. (Please give the maximal number and prove your conclusion.)

Second Day

January 12, 1999 — Time: 4.5 hours

4. Let m be a given integer. Prove that there exist integers a, b and k such that both a, b are not divisible by 2, $k \geq 0$, and

$$2m = a^{19} + b^{99} + k \cdot 2^{1999}\,.$$

5. Find the maximal real number λ such that, whenever

$$f(x) = x^3 + ax^2 + bx - c$$

is a real polynomial and all of its roots are non-negative real numbers, we always have

$$f(x) \geq \lambda(x-a)^3, \quad \forall x \geq 0\,.$$

When does equality hold?

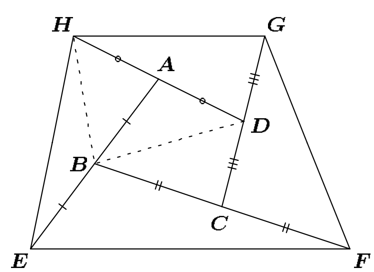
6. A big cube of dimensions $4 \times 4 \times 4$ consists of 64 unit cubes. Paint 16 of the unit cubes with the colour red in such a way that, in each cuboid of size $1 \times 1 \times 4$, $1 \times 4 \times 1$ or $4 \times 1 \times 1$ of the big cube, there exists exactly one red unit cube. What is the total number of ways to do this painting? Give your explanation.

Now we turn to solutions and comments by our readers to problems of the South African Mathematics Olympiad, Section B, September 1995 [1999: 392].

SECTION B.

1. The convex quadrilateral $ABCD$ has area 1, and AB is produced to E, BC to F, CD to G and DA to H, such that $AB = BE$, $BC = CF$, $CD = DG$ and $DA = AH$. Find the area of the quadrilateral $EFGH$.

Solutions by Christopher J. Bradley, Clifton College, Bristol, UK; and by Toshio Seimiya, Kawasaki, Japan. We give the solution by Seimiya.



We denote the area of n-gon $P_1P_2 \ldots P_n$ by $[P_1P_2 \ldots P_n]$. Since B and A are the mid-points of AE and DH respectively, we get $[HEB] = [HAB] = [ABD]$, so that

$$[AEH] = 2[ABD].$$

Similarly we have $[CFG] = 2[CBD]$. Thus,

$$\begin{aligned}[AEH] + [CFG] &= 2[ABD] + 2[CBD] \\ &= 2[ABCD] \\ &= 2.\end{aligned}$$

Similarly, we get

$$[BEF] + [DGH] = 2.$$

Thus, we have

$$\begin{aligned}[EFGH] &= [ABCD] + \{[AEH] + [CFG]\} + \{[BEF] + [DGH]\} \\ &= 1 + 2 + 2 \\ &= 5.\end{aligned}$$

2. Find all pairs (m,n) of natural numbers with $m < n$ such that m^2+1 is a multiple of n and n^2+1 is a multiple of m.

Solutions by Pierre Bornsztein, Pontoise, France; and by Christopher J. Bradley, Clifton College, Bristol, UK. We give Bornsztein's solution and remark.

We will prove that the solutions are the pairs of the form (f_{2n-1}, f_{2n+1}) for $n \in \mathbb{N}$, where $\{f_k\}$ is the Fibonacci sequence ($f_0 = 0$, $f_1 = 1$ and $f_{n+2} = f_n + f_{n+1}$).

Let (m,n) be such that m, $n \in \mathbb{N}^*$, $m < n$, and m^2+1 is a multiple of n and n^2+1 is a multiple of m.

If $m = 1$ then $n \geq 2$ and n divides 2. Thus, $n = 2$.

Conversely, $(1,2)$ is a solution.

Claim: (m,n) is a solution with $m > 1$ if and only if $\left(\frac{m^2+1}{n}, m\right)$ is a solution.

Proof of the Claim: First note that $(m,(m^2+1)) = 1$ and $(n,(n^2+1)) = 1$. It follows that

$$\begin{aligned}
& (m,n) \text{ is a solution} \\
\iff\ & \frac{(m^2+1)(n^2+1)}{mn} \text{ is an integer} \\
\iff\ & \frac{m^2+n^2+1}{mn} \text{ is an integer} \\
\iff\ & \text{there exists } k \in \mathbb{N}^* \text{ such that } m^2+n^2+1 \;=\; kmn \\
\iff\ & \text{there exists } k \in \mathbb{N}^* \text{ such that } n \text{ is a solution of the} \\
& \text{equation } (E_k) : X^2 - kmX + m^2 + 1 \;=\; 0\,.
\end{aligned}$$

In the same way: (m,n) is a solution $\iff$ there exists $k \in \mathbb{N}^*$ such that m is a solution of the equation $X^2 - knX + n^2 + 1 = 0$.

Since (E_k) is a quadratic equation with integer coefficients and leading coefficient equal to 1, one of its solutions is an integer if and only if the other is as well. Moreover n is a solution of (E_k) if and only if $\frac{m^2+1}{n}$ is a solution of (E_k) (from the product of the roots of (E_k)).

It follows that (m,n) is a solution $\iff$ there exists $k \in \mathbb{N}^*$ such that $\frac{m^2+1}{n}$ is a solution of (E_k).

- If $m = 1$ then $n = 2$ and $\frac{m^2+1}{n} = m$.
- If $m > 1$ then, since $m + 1 \leq n$, we have

$$mn \;\geq\; m(m+1) \;>\; m^2+1\,.$$

Thus, $\frac{m^2+1}{n} < m$.

We deduce that (m, n) is a solution with $m > 1 \iff \left(\frac{m^2+1}{n}, m\right)$ is a solution, and the claim is proved.

Let (m, n) be a solution, with $m > 1$. From the claim, we may find another solution (m', n') with $m' < n' = m < n$. Repeating this process, we construct a sequence (m_i, n_i) of solutions, and the sequence $\{n_i\}$ is a strictly decreasing sequence of positive integers. This sequence has to be finite. But the only way to stop the process is to obtain $m_i = 1$ (and then $n_i = 2$) for some k.

It follows that each solution of the problem is generated from $(1, 2)$ by using a finite number of applications of the function $(m, n) \to \left(n, \frac{n^2+1}{m}\right)$.

Conversely, if $m_1 = 1$ and $n_1 = 2$, and for $n \in \mathbb{N}^*$,

$$\begin{cases} m_{i+1} = n_i \,, \\ n_{i+1} = \frac{n_i^2+1}{m_i} \,. \end{cases} \tag{1}$$

From the claim, we know that (m_i, n_i) is a solution for all $i \geq 1$.

Then, the solutions of the problem are pairs (m_i, n_i) defined by (1), for all $i \geq 2$: $n_{i+1} = \frac{n_i^2+1}{m_i} = \frac{n_i^2+1}{n_{i-1}}$; that is

$$n_{i+1}n_{i-1} - 1 = n_i^2 \,. \tag{2}$$

It looks like a well-known property of the Fibonacci sequence

$$f_n^2 = f_{n+1}f_{n-1} + (-1)^{n-1}$$

where $f_0 = 0$, $f_1 = 1$, $f_{n+2} = f_n + f_{n+1}$. It is also well known that for all $n \geq 0$

$$f_n = \frac{1}{\sqrt{5}}\left(\varphi^n - \left(-\frac{1}{\varphi}\right)^n\right) \quad \text{where} \quad \varphi = \frac{1+\sqrt{5}}{2} \tag{3}$$

Using (3) it is not difficult to see that f_{2k+1} satisfies (2). Since $f_1 = m_1$ (if we define $n_0 = m_1$ we then have $n_0 = f_1$), and $f_3 = n_1$, $f_5 = n_2$, it follows that for all $i \geq 1$

$$n_i = f_{2i+1}$$

and then

$$m_i = n_{i-1} = f_{2i-1} \,,$$

And we are done.

Comments: A generalization of this problem may be found in [1]:

"Let $q \in \mathbb{N}^*$. Let $\mathcal{P}$ be the set of pairs (m, n) of coprime positive integers m, n such that m divides $n^2 + q^2$ and n divides $m^2 + q^2$, with $m \geq n$.

Let f_a be the a^{th} generalized Fibonacci number ($f_0 = 0$, $f_1 = 1$, $f_{n+2} = qf_{n+1} + pf_n$, where $p = 1$ in this case).

Then:

- if a is odd, $(f_a, f_{a-2}) \in \mathcal{P}$; and
- $\mathcal{P} = \{(f_a, f_{a-2}) : a \text{ odd}\}$ if and only if the only pairs $(m, n) \in \mathcal{P}$ with $n \le q^2$ are the pairs

$$(1, 1)\,, \quad (q^2 + 1, 1)\,."$$

Reference:

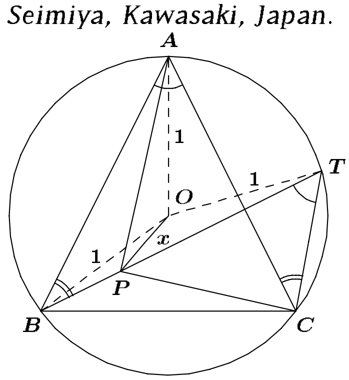
1. P. Hilton, J. Pedersen, *A fresh look at old favourites: the Fibonacci and Lucas sequences revisited*, *Australian Mathematical Society Gazette*, Vol 25, no 3 (August 1998), p. 146–160.

3. The circumcircle of $\triangle ABC$ has radius 1 and centre O, and P is a point inside the triangle such that $OP = x$. Prove that

$$AP \cdot BP \cdot CP \;\le\; (1 + x)^2 (1 - x)\,,$$

with equality only if $P = O$.

Solution by Toshio Seimiya, Kawasaki, Japan.



If we assume that

$$\angle PBA + \angle PCA \;<\; \angle BAC\,, \tag{1}$$

that

$$\angle PAB + \angle PCB \;<\; \angle ABC\,, \tag{2}$$

and that

$$\angle PAC + \angle PBC \;<\; \angle ACB\,, \tag{3}$$

then, adding (1) and (2) to (3), we have

$$\angle BAC + \angle ABC + \angle ACB \;<\; \angle BAC + \angle ABC + \angle ACB\,.$$

This is a contradiction. Therefore, (1), (2) and (3) do not hold simultaneously, so that at least one of them is not true.

Without loss of generality we may assume that (1) is not true, so that we have

$$\angle PBA + \angle PCA \;\geq\; \angle BAC\,.$$

Let T be the second intersection of BP with the circumcircle of $\triangle ABC$. Then, we have $\angle BTC = \angle BAC$ and $\angle ACT = \angle ABT$.

Thus, $\angle PCT = \angle ACP + \angle ACT = \angle ACP + \angle ABT = \angle ACP + \angle ABP \geq \angle BAC = \angle BTC = \angle PTC$; that is, $\angle PCT \geq \angle PTC$. Thus, we have $PT \geq CP$. Since

$$BP \cdot PT \;=\; OT^2 - OP^2 \;=\; 1 - x^2\,,$$

we have

$$BP \cdot CP \;\leq\; BP \cdot PT \;=\; 1 - x^2\,. \tag{4}$$

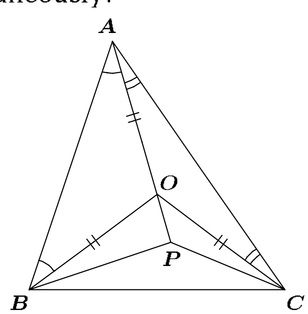
Since

$$AP \;\leq\; AO + OP \;=\; 1 + x\,, \tag{5}$$

we obtain, from (4) and (5), that

$$AP \cdot BP \cdot CP \;\leq\; (1+x)(1-x^2) \;=\; (1+x)^2(1-x)\,. \tag{6}$$

Next, we consider when equality in (6) holds. This is when equalities in (4) and (5) hold simultaneously.



In (5), equality holds if and only if P is a point on AO produced beyond O or $P = O$.

If $\triangle ABC$ is not an acute triangle, O is not an interior point of $\triangle ABC$, so that equality in (5) does not hold.

When $\triangle ABC$ is an acute triangle, let P be a point on AO produced beyond O. Then $\angle ABP + \angle ACP > \angle ABO + \angle ACO = \angle BAO + \angle CAO = \angle BAC$. Thus, $BP \cdot CP < 1 - x^2$. Thus, equality in (4) does not hold.

If $P = O$, in (4), (5) and (6), then all equalities hold.

Therefore, in (6) equality holds if and only if $P = O$.

Remark by Pierre Bornsztein, Pontoise, France.
This problem is solved in American Mathematical Monthly 1995, problem 10282, p. 468. In fact, the problem of the Monthly is, with notations as above:

"Show that $PA \cdot PB \cdot PC < \frac{32}{27}$."

by Paul Erdős.

Note that $\frac{32}{27}$ is the maximum value of $(1 + x)^2(1 - x)$ in the interval $[0, 1]$. The editor comments

"A generalization to an n–gon inscribed in a circle was obtained. The bound in this case is the maximum value of $(1-x)(1+x)^{n-1}$, which is $(\frac{2}{n})^n(n - 1)^{n-1}$."

Now we turn to solutions to problems of the Taiwan Mathematical Olympiad 1996 [1999 : 392–393].

1. Let the angles α, β, γ be such that $0 < \alpha, \beta, \gamma < \frac{\pi}{2}$ and $\alpha + \beta + \gamma = \frac{\pi}{4}$. Suppose that

$$\tan\alpha = \frac{1}{a}, \quad \tan\beta = \frac{1}{b}, \quad \tan\gamma = \frac{1}{c},$$

where a, b, c are positive integers. Determine the values of a, b, c.

Solutions by Pierre Bornsztein, Pontoise, France; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give Klamkin's solution.

Since

$$\begin{aligned} 1 = \tan\frac{\pi}{4} &= \tan(\alpha+\beta+\gamma) \\ &= \frac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1 - \tan\beta\tan\gamma - \tan\gamma\tan\alpha - \tan\alpha\tan\beta} \\ &= \frac{bc + ca + ab - 1}{abc - a - b - c}, \end{aligned}$$

we solve for b and get

$$b = 1 + \frac{2a + 2c}{(c-1)a - (c+1)}.$$

We now successively let $c = 2, 3, \ldots$, and determine a such that b is an integer. These are given only by permutations of

$$(c, a, b) = (2, 4, 13), \ (2, 5, 8), \quad \text{and} \quad (3, 3, 7).$$

2. Let a be a real number such that $0 < a \le 1$ and $a \le a_j \le \frac{1}{a}$, for $j = 1, 2, \ldots, 1996$. Show that for any non-negative real numbers λ_j ($j = 1, 2, \ldots, 1996$), with

$$\sum_{j=1}^{1996} \lambda_j \;=\; 1\,,$$

one has

$$\left(\sum_{i=1}^{1996} \lambda_i a_i\right)\left(\sum_{j=1}^{1996} \lambda_j a_j^{-1}\right) \;\le\; \frac{1}{4}\left(a+\frac{1}{a}\right)^2 .$$

Solutions by Michel Bataille, Rouen, France; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea. We give the solution by Bataille.

Let $n = 1996$ (or n be any positive integer, actually). From the AM–GM Inequality,

$$\begin{aligned}\left(\sum_{i=1}^{n} \lambda_i a_i\right)^{1/2}\left(\sum_{j=1}^{n} \lambda_j a_j^{-1}\right)^{1/2} &\le \frac{1}{2}\left(\sum_{i=1}^{n} \lambda_i a_i + \sum_{j=1}^{n} \lambda_j a_j^{-1}\right)\\ &= \frac{1}{2}\left(\sum_{j=1}^{n} \lambda_j\left(a_j+\frac{1}{a_j}\right)\right) .\end{aligned}$$

Now, if t satisfies $a \le t \le \frac{1}{a}$, then $\left(t+\frac{1}{t}\right) - \left(a+\frac{1}{a}\right) = (t-a)\left(1-\frac{1}{at}\right) \le 0$, so that $t+\frac{1}{t} \le a+\frac{1}{a}$. It follows that $a_j + \frac{1}{a_j} \le a + \frac{1}{a}$ ($j = 1, 2, \ldots, n$) and, since $\lambda_j \ge 0$, $\sum_{j=1}^{n} \lambda_j\left(a_j+\frac{1}{a_j}\right) \le \left(a+\frac{1}{a}\right)\sum_{j=1}^{n}\lambda_j = a+\frac{1}{a}$.

From this, we get

$$\left(\sum_{i=1}^{n} \lambda_i a_i\right)^{1/2}\left(\sum_{j=1}^{n} \lambda_j a_j^{-1}\right)^{1/2} \;\le\; \frac{1}{2}\left(a+\frac{1}{a}\right)$$

and the required result by squaring both sides.

Generalization: Replacing the hypothesis on the a_i by $0 < m \le a_i \le M$, we get the following inequality:

$$\left(\sum_{i=1}^{n} \lambda_i a_i\right)\left(\sum_{j=1}^{n} \lambda_j a_j^{-1}\right) \;\le\; \frac{1}{4}\frac{(m+M)^2}{mM} .$$

It suffices to remark that $\sqrt{\frac{m}{M}} \le \frac{a_i}{\sqrt{mM}} \le \sqrt{\frac{M}{m}}$ and apply the result above with $a = \sqrt{\frac{m}{M}}$ and $\frac{a_i}{\sqrt{mM}}$ instead of a_i.

We also give the "quick" solution of Klamkin.

The result follows immediately by applying the following inequality of Polya and Szego [1]:

$$4\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) \;\leq\; \left(\sqrt{\frac{M_1M_2}{m_1m_2}} \;+\; \sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2 \left(\sum_{k=1}^{n} a_kb_k\right)^2$$

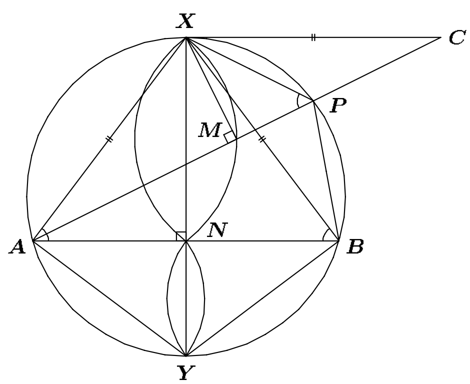
where $0 < m_1 \leq a_k \leq M_1$, $0 < m_2 \leq b_k \leq M_2$ for $k = 1, 2, \ldots, n$.

Reference:

1. D.S. Mitrinovic, *Analytic Inequalities*, Springer-Verlag, Heidelberg, 1970, p. 60.

3. Let A and B be two fixed points on a fixed circle. Let a point P move on this circle and let M be a corresponding point such that either M is on the segment PA with $AM = MP + PB$ or M is on the segment PB with $AP + MP = PB$. Determine the locus of such points P.

Solution by Toshio Seimiya, Kawasaki, Japan.



The perpendicular bisector of AB meets the circle at X and Y. Then $XA = XB$ and $YA = YB$.

Let P be a point on the minor arc XB. Then $PA \geq PB$.

Let C be a point on AP produced beyond P such that $PC = PB$. Since $AM = MP + PB = MP + PC = MC$, M is the mid-point of AC.

Since $\angle XPA = \angle XBA = \angle XAB$, we have

$$\angle XPC \;=\; 180^\circ - \angle XPA \;=\; 180^\circ - \angle XAB \;=\; \angle XPB\,.$$

Since $PC = PB$ and $XP = XP$ we get $\triangle XPC \equiv \triangle XPB$. Hence, $XC = XB = XA$.

Since M is the mid-point of AC we have $XM \perp AC$; that is, $\angle XMA = 90^\circ$.

Let N be the mid-point of AB. Then $\angle XNA = 90^\circ$.

If P moves on the minor arc BX from B to X, then M moves on the minor arc NX of the circle with diameter XA. Similarly, if P moves on the minor arc XA from X to A, then M moves on the minor arc XN of the circle with diameter XB.

And if P moves on the minor arc AY, then m moves on the minor arc NY of the circle with diameter BY.

And if P moves on the minor arc YB, then M moves on the minor arc NY of the circle with diameter AY.

Thus, the locus of M is the union of the four minor arcs as shown in the figure as a shape of a "figure-eight" loop.

4. Show that for any real numbers a_3, a_4, ..., a_{85}, the roots of the equation

$$a_{85}x^{85} + a_{84}x^{84} + \cdots + a_3x^3 + 3x^2 + 2x + 1 \;=\; 0$$

are not real.

Solutions by Pierre Bornsztein, Pontoise, France; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give Bornsztein's write-up.

Let $P(x) = a_{85}x^{85} + \cdots + a_3x^3 + 3x^2 + 2x + 1$. Since $P(0) = 1$, then 0 is not a root of P.

Let r_1, ..., r_{85} be the complex roots of P.

For $i = 1, \ldots, 85$, denote $s_i = \frac{1}{r_i}$. Then the s_i's are the complex roots of the polynomial $Q(y) = y^{85} + 2y^{84} + 3y^{83} + a_3y^{82} + \cdots + a_{84}y + a_{85}$. It follows that

$$\sum_{i=1}^{85} s_i \;=\; -2 \quad \text{and} \quad \sum_{i<j} s_is_j \;=\; 3\,.$$

Then

$$\sum_{i=1}^{85} s_i^2 \;=\; \left(\sum_{i=1}^{85} s_i\right)^2 - 2\sum_{i<j} s_is_j \;=\; -2 \;<\; 0\,.$$

Thus, the s_i's are not all real, and then the r_i's are not all real.

Remark. The conclusion holds for all real numbers a_0, a_1, a_2 such that $a_0 \neq 0$ and $a_1^2 < 2a_0a_2$.

6. Let q_0, q_1, q_2, ... be a sequence of integers such that

(a) for any $m > n$, $m - n$ is a factor of $q_m - q_n$, and

(b) $|q_n| \leq n^{10}$ for all integers $n \geq 0$.

Show that there exists a polynomial $Q(x)$ satisfying $Q(n) = q_n$ for all n.

Comment by Pierre Bornsztein, Pontoise, France.

This is a particular case of problem 4 of the 1995 USAMO, where it was asked:

> "Suppose q_0, q_1, ... is an infinite sequence of integers satisfying the following two conditions:
>
> (i) $m - n$ divides $q_m - q_n$ for $m > n \geq 0$.
>
> (ii) there is a polynomial P such that $|q_n| < P(n)$ for all n.
>
> Show there is a polynomial Q such that $q_n = Q(n)$ for all n."

Reference:

1. *Math. Magazine*, Vol 69, no 3, June 1996, p. 235.

To complete this number of the *Corner* we turn to solutions of problems of the Croatian National Mathematics Competition, IV Class [1999 : 393–394].

1. Is there any solution of the equation

$$\lfloor x\rfloor + \lfloor 2x\rfloor + \lfloor 4x\rfloor + \lfloor 8x\rfloor + \lfloor 16x\rfloor + \lfloor 32x\rfloor \;=\; 12345\,?$$

($\lfloor x\rfloor$ denotes the greatest integer which does not exceed x.)

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give Bornsztein's solution.

The equation

$$\lfloor x\rfloor + \lfloor 2x\rfloor + \lfloor 4x\rfloor + \lfloor 8x\rfloor + \lfloor 16x\rfloor + \lfloor 32x\rfloor \;=\; 12345 \qquad (1)$$

has no real solution. Suppose, for a contradiction, that $x \in \mathbb{R}$ satisfies (1).

Then $x > 0$ and we may write

$$x \;=\; N + \frac{a}{2} + \frac{b}{4} + \frac{c}{8} + \frac{d}{16} + \frac{e}{32} + f$$

where N is a non-negative integer,

$$a,\, b,\, c,\, d,\, e \in \{0,\, 1\} \quad \text{and} \quad f \in \left[0,\, \tfrac{1}{32}\right)\,.$$

From (1) we obtain

$$63N + 31a + 15b + 7c + 3d + e \;=\; 12345\,.$$

Thus,

$$63N \leq 12345 \leq 63N + 31 + 15 + 7 + 3 + 1\,.$$

That is,

$$\frac{12288}{63} \leq N \leq \frac{12345}{63}\,.$$

Then, $195 < N < 196$, which is impossible if N is supposed to be an integer. Thus, (1) has no solution, as claimed.

2. Determine all pairs of numbers λ_1, $\lambda_2 \in \mathbb{R}$ for which every solution of the equation

$$(x + i\lambda_1)^n + (x + i\lambda_2)^n = 0$$

is real. Find the solutions.

Solutions by Michel Bataille, Rouen, France; and by Pierre Bornsztein, Pontoise, France. We give Bataille's solution.

Let x be any complex number solution. Then, $(x+i\lambda_1)^n = -(x+i\lambda_2)^n$ so that

$$|x + i\lambda_1|^n = |x + i\lambda_2|^n \quad \text{and} \quad |x + i\lambda_1| = |x + i\lambda_2|\,. \qquad (1)$$

Denote by A_1, A_2 the points in the complex plane corresponding to $-i\lambda_1$, $-i\lambda_2$ respectively. The relation (1) means that $MA_1 = MA_2$, where M corresponds to x. Thus, M is on the perpendicular bisector Δ of segment A_1A_2. Note that Δ is a line perpendicular to the imaginary axis. It follows that every solution is a real number if and only if Δ coincides with the real axis; that is, $\lambda_1 + \lambda_2 = 0$.

Conversely, suppose $\lambda_1 = -\lambda_2 = \lambda \in \mathbb{R}$. The given equation becomes $(x+i\lambda)^n = -(x-i\lambda)^n$. If $\lambda = 0$, then $x = 0$ is the only solution. Assuming now that $\lambda \neq 0$, our equation is equivalent to:

$$\left(\frac{x + i\lambda}{x - i\lambda}\right)^n = -1 \quad \text{or} \quad \frac{x + i\lambda}{x - i\lambda} = u_k$$

where $u_k = \exp(i(\frac{\pi}{n} + \frac{2k\pi}{n}))$ for $k = 0, 1, \ldots, n-1$. This gives $x = i\lambda\frac{u_k+1}{u_k-1}$ or, by an easy computation $x = \lambda\cot\left(\frac{(2k+1)\pi}{2n}\right)$.

Thus, the solutions are the n real numbers $x_k = \lambda\cot\left(\frac{(2k+1)\pi}{2n}\right)$, $(k = 0, 1, \ldots, n-1)$.

3. Determine all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ continuous at 0, which satisfy the following relation

$$f(x) - 2f(tx) + f(t^2x) = x^2 \quad \text{for all} \quad x \in \mathbb{R}\,,$$

where $t \in (0, 1)$ is a given number.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; and by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea. We give the write-up by Lee.

First, we introduce the function $g(x) = f(x) - f(tx)$ defined on $\mathbb{R}$. Then g is also continuous at 0.

Since $g(tx) = f(tx) - f(t^2x)$, we easily get $g(x) - g(tx) = x^2$. Therefore, we have

$$g(x) - g(t^{n+1}x) \;=\; \sum_{i=1}^{n+1}\{g(t^{i-1}x) - g(t^ix)\} \;=\; \sum_{i=1}^{n+1}(t^{i-1}x)^2\,,$$

or

$$g(x) \;=\; g(t^{n+1}x) + \left(\sum_{i=1}^{n+1}(t^2)^{i-1}\right)x^2\,.$$

We note that $\lim_{n\to\infty} t^{n+1} = 0$ since $t \in (0,1)$. By the continuity of g at 0, we have

$$\begin{aligned} g(x) &= \lim_{n\to\infty} g(t^{n+1}x) + \lim_{n\to\infty}\left(\sum_{i=1}^{n+1}(t^2)^{i-1}\right)x^2 \\ &= g(0) + \frac{1}{1-t^2}x^2\,. \end{aligned}$$

Since $g(0) = f(0) - f(t\cdot 0) = 0$, we obtain $g(x) = \frac{1}{1-t^2}x^2$. Thus, we have $f(x) - f(tx) = \frac{1}{1-t^2}x^2$.

Also, we get

$$f(x) - f(t^{n+1}x) \;=\; \sum_{i=1}^{n+1}\{f(t^{i-1}x) - f(t^ix)\} \;=\; \frac{1}{1-t^2}\sum_{i=1}^{n+1}(t^{i-1}x)^2\,.$$

By the continuity of f at 0, we have

$$f(x) \;=\; \lim_{n\to\infty} f(t^{n+1}x) + \frac{1}{1-t^2}\lim_{n\to\infty}\sum_{i=1}^{n+1}(t^{i-1}x)^2$$

or

$$f(x) \;=\; f(0) + \frac{1}{(1-t^2)^2}x^2\,.$$

Therefore, $f(x) = \frac{1}{(1-t^2)^2}x^2 + C$, $(C \in \mathbb{R})$.

4. Let α and β be positive irrational numbers such that $\frac{1}{\alpha}+\frac{1}{\beta}=1$ and $A=\{\lfloor n\alpha\rfloor \mid n\in\mathbb{N}\}$, $B=\{\lfloor n\beta\rfloor \mid n\in\mathbb{N}\}$. Prove that $A\cup B=\mathbb{N}$ and $A\cap B=\emptyset$.

Remark: You can prove the following equivalent assertion: For a function $\pi:\mathbb{N}\to\mathbb{N}$ defined by

$$\pi(m) = \text{Card}\{k \mid k\in\mathbb{N},\, k\le m, k\in A\}+\text{Card}\{k \mid k\in\mathbb{N},\, k\le m,\, k\in B\}$$

one has $\pi(m)=m$, $\forall m\in\mathbb{N}$. ($\lfloor x\rfloor$ denotes the greatest integer which does not exceed x.)

Comment by Pierre Bornsztein, Pontoise, France.

This problem is well known as Beatty's problem. See [1], [2] or [3] for an elementary proof and related results.

It is also well known that the converse is true (see [3]).

For a positive real number x define $S(x)=\{\lfloor nx\rfloor : n\in\mathbb{N}^*\}$.

Then a necessary and sufficient condition for $\mathbb{N}^*$ to be the disjoint union of $S(\alpha)$ and $S(\beta)$ is that α and β are irrationals such that $\frac{1}{\alpha}+\frac{1}{\beta}=1$.

Moreover, Uspensky proved in 1927 that:

> "There do not exist three or more positive numbers α_1, α_2, ..., α_n such that $\mathbb{N}^*$ is the union of the pairwise disjoint sets $S(\alpha_1)$, $S(\alpha_2)$, ..., $S(\alpha_n)$."

For the special case $n=3$, see [4]. For the general case, see [3].

References:

1. Ross Honsberger, *Ingenuity in Mathematics*, M.A.A. p. 93–110.

2. A.M. Gleason, R.E. Greenwood, L.M. Kelly, *The William Lowell Putnam Mathematical Competition — Problems and Solutions 1938–1964*, M.A.A., Afternoon session 1959 problem 6, p. 513.

3. Joe Roberts, *Elementary Number Theory: A Problem Oriented Approach*, M.I.T. Press 1977, p. 38–45 and 475–585.

4. *The William Lowell Putnam Mathematical Competition 1995*, *American Mathematical Monthly* 1996, pp. 676–677 (problem B.6).

Comment by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

This problem has quite a history and appeared as problem 6 in the 20th Putnam Competition, 1959 [1]. Although $A\cap B=\emptyset$ was not asked for, it was part of the solution. At the end of the given solution is the following remark:

This is sometimes called Beatty's problem, after Samuel Beatty (1881–1970). In a slightly different form it appeared as Problem 3117, *American Mathematical Monthly*, Vol. 34 (1927), pp. 158–159. Howard Grossman, *A*

Set Containing All Integers, American Mathematical Monthly, Vol. 69 (1962), pp. 532–533, gives a proof by analyzing lattice points. A.S. Fraenkel, *The Bracket Function and Complementary Sets of Integers, Canadian Journal of Mathematics*, Vol. 21 (Jan. 1969), pp. 6–27, gives a history, a bibliography, and a generalization of the problem.

Reference:

1. A.M. Gleason, R.E. Greenwood, L.M. Kelly, *The William Lowell Putnam Mathematical Competition: Problems and Solutions 1938–1964*, M.A.A., Washington, D.C., 1980, pp. 513–514.

As a final solutions set this issue, we present the solutions to problems posed at the Additional Competition (Croatian National Mathematica Competition) for Selection of the IMO Team [1999 : 394].

1. (a) $n = 2k+1$ points are given in the plane. Construct an n–gon such that these points are mid-points of its sides.

(b) Arbitrary $n = 2k$, $k > 1$, points are given in the plane. Prove that it is impossible to construct an n–gon, in each case, such that these points are mid-points of its sides.

Solutions by Michel Bataille, Rouen, France; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give Bataille's solution.

We will denote by H_A the half-turn around point A and by $T_{\mathbf{U}}$ the translation with vector $\mathbf{U}$. Note the following formula: $H_B \circ H_A = T_{\mathbf{U}}$ where $\mathbf{U} = 2\overrightarrow{AB}$.

Suppose now that A_1, A_2, ..., A_n are the mid-points of the sides of the n–gon $M_1M_2\ldots M_n$ (with A_1 the mid-point of M_1M_2, etc. ...). Then

$$\begin{aligned} M_1 &= H_{A_n}(M_n) = H_{A_n} \circ H_{A_{n-1}}(M_{n-1}) \\ &= \cdots = H_{A_n} \circ H_{A_{n-1}} \circ \cdots \circ H_{A_1}(M_1) \end{aligned}$$

so that M_1 is invariant under the transformation $H = H_{A_n} \circ H_{A_{n-1}} \circ \cdots \circ H_{A_1}$.

If $n = 2k$, then $H = T_{\mathbf{U}_k} \circ T_{\mathbf{U}_{k-1}} \circ \cdots \circ T_{\mathbf{U}_1}$ where $\mathbf{U}_i = 2\overrightarrow{A_{2t-1}A_{2t}}$ $(i = 1, \ldots, k)$. Hence, H is the translation with vector $\mathbf{U} = \mathbf{U}_1 + \cdots + \mathbf{U}_k$. Unless $\mathbf{U} = \overrightarrow{0}$, H has no invariant point so that no n–gon $M_1M_2\ldots M_n$ can be obtained. (If $\mathbf{U} = \overrightarrow{0}$, M_1 can be any point in the plane and we obtain an infinity of solutions; see figure 2.)

If $n = 2k+1$, then

$$\begin{aligned} H &= H_{A_n} \circ T_{\mathbf{U}} \quad \text{(with the notation above)} \\ &= H_{A_n} \circ H_{A_n} \circ H_B \quad \text{where } B \text{ is such that } 2\overrightarrow{BA_n} = \mathbf{U} \\ &= H_B\,. \end{aligned}$$

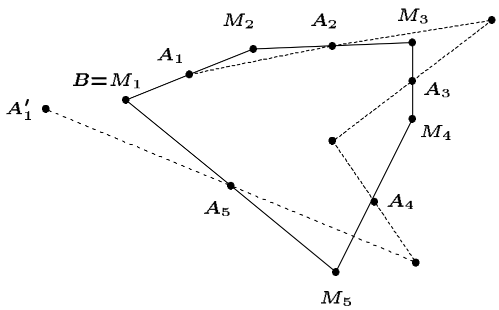


figure 1 (for $n = 5$)

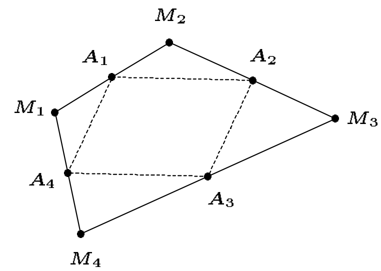


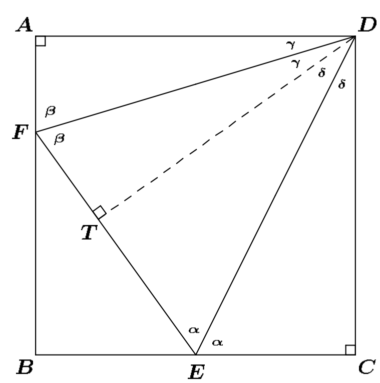
figure 2 (for $n = 4$ and $U = \overrightarrow{0}$)

Hence, H is the half-turn around B and, necessarily, $M_1 = B$. Note that B is easily constructed: just construct $A'_1 = H(A_1)$ and B is the mid-point of $A_1A'_1$.

Conversely, taking $M_1 = B$ and constructing successively the points $M_2 = H_{A_1}(M_1), \ldots, M_n = H_{A_{n-1}}(M_{n-1})$, we obtain a suitable n-gon $M_1M_2\ldots M_n$ (since $H_{A_n}(M_n) = H(M_1) = M_1$). Thus, when n is odd, there is a unique solution (see figure 1).

2. The side-length of the square $ABCD$ equals a. Two points E and F are given on sides $\overline{BC}$ and $\overline{AB}$ such that the perimeter of the triangle BEF equals $2a$. Determine the angle $\angle EDF$.

Solutions by Pierre Bornsztein, Pontoise, France; by Christopher J. Bradley, Clifton College, Bristol, UK; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; by Toshio Seimiya, Kawasaki, Japan. We give the solution by Seimiya.



Since $BA = BC = a = \frac{1}{2}(BE + EF + FB)$, the excircle of $\triangle BEF$ opposite to B touches BA and BC at A and C respectively. Since $DA \perp AB$

and $DC \perp BC$, D is the excentre, so that DF and DE are the bisectors of $\angle AFE$ and $\angle FEC$, respectively.

Let T be the foot of the perpendicular from D to EF. Since $\angle AFD = \angle TFD$, and $\angle DAF = \angle DTF$ $(= 90°)$, we get

$$\angle ADF = \angle TDF .$$

Similarly we have $\angle CDE = \angle TDE$.

Therefore, $\angle ADF + \angle CDE = \angle TDF + \angle TDE = \angle EDF$. Thus, $\angle ADC = 2\angle EDF$.

Therefore, $\angle EDF = \frac{1}{2}\angle ADC = 45°$.

3. Find all pairs of consecutive integers the difference of whose cubes is a full square.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; by Christopher J. Bradley, Clifton College, Bristol, UK; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We use Bornsztein's write-up.

Let a, b be two integers.

$$\begin{aligned} (a+1)^3 - a^3 = b^2 &\iff 3a^2 + 3a + 1 = b^2 && (1) \\ &\iff 3(4a^2 + 4a + 1) + 1 = 4b^2 \\ &\iff (2b)^2 - 3(2a+1)^2 = 1 \\ &\iff (2b, 2a+1) \quad \text{is a solution} \end{aligned}$$

of Pell's equation

$$X^2 - 3Y^2 = 1 . \qquad (2)$$

The minimal non-trivial solution of (2) is $(2, 1)$. It is then well known that the solutions of (2) are the pairs $(\pm x_n, \pm y_n)$ where $x_0 = 1$, $y_0 = 0$, and for all $n \geq 0$

$$\left\{ \begin{array}{lcl} x_{n+1} & = & 2x_n + 3y_n \\ y_{n+1} & = & 2y_n + x_n \end{array} \right. .$$

But we want only those with x_n even and y_n odd.

It is easy to see that if x_n is even and y_n is odd then x_{n+1} is odd and y_{n+1} is even, and then x_{n+2} is even and y_{n+2} is odd.

Thus, since $x_1 = 2$ and $y_1 = 1$, we consider only the pairs (x_{2n+1}, y_{2n+1}). Since, for all $n \geq 0$

$$\left\{ \begin{array}{lcl} x_{n+2} & = & 7x_n + 12y_n \\ y_{n+2} & = & 4x_n + 7y_n \end{array} \right. .$$

Then, the solutions of (1) are the pairs (a, b) of the form

$$\left(\frac{-1 \pm V_n}{2}, \pm \frac{U_n}{2} \right) \quad \text{where} \quad U_1 = 2 , V_1 = 1$$

and for all $n \geq 1$

$$\begin{cases} U_{n+1} &= 7U_n + 12V_n \\ V_{n+1} &= 4U_n + 7V_n \end{cases} .$$

For example, we first note that $(a+1)^3 - a^3 = b^2$ if and only if $(-a)^3 - (-a-1)^3 = b^2$. Then we give only the first positive values of a and b.

n	U_n	V_n	a	b
1	2	1	0	1
2	26	16	7	13
3	362	209	104	181
4	5042	2911	1455	2521
5	70226	40545	20272	35113
⋮				

4. Let A_1, A_2, ..., A_n be a regular n–gon inscribed in the circle of radius 1 with the centre at O. A point M is given on the ray OA_1 outside the n–gon. Prove that

$$\sum_{k=1}^{n} \frac{1}{|MA_k|} \geq \frac{n}{|OM|} .$$

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We first give the solution by Bataille.

We may suppose that a system of coordinates has been chosen so that the complex numbers associated to O, A_1, A_2, ..., A_n, M are respectively 0, 1, u, ..., u^{n-1}, r, where $u = \exp(\frac{2\pi i}{n})$ and r is a real number > 1.

Note that 1, u, ..., u^{n-1} are the n^{th} roots of unity. Hence we have the identity

$$z^n - 1 = (z-1)(z-u)\ldots(z-u^{n-1}) . \qquad (1)$$

Now,

$$\begin{aligned} \frac{1}{n}\sum_{k=1}^{n} \frac{1}{|MA_k|} &= \frac{1}{n}\sum_{k=0}^{n-1} \frac{1}{|r-u^k|} \\ &\geq \sqrt[n]{\frac{1}{|r-1|}\cdot\frac{1}{|r-u|}\cdot \cdots \cdot \frac{1}{|r-u^{n-1}|}} \quad \text{(by AM-GM)} \\ &= \sqrt[n]{\frac{1}{|r^n-1|}} \quad \text{(using (1))} \\ &= \sqrt[n]{\frac{1}{r^n-1}} > \sqrt[n]{\frac{1}{r^n}} = \frac{1}{r} = \frac{1}{|OM|} . \end{aligned}$$

The result follows. [Ed.: Note that the proof gives a strict inequality.]

Next we give Klamkin's solution that provides some interesting connections.

By applying the AM-GM Inequality, it suffices to show that

$$|OM|^n \;\geq\; \prod_{k=1}^{n} |MA_k|\,. \qquad (1)$$

This will follow from *de Moivre's* property; that is, "if $A_0A_1A_2 \ldots A_{n-1}$ is a regular polygon inscribed in a circle centre O, radius a, and P is a point such that $OP = x$, $\angle(OA_0, OP) = \theta$ then

$$\angle(OA_r, OP) \;=\; \theta + \frac{2r\pi}{n} \quad \text{and} \quad PA_r^2 \;=\; x^2 + a^2 - 2xa\cos\left(\theta + \frac{2r\pi}{n}\right).$$

Also,

$$PA_0^2 \cdot PA_1^2 \cdots PA_{n-1}^2 \;=\; \prod_{r=0}^{n-1}\left(x^2 - 2xa\cos\left(\theta + \frac{2r\pi}{n}\right) + a^2\right)$$

or

$$PA_0 \cdot PA_1 \cdots PA_{n-1} \;=\; \sqrt{x^{2n} - 2x^n a^n \cos n\theta + a^{2n}}\,.$$

If P lies on OA_0 so that $\theta = 0$, then $PA_0 \cdot PA_1 \cdots PA_{n-1} = |x^n - a^n|$. If OP bisects $\angle A_{n-1}OA_0$, so that $\theta = \frac{\pi}{n}$, then $PA_0 \cdot PA_1 \cdots PA_{n-1} = x^n + a^n$. These special results are called *Cotes'* properties".

Applying this to (1), we get $|OM|^n \geq |OM|^n - 1$, from which the result is now obvious.

For a simple derivation of the first *Cotes'* property using complex numbers, see the solution of problem 2 of the 15$^{\text{th}}$ Putnam Competition, 1955 [1].

Reference:

1. A.M. Gleason, R.E. Greenwood, L.M. Kelly, *The William Lowell Putnam Mathematical Competition: Problems and Solutions 1938–1964*, M.A.A., Washington, D.C., 1980, p. 403.

That completes the *Corner* for this number. Send me your contests, nice solutions, and comments.

BOOK REVIEWS

ALAN LAW

Mathematical Puzzle Tales, by Martin Gardner,
with a foreword by Isaac Asimov,
published by the Mathematical Association of America, 2000,
ISBN: 0-88385-533-X, softcover, 151+xiii pages, $22.50 (U.S.).
Reviewed by **Edward J. Barbeau**, *University of Toronto, Toronto, Ontario.*

Since its inaugural issue in the spring of 1977, each issue of *Isaac Asimov's Science Fiction Magazine* has contained a puzzle from Martin Gardner, dressed up as a brief science fiction tale. This book collects the first thirty-six. Most of them are mathematical, covering a variety of topics — numbers, geometry, games of strategy, combinatorics, simple logic — but a few involve word play and trivia (which name of a US state shares at least one letter in common with the name of each other state?).

Several of the problems are well-known chestnuts (such as the seventh about an even number of people shaking an odd number of hands in a social encounter, or the nineteenth, which is a version of the water-and-wine mixing problem); some are straightforward for a mathematical reader. But others are new and challenging. For example, the ninth puzzle concerns the organisms *toroidus klonefakus* that have the shape of a torus and divide like cells. Sometimes this division results in linked specimens; three, for example, may constitute a set of Boromean rings in which no two but all three are interlocked. The reader is asked to imagine how a larger colony could be linked in a circular chain in such a way that, if one is eaten by a predator, all the rest can swim free. Some puzzles touch on interesting research questions.

As you would expect in a Martin Gardner book, the author follows up the puzzles with solutions, comments from readers, discussions, generalizations, references to the literature, and ancillary problems. Readers turning to solutions in the "first" answer section will often be led to an investigation consummated in a "second" answer section, which in some cases, will lead to a "third" answer section. There is a two-page bibliography of other recreational books.

Apart from feeding my own pleasure, I anticipate using certain problems in my courses, particularly one for undergraduates who plan to go into school teaching. Seasoned with Gardner's wry whimsy and zest for the incongruous, this monograph would provide enjoyment for quite a general audience.

Machine Proofs in Geometry. Automated Production of Readable Proofs for Geometry Theorems,
by Shang-Ching Chou, Xiao-Shan Gao and Jing-Zhong Zhang,
published by the World Scientific (Series on Applied Mathematics, Vol. 6), 1994, ISBN 981-02-1584-3, hardcover, 461 + xvii pages, $96.00 (U.S.).
Reviewed by **Maria Hernandez Cifre**, *Universidad de Murcia, Spain.*

In this book, the authors Chou, Gao and Zhang present an automated method for proving theorems in Geometry. They have developed a technique which produces short and, most important, readable proofs for hundreds of geometric statements in plane and solid geometries. They have also implemented this method in a computer program which produces the full proof of the different theorems.

When we are trying to learn or teach Classic Geometry, the basic geometric concepts (such as points, lines, circles, angles, ...) are easily understandable; the main difficulties appear when we want to prove geometric results by using logical reasonings to justify them, because there exist no methods or algorithms that can be useful to solve most problems. Each proof requires particular tricks that lead to the solution. One of the main contributions of this book is precisely the fact mentioned above: to give a systematic method that produces *easily readable* proofs, even for extremely difficult theorems in Geometry.

The main tool used by the authors is the *area method*. This technique is one of the oldest and most effective in plane geometry; Pythagoras' Theorem was first proved using the areas of triangles. The authors recognize the generality of this method, and develop it into a systematic way for solving very different types of geometric problems.

The book starts introducing the main geometric concepts and results that will be necessary for the rest of the book at an elementary level: Signed Areas, Pythagoras Difference, Full-Angles, the Co-side and Co-angle Theorems and the classic Area Method. The authors present also in this first part many interesting examples, classical geometric theorems which are proved by the new method. One of the facts that makes this book attractive and readable is the use of many examples to help provide easier and more understandable text.

In the following parts, the authors mechanize the area method in an algorithmic way, introducing it step by step: first they consider geometric problems involving just collinearity and parallelism (that is, statements in affine geometry). Later, perpendicular lines and circles are included (statements in metric geometry). The last part of the book is dedicated to developing this *mechanical theorem proving method* in Geometry of dimension three, showing also that this method works not only for Euclidean Geometry, but also for non-Euclidean Geometries, such as the Geometry of Minkowski.

The book concludes with a collection of 400 theorems in plane geometry which have been proved by a computer program based on the method that the authors have developed in the first part of the book. The simplicity of the proofs obtained can be checked in this last chapter, illustrating the great efficiency and the power of their method. But at the same time, these mechanical proofs can be easily understandable. This fact helps the readers to design such proofs by themselves when they need to solve difficult problems in Geometry.

Farewell

My three year appointment as Book Reviews Editor ends with the publication of this month's Crux. It has been fun.

I would like to take this opportunity to acknowledge the many reviewers I have dealt with during my tenure; their efforts have provided us with quite a variety of interesting and useful reviews. I also wish to single out one person for special thanks — Jennifer Keir in the Computer Science Department at the University of Waterloo. Jennifer's expertise in LaTeX combined with her sharp proof-reading eye have made my job an easy one throughout: her continued support is much appreciated.

Alan Law

Welcome

We look forward to the next Book Reviews Editor, John Grant McLoughlin.

John has considerable experience in the Canadian problem solving community, with time spent in Waterloo, Labrador City, Corner Brook, Buffalo and Kelowna, before taking up his present appointment in the Faculty of Education at Memorial University. We take this opportunity of welcoming John "on board".

Bruce Shawyer

Summation of Finite Series of Integers

C.-S. Lin

If someone asks me how to verify the equality

$$\sum_{k=1}^{n} k^2 \;=\; \frac{n(n+1)(2n+1)}{6}\,,$$

immediately I would say it is easy by the Principle of Mathematical Induction, and that is true. If I am further asked how to get the equality in the first place, I would be probably hesitative for a while and I might fail to answer, unless I already knew a method. In fact, some strikingly original algebraic proofs are due to Archimedes and Fibonacci [1, p. 104 and 102]. In numerical analysis we use factorial polynomials and the telescoping method [5, Chap. 17], and a graphical expression (not a proof, though) can be found in [4, p. 77]. Mathematical induction is a great common tool used for checking an equality like the one above. But it is imperfect in the sense that we have to know the equality in question beforehand. In this article, motivated by the method of proof of the equality above due to Chorlton [2, p. 305], we shall use finite sums of sine and cosine functions to produce several types of formulas involving summations of finite series of integers by way of differentiation. The next result is our basic tool.

Lemma. For any integer $n \geq 1$, the following sine-series and cosine-series hold.

(1) $$\sin x + \sin 2x + \cdots + \sin(n-1)x + \sin nx = \frac{\cos\frac{1}{2}x - \cos\left(n+\frac{1}{2}\right)x}{2\sin\frac{1}{2}x}.$$

(2) $$\cos x + \cos 2x + \cdots + \cos(n-1)x + \cos nx = \frac{\sin\left(n+\frac{1}{2}\right)x - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x}.$$

(3) $$\sin x - \sin 2x + \sin 3x - \cdots \mp \sin(n-1)x \pm \sin nx = \frac{\sin\frac{1}{2}x \pm \sin\left(n+\frac{1}{2}\right)x}{2\cos\frac{1}{2}x}.$$

The upper and lower signs depend on whether n is odd or even, respectively.

(4) $$\cos x - \cos 2x + \cos 3x - \cdots \mp \cos(n-1)x \pm \cos nx = \frac{\cos\frac{1}{2}x \pm \cos\left(n+\frac{1}{2}\right)x}{2\cos\frac{1}{2}x}.$$

The upper and lower signs depend on whether n is odd or even, respectively.

Proof. First, we note that the technique of the proof of each equality is the same and is known [3, p. 289], but let us prove it for the sake of completeness.

(1) $$[\cos\tfrac{1}{2}x - \cos\tfrac{3}{2}x] + [\cos\tfrac{3}{2}x - \cos\tfrac{5}{2}x] + \cdots + [\cos\left(n-\tfrac{1}{2}\right)x - \cos\left(n+\tfrac{1}{2}\right)x] = 2\sin\tfrac{1}{2}x\,[\sin x + \sin 2x + \cdots + \sin(n-1)x + \sin nx],$$

and the required equality follows easily.

(2) was proved in [2] by the same method as above.

(3) Let n be odd first. Then

$$[\sin\tfrac{1}{2}x + \sin\tfrac{3}{2}x] - [\sin\tfrac{3}{2}x + \sin\tfrac{5}{2}x] + \cdots + [\sin(n-\tfrac{1}{2})x + \sin(n+\tfrac{1}{2})x] = 2\cos\tfrac{1}{2}x\,[\sin x - \sin 2x + \cdots - \sin(n-1)x + \sin nx],$$

and we have the case of odd n. The case of even n and equalities (4) are treated in a similar fashion, and we shall omit the details.

Proposition. For any integer $n \geq 1$ the following summations of integers hold.

(1) $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, and $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$, etc.

(2) $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$, and $\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$, etc.

(3) $$\sum_{k=1}^{n}(-1)^{k+1}k = \begin{cases} \dfrac{n+1}{2} & \text{if } n \text{ is odd, and} \\ \dfrac{-n}{2} & \text{if } n \text{ is even; and} \end{cases}$$

$$\sum_{k=1}^{n}(-1)^{k+1}k^3 = \begin{cases} \dfrac{(n+1)^2(2n-1)}{4} & \text{if } n \text{ is odd, and} \\ \dfrac{-n^2(2n+3)}{4} & \text{if } n \text{ is even, etc.} \end{cases}$$

(4) $$\sum_{k=1}^{n}(-1)^{k+1}k^2 = \begin{cases} \dfrac{n(n+1)}{2} & \text{if } n \text{ is odd, and} \\ \dfrac{-n(n+1)}{2} & \text{if } n \text{ is even; and} \end{cases}$$

$$\sum_{k=1}^{n}(-1)^{k+1}k^4 = \begin{cases} \dfrac{n(n+1)(n^2+n-1)}{2} & \text{if } n \text{ is odd, and} \\ \dfrac{-n(n+1)(n^2+n-1)}{2} & \text{if } n \text{ is even, etc.} \end{cases}$$

Proof. (1) Let $f(x) = \sin x + \sin 2x + \cdots + \sin(n-1)x + \sin nx$. Then

$f'(x) = \cos x + 2\cos 2x + \cdots + (n-1)\cos(n-1)x + n\cos nx$;

$f''(x) = -\left[\sin x + 2^2 \sin 2x + \cdots + (n-1)^2 \sin(n-1)x + n^2 \sin nx\right]$;

and

$f'''(x) = -\left[\cos x + 2^3 \cos 2x + \cdots + (n-1)^3 \cos(n-1)x + n^3 \cos nx\right]$,

which yield $f(0) = 0$, $f'(0) = \sum_1^n k$, $f''(0) = 0$, and $f'''(0) = -\sum_1^n k^3$. Now, we have from (1) in the Lemma,

$$(\alpha) \qquad 2\left(\sin \tfrac{1}{2}x\right) f(x) = \cos \tfrac{1}{2}x - \cos\left(n + \tfrac{1}{2}\right)x.$$

Straightforward computation shows that the first equality in (1) follows by differentiating the identity (α) twice and substituting 0 for x. If it is done four times at $x = 0$, then we get the second equality in (1). The similar further process can be used for $\sum_1^n k^5$ and $\sum_1^n k^7$, etc.

(2) Let $g(x) = \cos x + \cos 2x + \cdots + \cos(n-1)x + \cos nx$. Then $g(0) = n$, $g'(0) = 0$, $g''(0) = -\sum_1^n k^2$, $g'''(0) = 0$, and $g^{(4)}(0) = \sum\limits_{k=1}^{n} k^4$. We need the next equality, which is from (2) in the Lemma:

$$(\beta) \qquad 2\left(\sin \tfrac{1}{2}x\right) g(x) = \sin\left(n + \tfrac{1}{2}\right)x - \sin \tfrac{1}{2}x.$$

Repeated differentiation of the identity (β) three times, and letting $x = 0$, we get the first equality in (2). If it is done five times at $x = 0$, then we have the second equality in (2). The similar further process implies formulas for $\sum\limits_{k=1}^{n} k^6$ and $\sum\limits_{k=1}^{n} k^8$, etc.

(3) We use $h(x) = \sin x - \sin 2x + \sin 3x - \cdots \mp \sin(n-1)x \pm \sin nx$. Differentiate the identity obtained from (3) in the Lemma; that is,

$$(\gamma) \qquad 2\left(\cos \tfrac{1}{2}x\right) h(x) = \sin \tfrac{1}{2}x \pm \sin\left(n + \tfrac{1}{2}\right)x,$$

depending on odd or even n, we then obtain the formulas in (3), and the formulas for $\sum\limits_{k=1}^{n} (-1)^{k+1}k^5$ and $\sum\limits_{k=1}^{n} (-1)^{k+1}k^7$, etc., as well.

(4) The function $j(x) = \cos x - \cos 2x + \cos 3x - \cdots \mp \cos(n-1)x \pm \cos nx$, together with the identity from (4) in the Lemma (that is,

$$(\delta) \qquad 2\left(\cos \tfrac{1}{2}x\right) j(x) = \cos \tfrac{1}{2}x \pm \cos\left(n + \tfrac{1}{2}\right)x,$$

depending on odd or even n), lead to the desired conclusion, and for formulas of types $\sum\limits_{k=1}^{n} (-1)^{k+1}k^6$ and $\sum\limits_{k=1}^{n} (-1)^{k+1}k^8$, etc., as well.

Remarks.

(A) We see that the method used in this paper (that is, applying finite sums of sine and cosine functions and by way of differentiation) is a unified approach to sums of integers. This same idea can also be extended to other types of sum. For example, the sine-series $\sin x + \sin 3x + \cdots + \sin(2n-1)$ produces formulas of types $\sum_{k=1}^{n}(2k-1)$ and $\sum_{k=1}^{n}(2k-1)^3$, etc.

(B) Some calculations in the proof of the Proposition are relatively long and laborious. Therefore, it is recommended to make use of a computer to simplify, expand and factor expressions.

References

[1] D.M. Burton, *The History of Mathematics,* 2nd Edition, Wm.C. Brown Publishers, 1991.

[2] F. Chorlton, Math. Gazette, 71 (1987), 305–307.

[3] H.S. Hall and S.R. Knight, *Elementary Trigonometry*, MacMillan and Co., Ltd., 1952.

[4] R.B. Nelsen, *Proofs without Words*, Mathematical Association of America, 1993.

[5] F.Scheid, *Numerical Analysis*, Schaum's Outline Series, McGraw-Hill Book Company, 1968.

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A Couple of Pretty Mean Questions

Clifford Wagner

The general public uses the word average for a concept that mathematicians prefer to call the arithmetic mean. Fortunately, mathematicians bring more than jargon to a discussion of averages. This note discusses two questions about arithmetic means. See how well you can answer them before reading the hints and answers.

Question 1: Suppose the Internal Revenue Service reports that last year's returns with taxable income had an average adjusted gross income (AGI) of \$46,000 and an average effective tax rate of 15 percent. The average effective tax rate was obtained by taking each return's tax as a percentage of AGI and averaging these percentages. What can one say about the average tax paid per taxable return? Clearly state any assumptions.

Hint 1: The average tax paid is not necessarily \$6,900 (15 percent of \$46,000). If necessary, let the number of tax returns be $n = 3$ and create some examples that match the question.

Hint 2: Let us suppose that the various AGIs can be sorted to create a sequence, $a_1 \leq a_2 \leq \ldots \leq a_n$, with corresponding tax rates, $b_1 \leq b_2 \leq \ldots \leq b_n$. Although the tax code is said to be progressive, this pairing of ordered sequences must be considered an assumption.

Hint 3: One can use the Chebyshev Inequality for Arithmetic Means [Mitrinović, 1970], which states that given two increasing sequences $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$, with, $a_1 \leq a_2 \leq \ldots \leq a_n$ and $b_1 \leq b_2 \leq \ldots \leq b_n$, the mean of the product sequence $\{a_1b_1, a_2b_2, \ldots, a_nb_n\}$ is at least as great as the product of the means of the two given sequences. That is,

$$\frac{1}{n}\sum_{i=1}^{n} a_i b_i \;\geq\; \left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} b_i\right) .$$

This inequality is not to be confused with the other well-known Chebyshev Inequality concerning the variance of a random variable.

Answer to Question 1: By the Chebyshev Inequality, the mean tax is **at least** \$6,900.

Note: [Mitrinović, 1970] shows that Chebyshev's Inequality is equivalent to the inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n} (a_i - a_j)(b_i - b_j) \;\geq\; 0 .$$

Thus, when both sequences are increasing (or both decreasing), the observation

$$(a_i - a_j)(b_i - b_j) \geq 0 \text{ for all } i, j = 1, \ldots, n$$

immediately leads to Chebyshev's Inequality, and also shows that equality occurs if and only if at least one sequence is constant.

Question 2: Suppose I surveyed prices for regular grade gasoline at three competing gas stations on a summer weekend. I observed that the average posted price was \$1.50 per gallon, and I determined that the average sales volume for regular gasoline was $5,000$ gallons per station. What can one say about the average revenue from regular gasoline at these three stations on that particular weekend? Clearly state any assumptions.

Hint 1: The average revenue is not necessarily \$7,500 ($5,000$ gallons at \$1.50 per gallon). If necessary, create some examples that match the question.

Hint 2: Assume that the various sales volumes can be sorted so as to constitute a sequence, $a_1 \leq a_2 \leq a_3$, with corresponding prices $b_1 \geq b_2 \geq b_3$. The assumption here is that sales volume and price charged are inversely related.

Hint 3: Use the previous note regarding the Chebyshev Inequality to recognize that when one sequence is increasing and the other decreasing we have

$$(a_i - a_j)(b_i - b_j) \leq 0 \text{ for all } i, j = 1, \ldots, n,$$

and this causes a reversal of the sign in the Chebyshev Inequality.

Answer to Question 2: By the reversed Chebyshev Inequality, the mean revenue is **at most** \$7,500.

Reference

1. Mitrinović, D.S., *Analytic Inequalities*, Springer-Verlag, 1970, pp. 36–37.

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THE SKOLIAD CORNER

No. 58

Shawn Godin

Solutions may be sent to Shawn Godin, Cairine Wilson S.S., 975 Orleans Blvd., Orleans, ON, CANADA, K1C 2Z5, or emailed to

`mayhem-editors@cms.math.ca`.

Please include on any correspondence your name, school, grade, city, province or state and country. We are especially looking for solutions from high school students. Please send your solutions to the problems in this edition by *1 May 2002*. Look for prizes for solutions in the new year.

Our first item is the 2001 Concours De Mathématiques Du Nouveau-Brunswick. This contest is written every year by students in New Brunswick and hosted by the University of Moncton and the University of New Brunswick. Solutions are always accepted in both English and French. My thanks go to Bob McKellar and Daryl Tingley at the University of New Brunswick for forwarding the material to me.

UNIVERSITÉ DE MONCTON
et
UNIVERSITY OF NEW BRUNSWICK

20^e CONCOURS DE MATHÉMATIQUES DU NOUVEAU-BRUNSWICK

le vendredi 11 mai 2001

9^e année

PARTIE A

1. Évaluez l'expression suivante : $\dfrac{\frac{1}{3}+\frac{1}{4}}{\frac{1}{3}-\frac{1}{4}}$.

   (A) $\dfrac{7}{12}$ (B) $\dfrac{12}{7}$ (C) 7 (D) 12 (E) Non définie

2. Si a est 50% plus grand que c et b est 25% plus grand que c, dites de combien a est plus grand que b en pourcentage ?

   (A) 10% (B) 20% (C) 25% (D) 31% (E) Aucune de ces réponses

3. Déterminez la valeur de $\dfrac{x+y}{x-y}$ si $x = \dfrac{3}{4}$ et $y = \dfrac{2}{3}$.

   (A) $\dfrac{5}{3}$ (B) 5 (C) 6 (D) 17 (E) Aucune de ces réponses

4. Combien de nombres de quatre chiffres peut-on obtenir en ordonnant les chiffres $1, 2, 3, 3$?

(A) 4 (B) 6 (C) 12 (D) 24 (E) Aucune de ces réponses

5. Il y a quelques années, les conducteurs de bétonnières ont fait une grève de 46 jours. Avant la grève ces chauffeurs gagnaient \$7,50 de l'heure et travaillaient 8 heures par jour, 260 jours par année. Quel est, en pourcentage, l'augmentation annuelle de salaire nécessaire pour récupérer en un an le montant de salaire perdu lors de cette grève ?

(A) $\frac{23}{1040} \times 100\%$ (B) $7,5\%$ (C) $\frac{23}{130} \times 100\%$ (D) $\frac{69}{52} \times 100\%$
(E) Aucune de ces réponses

6. Une automobile va du point A au point B à une vitesse de 40 km/h. À quelle vitesse doit-on conduire la voiture du point B au point A pour que la vitesse moyenne pour le trajet aller-retour, soit de 50 km/h ?

(A) 50 km/h (B) 58 km/h (C) 60 km/h (D) $66\frac{2}{3}$ km/h
(E) Information insuffisante

7. Pour une fête, Justin achète une pizza et la coupe en 24 morceaux. Marc mange le $\frac{1}{6}$ de la pizza. Claudine mange $\frac{1}{4}$ de ce qui reste et Sylvie $\frac{1}{3}$ de ce qui reste après que Claudine et Marc soient servis. Si Justin mange le restant, quelle fraction de la pizza Justin n'a pas mangée ?

(A) $\frac{1}{2}$ (B) $\frac{5}{12}$ (C) $\frac{7}{12}$ (D) $\frac{2}{3}$
(E) Aucune de ces réponses

8. Si on multiplie un nombre donné par 4 et on soustrait ensuite 12, on obtient un résultat 2 fois plus grand que si l'on soustrait d'abord 12 à ce nombre et que l'on multiplie ensuite par 4. Quelle est la somme des chiffres de ce nombre ?

(A) 3 (B) 4 (C) 5 (D) 7 (E) 9

9. 5^{10} est un nombre à n chiffres. Quelle est la valeur de n ?

(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

10. Alphonse a 3 fois plus de billes que Béatrice. S'il lui en donnait 15, il en aurait 2 fois plus qu'elle. Combien devrait-il lui en donner pour qu'ils en aient tous deux le même nombre ?

(A) 30 (B) 45 (C) 60 (D) 90 (E) Information insuffisante

PARTIE B

11. On définit l'opération * de la façon suivante : $a^*b = 3a - 2b$. Quelle est la valeur de $(1^*(-2))^*(3^*4)$?

    (A) -24 (B) -5 (C) 5 (D) 19 (E) Aucune de ces réponses

12. Les villes de Artin et Balin sont à une distance de 500 km l'une de l'autre. Un avion fait régulièrement la navette entre les deux. Le vent souffle à une vitesse constante de Artin vers Balin et la vitesse de l'avion sans tenir compte du vent est de 900 km/h. Si le voyage de Artin vers Balin prend 30 minutes et le voyage dans le sens contraire prend $37,5$ minutes, quelle est la vitesse du vent ?

    (A) 80 km/h (B) 100 km/h (C) 120 km/h (D) 200 km/h
    (E) Aucune de ces réponses

13. Dans une partie de basketball, une équipe peut compter un, deux ou trois points en lançant le ballon dans le panier. Notre équipe lance le ballon 50 fois dans le panier et marque 80 points. Quel est le nombre maximal de lancer à trois points que notre équipe a compté ?

    (A) 5 (B) 10 (C) 15 (D) 20 (E) Pas assez d'information

14. Un héritage est partagé entre 5 frères. Le premier reçoit la moitié de l'héritage plus 1\$. Le deuxième reçoit la noitié du reste plus 2\$. Le troisième reçoit la moitié du reste plus 3\$. Le quatrième reçoit la moitié du reste plus 4\$. Le cinquième frère reçoit 500\$. Quelle est la valeur de cet héritage ?

    (A) 7098\$ (B) 7598\$ (C) 8098\$ (D) 8598\$ (E) 9098\$

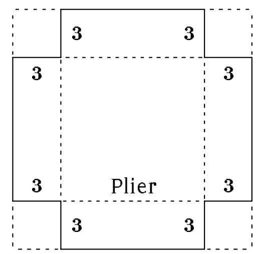
15. Dans la suite $1, 3, 3, 3, 5, 5, 5, 5, 5, 7, 7, \ldots$, le $100^{\text{ième}}$ nombre est

    (A) 10 (B) 19 (C) 20 (D) 21 (E) Aucune de ces réponses

16. X est l'entier supérieur à 1 le plus petit tel que si je divise X par deux, trois, quatre, cinq ou six, j'obtiens un reste de 1. La somme des chiffres de X est :

    (A) 4 (B) 5 (C) 6 (D) 7 (E) 10

17. Une compagnie fabrique un emballage pour un nouveau produit. Une partie de cet emballage est une boîte ouverte obtenue d'une pièce carrée d'aluminium en découpant des carrés de 3 cm de côté à chaque coin. (Voir figure). Si la boîte doit avoir un volume de 75 cm^3, quelles sont les dimensions en cm^2 de la pièce d'aluminium qui doit être utilisée ?



(A) 6×6 (B) 9×9 (C) 10×10 (D) 11×11 (E) Aucune de ces réponses

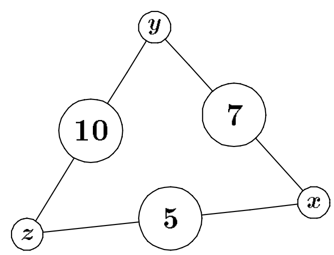
18. Quel est le $2001^{\text{ième}}$ nombre de la suite : $2, 5, 8, 11, \ldots$?

(A) 5996 (B) 5999 (C) 6000 (D) 6001 (E) 6002

19. Combien de chiffres sont nécessaires pour écrire tous les nombres entiers de 1 à 1000 inclusivement ? Par exemple, pour écrire les nombres de 1 à 10, inclusivement, on a besoin de 11 chiffres.

(A) 2889 (B) 2892 (C) 2893 (D) 2899 (E) 2989

20. Les nombres dans les grands cercles sont obtenus en additionnant les deux nombres dans les petits cercles attachés au grand cercle. Déterminez la somme des nombres dans les petits cercles.



(A) 9 (B) 11 (C) 13 (D) 20 (E) Aucune de ces réponses

PARTIE C

21. Calculez la valeur de $\dfrac{2^{2001} + 2^{1999}}{2^{2000} - 2^{1998}}$

(A) 2 (B) $\dfrac{10}{3}$ (C) $2^{1000} + 1$ (D) $2^{2000} + 1$ (E) Aucune de ces réponses

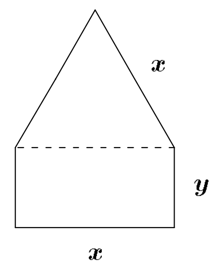
22. Sachant que dans un polygone régulier tous les côtés ont la même longueur et tous les angles internes sont égaux, quel est le nombre de diagonales dans un polygone régulier de 12 côtés ? Une diagonale est un segment qui relie deux sommets non consécutifs du polygone.

(A) 27 (B) 35 (C) 44 (D) 54 (E) 65

23. Si on définit l'inverse d'un nombre entier à deux chiffres comme le nombre obtenu en permutant les deux chiffres qui composent le nombre initial (exemple : 34 est l'inverse de 43). Combien de nombres entiers à deux chiffres donnent un carré parfait lorsqu'ils sont additionnés à leur inverse ?

(A) 1 (B) 4 (C) 8 (D) 9 (E) Aucune de ces réponses

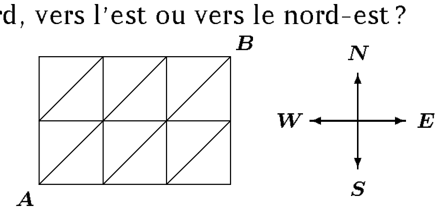
24. Une fenêtre a la forme d'un rectangle surmonté d'un triangle équilatéral. Si son périmètre est de $6 - \sqrt{3}$ et sa superficie est de $\dfrac{6-\sqrt{3}}{4}$, déterminer $x + y$.



(A) $\dfrac{1+\sqrt{3}}{2}$ (B) $\dfrac{6+\sqrt{3}}{4}$ (C) $\dfrac{5-\sqrt{3}}{2}$

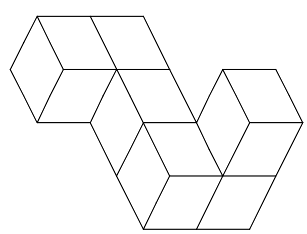
(D) $\dfrac{5+\sqrt{3}}{2}$ (E) Aucune de ces réponses

25. Combien de chemins vont de A à B si les seuls directions possible sont d'aller vers le nord, vers l'est ou vers le nord-est ?



(A) 15 (B) 20 (C) 25 (D) 30 (E) Aucune de ces réponses

26. Quelle est l'aire en cm^2 du solide illustré si chaque cube mesure 1 cm de côté ?



(A) 24 (B) 25 (C) 29 (D) 30 (E) Aucune de ces réponses

Our next entry is the 2001 Maritime Mathematics Contest written each year by students in Nova Scotia, New Brunswick and Prince Edward Island. My thanks go to David Horrocks at the University of Prince Edward Island for forwarding the material to me.

2001 Maritime Mathematics Contest

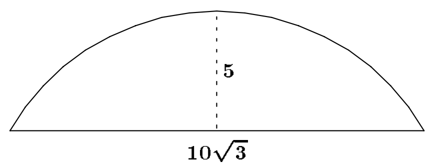
1. Alice and Bob were comparing their stacks of pennies. Alice said "If you gave me a certain number of pennies from your stack, then I'd have six times as many as you, but if I gave you that number, you'd have one-third as many as me." What is the smallest number of pennies that Alice could have had?

2. The infinite sequence

$$1\,2\,3\,4\,5\,6\,7\,8\,9\,1\,0\,1\,1\,1\,2\,1\,3\,1\,4\,1\,5\,1\,6\,1\,7\,1\,8\,1\,9\,2\,0\,2\,1\,2\,2\,2\,3\ldots$$

   is obtained by writing the positive integers in order. What is the 2001^{st} digit in this sequence?

3. The maximum height of a railway tunnel is 5 metres and the width of the tunnel is $10\sqrt{3}$ metres. The outline of the tunnel is in the form of a segment of a circle as shown below. Determine the area of a cross-section of the tunnel.



4. Which of the following numbers is greater?

$$A = \frac{2.0000004}{(1.0000004)^2 + 2.0000004} \quad \text{or} \quad B = \frac{2.0000002}{(1.0000002)^2 + 2.0000002}$$

5. Alice and Bob play the following game with a pile of 2001 beans. A move consists of removing one, two, or three beans from the pile. The players move alternately, beginning with Alice. The person who takes the last bean in the pile is the winner. Which player has a winning strategy for this game and what is that strategy?

6. Show that, regardless of what integers are substituted for x and y, the expression

$$x^5 - x^4y - 13x^3y^2 + 13x^2y^3 + 36xy^4 - 36y^5$$

   is never equal to 77.

Well that ends another year of the Skoliad corner. We would like to hear any feedback about what we are doing with the corner. In the new year there are a number of prizes that will be awarded for solutions that are sent in. Send us your contest material and solutions!

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to **Mathematical Mayhem, Cairine Wilson Secondary School, 975 Orleans Blvd., Gloucester, Ontario, Canada. K1C 2Z5**, or to **Mathematical Mayhem, c/o Faculty of Mathematics, University of Waterloo, 200 University Avenue West, Waterloo, Ontario. N2L 3G1**. The electronic address is

`mayhem-editors@cms.math.ca`

The Assistant Mayhem Editor is Chris Cappadocia (University of Waterloo). The rest of the staff consists of Adrian Chan (Harvard University), Jimmy Chui (University of Toronto), Donny Cheung (University of Waterloo), and David Savitt (Harvard University).

Editorial

Shawn Godin

Another year has come to an end and with it another volume of Crux with Mayhem. It is hard to believe that I have been at this job for a year; it feels like ten! (I have earned enough new grey hairs to count for ten years). Over the past year we have started changing the face of Mayhem and we hope that, as we iron out the wrinkles of our new format, Crux with Mayhem will become accessible to a broader audience. The current changes should be finished by early in 2002, and we should be in our new format for the better part of the next volume.

As our format changed, there were a few solutions that slipped though the cracks. The following solutions were received after the solutions were published due to being sent to the old problem editor and then forwarded to the interim problem editor (yours truly): JOSE LUIS DIAZ (H273), ROBERT BILINSKI (H273 and H274) and PAUL JEFFREYS (H281 and H282). My apologies to these, and any other readers who had their solutions not acknowledged.

At this point I should thank some people who have made my first year go a bit smoother. First and foremost a big thank you to BRUCE SHAWYER for all his help, advice and guidance in this year of a steep learning curve. He always was quick to answer the email, or phone and calm my shattered nerves, as well as not being too harsh when deadlines came and went (like this one!).

Another big asset was NAOKI SATO, the previous Mayhem editor. Naoki has been very helpful, meeting with me, sending me material and giving me advice and help when I needed it. I hope that he will continue to contribute to Mayhem over the coming years.

At CMS headquarters in Ottawa is GRAHAM WRIGHT, who I am sure has put in a new phone line just for my calls. Graham helps with all of my problems from getting stationery and back issues of the journal to helping secure some funds for prizes in our 2002 year of prizes.

The Mayhem staff: ADRIAN CHAN, DONNY CHEUNG, JIMMY CHUI and DAVID SAVITT have made my job that much easier. Their hard work and dedication have really helped shape Mayhem into the great journal that it is. Since we were planning on changing the format, Adrian, Donny and David decided that it was time for them to move on to other things. We will miss your work, and appreciate all the time you have put in over the years. We wish you the best, and hope you'll send us some material from time to time. The new Mayhem assistant editor, CHRIS CAPPADOCIA, has been there when I needed him, and is taking over a larger role in the upcoming year looking after the problem section.

Some other people that have been very helpful over the year that have to be mentioned for their continuous help are: ARLENE ANGEL, DAVID BRIGGS, BILL CLARKE, ELIZABETH ELTON, EDWARD WANG, RICHARD HOSHINO, and CYRUS HSIA.

The year 2002 will be a year of prizes at Mayhem. The Endowment Fund of the CMS has graciously provided us with funds to provide prizes for solutions for the upcoming year. Look for books, past volumes of Crux and Mayhem as well as subscriptions in the next volume. Prizes will be for individuals as well as schools. Look for a complete description in the next issue.

Mayhem Problems

Proposals and solutions may be sent to **Mathematical Mayhem, c/o Faculty of Mathematics, University of Waterloo, 200 University Avenue West, Waterloo, Ontario, N2L 3G1**, or emailed to

`mayhem-editors@cms.math.ca`

Please include in all correspondence your name, school, grade, city, province or state and country. We are especially looking for solutions from high school students. Please send your solutions to the problems in this edition by *1 May 2002*. Look for prizes for solutions in the new year.

M22. *Proposed by the Mayhem staff.*

George is walking across a bridge on the train track. When he is $\frac{5}{12}$ of the way across the bridge he notices a train bearing down on him at 90 km/h. If he can just escape death by running in either direction, how fast can George run?

M23. *Proposed by José Luis Díaz-Barrero and Juan José Egozcue, Barcelona, Spain.*

Find all complex solutions of the following system of equations

$$\begin{aligned} x^3+y^3+z^3+t^3 &= 12 \\ x^2+y^2+z^2+t^2 &= 0 \\ xy+zt+(x+y)(z+t) &= 0 \\ xyzt &= 3\,. \end{aligned}$$

M24. *Proposed by the Mayhem staff.*

A school math club is deciding on a name for its mascot, a stuffed rabbit. They have narrowed the choices down to three: Euler, Galois and Ramanujan. To pick the name they have each of the 100 club members rank the names in order of preference. When the polls were totalled it was found that 60 people preferred Galois over Ramanujan and 62 preferred Ramanujan over Euler. It was suggested, by a Galois supporter, that Euler should be dropped. A staunch Eulerist objected and demanded the counting continue. When the final totals came in it was found that 68 preferred Euler over Galois! If each possible ranking was picked by at least one member, how many picked each name as their first choice?

M25. *Proposed by the Mayhem staff.*

What is the smallest number with the property that when the first digit (leftmost) is moved to the rightmost position, the new number is three times the original?

M26. *Proposed by the Mayhem staff.*

Find two isosceles triangles, with two sides 106 units long and the other side an integer, that have the same area.

M27. *Proposed by the Mayhem staff.*

Find $\sqrt{ab+1}$ where $a=\overbrace{111\cdots 11}^{\text{2002 1's}}$ and $b=1\overbrace{00\cdots 00}^{\text{2001 0's}}5$.

M28. *Proposed by the Mayhem staff.*

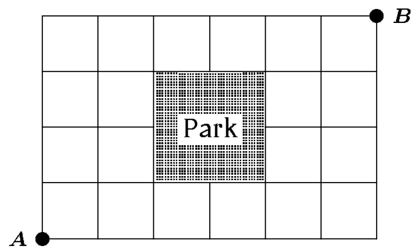
Shawn tosses 2001 fair coins and Bruce tosses 2002 fair coins. What is the probability that Bruce gets more heads than Shawn?

Problem of the Month

Jimmy Chui, student, University of Toronto

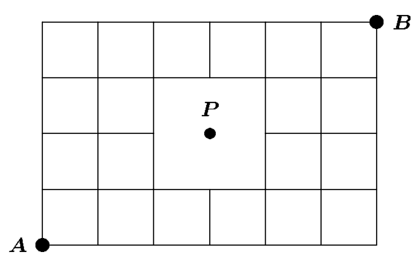
Problem.

Problem. A road map of Grid City is shown in the diagram. The perimeter of the park is a road but there is no road through the park. How many different shortest road routes are there from point A to point B?



(1996 COMC, Problem A5)

Solution 1. Point B is situated 6 blocks east and 4 blocks north of point A. Anyone can walk from point A to point B any way he or she chooses, except there is no pathway through the park. Now suppose a man by the name of Max Power wants to walk from point A to point B in the least time possible. And he absolutely refuses to walk through the park.



Let us call the centre of the park, point P. Now, point P is 3 blocks east and 2 blocks north of point A, and point B is also 3 blocks east and 2 blocks north of point P.

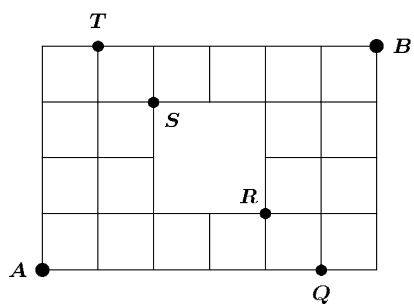
How can we count the total number of ways that Max can walk from point A to point B?

Method I. We can do a count as follows. If the park wasn't there, then Max is free to walk the shortest route (10 blocks) any way he chooses. As long as he chooses to go north 4 times and east 6 times, then he will reach his destination. This is exactly to $\binom{10}{4}$, or equivalently, $\binom{10}{6}$, which has the value of 210.

But in this calculation, the park disappeared and Max was free to walk through it. The only paths that are affected are the ones that pass through the park. The number of ways that Max can get from point A to point P would be $\binom{5}{2}$, and the number of ways that he could travel from point P to point B would also be $\binom{5}{2}$. Whatever choice of path Max takes to get to the park does not have anything to do with the choice from the park to point B. So the total number of paths that connect point A to point B, through point P, is $\binom{5}{2}\binom{5}{2} = 100$.

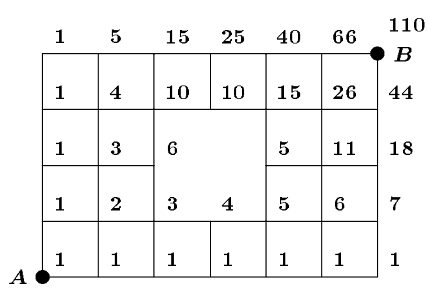
These 100 paths are impossible for Max to take. The number of paths feasible for Max to take would be $210 - 100 = 110$.

Method II. Another way is to note that there are a certain number of points that Max has to pass through. In the above diagram, Max **must** pass through one and only one of the points Q, R, S, and T.



The number of paths through point Q is $\binom{5}{0}\binom{5}{4} = 5$. Similarly, the number of paths through points R, S, and T, are respectively $\binom{5}{1}\binom{5}{3} = 50$, $\binom{5}{3}\binom{5}{1} = 50$, and $\binom{5}{4}\binom{5}{0} = 5$. And this means that Max has a total choice of $5 + 50 + 50 + 5 = 110$ paths.

Solution 2. We can do a manual count in a Pascal Triangle-esque manner. The diagram below shows the count.



Polya's Paragon

Shawn Godin

As a small child, I used to love playing with counting problems. I would spend hours on end enumerating this and counting that. I distinctly remember the moment when I realized that removing my shoes and socks allowed me to deal with even larger numbers. It was not until the end of my high school career that I ran into techniques that allowed me to count things without having to count them.

Two very basic ideas open doors to more complex situations.

The Fundamental Counting Theorem: If we can perform a certain action in a ways, a second action in b ways, a third action in c ways $\ldots$, then the number of ways of doing the first action **and** the second action **and** the third action **and** $\ldots$ is $a \times b \times c \times \ldots$

The Rule of Sum: If we can perform a certain action in a ways, a second action in b ways, a third action in c ways $\ldots$, and none of the actions can be performed simultaneously, then the number of ways of doing the first action **or** the second action **or** the third action **or** $\ldots$ is $a + b + c + \ldots$

Let us take a look at these two very simple ideas at work:

How many ways can 10 *people be lined up to take a club photo?*

By our first idea we have $10 \times 9 \times \cdots \times 2 \times 1 = 3628800$.

How many ways can a president, vice-president and treasurer be picked from a group of 10 *people?*

Similarly we have $10 \times 9 \times 8 = 720$.

These two types of problems arise often, so we have special notation for each of them.

Definition: The total number of ways to arrange n distinct objects is $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$, where $n!$ is read n **factorial**.

Thus for our first problem the result is $10!$. In the second case we have the first three terms of $10!$, which we will call $P(10,3)$. In general we have $P(n,r) = \dfrac{n!}{(n-r)!}$ where $P(n,r)$ is the number of arrangements of r of the n objects in a specific order. What happens if order is not important?

How many ways can a group of 3 *people be chosen from a group of* 10 *people?*

In this case the $P(10,3)$ from above counts 123, 132, 213, 231, 312, and 321 as different arrangements, but they represent the same group of three. Thus each group of three is repeated $3!$ times in our case above, so that the number of groups of three must be $\dfrac{P(10,3)}{3!} = \dfrac{10!}{7!3!} = 120$.

In general the number of ways to pick r of n objects, without regard to order is $\dbinom{n}{r} = \dfrac{n!}{(n-r)!r!}$. The symbol $\dbinom{n}{r}$ is read n **choose** r, and the collection of these objects is called the binomial coefficients since they show up in expanding powers of binomials.

We can put everything together with some problems.

1. How many unique arrangements are there of the letters in the word $EULER$?
2. How many ways can you arrange the letters $ABCDEFG$ such that the A and B are beside each other?
3. How many ways can a class of 20 people be divided into 4 teams of 5 people?

Who said the following?

Even fairly good students, when they have obtained the solution of the problem and written down neatly the argument, shut their books and look for something else. Doing so, they miss an important and instructive phase of the work. . . . A good teacher should understand and impress on his students the view that no problem whatever is completely exhausted. One of the first and foremost duties of the teacher is not to give his students the impression that mathematical problems have little connection with each other, and no connection at all with anything else. We have a natural opportunity to investigate the connections of a problem when looking back at its solution.

In order to translate a sentence from English into French, two things are necessary. First, we must understand thoroughly the English sentence. Second, we must be familiar with the forms of expression peculiar to the French language. The situation is very similar when we attempt to express in mathematical symbols a condition proposed in words. First, we must understand thoroughly the condition. Second, we must be familiar with the forms of mathematical expression.

Symmetric Polynomial Inequalities

Naoki Sato

Consider the following problem:

> Let a, b, and c be non-negative reals such that $a + b + c = 1$. Show that $ab + ac + bc \le 1/3$.

In this inequality problem, the variables satisfy a given constraint (their sum must be 1), and the expression in the inequality is a symmetric polynomial in those variables. In this article, we discuss a method of analyzing and proving certain inequalities of this type.

Let x_1, x_2, ..., x_n be n non-negative real numbers, and let s_k denote the sum of the products of the x_i, taken k at a time. For example, for $n = 3$,

$$\begin{aligned} s_1 &= x_1 + x_2 + x_3\,, \\ s_2 &= x_1x_2 + x_1x_3 + x_2x_3\,, \quad \text{and} \\ s_3 &= x_1x_2x_3\,. \end{aligned}$$

We are interested in inequality problems where the condition is $s_1 = 1$, and the inequality itself is of the form

$$f(x_1, x_2, \ldots, x_n) \;:=\; c_0 + c_2s_2 + c_3s_3 + \cdots + c_ns_n \;\ge\; 0\,, \qquad (*)$$

where the c_i are constants (any terms involving s_1 can be absorbed into the constant c_0, since $s_1 = 1$).

Let T_n denote the set of n-tuples $(x_1, x_2, \ldots, x_n)$ where $x_i \ge 0$ for all i and $s_1 = 1$. Let A_n denote the set of n-tuples which have $1/k$ in k coordinates, and 0 in all others, for $1 \le k \le n$, so for example,

$$\begin{aligned} A_3 = \{&(1,0,0), (0,1,0), (0,0,1), (1/2,1/2,0), (1/2,0,1/2), \\ &(0,1/2,1/2), (1/3,1/3,1/3)\}\,, \end{aligned}$$

so that A_n is a subset of T_n.

We can then view $f(x_1, x_2, \ldots, x_n)$ as a function over T_n. A result in analysis states that a continuous function over a compact set achieves its minimum (and maximum) values at certain points, which is the case here. What this means is that we can assume that there is a point in T_n at which f has a minimum value. This does not hold in general; for example, the function $1/x$ for $x > 0$ has no minimum value.

We use an optimization argument to determine where these points must lie. The idea is this: let $P = (x_1, x_2, \dots, x_n)$ be a point where f has a minimum, and select two of its coordinates, say x_1 and x_2, and let $s = x_1 + x_2$. Now consider the function $f(t, s-t, x_3, \dots, x_n)$, where t varies from 0 to s. This function must have a minimum at $t = x_1$, by definition of P. This will tell us what x_1 and x_2 are, and by extension, where to seek such points P.

Now, to do the calculations, note that f as in $(*)$ can be written in the form

$$d_0 + d_1(x_1 + x_2) + d_2 x_1 x_2$$

for some constants d_0, d_1, and d_2 (these constants include the values of x_3, x_4, ..., x_n). Substituting $x_1 = t$ and $x_2 = s - t$, we get

$$d_0 + d_1 s + d_2 t(s-t) .$$

Since s is constant, the problem becomes minimizing $d_2 t(s-t)$. If $d_2 > 0$, then the minimum occurs at $t = 0$ or $t = s$. If $d_2 < 0$, then the minimum occurs at $t = s/2$. Finally, if $d_2 = 0$, then the function is constant, and we may choose any value of x_1 we like. Therefore, the coordinates x_1 and x_2 of P must satisfy $x_1 = 0$, $x_2 = 0$, or $x_1 = x_2$.

The coordinates x_1 and x_2 were chosen arbitrarily, so these results hold for any pair of coordinates. This means that the minimum value of f must occur at a point (or points) where for any pair of coordinates x_i and x_j, $x_i = 0$, $x_j = 0$, or $x_i = x_j$. These conditions imply that all positive coordinates are equal in value, which describe precisely the points in A_n. Therefore, the minimum (and maximum) value of f must occur at a point in A_n.

Example 1. Let a, b, c be non-negative reals such that $a + b + c = 1$. Show that $ab + ac + bc \geq 9abc$.

Solution. Let $f(a,b,c) = ab + ac + bc - 9abc$. We need to check only the values of f over A_3. Because of symmetry, we need to check only the following points:

$$\begin{aligned} f(1,0,0) &= 0\,, \\ f(1/2,1/2,0) &= 1/4\,, \quad \text{and} \\ f(1/3,1/3,1/3) &= 0\,. \end{aligned}$$

Therefore, for $a + b + c = 1$, $f(a,b,c) \geq 0$.

Example 2. Prove that $0 \leq yz + zx + xy - 2xyz \leq 7/27$, where x, y and z are non-negative real numbers for which $x + y + z = 1$.
(1984 IMO)

Solution. Let $f(x,y,z) = xy + xz + yz - 2xyz$. Then

$$\begin{aligned} f(1,0,0) &= 0\,, \\ f(1/2,1/2,0) &= 1/4\,, \quad \text{and} \\ f(1/3,1/3,1/3) &= 7/27\,. \end{aligned}$$

Therefore, for $x + y + z = 1$,

$$\min\{0,1/4,7/27\} \;=\; 0 \;\leq\; f(x,y,z) \;\leq\; \max\{0,1/4,7/27\} \;=\; 7/27\,.$$

Example 3. Let a, b, c, and d be non-negative real numbers such that $a + b + c + d = 1$. Prove that

$$abc + abd + acd + bcd \;\leq\; \frac{1}{27} + \frac{176}{27}\,abcd\,.$$

(1993 IMO Proposal)

Solution.
Let $f(a,b,c,d) = 1/27 - (abc + abd + acd + bcd) + 176abcd/27$. Then

$$\begin{aligned} f(1,0,0,0) &= 1/27\,, \\ f(1/2,1/2,0,0) &= 1/27\,, \\ f(1/3,1/3,1/3,0) &= 0\,, \text{ and} \\ f(1/4,1/4,1/4,1/4) &= 0\,. \end{aligned}$$

Therefore, for $a + b + c + d = 1$, $f(a,b,c,d) \geq 0$.

How many points in A_n can f be zero at? If f is zero at every point in A_n, then f is zero everywhere on T_n, and all the coefficients must be zero. There is an f which is zero at every point in A_n, except $(1/n, 1/n, \ldots, 1/n)$. We leave it as a problem for the reader to find this f and the corresponding inequality.

Another way is to divide A_n into two sets B_n and C_n, where B_n is the set of n-tuples which have $1/k$ in k coordinates and 0 in all others, for $2 \leq k \leq n$, and C_n is the set of n-tuples which have 1 in one coordinate and 0 in all others. Then we can find a function f which is zero at every point in B_n, and the answer may be surprising.

Let us first look at some small cases. For $n = 2$,

$$f_2(x_1,x_2) \;:=\; 1 - 4x_1x_2$$

is zero at $(x_1,x_2) = (1/2,1/2)$. For $n = 3$,

$$f_3(x_1,x_2,x_3) \;:=\; 1 - 4(x_1x_2 + x_1x_3 + x_1x_3) + 9x_1x_2x_3$$

is zero at $(x_1,x_2,x_3) = (1/2,1/2,0)$, $(1/2,0,1/2)$, $(0,1/2,1/2)$, and $(1/3,1/3,1/3)$. It makes sense that the same coefficients of 1 and -4 appear: f_3 must be zero at $(1/2,1/2,0)$, and setting $x_3 = 0$ in f_3 gives us a symmetric polynomial in x_1 and x_2, which effectively reduces the problem to the $n = 2$ case. We find that not only do the coefficients continue, but the pattern suggested by 1, -4, and 9 does as well.

Theorem. The function

$$
\begin{aligned}
f_n(x_1, x_2, \dots, x_n) &:= 1 - 4s_2 + 9s_3 - \cdots + (-1)^{n-2}(n-1)^2 s_{n-1} \\
&\qquad + (-1)^{n-1} n^2 s_n \\
&= 1 + \sum_{i=2}^{n} (-1)^{i-1} i^2 s_i
\end{aligned}
$$

is zero at every point in B_n. It is also equal to 1 at every point in C_n, so $0 \leq f_n(x_1, x_2, \dots, x_n) \leq 1$ for all $(x_1, x_2, \dots, x_n) \in T_n$.

Proof. We prove the result using generating functions. Let

$$
\begin{aligned}
g(t) &= (1 + x_1 t)(1 + x_2 t) \cdots (1 + x_n t) \\
&= 1 + s_1 t + s_2 t^2 + \cdots + s_n t^n \\
&= 1 + \sum_{i=1}^{n} s_i t^i .
\end{aligned}
$$

Then

$$
\begin{aligned}
& g'(t) &&= \sum_{i=1}^{n} i s_i t^{i-1} \\
\Longrightarrow\quad & t g'(t) &&= \sum_{i=1}^{n} i s_i t^{i} \\
\Longrightarrow\quad & g'(t) + t g''(t) &&= \sum_{i=1}^{n} i^2 s_i t^{i-1} \\
\Longrightarrow\quad & t g'(t) + t^2 g''(t) &&= \sum_{i=1}^{n} i^2 s_i t^{i} .
\end{aligned}
$$

Setting $t = -1$, we obtain

$$
-g'(-1) + g''(-1) = \sum_{i=1}^{n} (-1)^i i^2 s_i .
$$

We can now express f_n in terms of g as follows:

$$
\begin{aligned}
f_n(x_1, x_2, \dots, x_n) &= 1 + \sum_{i=2}^{n} (-1)^{i-1} i^2 s_i \\
&= \sum_{i=1}^{n} (-1)^{i-1} i^2 s_i \\
&= -\sum_{i=1}^{n} (-1)^{i} i^2 s_i \\
&= -[-g'(-1) + g''(-1)] \\
&= g'(-1) - g''(-1) .
\end{aligned}
$$

Now, from $g(t) = (1+x_1t)(1+x_2t)\cdots(1+x_nt)$, we can derive that

$$\begin{aligned} \frac{g'(t)}{g(t)} &= \sum_{i=1}^{n} \frac{x_i}{1+x_it} \\ \Longrightarrow \quad \frac{g''(t)g(t)-g'(t)^2}{g(t)^2} &= -\sum_{i=1}^{n} \frac{x_i^2}{(1+x_it)^2}\,. \end{aligned}$$

Set $x_1 = x_2 = \cdots = x_k = 1/k$ and $x_{k+1} = x_{k+2} = \cdots = x_n = 0$, $2 \le k \le n$, and $t = -1$. Then

$$\frac{g'(-1)}{g(-1)} = \sum_{i=1}^{k} \frac{1/k}{1-1/k} = \frac{k}{k-1}\,,$$

and

$$\begin{aligned} \frac{g''(-1)g(-1)-g'(-1)^2}{g(-1)^2} &= -\sum_{i=1}^{k} \frac{1/k^2}{(1-1/k)^2} = -\frac{k}{(k-1)^2} \\ \Longrightarrow \qquad \frac{g''(-1)}{g(-1)} &= \frac{g'(-1)^2}{g(-1)^2} - \frac{k}{(k-1)^2} \\ &= \frac{k^2}{(k-1)^2} - \frac{k}{(k-1)^2} \\ &= \frac{k}{k-1}\,. \end{aligned}$$

Therefore, $g'(-1) = g''(-1)$, so that

$$f_n(x_1, x_2, \ldots, x_n) = g'(-1) - g''(-1) = 0\,.$$

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PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a proposal is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was proposed without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 April 2002**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in LaTeX). Graphics files should be in LaTeX format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

We regret to have to admit that problem 2647 is a repeat of 2611. It must be a good problem if we should want to use it twice!

2689. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Given $\triangle ABC$ and a point P not on it, draw $BD \| AC$ such that D lies on AP, draw $AE \| CB$ such that E lies on CP, and draw $CF \| AB$ such that F lies on BP. Let X be the point of intersection of the lines AB and CD, let Y be the point of intersection of the lines AC and BE, and let Z be the point of intersection of the lines BC and AF. Prove that X, Y and Z are collinear.

2690. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let $\triangle ABC$ be such that $\angle A$ is the largest angle. Let r be the inradius and R the circumradius. Prove that

$$A \gtrless 90^\circ \iff R + r \gtrless \frac{b+c}{2}\,.$$

2691. *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

The length of one base of an isosceles trapezoid, the equal sides, and the equal diagonals, are all odd integers. Show that if the remaining base also has integer length, then it is divisible by 8.

2692. *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

Let PQ be the distance between the mid-points of the diagonals of quadrilateral $ABCD$ with sides a, b, c, d and diagonals p and q. Give an example of such a quadrilateral where a, b, c, d, p, q and PQ are all positive integers.

2693. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Given triangle ABC and a point P, the line through P parallel to BC, intersects AC, AB at Y_1, Z_1 respectively. Similarly, the parallel to CA intersects BC, AB at X_2, Z_2, and the parallel to AB intersects BC, AC at X_3, Y_3. Locate the point P for which the sum $Y_1P \cdot PZ_1 + Z_2P \cdot PX_2 + X_3P \cdot PY_3$ of products of *signed* lengths is maximal.

2694. *Proposed by Aaron Lee and Jason Wilson, students, Biola University, La Mirada, CA, USA.*

Given a line segment AB, construct a square $ABCD$ using four or fewer circular arcs and a straightedge. The construction should use fewer arcs than those usually given in texts.

2695. *Proposed by Aaron Lee and Jason Wilson, students, Biola University, La Mirada, CA, USA.*

Given a line ℓ and a point P not on it, construct the line through P parallel to ℓ, using two or fewer circular arcs and a straightedge. The construction should use fewer arcs than those usually given in texts.

2696. *Proposed by Aaron Lee and Jason Wilson, students, Biola University, La Mirada, CA, USA.*

Using only a straightedge, construct the tangents from a point outside a given circle (and its centre).

2697. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Find a closed form for $\sum_{k=1}^{n} k\sin^2(kx)$.

2698. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands (adapted by the editor).*

The perimeter of a right triangle with integer sides is a perfect square. The area of the triangle is the cube of an integer. Find the smallest triangle satisfying these conditions.

[Ed. Smeenk asked the case when the hypotenuse has length 240.]

2699. *Proposed by Maureen P. Cox and Albert White, St. Bonaventure University, St. Bonaventure, NY, USA.*

Evaluate $$\prod_{k=1}^{n}\left(\frac{4k+4n-3}{4n}\right)^{\frac{4n+4k-3}{4n^2}}.$$

2700. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Let n be a positive integer. Show that

$$\sum_{k=1}^{n}\frac{k}{n+k}\binom{n}{k} \;<\; \sum_{k=1}^{n}\binom{n}{k}\log\left(\frac{n+k}{n}\right) \;<\; 2^{n-1}.$$

[Ed. "log" is, of course, the natural logarithm.]

2700A. *Proposed by Paul Bracken, CRM, Université de Montréal, Montréal, Québec.*

Show that the function e^{-xn^2} can be written in the following form,

$$\begin{aligned} e^{-xn^2} &= \sum_{k=0}^{n-1}(-1)^k\frac{x^k n^{2k}}{k!} + (-1)^n\frac{x^n n^{2n}}{n!}\phi_x(n)\,, \quad \text{where} \\ \phi_x(n) &= 1-\int_0^{xn^2} e^{-t}\left(1-\frac{t}{xn^2}\right)^n dt\,. \end{aligned}$$

Determine the leading large n behaviour of $\phi_x(n)$, and show that

$$\lim_{n\to\infty} n\phi_x(n) \;=\; 1/x\,.$$

The solution of problems is one of the lowest forms of mathematical research, ... yet its educational value cannot be overestimated. It is the ladder by which the mind ascends into higher fields of original research and investigation. Many dormant minds have been aroused into activity through the mastery of a single problem.

Benjamin Franklin Finkel, in "The American Mathematical Monthly", no. 1.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2577. [2000 : 429] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

Let a_1, a_2, ..., a_n $(n \geq 2)$ be positive integers. Determine the values of n and k $(2 \leq k \leq n)$ for which the following identity holds:

$$\gcd_{1\leq i_1<\ldots<i_k\leq n}(\operatorname{lcm}\{a_{i_1},\ \ldots,\ a_{i_k}\}) \;=\; \operatorname{lcm}_{1\leq i_1<\ldots<i_k\leq n}(\gcd\{a_{i_1},\ \ldots,\ a_{i_k}\})\,.$$

Solution by Michel Bataille, Rouen, France.

We show that the given identity holds if and only if n is odd and $k = \frac{1}{2}(n+1)$.

Suppose first that the equation

$$\gcd_{1\leq i_1<\ldots<i_k\leq n}(\operatorname{lcm}\{a_{i_1},\ \ldots,\ a_{i_k}\}) \;=\; \operatorname{lcm}_{1\leq i_1<\ldots<i_k\leq n}(\gcd\{a_{i_1},\ \ldots,\ a_{i_k}\})$$

holds for all n–tuples $(a_1, a_2, \ldots, a_n)$ of positive integers.

Then it must hold, in particular, for $a_i = 2^i$ $(1 \leq i \leq n)$. In this case, $\operatorname{lcm}_{1\leq i_1<\ldots<i_k\leq n}\{a_{i_1},\ \ldots,\ a_{i_k}\} = 2^{i_k} \geq 2^k$ (since $i_k \geq k$) with equality when $(i_1, i_2, \ldots, i_k) = (1, 2, \ldots, k)$.

Hence, $\gcd_{1\leq i_1<\ldots<i_k\leq n}(\operatorname{lcm}\{a_{i_1},\ \ldots,\ a_{i_k}\}) = 2^k$.

Similarly, $\gcd_{1\leq i_1<\ldots<i_k\leq n}\{a_{i_1},\ \ldots,\ a_{i_k}\} = 2^{i_1} \leq 2^{n-(k-1)}$ with equality when $(i_1, i_2, \ldots, i_k) = (n-k+1, \ldots, n)$.

Hence, $\operatorname{lcm}_{1\leq i_1<\ldots<i_k\leq n}(\gcd\{a_{i_1},\ \ldots,\ a_{i_k}\}) = 2^{n-(k-1)}$.

It follows that $2^k = 2^{n-(k-1)}$, or $n = 2k-1$; that is, n is odd and $k = \frac{1}{2}(n+1)$.

Conversely, suppose that $n = 2m+1$ and $k = m+1$. When q is a prime integer, we will denote by $v_q(a)$ the exponent of the highest power of q which divides a.

Let $(a_1, a_2, \ldots, a_{2m+1})$ be any family of $n = 2m+1$ positive integers. We have to prove that

$$\begin{aligned} L &= \gcd_{1\leq i_1<\ldots<i_{m+1}\leq 2m+1}(\operatorname{lcm}\{a_{i_1},\ \ldots,\ a_{i_{m+1}}\}) \\ &= \operatorname{lcm}_{1\leq i_1<\ldots<i_{m+1}\leq 2m+1}(\gcd\{a_{i_1},\ \ldots,\ a_{i_{m+1}}\}) \;=\; R \end{aligned}$$

Fix an arbitrary prime p. We may assume, by changing the numbering if necessary, that $v_p(a_1) \leq v_p(a_2) \leq \ldots \leq v_p(a_{2m+1})$.

In any sub-family $(a_{i_1}, \ldots, a_{i_{m+1}})$, there is at least one number from $\{a_{m+1}, \ldots, a_{2m+1}\}$ such that $v_p(\text{lcm}\{a_{i_1}, \ldots, a_{i_{m+1}}\}) \geq v_p(a_{m+1})$.

Moreover, this value $v_p(a_{m+1})$ is attained for the sub-family $(a_1, a_2, \ldots, a_{m+1})$. It follows that $v_p(L) = v_p(a_{m+1})$.

Now, in $(a_{i_1}, \ldots, a_{i_{m+1}})$, there is also a number coming from $\{a_1, a_2, \ldots, a_{m+1}\}$. Hence, $v_p(\gcd\{a_{i_1}, \ldots, a_{i_{m+1}}\}) \leq v_p(a_{m+1})$, with equality for the sub-family $(a_{m+1}, \ldots, a_{2m+1})$. It follows that $v_p(R) = v_p(a_{m+1})$.

Thus, $v_p(L) = v_p(R)$, and since this result may be obtained for each prime integer p, we have $L = P$.

Also solved by CON AMORE PROBLEM GROUP, The Danish University of Education, Copenhagen, Denmark; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GERRY LEVERSHA, St. Paul's School, London, England; CHRIS WILDHAGEN, Rotterdam, the Netherlands; and the proposer.

The solutions offered by Janous, Leversha, Wildhagen and the proposer were very terse. The editor preferred to highlight a fuller solution for the sake of our younger readers. Janous ended his solution with the comment "Neat!".

2578. [2000 : 429] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

For each integer n, determine the hundreds and the units digits of the number $\dfrac{1+5^{2n+1}}{6}$.

Solution by Mihai Cipu, IMAR, Bucharest, Romania

First we note that n must be non-negative in order that

$$x_n := \frac{1+5^{2n+1}}{6} \geq 1 .$$

Thus for $n < 0$ the integer part of x_n is 0 and the hundreds and units digits are both 0. For $n = 0$ and $n = 1$ one gets $x_0 = 1$ and $x_1 = 21$, respectively. [*Ed.:* which means that the units digits is 1 and the hundreds digit is 0.] Let us suppose now that $n \geq 2$. Then

$$\begin{aligned} x_n &= 5^{2n} - 5^{2n-1} + 5^{2n-2} - \cdots - 5 + 1 = 4 \cdot 5^{2n-1} + x_{n-1} \\ &= 100 \cdot 5^{2n-3} + x_{n-1} . \end{aligned}$$

Hence, $x_n \equiv 500 + x_{n-1} \pmod{1000}$, and the hundreds digit is repeated with period 2, while the last digit is constant: $x_{2n} \equiv 21 \pmod{1000}$ and $x_{2n+1} \equiv 521 \pmod{1000}$.

Also solved by ANGELO STATE UNIVERSITY PROBLEM SOLVING GROUP, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; BRIAN BEASLEY, *Presbyterian College, Clinton, SC, USA; ROBERT BILINSKI, Outremont, Québec; PIERRE BORNSZTEIN, Pontoise, France; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; LAKE SUPERIOR STATE UNIVERSITY PROBLEM SOLVING GROUP; KEE-WAI LAU, Hong Kong, China; GERRY LEVERSHA, St. Paul's School, London, England; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; CARL LIBIS, Richard Stockton College of New Jersey, Pomona, NJ, USA; HENRY LIU, student, Trinity College, Cambridge, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; HENRY J. PAN, student, East York Collegiate Institute, Toronto, Ontario; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; D.J. SMEENK, Zaltbommel, the Netherlands; SOUTHWEST MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; and CHRIS WILDHAGEN, Rotterdam, the Netherlands.*

Many solvers pointed out that it appeared strange to not ask for the tens digit as well, since it was there for the asking. Indeed, the proposer's original submission asked for only the last two digits of the expression for $n > 0$. Several solvers, including the proposer, solved this problem instead of the one in print. Those solvers were ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; LI ZHOU, Polk Community College, Winter Haven, FL; and the proposer. There was also one incorrect solution.

2579. [2000 : 430] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

The excircle on the side BC of triangle ABC touches AC and AB, respectively at Y_A and Z_A. Likewise, the one on CA touches BC and BA at X_B and Z_B, and the one on AB touches CA and CB at Y_C and X_C. Let A' be the intersection of Z_BX_B and X_CY_C, B' be that of X_CY_C and Y_AZ_A, and C' be that of Y_AZ_A and Z_BX_B. Show that AA', BB' and CC' are concurrent. What is the point of intersection of these three lines?

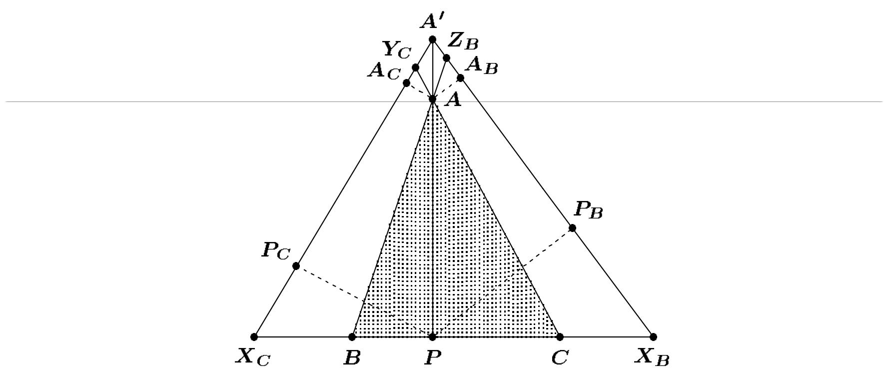
Solution by Niels Bejlegaard, Copenhagen, Denmark.

The sides of $\triangle A'B'C'$ intercept lengths equal to $s = \frac{a+b+c}{2}$ from each vertex of $\triangle ABC$; more precisely $s = CX_C = CY_C = BX_B = BZ_B$, so that

$$AY_C = s - b\,, \quad AZ_B = s - c\,, \text{ and } BX_C = CX_B = s - a\,.$$

[See figure on page 540.]

Moreover, because the triangles CX_CY_C and BX_BZ_B are isosceles, X_CY_C is perpendicular to the bisector of $\angle C$ while X_BZ_B is perpendicular to the bisector of $\angle B$. Call P the point where AA' intersects BC; we shall see that P is the foot of the altitude from A in $\triangle ABC$. In the figure, A_C and P_C are the respective feet of the perpendiculars to X_CY_C from A and from P, while A_B and P_B are the feet of the perpendiculars to X_BZ_B from A and



from P. Hence, $AA_C = (s-b)\cos\frac{C}{2}$ and $AA_B = (s-c)\cos\frac{B}{2}$, so that for some $k > 0$, $PP_C = k(s-b)\cos\frac{C}{2}$ and $PP_B = k(s-c)\cos\frac{B}{2}$. As a consequence,

$$PB = \frac{PP_C}{\cos\frac{C}{2}} - (s-a) = k(s-b) - (s-a)\,,$$

and similarly,

$$PC = k(s-c) - (s-a)\,.$$

Since $a = PB + PC$, we deduce that $k = \frac{b+c}{a}$ and

$$\begin{aligned} PB &= \frac{b+c}{a}\cdot\frac{a+c-b}{2} - \frac{b+c-a}{2} \\ &= \frac{a^2+c^2-b^2}{2a} = \frac{2ac\cos B}{2a} = c\cos B\,. \end{aligned}$$

This indicates that P is the foot of the altitude from A in $\triangle ABC$, as claimed. Similarly, BB' and CC' are the other altitudes of $\triangle ABC$, so the three given lines concur at the orthocentre of $\triangle ABC$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GERRY LEVERSHA, St. Paul's School, London, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

2580. [2000 : 430] *Proposed by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

Suppose that a, b and c are positive real numbers. Prove that

$$\frac{b+c}{a^2+bc}+\frac{c+a}{b^2+ac}+\frac{a+b}{c^2+ab} \;\le\; \frac{1}{a}+\frac{1}{b}+\frac{1}{c}\,.$$

Solution by Richard Eden, Ateneo de Manila University, Philippines; John Fremlin, student, Colchester Royal Grammar School, Colchester, England; and Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Let $D = abc(a^2+bc)(b^2+ac)(c^2+ab)$. Clearly, $D > 0$. We have

$$\begin{aligned}
&\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{b+c}{a^2+bc}-\frac{c+a}{b^2+ac}-\frac{a+b}{c^2+ab}\\
&\qquad= \frac{a^4b^4+b^4c^4+c^4a^4-a^4b^2c^2-b^4c^2a^2-c^4a^2b^2}{D}\\
&\qquad= \frac{(a^2b^2-b^2c^2)^2+(b^2c^2-c^2a^2)^2+(c^2a^2-a^2b^2)^2}{2D} \;\ge\; 0\,,
\end{aligned}$$

which shows that the given inequality is true. Equality holds if and only if $a = b = c$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina(2 solutions); MICHEL BATAILLE, Rouen, France; NIELS BEJLEGAARD, Copenhagen, Denmark; MIHÁLY BENCZE, Brasov, Romania; PIERRE BORNSZTEIN, Pontoise, France; PAUL BRACKEN, CRM, Université de Montréal, Montréal, Québec; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; MIHAI CIPU, IMAR, Bucharest, Romania; CON AMORE PROBLEM GROUP, The Danish University of Education, Copenhagen, Denmark; JOSÉ LUÍS DIAZ-BARRERO, Universitat Politècnica de Catalunya, Terrassa, Spain; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; THOMAS JANG, Southwest Missouri State University, Springfield, MO, USA; D. KIPP JOHNSON, Valley Catholic High School, Gaston, OR, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; DAVID LOEFFLER, student, Cotham School, Bristol, UK; PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; JOEL TAY, Anglo-Chinese School, Singapore; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2581. [2000 : 430] *Proposed by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

Suppose that a, b and c are positive real numbers. Prove that

$$\frac{ab+c^2}{a+b}+\frac{bc+a^2}{b+c}+\frac{ca+b^2}{c+a} \;\ge\; a+b+c\,.$$

Solution by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.

Let $D = (a+b)(b+c)(c+a)$. Clearly, $D > 0$. We show that the difference between the left-hand side and the right-hand side of the inequality is non-negative.

$$\begin{aligned}
&\frac{ab+c^2}{a+b} - c + \frac{bc+a^2}{b+c} - a + \frac{ca+b^2}{c+a} - b \\
&= \frac{c^2+ab-ac-bc}{a+b} + \frac{a^2+bc-ab-ac}{b+c} + \frac{b^2+ac-ab-bc}{c+a} \\
&= \frac{(c-a)(c-b)}{a+b} + \frac{(a-b)(a-c)}{b+c} + \frac{(b-a)(b-c)}{c+a} \\
&= \frac{(c^2-a^2)(c^2-b^2)+(a^2-b^2)(a^2-c^2)+(b^2-a^2)(b^2-c^2)}{D} \\
&= \frac{a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2}{D} \\
&= \frac{[(a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2]}{2D} \geq 0\,.
\end{aligned}$$

Equality holds if and only if $a = b = c$.

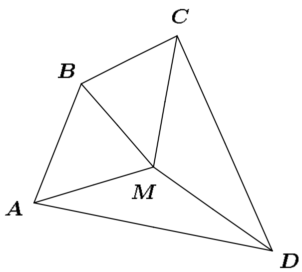
Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (four other solutions); MICHEL BATAILLE, Rouen, France; NIELS BEJLEGAARD, Denmark; MIHÁLY BENCZE, Brasov, Romania; PIERRE BORNSZTEIN, Pontoise, France; PAUL BRACKEN, CRM, Université de Montréal, Montréal, Québec; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; MIHAI CIPU, IMAR, Bucharest, Romania; CON AMORE PROBLEM GROUP, The Danish University of Education, Copenhagen, Denmark; JOSÉ LUÍS DIAZ-BARRERO, Universitat Politècnica de Catalunya, Terrassa, Spain; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD EDEN, Ateneo de Manila University, Philippines; JOHN FREMLIN, student, Colchester Royal Grammar School, Colchester, England; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA (two solutions); THOMAS JANG, Southwest Missouri State University, Springfield, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Valley Catholic High School, Gaston, OR, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; HENRY LIU, Trinity College, Cambridge, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; JOEL TAY, Anglo-Chinese School, Singapore; PANOS E. TSAOUSSOGLOU, Athens, Greece; CHRIS WILDHAGEN, Rotterdam, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Most solutions were similar to the above one. Bornsztein and Klamkin noted that the given inequality can be reduced to particular cases of Schur's Inequality

$$x^t(x-y)(x-z) + y^t(y-z)(y-x) + z^t(z-x)(z-y) \;\geq\; 0\,,$$

where t is a real number and x, y, z are non-negative real numbers.

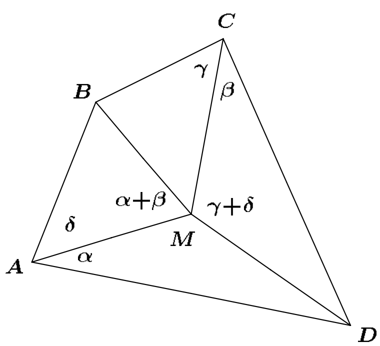
2583. [2000 : 430] *Proposed by Nairi M. Sedrakyan, Yerevan, Armenia.*

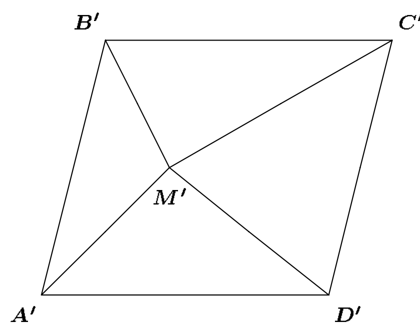


Given a point M inside the convex quadrangle (see diagram), such that $\angle AMB = \angle MAD + \angle MCD$, $\angle CMD = \angle MCB + \angle MAB$ and $MA = MC$.

Prove that $AB \cdot CM = BC \cdot MD$.

I. *Solution by the proposer.*





Consider the points A', B', C', D', and M' such that the following conditions are satisfied (see the diagram above): $A'B' = AB \cdot CM$, $B'M' = BM \cdot CM$, $A'M' = AM \cdot CM$, $B'C' = BM \cdot AD$, $C'M' = BM \cdot MD$, $C'D' = BC \cdot MD$, $M'D' = MA \cdot MD$, $A'D' = MA \cdot CD$.

Then we have $\triangle ABM \sim \triangle A'B'M'$, $\triangle BMC \sim \triangle C'M'D'$, $\triangle CMD \sim \triangle A'M'D'$, and $\triangle AMD \sim \triangle B'M'C'$. Hence,

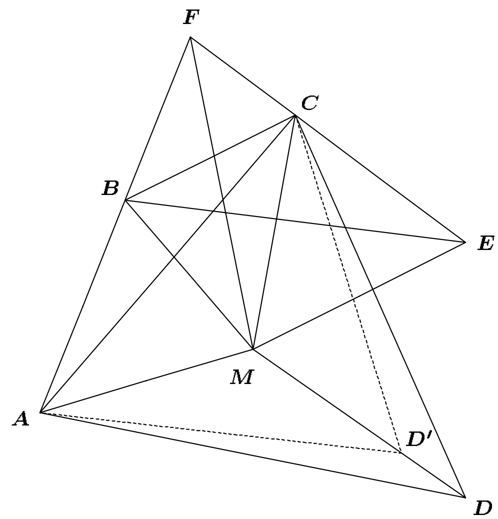
$$\angle A'M'B' = \angle AMB = \angle MAD + \angle MCD = \angle M'B'C' + \angle M'A'D',$$

from which it follows that $\angle D'A'B' + \angle A'B'C' = 180^\circ$, implying that $B'C' \parallel A'D'$. We also have

$$\angle A'M'D' = \angle CMD = \angle MCB + \angle MAB = \angle M'D'C' + \angle M'A'B',$$

from which we see that $\angle B'A'D' + \angle A'D'C' = 180^\circ$, implying that $A'B' \parallel C'D'$. Therefore, the quadrilateral $A'B'C'D'$ is a parallelogram. Thus, $A'B' = C'D'$, from which follows $AB \cdot CM = BC \cdot MD$.

II. *Solution by Nikolaos Dergiades, Thessaloniki, Greece.*



On MD we take the point D' such that

$$\frac{MD'}{CM} = \frac{AB}{BC}. \tag{1}$$

We will prove that $D' \equiv D$. See the diagram above. Since

$$\begin{aligned} \angle AMB + \angle BMC &= \angle MAD + \angle MCD + \angle ADC \\ \text{and} \quad \angle CMD + \angle AMD &= \angle MCB + \angle MAB + \angle ABC \end{aligned}$$

we get from the hypothesis that

$$\angle BMC = \angle ADC \tag{2}$$
$$\text{and} \quad \angle AMD = \angle ABC \tag{3}$$

By rotation around M we move the point A onto C and $\triangle MAB$ onto $\triangle MCE$, and we have

$$\angle BCE = \angle CMD. \tag{4}$$

Let AB and EC meet at F. Since $\angle MAB = \angle MCE$, the points M, A, F, C are concyclic. A similar argument applies to the points M, B, F, E. From (1), (3) and $MA = MC$ we have $\triangle D'MA \sim \triangle ABC$, which implies

$$\angle MD'A = \angle BAC = \angle FMC. \tag{5}$$

From (1) and (4) we have $\triangle BCE \sim \triangle CMD'$, implying

$$\angle MD'C = \angle BEC = \angle FMB. \tag{6}$$

From (2), (5), and (6) we get $\angle AD'C = \angle BMC = \angle ADC$ which implies $D' \equiv D$, and by (1) we have $AB \cdot CM = BC \cdot MD$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina. His solution was essentially the same as solution I *above.*

2584. [2000 : 430] *Proposed by Nairi M. Sedrakyan, Yerevan, Armenia.*

You are given that X, Y, Z and T are points on the chord AB of the circle Γ. Circles Γ_1 and Γ_2 pass through the points X and Y, and touch the circle Γ at points P and S, respectively, while the circles Γ_3 and Γ_4 pass through the points Z and T, and touch the circle Γ at points Q and R, respectively. Also, Q belongs to the arc APB and the segments XY and ZT do not have common points. Prove that the segments PR, QS and AB intersect at the same points.

Solution by Nikolaos Dergiades, Thessaloniki, Greece.

Let the tangents to Γ at P and S intersect at K. [If these tangents are parallel then we shall say that K is at infinity.] Then K is the radical centre of Γ, Γ_1, Γ_2, and it therefore lies on the extension of the chord AB. Similarly, if M is the intersection point of the tangents to Γ at R and Q, then M also lies on AB. In other words, if N is the point where KP meets MQ and L is the point where KS meets MR, then AB is a portion of the diagonal KM of the quadrilateral $KLMN$. Since Γ is inscribed in the hexagon $KLRMNP$, Brianchon's theorem implies that the diagonals KM, LN, PR pass through the same point. Similarly QS passes through the point of intersection of KM and LN; therefore PR, QS, and AB intersect in the same point.

Editor's comment. From the first lines of Dergiades' argument we see that our problem reduces to the familiar projective result,

> *If the vertices of a complete quadrangle lie on a conic, the tangents at a pair of vertices meet in a point of the line joining the diagonal points of the quadrangle that are not on the side joining the two vertices.*

Dergiades shows that this is simply a degenerate case of Brianchon's theorem. Woo provides another quick argument: there is a projective transformation that takes $PQRS$ to a rectangle inscribed in a circle [where AB is part of some diameter], in which case the desired conclusion immediately follows.

Also solved by MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; PETER Y. WOO, Biola University, La Mirada, CA, USA (3 proofs); and the proposer.

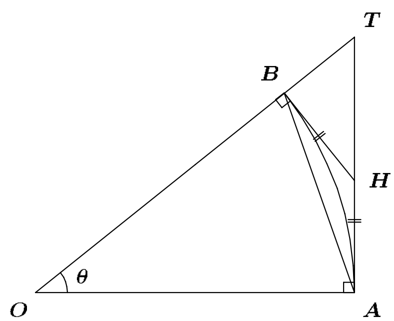
2585. [2000 : 430] *Proposed by Vedula N. Murty, Visakhapatnam, India.*

Prove that, for $0 < \theta < \pi/2$,

$$\tan\theta + \sin\theta \;>\; 2\theta\,.$$

I. *Solution by Toshio Seimiya, Kawasaki, Japan.*

Let OAB be a sector with centre O, radius 1 and $\angle AOB = \theta$, where $0 < \theta < \frac{\pi}{2}$.



Let T be the point of intersection of OB and the tangent to the arc AB at A. Let H be the point of intersection of AT and the tangent to the arc AB at B. Then $\angle OAH = \angle OBH = 90^\circ$ and $HA = HB$. Since $\angle TBH = 90^\circ$, $TH > HB = HA$. Let $[F]$ denote the area of a plane figure F. Then $HA < TH$ implies

$$[HAB] \;<\; [HBT]\,,$$

and therefore,

$$[\text{sector } OAB] - [OAB] < [OAT] - [\text{sector } OAB]\,.$$

The last inequality can be written as

$$\frac{1}{2}\theta - \frac{1}{2}\sin\theta \;<\; \frac{1}{2}\tan\theta - \frac{1}{2}\theta\,,$$

or

$$2\theta \;<\; \tan\theta - \sin\theta\,,$$

which is the desired inequality.

II. *Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

More generally, we show that for $0 < \theta < \frac{\pi}{2}$ and any pair of real numbers a and b with $a^2 + b^2 > 0$,

$$a^2\tan\theta + b^2\sin\theta \;>\; 2ab\theta\,.$$

Consider the function $f(\theta) = a^2 \tan\theta + b^2 \sin\theta - 2ab\theta$ on the interval $(0, \frac{\pi}{2})$. We have

$$\begin{aligned} f'(\theta) &= a^2 \sec^2\theta + b^2 \cos\theta - 2ab \\ &> a^2 \sec^2\theta + b^2 \cos^2\theta - 2ab \\ &= (a\sec\theta - b\cos\theta)^2 \;\geq\; 0\,. \end{aligned}$$

Thus $f(\theta)$ is strictly increasing on the interval $(0, \frac{\pi}{2})$. In addition, $f(0) = 0$ and $f(\theta)$ is increasing and continuous on $[0, \frac{\pi}{2})$. Therefore, $f(\theta) > f(0) = 0$ for every θ in $(0, \frac{\pi}{2})$, which gives the desired inequality.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (three solutions); MICHEL BATAILLE, Rouen, France; BRIAN BEASLEY, Presbyterian College, Clinton, SC, USA; MIHÁLY BENCZE, Brasov, Romania; PIERRE BORNSZTEIN, Pontoise, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIHAI CIPU, IMAR, Bucharest, Romania; CON AMORE PROBLEM GROUP, The Danish University of Education, Copenhagen, Denmark; NIKOLAOS DERGIADES, Thessaloniki, Greece; JOSÉ LUÍS DIAZ-BARRERO, Universitat Politècnica de Catalunya, Terrassa, Spain; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; CHARLES HENDEE, Southwest Missouri State University, Springfield, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Valley Catholic High School, Gaston, OR, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong; HENRY LIU, Trinity College, Cambridge, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK (two solutions); PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA; HENRY J. PAN, student, East York Collegiate Institute, Toronto, Ontario; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; NICHOLAS THAM, Raffles Junior College, Singapore; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; CHRIS WILDHAGEN, Rotterdam, the Netherlands; CALVIN ZHIWEI, Singapore; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There were two incorrect solutions submitted.

Seimiya was the only solver who submitted a geometric solution. Janous noted that the following known fact can be used to generalize the given inequality: If a and b are positive real numbers, then

$$M_\lambda(a,b) \;=\; \begin{cases} (a^\lambda + b^\lambda)^{\frac{1}{\lambda}}\,, & \lambda \neq 0 \\ \sqrt{ab}\,, & \lambda = 0 \end{cases}$$

is an increasing function of λ. The given inequality is $M_1(\tan x, \sin x) > x$ for $x \in (0, \frac{\pi}{2})$. Janous then proceeded to show the stronger inequality $M_0(\tan x, \sin x) > x$ for $x \in (0, \frac{\pi}{2})$ and conjectured that $M_\lambda(\tan x, \sin x) > x$ for all $x \in (0, \frac{\pi}{2})$ if and only if $\lambda \geq -(\log 2)/\log(\frac{\pi}{2})$. The inequality $M_0(\tan x, \sin x) > x$ was also proven by Arslanagić in one of his solutions.

2586. [2000 : 431] *Proposed by Michael Lambrou, University of Crete, Crete, Greece.*

Find all (real or complex) solutions of the system

$$\begin{aligned} 3x + x^3 &= y(1+3x^2)\,, \\ 3y + y^3 &= z(1+3y^2)\,, \\ 3z + z^3 &= w(1+3z^2)\,, \\ 3w + w^3 &= x(1+3w^2)\,. \end{aligned}$$

Solution by the Southwest Missouri State University Problem Solving Group, Springfield, MO, USA.

Since $3u+u^3$ and $1+3u^2$ cannot vanish simultaneously, we may rewrite the system as

$$y = \frac{3x+x^3}{1+3x^2}\,,\quad z = \frac{3y+y^3}{1+3y^2}\,,\quad w = \frac{3z+z^3}{1+3z^2}\,,\quad x = \frac{3w+w^3}{1+3w^2}\,.$$

It is an elementary fact from hyperbolic trigonometry that

$$\tanh 3t \;=\; \frac{3\tanh t + \tanh^3 t}{1+3\tanh^2 t}\,.$$

Now the hyperbolic tangent function takes on all complex values except 1 and -1. From the system above, it is immediate that if $x = 1$, then $y = z = w = 1$ and if $x = -1$, then $y = z = w = -1$. For any other value of x, let $x = \tanh t$. Applying our hyperbolic trigonometric identity to the system above yields: $y = \tanh 3t$, $z = \tanh 9t$, $w = \tanh 27t$, and $x = \tanh 81t$. This implies that $\tanh t = \tanh 81t$. Since the period of the hyperbolic tangent is πi, this forces $t + i\pi k = 81t$, where k is an integer, and hence $t = i\pi k/80$. Since $\tanh iu = i\tanh u$ and it suffices to take t over a full period of the hyperbolic tangent function, the set of all solutions to the original system is

$$\begin{aligned} \{x,\,y,\,z,\,w\} \;=\; \pm\{1,\,1,\,1,\,1\} \quad &\text{or} \\ \{i\tan(\pi k/80), i\tan(3\pi k/80), i\tan(9\pi k/80&, i\tan(27\pi k/80)\} \\ \text{for } k = -39,\,\ldots,\,39. \end{aligned}$$

Note that a similar argument will work for analogous systems in any number of variables.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES R. DIMINNIE and ROGER ZARNOWSKI, Angelo State University, San Angelo, TX, USA; RICHARD B. EDEN, Ateno de Manila University, Philippines; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; CHRIS WILDHAGEN, Rotterdam, the Netherlands; and the proposer.

Most submissions were variations of the above one. Of the 81 *solutions, three are real. Several solvers missed the solutions* $(1,1,1,1)$ *and* $(-1,-1,-1,-1)$*, which may be found by inspection. There was one incomplete solution.*

2587. [2000 : 431] *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

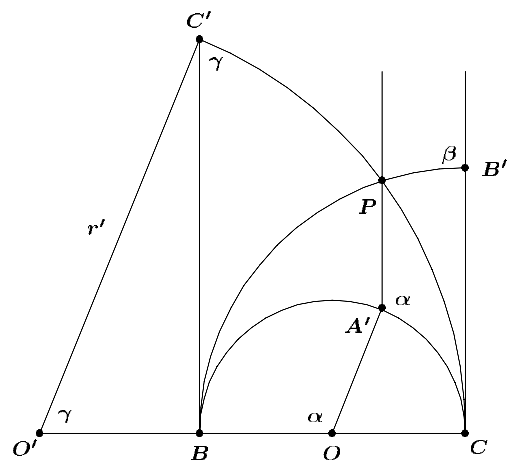
In the half plane $Z = \{(x,y) : y \geq 0\}$, let f be the union of the set of all semicircles lying in Z with diameters on the x–axis, with the set of all lines in Z perpendicular to the x–axis.

Denote by f_{XY} the unique member of f that goes through any two points X and Y in Z. For any three points A, B and C in Z, denote by $\triangle ABC$ the curvilinear triangle formed by the arcs f_{AB}, f_{BC} and f_{CA}.

Let A, B and C be any three points on the x–axis. Let P be any point in the interior of $\triangle ABC$. Let $A' = f_{AP} \cap f_{BC}$, $B' = f_{BP} \cap f_{CA}$ and $C' = f_{CP} \cap f_{AB}$. Let α be the angle at A', interior to $\triangle CAA'$, let β be the angle at B' interior to $\triangle ABB'$, and let γ be the angle at C' interior to $\triangle BCC'$.

Prove that $\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)\tan\left(\frac{\gamma}{2}\right) = 1$.

I. *Solution by the proposer.*



Invert the figure in a circle whose centre is A to get the accompanying diagram — the point A goes to infinity and the semicircles f_{AP}, f_{AB}, and f_{AC} all become halflines perpendicular to BC. Introduce coordinates so that the origin O is the mid-point of BC, and B is $(-1,0)$ while C is $(1,0)$. Let O' and r' be the centre and radius of the arc $f_{CC'}$. Then $\angle CO'C' = \gamma$, $O'B = r'\cos\gamma$, and $BC = 2 = r'(1-\cos\gamma)$. Therefore,

$$r' = \frac{2}{1-\cos\gamma} = \csc^2\frac{\gamma}{2}.$$

If x' is the x–coordinate of O', then

$$-x' = OO' = CO' - CO = r' - 1 = \csc^2\frac{\gamma}{2} - 1 = \cot^2\frac{\gamma}{2},$$

and the points of $f_{CC'}$ satisfy

$$\left(x+\cot^2\frac{\gamma}{2}\right)^2+y^2 \;=\; \left(\csc^2\frac{\gamma}{2}\right)^2 ,$$

or

$$x^2+y^2+2x\cot^2\frac{\gamma}{2} \;=\; \csc^4\frac{\gamma}{2}-\cot^4\frac{\gamma}{2} \;=\; \csc^2\frac{\gamma}{2}+\cot^2\frac{\gamma}{2} ;$$

that is

$$x^2+y^2+2x\cot^2\frac{\gamma}{2} \;=\; 1+2\cot^2\frac{\gamma}{2} .$$

Similarly, $f_{BB'}$ satisfies

$$x^2+y^2-2x\tan^2\frac{\beta}{2} \;=\; 1+2\tan^2\frac{\beta}{2} .$$

Subtracting, we find that the x–coordinate x'' of P satisfies

$$x'' \;=\; \frac{\cot^2\frac{\gamma}{2}-\tan^2\frac{\beta}{2}}{\cot^2\frac{\gamma}{2}+\tan^2\frac{\beta}{2}} .$$

From the diagram we see that

$$x'' \;=\; -\cos\angle A'OB \;=\; -\cos\alpha .$$

Therefore,

$$\cot^2\frac{\alpha}{2} \;=\; \frac{1+\cos\alpha}{1-\cos\alpha} \;=\; \frac{1-x''}{1+x''} \;=\; \frac{\tan^2\frac{\beta}{2}}{\cot^2\frac{\gamma}{2}} .$$

Consequently, $\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)\tan\left(\frac{\gamma}{2}\right) \;=\; 1$ as desired.

II. *Outline of the solution by J. Chris Fisher, University of Regina, Sakatchewan.*

In the half-plane model of the hyperbolic plane (with BC as the line at infinity), the curvilinear triangle of the problem becomes the trebly asymptotic triangle ABC of hyperbolic geometry. This means that each side of the triangle is parallel to the other two, so that the three sides meet in pairs at the improper points A, B, and C. We can define a *cevian* of an asymptotic triangle to be a halfline from a point on a side to the opposite improper vertex. Our problem (extended to include the converse) requires proving a hyperbolic version of Ceva's theorem for asymptotic triangles; specifically,

> *The three cevians of a trebly asymptotic triangle intersect in a point if and only if the equally oriented angles α,β,γ that the sides make with their cevians satisfy*
>
> $$\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)\tan\left(\frac{\gamma}{2}\right) \;=\; 1 .$$

Our result makes use of Lobachevsky's formula for the relationship between the acute angles U and V of a right triangle, namely

$$\cos U \;=\; \frac{\sin V}{\sin \Pi(u)}\,,$$

or equivalently,

$$\tan\frac{U}{2} \;=\; \sqrt{\frac{\sin\Pi(u)-\sin V}{\sin\Pi(u)+\sin V}}\,.$$

Here, $\Pi(u)$ denotes the angle of parallelism that corresponds to the length u of the side opposite the angle U. Assume that the cevians all intersect at the point P. Denote the feet of the perpendiculars from P to the sides of the given asymptotic triangle by A^*, B^*, C^*, and let the distances be $a = PA^*$, $b = PB^*$, and $c = PC^*$. We now have three right triangles. In the right triangle $PA'A^*$, for example, the angle at A' is either α or its supplement (whichever is acute), and the side opposite has length a. To compute $\tan\frac{\alpha}{2}$, it suffices to show that the angle at P is $\left|\Pi(b)-\Pi(c)\right|$. This follows quickly from the fact that the angles formed by the half lines that meet at P are $2\Pi(a)$, $2\Pi(b)$, and $2\Pi(c)$ (and consequently, $\Pi(a)+\Pi(b)+\Pi(c) \;=\; \pi$). Thus, for the first triangle we have

$$\tan\frac{\alpha}{2} \;=\; \sqrt{\frac{\sin\Pi(a)\pm\sin(\Pi(b)-\Pi(c))}{\sin\Pi(a)\mp\sin(\Pi(b)-\Pi(c))}}\,.$$

Multiplying this by the similar equations for the other two tangents produces an expression that reduces to 1.

No other solutions were submitted.

2588. [2000 : 431] *Proposed by Niels Bejlegaard, Copenhagen, Denmark.*

Each positive whole integer a_k $(1 \le k \le n)$ is less than a given positive integer N. The least common multiple of any two of the numbers a_k is greater than N.

(a) Show that $\displaystyle\sum_{k=1}^{n}\frac{1}{a_k} \;<\; 2$.

(b)* Show that $\displaystyle\sum_{k=1}^{n}\frac{1}{a_k} \;<\; \frac{6}{5}$.

(c)* Find the smallest real number γ such that $\displaystyle\sum_{k=1}^{n}\frac{1}{a_k} \;<\; \gamma$.

Editor's comment.

There was one solution to part (a) submitted, by Michel Bataille, Rouen, France. Pierre Bornsztein, Pontoise, France, and Walther Janous, Innsbruck, Austria, both pointed out that part (a) was proposed by Paul Erdős as Problem 4365 in the American Mathematical Monthly, Vol. 56 (1949). Part (b) was settled by R. S. Lehman in the same publication, Vol. 58 (1951), pp. 345–346. Janous also noted that part (c) is entirely solved. Indeed, $\sum_{k=1}^{n}\frac{1}{a_k} \le \frac{31}{32}$, with $\frac{31}{32}$ best possible. This was proved in A. Schinzel et G. Szeueres, Sur un problème de M. Paul Erdős, Acta Sci Math. (Szeged) Vol. 20 (1959), pp. 221–229. Finally, Janous noted that several further results "in the vicinity" of the Erdős one can be found in D. S. Mitrinović and M. S. Popadić, Inequalities in Number Theory, Naučni Podmladak, Nič 1978, Chapter 2, pp. 26–37.

2589. [2000 : 497] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

For $n = 2, 3, \ldots$, evaluate $\displaystyle\sum_{k=1}^{\lfloor n/2\rfloor}\binom{n}{k}\binom{n}{k-1}$.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let S_n denote the given sum. Note first that $S_n = \displaystyle\sum_{k=1}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{k}\binom{n}{n+1-k}$.

Since $(1+x)^{2n} = (1+x)^n(1+x)^n = \left(\displaystyle\sum_{i=0}^{n}\binom{n}{i}x^i\right)\left(\displaystyle\sum_{j=0}^{n}\binom{n}{j}x^j\right)$,

we have, by comparing the coefficients of the term x^{n+1}, that

$$\binom{2n}{n+1} = \sum_{k=1}^{n}\binom{n}{k}\binom{n}{n+1-k}. \qquad (1)$$

If $n = 2m$ is even, where $m \ge 1$, then by letting $j = n-k+1$ we have

$$\sum_{k=m+1}^{n}\binom{n}{k}\binom{n}{n+1-k} = \sum_{j=1}^{m}\binom{n}{n+1-j}\binom{n}{j}. \qquad (2)$$

From (1) and (2) we then get

$$\begin{aligned}\binom{2n}{n+1} &= \sum_{k=1}^{m}\binom{n}{k}\binom{n}{n+1-k} + \sum_{k=m+1}^{n}\binom{n}{k}\binom{n}{n+1-k}\\ &= 2\sum_{k=1}^{m}\binom{n}{k}\binom{n}{n+1-k},\end{aligned}$$

whence $S_n = \frac{1}{2}\binom{2n}{n+1}$.

If $n = 2m+1$ is odd, where $m \geq 0$, then the same change of index $j = n - k + 1$ would yield

$$\sum_{k=m+2}^{n}\binom{n}{k}\binom{n}{n+1-k} = \sum_{j=1}^{m}\binom{n}{n+1-j}\binom{n}{j}. \qquad (3)$$

From (1) and (3) we then get

$$\begin{aligned}\binom{2n}{n+1} &= 2\sum_{k=1}^{m}\binom{n}{k}\binom{n}{n+1-k} + \binom{n}{m+1}\binom{n}{n-m}\\ &= 2\sum_{k=1}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{k}\binom{n}{n+1-k} + \binom{n}{m+1}^2,\\ \text{whence } S_n &= \frac{1}{2}\left(\binom{2n}{n+1} - \binom{n}{\frac{n+1}{2}}^2\right).\end{aligned}$$

Taking both cases into account, S_n can be expressed as follows:

$$S_n = \frac{1}{2}\left(\binom{2n}{n+1} - \frac{1-(-1)^n}{2}\binom{n}{\frac{n+1}{2}}^2\right).$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OLEG IVRII, (grade 8) student, Cummer Valley Middle School, North York, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GERRY LEVERSHA, St. Paul's School, London, England; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; HENRY LIU, student, Trinity College, Cambridge, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; WILLIAM MOSER, McGill University, Quebec; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; CHRIS WILDHAGEN, Rotterdam, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and proposer.

All the submitted solutions are either very similar to, or are minor variations of, the one given above, though many solvers made claims, without any justifications, like "$S_n = \frac{1}{2}\sum_{k=1}^{n}\binom{n}{k}\binom{n}{k-1}$, if n is even." The identity in (1) above is of course just a special case of the Vandermonde's Convolution Formula. This was pointed out by Bradley and

Seiffert (and used directly in their proofs). Lewis, Moser, and Zhou all pointed out that this identity can also be established by a simple and well-known combinatorial argument. For the binomial coefficient $\binom{n}{\frac{n(n+1)}{2}}$ given in the solution above (for the case when n is odd), three different, but equivalent forms were given by various solvers: namely, $\binom{n}{\frac{n-1}{2}}$, $\binom{n}{\lfloor\frac{n}{2}\rfloor}$ and $\binom{n}{\lfloor\frac{n}{2}\rfloor+1}$.

2590. [2000 : 497] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

For $n = 1, 2, \ldots$, prove that $\displaystyle\prod_{k=1}^{n}\binom{n}{k}^2 \le \left(\frac{1}{n+1}\binom{2n}{n}\right)^n$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

By the solution to problem #2589 [Ed: see equation (1) of the published solution to #2589 above] and the AM–GM inequality, we get that

$$\begin{aligned}\frac{1}{n+1}\binom{2n}{n} &= \frac{1}{n}\binom{2n}{n+1} = \frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}\binom{n}{k-1}\\ &\ge \left[\prod_{k=1}^{n}\binom{n}{k}\binom{n}{k-1}\right]^{\frac{1}{n}} = \left[\prod_{k=1}^{n}\binom{n}{k}^2\right]^{\frac{1}{n}}\end{aligned}$$

which completes the proof.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, Trinity College, Cambridge, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; STEFFEN WEBER, Merseburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Besides Zhou, essentially the same solution based on the proof of #2589 was also given by Arslanagić, Bataille, Loeffler, and the proposer. It is easy to see that equality holds if and only if $n = 1$ or 2. This was pointed out by Arslanagić, Diminnie, and Loeffler.

2591. [2000 : 497] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

Two players, A and B, each toss n fair coins, and two other players, C and D, toss $n-1$ and $n+1$ fair coins, respectively.

For each $n = 2, 3, \ldots$, prove that the two events:

A gets exactly one head more than B

and

C and D get exactly the same number of heads

are equally likely.

Find the probability of these events.

Solution by Gerry Leversha, St. Paul's School, London, England.
The probability that A gets exactly one head more than B is

$$\frac{1}{2^{2n}}\sum_{k=1}^{n}\binom{n}{k}\binom{n}{k-1} = \frac{1}{2^{2n}}\sum_{k=1}^{n}\binom{n}{k}\binom{n}{n-k+1} = \frac{1}{2^{2n}}\binom{2n}{n+1},$$

where the second equality is obtained by considering the coefficient of the term x^{n+1} on both sides of the identity $(1+x)^n(1+x)^n \equiv (1+x)^{2n}$.

The probability that C and D get exactly the same number of heads is

$$\begin{aligned}\frac{1}{2^{2n}}\sum_{k=1}^{n-1}\binom{n-1}{k}\binom{n+1}{k} &= \frac{1}{2^{2n}}\sum_{k=1}^{n-1}\binom{n-1}{k}\binom{n+1}{n+1-k}\\ &= \frac{1}{2^{2n}}\binom{2n}{n+1},\end{aligned}$$

where the second equality is obtained by considering the coefficient of the term x^{n+1} on both sides of the identity $(1+x)^{n-1}(1+x)^{n+1} \equiv (1+x)^{2n}$.

The events are thus equally likely.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; OLEG IVRII, (grade 8) student, Cummer Valley Middle School, North York, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; HENRY LIU, Trinity College, Cambridge, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2592. [2000 : 498] *Proposed by Nairi M. Sedrakyan, Yerevan, Armenia.*

Describe all numbers, which can be represented in the form of $\dfrac{a^3+b^3}{c^3+d^3}$, where a, b, c, d are natural numbers.

Amalgamated solutions of Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina and the proposer.
We first claim that every rational number from the interval $(1,2)$ can be represented in the form $\dfrac{a^3+b^3}{a^3+d^3}$. Indeed, let $\dfrac{m}{n}\in(1,2)$, where m and n are

natural numbers. We will choose a, b, d such that $b \neq d$ and $a^2 - ab + b^2 = a^2 - ad + d^2$; that is $b + d = a$. In that case

$$\frac{a^3 + b^3}{a^3 + d^3} = \frac{a+b}{a+d} = \frac{a+b}{2a-b}.$$

Taking now $a+b = 3m$ and $2a-b = 3n$ (that is, $a = m+n$ and $b = 2m-n$) the claim is proved.

Now we will prove that any positive rational number $r > 0$ can be represented in the given form. Let $r > 0$ be any positive rational number. Select positive integers p and q such that

$$1 < \frac{p^3}{q^3} r < 2.$$

From the above, there exist positive integers a, b, d such that

$$\frac{p^3}{q^3} r = \frac{a^3 + b^3}{a^3 + d^3}.$$

Hence,

$$r = \frac{(aq)^3 + (bq)^3}{(ap)^3 + (dp)^3}.$$

Also solved by MOHAMMED AASSILA, Strasbourg, France; PIERRE BORNSZTEIN, Pontoise, France; and the SOUTHWEST MISSOURI STATE PROBLEM SOLVING GROUP. There was one incorrect solution.

2593. [2000 : 498] *Proposed by Nairi M. Sedrakyan, Yerevan, Armenia.*

Let $S(a)$ denote the sum of the digits of the natural number a. Let k and n be natural numbers with $(n, 3) = 1$. Prove that there exists a natural number m which is divisible by n and $S(m) = k$ if either

(a) $k > n - 2$; or

(b) $k > S^2(n) + 7S(n) - 9$.

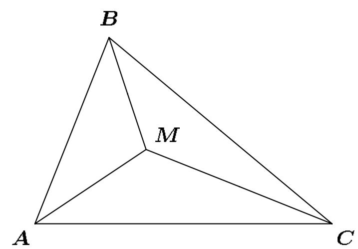
Editor's comment.

There have been no solutions submitted except that of the proposer. The editor has had some difficulty easily reading it, the problem is non-trivial, and the proposer's solution is difficult to follow. Since there were no other solutions submitted, we are keeping the problem open in the hope that you, our readers, will find a simpler solution.

2594. [2000 : 498] *Proposed by Nairi M. Sedrakyan, Yerevan, Armenia.*

Given a point M inside the triangle ABC (see diagram), prove that

$$\min(MA, MB, MC) + MA + MB + MC \;<\; AB + BC + AC\,.$$



Editor's Note: A number of correspondents noted that this is a known problem that has appeared in a number of places. Christopher J. Bradley, Clifton College, Bristol, UK, recalls that the problem was submitted to the IMO in Bucharest, but was not selected by the jury. Walther Janous, Ursulinengymnasium, Innsbruck, Austria, notes further that this was used as a problem at the Romanian Olympiad, and provides a reference to a recent article which uses complex numbers to give a proof (New Geometrical Inequality for Interior Point of a Triangle; M. Diuca and M. Bencze; Octogon, vol. 9, No. 1, 2001, pp. 437–440). Finally, as was pointed out by David Loeffler, student, Cotham School, Bristol, UK, this was Problem 2 in the final selection round of the 2000 British Mathematics Olympiad.

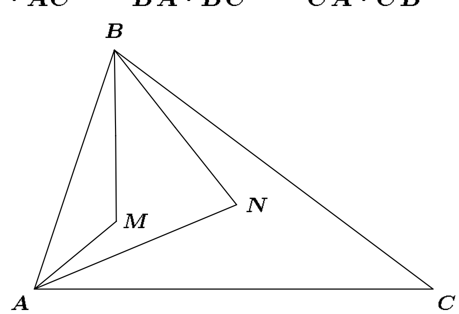
We refer any interested readers to these sources for a solution.

Solutions were submitted by MOHAMMED AASSILA, Strasbourg, France; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; PIERRE BORNSZTEIN, Pontoise, France; C. FESTRAETS-HAMOIR, Brussels, Belgium; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

2595. [2000 : 498] *Proposed by Nairi M. Sedrakyan, Yerevan, Armenia.*

Given that M and N are points inside the triangle ABC such that $\angle MAB = \angle NAC$ and $\angle MBA = \angle NBC$, prove that

$$\frac{AM\cdot AN}{AB\cdot AC} + \frac{BM\cdot BN}{BA\cdot BC} + \frac{CM\cdot CN}{CA\cdot CB} \;=\; 1\,.$$



Solution by Pierre Bornsztein, Pontoise, France.

Let K be the point on the half-line from B towards N such that $\angle BCK = \angle BMA$. Since M is inside $\triangle ABC$, we have $\angle BMA > \angle BCA$, which implies that K is outside the triangle. Moreover, we have that $\angle MBA = \angle NBC = \angle KBC$. It follows that $\triangle ABM$ and $\triangle KBC$ are similar. Thus,

$$\frac{AB}{KB} = \frac{BM}{BC} = \frac{AM}{KC}. \qquad (1)$$

Since N is inside $\triangle ABC$ we have

$$\begin{aligned} \angle ABK &= \angle ABN = \angle ABC - \angle NBC \\ &= \angle ABC - \angle MBA \\ &= \angle MBC; \end{aligned}$$

together with $\frac{AB}{KB} = \frac{BM}{BC}$ this implies that $\triangle ABK$ is similar to $\triangle MBC$ so that

$$\frac{AB}{MB} = \frac{BK}{BC} = \frac{AK}{MC}. \qquad (2)$$

Since in the first pair of similar triangles $\angle BKC = \angle BAM$, we have $\angle NKC = \angle BKC = \angle MAB = \angle NAC$. It follows that $AKCN$ is cyclic, so that Ptolemy's Theorem leads to $AC \cdot NK = AK \cdot NC + AN \cdot CK$; that is,

$$AC \cdot (BK - BN) = AK \cdot NC + AN \cdot CK. \qquad (3)$$

From (1) and (2) we have

$$AK = \frac{AB \cdot MC}{BM}, \quad BK = \frac{AB \cdot BC}{BM}, \quad CK = \frac{AM \cdot BC}{BM}.$$

Substituting these expressions in (3), and multiplying through by $\frac{BM}{AB \cdot BC \cdot CA}$, we obtain

$$1 - \frac{BM \cdot BN}{BA \cdot BC} = \frac{AM \cdot AN}{AB \cdot AC} + \frac{CM \cdot CN}{CA \cdot CB}.$$

The desired conclusion follows immediately.

Remark. This problem was proposed but not used by the jury at the IMO in 1998.

Also solved by MOHAMMED AASSILA, Strasbourg, France; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; HENRY LIU, student, Cambridge, England; TOSHIO SEIMIYA, Kawasaki, Japan; and by the proposer.

Arslanagić and the proposer both provided solutions that were essentially identical to the featured solution. Two others exploited the fact that M and N are isogonal conjugates with respect to $\triangle ABC$, but this observation did not provide any substantial advantage.

2596. [2000 : 498] *Proposed by Clark Kimberling, University of Evansville, Evansville, IN, USA.*

Write $r \prec s$ if there is an integer k satisfying $r < k < s$. Find, as a function of n ($n \geq 2$) the least positive integer k satisfying

$$\frac{k}{n} \prec \frac{k}{n-1} \prec \frac{k}{n-2} \prec \ldots \prec \frac{k}{2} \prec k\,.$$

This problem has actually appeared before! We printed it as problem 2272★ [1997 : 365]. At that time, the editor had a piece of paper without the name of the proposer (and no solution). We printed a solution in [1998 : 438]. Thanks to Michel Bataille for pointing this out. Now we know that the "anonymous" proposer is Clark Kimberling!

2597. [2000 : 499] *Proposed by Michael Lambrou, University of Crete, Crete, Greece.*

Let P be an arbitrary interior point of an equilateral triangle ABC. Prove that $|\angle PBC - \angle PCB| \;\leq\; \arcsin\left(2\sin\left(\frac{|\angle PAB - \angle PAC|}{2}\right)\right) - \left(\frac{|\angle PAB - \angle PAC|}{2}\right) \;\leq\; |\angle PAB - \angle PAC|$.

Show that the left inequality cannot be improved in the sense that there is a position Q of P on the ray AP giving an equality.

(Thus, the inequality in **2255** is improved.)

Editor's Note: No solutions (other than the proposer's) have been received. We await the answer to this challenge from you, our readers!

2598. [2000 : 499, 2001 : 267] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Suppose that AD, BE and CF are the internal angle bisectors of $\triangle ABC$, with D on BC, E on CA and F on AB. Write $a = BC$, $b = CA$, $c = AB$, $x = AE$ and $y = AF$. We are given that $x + y = a$. Prove that:

(a) $a^2 = bc$;

(b) $\frac{1}{x} - \frac{1}{y} \;=\; \frac{1}{b} - \frac{1}{c}$;

(c) $\frac{1}{x} + \frac{1}{y} \;=\; \left(\frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}}\right)^2$;

(d) $AD < a$.

Solution de C. Festraets-Hamoir, Brussels, Belgium.

$$\frac{x}{b-x} = \frac{c}{a} \iff ax = bc - cx \iff x = \frac{bc}{a+c}.$$

$$\frac{y}{c-y} = \frac{b}{a} \iff ay = bc - by \iff y = \frac{bc}{a+b}.$$

(a)

$$x + y = a = \frac{bc}{a+c} + \frac{bc}{a+b}.$$

$$\begin{aligned} a^3 + a^2b + a^2c + abc &= bca + b^2c + bca + bc^2, \\ a^2(a+b+c) &= bc(a+b+c), \\ a^2 &= bc. \end{aligned}$$

(b)

$$\frac{1}{x} = \frac{a+c}{bc} \quad \text{et} \quad \frac{1}{y} = \frac{a+b}{bc}.$$

$$\frac{1}{x} - \frac{1}{y} = \frac{c-b}{bc} = \frac{1}{b} - \frac{1}{c}.$$

(c)

$$\frac{1}{x} + \frac{1}{y} = \frac{2a+b+c}{bc} = \frac{2\sqrt{bc}+b+c}{bc} = \left(\frac{\sqrt{b}+\sqrt{c}}{\sqrt{bc}}\right)^2 = \left(\frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}}\right)^2.$$

(d)

$$\begin{aligned} AD &= \frac{2bc}{b+c}\cos\left(\tfrac{A}{2}\right) \\ &\leq \sqrt{bc}\cos\left(\tfrac{A}{2}\right) && \text{(MH<MG)} \\ &< \sqrt{bc} = a && \text{car } \cos\left(\tfrac{A}{2}\right) < 1. \end{aligned}$$

Parts (a), (b) and (c) were also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; PIERRE BORNSZTEIN, Pontoise, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, Trinity College, Cambridge, UK; DAVID LOEFFLER; student, Cotham School, Bristol, UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; PANOS E. TSAOUSSOGLOU, *Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.*

All solvers realized that (d) *was incorrect as originally printed. Some gave actual counter-examples. Bataille, Bornsztein, Bradley, Engelhaupt, Leversha, Liu, Loeffler, Seimiya and Zhou all proved the correct version.*

The editor feels that this is, by far, the most minimally French solution ever to have been submitted in the history of ***CRUX with MAYHEM****!*

2599. [2000 : 499] *Proposed by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

Let P be a point inside the triangle ABC and let AP, BP, CP meet the sides BC, CA, AB at L, M, N, respectively. Show that the following two conditions are equivalent:

$$\frac{1}{AP}+\frac{1}{PL} = \frac{1}{BP}+\frac{1}{PM} = \frac{1}{CP}+\frac{1}{PN};$$

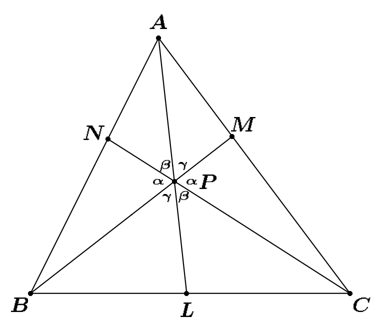
$$\angle APN = \angle NPB = \angle BPL = \angle LPC = \angle CPM = \angle MPA = 60^\circ .$$

Solution by Toshio Seimiya, Kawasaki, Japan.

Let $\angle BPN = \angle CPM = \alpha$, $\angle CPL = \angle APN = \beta$ and $\angle APM = \angle BPL = \gamma$. Then $\alpha+\beta+\gamma=\pi$.

Let $[F]$ denote the area of a plane figure F.

Since $[PBC] = [PBL] + [PCL]$, we have



$$\frac{1}{2}PB\cdot PC\sin(\beta+\gamma) = \frac{1}{2}PB\cdot PL\sin\gamma + \frac{1}{2}PC\cdot PL\sin\beta .$$

Dividing both sides by $\frac{PB\cdot PC\cdot PL}{2}$, we obtain

$$\frac{\sin(\beta+\gamma)}{PL} = \frac{\sin\gamma}{PC}+\frac{\sin\beta}{PB} .$$

Since $\alpha+\beta+\gamma=\pi$, we have $\sin(\beta+\gamma)=\sin\alpha$, and we can rewrite the above equality as

$$\frac{\sin\alpha}{PL} = \frac{\sin\gamma}{PC}+\frac{\sin\beta}{PB} .$$

Adding $(\sin\alpha/AP)$ to both sides, we get

$$\left(\frac{1}{AP}+\frac{1}{PL}\right)\sin\alpha \;=\; \frac{\sin\alpha}{AP}+\frac{\sin\beta}{BP}+\frac{\sin\gamma}{CP}\,.$$

Similarly,

$$\left(\frac{1}{BP}+\frac{1}{PM}\right)\sin\beta \;=\; \frac{\sin\alpha}{AP}+\frac{\sin\beta}{BP}+\frac{\sin\gamma}{CP}$$

and

$$\left(\frac{1}{CP}+\frac{1}{PN}\right)\sin\gamma \;=\; \frac{\sin\alpha}{AP}+\frac{\sin\beta}{BP}+\frac{\sin\gamma}{CP}\,.$$

Thus,

$$\left(\frac{1}{AP}+\frac{1}{PL}\right)\sin\alpha \;=\; \left(\frac{1}{BP}+\frac{1}{PM}\right)\sin\beta \;=\; \left(\frac{1}{CP}+\frac{1}{PN}\right)\sin\gamma\,.$$

Therefore,

$$\begin{aligned}
\frac{1}{AP}+\frac{1}{PL} \;=\; \frac{1}{BP}+\frac{1}{PM} \;&=\; \frac{1}{CP}+\frac{1}{PN}\\
\iff\; \sin\alpha \;=\; \sin\beta \;&=\; \sin\gamma\\
\iff\; \alpha \;=\; \beta \;&=\; \gamma\\
\iff\; \angle APM \;=\; \angle BPL \;=\; \angle BPN \;&=\; \angle CPM\\
\;=\; \angle CPL \;&=\; \angle APN\,.
\end{aligned}$$

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Cotham School, Bristol, UK; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

All solvers noticed the typo in the original statement of the problem ($1/PN$ was printed $1/CN$) and then solved the problem with the correct condition.

2600. [2000 : 499] *Proposed by Svetlozar Doichev, Stara Zagora, Bulgaria.*

Find all real numbers x such that, if a and b are the lengths of the sides of a triangle with medians from the mid-points of these sides of lengths m_a and m_b, respectively, then the equalities $a + xm_a = b + xm_b$ and $a = b$ are equivalent.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA (adapted by the editor).

Clearly, if $a = b$, then $m_a = m_b$, so that $a = b$ implies $a + xm_a = b + xm_b$ for all real x. Thus, $a + xm_a = b + xm_b$ and $a = b$ are equivalent if and only if $a + xm_a = b + xm_b$ implies $a = b$.

By the Cosine Law,

$$m_a^2 \;=\; b^2 + \frac{a^2}{4} - ba\cos C$$

and

$$m_b^2 \;=\; a^2 + \frac{b^2}{4} - ab\cos C\,,$$

so that

$$m_a^2 - m_b^2 \;=\; \frac{3}{4}(b^2 - a^2)\,.$$

Now,

$$\begin{aligned}
a + xm_a \;&=\; b + xm_b \\
\iff\quad a - b \;&=\; x(m_b - m_a) \\
\iff\quad (a-b)(m_b + m_a) \;&=\; x(m_b^2 - m_a^2) \\
\iff\quad (a-b)(m_b + m_a) \;&=\; \frac{3}{4}x(a^2 - b^2) \\
\iff\quad (a-b)\left(m_a + m_b - \frac{3}{4}x(a+b)\right) \;&=\; 0 \\
\iff\quad (a-b)\left(x - \frac{4(m_a+m_b)}{3(a+b)}\right) \;&=\; 0\,. \qquad (1)
\end{aligned}$$

By the Triangle Inequality,

$$b - \frac{a}{2} \;<\; m_a \;<\; b + \frac{a}{2}$$

and

$$a - \frac{b}{2} \;<\; m_b \;<\; a + \frac{b}{2}\,.$$

Hence,

$$\frac{2}{3} \;<\; \frac{4(m_a+m_b)}{3(a+b)} \;<\; 2\,.$$

We next show that the ratio $4(m_a+m_b)/(3(a+b))$ can take any value in the interval $(2/3, 2)$. Consider the right triangle $(a,b,c) = (t, \sqrt{t^2+1}, 1)$. We have $m_a = (1/2)\sqrt{t^2+4}$ and $m_b = (1/2)\sqrt{t^2+1}$, so that

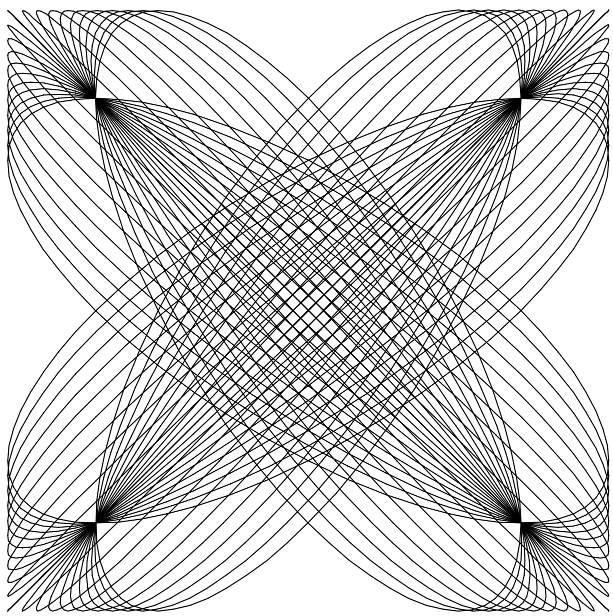
$$\frac{4(m_a+m_b)}{3(a+b)} \;=\; \frac{2(\sqrt{t^2+4} + \sqrt{t^2+1})}{3(t + \sqrt{t^2+1})}\,.$$

Let $f(t)$ denote the expression on the right side of the last equality. The function $f(t)$ is continuous for $t > 0$, $\lim_{t\to 0^+} f(t) = 2$ and $\lim_{t\to+\infty} f(t) = 2/3$. Hence the range of the function $f(t)$ is $(2/3, 2)$.

Therefore, the equality (1) implies $a = b$ if and only if

$$x \in (-\infty, 2/3] \cup [2, +\infty) .$$

Also solved by MICHEL BATAILLE, Rouen, France; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer. There was one incorrect solution submitted.



YEAR END FINALE

Again, a year has flown by! It is difficult to realize that I have now done this job for six years. My term has one year to go until the end of 2002 (I retire from Memorial University in Summer 2002), when you will have a new Editor-in-Chief.

There are many people that I wish to thank most sincerely for particular contributions. I thank most sincerely, ILIYA BLUSKOV, ROLAND EDDY, CHRIS FISHER, CLAYTON HALFYARD, GEORG GUNTHER, BILL SANDS, JIM TOTTEN, and EDWARD WANG, for their regular yeoman service in assessing the solutions; BRUCE GILLIGAN, for ensuring that we have quality articles; ALAN LAW, for ensuring that we have quality book reviews, ROBERT WOODROW, who has carried the heavy load of two corners, (but has now relinquished the Skoliad Corner to SHAWN GODIN) and RICHARD GUY for sage advice whenever necessary.

The editors of the ***MATHEMATICAL MAYHEM*** section, SHAWN GODIN, CHRIS CAPPADOCIA, ADRIAN CHAN, DONNY CHEUNG, JIMMY CHUI and DAVID SAVITT, all do a sterling job.

I also thank all the editors who assist with proofreading. As well, I thank MOHAMMED AASSILA for helping out in this task. The quality of the work of all these people is a vital part of what makes ***CRUX with MAYHEM*** what it is. Thank you one and all.

As well, I would like to give special thanks to our Associate Editor, CLAYTON HALFYARD, for continuous sage advice, and for keeping me from printing too many typographical and mathematical errors; and to my colleagues, YURI BAHTURIN, HERMAN BRUNNER, ROLAND EDDY, ANDY FOSTER, EDGAR GOODAIRE, MAURICE OLESON, MIKE PARMENTER, DAVID PIKE, DONALD RIDEOUT, NABIL SHALABY, BRUCE WATSON, in the Department of Mathematics and Statistics at Memorial University, and to JOHN GRANT McLOUGHLIN, Faculty of Education, Memorial University (who will succeed Alan Law as Book Reviews Editor for 2002), for their occasional sage advice. I have also been helped by some Memorial University students, RENEÉ BARTER, ANDREA BURGESS, KARELYN DAVIS, MARK DAVIS, ALASDAIR GRAHAM, JONATHAN KAVANAGH, SHANNON SULLIVAN, as well as WISE Summer students, JILLIAN CLARKE, WENDY KELLY and STEPHANIE LINEHAN.

As noted earlier, ALAN LAW has completed his term as Book Reviews Editor. I have really enjoyed my association with Alan over the past eight years. He was one of the people who persuaded me to take on this job, and I was delighted to repay the compliment, and have him as Book Reviews Editor. Thanks, Alan, and all the very best.

The staff of the Department of Mathematics and Statistics at Memorial University deserve special mention for their excellent work and support: ROS ENGLISH, MENIE KAVANAGH, WANDA HEATH, and LEONCE MORRISSEY; as well as the computer and networking expertise of DWAYNE HART, CRAIG SQUIRES and MARIAN WISSINK.

Also the LaTeX expertise of JOANNE LONGWORTH at the University of Calgary, JUNE ALEONG at Wilfred Laurier University, the ***MAYHEM*** staff, and all others who produce material, is much appreciated.

Thanks to GRAHAM WRIGHT, the Managing Editor, who keeps me on the right track, and to the U of T Press, who continue to print a fine product.

The online version of ***CRUX with MAYHEM*** continues to attract attention. We recommend it highly to you. Thanks are due to LOKI JORGENSON, JUDI BORWEIN, and the rest of the team at SFU who are responsible for this.

Finally, I would like to express real and heartfelt thanks to the Head of my Department, HERBERT GASKILL, and to the Dean of Science, BOB LUCAS. Without their support and understanding, I would not be able to do the job of Editor-in-Chief.

Last but not least, I send my thanks to you, the readers of ***CRUX with MAYHEM***. Without you, ***CRUX with MAYHEM*** would not be what it is. We receive about 150 proposals each year, and, as you know, we publish 100 problem proposals in each volume. Of course, we receive hundreds of solutions, as you will see in the index that follows. Every year, we receive solutions from new readers. This is very pleasing. More and more proposals and solutions are arriving by email. The editor is able to process files sent in LaTeX, in WORD and in WordPerfect formats, as well (although less desirable) image files. One small reminder: please ensure that your name and address is on EVERY problem or proposals, and that each starts on a fresh sheet of paper. Otherwise, there may be filing errors, resulting in a submitted solution being lost.

Keep those contributions and letters coming in. We need your ARTICLES, PROPOSALS and SOLUTIONS to keep ***CRUX with MAYHEM*** alive and well. I do enjoy knowing you all.

Wishing all our readers the compliments of the season, and a very happy, peaceful and prosperous year 2002.

Bruce Shawyer

Awards of Subscriptions to *CRUX with MAYHEM*

Through the generosity of a regular subscriber to ***CRUX with MAYHEM***, the Canadian Mathematical Society is very grateful to be able to award complimentary subscriptions to assist students in some developing countries. The criterion for the award is good performance by a student from that country at the IMO.

Two subscriptions for a three-year period (2001 to 2003) have been awarded to the schools attended by students from Peru and Uruguay. We wish the teachers and students from these schools "Happy Problem Solving", and good luck for the 2002 IMO.

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The Editorial Board of ***Crux Mathematicorum with Mathematical Mayhem*** expresses its thanks to the donor, congratulates the students involved, and welcomes these schools to the ***CRUX with MAYHEM*** family.

Miscellaneous

Johannes Waldmann reports an investigation on problem 2548 — he has more data. He checked that at step 953494 of the sequence, all numbers up to 22949 have occured, and the maximal element up to that point is 1655261568 (at step 688664).

In problem 2545, we inadvertantly attributed a solution to Andy Liu instead of Henry Liu. Apologies to both.

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