THE ACADEMY CORNER

No. 32

Bruce Shawyer

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We present readers' solutions to some of the questions of the 1999 Atlantic Provinces Council on the Sciences Annual Mathematics Competition, held this year at Memorial University, St. John's, Newfoundland [1999 : 452].

3. Prove that $\sin^2(x+\alpha)+\sin^2(x+\beta)-2\cos(\alpha-\beta)\sin(x+\alpha)\sin(x+\beta)$ is a constant function of x.

   Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

   Let $f(x)$ denote the given function. Differentiating and using the familiar formula: $\sin a + \sin b = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$, we have

$$\begin{aligned} f'(x) &= \sin(2(x+\alpha)) + \sin(2(x+\beta)) \\ &\quad -2\cos\big(\sin(x+\alpha)\cos(x+\beta) + \cos(x+\alpha)\sin(x+\beta)\big) \\ &= 2\sin(2x+\alpha+\beta)\cos(\alpha-\beta) \\ &\quad -2\cos(\alpha-\beta)\sin(2x+\alpha+\beta) \\ &= 0\,. \end{aligned}$$

   Hence $f(x) = C$ for some constant C. Since $f(-\beta) = \sin^2(\alpha-\beta)$, we conclude that $f(x) = \sin^2(\alpha-\beta)$.

4. In Scottish Dancing, there are three types of dances, two of which are fast rhythms, Jigs and Reels, and one is a slow rhythm, Strathspey.

   A Scottish Dance program always starts with a Jig. The following dances are selected (by type) according to the following rules:

   (i) the next dance is always of a different type from the previous one,

   (ii) no more than two fast dances can be consecutive.

   Find how many different arrangements of Jigs, Reels and Strathspeys are possible in a Scottish Dance list which has (a) seven dances, (b) fifteen dances.

Solution to (a) by Michael Parmenter, Memorial University of Newfoundland, St. John's, Newfoundland.

We list the cases according to which of the seven dances are Strathspeys, and count the number of arrangements in each case:

Strathspey positions		Number
3, 5		$2 \times 2 = 4$
3, 6		$2 \times 2 = 4$
2, 5		$2 \times 2 = 4$
3, 5, 7		$2 \times 2 = 4$
2, 5, 7		$2 \times 2 = 4$
2, 4, 7		$2 \times 2 = 4$
2, 4, 6		$2 \times 2 \times 2 = 8$
	Total	32

Parmenter also solved (b).

Solution and generalisation by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

We consider the more general problem when there are n dances. Let $f(n)$, $g(n)$, respectively, denote the number of different arrangements satisfying conditions (i) and (ii), starting with a Jig, Reel, respectively. It is easy to see that $f(1) = 1$, $f(2) = 2$, and $f(3) = 3$.

For $n > 3$, we claim that $f(n)$ satisfies the 3rd–order recurrence relation

$$f(n) \;=\; 2\big(f(n-2) + f(n-3)\big)\,.$$

For convenience of notation, we denote Jig, Reel and Strathspey by J, R, and S respectively. Since the 1st dance is a J, the 2nd dance must be an S or an R. Among all those arrangements starting with J, S, there are $f(n-2)$ of them which start with J, S, J, and $g(n-2)$ of them which start with J, S, R. On the other hand, if the 2nd dance is an R, then the 3rd dance must be an S. Among all those arrangements starting with J, R, S, there are $f(n-3)$ of them which start with J, R, S, J, and $g(n-3)$ of them which start with J, R, S, R. Hence we get

$$f(n) \;=\; f(n-2) + g(n-2) + f(n-3) + g(n-3)\,.$$

Note that for any admissible arrangement, if we replace any J by an R and *vice versa*, we obtain an arrangement for which conditions (i) and (ii) still hold. This one-to-one correspondence shows that $f(n) = g(n)$ for all n. Therefore, we obtain the recurrence relation $f(n) = 2f(n-2) + 2f(n-3)$.

Using the initial values of $f(1)$, $f(2)$, and $f(3)$, and iterating the recurrence relation, we easily find that $f(4) = 6$, $f(5) = 10$, $f(6) = 18$, $f(7) = 32$, $f(8) = 56$, $f(9) = 100$, $f(10) = 176$, $f(11) = 312$, $f(12) = 552$, $f(13) = 976$, $f(14) = 1728$, and $f(15) = 3056$.

Comments:

(a) This is indeed a very interesting problem.

(b) To obtain an explicit formula for the value of $f(n)$ for arbitrary n seems to be difficult, since the characteristic equation of the recurrence relation is the cubic equation $x^3 - 2x - 2 = 0$. Standard theory tells us that if the three roots of this equation are denoted by r_1, r_2 and r_3, then the general solution is given by $c_1 r_1^n + c_2 r_2^n + c_3 r_3^n$ for some constants c_1, c_2 and c_3.

5. Find all differentiable functions $f(x)$ which satisfy the integral equation

$$\big(f(x)\big)^{2000} = \int_1^x \big(f(t)\big)^{1999}\,dt\,.$$

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

The only such functions are $f(x) \equiv 0$ and $f(x) = \frac{x-1}{2000}$.

Clearly, the function $f(x) \equiv 0$ satisfies the given equation. Suppose then that $f(x) \not\equiv 0$. Differentiating both sides of the given equation yields $2000\big(f(x)\big)^{1999} f'(x) = \big(f(x)\big)^{1999}$, and so $f'(x) = \frac{1}{2000}$. Hence, $f(x) = \frac{x}{2000} + C$ for some constant C. Setting $x = 1$ in the given equation yields $f(1) = 0$, giving $C = -\frac{1}{2000}$, and so $f(x) = \frac{x-1}{2000}$.

Also solved by Richard Tod, The Royal Forest of Dean, Gloucestershire, England.

7. Pat has a method for solving quadratic equations. For example, Pat solves $6x^2 + x - 2 = 0$ as follows:

Step 1. Pat multiplies the leading coefficient by the constant, and solves the simpler equation $x^2 + x - 12 = 0$ to get $(x+4)(x-3) = 0$.

Step 2. Pat then replaces each x by $6x$ (x times the leading coefficient) to get $(6x+4)(6x-3) = 0$.

Step 3. Pat then simplifies this equation to get $(3x+2)(2x-1) = 0$, which solves the original equation.

Prove or disprove that Pat's method always works.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Pat's method always works. Let $ax^2 + bx + c = 0$ be an arbitrary quadratic equation with $a \neq 0$. Then step 1 yields $x^2 + bx + ac = 0$.

Suppose that $x^2 + bx + ac = (x-\alpha)(x-\beta)$. Then $\alpha + \beta = -b$ and $\alpha\beta = ac$.

Step 2 then gives $(ax-\alpha)(ax-\beta) = 0$, or $a^2x^2 - a(\alpha+\beta)x + \alpha\beta = 0$. Hence, $a^2x^2 + abx + ac = 0$. Since $a \neq 0$, step 3 finally yields $ax^2 + bx + c = 0$.

THE OLYMPIAD CORNER

No. 205

R.E. Woodrow

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We begin this number with a whirlwind tour of contests from the four corners of the globe. My thanks go to Richard Nowakowski, Canadian Team Leader at the IMO in Buenos Aires, who collected all four sets for our use. We start the tour in Finland.

FINNISH HIGH SCHOOL MATHEMATICS CONTEST

Final Round

Varkaus, January 25, 1997

1. Determine all numbers a for which the equation

$$a3^x + 3^{-x} = 3$$

has a unique solution x.

2. Two circles, of radii R and r, $R > r$, are externally tangent. Consider the common tangent of the circles, not passing through their common point. Determine the maximal radius of a circle drawn in the domain bounded by this tangent line and the circles.

3. Twelve knights sit around a round table. Every knight hates the two knights sitting next to him, but none of the other nine knights. A task group of five knights is to be sent to save a princess in trouble. No two knights who hate each other can be included in the group. In how many ways can the group be selected?

4. Determine the sum of all four-digit numbers, all the digits of which are odd.

5. Let $n \geq 3$. Find a configuration of n points in the plane such that the mutual distance of no pair of points exceeds one and exactly n pairs of points have a mutual distance equal to one.

Now we fly east to Georgia for XI and X form contests.

GEORGIAN MATHEMATICAL OLYMPIAD
May 1997, Final Round
XI Form

1. Consider the following sequence of functions:

$$f_1(x) = \log_{\sqrt{5}}(x)\,,\ f_2(x) = f_1(f_1(x))\,,\ \ldots\,,\ f_{n+1}(x) = f_1(f_n(x))\,,\ \ldots\,.$$

Find the smallest natural value of k such that $f_k(k)$ is not defined.

2. Two positive numbers are written on a board. At each step you must perform one of the following:

(i) choose one of the numbers, say a, already written on the board and write down either a^2 or $\frac{1}{a}$ on the board;

(ii) choose two numbers, say a and b, on the board and write down either $a + b$ or $|a - b|$ on the board.

Obviously, after each step the quantity of numbers on the board increases. How should you proceed in order that the product of the two initial numbers will eventually be written on the board?

3. Given a convex quadrilateral with sides not exceeding 20, prove that the distance from any interior point to the nearest vertex does not exceed 15.

4. We say that there is an algebraic operation defined on the closed interval $[0, 1]$ if there is a rule that corresponds to every pair (a, b) of numbers from this interval a new number c from the same interval. We denote it by $c = a \otimes b$. Find all positive k with the property that there exists an algebraic operation defined on $[0, 1]$ such that for any x, y, z from $[0, 1]$ the following equalities hold:

(i) $x \otimes 1 = 1 \otimes x = x$,

(ii) $x \otimes (y \otimes z) = (x \otimes y) \otimes z$,

(iii) $(zx) \otimes (zy) = z^k(x \otimes y)$.

For all such k define the corresponding algebraic operation.

5. In the rectangular parallelepiped $ABCDA'B'C'D'$ the points N and P are the centres of the faces $ABB'A'$ and $ADD'A'$ respectively. Let M be a point on the diagonal $A'C$, such that $A'M = \frac{1}{3}A'C$. Prove that $MN \perp AB'$ and $MP \perp AD'$ if and only if the given parallelepiped is a cube.

X Form

1. Find all triples (x, y, z) of integers satisfying the inequality:

$$x^2 + y^2 + z^2 + 3 \ < \ xy + 3y + 2z\,.$$

2. Determine whether or not it is possible to fill an $n \times n$ table with entries equal to 0, -1, or 1 so that when calculating the sums of the entries along the rows and the columns, one gets 20 different numbers.

3. See XI.2.

4. The area of a given trapezoid is 2 cm^2 and the sum of its diagonals equals 4 cm. Find the altitude of the trapezoid.

5. Prove that in any triangle the following inequality holds: $pR \geq 2S$, where p, R, S are respectively the semiperimeter, the radius of the circumcircle and the area of the triangle.

Continuing east (but backing up in time!) we catch the three parts of the 6$^{\text{th}}$ Republic of China Mathematical Olympiad.

6$^{\text{th}}$ ROC (TAIWAN) MATHEMATICAL OLYMPIAD

Part I

April 14, 1997 (Time: 4.5 hours)

Each problem is worth 7 points.

1. Let a be a rational number, b, c, d be real, and the function $f : R \to [-1, 1]$ satisfy

$$f(x + a + b) - f(x + b) = c \cdot \left[x + 2a + [x] - 2[x + a] - [b]\right] + d$$

for each $x \in \mathbb{R}$, where $[t]$ denotes the largest integer that is less than or equal to t. Show that f is a periodic function (that is, there is a positive number p such that $f(x + p) = f(x)\ \forall x \in \mathbb{R}$).

2. Let AB be a given line segment. Find all possible points C in the plane such that in $\triangle ABC$, the height from the vertex A and the length of the median from the vertex B are equal.

3. Let $n \geq 3$. Suppose that the sequence a_1, a_2, $\ldots$, a_n are positive real numbers satisfying $a_{i-1} + a_{i+1} = k_i a_i$, $\forall i = 1, 2, \ldots, n$, where each k_i is a positive integer, $a_0 = a_n$, $a_{n+1} = a_1$. Show that

$$2n \leq k_1 + k_2 + \cdots + k_n \leq 3n\,.$$

Part II

May 11, 1997 (Time: 4.5 hours)

Each problem is worth 7 points.

1. Let $k = 2^{2^n} + 1$ for some positive integer n. Show that k is a prime if and only if k is a factor of $3^{(k-1)/2} + 1$.

2. Let $ABCD$ be a tetrahedron. Show that:

(i) if $AB = CD$, $AD = BC$, $AC = BD$, then $\triangle ABC$, $\triangle ACD$, $\triangle ABD$, and $\triangle BCD$ are acute triangles;

(ii) if the area of $\triangle ABC$, $\triangle ACD$, $\triangle ABD$, and $\triangle BCD$ are the same, then $AB = CD$, $AD = BC$, $AC = BD$.

3. Let X be the set consisting of elements of the form

$$a_{2k} \cdot 10^{2k} + a_{2k-2} \cdot 10^{2k-2} + \cdots + a_2 \cdot 10^2 + a_0 \,,$$

where $k = 0, 1, 2, \ldots$, and each $a_{2i} \in \{1, 2, 3, \ldots, 9\}$. Show that every integer of the form $2^p \cdot 3^q$, where p, q are non-negative integers, is a factor of some element in X.

Part III

June 25, 1997 (Time: 4.5 hours)

Each problem is worth 7 points.

1. Determine all the possible integers k such that there is a function $f : \mathbb{N} \longrightarrow \mathbb{Z}$ such that

(i) $f(1997) = 1998$,

(ii) $f(ab) = f(a) + f(b) + k \cdot f(d(a,b))$, $\forall a, b \in \mathbb{N}$, where $d(a,b)$ denotes the greatest common divisor of a and b.

2. Let $\triangle ABC$ be an acute triangle with circumcentre O and circumradius R. Show that if AO meets the circle OBC again at D, BO meets the circle OCA again at E, and CO meets the circle OAB again at F, then $OD \cdot OE \cdot OF \geq 8R^3$.

3. Let $X = \{1, 2, 3, \ldots, n\}$, $n \geq k \geq 3$, and let F_k be a family of subsets of X with k elements, so that any two subsets in F_k have at most $k - 2$ common elements. Show that for each $k \geq 3$ there exists a subset M_k of X with at least $\lfloor \log_2 n \rfloor + 1$ elements such that it does not contain any subset in F_k.

Finally we give the problems of the 11^{th} Iberoamerican Mathematical Olympiad.

11^{th} IBEROAMERICAN MATHEMATICAL OLYMPIAD

September 24–25, 1996, Costa Rica

First Day — Time Allowed: 4.5 hours

1. (Brazil): Let n be a natural number. A cube of side n can be split into 1996 cubes. The sides of these cubes are also natural numbers. Determine the minimum possible value of n.

2. (Spain): Let M be the mid-point of the median AD of the triangle ABC (D belongs to the side BC). The line BM meets the side AC at the point N. Show that AB is tangent to the circumcircle of the triangle NBC if and only if the equality

$$\frac{BM}{MN} = \frac{BC^2}{BN^2}$$

holds.

3. (Spain): We have a chessboard of size $(k^2 - k + 1) \times (k^2 - k + 1)$, with $k = p + 1$, p being a prime number.

For each prime number p, give a method of distribution of the numbers 0 and 1, one number in each square of the chessboard, in such a way that in each row, there are exactly k zeros; in each column, there are exactly k zeros; and moreover, no rectangle with sides parallel to the sides of the chessboard has a number 0 on the vertices.

Second Day — Time Allowed: 4.5 hours

4. (Brazil): Given a natural number $n \geq 2$, all the fractions of the form $\frac{1}{ab}$, with a and b natural numbers, coprime, and such that

$$a < b \leq n, \qquad a + b > n,$$

are considered. Show that the sum of all these fractions equals $\frac{1}{2}$.

5. (Peru): Three coins, A, B and C are situated one at each vertex of an equilateral triangle of side n. The triangle is divided into little equilateral triangles of side 1 by lines parallel to the sides.

At the beginning, all the lines of the figure are blue. The coins move along the lines, painting in red their trajectory, following the two rules:

(i) First coin to move is A, then B, then C, then again A, and so on. At each turn, each coin paints exactly one side of one of the little triangles.

(ii) No one coin can move along a side of a triangle which is already painted red; but that coin can stay at the end of a painted segment, alone or with another coin waiting its turn at moving.

Show that, for all integers $n > 0$, it is possible to paint all the sides of all the little triangles red.

6. (Spain): We have n distinct points $A_1, \ldots, A_n$ in the plane. To each point A_i a real number $\lambda_i \neq 0$ is assigned, in such a way that

$$\overline{A_iA_j}^2 = \lambda_i + \lambda_j\,, \quad \text{for all } i,\, j \text{ with } i \neq j\,.$$

Show that:

(a) $n \leq 4$;

(b) if $n = 4$, then

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4} = 0\,.$$

Next we turn to readers' solutions to problems given in the November 1998 number of the *Corner*.

3. [1998: 385] *18th Austrian-Polish Mathematics Competition*

Let $P(x) = x^4 + x^3 + x^2 + x + 1$. Show that there exist polynomials $Q(y)$ and $R(y)$ of positive degrees, with integer coefficients, such that $Q(y) \cdot R(y) = P(5y^2)$ for all y.

Solutions by Mohammed Aassila, Strasbourg, France; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, whose solution we give.

Since $P(5y^2) = 5^4y^8 + 5^3y^6 + 5^2y^4 + 5y^2 + 1$, we try factors of the form

$$(25y^4 + ay^3 + by^2 + cy + 1)(25y^4 - ay^3 + by^2 - cy + 1)\,.$$

On expanding out, these are factors: $a = 25$, $b = 15$, and $c = 5$.

Now we turn to solutions to four problems of the Georgian Mathematical Olympiad 1995, Final Round (see [1998: 388]).

3. (Grade IX). Prove that if the product of three positive numbers is 1 and their sum is more than the sum of their reciprocals, then only one of these numbers can be more than 1.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let a, b, and c denote the three numbers and let $m = \max\{a, b, c\}$. Since $abc = 1$, clearly $m \geq 1$ and $m = 1$ if and only if $a = b = c = 1$, in which case we have

$$a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\,,$$

a contradiction. Thus $m > 1$. Suppose two of the a, b, c are greater than 1, say, a and b. Then substituting $c = \frac{1}{ab}$ into $a + b + c > \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ we get

$$a + b + \frac{1}{ab} \;>\; \frac{1}{a} + \frac{1}{b} + ab \;=\; \frac{a+b}{ab} + ab$$

or

$$ab(a+b) + 1 \;>\; a + b + a^2b^2$$

or

$$(a+b)(ab-1) > (ab-1)(ab+1).$$

Thus $a + b > ab + 1$ or $(a-1)(b-1) < 0$, which is clearly a contradiction, and our proof is complete.

Remark: In fact, the proof above shows that if $m = a$, then b and c must both be *strictly* smaller than 1 since, for example, if $b = 1$, then $ac = 1$ would imply $c = \frac{1}{a}$ and so $a+b+c = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$, contradicting the assumption.

5. (Grade IX). A set M of integers has the following property: if the numbers a and b are in M, then $a + 2b$ also belongs to M. It is known that the set contains positive as well as negative numbers. Prove that if the numbers a, b and c are in M, then $a + b - c$ is also in M.

Comment by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

The problem, as stated, is clearly *incorrect* unless $a = b$. A simple counterexample is $M = \{-1, 1, 3, 5, 7, \dots\}$.

1. (Grade X).

(a) Five different numbers are written in one line. Is it always possible to choose three of them placed in increasing or decreasing order?

(b) Is it always possible to do the same, if we have to choose four numbers from nine?

Comment by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

The answer to (a) is YES. In fact, these are special cases of a well-known theorem due to Erdős and Szekeres, which states that if $m, n \in \mathbb{N}$, then any sequence of $mn + 1$ real numbers contains a monotonically increasing subsequence of $m + 1$ terms or a monotonically decreasing subsequence of $n + 1$ terms, or both. Here, $5 = 2 \cdot 2 + 1$. (See for example, *Introduction to Combinatorics* by Martin J. Erickson, Wiley-Interscience Series; p. 41.) [*Ed.* The answer to (b) is NO, as shown by the sequence 321654987.]

Since $5 = 2 \times 2 + 1$ and $9 = 2 \times 4 + 1$, both answers follow immediately.

3. (Grade X). Prove that for any natural number n, the average of all its factors lies between the numbers $\sqrt{n}$ and $\frac{n+1}{2}$.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let $d_1, d_2, \ldots, d_k$ denote the positive divisors of n where $k = \tau(n)$ is the number of (positive) divisors of n. We are to show that

$$\sqrt{n} \;\leq\; \frac{1}{k}\sum_{i=1}^{k} d_i \;\leq\; \frac{n+1}{2}\,. \tag{1}$$

To establish the right inequality in (1) we first show that

$$d + \frac{n}{d} \;\leq\; n+1 \tag{2}$$

for all divisors d of n.

This is clearly true if $d = 1$. If $d > 1$, then

$$\begin{aligned} d + \frac{n}{d} \leq n+1 \quad &\Longleftrightarrow \quad d^2 + n \leq d(n+1) \\ &\Longleftrightarrow \quad d(d-1) \leq (d-1)n \;\Longleftrightarrow\; d \leq n\,, \end{aligned}$$

which clearly holds. Since $d \mid n \Longleftrightarrow \frac{n}{d} \mid n$ we have, from (2) that

$$2\sum_{i=1}^{k} d_i \;=\; \sum_{i=1}^{k}\left(d_i + \frac{n}{d_i}\right) \;\leq\; k(n+1)$$

from which $\frac{1}{k}\sum_{i=1}^{k} d_i \leq \frac{n+1}{2}$ follows.

If equality holds, then for *any* divisor d of n, either $d = 1$ or $\frac{n}{d} = 1$. Thus $k = 2$ and n must be a prime. Conversely, if n is a prime, then $k = 2$ and $\frac{1}{k}\sum_{i=1}^{k} d_i = \frac{n+1}{2}$. To show the left inequality in (1) we use the Arithmetic-Geometric-Mean Inequality. Note that if n is not a perfect square, then for all divisors d of n we have $d \neq \frac{n}{d}$ and so k must be even. Pairing off d with $\frac{n}{d}$, we obtain $\prod_{i=1}^{k} d_i = n^{k/2}$. If $n = q^2$ is a perfect square, then k is odd. Again, pairing off d with $\frac{n}{d}$ for all $d \neq q$, we find that

$$\prod_{i=1}^{k} d_i \;=\; q \cdot n^{(k-1)/2} \;=\; n^{1/2} \cdot n^{(k-1)/2} \;=\; n^{k/2}\,,$$

which is the same as above.

Hence in both cases, we have

$$\frac{1}{k}\sum_{i=1}^{k} d_i \;\geq\; \left(\prod_{i=1}^{k} d_i\right)^{1/k} \;=\; (n^{k/2})^{1/k} \;=\; \sqrt{n}\,.$$

Clearly, equality holds in this inequality if and only if $n = 1$.

Now we turn to the December 1998 *Corner* and readers' comments and solutions regarding the problems of the Bi-National Israel-Hungary Competition 1995, [1998: 452].

BI-NATIONAL ISRAEL-HUNGARY COMPETITION 1995

1. Denote the sum of the first n prime numbers by S_n. Prove that there exists a whole square between S_n and S_{n+1}.

Comment by Mohammed Aassila, Strasbourg, France.

The problem appeared in ***CRUX*** in 1984 as problem 874. It was proposed by the COPS of Ottawa and solved by Walther Janous.

Solutions by Michel Bataille, Rouen, France; and by Pierre Bornsztein, Courdimanche, France. We give Bataille's write-up.

We have $S_1 < 2^2 < S_2 < 3^2 < S_3 < 4^2 < S_4 < 5^2 < S_5$.

Suppose that $n \geq 5$, so that $p_n \geq 11$. Let a_n be the integer ≥ 6 defined by $p_n = 2a_n - 1$, and S'_n the sum of all odd integers from 1 to p_n, inclusive. It is well known that $S'_n = a_n^2 = \left(\frac{p_n+1}{2}\right)^2$ and easily seen that $S'_n > S_n$, because $n \geq 5$. Now let us assume that there is no square between S_n and S_{n+1}. Then there would exist an integer k such that $k^2 \leq S_n < S_{n+1} \leq (k+1)^2$ and we would have

$$S_{n+1} - S_n \;\leq\; 2k+1\,; \quad \text{that is } p_{n+1} \;\leq\; 2k+1\,.$$

From this we could write successively:

$$p_n \;\leq\; 2k-1, \quad k \;\geq\; \frac{p_n+1}{2} \quad \text{and} \quad S_n \;\geq\; \left(\frac{p_n+1}{2}\right)^2$$

which contradicts $S'_n > S_n$. The result follows.

2. Let P, P_1, P_2, P_3, P_4 be five points on a circle. Denote the distance of P from the line P_iP_k by d_{ik}. Prove that $d_{12}d_{34} = d_{13}d_{24}$.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; and by Geoffrey A. Kandall, Hamden, CT, USA. We give Kandall's solution.

Let D denote the diameter of the circle.

We have $d_{ik} \cdot P_iP_k = 2[PP_iP_k] = PP_i \cdot PP_k \cdot \sin\angle P_iPP_k$. Hence

$$d_{ik} \;=\; \frac{PP_i \cdot PP_k \cdot \sin\angle P_iPP_k}{P_iP_k} \;=\; \frac{PP_i \cdot PP_k}{D}\,.$$

It is now obvious that

$$d_{12}d_{34} \;=\; d_{13}d_{24} \;=\; (d_{14}d_{23}) \;=\; \frac{1}{D^2}\prod_{i=1}^{4} PP_i\,.$$

3. Consider the polynomials $f(x) = ax^2 + bx + c$ which satisfy $|f(x)| \leq 1$ for all $x \in [0, 1]$. Find the maximal value of $|a| + |b| + |c|$.

Comment by Mohammed Aassila, Strasbourg, France. Solved by Pierre Bornsztein, Courdimanche, France. We give the remark of Aassila.

The solution to this problem appeared in ***CRUX*** in 1984. It was given at the 1980 Leningrad Mathematical Olympiad [1983: 304]. It was corrected and solved by M.S. Klamkin [1984: 287].

As a last solution set this issue we give readers' comments and solutions to problems of the 31st Spanish Mathematical Olympiad, First Round, given in [1998: 452–453].

1. Let a, b, c be distinct real numbers and $P(x)$ a polynomial with real coefficients. If:

- the remainder on division of $P(x)$ by $x - a$ equals a,
- the remainder on division of $P(x)$ by $x - b$ equals b,
- and the remainder on division of $P(x)$ by $x - c$ equals c;

determine the remainder on division of $P(x)$ by $(x - a)(x - b)(x - c)$.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Geoffrey A. Kandall, Hamden, CT, USA; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bataille's solution.

As is well known, the remainder on division of $P(x)$ by $x - a$ is $P(a)$. So, the hypotheses imply: $P(a) = a$, $P(b) = b$, $P(c) = c$.

Let $R(x)$ be the remainder on division of $P(x)$ by $(x-a)(x-b)(x-c)$, so that the degree of $R(x)$ is ≤ 2 and $P(x) = (x - a)(x - b)(x - c)Q(x) + R(x)$ for a polynomial $Q(x)$.

We remark that $R(a) = P(a) = a$ and similarly $R(b) = b$ and $R(c) = c$. From this observation, we may conclude through one of the three following ways:

(1) the polynomial $R(x) - x$ has degree ≤ 2 and three distinct zeros a, b, c. Hence $R(x) - x$ is the zero polynomial and $R(x) = x$.

(2) $R(x)$ has the form $ux^2 + vx + w$ where (u, v, w) is the solution of the system

$$\begin{cases} ua^2 + va + w = a \\ ub^2 + vb + w = b \\ uc^2 + vc + w = c\,. \end{cases} \qquad \text{(S)}$$

The determinant of (S) is a Vandermonde determinant and is not zero (since a, b, c are distinct), so (S) has a unique solution, which clearly is $u = 0$, $v = 1$, $w = 0$. Thus $R(x) = x$ again.

(3) $R(x)$ is the Lagrange's interpolation polynomial:

$$R(x) \;=\; a\,\frac{(x-b)(x-c)}{(a-b)(a-c)} + b\,\frac{(x-a)(x-c)}{(b-a)(b-c)} + c\,\frac{(x-a)(x-b)}{(c-a)(c-b)}\,.$$

Multiplying out and grouping similar terms, a lengthy but easy calculation provides $R(x) = x$ again.

2. Show that, if $(x+\sqrt{x^2+1})(y+\sqrt{y^2+1}) = 1$, then $x+y=0$.

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Geoffrey A. Kandall, Hamden, CT, USA; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We first give Aassila's solution and generalization.

We prove, more generally, that if $(x+\sqrt{x^2+1})(y+\sqrt{y^2+1}) = p$, then $x+y = \frac{p-1}{\sqrt{p}}$.

Set $z = x+\sqrt{x^2+1}$. Then $z > 0$ and $x = \frac{z^2-1}{2z}$. Consequently, we obtain

$$y+\sqrt{y^2+1} \;=\; \frac{p}{z}$$

and then

$$y \;=\; \frac{(p/z)^2-1}{(2p/z)} \;=\; \frac{p^2-z^2}{2pz}\,.$$

Hence,

$$\begin{aligned} x+y \;&=\; \frac{z^2-1}{2z} + \frac{p^2-z^2}{2zp} \;=\; \frac{p-1}{2p}\left(z+\frac{p}{z}\right) \\ &\geq\; \frac{p-1}{p}\sqrt{z\cdot\frac{p}{z}} \;=\; \frac{p-1}{\sqrt{p}}\,. \end{aligned}$$

Equality occurs for $x = y = \frac{p-1}{2\sqrt{p}}$.

Next we give a nice solution of Bataille.

Taking logarithms, the hypothesis implies

$$\ln(x+\sqrt{x^2+1}) \;+\; \ln(y+\sqrt{y^2+1}) \;=\; 0 \quad \text{or}$$

$$\sinh^{-1}(x) \;+\; \sinh^{-1}(y) \;=\; 0\,.$$

Since the function $\sinh^{-1}$ is odd we obtain

$$\sinh^{-1}(x) \;=\; \sinh^{-1}(-y) \quad \text{and} \quad x = -y\,,$$

as required.

And for yet another method we turn to Wang's solution.

Since

$$\left(x-\sqrt{x^2+1}\right)\left(x+\sqrt{x^2+1}\right)\left(y-\sqrt{y^2+1}\right)\left(y+\sqrt{y^2+1}\right) = (-1)^2 = 1\,,$$

the given condition implies that

$$\left(x-\sqrt{x^2+1}\right)\left(y-\sqrt{y^2+1}\right) \;=\; 1\,.$$

Hence

$$x+\sqrt{x^2+1} \;=\; \frac{1}{y+\sqrt{y^2+1}} \;=\; -\left(y-\sqrt{y^2+1}\right)$$

and

$$x-\sqrt{x^2+1} \;=\; \frac{1}{y-\sqrt{y^2+1}} \;=\; -\left(y+\sqrt{y^2+1}\right)\,.$$

Adding, we get

$$2x \;=\; -2y \quad \text{or} \quad x+y \;=\; 0\,.$$

3. The squares of the sides of a triangle ABC are proportional to the numbers 1, 2, 3.

(a) Show that the angles formed by the medians of ABC are equal to the angles of ABC.

(b) Show that the triangle whose sides are the medians of ABC is similar to ABC.

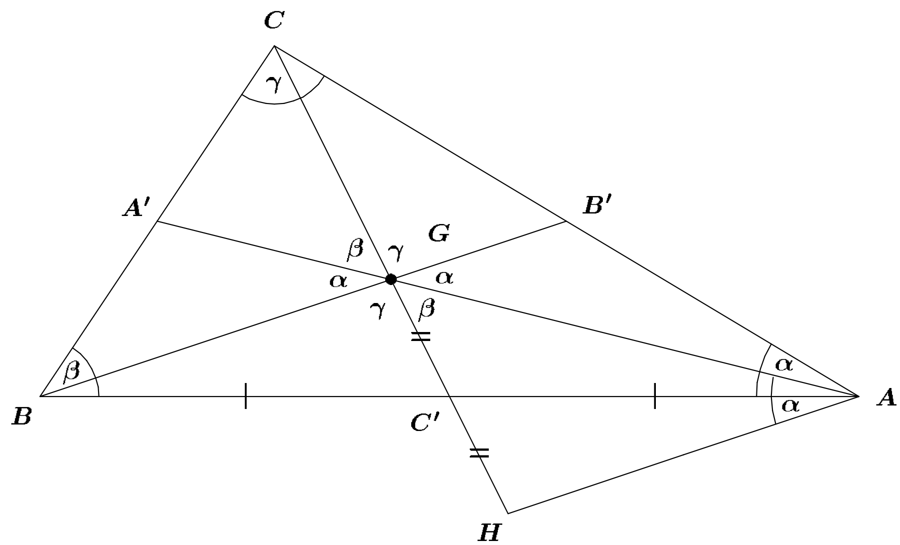
Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Geoffrey A. Kandall, Hamden, CT, USA; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give Kandall's solution.

(b) Let us say that $a^2 : b^2 : c^2 = 1 : 2 : 3$; that is, $a^2 = t$, $b^2 = 2t$, $c^2 = 3t$. Let m_a, m_b, m_c denote the medians passing through A, B, C respectively.

Then $4m_a^2 = 2b^2+2c^2-a^2 = 9t$, $4m_b^2 = 2a^2+2c^2-b^2 = 6t$, $4m_c^2 = 2a^2+2b^2-c^2 = 3t$. Thus, $m_c^2 : m_b^2 : m_a^2 = 1:2:3 = a^2 : b^2 : c^2$. Hence $m_c : m_b : m_a = a : b : c$. This proves (b).

(a) Let A', B', C' be the mid-points of BC, AC, AB, and let G be the centroid of ABC. Extend GC' to H so that $GC' = C'H$.

Then $AG = \frac{2}{3}m_a$, $AH = BG = \frac{2}{3}m_b$, $GH = \frac{2}{3}m_c$. Hence, by what was proved in (b), $ABC \sim AGH$. Consequently, $\angle GAH = \angle BAC \equiv \alpha$, $\angle HGA = \angle CBA \equiv \beta$, $\angle AHG = \angle ACB \equiv \gamma$. We can now fill in our diagram as follows:



This proves (a).

4. Find the smallest natural number m such that, for all natural numbers $n \geq m$, we have $n = 5a + 11b$, with a, b integers ≥ 0.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's more general presentation of the solution.

We prove the more general result that if $p, q \in \mathbb{N}$ are relatively prime, then the smallest integer m such that, for all $N \geq m$, n can be written as a non-negative integer linear combination of p and q is $m = pq - p - q + 1$. For the present problem $\{p, q\} = \{5, 11\}$ and so the answer is $m = 40$.

For convenience, call an integer n "*expressible*" if $n = ap + bq$ for some non-negative integers a and b. We prove the following result which was first obtained by J.J. Sylvester in 1884 in response to a problem proposed earlier by G. Frobenius.

Theorem. The smallest integer m such that n is expressible for all $n \geq m$ is

$$m = pq - p - q + 1 .$$

Proof. We first show that $m - 1$ is not expressible. Suppose, to the contrary, that $pq - p - q = ap + bq$ where a, b are non-negative integers. Then we have $pq = (1+a)p + (1+b)q$. Hence $p \mid 1+b$ and $q \mid 1+a$. Letting $1 + a = a'q$ and $1 + b = b'p$ where $a' \geq 1$, $b' \geq 1$ we get $pq = (a' + b')pq$ and so $a' + b' = 1$, which is clearly a contradiction.

Next we show that m is expressible. Since $(p, q) = 1$, there exist integers x and y such that $xp + yq = 1$ and thus $(x - kq)p + (y + kp)q = 1$

for all integers k. Since clearly $p \not| y$, we can choose an appropriate k so that $-p < y + kp < 0$. Then clearly $x - kq > 0$. Let $x_0 = x - kq$ and $y_0 = y + kp$. Then we have $-p < y_0 < 0 < x_0$, $x_0p + y_0q = 1$ and so

$$m \;=\; pq - p - q + (x_0p + y_0q) \;=\; (x_0 - 1)p + (p + y_0 - 1)q\,,$$

showing that m is indeed expressible.

Now we show by induction that n is expressible for all $n \geq m$. Suppose n_0 is expressible for some $n_0 \geq m$. Then $n_0 = \alpha p + \beta q$ for some non-negative integers α and β, and so

$$n_0 + 1 \;=\; (\alpha + x_0)p + (\beta + y_0)q\,.$$

If $\beta + y_0 \geq 0$, then we are done. Suppose $\beta + y_0 < 0$. Then we write $n_0 + 1 = (\alpha + x_0 - q)p + (\beta + y_0 + p)q$. Since $\beta + y_0 + p > 0$, it remains to show that $\alpha + x_0 - q \geq 0$. If $\alpha + x_0 - q < 0$, then $\alpha + x_0 - q \leq -1$ and thus

$$\alpha + x - kq - q + 1 \;\leq\; 0\,. \tag{1}$$

On the other hand, since $\beta + y_0 < 0$, we have $\beta + y_0 \leq -1$ and thus

$$\beta + y + kp + 1 \;<\; 0\,. \tag{2}$$

From $p \times (1) + q \times (2)$ we get

$$\alpha p + \beta q + xp + yq - pq + p + q \;\leq\; 0$$

or

$$n_0 + 1 \;\leq\; pq - p - q \;=\; m - 1\,,$$

a contradiction. Therefore, $\alpha + x_0 - q \geq 0$ and thus $n_0 + 1$ is expressible, completing our proof.

5. A subset $A \subseteq M = \{1, 2, 3, \ldots, 11\}$ is good if it has the following property:

"If $2k \in A$, then $2k - 1 \in A$ and $2k + 1 \in A$".

(The empty set and M are good). How many good subsets has M?

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bornsztein's solution.

We will prove that the number N of good subsets is $N = 233$.

Let $n(A)$ be the number of even numbers which belong to A.

Case 0. $n = 0$. We only have to determine the odd numbers in A. There are 6 odd numbers in M. For each we have two possibilities, so we have 2^6 good subsets A with $n(A) = 0$.

Case 1. $n = 1$. There are 5 possibilities for the choice of the even number. For each choice, 2 odd numbers are necessarily in A. The remaining odd numbers can be determined in 2^4 ways. Thus we have $5 \times 2^4 = 80$ good sets A with $n(A) = 1$.

Case 2. $n = 2$.

Subcase (i). The even numbers in good subsets are consecutive. We have 4 choices for the two consecutive even numbers. Each choice decides 3 odd numbers, leaving 2^3 choices. There are thus 4×2^3 good subsets A under this subcase.

Subcase (ii). The two even numbers in A are not consecutive. We have $\binom{5}{2} - 4 = 6$ choices for even numbers. Since for each choice, 4 odd numbers are decided, this leaves 2^2 choices. The total here is then 6×2^2.

The total number of good subsets A with $n(A) = 2$ is 56.

Case 3. $n = 3$.

Subcase (i). The even numbers in A are consecutive. This gives 3 possibilities, each deciding 4 odd numbers and leaving 2^2 choices.

Thus we have 3×2^2 such good sets.

Subcase (ii). No two even numbers of A are consecutive. This gives only 1 choice for even numbers 2, 6, 10. Then $A = \{1, 2, 3, 5, 6, 7, 9, 10, 11\}$, a unique choice.

Subcase (iii). Exactly two of the 3 even numbers of A are consecutive. We have $\binom{5}{3} - 4 = 6$ choices for the even numbers. For each one, 5 odd numbers are fixed. This leaves 2 choices for each of $6 \times 2 = 12$ good subsets.

Thus we have $3 \times 2^2 + 1 + 12 = 25$ good subsets A with $n(A) = 3$.

Case 4. $n = 4$.

Subcase (i). $2 \notin A$ or $10 \notin A$. This gives 2 possibilities for the even numbers, and each leaves a choice for only 1 odd number, a total of 4 good sets.

Subcase (ii). $2 \in A$ and $10 \in A$.

We have 3 choices for the even number which is not an element of A. For each, the odd numbers are all in A. This gives 3 good sets.

Thus we have $4 + 3 = 7$ good subsets A with $n(A) = 4$.

Case 5. $n = 5$, then $A = M$, 1 possibility.

Finally the total number of good subsets is

$$2^6 + 5 \times 2^4 + 56 + 25 + 7 + 1 \;=\; 233\,.$$

That completes the *Corner* for this issue. It is Olympiad Season. Send me your contest materials along with your nice solutions!

BOOK REVIEWS

ALAN LAW

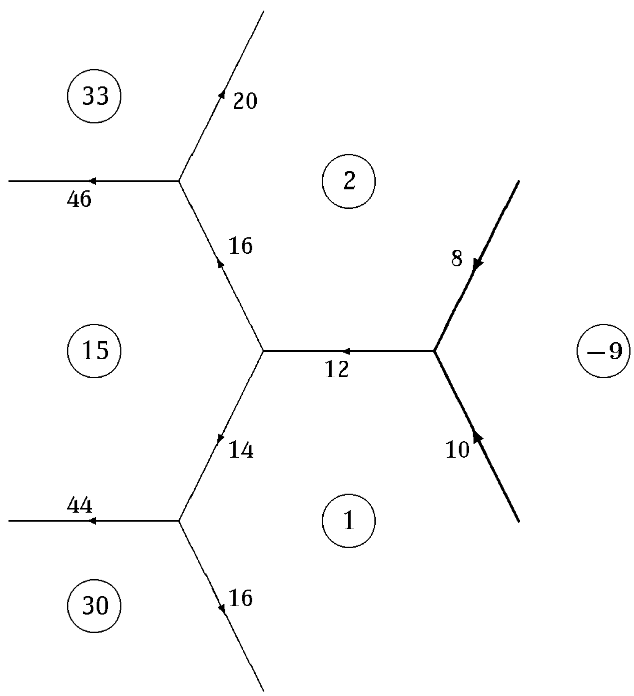
The Sensual (Quadratic) Form by John Horton Conway, assisted by Francis Y.C. Fung,
published by the Mathematical Association of America, 1997, The Carus Mathematical Monographs, Number **26**.
ISBN # 0-88385-030-3, softcover, xiv+152 pages, $35.95 (US).
Reviewed by ***Richard K. Guy****, University of Calgary, Calgary, Alberta.*

I am proud that during my service on the Hedrick Lecturers Committee, we secured the services, along with those of Sir Michael Atiyah and Ron Graham, of John Horton Conway. The lectures had to be heard, seen, felt, smelled and tasted in order to appreciate them fully, but now we have the next best thing, a written (and thankfully expanded) record.

If you know nothing about quadratic forms, or if you are an expert, you should read this book. There is no shortage of books on quadratic forms [1, 2, 3, 5, 6, 7, 9, 10, 12], so something unusual is needed to excuse another. As we have come to expect, Conway provides us with the unusual. He gives us new insights, insounds, inscents; he puts us in touch while writing tastefully. To do justice to it, we would need to copy out the whole book. I content myself with describing one item, Conway's river, and leave you to read about conorms and vonorms, Farey fractions and $PSL_2(\mathbb{Z})$, isospectral lattices (why you cannot hear their shape), gluing, the little Methuselah form, the fifteen theorem, the quadratic form as a bouquet of flowers – each flower from a finite field, and much more, in this remarkably informative yet concise little book.

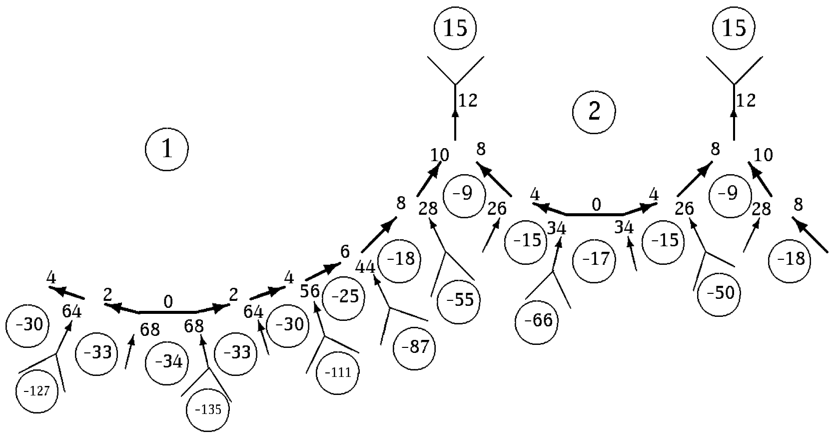
Binary quadratic forms have been largely understood since the work of Legendre [8] and Gauss [4]. But Conway can always make things clearer. His **river**, which is just a simple or double **well** for a definite binary quadratic form, separates the positive and negative values of an indefinite form, and is periodic unless the form represents zero, in which case it ends in **lakes** – there is a special case in which the river is of zero length and degenerates into a **weir** between the two lakes.

Start from your favourite form, say $2x^2 + 12xy + y^2$, and draw a **topograph**, a trivalent graph in which the edges represent **bases** and the vertices **superbases**. The regions will represent the values taken by the form and by equivalent forms. For example, $(1, 0)$ and $(0, 1)$ form a familiar base, and we throw in their sum, $(1, 1)$, and get a superbase. Draw a trivalent vertex, and label the three regions with the values of your form at these three vectors, 2, 1 and 15. To extend the topograph, make the other ends of the edges trivalent, and calculate the numbers to put on the edges and to label the new regions by the **Arithmetic Progression Rule**. In going from region 1 along the edge between regions 2 and 15 we increase from 1 to $2 + 15 = 17$; that is, by 16, which we write on the edge. Continue into the new region by adding another 16 to give 33. This is the value of our form for $(x, y) = (2, 1)$, in fact for $(\pm 2, \pm 1)$. Similarly, from region 2, between $1 + 15$, we increase by 14, and a further 14 gives 30. When we go from 15 between $2 + 1$, we *decrease* by 12, and a further decrease gives -9, a negative value, taken by our form at $(\pm 1, \mp 1)$, and we have discovered two bits of the river, the heavy lines on the right of the figure.

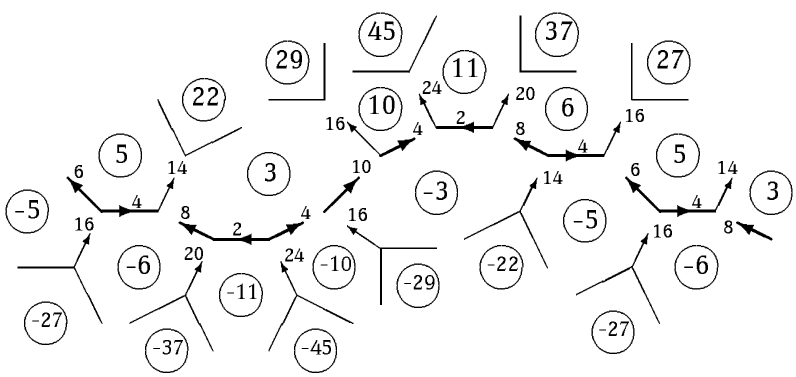


A useful check is that, although Kirchhoff's first law does not hold, the algebraic sum of the flows from a vertex is the sum of the values of the three regions, $16 + 14 - 12 = 1 + 2 + 15$, $12 - 8 - 10 = -9 + 1 + 2$. Notice also that as you go to each next edge round a region, the flow increases by $2v$, where v is the value of the region.

Continue to extend the topograph along the river. If you wander away from the river, the values in the regions increase in size, so the smallest values are always next to the river. After a while you will discover that the river is periodic. In the picture, we have only drawn a bit more than half the period, because it is not only periodic, but is its own reflection in the perpendicular bisectors of the horizontal, zero, edges.



Gaze at it and learn! If you go from region a across edge b into region c, then you have the form $ax^2 + bxy + cy^2$, for example, $-9x^2 + 8xy + 2y^2$, which takes the values 1, 2, -9 and -15 at $(1,1)$, $(0,1)$, $(1,0)$ and $(1,-1)$. All the equivalent quadratic forms are there, if you care to wander far enough away from the river. Are all the quadratic forms of discriminant 136 there? No, because the class number is 2, and the topograph illustrates only one class. You can easily find a form which is not there. There is no edge 14 and $136 = 14^2 - 4 \times 3 \times 5$, so we can draw another topograph starting from $3x^2 + 14xy + 5y^2$.



This is also periodic, but the extra symmetry this time is by rotation through $180°$ about the mid-points of the 6–edges and the 10–edges, while changing the signs of the values of the regions. Exercises for the reader: pick out the 26 forms (reduced in the narrow sense) with $a > 0$, $c > 0$, $b > a + c$, and the 2×26 simple forms, with $a > 0 > c$. Hint: 16 from the first river, 10 from the second; wander along the banks, or keep jumping across.

This is just one of the many delights in this book. Get it and read it. Mathematics is a difficult subject, and as usual, you must work as you read, but Conway makes the work unusually untedious.

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An Asymptotic Approximation for the Birthday Problem

S. Ejaz Ahmed and Richard J. McIntosh

ABSTRACT. It is known that for a class of 23 students the probability that at least two students have the same birthday is more than 0.5. Suppose that the number of days in the calendar tends to infinity. For a fixed number p with $0 < p < 1$ we give an asymptotic formula and a simple proof, not using Stirling's formula, for the minimum class size to ensure a probability of at least p that two or more students have the same birthday.

There were two U.S. Presidents born on November 2 — James K. Polk, 1795, and Warren G. Harding, 1865. Three U.S. Presidents died on the 4$^{\text{th}}$ of July — John Adams, 1826, Thomas Jefferson, 1826, and James Munroe, 1831. Given a small collection of people there is good chance that two or more individuals will have the same birthday. It is not difficult to show that 23 is the minimum number of students required in a class to ensure a probability of at least 0.5 that two or more students will have the same birthday (see for example, section II.3, p. 33 of W. Feller, An Introduction to Probability Theory and Its Applications, vol. 1, Wiley, New York, 1968).

The number of ways of choosing a sequence of k days from a calendar with n days is n^k because for each day selected we have n choices. If we require our choice of k days to be distinct, then the number of ways of doing this is reduced to $n(n-1)(n-2)\cdots(n-(k-1))$ because in our selection process we must avoid the days already chosen. Therefore if k days are chosen at random, then the probability that they will be distinct is equal to

$$\frac{n(n-1)(n-2)\cdots(n-(k-1))}{n^k}\,.$$

We will let the k days chosen be the birthdays of students in a class of size k. So the probability of two or more students having the same birthday is equal to

$$1-\frac{n(n-1)(n-2)\cdots(n-(k-1))}{n^k}\,. \qquad (1)$$

Now let $0 < p < 1$ and define k (as a function of n) to be the minimum class size to ensure a probability of at least p that two or more students will have the same birthday. We immediately see that k is the smallest positive integer making (1) greater than or equal to p. It follows that for this definition of k,

$$\lim_{n\to\infty}\left(1-\frac{n(n-1)(n-2)\cdots(n-(k-1))}{n^k}\right) = p\,,$$

or equivalently,

$$\lim_{n\to\infty}\frac{n(n-1)(n-2)\cdots(n-(k-1))}{n^k} = 1-p\,. \qquad (2)$$

In working with limits as $n \to \infty$ the concept of asymptotic functions is very useful. Two functions $f(n)$ and $g(n)$ are said to be *asymptotic* if

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = 1\,,$$

in which case we write $f(n) \sim g(n)$. With this notation in mind we now state our main theorem:

Theorem. *Fix $0 < p < 1$ and let the number of days n in the calendar tend to infinity. Then the minimum class size to ensure a probability of at least p that two or more students will have the same birthday is given asymptotically by*

$$k \sim \sqrt{2n\ln\frac{1}{1-p}}\,. \qquad (3)$$

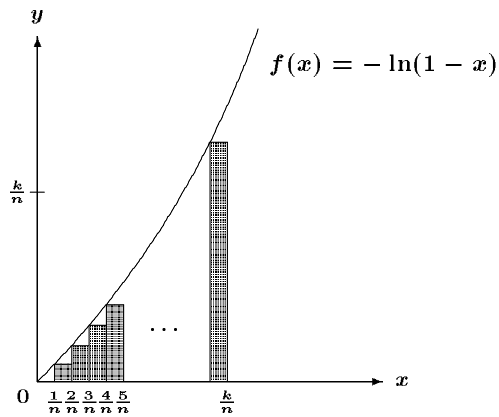
Proof. It is clear that as n tends to infinity so does k. Equation (2) can be rewritten in the form

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) = 1-p\,.$$

Taking logarithms and multiplying by $-\frac{1}{n}$ we obtain

$$\sum_{j=0}^{k-1}\frac{1}{n}\left(-\ln\left(1-\frac{j}{n}\right)\right) \sim -\frac{1}{n}\ln(1-p)\,. \qquad (4)$$

The sum on the left side of (4) is a lower Riemann sum for the function $f(x) = -\ln(1-x)$ on the interval $[0, \frac{k}{n}]$.



From the right side of (4) we see that the value of this Riemann sum is asymptotic to $-\frac{1}{n}\ln(1-p)$, which tends to 0 as $n \to \infty$. This implies that $k/n \to 0$ as $n \to \infty$. Since the slope $f'(0) = 1$, we see that as $n \to \infty$ the graph of the Riemann sum approximates the shape of a 45 degree right triangle with legs $\frac{k}{n}$ and area $-\frac{1}{n}\ln(1-p)$. By the well-known formula for the area of a triangle, it follows that

$$\frac{1}{2}\left(\frac{k}{n}\right)^2 \sim -\frac{1}{n}\ln(1-p)\,.$$

Therefore

$$\frac{k}{n} \sim \sqrt{-\frac{2}{n}\ln(1-p)}$$

and so

$$k \sim \sqrt{2n\ln\frac{1}{1-p}}\,,$$

which completes the proof.

The difference between k and our asymptotic approximation of k is not uniformly bounded for $0 < p < 1$ because the logarithm term in (3) tends to infinity as $p \to 1$, but $k \le n+1$. To explore the behaviour of this asymptotic approximation we have computed its value for certain values of p and n, listed in the tables below. In the situations illustrated in these tables we see that

$$\left|\left\lceil\sqrt{2n\ln\frac{1}{1-p}}\,\right\rceil - k\right| \le 1\,, \tag{5}$$

where $\lceil x\rceil$ denotes the smallest integer $\ge x$. Further calculations suggest that (5) holds for $0 < p \le 0.98$ and $n \ge 1$.

Acknowledgement. Support by the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged.

Table I: $p = 0.1$

n	k	$\sqrt{2n \ln \frac{1}{1-p}}$
1	2	0.459
2	2	0.649
3	2	0.795
4	2	0.918
5	2	1.026
6	2	1.124
7	2	1.215
8	2	1.298
9	2	1.377
10	2	1.452
20	3	2.053
30	4	2.514
40	4	2.903
50	4	3.246
60	5	3.556
70	5	3.841
80	5	4.106
90	5	4.355
100	6	4.590
200	7	6.492
300	9	7.951
365	10	8.770
400	10	9.181
500	11	10.265
600	12	11.244
700	13	12.145
800	14	12.984
900	15	13.771
1000	15	14.516
2000	21	20.529
5000	33	32.459
10000	47	45.904
20000	66	64.919
50000	104	102.645
100000	146	145.162
200000	206	205.291
500000	326	324.593
1000000	460	459.044

Table II: $p = 0.5$

n	k	$\sqrt{2n \ln \frac{1}{1-p}}$
1	2	1.177
2	2	1.665
3	3	2.039
4	3	2.355
5	3	2.633
6	4	2.884
7	4	3.115
8	4	3.330
9	4	3.532
10	5	3.723
20	6	5.266
30	7	6.449
40	8	7.447
50	9	8.326
60	10	9.120
70	11	9.851
80	11	10.531
90	12	11.170
100	13	11.774
200	17	16.651
300	21	20.393
365	23	22.494
400	24	23.548
500	27	26.328
600	30	28.841
700	32	31.151
800	34	33.302
900	36	35.322
1000	38	37.233
2000	53	52.655
5000	84	83.255
10000	119	117.741
20000	167	166.511
50000	264	263.277
00000	373	372.330
200000	527	526.554
500000	833	832.555
1000000	1178	1177.410

Table III: $p = 0.9$		
n	k	$\sqrt{2n \ln \frac{1}{1-p}}$
1	2	2.146
2	3	3.035
3	4	3.717
4	4	4.292
5	5	4.799
6	5	5.257
7	6	5.678
8	6	6.070
9	7	6.438
10	7	6.786
20	10	9.597
30	12	11.754
40	14	13.572
50	15	15.174
60	17	16.623
70	18	17.954
80	19	19.194
90	21	20.358
100	22	21.460
200	31	30.349
300	37	37.169
365	41	40.999
400	43	42.919
500	48	47.985
600	53	52.565
700	57	56.777
800	61	60.697
900	65	64.379
1000	68	67.861
2000	96	95.971
5000	152	151.743
10000	215	214.597
20000	304	303.485
50000	480	479.853
100000	679	678.614
200000	960	959.705
500000	1518	1517.427
1000000	2146	2145.966

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THE SKOLIAD CORNER

No. 45

R.E. Woodrow

This number we give the problems of the Kangourou Des Mathématiques, Épreuve EUROPÉENNE Cadets ($4^{\text{ième}}$-$3^{\text{ième}}$), written Friday, March 21, 1997. My thanks go to Richard Nowakowski, Canadian Team Leader to the IMO at Buenos Aires for collecting the contest questions and forwarding them to me. Readers should note that expressions such as $0,3$ would be 0.3 in English speaking countries.

KANGOUROU DES MATHÉMATIQUES
Épreuve EUROPÉENNE Cadets ($4^{\text{ième}}$-$3^{\text{ième}}$)
Vendredi 21 mars 1997 — Durée : 1h 15

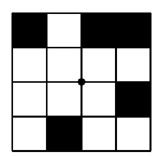
Les questions 1 à 10 valent 3 points chacune.

1. L'année dernière, $1\ 100\ 000$ jeunes de 22 pays ont participé au concours Kangourou. Combien de milliers de participants y a-t-il eu à ce Kangourou ?

(a) 110 **(b)** 1010 **(c)** 1100 **(d)** 1001 **(e)** $11\ 000$

2. Quel nombre minimum de petits carrés faut-il noircir pour que le grand carré ait un centre de symétrie ?

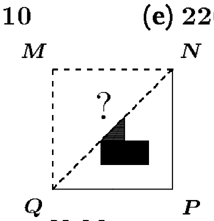
(a) 1 **(b)** 2 **(c)** 3 **(d)** 4 **(e)** 5

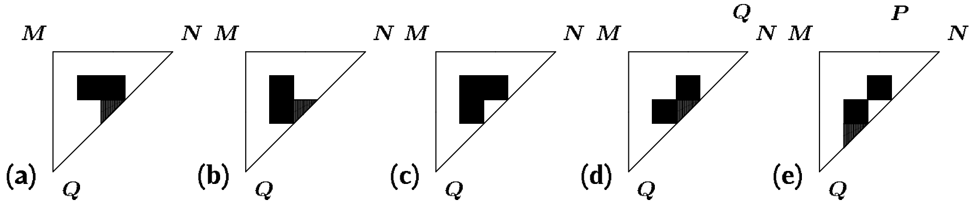


3. Au Kangouland, il y a eu, en 1996, $100\ 000$ participants au concours Kangourou. Le nombre de participants au concours y double chaque année. À partir de quelle année dépassera-t-on le million de participants au Kangouland ?

(a) 1998 **(b)** 1999 **(c)** 2000 **(d)** 2010 **(e)** 2200

4. Quel triangle MNQ faut-il choisir pour que la diagonale $[NQ]$ soit axe de symétrie de la figure obtenue ?



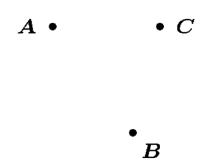


5. Christophe saute du plongeoir. Il s'élève d'un mètre en l'air, redescend de cinq mètres puis effectue une remontée de deux mètres pour atteindre la surface.

À quelle hauteur au-dessus de l'eau se trouve le plongeoir ?

(a) 1 m **(b)** 2 m **(c)** 3 m **(d)** 4 m **(e)** le plongeoir est sous l'eau

6. A, B et C sont trois points. On veut ajouter un quatrième point de façon à ce que les quatre points soient les sommets d'un parallélogramme. Combien de points conviennent ?



(a) 1 **(b)** 2 **(c)** 3 **(d)** 4 **(e)** 6

7. Le ticket d'entrée au *Palais des Sciences* coûte 50 centimes pour les enfants et 1 franc pour les adultes. Dimanche dernier, 50 personnes ont visité le Palais et la recette totale a été de 35 francs. Combien y avait-il d'adultes parmi les visiteurs ?

(a) 18 **(b)** 20 **(c)** 25 **(d)** 40 **(e)** 45

8. Quel est le quotient de 111 111 111 par 9 ?

(a) 99 **(b)** 12345678 **(c)** 12312312 **(d)** 1111111 **(e)** 12345679

9. Marie a 5 stylos. Michel a moins de stylos que Marie. Et leur petite sœur a autant de stylos à elle toute seule qu'eux deux réunis. À eux trois, ils peuvent avoir :

(a) 8 stylos **(b)** 11 stylos **(c)** 13 stylos **(d)** 14 stylos **(e)** 20 stylos

10. Ce matin, Laura, en faisant sa toilette, aperçoit dans le miroir les aiguilles de la pendule placée derrière elle. "Tiens", dit-elle, "la pendule est arrêtée : elle marque quatre heures moins cinq." Laura se trompe ! Quelle heure est-il en réalité ?

(a) 8h 05 **(b)** 4h 55 **(c)** 7h 55 **(d)** 8h 55 **(e)** 4h 05

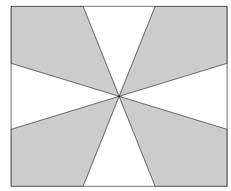
Les questions 11 à 20 valent 4 points chacune.

11. Parmi les nombres proposés, quel est le plus proche du nombre

$$\frac{21 \times 0,3 \times 1997}{10\ 000} \ ?$$

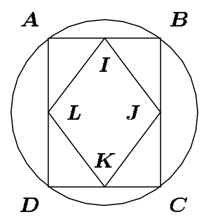
(a) $0,001$ **(b)** $0,01$ **(c)** $0,1$ **(d)** 1 **(e)** 10

12. Chaque côté d'un rectangle est divisé en trois segments de même longueur ; les points obtenus sont reliés de manière à obtenir le drapeau ci-contre. Combien vaut le quotient de l'aire de la partie blanche par celle de la partie grisée ?



(a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\frac{1}{4}$ (e) $\frac{2}{3}$

13. Dans un cercle de rayon 3 cm, on a inscrit un rectangle $ABCD$. Soient I, J, K et L les milieux de ses côtés. Quel est, en centimètres, le périmètre du losange $IJKL$?

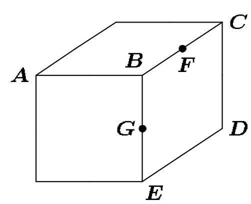


(a) 6 (b) 9 (c) 12 (d) $4\sqrt{3}$ (e) cela dépend du rectangle

14. Papa a acheté des fruits : des pommes, des poires, des bananes et des oranges. Il a 44 fruits dans son panier. Il y a 2 pommes de plus que de poires. Il y a 8 poires de plus que de bananes. Il y a 2 bananes de plus que d'oranges. Dans son panier, papa a :

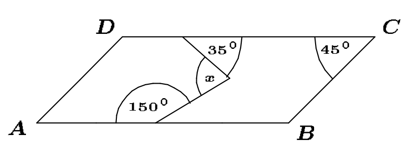
(a) 12 poires (b) 14 poires (c) 15 poires (d) 16 poires (e) 18 poires

15. Les points F et G sont les milieux des arêtes $[BC]$ et $[BE]$ du cube dessiné. Parmi les lignes brisées ci-dessous, qui joignent le sommet D au sommet A, quelle est la plus courte ?



(a) D-B-A (b) D-C-A (c) D-F-A (d) D-E-A (e) D-G-B-A

16. $ABCD$ est un parallélogramme. Les angles ayant les valeurs indiquées sur le dessin, combien vaut x ?

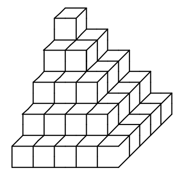


(a) 50° (b) 60° (c) 65° (d) 70° (e) 75°

17. Une montre digitale affiche l'heure 19 :57 :33.
Quel est le nombre minimum de secondes qui s'écouleront avant que tous les chiffres de l'affichage aient changé ?

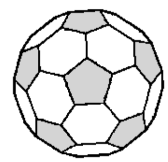
(a) 147 (b) 1 (c) 27 (d) 60 (e) 120

18. Cette pyramide est formée de petits cubes identiques de côté 1. Combien de petits cubes doit-on ajouter à la pyramide pour obtenir un grand cube de côté 5 ?



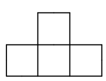
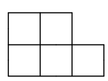
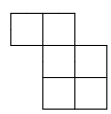
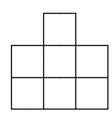
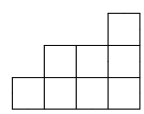
(a) 24 **(b)** 36 **(c)** 50 **(d)** 70 **(e)** 90

19. Un polyèdre en forme de ballon de football possède 32 faces : 20 sont des hexagones réguliers et 12 sont des pentagones réguliers. Combien ce solide a-t-il de sommets ?



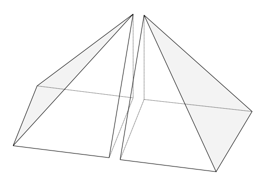
(a) 72 **(b)** 90 **(c)** 60 **(d)** 56 **(e)** 54

20. Avec quatre pièces de puzzle prises parmi les cinq ci-dessous, on peut former un carré. Quelle est la pièce qu'il ne faut pas utiliser ?

(a)  **(b)**  **(c)**  **(d)**  **(e)** 

Les questions 21 à 30 valent 5 points chacune.

21. Une pyramide régulière à base carrée est coupée en deux par un plan. On rassemble les deux moitiés en collant l'un contre l'autre les deux triangles isocèles grisés. Le nouveau solide obtenu possède :



(a) 5 faces **(b)** 6 faces **(c)** 4 faces **(d)** 7 faces **(e)** 8 faces

22. Un examen comporte 8 épreuves, chacune notée sur 5. Après les six premières épreuves, Anne a une moyenne de $3,5$. Quelle doit être sa moyenne aux deux dernières épreuves pour que sa moyenne finale soit de 4 ?

(a) 5 **(b)** c'est impossible **(c)** $4,5$ **(d)** 4 **(e)** $3,5$

23. On divise par 15 le nombre $\ll 10\ldots0\ldots\ldots0 \gg$ dont l'écriture décimale est un 1 suivi de 1997 zéros. Quel reste obtient-on ?

(a) 1 **(b)** 6 **(c)** 9 **(d)** 10 **(e)** 12

24. Quel nombre est le plus grand ?

(a) 2^{12} **(b)** 4^{15} **(c)** 8^{11} **(d)** 16^{8} **(e)** 32^{6}

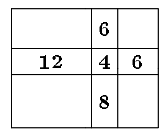
25. K est égal à 10% de L. L est égal à 20% de M. M est égal à 30% de N. Et P est égal à 40% de N. Alors, le rapport $\frac{K}{P}$ est égal à :

(a) 7 **(b)** $\frac{3}{2}$ **(c)** $\frac{2}{300}$ **(d)** $\frac{3}{200}$ **(e)** $\frac{1}{250}$

26. On plie soigneusement en deux une feuille de papier rectangulaire, cinq fois de suite, en pliant à chaque fois suivant un pli perpendiculaire au pli précédent. Après cela, on déchire les quatre coins du (petit) rectangle de papier obtenu. Ceci fait, on déplie la feuille. Combien de vrais trous voit-on alors à l'intérieur de la feuille de papier ?

(a) 4 **(b)** 9 **(c)** 18 **(d)** 20 **(e)** 21

27. Un grand rectangle est divisé en 9 petits rectangles, comme le montre le dessin. A l'intérieur de certains petits rectangles est inscrit leur *périmètre* en cm. Quel est le périmètre (en cm) du grand rectangle ?

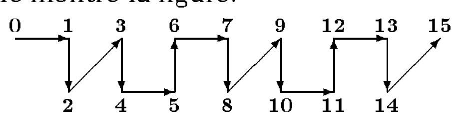


(a) 26 **(b)** 28 **(c)** 36 **(d)** 30 **(e)** 24

28. Pinocchio a une collection fantastique de calendriers des années précédentes. Mais il n'a pas de quoi s'offrir le calendrier 1997. Quel calendrier d'une année précédente doit-il utiliser pour que chaque date corresponde au bon jour de la semaine ?

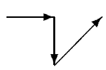
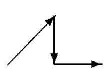
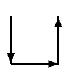
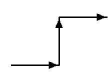
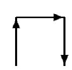
(a) 1986 **(b)** 1987 **(c)** 1989 **(d)** 1990 **(e)** 1996

29. Les nombres entiers de 0 à 2000 ont été reliés par des flèches comme le montre la figure.

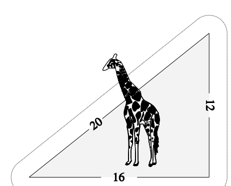


et ainsi de suite...

Quelle est la succession de flèches qui relie le nombre 1997 au nombre 2000 ?

(a)  **(b)**  **(c)**  **(d)**  **(e)** 

30. Une girafe est installée dans un curieux pré triangulaire, clôturé. Les côtés du pré mesurent 20 m, 16 m et 12 m. Grâce à son long cou, la girafe peut brouter la délicieuse herbe verte qui pousse à l'extérieur de la clôture jusqu'à une distance de 2 mètres. Soit S l'aire, en m^2, d'herbe verte qu'elle pourra brouter à l'extérieur de son pré. Parmi ces nombres, quelle est la meilleure approximation de S ?



(a) 96 **(b)** $99,14$ **(c)** $102,28$ **(d)** $105,42$ **(e)** $108,56$

Last number we gave the problems of the 1999 Maritimes Mathematics Competition, written March 11, 1999. Here are solutions.

1. Let natural numbers be assigned to the letters of the alphabet as follows: $A = 1$, $B = 2$, $C = 3, \ldots, Z = 26$. The value of a word is defined to be the product of the numbers assigned to the letters in that word. For example, the value of $MATH$ is $13 \times 1 \times 20 \times 8 = 2080$. Find a word whose value is 285.

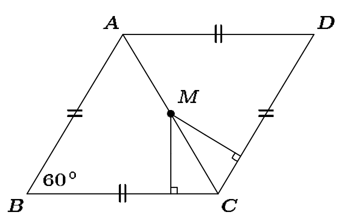
Solution. Factoring,

$$\begin{aligned} 285 &= 1 \times 3 \times 5 \times 19 = 1 \times 15 \times 19 \\ &= 3 \times 5 \times 19 = 15 \times 19 \,. \end{aligned}$$

Now 15 corresponds to O and 19 to S, and the value of SO is 285. The other possible choices of letters, $\{A, O, S\}$, $\{C, E, S\}$ and $\{A, C, E, S\}$ (even repeating A) do not seem to give English words other than $ACES$ and $CASE$. (Luckily, OS, $ACES$ and $CASE$ are answers in French!)

2. A rhombus is a parallelogram with all four sides having the same length. If one of the interior angles of a rhombus is $60°$, find the ratio of the area of the rhombus to the area of the inscribed circle.

Solution.



Label the vertices $ABCD$ with $\angle ABC = 60°$. Then $\angle CDA = 60°$. Draw diagonal AC. Then

$$\begin{aligned} \angle BAC &= \angle BCA = 60° \\ &= \angle DAC = DCA \,. \end{aligned}$$

Let M be the mid-point of AC. Now the perpendicular distances from M to BC, from M to CD, from M to AD and from M to AB are all equal to $\frac{1}{2}\overline{AB}\sin 60°$, that is the radius of the inscribed circle.

The area of the inscribed circle is $\pi(\frac{1}{2}\overline{AB}\sin 60°)^2$. Thus

$$\frac{\text{Area of rhombus}}{\text{Area of inscribed circle}} = \frac{\overline{AB}^2 \sin 60°}{\pi(\frac{1}{2}\overline{AB}\sin 60°)^2} = \frac{4}{\pi \sin 60°} = \frac{8}{\pi\sqrt{3}} \,.$$

3. A straight line cuts the asymptotes of a hyperbola in points A and B and cuts the curve at points P and Q. Prove that $AP = BQ$. (Hint: Use the fact that every hyperbola can be rotated, translated and scaled so that it is given by the equation $xy = 1$, and the asymptotes in this case are just the x–axis and the y–axis.)

Solution. From the hint we can assume, without loss of generality, that the hyperbola is given by $xy = 1$. We may do this because rotations and translations do not affect lengths while scaling changes the lengths of segments of a fixed line by the same scale factor, no matter where they are located on the line. Now we may suppose that A is the point $(a, 0)$ and B is $(0, b)$ on the x and y axes respectively. If $a = 0$ then $b = 0$ and P, Q have the form $(\frac{1}{\sqrt{m}}, \sqrt{m})$, $(\frac{-1}{\sqrt{m}}, \sqrt{m})$ which are equidistant from $A = B = (0,0)$. So suppose $AB \neq 0$. Note that $ab \neq 0$. An equation of the line is then $bx + ay = ab$. From $y = \frac{1}{x}$, solving for P, Q

$$bx + \frac{a}{x} = ab\,, \qquad bx^2 - abx + a = 0\,, \qquad (x \neq 0)\,,$$

$$x = \frac{ab \pm \sqrt{a^2b^2 - 4ab}}{2b}\,,$$

$$\begin{aligned} y = \frac{1}{x} &= \frac{2b}{ab \pm \sqrt{a^2b^2 - 4ab}} \\ &= \frac{2b(ab \mp \sqrt{a^2b^2 - 4ab})}{a^2b^2 - (a^2b^2 - 4ab)} = \frac{ab \mp \sqrt{a^2b^2 - 4ab}}{2a}\,. \end{aligned}$$

So the two points of intersection are

$$P = \left(\frac{ab + \sqrt{a^2b^2 - 4ab}}{2b}\,, \quad \frac{ab - \sqrt{a^2b^2 - 4ab}}{2a}\right)$$

and

$$Q = \left(\frac{ab - \sqrt{a^2b^2 - 4ab}}{2b}\,, \quad \frac{ab + \sqrt{a^2b^2 - 4ab}}{2a}\right)\,.$$

The squares of the distances in question become

$$\begin{aligned} AP^2 &= \left(\frac{ab + \sqrt{a^2b^2 - 4ab}}{2b} - a\right)^2 + \left(\frac{ab - \sqrt{a^2b^2 - 4ab}}{2a}\right)^2 \\ &= \left(\frac{\sqrt{a^2b^2 - 4ab} - ab}{2b}\right)^2 + \left(\frac{ab - \sqrt{a^2b^2 - 4ab}}{2a}\right)^2\,, \end{aligned}$$

$$\begin{aligned} BQ^2 &= \left(\frac{ab - \sqrt{a^2b^2 - 4ab}}{2b}\right)^2 + \left(\frac{ab + \sqrt{a^2b^2 - 4ab}}{2a} - b\right)^2 \\ &= \left(\frac{ab - \sqrt{a^2b^2 - 4ab}}{2b}\right)^2 + \left(\frac{\sqrt{a^2b^2 - 4ab} - ab}{2a}\right)^2\,. \end{aligned}$$

So $AP = BQ$ as required.

4. Find the largest number n with the property that the sum of the cubes of its digits (in base 10) is greater than n.

Solution. Denote $n = d_k \dots d_1 = d_k 10^{k-1} + \dots + d_1 10^0$ in base 10. Then, $d_k \neq 0$ and $0 \le d_i \le 9$ for $i = 1, 2, \dots, k$.

Thus $\sum_{i=1}^{k} d_i^3 \le 9^3 \cdot k = 729k$. However $n \ge 1 \cdot 10^{k-1}$.

Now $729k < 10^{k-1}$ for $k \ge 5$, by induction on k, so $k \le 4$.

Exploring $k = 4$, since $729 \times 4 = 2916$, we must in fact have $d_4 = 1$ or $d_4 = 2$ (and $\sum d_i^3 \le 2187 + 1$ or $2187 + 8$).

If $d_4 = 2$, we must have $d_3 = 0$ or $d_3 = 1$ and $\sum d_i^3 \le 1458 + 8 + 0$ or $1458+8+1$, so $d_4 \neq 2$. With $d_4 = 1$, we have 1999, which has $\sum d_i^3 = 2188$. The largest such number is 1999.

5. Find all non-negative numbers x, y and z such that

$$\begin{aligned} z^x &= y^{2x} \\ 2^z &= 2 \cdot 4^x \\ x + y + z &= 16\,. \end{aligned}$$

Solution. Note that $x, y, z \ge 0$.

With $x = 0$, $2^z = 2 \cdot 4^0 = 2$, giving $z = 1$ and $y = 15$, a solution.

So we suppose that $x > 0$. Then $z^x = y^{2x}$ implies (with $z, y \ge 0$) $z = y^2$, and $y = \sqrt{z}$.

From $2^z = 2 \cdot 4^x = 2^{2x+1}$ we have $z = 2x + 1$ and $y^2 = 2x + 1$.

Now $x + y + z = 16$ yields

$$\frac{y^2 - 1}{2} + y + y^2 = 16\,,$$

or

$$\begin{aligned} 3y^2 + 2y - 33 &= 0\,, \\ (3y + 11)(y - 3) &= 0\,, \\ y &= \frac{-11}{3}\,, \qquad y = 3\,, \quad \text{giving } y = 3 \text{ since } y \ge 0\,. \end{aligned}$$

Then $z = y^2 = 9$ and $x = \frac{y^2-1}{2} = 4$. The two solutions are therefore $(0, 15, 1)$ and $(4, 3, 9)$.

6. The following symmetric table is known as Sundaram's sieve. The first row and column is the arithmetic progression $4, 7, 10, 13, \ldots$. Successive rows are also arithmetic progressions, the common differences, respectively being the odd integers $3, 5, 7, 9, \ldots$. Show that for every positive integer n, $2n + 1$ is prime if and only if n is not in the table.

4	7	10	13	16	19	22	$\cdots$
7	12	17	22	27	32	37	$\cdots$
10	17	24	31	38	45	52	$\cdots$
13	22	31	40	49	58	67	$\cdots$
16	27	38	49	60	71	82	$\cdots$
19	32	45	58	71	84	97	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Solution. Consider the entry in row i and column j of the table. The lead entry in row i is $1 + 3i$, and the common difference for that row $d_i = (1 + 2i)$, so the entry

$$\begin{aligned} a_{ij} &= (1 + 3i) + (1 + 2i)(j - 1) \\ &= i + j + 2ij \,. \end{aligned}$$

Now suppose first that n appears in the table, so there are i and j with $n = i + j + 2ij$. Then $2n + 1 = 2i + 2j + 4ij + 1 = (2i + 1)(2j + 1)$, which is not prime since $i, j \geq 1$.

Conversely, suppose that $2n + 1$ is not prime. As it is an odd number we have $2n + 1 = (2i + 1)(2j + 1)$ for some $i, j \geq 1$.

But then $n = i + j + 2ij = a_{ij}$ appears in the table.

That completes the *Skoliad Corner* for this number. I need contest materials at a suitable level, and I welcome your suggestions for features of future *Skoliad Corners*.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to Mathematical Mayhem, Department of Mathematics, University of Toronto, 100 St. George St., Toronto, Ontario, Canada. M5S 3G3. The electronic address is

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The Assistant Mayhem Editor is Cyrus Hsia (University of Western Ontario). The rest of the staff consists of Adrian Chan (Harvard University), Jimmy Chui (University of Toronto), and David Savitt (Harvard University)

Mayhem Problems

The Mayhem Problems editors are:

Adrian Chan	*Mayhem High School Problems Editor,*
Donny Cheung	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only problems — the next issue will feature only solutions.

We warmly welcome proposals for problems and solutions. With the schedule of eight issues per year, we request that solutions from this issue be submitted in time for issue 4 of 2001.

High School Problems

Editor: Adrian Chan, 1195 Harvard Yard Mail Center, Cambridge, MA, USA 02138-7501 <ahchan@fas.harvard.edu>

H269. Find the lengths of the sides of a triangle with 20, 28, and 35 as the lengths of its altitudes.

H270. Find all triangles ABC that satisfy

$$\sin(A-B)+\sin(B-C)+\sin(C-A) \;=\; 0\,.$$

H271. *Proposed by Ho-Joo Lee, student, Kwangwoon University, Seoul, South Korea.*

Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Let $n = \lfloor 1/(a - \lfloor a \rfloor) \rfloor$ for some positive real number a.

Show that $\lfloor (n+1)a \rfloor \equiv 1 \pmod{n+1}$.

H272. *Proposed by Ho-Joo Lee, student, Kwangwoon University, Seoul, South Korea.*

Let $\{a_1, a_2, a_3, \ldots, a_n\}$ be a set of real numbers. Let $s = a_1 + a_2 + \cdots + a_n$. Show that $\sum_{i=1}^{n} \sum_{k=1}^{n} (a_k - a_i)(s - a_i) \geq 0$.

Advanced Problems

Editor: Donny Cheung, c/o Conrad Grebel College, University of Waterloo, Waterloo, Ontario, Canada. N2L 3G6 <dccheung@uwaterloo.ca>

A245. Show that a polygon with fixed side lengths has maximal area when it can be inscribed in a circle.

A246. *Proposed by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.*

Given a triangle with angles A, B, C, circumradius R, and inradius r, prove that

$$1 + \frac{r}{R} \leq \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \leq \frac{17}{12} + \frac{r}{6R}.$$

A247. There are n flat convex planar surfaces in 3–space, and the sum of their areas is 1. Prove that there exists a plane so that the sum of the areas of their projections to the plane is less than $1/2$.

A248.

(a) Prove that in every sequence of 79 consecutive positive integers written in the decimal system, there is a positive integer whose sum of digits is divisible by 13.

(1997 Baltic Way)

(b) Give a sequence of 78 consecutive positive integers each with a sum of digits not divisible by 13.

Challenge Board Problems

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

C98. Let H be a subset of the positive integers with the property that if x, $y \in H$, then $x + y \in H$. Define the *gap sequence* G_H of H to be the set of positive integers not contained in H.

(a) Prove that if G_H is a finite set, then the arithmetic mean of the integers in G_H is less than or equal to the number of elements in H.

(b) Determine all sets H for which equality holds in part (a).

C94. *Proposed by Edward Crane and Russell Mann, graduate students, Harvard University, Cambridge, MA, USA.*

Suppose that V is a k–dimensional vector subspace of the Euclidean space $\mathbb{R}^n$ which is defined by linear equations with coefficients in $\mathbb{Q}$. Let Λ be the lattice in V given by the intersection of V with the lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$, and let $\Lambda^\perp$ be the lattice given by the intersection of the perpendicular vector space $V^\perp$ with $\mathbb{Z}^n$. Show that the (k–dimensional) volume of Λ is equal to the (($n-k$)-dimensional) volume of $\Lambda^\perp$.

Problem of the Month

Jimmy Chui, student, University of Toronto

Problem. Let $x_1, x_2, \ldots, x_{n+1}$ be positive real numbers such that

$$\frac{1}{1+x_1} + \frac{1}{1+x_2} + \cdots + \frac{1}{1+x_{n+1}} \;=\; 1\,.$$

Show that

$$x_1 x_2 \cdots x_{n+1} \;\geq\; n^{n+1}\,.$$

(1998 Canadian IMO Training)

Solution. Let us make the substitution $y_i = 1/(1+x_i)$ for $i = 1, 2, \ldots, n+1$. Then, we want to show that

$$\prod_{i=1}^{n+1} \left(\frac{1-y_i}{y_i}\right) \;\geq\; n^{n+1}\,,$$

where $y_1 + y_2 + \cdots + y_{n+1} = 1$.

Let

$$s_i \;=\; \sum_{\substack{j=1\\ j\neq i}}^{n+1} y_j\,, \quad \text{and} \quad p_i \;=\; \prod_{\substack{j=1\\ j\neq i}}^{n+1} y_j\,.$$

Observe that

$$\frac{1-y_i}{y_i} \;=\; \frac{s_i}{y_i} \geq \frac{n\sqrt[n]{p_i}}{y_i}$$

by the AM-GM Inequality.

Note that

$$\prod_{i=1}^{n+1} \sqrt[n]{p_i} \;=\; \prod_{i=1}^{n+1} y_i$$

since each y_i appears exactly n times in the product. Then we have that

$$\prod_{i=1}^{n+1}\left(\frac{1-y_i}{y_i}\right) \;\geq\; \prod_{i=1}^{n+1}\frac{n\sqrt[n]{p_i}}{y_i} \;=\; n^{n+1}\cdot\frac{\prod_{i=1}^{n+1} y_i}{\prod_{i=1}^{n+1} y_i} \;=\; n^{n+1}\,,$$

QED.

The key step in this solution is the very clever substitution $1 - y_i \;=\; s_j$. Literally, this is a very neat five-line solution to an otherwise difficult problem. Now, as a follow-up, try to solve Problem 3 of the 1998 USAMO:

Let a_0, a_1, ..., a_n be numbers from the interval $(0, \pi/2)$ such that

$$\tan\left(a_0 - \frac{\pi}{4}\right) + \tan\left(a_1 - \frac{\pi}{4}\right) + \cdots + \tan\left(a_{n+1} - \frac{\pi}{4}\right) \;\geq\; n-1\,.$$

Prove that

$$\tan a_0 \tan a_1 \cdots \tan a_{n+1} \;\geq\; n^{n+1}\,.$$

J.I.R. McKnight Problems Contest 1995

1. Solve for x, given

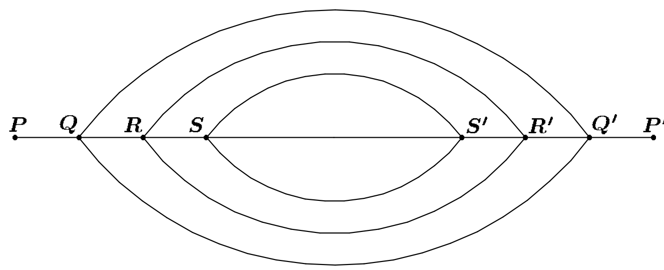
   (a) $(\log_2 x)(\log_2 x^2) - \log_2 x^3 - 9 = 0$,

   (b) $(\log_2(35 - x^3))/(\log_2(5 - x)) = 3$.

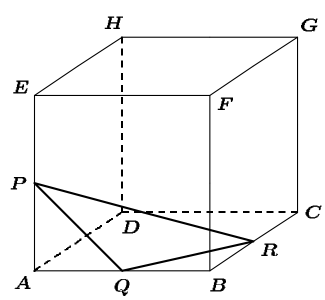
2. Solve for x, given the following equation:

$$\cos^{-1} x - \sin^{-1} x \;=\; \frac{\pi}{6}\,.$$

3. In the diagram, $PQ = P'Q'$, $QR = Q'R'$, and $RS = R'S'$. Albert Mouse leaves from P to go to P' while Betty Mouse leaves from P' to go to P. Both mice start out at the same time and proceed at the same speed. If the chances are even of picking any route at each intersection, then what is the probability that Albert and Betty will not meet? (Back tracking is not permitted).



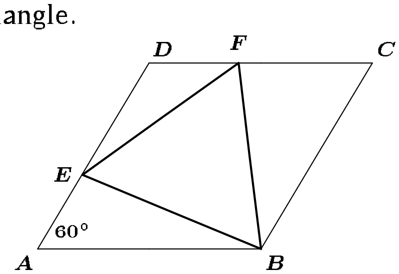
4. Given P, Q, R are the mid-points of the edges of a cube of side 2, as shown.



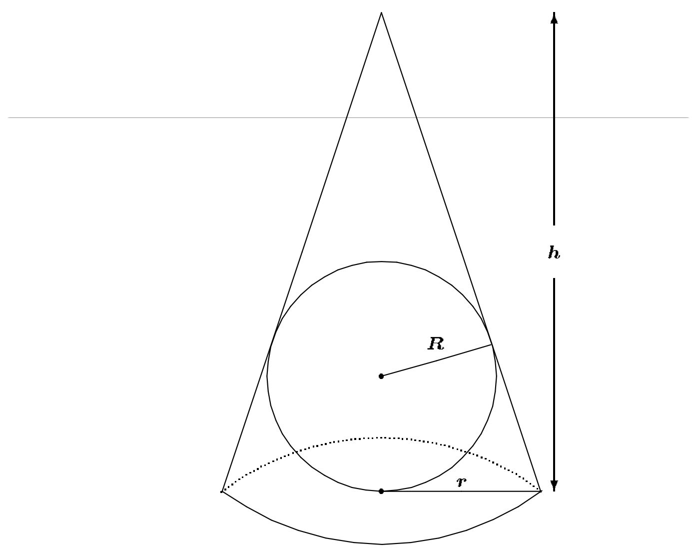
(a) Find the angle between the two planes determined by triangle PQR and square $ABCD$, and express it in degree measure rounded to 1 digit after the decimal point.

(b) Find the area of triangle PQR.

5. Given a rhombus $ABCD$, such that angle $DAB = 60°$. Point E is on AD and point F on DC such that $AE = DF$. Show that triangle BEF is an equilateral triangle.



6. Find the radius r and the height h of the right circular cone of minimum volume which can be circumscribed about a sphere of fixed radius R. Express both quantities r and h in terms of R.

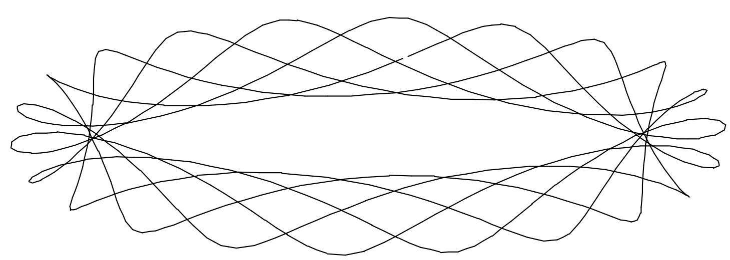


7. Find the sum of the following:

$$S_n = \frac{2^2+1}{2^2-1} + \frac{3^2+1}{3^2-1} + \frac{4^2+1}{4^2-1} + \cdots + \frac{(n+1)^2+1}{(n+1)^2-1}\,.$$

8. Find $f(x)$ such that:

$$f(x) + f\left(\frac{x-1}{x}\right) = 1 + x\,.$$



How to Solve the Cubic

Naoki Sato

The simple and usual method of solving the quadratic polynomial is "completing the square"; however, as anyone who has tried knows, this is woefully inadequate for the cubic polynomial. And, unlike the quadratic, the solution of the cubic is not a part of the standard high-school curriculum, but is still quite accessible to young students. All that is required is some background in basic polynomial algebra, which we will cover. So now, you too can solve a cubic.

Polynomials: Roots and Coefficients

Let the roots of the polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be $r_1, r_2, \ldots, r_n$. By the Factor Theorem,

$$q(x) \;=\; (x - r_1)(x - r_2)\cdots(x - r_n)$$

divides $p(x)$. However, both p and q have degree n, so one is a constant multiple of the other. The leading coefficients (the coefficients of x^n) in p and q are a_n and 1 respectively, so $p(x) = a_n q(x)$. Let s_k be the sum of the products of the r_i taken k at a time, so that

$$\begin{aligned} s_1 &= r_1 + r_2 + \cdots + r_n\,, \\ s_2 &= r_1 r_2 + r_1 r_3 + \cdots + r_{n-1} r_n\,, \\ &\cdots \\ s_n &= r_1 r_2 \cdots r_n\,. \end{aligned}$$

Then

$$\begin{aligned} &a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \\ &\quad= a_n(x - r_1)(x - r_2)\cdots + (x - r_n) \\ &\quad= a_n\big[x^n - (r_1 + r_2 + \cdots + r_n)x^{n-1} \\ &\qquad\quad +(r_1 r_2 + r_1 r_3 + \cdots + r_{n-1} r_n)x^{n-2} \\ &\qquad\quad - \cdots + (-1)^n r_1 r_2 \cdots r_n\big] \\ &\quad= a_n(x^n - s_1 x^{n-1} + s_2 x^{n-2} - \cdots + (-1)^n s_n)\,. \end{aligned}$$

Equating coefficients, we conclude that

$$s_k \;=\; (-1)^k \cdot \frac{a_{n-k}}{a_n}$$

for $k = 1, 2, \ldots, n$. The s_k are called the *elementary symmetric polynomials* in the r_i, and by the derivation above, they are determined by the coefficients of p.

For example, let a and b be the roots of $x^2 - 4x + 2 = 0$. Then we need not solve for the roots to tell that $a + b = 4$ and $ab = 2$; we can simply read off the coefficients. As a simple exercise, calculate $a^2 + b^2$ and $a^3 + b^3$. We are now ready to attack the cubic polynomial.

An Algebraic Approach

We begin with the general cubic polynomial

$$a_3x^3 + a_2x^2 + a_1x + a_0 \;=\; 0\,.$$

Divide by a_3 to get the monic cubic polynomial

$$x^3 + ax^2 + bx + c \;=\; 0\,.$$

We perform the substitution $x = t + s$, and express the cubic in terms of the new variable t:

$$\begin{aligned} &x^3 + ax^2 + bx + c \\ &\quad= \; (t+s)^3 + a(t+s)^2 + b(t+s) + c \\ &\quad= \; t^3 + 3st^2 + 3s^2t + s^3 + at^2 + 2ast + as^2 + bt + bs + c \\ &\quad= \; t^3 + (a+3s)t^2 + (2as + b + 3s^2)t + s^3 + as^2 + bs + c\,. \qquad (1) \end{aligned}$$

The correct value of s will turn (1) into a cubic with no t^2 term, called a *depressed* cubic. Solving: $a + 3s = 0$, or $s = -a/3$. Let our new cubic be

$$t^3 + pt + q\,, \qquad (2)$$

where p and q are functions of a, b, and c.

Let ω be a primitive cube root of unity; that is, $\omega^3 = 1$, but $\omega \neq 1$ and $\omega^2 \neq 1$. These imply that $\omega^3 - 1 = (\omega - 1)(\omega^2 + \omega + 1) = 0$, so $\omega^2 + \omega + 1 = 0$. If we solve this quadratic, then we obtain the roots

$$\omega = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\,i\,.$$

Check that $\omega^3 = 1$. We will see that it does not matter which root we take ω to be.

We now utilize the wonderful identity

$$\begin{aligned} &(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z) \\ &\quad= \; (x+y+z)(x^2+y^2+z^2-xy-xz-yz) \\ &\quad= \; x^3+y^3+z^3-3xyz\,. \end{aligned}$$

Verify this identity. Note that the right-hand side is a cubic in x with no term in x^2; we wish to emulate our depressed cubic with this expression. Substitute $x = t$, $y = -\alpha$ and $z = -\beta$, to obtain

$$(t - \alpha - \beta)(t - \omega\alpha - \omega^2\beta)(t - \omega^2\alpha - \omega\beta) = t^3 - 3\alpha\beta t - \alpha^3 - \beta^3 . \quad (3)$$

Note that the roots of (3) are $\alpha + \beta$, $\omega\alpha + \omega^2\beta$, and $\omega^2\alpha + \omega\beta$. Our depressed cubic is $t^3 + pt + q$, so we want to find (solve for) α and β such that

$$\alpha\beta \;=\; -\frac{p}{3} \quad \text{and} \quad \alpha^3 + \beta^3 \;=\; -q .$$

Then $\alpha^3\beta^3 = -p^3/27$. We conclude that α^3 and β^3 are the roots of

$$u^2 + qu - \frac{p^3}{27} \;=\; 0 .$$

Therefore, α and β are

$$\sqrt[3]{\frac{-q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2}} \;=\; \sqrt[3]{-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} .$$

The roots of (2) are $\alpha + \beta$, $\omega\alpha + \omega^2\beta$, and $\omega^2\alpha + \omega\beta$, as stated above. That, technically, does it.

But if $q^2/4 + p^3/27 < 0$, then we run into difficulties and the formula above is not very practical (see Problem 3). In such a case, we must try another mode of attack.

A Trigonometric Approach

Assume that $q^2/4 + p^3/27 < 0$. Then $p < 0$. This time, we utilize the trigonometric identity

$$\cos 3\theta \;=\; 4\cos^3\theta - 3\cos\theta .$$

Note that the right-hand side is a cubic in $\cos\theta$, with no quadratic term. We begin with the depressed cubic in (2), and make the substitution $t = r\cos\theta$, where r is a constant to be determined:

$$t^3 + pt + q \;=\; r^3\cos^3\theta + rp\cos\theta + q .$$

We wish the expression $r^3\cos^3\theta + rp\cos\theta$ to mimic the expression $4\cos^3\theta - 3\cos\theta$. We can achieve this by setting r to be a value which makes the two proportional, so we solve

$$\frac{r^3}{4} \;=\; -\frac{rp}{3} \implies r^2 \;=\; -\frac{4}{3}p \implies r \;=\; \pm 2\sqrt{-\frac{p}{3}} .$$

Then

$$\begin{aligned}
t^3 + pt + q &= r^3 \cos^3\theta + pr\cos\theta + q \\
&= r(r^2\cos^3\theta + p\cos\theta) + q \\
&= r\left(-\frac{4}{3}\,p\cos^3\theta + p\cos\theta\right) + q \\
&= -\frac{rp}{3}(4\cos^3\theta - 3\cos\theta) + q \\
&= -\frac{rp}{3}\cos 3\theta + q \\
\Longrightarrow \quad \cos 3\theta &= \frac{3q}{rp}\,. \qquad (4)
\end{aligned}$$

Let θ_0 be a solution of (4). Then $\theta_1 := \theta_0 + 120°$ and $\theta_2 := \theta_0 + 240°$ are also solutions of (4). Then the roots of (2) are $r\cos\theta_0$, $r\cos\theta_1$, and $r\cos\theta_2$.

Problems.

1. In (1), express p and q in terms of a, b, and c.

2. Solve the following cubics. You may be required to use different methods.

   (a) $x^3 - 7x - 6 = 0$.

   (b) $2x^3 - 30x^2 + 162x - 350 = 0$.

   (c) $x^3 - 3x^2 + 3x - 2 = 0$.

   (d) $x^3 + x^2 + x + 1/3 = 0$.

3. According to the algebraic approach, the "real root" of $x^3 - 15x - 4 = 0$ is
$$\sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i}\,.$$
   However, we can see that the real root is 4. Explain.

4. Find a geometrical interpretation of the trigonometric approach.

5. Solve the following system of equations:
$$\begin{aligned}
x + y + z &= 1\,, \\
x^2 + y^2 + z^2 &= 7\,, \\
x^3 + y^3 + z^3 &= 13\,.
\end{aligned}$$
   Hint: Find the values of $xy + xz + yz$ and xyz.

6. Let r_1, r_2, and r_3 be the roots of the depressed cubic $t^3 + pt + q = 0$. The *discriminant* of this cubic is defined to be

$$D := [(r_1 - r_2)(r_1 - r_3)(r_2 - r_3)]^2 .$$

   (a) Show that $D = -4p^3 - 27q^2$.
   Hint: Remember that $r_1 + r_2 + r_3 = 0$ and $r_i^3 + pr_i + q = 0$ for all i.

   (b) Assume that r_1 is non-real, say $u + vi$, where $u, v \in \mathbb{R}$, $v \neq 0$. Then a result in algebra states that $u - vi$ must also be a root, so let $r_2 = u - vi$. Since the sum of the roots is zero, $r_3 = -2u$. Show that $D = -4v^2(9u^2 + v^2)^2 < 0$.

   (c) Conclude that

$$D \begin{cases} > 0 & \text{if and only if the roots are real and distinct,} \\ = 0 & \text{if and only if the roots are real and at least two are equal, and} \\ < 0 & \text{if and only if one root is real and the other two are non-real.} \end{cases}$$

   Thus, D indicates the nature of the roots.

7. The real numbers α, β satisfy the equations

$$\begin{aligned} \alpha^3 - 3\alpha^2 + 5\alpha - 17 &= 0 \\ \beta^3 - 3\beta^2 + 5\beta + 11 &= 0 . \end{aligned}$$

   Find $\alpha + \beta$.

   (1993 Irish Mathematical Olympiad)

8. Prove that if c is a rational number, then the equation

$$x^3 - 3cx^2 - 3x + c = 0$$

   has at most one rational solution.

9. Prove that $\sin(\pi/14)$ is a root of the equation

$$8x^3 - 4x^2 - 4x + 1 = 0 .$$

   What are the other two roots?

10. Let the cubic equation $x^3 + ax^2 + bx + c = 0$ have the three real roots r_1, r_2, and r_3, such that $r_1 \leq r_2 \leq r_3$.

   (a) Show that $a^2 \geq 3b$.

   (b) Show that

$$\frac{-a - 2\sqrt{a^2 - 3b}}{3} \leq r_1 \leq \frac{-a - \sqrt{a^2 - 3b}}{3},$$

   and that

$$\frac{-a + \sqrt{a^2 - 3b}}{3} \leq r_3 \leq \frac{-a + 2\sqrt{a^2 - 3b}}{3}.$$

(c) Show that

$$\sqrt{a^2 - 3b} \;\le\; r_3 - r_1 \le \frac{2\sqrt{3}}{3}\sqrt{a^2 - 3b}\,.$$

11. Let $f(x)$ be a cubic polynomial in x with roots r_1, r_2, and r_3. If

$$\frac{f(\frac{1}{2}) + f(-\frac{1}{2})}{f(0)} \;=\; 997\,,$$

then find

$$\frac{1}{r_1 r_2} + \frac{1}{r_2 r_3} + \frac{1}{r_3 r_1}\,.$$

(***Mathematical Mayhem***, **H46**)

12. (a) Let x, y, and z be real numbers, such that $x+y+z$, $xy+xz+yz$, and xyz are all positive. Prove that x, y, and z are also positive.

    (b) Does the assertion in (a) still hold if x, y, and z are allowed to be complex?

(***Mathematical Mayhem***, **A119**)

13. How many triples a, b, c of real numbers are there such that a, b, c are the roots of the equation $x^3 + ax^2 + bx + c = 0$?

(1993 Descartes Contest, Problem 9)

14. Find necessary and sufficient conditions on the coefficients of the cubic $x^3 + ax^2 + bx + c = 0$ for the roots to be in arithmetic progression, and in geometric progression.

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PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk (⋆) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ *or* A4 *sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than* **1 November 2000**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in* LaTeX*). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

For the information of proposers who submitted problems in 1998, we have either used them, or will not be using them.

2525. *Proposed by Antreas P. Hatzipolakis, Athens, Greece, and Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

In $\triangle ABC$, we have $B = 135^\circ - \frac{A}{2}$ and $C = 45^\circ - \frac{A}{2}$. Show that

(a) the centre V of the nine-point circle of $\triangle ABC$ lies on the side BC;

(b) if $A = 60^\circ$, then AV bisects angle A.

2526. *Proposed by K.R.S. Sastry, Dodballapur, India.*

In a triangle, prove that an internal angle bisector trisects an altitude if and only if the bisected angle has the measure $\pi/3$ or $2\pi/3$.

2527. *Proposed by K.R.S. Sastry, Dodballapur, India.*

Let AD, BE and CF be concurrent cevians of $\triangle ABC$. Assume that:

(a) AD is a median; (b) BE bisects $\angle ABC$; (c) BE bisects AD.

Prove that $CF > BE$.

2528. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Prove that every rectifiable centrosymmetric curve on a unit sphere in $\mathbb{E}^3$ has length greater than or equal to 2π.

2529. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let $G = \{A_1, A_2, \ldots, A_n\}$ be a set of points on a unit hemisphere. Let $\widehat{A_iA_j}$ be the spherical distance between the points A_i and A_j. Suppose that $\widehat{A_iA_j} \geq d$. Find $\max d$.

2530. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let F be a compact convex set in $\mathbb{E}^3$, let T be the translation along a vector $\vec{a}$, and let $F' = T(F)$.

Prove that the intersection of the boundary of F and the boundary of F' is connected.

2531. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let F be a convex plane set and AB its diameter. The points A and B divide the perimeter of F into two parts, L_1 and L_2, say. Prove that

$$\frac{1}{\pi - 1} < \frac{L_1}{L_2} < \pi - 1\,.$$

2532. *Proposed by Ho-joo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

Suppose that a, b and c are positive real numbers satisfying $a^2 + b^2 + c^2 = 1$. Prove that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq 3 + \frac{2(a^3 + b^3 + c^3)}{abc}\,.$$

2533. *Proposed by K.R.S. Sastry, Bangalore, India.*

In the integer sided $\triangle ABC$, let e denote the length of the segment of the Euler line between the orthocentre and the circumcentre.

Prove that $\triangle ABC$ is right angled if and only if e equals one half of the length of one of the sides of $\triangle ABC$.

Compare problem 2433. [1999 : 173, 2000 : 187]

2534. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Suppose that a is an integer and x and y natural numbers. Define $z_a(x,y) = \dfrac{x^2 + y^2 + a}{xy}$.

1. Show that there exist infinitely many values of a such that $z_a(x,y)$ is an integer for infinitely many pairs $(x,y) \in \mathbb{N}^2$.

2.* Is the set $E(a)$ of integers $z_a(x,y)$ as obtained above necessarily infinite? If the answer is "no", determine those a's which determine finite sets $E(a)$.

2535*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

1. Prove that neither of the integers $a(n) = 3n^2 + 3n + 1$ and $b(n) = n^2 + 3n + 3$ $(n \geq 1)$ has a divisor k such that $k \equiv 2 \pmod 3$.
2. Prove or disprove that both of the sequences $\{a(n)\}$ and $\{b(n)\}$ $(n \geq 1)$ contain infinitely many primes.

2536. *Proposed by Cristinel Mortici, Ovidius University of Constanta, Romania.*

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and periodic function such that for all positive integers n the following inequality holds:

$$\frac{|f(1)|}{1} + \frac{|f(2)|}{2} + \ldots + \frac{|f(n)|}{n} \leq 1 .$$

Prove that there exists $c \in \mathbb{R}$ such that $f(c) = 0$ and $f(c+1) = 0$.

2537. *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

Find the exact value of $\cot\left(\frac{\pi}{7}\right) + \cot\left(\frac{2\pi}{7}\right) - \cot\left(\frac{3\pi}{7}\right)$.

2538. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

On a recent calculus test, students were asked to compute the arc length of a curve represented by a certain function $f(x)$, for $x = a$ to $x = b$, $a < b$. One of the students, a Mr. Fluke, simply calculated $f'(b) - f'(a)$, and obtained the correct answer.

Determine all real functions, $f(x)$, differentiable on some open interval I, such that, for all a, b satisfying $(a, b) \subset I$, the arc length of the curve $y = f(x)$, from $x = a$ to $x = b$ is equal to $f'(b) - f'(a)$.

2488. [1999: 431] **ADDENDUM**

The proposer, G. Tsintsifas, Thessaloniki, Greece, observes that we need to point out the necessary condition for a convex cover: $\sum_{k=1}^{n+1} \mu_k = 1$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2426. [1999: 172] *Proposed by Mohammed Aassila, Strasbourg, France.*

(a) Show that there are two polynomials, $p(x)$ and $q(x)$, both having three integer roots and such that $p(x) - q(x)$ is a non-zero constant.

(b)⋆ Do there exist two polynomials, $p(x)$ and $q(x)$, both having $n > 3$ integer roots and such that $p(x) - q(x)$ is a non-zero constant?

Combination of solutions by Michael Lambrou, University of Crete, Crete, Greece, and Kenneth M. Wilke, Topeka, KS, USA.

This problem is essentially a special case of the Tarry–Escott or Prouhet–Tarry–Escott problem, which requires finding solutions of the system of equations

$$\begin{aligned} a_1 + a_2 + \cdots + a_n &= b_1 + b_2 + \cdots + b_n \\ a_1^2 + a_2^2 + \cdots + a_n^2 &= b_1^2 + b_2^2 + \cdots + b_n^2 \\ &\vdots \\ a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1} &= b_1^{n-1} + b_2^{n-1} + \cdots + b_n^{n-1} \end{aligned} \qquad (1)$$

in integers $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$, where $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are not a permutation of each other. By setting

$$p(x) = (x - a_1)(x - a_2)\cdots(x - a_n) = x^n + A_1 x^{n-1} + \cdots + A_{n-1}x + A_n$$

and

$$q(x) = (x - b_1)(x - b_2)\cdots(x - b_n) = x^n + B_1 x^{n-1} + \cdots + B_{n-1}x + B_n ,$$

we can use Newton's relations regarding the roots of an equation to determine the sums of the powers of the roots of $p(x)$; that is, putting

$$S_j = a_1^j + a_2^j + \cdots + a_n^j \qquad \text{for} \quad 1 \le j \le n ,$$

we have

$$\begin{aligned} S_1 + A_1 &= 0 \\ S_2 + A_1 S_1 + 2A_2 &= 0 \\ &\vdots \\ S_n + A_1 S_{n-1} + \cdots + A_{n-1} S_1 + nA_n &= 0 . \end{aligned}$$

[See for example Ed Barbeau's book *Polynomials*, page 199, exercise 6.—*Ed.*]
Similarly we have for $q(x)$:

$$\begin{aligned} T_1 + B_1 &= 0 \\ T_2 + B_1T_1 + 2B_2 &= 0 \\ &\vdots \\ T_n + B_1T_{n-1} + \cdots + B_{n-1}T_1 + nB_n &= 0\,, \end{aligned}$$

where

$$T_j \;=\; b_1^j + b_2^j + \cdots + b_n^j \qquad \text{for} \quad 1 \le j \le n\,.$$

The condition that $p(x) - q(x)$ is a non-zero constant amounts to saying that $A_1 = B_1$, $A_2 = B_2$, ..., $A_{n-1} = B_{n-1}$. It is easy to see that this reduces to finding solutions of the equations $S_i = T_i$ for $1 \le i \le n-1$; that is, (1). Conversely if (1) holds, then so do $A_i = B_i$ for $1 \le i \le n-1$. Moreover $p \ne q$ requires that the roots a_1, ..., a_n of p and b_1, ..., b_n of q are not a permutation of each other.

An excellent summary of the known results regarding this problem may be found on the Internet at

http://mathworld.wolfram.com/Prouhet-Tarry-EscottProblem.html

(which also contains a large bibliography) or on Chen Shuwen's webpage at

http://member.netease.com/~chin/eslp/TarryPrb.htm

[Readers without access to the Internet could consult Hardy and Wright's *An Introduction to the Theory of Numbers*, 5th ed. (1979), § 21.9 and 21.10, pp. 328–332, 338.]

For example, to answer part (a), let

$$p(x) \;=\; (x-a)(x-b)(x+a+b) \quad \text{and} \quad q(x) \;=\; (x+a)(x+b)(x-a-b)\,,$$

where a and b are arbitrarily chosen non-zero integers such that $a \ne -b$. Then $p(x) - q(x) = 2ab(a+b)$, so that we have an infinite family of solutions. However, these are not the only solutions possible for part (a).

Solutions for (1), and hence the present problem, are known for all n from 3 to 10 (see the second website given above). For example, for $n = 6$ we could use

$$p(x) \;=\; x^2(x^2-49)^2\,, \qquad q(x) \;=\; (x^2-9)(x^2-25)(x^2-64)\,,$$

so that $p(x) - q(x) = 14400$. Recently (1999) Chen Shuwen gave an example with $n = 12$, from which we may write

$$\begin{aligned} p(x) \;=\; & x(x-11)(x-24)(x-65)(x-90)(x-129)(x-173) \\ & \times (x-212)(x-237)(x-278)(x-291)(x-302) \end{aligned}$$

and

$$q(x) = (x-3)(x-5)(x-30)(x-57)(x-104)(x-116)(x-186) \times (x-198)(x-245)(x-272)(x-297)(x-299).$$

No example is known for $n = 11$ or for n greater than 12. However, it has been conjectured that there is a solution for any integer $n \geq 3$.

Also solved by FEDERICO ARBOLEDA and BERNARDO RECAMÁN SANTOS, Bogotá, Colombia; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES R. DIMINNIE and TREY SMITH, Angelo State University, San Angelo, Texas, USA; DOUGLASS L. GRANT, University College of Cape Breton, Sydney, Nova Scotia; G.P. HENDERSON, Garden Hill, Campbellcroft, Ontario; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; PETER HURTHIG, Columbia College, Vancouver, BC; and DIGBY SMITH, Mount Royal College, Calgary, Alberta. All these gave a solution for at least $n = 3$ and 4. Part (a) *only was solved by HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer.*

WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria, and MAX SHKARAYEV and MARK LYON, students, University of Arizona, Tuczon, AZ, USA, gave solutions in which $p(x)$ and $q(x)$ have degrees larger than n, which was not explicitly prohibited in the statement of the problem (but should have been). Their solutions are almost the same, namely they let $p(x)$ be a polynomial (which can be of degree $2n-1$) satisfying

$$p(0) = p(1) = \cdots = p(n-1) = 0, \quad p(n) = p(n+1) = \cdots = p(2n-1) = 1,$$

and let $q(x) = p(x) - 1$ (Diminnie and Smith also mention this interpretation). Can anyone give (for all $n \geq 3$) polynomials $p(x)$ and $q(x)$, of degree smaller than $2n-1$, each with (at least) n integer roots, and so that $p(x) - q(x)$ is a non-zero constant?

Many solvers pointed out that if there is a solution for any value of n, then there are infinitely many solutions, because the polynomials can be translated by an arbitrary integer: if $p(x)$ and $q(x)$ work, so do $p(x-c)$ and $q(x-c)$ for any integer c. Thus we could assume that $p(0) = 0$, say. Even with this restriction, infinite families of solutions are known for some values of n. The above solution for part (a) *is one such, and Henderson found such families for $n = 3, 4, 5, 6$. His solution for $n = 6$ contains the solution given above:*

$$p(x) = x^2[x^2 - (3r^2 + s^2)^2]^2,$$

$$q(x) = (x^2 - 16r^2s^2)[x^2 - (3r^2 - 2rs - s^2)^2][x^2 - (3r^2 + 2rs - s^2)^2],$$

for which $p(x) - q(x) = 16r^2s^2(r^2 - s^2)^2(9r^2 - s^2)^2$. This is a solution for all distinct positive integers r, s for which $s \neq 3r$. The solution given above is the case $r = 1$, $s = 2$. Other solvers found infinite families for $n = 3$ and 4; for example, for $n = 4$ we could use

$$p(x) = x^2(x^2 - a^2), \qquad q(x) = (x^2 - b^2)(x^2 - c^2)$$

where (a, b, c) is any Pythagorean triple*; that is, positive integers satisfying $a^2 = b^2 + c^2$.*

2427. [1999: 172] *Proposed by Toshio Seimiya, Kawasaki, Japan.*
Suppose that G is the centroid of triangle ABC, and that

$$\angle GAB + \angle GBC + \angle GCA = 90^\circ.$$

Characterize triangle ABC.

Solution by Nikolaos Dergiades, Thessaloniki, Greece.

Let $x = \cot A$, $y = \cot B$, $z = \cot C$. Since $A + B + C = 180^\circ$, then $\cot(B + C) = \cot(180^\circ - A)$, which can be written as

$$\frac{\cot B \cot C - 1}{\cot B + \cot C} = -\cot A\,,$$

or

$$xy + yz + zx = 1\,. \qquad (1)$$

Let $\angle GAB = \alpha$, $\angle GBC = \beta$, $\angle GCA = \gamma$. Then $\alpha + \beta + \gamma = 90^\circ$ implies $\cot(\beta + \gamma) = \cot(90^\circ - \alpha)$, which can be written as

$$\frac{\cot\beta \cot\gamma - 1}{\cot\beta + \cot\gamma} = \frac{1}{\cot\alpha}\,,$$

or

$$\cot\alpha + \cot\beta + \cot\gamma = \cot\alpha \cot\beta \cot\gamma\,. \qquad (2)$$

Let M be the mid-point of the segment BC. By the Law of Sines for $\triangle ABM$ and $\triangle ABC$,

$$\frac{BM}{\sin\alpha} = \frac{AB}{\sin(B + \alpha)} \quad \text{and} \quad \frac{BC}{\sin A} = \frac{AB}{\sin(B + A)}\,.$$

Since $BC = 2BM$ we obtain

$$\frac{\sin(B + \alpha)}{\sin B \sin\alpha} = \frac{2\sin(B + A)}{\sin B \sin A}\,.$$

This implies $\cot B + \cot\alpha = 2(\cot B + \cot A)$; that is, $\cot\alpha = 2x + y$. Similarly, $\cot\beta = 2y + z$, and $\cot\gamma = 2z + x$. Substitution in (2) yields

$$3(x + y + z) = (2x + y)(2y + z)(2z + x)\,.$$

Using the equality (1),

$$3(x + y + z)(xy + yz + zx) = (2x + y)(2y + z)(2z + x)\,,$$

which transforms to

$$(x - y)(y - z)(z - x) = 0\,.$$

It follows that $x = y$, or $y = z$, or $z = x$. Therefore, the triangle is isosceles.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; VEDULA N. MURTY, Dover, PA, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There were also two incorrect solutions submitted.

2428. [1999: 172] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given triangle ABC with $\angle BAC = 90°$. The incircle of triangle ABC touches AB and AC at D and E respectively. Let M be the mid-point of BC, and let P and Q be the incentres of triangles ABM and ACM respectively. Prove that

1. $PD \parallel QE$;
2. $PD^2 + QE^2 = PQ^2$.

Solution by Nikolaos Dergiades, Thessaloniki, Greece.

Let the incircle of $\triangle ABC$ touch BC at the point F. Then $BD = BF$ and since BP bisects the angle ABC, $PD = PF$ and $\angle PDB = \angle PFB$. Similarly, $QE = QF$ and $\angle QEC = \angle QFC$.

1. Let PD meet AC at G. We shall prove that $\angle QEC = \angle DGA$. Since $\triangle ABM$ is isosceles, MP is the perpendicular bisector of the segment AB. Thus MP meets AB at the point K, where $AK = KB$, and we obtain

$$PK = BK \tan\frac{B}{2} = BM \tan\frac{B}{2}\cos B = BC \tan\frac{B}{2}\sin\frac{C}{2}\cos\frac{C}{2}.$$

We know that

$$BD = \frac{AB + BC - AC}{2},$$

so that

$$KD = \frac{BC - AC}{2} = \frac{BC(1-\cos C)}{2} = BC\sin^2\frac{C}{2}.$$

Let $\omega = \angle PDB$ and $\varphi = \angle QEC$. Then

$$\tan\omega = \frac{PK}{KD} = \frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}.$$

Similarly,

$$\tan\varphi = \frac{\tan\frac{C}{2}}{\tan\frac{B}{2}}.$$

Then $\tan\omega\tan\varphi = 1$ and therefore $\omega + \varphi = 90°$.

Finally, $\angle QEC = \varphi = 90° - \omega = \angle DGA$, which shows that $PD \parallel QE$.

2. Since $\angle PFQ = 180° - \angle PFM - \angle QFC = 180° - \omega - \varphi = 90°$, then

$$PD^2 + QE^2 = PF^2 + QF^2 = PQ^2,$$

which completes the proof.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; SAM BAETHGE, Nordheim, TX, USA; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER

J. BRADLEY, Clifton College, Bristol, UK; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; PARAGIOU THEOKLITOS, Limassol, Cyprus, Greece; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

2429. [1999: 172] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Suppose that D, E and F are points on the side AB (or its production) of triangle ABC. Suppose further that CD is a median, that CE is the bisector of $\angle ACB$, and that CF is its external bisector.

The circumcircle, Γ, of triangle EFC intersects CD again at P. Suppose that Γ_A and Γ_B are the circumcircles of triangles CPA and CPB respectively.

Show that Γ_A and Γ_B are tangent to AB at A and B respectively.

Solution by Toshio Seimiya, Kawasaki, Japan.

We may assume that $CA > CB$. [If $CA = CB$, then $D = E$, and F is at infinity.] Since CE and CF are the interior and exterior bisectors of $\angle ACB$ we have

$$AE : EB \;=\; CA : CB \;=\; AF : BF\,. \tag{1}$$

We put $AD = DB = x$, $DE = e$, and $DF = f$. In this notation (1) becomes

$$(x+e) : (x-e) \;=\; (x+f) : (f-x)\,;$$

that is, $(x+e)(f-x) = (x-e)(x+f)$. It follows that

$$x^2 = ef\,. \tag{2}$$

Since C, P, E, F are concyclic we get $DP \cdot DC = DE \cdot DF = ef$. Hence we have from (2)

$$DA^2 \;=\; DP \cdot DC\,, \tag{3}$$

and

$$DB^2 \;=\; DP \cdot DC\,. \tag{4}$$

From (3) $\Gamma_A = CPA$ touches AB at A, and from (4) $\Gamma_B = CPB$ touches AB at B.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

2430. [1999: 172] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Points A and B lie outside circle Γ. Find a point C on Γ with the following property:

AC and BC intersect Γ again at D and E respectively, with $DE \parallel AB$.

Solution by Eckard Specht, Magdeburg, Germany.

Let u and v be the respective lengths of the tangential segments to Γ from A and from B. By Euclid III.36 we know that $u^2 = AC \cdot AD$ and $v^2 = BC \cdot BE$. It follows that

$$\frac{u^2}{v^2} = \frac{AC}{BC} \cdot \frac{AD}{BE}. \tag{1}$$

Since $DE || AB$ we have $\dfrac{AC}{BC} = \dfrac{AD}{BE}$, and with (1),

$$\frac{AC}{BC} = \frac{u}{v} := q.$$

A, B, and Γ are fixed, so that q is constant. The locus of points C whose distances to given points A, B are in the same ratio is known as the ***circle of Apollonius***. So we have to divide the segment AB internally and externally in the ratio q to get the points X and Y as a diameter of the Apollonian circle. The intersection of the circle with Γ determines the two possible locations of the desired point C.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; TOSHIO SEIMIYA, Kawasaki, Japan; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Several solvers noted that since C is easily seen to be on a circle through A and B that is tangent to Γ, our problem can quickly be reduced to that special case of the problem of Apollonius. *It was an amusing coincidence that our featured solution made use of another idea that is due to Apollonius. There is a vast literature on all aspects of the Apollonius problem, where a variety of solutions to our special case can be found.*

2432. [1999: 173] *Proposed by K.R.S. Sastry, Bangalore, India.*

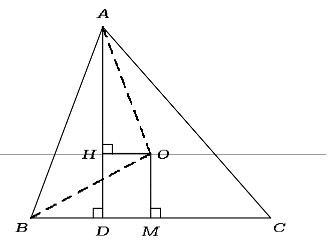
In $\triangle ABC$, we use the standard notation: O is the circumcentre, H is the orthocentre. Let M be the mid-point of BC, $OH = m$, $OM = n$ ($m, n \in \mathbb{N}$), and suppose that $OH \parallel BC$.

How many sides of $\triangle ABC$ can have integer lengths?

Solution by the proposer.

Answer: at most two sides.

It is known that $AH = 2OM$. If a, b, c are all integers, then $[ABC] = \frac{1}{2}BC \cdot AD$ is rational, so that all of $\sin A$, $\sin B$, $\sin C$, $\cos A$, $\cos B$ and $\cos C$ are rational; that is, $\triangle ABC$ is a Heron triangle.



$OH = m \quad OM = n$
$BO = AO = R = \sqrt{m^2 + 4n^2}$
$BM = \sqrt{m^2 + 3n^2}$
$AH = 2n$

Since $a = 2R\sin A$, the circumradius R is rational as well.

We have $R^2 = m^2 + (2n)^2$. Therefore $R = u^2 + v^2$, $2n = 2uv$ and $m = u^2 - v^2$ by the well-known Pythagorean solution.

But then, $BM^2 = m^2 + 3n^2 = (u^2 - v^2)^2 + 3u^2v^2 = u^4 + u^2v^2 + v^4$ must be the square of an integer. This is known to be impossible. See, for example, L.E. Dickson, *History of the Theory of Numbers*, Vol. II, Chelsea, NY (1971) p. 636.

Hence all three side lengths cannot be integers.

However, two of these can be integral. To see this, set $m = 23$, $n = 40$. Then $BM = 73$, $BC = 146$, $AB^2 = AD^2 + BD^2 = 120^2 + (73 - 23)^2 = 130^2$, so that $AB = 130$.

Also solved by MICHAEL LAMBROU, University of Crete, Crete, Greece; AND GERRY LEVERSHA, St. Paul's School, London, England. There was one incomplete solution.

Lambrou's solution was essentially the same as the proposer's, but took twice as long. Lambrou remarked that a parametric family of triangles can be taken with

$$m = 14(st)^2 - s^4 - t^4, \qquad n = 4(s^2 - t^2)st,$$

giving $a = 2(s^4 + 10(st)^2 + t^4)$ *and* $b = 12st(s^2 + t^2)$. *Again, we have* $OH \| BC$.

WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria, commented on a characterization of triangles ABC *satisfying* $OH \| BC$, *using the (unfortunately not too well-known) fact:*

> *If* O *is the centre of the circumcircle of* $\triangle ABC$ *and also the origin of a rectangular system of coordinates, then* H *is given by* $H = A + B + C$.

Thus, $OH \| BC$ *is equivalent to the existence of a real number* λ *such that* $\overrightarrow{OH} = \lambda\overrightarrow{BC}$, *which, in turn, is equivalent to* $A + B + C = \lambda(C - B)$, *or*

$$A = (\lambda - 1)C - (\lambda + 1)B. \qquad \star$$

Therefore, if $\triangle ABC$ *is in "general position", we have to replace* A, B, C *in* ($\star$) *by* $A - O$, $B - O$, $C - O$ *respectively, yielding* $OH \| BC$ *if and only if there exists a* $\lambda \in \mathbb{R}$ *such that*

$$A = (\lambda - 1)C - (\lambda + 1)B + 3(O).$$

2433. [1999: 173] *Proposed by K.R.S. Sastry, Bangalore, India.*

In $\triangle ABC$, let e denote the length of the segment of the Euler line between the orthocentre and the circumcentre.

Prove or disprove that $\triangle ABC$ is right angled if and only if e equals one half of the length of one of the sides of $\triangle ABC$.

Solution by Christopher J. Bradley, Clifton College, Bristol, UK.

We use the well-known formula

$$e^2 = OH^2 = R^2(1 - 8\cos A\cos B\cos C).$$

[Ed. O is the circumcentre, H is the orthocentre and R is the circumradius.]

If $A = 90^\circ$, then $\cos A = 0$, $\sin A = 1$ and $e = R = \frac{a}{2\sin A} = \frac{a}{2}$.

Similarly, if either B or $C = 90^\circ$, then $e = \frac{b}{2}$ or $\frac{c}{2}$ respectively.

On the other hand, if $e = \frac{a}{2}$, then $e^2 = \frac{a^2}{4} = R^2\sin^2 A$.

Hence, $\sin^2 A = 1 - 8\cos A\cos B\cos C = 1 - \cos^2 A$.

Therefore $\cos A = 0$ or $\cos B\cos C = \frac{1}{8}\cos A$.

Thus, the right angle is sufficient but not necessary for e to be equal to one half of the length of one of the sides of triangle ABC.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

*The proposer notes that the claim would be true if we insist that triangle ABC has integer sides. He provides a proof, but the editor would like to see if another reader can solve this — see problem **2533**.*

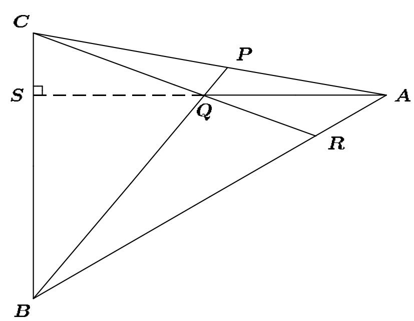
2434. [1999: 173] *Proposed by K.R.S. Sastry, Bangalore, India.*

In $\triangle ABC$, let $\angle ABC = 60^\circ$. Point P is on the line segment AC such that $\angle CBP = \angle BAC$. Point Q is on the line segment BP such that $BQ = BC$.

Prove that Q lies on the altitude through A of $\triangle ABC$ if and only if $\angle BAC = 40^\circ$.

I. *Solution by the proposer.*

Extend CQ to meet AB at R. Let AS be the altitude from A. We will show that AS contains the point Q.



We calculate the following angles:

$\angle BAS = 30^\circ$, $\angle SAC = A - 30^\circ$, $\angle ABP = 60^\circ - A$, $\angle BCR = 90^\circ - \frac{A}{2}$.

Now, using $[XYZ]$ to denote the area of $\triangle XYZ$, etc.,

$$\frac{BS}{SC} = \frac{[ABS]}{[ASC]} = \frac{\frac{1}{2}AB \cdot AS \sin 30^\circ}{\frac{1}{2}AC \cdot AS \sin(A - 30^\circ)} = \frac{AB \sin 30^\circ}{AC \sin(A - 30^\circ)} .$$

Likewise,

$$\frac{CP}{PA} = \frac{BC \sin A}{AB \sin(60^\circ - A)} , \quad \text{and} \quad \frac{AR}{RB} = \frac{AC \sin(30^\circ - \frac{A}{2})}{BC \cos \frac{A}{2}} .$$

By Ceva's Theorem, AS, BP and CR are concurrent if and only if $\frac{BS}{SC} \cdot \frac{CP}{PA} \cdot \frac{AR}{RB} = 1$. That is,

$$(\sin 30^\circ)(\sin A)\left(\sin(30^\circ - \tfrac{A}{2})\right) = \left(\sin(A - 30^\circ)\right)\left(\sin(60^\circ - A)\right)\left(\cos \tfrac{A}{2}\right) ,$$

which becomes

$$\left(\tfrac{1}{2}\right)\left(2 \sin \tfrac{A}{2} \cos \tfrac{A}{2}\right) = \sin(A - 30^\circ)\left(2 \sin(30^\circ - \tfrac{A}{2}) \cos(30^\circ - \tfrac{A}{2})\right) \cos \tfrac{A}{2} .$$

On simplification, this becomes

$$\sin \tfrac{A}{2} = 2 \sin(A - 30^\circ) \cos\left(30^\circ - \tfrac{A}{2}\right) = \sin \tfrac{A}{2} + \sin\left(\tfrac{3}{2}A - 60^\circ\right) .$$

Hence we conclude that the lines AS, BP and CR are concurrent if and only if $\sin\left(\frac{3}{2}A - 60^\circ\right) = 0$; that is, if and only if $\frac{3}{2}A - 60^\circ = 0^\circ$ or 180°; that is, if and only if $\angle BAC = 40^\circ$ or 160°. However, $\angle ABC = 60^\circ$, forcing $\angle BAC = 40^\circ$.

II. *Solution by D.J. Smeenk, Zaltbommel, the Netherlands.*

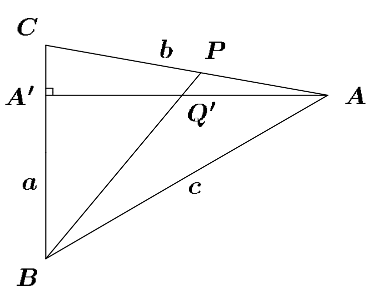


Figure 1.

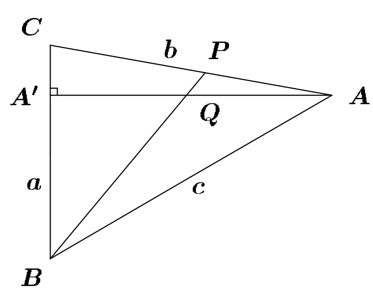


Figure 2.

First assume that $\angle BAC = 40^\circ$ (Figure 1.). Since $\angle ABC = 60^\circ$, we know that $\angle ACB = 80^\circ$. We denote by A' the foot of the perpendicular from A to BC, and let Q' be the intersection of AA' and BP.

Now, in $\triangle BA'Q'$, we have $BA' = \frac{1}{2}c$ and $BQ' = \frac{c}{2\cos 40^\circ}$.

Also, in $\triangle ABC$, we have $\frac{c}{a} = \frac{\sin 80^\circ}{\sin 40^\circ}$, which implies that $c = 2a\cos 40^\circ$. Hence $BQ' = a$, so that $Q' = Q$, telling us that the point Q lies on AA'.

Now assume that Q lies on AA' (Figure 2). We need to show that $\angle BAC = 40$.

From $BA' = \frac{1}{2}c$ and $BQ = \frac{c}{2\cos A}$, we obtain, using the Law of Sines, that $\sin C = 2\sin A\cos A = \sin 2A$.

Hence either $C = 2A$, in which case (since $B = 60^\circ$), we obtain $A = 40^\circ$, or $C = 180^\circ - 2A$ (in which case we obtain $A = B = C = 60^\circ$).

Editor's comment.

In the original proposal of this problem by SASTRY, the wording describing the position of point P was different: "The point P is in AC ... ". This clearly suggests that the proposer thought of P as lying strictly between A and C. The wording of 2434 was changed to read "Point P is on the line segment AC ... ", which allows for P to assume either end-point position $P = A$ or $P = B$. Unfortunately, the wording in CRUX, makes the problem incorrect, for in this case, when $BAC = 60^\circ$ (thus making the triangle equilateral), it happens that $P = Q = A$, and so the altitude through A naturally passes through Q.

Both MICHAEL LAMBROU, University of Crete, Crete, Greece, and NIKOLAOS DERGIADES, Thessaloniki, Greece, were explicit in their solution in pointing out this error in the wording. As well, SMEENK, WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria, SAM BAETHGE, Nordheim, TX, USA, and GERRY LEVERSHA, St. Paul's School, London, England, provided correct solutions in which they recognized that the equilateral triangle provided a trivial case in which the result of the problem held.

The solutions submitted by HEINZ-JÜRGEN SEIFFERT, Berlin, Germany, ÀNGEL JOVAL ROQUET, La Seu d'Urgell, Spain, KEE-WAI LAU, Hong Kong, and TOSHIO SEIMIYA, Kawasaki, Japan, correctly solved the problem as it had been intended by the proposer, even though they either failed to recognize the degenerate case, or dismissed it out of hand. Partial solutions were submitted by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK, VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA, and PETER Y. WOO, Biola University, La Mirada, CA, USA.

Most of the solutions used the Law of Sines in one form or another. The exception to this was the solution by the proposer, which uses Ceva's Theorem.

2435. [1999: 173] *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

Show that, for $x > 0$, the following functions are increasing:

$$f(x) = \frac{\left(1+\frac{1}{x}\right)^x}{(1+x)^{\frac{1}{x}}} \quad \text{and} \quad g(x) = \left(1+\frac{1}{x}\right)^x - (1+x)^{\frac{1}{x}}.$$

Composite of the solutions by Walther Janous, Ursulinengymnasium, Innsbruck, Austria; and by Phil McCartney, Northern Kentucky University, Highland Heights, KY, USA.

Define $h(x) := \left(1+\frac{1}{x}\right)^x$ and $d(x) := \ln\left(1+\frac{1}{x}\right) - \frac{1}{1+x}$ for $x > 0$. Note that $h'(x) = h(x)\cdot d(x)$ and $h(x) > 0$. But $d'(x) = \frac{-1}{x(1+x)^2} < 0$ so

$d(x)$ decreases on $(0, \infty)$. However, $\lim_{x \to \infty} d(x) = 0$, and so $d(x) > 0$ for all $x \in (0, \infty)$ which implies that $h'(x) > 0$ on $(0, \infty)$. Therefore, $h(x)$ is increasing and so $h\left(\frac{1}{x}\right)$ is decreasing on $(0, \infty)$. Since $f(x) = \frac{h(x)}{h\left(\frac{1}{x}\right)}$ and $g(x) = h(x) - h\left(\frac{1}{x}\right)$, both conclusions following immediately.

Also solved by MICHEL BATAILLE, Rouen, France; PAUL BRACKEN, CRM, Université de Montréal, Montréal, Québec; NIKOLAOS DERGIADES, Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; VEDULA N. MURTY, Visakhapatnam, India; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; MAX SHKARAYEV and MARK LYON, students, University of Arizona, Tucson, AZ; and the proposer.

Of course, it is a well-known fact in calculus that the function $\left(1 + \frac{1}{x}\right)^x$ *is strictly increasing on* $(0, \infty)$. *This was explicitly pointed out by Lambrou.*

2436. [1999: 173] *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

Find all real solutions of

$$2\cosh(xy) + 2^y - \left[(2\cosh(x))^y + 2\right] = 0\,.$$

Solution by Nikolaos Dergiades, Thessaloniki, Greece; and by Heinz-Jürgen Seiffert, Berlin, Germany (combined by the editor).

Let $t = e^x$ and $p = y$. Then the given equation becomes

$$t^p + t^{-p} + 2^p = \left(t + t^{-1}\right)^p + 2\,.$$

This equality occurs in Michael Lambrou's solution of **2329**★ [1999 : 241].

Hence the only real solutions of the given equation are given by

1. $x = 0$, y arbitrary;
2. $y = 0$, x arbitrary;
3. $y = 1$, x arbitrary; and
4. $y = 2$, x arbitrary.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; and the proposer.

Lambrou, in fact, gave a complete self-contained proof, as well as referring to his previous solution.

2437. [1999: 173] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Let P be a point in the plane of triangle ABC. If the mid-points of the line segments AP, BP, CP all lie on the nine-point circle of triangle ABC, prove that P must be the orthocentre of triangle ABC.

Solution by Jeremy Young, student, Nottingham High School, Nottingham, UK.

P is the external centre of similitude of the circumcircle and the nine-point circle of $\triangle ABC$ (since A, B, C lie on the circumcircle while the mid-points of AP, BP, CP are assumed to lie on the nine-point circle). H (the orthocentre of $\triangle ABC$) is also the external centre of similitude of these two circles. (See, for example, H.S.M. Coxeter, *Introduction to Geometry*, section 5.2, p. 72; it is also implicit in Coxeter and S.L. Greitzer, *Geometry Revisited*, section 1.8, p. 20.) Because a pair of circles can have at most one external centre of similitude (*Introduction to Geometry*, p. 70), it follows that $P = H$, as desired.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.