THE OLYMPIAD CORNER

No. 204

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

We begin this number with a different feature. Arthur Baragar has provided us with a personal account of his experiences as Deputy Leader to the 40th IMO in Romania. He is a professor at the University of Nevada, Las Vegas, and was a member of the Canadian 1981 IMO team, a coordinator in 1995 and leader observer in 1998.

SNAPSHOTS OF THE '99 CANADIAN IMO TEAM

Arthur Baragar, Deputy Leader

⋆ ⋆ ⋆ July 13th ⋆ ⋆ ⋆

I'm sitting at a table in a Frankfurt airport waiting room, playing bridge with three high school students. Slung about my waist is a fanny pack stuffed with passports and plane tickets for eight. A cacophony of Greek buzzes about us, as tired travellers wait to board the plane to Athens. Among them is our contingent, easily distinguished by our bright red golf shirts emblazoned with red maple leaves. We are exhausted after our flight from Toronto, and a little envious of the travellers who are boarding. This is our gate, but it isn't our flight. Ours is the next plane, bound for Bucharest, Romania.

My partner at the table is David Arthur, who has just completed grade 12 at Upper Canada College in Toronto. He and the five other high school students in our group have been selected to represent Canada at the 40th International Mathematical Olympiad. He is playing the hand, which doesn't seem too unusual. Somehow, I'm often the dummy. Defending the hand are David Pritchard of Woburn Collegiate Institute in Scarborough, and David Nicholson of Fenelon Falls Secondary School in Fenelon Falls, Ontario.

As the waiting room slowly empties, we begin to realize that not all the travellers are bound for Athens, but that some are young and would also be on our flight. Jessie Lei, from Vincent Massey Secondary School in Windsor, has come prepared to trade, and is soon exchanging coins with the team from Mexico. She discovered the joy of souvenirs last year, when she first represented Canada at the IMO in Taiwan. The other veteran amongst us is our Team Captain, Jimmy Chui, from Earl Haig Secondary School in North York, who is competing in the IMO for the third time. The sixth member of our team is James Lee, from Eric Hamber Secondary School in Vancouver.

We pack in the cards, and I join my associate, Dorette Pronk, a professor from Calvin College in Michigan. She was first invited to escort the Canadian team to the IMO in '98, while she was working at Dalhousie University. She is in a conversation with Michael Albert, the Deputy Leader from New Zealand and a fellow Canadian who graduated from the University of Waterloo. Their team is well rested after three days in Germany recovering from jet lag, but their trip has already run afoul of Murphy's law. Their German stay was spent replacing a lost passport, a reminder of the awkward burden around my waist. They too had exercised the same precautions, and swore the student had been trusted with it for no more than fifteen minutes. My hand went to my waist — our documents were still there.

⋆ ⋆ ⋆ **July 15th** ⋆ ⋆ ⋆

I toss the frisbee back into the crowd on the field. My aim is not too good, and the disc floats to someone other than the intended target. We are working out some of our anxiety with the American team by throwing several frisbees around. We've been joined by a few Australians, as well as a couple of very young Roma, better known as gypsies. Carmen, our Romanian guide, discreetly voices her concern about our uninvited guests, and cautions us to watch our belongings. The fanny pack is still uncomfortably slung about my waist, but we have little else to worry about. I am happier that we can connect on some level with the locals, and that the students are working out some of their pre-exam jitters.

Our arrival in Bucharest was recorded on both print film and video, and played back to the audience at the opening ceremonies earlier today. Our red shirts no doubt attracted the attention of the reporters. Fortunately, we didn't look nearly as bad as our 18 hours of travel made us feel. I'm sure the images brought a sense of pride and longing to our leader, Prof. Ed Barbeau of the University of Toronto. Ed is a veteran coach of IMO teams, and was my coach when I represented Canada at the '81 IMO in Washington. This year, he was with us for the first week of coaching in Waterloo before departing for Romania. Every team sends their leader a few days early to serve on the jury which selects the questions for the competition. Because Ed has already seen the exam, he and the rest of the jury are sequestered. We waved hello to him in his balcony seat, but that is all the contact we will have with him until after the exam.

The first two days in Bucharest were mostly spent recovering from jet lag, getting to know our fellow competitors, and seeing some of the sites. Bucharest might best be described as a city of faded glory. There are many grand buildings from before the war mixed in with the stark architecture of the communist era. But most are in a state of disrepair — broken fences, minimally tended gardens and torn-up streets. A street-car with numerous rusty patches rumbles by our playing field. The weather has been warm and humid, and water is slowly seeping through my shoes as I race to catch a disc in the wet, shin-high grass.

Playing Frisbee with the American team is something of a tradition for us. The Americans are very friendly and seem particularly excited to be with us. They have already spent two weeks in the country training with the Romanian team, and are happy to be amongst native English speakers again (even though we say "zed" instead of "zee").

⋆ ⋆ ⋆ **July 18th** ⋆ ⋆ ⋆

My real work has begun. I've been reunited with Ed, and we are sitting across from the coordinators for question three. The exam was tough this year, and the students were a bit discouraged with their performance. They shouldn't have been. They were not the only ones who found the exam tough, and I am proud of their performance. Question three was the hard question in the first session, and the first on our grading schedule. No one on our team has a complete solution, but several found a key result. Since we are going for part marks, some of the work is hard to find. After each session, I interviewed the students, taking simple notes on what each student did. During this interview the students were able to point out what they thought was important in their work, now that they knew how to solve the questions. I therefore know where to look for their most promising work. The coordination for the first five students goes better than expected.

Let me elaborate by briefly describing the mechanics of the coordination procedure used to award grades. The leaders of each team come before a pair of Romanian judges called coordinators, and present a question for grading. The leaders usually propose the grades, and argue what progress the student has made on the problem to warrant that grade. The coordinators judge the progress with that made by other students and between the four of us, a grade is decided upon. On this question, we got the top of what we thought would be a fair grade for five of the six students. The sixth student was David Arthur. In my interview with him, he gave an outline of how to solve the problem, and described the details of all but the last step. A little while later, he filled in the details of the last step. His verbal description was a little more refined than what he managed on the exam, but much of it was there.

I began by pointing out where each of the steps was, and on a separate page, the missing step. I argued that the student was on the verge of a complete solution. They were not satisfied, and pointed out that even with the last step, only one direction would be established. I realized that they didn't fully understand this particular proof, so I presented the full proof in my own way — kind of like reading the rule book to an official. We then went back over the paper — a slow-motion review of the play. They still were not happy. David didn't even have a correct asymptotic result, which many other students found. Such results, by themselves, were not relevant to any proof, and unfortunately, didn't naturally fall out in this proof until the last step. We soon reached an impasse, and in the end, we had to settle for a low grade. A frustrating, but not unfair result — a little like a puck ringing off a goal post. Close, but nothing to show for it on the scoreboard.

Coordination for the rest of the questions went well. The Romanian coordinators had a pleasant attitude towards coordinating, and for the most part, seemed to want to give points away. They only demanded from us adequate reasons to award them. On one occasion, they even seemed as eager as us to excavate a student's work to make sure full credit was awarded. "Excavating" is an appropriate description. Our students are fairly good at expressing their thoughts when they have a complete solution, but are not very good at conveying the ideas which they do not yet know how to use. Some of their work which was worthy of note was hidden among reams of scrap, some of which was inserted with other questions. The interviews provided invaluable help with this archaeological dig.

⋆ ⋆ ⋆ **July 21st** ⋆ ⋆ ⋆

I'm sitting on the front seat of a coach travelling up a windy road which follows a mountain stream. We have a police escort three buses ahead. The car races ahead, lights flashing and siren blaring, waving traffic in both directions out of the way so that our VIP caravan can pass. Without the escort, the trip would take twice as long. I have been impressed by how important the country considers this competition. We have seen IMO posters on the public buses and throughout the city; we have attended receptions at some of the country's grandest government buildings; but nothing has impressed me more with our importance than this police escort. As we cross into another jurisdiction, the police car ahead pulls over and another takes its place. Our escort for the last hour waves to our entourage as we pass by.

We are on our way to Castle Brun, a medieval fort once used to collect tolls at the entrance of a mountain pass, and rumoured to be the home of Count Dracula. Transylvania, (literally "between hills") is a very beautiful region, reminiscent of the alpine valleys of Switzerland and Austria. The castle is a treasure, beautifully situated with marvellous views, cosy rooms, secret passageways and a quaint courtyard. The atmosphere invites the imagination to run wild. I could spend a lot of time here, but we do not have much — we have more places to see.

Our escorts whisk us back into Bucharest. Stuffed in my bag are table cloths and souvenirs of our visit. I had recognized an opportunity to get some Christmas shopping done, and even borrowed money from one of my fellow travellers. My fanny pack with most of my money was no longer about my waist, but safely stored in the hotel safe — one of the luxuries enjoyed when I joined the leaders after the exam.

In the city streets which are several lanes wide, cars are able to squeeze in between the buses, and our relation to the police escort becomes less obvious. A car in front of us dutifully stops at a light, bringing the tail end of our convoy to a screeching halt. The occupant of the car is unimpressed with how well our driver can lean on his horn, and our escort soon disappears ahead of us. We are left to negotiate city traffic at a rush hour pace. It's not a big deal — we're almost home, and we got almost full value out of the escort.

Tomorrow is a shopping day, followed by a sumptuous dinner and disco. In two days, we leave, proud of our accomplishments and happy to have been here. The Romanians have been wonderful hosts and organized an excellent competition.

There is so much more I haven't described — the tomatoes (fresh, juicy, and a delight with every meal); the cheese (not for me); the desserts; the bonfire; the passing of the Canadian designed IMO flag; tight connections; interviews with the press; our trips to Niagara Falls and Elora during training; the logo; and so on.

⋆ ⋆ ⋆ **Coda** ⋆ ⋆ ⋆

Results: David Arthur, Jimmy Chui and David Pritchard won Bronze medals. As a team, Canada tied with the Dutch team for 31st place. China and Russia tied for 1st, Vietnam was 3rd, Romania was 4th, and the American team placed 10th.

The Exam: Here are the problems of the 40th IMO Competition.

40th INTERNATIONAL MATHEMATICAL OLYMPIAD
Bucharest
Day I — July 16, 1999

1. (Luxembourg) Determine all finite sets S of at least three points in the plane which satisfy the following condition:

for any two distinct points A and B in S, the perpendicular bisector of the line segment AB is an axis of symmetry for S.

2. (India) Let n be a fixed integer, with $n \geq 2$.

(a) Determine the least constant C such that the inequality

$$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \;\leq\; C \left(\sum_{1 \leq i \leq n} x_i \right)^4$$

holds for all real numbers $x_1, \cdots, x_n \geq 0$.

(b) For this constant C, determine when equality holds.

3. (Belarus) Consider an $n \times n$ square board, where n is a fixed even positive integer. The board is divided into n^2 unit squares. We say that two different squares on the board are adjacent if they have a common side.

N unit squares on the board are marked in such a way that every square (marked or unmarked) on the board is adjacent to at least one marked square.

Determine the smallest possible value of N.

Day II — July 17, 1999

4. (UK) Determine all pairs (n, p) of positive integers such that

p is a prime,
$n \leq 2p$, and
$(p-1)^n + 1$ is divisible by n^{p-1}.

5. (IMO) Two circles G_1 and G_2 are contained inside the circle G, and are tangent to G at the distinct points M and N, respectively. G_1 passes through the centre of G_2. The line passing through the two points of intersection of G_1 and G_2 meets G at A and B. The lines MA and MB meet G_1 at C and D, respectively.

Prove that CD is tangent to G_2.

6. (Bulgaria) Determine all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

for all real numbers x, y.

Acknowledgments: I should acknowledge the work of Graham Wright, who might be thought of as our team manager. Our sponsors include: Alberta Education, the Bank of Montreal, the Canadian Mathematical Society, the Centre for Education in Mathematics and Computing (University of Waterloo), the Fields Institute for Research in the Mathematical Sciences, Industry Canada, the New Brunswick Department of Education, the Newfoundland and Labrador Department of Education, the Northwest Territories Department of Education, the Ontario Ministry of Education, the Quebec Ministry of Education, the Samuel Beatty Fund, the Saskatchewan Department of Education, the Senator Norman M. Paterson Foundation, the Sun Life Assurance Company of Canada, the University of Calgary, the University of New Brunswick at Fredericton, the University of Ottawa, the University of Toronto and Waterloo Maple Inc.

In memoriam: Jessie Lei was killed in an automobile accident over the New Year holidays. Our sympathy goes out to her family and friends. She will be dearly missed.

As a first *Olympiad* Problem Set for this number we give the problems of the 10^{th} Mexican Mathematics Olympiad National Contest of November 1996. My thanks go to Professor Richard Nowakowski, Canadian Team Leader to the IMO at Mar del Plata, Argentina, for collecting the problems for us.

10^{th} MEXICAN MATHEMATICS OLYMPIAD NATIONAL CONTEST, November, 1996

First Day

1. Let $ABCD$ be a quadrilateral and let P and Q be the trisecting points of the diagonal BD (that is, P and Q are the points on the line segment BD for which the lengths BP, PQ and QD are all the same). Let E be the intersection of the straight line through A and P with BC and let F be the intersection of the straight line through A and Q with DC. Prove the following:

(i) If $ABCD$ is a parallelogram, then E and F are the mid-points of BC and CD, respectively.

(ii) If E and F are the mid-points of BC and CD, respectively, then $ABCD$ is a parallelogram.

2. There are 64 booths around a circular table and in each one there is a chip. The chips and the booths are numbered 1 to 64 sequentially (each chip has the same number as the booth it is in). At the centre of the table there are 1996 light bulbs turned off. Each minute the chips move simultaneously in a circular way (following the numbering sense) as described: chip #1 moves one booth, chip #2 moves two booths, chip #3 moves three booths, etc., so that more than one chip can be in the same booth at a given minute. For each minute that a chip shares a booth with chip #1, a bulb is lit (one for each chip sharing position at that moment with chip #1). Where is chip #1 on the first minute in which all bulbs are lit?

3. Prove that it is not possible to cover a 6 cm $\times$ 6 cm square board with eighteen 2 cm $\times$ 1 cm rectangles, in such a way that each one of the interior 6 cm lines that form the squaring goes through the middle of at least one of the rectangles. Prove also that it is possible to cover a 6 cm $\times$ 5 cm square board with fifteen 2 cm $\times$ 1 cm rectangles, in such a way that each one of the interior 6 cm lines that form the squaring and each one of the interior 5 cm lines that form the squaring goes through the middle of at least one of the rectangles.

Second Day

4. For which integers $n \geq 2$ can the numbers 1 to 16 be written each in one square of a squared 4×4 paper (no repetitions allowed) such that each of the 8 sums of the numbers in rows and columns is a multiple of n, and all of these 8 multiples of n are different from one another?

5. In an $n \times n$ squared paper, the numbers 1 to n^2 are written in the usual ordering (from left to right and then down as shown in the picture for the case $n = 3$).

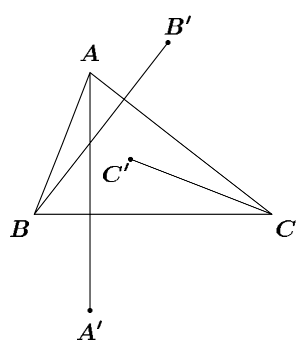
1	2	3
4	5	6
7	8	9

Any sequence of steps from a square to an adjacent one (sharing a side) starting at square number 1 and ending at square number n^2 is called a *path*. If $\mathcal{C}$ is a path, denote by $\mathcal{L}(\mathcal{C})$ the sum of the numbers through which $\mathcal{C}$ goes:

(i) For a fixed n, let M be the largest $\mathcal{L}(\mathcal{C})$ that can be obtained and let m be smallest $\mathcal{L}(\mathcal{C})$ possible. Prove that $M - m$ is a perfect square.

(ii) Prove that there is no n for which one can find a path $\mathcal{C}$ satisfying $\mathcal{L}(\mathcal{C}) = 1996$.

6. The picture below shows a triangle $\triangle ABC$ in which the length AB is smaller than that of BC, and the length of BC is smaller than that of AC. The points A', B' and C' are such that AA' is perpendicular to BC and the length of AA' equals that of BC; BB' is perpendicular to AC and the length of BB' equals that of AC; CC' is perpendicular to AB and the length of CC' equals that of AB. Moreover $\angle AC'B$ is a $90°$ angle. Prove that A', B' and C' are collinear.



The final contest set we give this issue is the Bi-National Israel-Hungary Competition, 1996. Thanks go to J.P. Grossman for collecting these problems and forwarding them to me when he was Canadian Team Leader to the IMO at Mumbai.

THE BI-NATIONAL ISRAEL-HUNGARY COMPETITION, 1996

Technion IIT, Israel

March 27, 1996 (Time: 4 hours)

Each problem is worth 7 points.

1. Find all sequences of integers $x_1, x_2, \ldots, x_{1997}$ such that

$$\sum_{k=1}^{1997} 2^{k-1}(x_k)^{1997} = 1996 \prod_{k=1}^{1997} x_k .$$

2. Let $n > 2$ be an integer, and suppose that n^2 can be represented as the difference of the cubes of two consecutive positive integers. Prove that n is the sum of two squares. Prove that such an n really exists.

3. A given convex polyhedron has no vertex which is incident with exactly 3 edges. Prove that the number of faces of the polyhedron which are triangles, is at least 8.

4. Let $a_1, a_2, \ldots, a_n$ be arbitrary real numbers and $b_1, b_2, \ldots, b_n$ real numbers satisfying the condition $1 \geq b_1 \geq b_2 \geq \cdots \geq b_n \geq 0$. Prove that there is a positive integer $k \leq n$ for which the inequality $|a_1b_1 + a_2b_2 + \cdots + a_nb_n| \leq |a_1 + a_2 + \cdots + a_k|$ holds.

We next turn to readers' solutions to problems of the 4^{th} Mathematical Olympiad of the Republic of China (Taiwan) [1998: 322–323].

1. Let $P(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + a_nx^n$ be a polynomial with complex coefficients. Suppose the roots of $P(x)$ are $\alpha_1, \alpha_2, \ldots, \alpha_n$ with $|\alpha_1| > 1$, $|\alpha_2| > 1$, $\ldots$, $|\alpha_j| > 1$, and $|\alpha_{j+1}| \leq 1$, $\ldots$, $|\alpha_n| \leq 1$. Prove:

$$\prod_{i=1}^{j} |\alpha_i| \leq \sqrt{|a_0|^2 + |a_1|^2 + \cdots + |a_n|^2}\,.$$

Solution by Mohammed Aassila, Strasbourg, France.

Let
$$\begin{aligned} Q(x) &= a_0x^m + a_1x^{m-1} + \cdots + a_m \\ R(x) &= b_0x^n + b_1x^{n-1} + \cdots + b_n \\ Q(x)R(x) &= c_0x^{n+m} + c_1x^{n+m-1} + \cdots + c_{n+m} \end{aligned}$$
and
$$Q(x)\overline{R}\left(\tfrac{1}{x}\right) = d_{-m}x^m + \cdots + d_nx^{-n}$$

with $a_0 = b_0 = 1$.

We claim that
$$\sum_{i=0}^{m+n} c_i^2 \;=\; \sum_{k=-m}^{n} d_k^2\,. \tag{1}$$

Indeed, d_k is the sum of $a_\alpha b_\beta$ with $\alpha - \delta = k$, so that d_k^2 is the sum of $a_\alpha b_\beta a_\gamma b_\beta$ with $\alpha - \delta = \gamma - \beta = k$, and summing over k means we take all α, β, γ, δ with $\alpha - \delta = \gamma - \beta$. Hence

$$\begin{aligned}\sum_{i=0}^{m+n} c_i^2 &= \sum_{\alpha+\beta=\gamma+\delta} a_\alpha b_\beta a_\gamma b_\delta \\ &= \sum_{\alpha-\delta=\beta-\gamma} a_\alpha b_\delta a_\gamma b_\beta \;=\; \sum_{k=-m}^{n} d_k^2\,.\end{aligned}$$

Hence, we set

$$\begin{aligned}Q(x) &= (x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_j)\,,\\ R(x) &= (x-\alpha_{j+1})(x-\alpha_{j+2})\dots(x-\alpha_n)\,.\end{aligned}$$

Because of (1), we know that $|a_0|^2 + \cdots + |a_n|^2$ is equal to the sum of the squares of the absolute values of the coefficients of $Q(x)\overline{R}(\frac{1}{x})$, and in particular we get $|\alpha_1 \dots \alpha_j|^2 \le |a_0|^2 + \cdots + |a_n|^2$.

2. Given a sequence of integers: x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, one constructs a second sequence: $|x_2 - x_1|$, $|x_3 - x_2|$, $|x_4 - x_3|$, $|x_5 - x_4|$, $|x_6 - x_5|$, $|x_7 - x_6|$, $|x_8 - x_7|$, $|x_1 - x_8|$. Such a process is called a single operation. Find all the 8–term integral sequences having the following property: after finitely many applications of the single operation, the sequence becomes an integral sequence with all terms equal.

Comment by Mohammed Aassila, Strasbourg, France.

This problem is a special case of problem 3 by Wai Ling Yee, [1998 : 34].

3. Suppose n persons meet in a meeting. Every one among them is familiar with exactly eight other participants of that meeting. Furthermore suppose that each pair of two participants who are familiar with each other has four acquaintances in common at that meeting, and each pair of two participants who are not familiar with each other has only two acquaintances in common. What are the possible values of n?

Solution by Pierre Bornsztein, Courdimanche, France.

Such a situation cannot occur.

Suppose, for a contradiction, that such a meeting is possible. Let x be a participant, and let x be familiar with x_1, x_2, $\dots$, x_8.

As x is familiar with x_8 they have four acquaintances in common, among x_1, $\dots$, x_7, say x_1, x_2, x_3, x_4. Then x_7 and x_8 are not familiar.

Now x is familiar with x_7, so they have four acquaintances in common among x_1, x_2, ..., x_6. So at least two must be among x_1, x_2, x_3, x_4, say x_1 and x_2.

But then x_7 and x_8, who are not familiar, have x_1, x_2 and x in common, and that is too many, a contradiction.

4. Given n distinct integers m_1, m_2, ..., m_n, prove that there exists a polynomial $f(x)$ of degree n and with integral coefficients which satisfies the following conditions:

(1) $f(m_i) = -1$, for all i, $1 \leq i \leq n$.

(2) $f(x)$ cannot be factorized into a product of two non-constant polynomials with integral coefficients.

Solutions by Mohammed Aassila, Strasbourg, France; and by Pierre Bornsztein, Courdimanche, France. The solutions were similar. We give Bornsztein's solution.

It suffices to show that $f(x) = (x - m_1)\dots(x - m_n) - 1$ satisfies condition (2).

Suppose, for a contradiction, that $f = QR$, with Q, $R \in \mathbb{Z}[x]$, and Q, R, with degree at most $n - 1$.

For all $i \in \{1, \dots, n\}$, since $f(m_i) = -1 = Q(m_i)R(m_i)$, with $Q(m_i)$, $R(m_i)$ integers, then

$$\begin{array}{lllll} Q(m_i) = 1 & \text{and} & R(m_i) = -1\,, & \text{or} \\ Q(m_i) = -1 & \text{and} & R(m_i) = 1\,. \end{array}$$

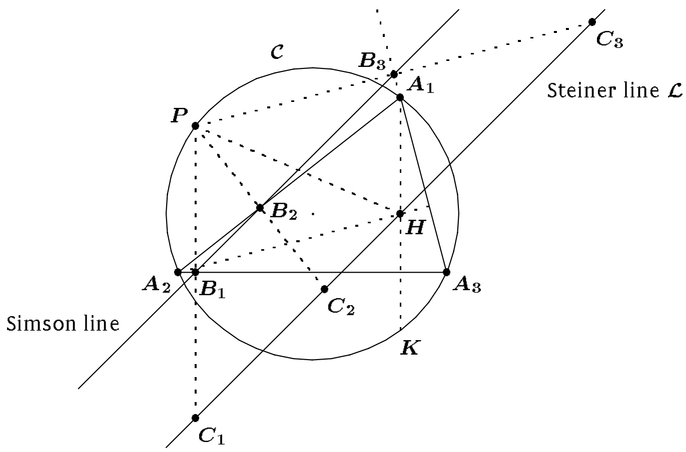
So, $Q+R$ is a polynomial with degree $\leq n-1$ and with n distinct roots, then $Q + R \equiv 0$; that is, $Q \equiv -R$. Thus $f = -Q^2$, which is impossible since the leading coefficient of f is 1, a contradiction. Thus f is irreducible in $\mathbb{Z}[X]$.

5. Let P be a point on the circumscribed circle of $\triangle A_1A_2A_3$. Let H be the orthocentre of $\triangle A_1A_2A_3$. Let B_1 (B_2, B_3 respectively) be the point of intersection of the perpendicular from P to A_2A_3 (A_1A_2, A_2A_3 respectively). It is known that the three points B_1, B_2, B_3 are collinear. Prove that the line $B_1B_2B_3$ passes through the mid-point of the line segment PH.

Solution by Pierre Bornsztein, Courdimanche, France.

The line $B_1B_2B_3$ is known as the Simson line.

Let C_1, C_2, C_3 be such that B_1, B_2, B_3 are the mid-points of PC_1, PC_2, PC_3, respectively.



It is known that C_1, C_2, C_3 are collinear. The line $C_1C_2C_3$ is the Steiner line, $\mathcal{L}$. (The Steiner line is the image of the Simson line by the homothetic transformation with centre P and ratio 2.)

It suffices to show that H is on the Steiner line.

If P is one of A_1, A_2, A_3, then $\mathcal{L}$ is an altitude in $\triangle A_1A_2A_3$ and $H \in \mathcal{L}$.

If $P \not\in \{A_1, A_2, A_3\}$, let K be symmetric with H with respect to A_2A_3. It is known that K is on the circle $\mathcal{C}$.

If H is among C_1, C_2, C_3, clearly $H \in \mathcal{L}$. Otherwise, with angles evaluated modulo π: [Ed. $(AB;CD)$, means the angle from the line segment AB to the line segment CD.]

$$\begin{aligned} (C_1H;C_2H) &= (C_1H;C_1P) + (C_1P;C_2P) + (C_2P;C_2H) \\ (C_1H;C_1P) &= (KH;KP) \quad \text{symmetry in } A_2A_3 \\ &= (KA_1;KP) \\ &= (A_2A_1;A_2P) \quad \text{concyclic.} \end{aligned}$$

Similarly

$$(C_2P;C_2H) = (A_1P;A_1A_2).$$

Thus

$$\begin{aligned} (C_1H;C_2H) &= (A_2A_1;A_2P) + (C_1P;C_2P) + (A_1P;A_1A_2) \\ &= (A_1P;A_2A_1) + (A_2A_1;A_2P) + (C_1P;C_2P) \\ &= (PA_1;PA_2) + (C_1P;C_2P). \end{aligned}$$

Moreover

$$\begin{aligned} (C_1P;C_2P) &= (B_1P;B_2P) \\ &= (B_1P;PA_2) + (PA_2;PA_1) + (PA_1;B_2P). \end{aligned}$$

Then

$$\begin{array}{rcl} (C_1H; C_2H) & = & (B_1P; PA_2) + (PA_1; B_2P) \\ & = & \underbrace{(B_1P; B_1A_2)}_{(\pi/2)} + (B_1A_2; PA_2) \\ & & \qquad + (PA_1; A_1B_2) + \underbrace{(A_1B_2; B_2P)}_{(\pi/2)} \\ \hline & = & (A_3P; PA_2) + (PA_1; A_1A_3) \\ & = & (A_2A_3; A_2P) + (A_2P; A_2A_1) \quad \text{(concyclic)} \\ & = & 0 \pmod{\pi}\,. \end{array}$$

Then C_1, C_2, H are collinear and in all cases $H \in \mathcal{L}$.

6. Let a, b, c, d be integers such that $ad - bc = k > 0$, $(a, b) = 1$, and $(c, d) = 1$. Prove that there are exactly k ordered pairs of real numbers (x_1, x_2) satisfying $0 \le x_1, x_2 < 1$ and for which both $ax_1 + bx_2$ and $cx_1 + dx_2$ are integers.

Solution by Mohammed Aassila, Strasbourg, France.

Because of Pick's Theorem, we know that the area of $P = \{(ax_1 + bx_2, cx_1 + dx_2),\ 0 \le x_1,\ x_2 \le 1\}$ is equal to the sum of the number of interior lattice points and half the number of boundary points. Since (x_1, x_2) is a lattice point if and only if $(1 - x_1, 1 - x_2)$ is as well, we conclude that the number of lattice points in P is exactly the area, which is $ad - bc = k$.

We next turn to solutions to problems of the XI Italian Mathematical Olympiad 1995, [1998 : 323–324].

2. In a class of 20 students no two of them have the same ordered pair (written and oral examinations) of scores in mathematics. We say that student A is better than B if his two scores are greater than or equal to the corresponding scores of B. The scores are integers between 1 and 10.

(a) Show that there exist three students A, B and C such that A is better than B and B is better than C.

(b) Would the same be true for a class of 19 students?

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

(a) Let the ordered pairs be denoted by (a_i, b_i), $i = 1, 2, \ldots, 20$, where a_i and b_i denote, respectively, the scores of the i^{th} student. For convenience of notation, write $(a_i, b_i) < (a_j, b_j)$ if the j^{th} student is better than the i^{th} student. Thus $(a_i, b_i) < (a_j, b_j)$ if and only if $a_i \le a_j$ and $b_i \le b_j$. Since there are 20 a_i's which are all in $\{1, 2, \ldots, 10\}$ we must have either

(1): For some $m \in \{1, 2, \ldots, 10\}$, $a_i = a_j = a_k = m$ where $i \ne j \ne k \ne i$, or

(2): Every $m \in \{1, 2, \ldots, 10\}$ appears exactly twice as the first component in the ordered pairs.

Similarly, either

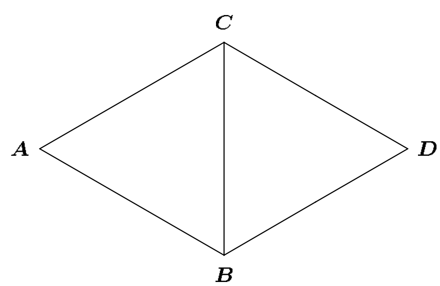
(3): For some $n \in \{1, 2, \ldots, 10\}$, $b_i = b_j = b_k = n$ where $i \neq j \neq k \neq i$, or

(4): Every $n \in \{1, 2, \ldots, 10\}$ appears exactly twice as the second component in the ordered pairs.

In case (1), we would have three ordered pairs of the form (m, b_i), (m, b_j), and (m, b_k). Since b_i, b_j, and b_k must all be distinct, we may assume, without loss of generality, that $b_i < b_j < b_k$, and then $(m, b_i) < (m, b_j) < (m, b_k)$. Similarly, in case (3), we would have three ordered pairs of the form (a_i, n), (a_j, n), and (a_k, n). Assuming $a_i < a_j < a_k$, we then have $(a_i, n) < (a_j, n) < (a_k, n)$. If neither (1) nor (3) holds, then both (2) and (4) must hold and so every $l \in \{1, 2, \ldots, 10\}$ appears exactly twice as the first component and exactly twice as the second component in the ordered pairs. In particular, there must be ordered pairs of the form $(1, b_i)$ and $(1, b_j)$ for some $b_i < b_j$. Since 10 must appear twice as the second component and $b_i \neq b_j$, there must be at least one ordered pair of the form $(a_k, 10)$ for some $a_k \neq 1$. Then we have $(1, b_i) < (1, b_j) < (a_k, 10)$ and the conclusion follows.

(b) No. For example, if the 19 ordered pairs are $(1, 10)$, $(2, 9)$, $(3, 8)$, $(4, 7)$, $(5, 6)$, $(6, 5)$, $(7, 4)$, $(8, 3)$, $(9, 2)$, $(10, 1)$ and $(2, 10)$, $(3, 9)$, $(4, 8)$, $(5, 7)$, $(6, 6)$, $(7, 5)$, $(8, 4)$, $(9, 3)$, $(10, 2)$, then no two ordered pairs from the first ten are comparable and no two ordered pairs from the other nine are comparable. Hence the required "chain" of three students does not exist.

3. In a town there are 4 pubs, A, B, C and D, connected as shown in the picture.



A drunkard wanders about the pubs starting with A and, after having a drink, goes to any of the pubs directly connected with equal probability.

(a) What is the probability that the drunkard is at pub C at his fifth drink?

(b) Where is the drunkard more likely to be after n drinks? $(n > 5)$

Solution by Pierre Bornsztein, Courdimanche, France.

Let a_n, b_n, c_n, d_n be the probabilities for the drunkard to be in pub A, B, C, D, respectively for drink number n $(n \geq 1)$. Let $(a_{n+1}; b_n)$ (etc.)

be the probability that the drunkard is in A for the $(n+1)^{\text{st}}$ drink knowing that he was in B for the n^{th} drink.

We have $a_1 = b_1 = c_1 = d_1 = 0$ and $a_2 = d_2 = 0$, $b_2 = c_2 = 1/2$.

For $n \geq 1$ we have

$$\begin{aligned} a_{n+1} &= a_n \times (a_{n+1}; a_n) + b_n \times (a_{n+1}; b_n) \\ &\quad + c_n \times (a_{n+1}; c_n) + d_n \times (a_{n+1}; d_n) \end{aligned}$$

with $(a_{n+1}; a_n) = 0 = (a_{n+1}; d_n)$ and $(a_{n+1}; b_n) = (a_{n+1}; c_n) = 1/3$. Then $a_{n+1} = \frac{1}{3}b_n + \frac{1}{3}c_n$.

Doing the same for b_{n+1}, c_{n+1}, and d_{n+1}, we obtain

$$\begin{aligned} a_{n+1} &= \tfrac{1}{3}b_n + \tfrac{1}{3}c_n = d_{n+1}\,, \\ b_{n+1} &= \tfrac{1}{2}a_n + \tfrac{1}{3}c_n + \tfrac{1}{2}d_n\,, \\ c_{n+1} &= \tfrac{1}{2}a_n + \tfrac{1}{3}b_n + \tfrac{1}{2}d_n\,. \end{aligned}$$

Because $b_1 = c_1$ and $b_2 = c_2$, we deduce by induction that $b_n = c_n$ for every $n \geq 1$.

Moreover $a_n + b_n + c_n + d_n = 1$.

$$\begin{aligned} &\text{Thus, for } n \geq 2\,, & a_n + b_n &= \tfrac{1}{2}\,, \\ &\text{and, for } n \geq 1\,, & a_{n+1} &= \tfrac{2}{3}b_n\,. \\ &\text{Thus, for } n \geq 2\,, & a_{n+1} &= \tfrac{1}{3} - \tfrac{2}{3}a_n \qquad (*) \end{aligned}$$

Let $u_n = a_n - \frac{1}{5}$. Using (*), we obtain $u_{n+1} = -\frac{2}{3}u_n$, for $n \geq 2$. Then, for $n \geq 2$, $u_n = \left(-\frac{2}{3}\right)^{n-2} u_2$ with $u_2 = a_2 - \frac{1}{5} = -\frac{1}{5}$. Thus,

$$u_n = -\frac{1}{5}\left(-\frac{2}{3}\right)^{n-2} \text{ for } n \geq 2$$

and

$$a_n = \frac{1}{5} - \frac{1}{5}\left(-\frac{2}{3}\right)^{n-2} = d_n\,.$$

Thus $b_n = c_n = \frac{1}{2} - a_n = \frac{3}{10} + \frac{1}{5}(-\frac{2}{3})^{n-2}$.

(a) We deduce that $c_5 = \frac{3}{10} - \frac{8}{5 \times 27} = \frac{13}{54}$.

(b) For $n > 5$, $a_n < b_n \iff -1 < 4(-\frac{2}{3})^{n-2}$.

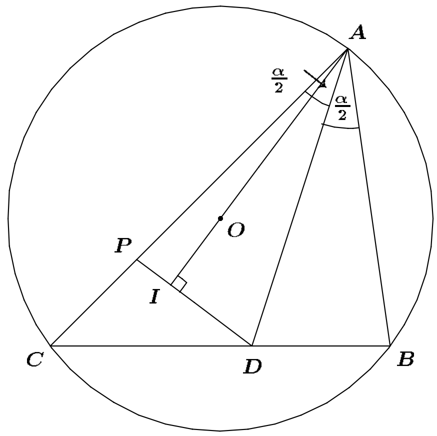
This is true for n even.

For odd n, $4(-\frac{2}{3})^{n-2}$ forms an increasing sequence with limit 0, and for $n = 7$ we have $4(-\frac{2}{3})^7 - 2 > -1$. Thus $d_n = a_n < b_n = c_n$ for $n > 5$.

Thus for $n > 5$, the drunkard is more likely to be in B than C.

4. An acute-angled triangle ABC is inscribed in a circle with centre O. Let D be the intersection of the bisector of A with BC and suppose that the perpendicular to AO through D meets the line AC in a point P interior to the segment AC. Show that $AB = AP$.

Solution by Pierre Bornsztein, Courdimanche, France.



Since ABC is acute, O is interior to ABC, $P \in AC$, so O is interior to ADC. We have $\angle AOC = 2\beta$ and $OA = OC$, therefore $\angle CAO = \frac{1}{2}(\pi - 2\beta) = \frac{\pi}{2} - \beta$. Thus

$$\angle OAD = \frac{\alpha}{2} - \angle CAO = \frac{\alpha}{2} + \beta - \frac{\pi}{2}.$$

Let $I = PD \cap AO$. We have that $\triangle AID$ is right-angled at I.

Then

$$\begin{aligned}\angle PDA &= \angle IDA = \frac{\pi}{2} - \angle OAD \\ &= \pi - \frac{\alpha}{2} - \beta = \angle BDA.\end{aligned}$$

We deduce that triangles ABD and APD are similar with AD in common. Therefore triangles ABD and APD are isometric, so that $AP = AB$.

6. Find all pairs of positive integers x, y such that

$$x^2 + 615 = 2^y.$$

Solutions by Mohammed Aassila, Strasbourg, France; Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.

The only solution is $x = 59$ and $y = 12$. Note first that for all non-negative integers k

$$2^{2k+1} \equiv 4^k \cdot 2 \equiv (-1)^k \cdot 2 \equiv 2 \text{ or } 3 \pmod 5.$$

Since $x^2 \equiv 0, 1$ or $4 \pmod 5$, y must be even. Letting $y = 2z$, then $x^2 + 615 = 2^y$ becomes $(2^z - x)(2^z + x) = 615$. Since $615 = 3 \times 5 \times 41$ there are only 4 cases:

(1) $2^z + x = 615, \quad 2^z - x = 1;$

(2) $2^z + x = 205, \quad 2^z - x = 3;$

(3) $2^z + x = 123, \quad 2^z - x = 5;$

(4) $2^z + x = 41, \quad 2^z - x = 15.$

Cases (1), (2) and (4) clearly yield no solution, since $2^{z+1} = 616$, 208 or 56, none of which is a power of two. Finally, case (3) yields $2^{z+1} = 128$ or $z = 6$, from which it follows that $x = 59$ and $y = 12$.

We next turn to the Yugoslav Federal Competition 1995, Third and Fourth Grade [1998: 325].

1. Let p be a prime number. Prove that the number

$$11\cdots122\cdots2\cdots99\cdots9 - 123456789$$

is divisible by p, where dots indicate that the corresponding digit appears p times consecutively.

Solutions by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.

Let n denote the given number and let $c = 123456789$. First note that the conclusion clearly holds if $p = 3$, since in this case both terms of n are divisible by 3. Assume that $p \neq 3$. By assumption, we have

$$n = \sum_{k=0}^{p-1} 10^{8p+k} + 2\sum_{k=0}^{p-1} 10^{7p+k} + \cdots + 9\sum_{k=0}^{p-1} 10^{k} - c\,. \qquad (1)$$

For $m = 0, 1, 2, \ldots, 8$ we get from (1) that

$$\begin{aligned} n &= \frac{1}{9}(10^p - 1)(10^{8p} + 2\times 10^{7p} + \cdots + 8\times 10^p + 9\times 1) - c \\ &= \frac{1}{9}(10^{9p} + 10^{8p} + \cdots + 10^p - 9) - c\,. \qquad (2) \end{aligned}$$

Since $p \mid n$ if and only if $9p \mid 9n$, it suffices to show that [Ed. because $p \neq 3$]

$$9p \mid 10^{9p} + 10^{8p} + \cdots + 10^p - 9 - 9c\,. \qquad (3)$$

Since $9 + 9c = 1,111,111,110 = 10^9 + 10^8 + \cdots + 10$, (3) is equivalent to

$$9p \mid (10^{9p} + 10^{8p} + \cdots + 10^p) - (10^9 + 10^8 + \cdots + 10)\,. \qquad (4)$$

By Fermat's Little Theorem, we have for all $m = 1, 2, \ldots, 9$, $10^{mp} \equiv (10^m)^p \equiv 10^m \pmod{p}$. Furthermore, as $10 \equiv 1 \pmod{9}$, we have $10^{mp} \equiv 10^m \pmod 9$ for all $m = 1, 2, \ldots, 9$. Since $p \neq 3$, $(p, 9) = 1$. Thus $10^{mp} \equiv 10^m \pmod{9p}$ and (4) follows.

This completes the proof.

Comment: Very interesting problem, indeed!

2. A polynomial $P(x)$ with integer coefficients is said to be divisible by a positive integer m if and only if the number $P(k)$ is divisible by m for all $k \in \mathbb{Z}$. If the polynomial

$$P(x) \;=\; a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$$

is divisible by m, prove that $a_n n!$ is divisible by m.

Solution by Pierre Bornsztein, Courdimanche, France. Comment by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's comment.

Assuming there is no error in the statement, then this problem seems trivial, since if $m \mid p(k)$ for all $k \in \mathbb{Z}$, then $m \mid p(0)$; that is, $m \mid a_n$, and so $m \mid a_n \cdot n!$.

3. A chord AB and a diameter $CD \perp AB$ of a circle k intersect at a point M. Let P lie on the arc ACB and let $a \not\in \{A, B, C\}$. Line PM intersects the circle k at P and $Q \neq P$, and line PD intersects chord AB at R. Prove that $RD > MQ$.

Comment by Pierre Bornsztein, Courdimanche, France.

This result is due to P. Erdős. See a solution in Ross Honsberger's "Mathematical Morsels".

Finally we discuss reader submissions about the Yugoslav Federal Competition 1995, Selection of the IMO Team [1998: 325].

1. Find all the triples (x, y, z) of positive rational numbers such that $x \leq y \leq z$ and

$$x + y + z\,,\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\,,\ xyz \in \mathbb{Z}\,.$$

Solution by Pierre Bornsztein, Courdimanche, France. Comment by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's comment.

This problem was the same as problem #1 of the 45th Mathematical Olympiad in Poland (Final Round) and has appeared in [1997 : 323]. A solution by Murray S. Klamkin appeared in [1998 : 394–395].

2. Let n be a positive integer having exactly 1995 1's in its binary representation. Prove that 2^{n-1995} divides $n!$.

Solutions by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bornsztein's solution.

We will prove that if $n \in \mathbb{N}^*$ and $\lambda(n)$ is the number of 1's in the binary representation of n, then $2^{n-\lambda(n)}$ divides $n!$.

Lemma. Let $p \in \mathbb{N}^*$. The greatest power of 2 which divides $(2^p)!$ is 2^N with $N = 2^p - 1$.

Proof of the lemma. It is well known that, with the notation of the lemma

$$N = \left\lfloor \frac{2^p}{2} \right\rfloor + \left\lfloor \frac{2^p}{2^2} \right\rfloor + \left\lfloor \frac{2^p}{2^3} \right\rfloor + \cdots$$

(where $\lfloor\ \rfloor$ is the integer part).

Then $N = 2^{p-1} + 2^{p-2} + \cdots = 2^p - 1$. ∎

We prove the main result by induction on $\lambda(n) = k \in \mathbb{N}^*$.

For $k = 1$, then n is a power of 2, $n = 2^p$, $p \in \mathbb{N}$ and we conclude with the lemma.

Suppose the result for a fixed $k \geq 1$.

Then $n = 2^p + m$ where $m \in \mathbb{N}^*$, $m < 2^p$ and $\lambda(m) = k$, and we have

$$n! = (2^p)!(2^p + 1)(2^p + 2)\cdots(2^p + m)\,.$$

But, by the lemma, we have $(2^p)! \equiv 0 \pmod{2^{2^p-1}}$. Moreover $(2^p+1)(2^p+2)\ldots(2^p+m)$ is the product of m consecutive positive integers, so that $(2^p+1)(2^p+2)\ldots(2^p+m)$ is divisible by $m!$ And, by the induction hypothesis, $m! \equiv 0 \pmod{2^{m-k}}$.

Then $(2^p + 1)\ldots(2^p + m) \equiv 0 \pmod{2^{m-k}}$. And so $n!$ is divisible by $2^{2^p-1} \times 2^{m-k} = 2^{2^p+m-(k+1)} = 2^{n-\lambda(n)}$.

Question: Is $2^{n-\lambda(n)}$ the greatest power of 2 which divides $n!$?

That completes the *Corner* for this issue. Please send me your Olympiad Contests and nice solutions.

BOOK REVIEWS

ALAN LAW

Logic as Algebra, by Paul Halmos and Steven Givant,
published by the Mathematical Association of America, 1998
ISBN# 0-88385-327-2, softcover, 134+ pages, $27.00 (U.S.).
Reviewed by **Maria Losada**, *Universidad Antonio Nariño, Bogotá, Colombia*.

This book is No. 21 of the Dolciani Mathematical Expositions, a series including many books with challenging problems for math lovers, which often approach a subject without formally introducing the reader to it. *Logic as Algebra* does not fall into this category. Instead it has a very consistent approach to propositional logic and monadic predicate calculus using algebraic structures and, although there are many bits left to the reader to fill in, the fundamental aspects needed are clearly and completely developed as befits the subject. Additionally there is not a single problem stated as such in the book, although a first time reader should provide most of the missing proofs and do the exercises implicitly left to the reader to fully grasp the subject.

The book is written in a very agreeable style which gives the impression of informality in a very formal subject. The authors have an efficient and clear approach to proofs and explanations that allows the book to be read easily. Still, its content is ample enough for a course in logic with the requirement that the logic student be familiar with certain elementary concepts of algebra, mainly notions of algebraic structures and their morphisms.

The first chapter gives a lovely introduction to syntax and semantics by examining a formal language with a small alphabet. The second introduces the reader to the larger and more complicated system of propositional calculus (see also [1] and [2]). Here the authors introduce the necessity of the algebraic approach very clearly in spite of the following conclusion: "The result of all the manipulations is that something we were driving at all along (to be discussed soon) follows from something else. This is not at all surprising — that, after all, is the only reason why this particular something else is considered. If the initially desired result had not followed, we would happily have changed the something else. In other words: we knew the answer all along, and all that has been happening is without a blunder." The following chapters pick up the algebraic language of Boolean algebras, Boolean logics and lattices (see also [4]). Someone familiar with the subject might want to start reading at Chapter 5 and use the earlier chapters for reference. The final chapter on monadic predicate calculus is, however, in my opinion, a must-read, for it contributes to the heart and goal of the book, wrapping up the subject nicely. It introduces quantifiers in a very streamlined algebraic setting and takes a good look at the classical syllogisms. Here also the reader is referred to [3] for more on the subject.

Logic as Algebra is designed to provide some fundamental insights in logic and its study using algebraic structures while allowing the reader to choose an appropriate level and follow a comfortable pace. It is a relevant stepping stone to the study of some beautiful areas of mathematics.

References

[1] D. Hilbert and W. Ackermann, *Principles of Mathematical Logic*, Chelsea Publishing Company, 1950.

[2] Paul C. Rosenbloom, *The Elements of Mathematical Logic*, Dover Publications, 1950.

[3] P. R. Halmos, *Algebraic Logic*, New York Chelsea Publishing Company, 1962.

[4] R.L. Goodstein, *Boolean Algebra*, The Commonwealth and International Library of Science, Technology, Engineering and Liberal Studies, 1963.

π curiosity.

Did you know that

$$\begin{aligned}\pi &= 128\tan^{-1}\left(\frac{1}{40}\right) - 4\tan^{-1}\left(\frac{1}{239}\right) - 16\tan^{-1}\left(\frac{1}{515}\right) \\ &\quad -32\tan^{-1}\left(\frac{1}{4030}\right) - 64\tan^{-1}\left(\frac{1}{32060}\right)\ ?\end{aligned}$$

Can *you* prove it?

Nikolaos Dergiades, Thessaloniki, Greece

The Existence of Looped Langford Sequences

Nabil Shalaby and Tara Stuckless (student)

Introduction

The problem to be discussed came out of the one originally posed by C. Dudley Langford [4], after observing his son's games with blocks. Langford noticed that the young boy had placed his blocks so that between the two red blocks, there was one block, between the two blue blocks, there were two blocks, and between the two yellow blocks there were three blocks. Further, he noticed that after some rearrangement, he could keep these properties, and add a pair of green blocks with four blocks between them.

This arrangement is an example of a Langford sequence of order $n = 5$, and defect $d = 2$. We write $L = (3, 4, 5, 3, 2, 4, 2, 5)$, where the number 2 represents the red blocks, 3 the blue, 4 the yellow, and 5 the green blocks. The order is the largest element in the sequence, while the defect is the smallest. Note that for every $k \in \{d, d+1, \ldots, n\}$, k occurs exactly twice in the sequence, with $k - 1$ numbers between the pair. When Langford presented this problem, in the same paper, he also gave sequences for $n = 3, 4, 5, 7, 8, 11, 12$, and 15, when $d = 2$, and asked for a theoretical treatment of the problem. Later, C.J. Priday [6] showed that for $d = 2$, if a Langford sequence of order n does not exist, then if a blank space, or hook, is added in the next to last position of the sequence, it is possible to arrange the $2n$ numbers in the required manner. Priday referred to a Langford sequence with no hooks as a *perfect* sequence, while this second type was called a *hooked* sequence. For example, $(2, 3, 2, *, 3)$ and $(6, 10, 11, 12, 4, 7, 6, 9, 4, 5, 8, 10, 7, 11, 5, 12, 9, *, 8)$ are hooked Langford sequences or orders 3 and 12, with defects 2 and 4 respectively. Notice that the two can be 'hooked' together, giving the perfect sequence $(6, 10, 11, 12, 4, 7, 6, 9, 4, 5, 8, 10, 7, 11, 5, 12, 9, 3, 8, 2, 3, 2)$.

To prove that for $d = 2$, a sequence of order n is either perfect or hooked, Priday used the concept of a *looped* set. Using a slightly different notation, he said that the set $\{d, d+1, \ldots, n\}$ is looped if there exist two Langford sequences of order n and defect d, one with hooks two places from the end of the sequence, and the other with a hook two places and one place from the end. We shall refer to these two sequences collectively as a looped sequence. For example, we have: $(2, 4, 2, 5, 3, 4, *, 3, 5)$; $(3, 5, 2, 3, 2, 4, 5, *, *, 4)$.

We can see that if the two sequences had no elements in common, we could loop the two together to obtain a perfect sequence. Consider the examples of $d = 2$, $n = 3$, and $d = 5$, $n = 9$, the sequences: $(2, 4, 2, 3, *, 4, 3)$; $(2, 3, 2, 4, 3, *, *, 4)$, and $(12, 10, 8, 5, 9, 13, 11, 7, 5, 6, 8, 10, 12, 9, 7, 6, *, 11, 13)$; $(8, 13, 11, 9, 7, 5, 10, 12, 8, 6, 5, 7, 9, 11, 13, 6, 10, *, *, 12)$ testify that the sets $\{2, 3, 4\}$ and $\{5, 6, 7, 8, 9, 10, 11, 12, 13\}$ are looped. We can combine the two using the single hooked portion of the first example, and the double hooked part of the second example, and obtain a perfect sequence of order 13, and defect 2, (8, 13, 11, 9, 7, 5, 10, 12, 8, 6, 5, 7, 9, 11, 13, 6, 10, 3, 4, 12, 3, 2, 4, 2). For the original problem of the existence of perfect Langford sequences, Davies [3] gave a complete solution for when $d = 2$. Priday [6] used looped sequences to show that the necessary conditions for the existence of a perfect sequence are $n \equiv 0, 1 \pmod 4$. Following this, the combined works of Bermond, Brouwer, and Germa [2], and Simpson [8] showed that for all $d \geq 2$, and for all admissible n, the necessary conditions for the existence of a perfect or hooked Langford sequence were also sufficient. Langford sequences stimulated the interest of researchers in other areas of combinatorics, and can be used as a tool in combinatorial designs. More can be seen on this in Shalaby [6].

The problem in question is to determine the necessary and sufficient conditions for the existence of looped Langford sequences of order n. The necessary conditions are proven easily using a method attributed to Bang [1], and used by Skolem [9] in solving an equivalent case of a perfect sequence with $d = 1$. In this paper we will show that the necessary conditions for a looped Langford sequence with defect $d = 2$, are also sufficient, and discuss the generalization of d.

Definitions

Definition 1 A *Langford sequence* of order n and defect d, $n \geq d$, is a sequence $L = (l_1, l_2, \ldots, l_{2(n-d+1)})$ of integers satisfying the conditions:

1. for every $k \in \{d, d+1, \ldots, n\}$ there exist exactly two elements $l_i, l_j \in L$ such that $l_i = l_j = k$,
2. if $l_i = l_j = k$ with $i \leq j$, then $j - i = k$.

Such a sequence is said to be a *perfect* Langford sequence.

Definition 2 A *hooked Langford sequence* of order n and defect d, $n \geq d$, is a sequence $L = (l_1, l_2, \ldots, l_{2(n-d+1)+1})$ of integers (note: an odd number of terms) satisfying the conditions:

1. for every $k \in \{d, d+1, \ldots, n\}$ there exist exactly two elements $l_i, l_j \in L$ such that $l_i = l_j = k$,
2. if $l_i = l_j = k$ with $i \leq j$, then $j - i = k$,
3. $l_{2n} = 0$.

We often denote the $2n^{\text{th}}$ place in the sequence with a $*$ instead of 0, and call it a hook.

Definition 3 A *looped Langford set* is a pair of sequences (L, K) of order n and defect d, $n \geq d$, $L = (l_1, l_2, \ldots, l_{2(n-d+1)+1})$, and $K = (k_1, k_2, \ldots, k_{2(n-d+1)+2})$ both satisfying conditions (1) and (2) of a perfect Langford sequence, and also,

1. $l_{2n-1} = 0$,
2. $k_{2n} = k_{2n+1} = 0$.

If such a set exists, we shall refer to it collectively as a *looped Langford sequence*.

A Solution for $d = 2$

Theorem 1 A looped Langford set of order n and defect 2 exists if and only if $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$.

Necessity.

The necessity is fairly straightforward to prove, employing Bang's method. Let a_r denote the position in which the element r first appears. Then for the sequence K to be looped, the element r appears again in the $(a_r + r)^{\text{th}}$ position. Thus we can sum the positions 1 through $2n-3$ and $2n$, giving us the following equality:

$$\sum_{r=2}^{n}(a_r + a_r + r) \;=\; \sum_{i=1}^{2n-3} i + 2n \;.$$

Using the formula for the sum of the first n and $2n-3$ integers, we obtain

$$\sum_{r=2}^{n} a_r \;=\; \frac{3n^2 - 7n + 8}{4} \;.$$

Thus, if a looped sequence exists for some n, we know that $4|(3n^2 - 7n + 8)$, or equivalently, $4|n(3n-7)$. Now, if $3n-7$ is odd, then n is even, and vice versa. Thus 2 cannot divide both terms. If $4|n$, then $n \equiv 0 \pmod 4$. Alternately if $4|(3n-7)$, then $3n \equiv 7 \pmod 4$, or equivalently, $n \equiv 1 \pmod 4$.

Sufficiency.

The proof for sufficiency involves taking separate cases for n congruent to 0, 1, 4, and 5 modulo 8, and for each case, presenting the looped sequence.

The Case for $n \equiv 0 \pmod 8$:

When $n \equiv 0 \pmod 8$, we use the following tables to obtain a solution. The first table gives the sequence with one hook, and the second constructs a looped sequence. Here, i and j are the positions of the element k in the sequence. Thus, each of the elements in column k will give the differences 2 through n, where $n = 8m$, $m \geq 2$.

i	j	k	
$8m-1+t$	$16m-1-t$	$8m-2t$,	$0 \leq t \leq 1$ and $3 \leq t \leq 4m-2$
$2m-3-2t$	$6m+2t$	$4m+3+4t$,	$0 \leq t \leq m-3$, and $m > 2$
$2m-2t$	$6m+1+2t$	$4m+1+4t$,	$0 \leq t \leq m-2$
$4m-1-t$	$4m+2+t$	$3+2t$,	$0 \leq t \leq 2m-3$
$4m+t$	$12m-1-t$	$8m-1-2t$	$0 \leq t \leq 1$
$8m+1$	$12m$	$4m-1$	
2	$8m-2$	$8m-4$	
1	$8m-4$	$8m-5$	
$2m-1$	$2m+1$	2	

i	j	k	
$2m+2+t$	$6m-2-t$	$4m-4-2t$,	$0 \leq t \leq 2m-4$
$8m+2+t$	$16m-3-t$	$8m-5-2t$,	$0 \leq t \leq 4m-4$
$4+2t$	$8m-4-2t$	$8m-8-4t$,	$0 \leq t \leq m-2$
$5+2t$	$8m-1-2t$	$8m-6-4t$,	$0 \leq t \leq m-2$
$4m+1$	$12m-1$	$8m-2$	
$8m+1$	$16m$	$8m-1$	
1	$4m-1$	$4m-2$	
2	$8m-2$	$8m-4$	
3	$8m$	$8m-3$	
$4m$	$12m$	$8m$	
$6m-1$	$6m+1$	2	

For the case of $m = 1$, we have the following solution: (2, 4, 2, 7, 5, 4, 8, 6, 3, 5, 7, 3, $*$, 6, 8); (2, 4, 2, 7, 5, 4, 6, 8, 3, 5, 7, 3, 6, $*$, $*$, 8).

To verify that the above tables will give a looped Langford set, we check that every difference from 1 to $n = 8m$ is present exactly once in each table.

In the first table, together rows nine, one, and seven give all of the even differences 2, 4, 6, ..., $8m - 2$, $8m$. The fourth row gives the differences 3, 5, 7, ..., $4m - 3$, row six gives $4m - 1$, and rows two and three together give the differences $4m + 1$, $4m + 3$, ..., $8m - 9$, $8m - 7$. The remaining odd differences, $8m - 5$, $8m - 3$, and $8m - 1$ are given by rows 8 and 5, respectively.

In the second table, rows eleven and one give the differences 2, 4, 6, ..., $4m-4$. Row seven gives $4m-2$, and rows three and four together give the differences $4m$, $4m + 2$, $4m + 4$, ..., $8m - 6$, $8m - 8$. The three remaining even differences, $8m - 4$, $8m - 2$, and $8m$ are obtained from rows eight, five, and ten, respectively. Row two gives the differences 3, 5, 7, ..., $8m - 5$, and rows nine and six give the differences $8m - 3$, and $8m - 1$.

Thus in both tables we can see that each of the differences 2, 3, ..., n appears exactly once, and so we have a looped Langford set. In subsequent proofs we will simply give the tables, and skip this process of verification.

The Case for $n \equiv 1 \pmod 8$:

Here we have $n = 8m + 1$, $m \geq 2$, and i, j, and k are defined as in the previous case.

i	j	k	
$8m + t$	$16m + 1 - t$	$8m + 1 - 2t$,	$0 \leq t \leq 1$ and $3 \leq 4m - 1$
$4m + 1 + t$	$12m + 1 - t$	$8m - 2t$,	$0 \leq t \leq 1$
$4m - 1 - t$	$4m + 3 + t$	$4 + 2t$,	$0 \leq t \leq 2m - 2$
$2m - 1 - 2t$	$6m + 3 + 2t$	$4m + 4 + 4t$,	$0 \leq t \leq m - 2$
$2m - 4 - 2t$	$6m + 2 + 2t$	$4m + 6 + 4t$,	$0 \leq t \leq m - 3$, and $m > 2$
$4m$	$8m + 2$	$4m + 2$	
1	$8m - 2$	$8m - 3$	
$2m - 2$	$2m$	2	

i	j	k	
$12m - 1 - t$	$12m + 3 + t$	$4 + 2t$,	$0 \leq t \leq 4m - 4$
$2 + 2t$	$8m - 3 - 2t$	$8m - 5 - 4t$,	$0 \leq t \leq m - 2$
$3 + 2t$	$8m - 2t$	$8m - 3 - 4t$	$0 \leq t \leq m - 1$
$4m - t$	$4m + 3 + t$	$3 + 2t$,	$0 \leq t \leq 2m - 3$
$4m + 1 + t$	$12m + 2 - t$	$8m + 1 - 2t$,	$0 \leq t \leq 1$
$8m + 1$	$12m$	$4m - 1$	
$8m + 2$	$16m + 2$	$8m$	
1	$8m - 1$	$8m - 2$	
$2m$	$2m + 2$	2	

For the case of $m = 1$, we have the following solution: (2, 4, 2, 6, 8, 4, 5, 9, 7, 6, 3, 5, 8, 3, $*$, 7, 9); (7, 2, 4, 2, 8, 6, 4, 7, 9, 5, 3, 6, 8, 3, 5, $*$, $*$, 9).

In this case, the solution for $m = 1$ can be obtained from the tables.

The Case for $n \equiv 4 \pmod 8$:

Here we have $n = 8m + 4$, $m \geq 2$, and i, j, and k are defined as in the previous cases.

i	j	k	
$8m + 3 + t$	$16m + 7 - t$	$8m + 4 - 2t$,	$0 \leq t \leq 1$ and $3 \leq t \leq 4m$
$4m + 2 + t$	$12m + 5 - t$	$8m + 3 - 2t$	$0 \leq t \leq 1$
$4m + 1 - t$	$4m + 4 + t$	$3 + 2t$,	$0 \leq t \leq 2m - 2$
$2m + 1 - 2t$	$6m + 4 + 2t$	$4m + 3 + 4t$,	$0 \leq t \leq m - 1$
$2m - 2 - 2t$	$6m + 3 + 2t$	$4m + 5 + 4t$,	$0 \leq t \leq m - 2$
1	$8m + 1$	$8m$	
$8m + 5$	$12m + 6$	$4m + 1$	
$2m$	$2m + 2$	2	

i	j	k	
$12m + 3 - t$	$12m + 7 + t$	$4 + 2t$	$0 \leq t \leq 4m - 2$
$4m + 1 - t$	$4m + 4 + t$	$3 + 2t$,	$0 \leq t \leq 2m$
$2m - 1 - 2t$	$6m + 6 + 2t$	$4m + 7 + 4t$,	$0 \leq t \leq m - 2$
$2m - 4 - 2t$	$6m + 5 + 2t$	$4m + 9 + 4t$,	$0 \leq t \leq m - 3$, and $m > 2$
$4m + 2 + t$	$12m + 5 - t$	$8m + 3 - 2t$	$0 \leq t \leq 1$
1	$8m + 3$	$8m + 2$	
$8m + 4$	$16m + 8$	$8m + 4$	
$8m + 1$	$12m + 6$	$4m + 5$	
$2m - 2$	$2m$	2	

For the case of $m = 1$, we have the solution: $(8, 2, 7, 2, 3, 11, 9, 3, 8, 7, 12, 10, 5, 6, 4, 9, 11, 5, 4, 6, *, 10, 12)$; $(7, 2, 10, 2, 12, 6, 9, 7, 11, 8, 4, 6, 10, 5, 4, 9, 12, 8, 5, 11, 3, *, *, 3)$.

The Case for $n \equiv 5 \pmod 8$:

Here $n = 8m + 5$, $m \geq 2$, and i, j, and k are defined as in the previous cases.

i	j	k	
$8m + 4 + t$	$16m + 9 - t$	$8m + 5 - 2t$,	$0 \leq t \leq 1$ and $3 \leq t \leq 4m + 1$
$4m + 3 + t$	$12m + 7 - t$	$8m + 4 - 2t$,	$0 \leq t \leq 1$
$4m + 1 - t$	$4m + 5 + t$	$4 + 2t$,	$0 \leq t \leq 2m - 1$
$2m - 2t$	$6m + 6 + 2t$	$4m + 6 + 4t$,	$0 \leq t \leq m - 2$
$2m - 3 - 2t$	$6m + 5 + 2t$	$4m + 8 + 4t$,	$0 \leq t \leq m - 2$
$4m + 2$	$8m + 6$	$4m + 4$	
2	$8m + 3$	$8m + 1$	
$2m - 1$	$2m + 1$	2	

i	j	k	
$1 + t$	$8m + 5 - t$	$8m + 4 - 2t$,	$0 \leq t \leq 4m$
$4m + 2 + t$	$12m + 7 - t$	$8m + 5 - 2t$,	$0 \leq t \leq 2$
$8m + 7 + 2t$	$16m + 6 - 2t$	$8m - 1 - 4t$,	$0 \leq t \leq m$
$8m + 6 + 2t$	$16m + 3 - 2t$	$8m - 3 - 4t$,	$0 \leq t \leq m - 1$
$12m + 4 - t$	$12m + 9 + t$	$5 + 2t$,	$0 \leq t \leq 2m - 5$, and $m > 2$
$16m + 7$	$16m + 10$	3	
$12m + 8$	$16m + 5$	$4m - 3$	
$10m + 6$	$10m + 8$	2	

When $m = 1$, we have: (2, 9, 2, 6, 4, 8, 12, 10, 4, 6, 9, 13, 11, 8, 7, 5, 3, 10, 12, 3, 5, 7, ∗, 11, 13); (12, 10, 8, 6, 4, 13, 11, 9, 4, 6, 8, 10, 12, 2, 7, 2, 9, 11, 13, 3, 5, 7, 3, ∗, ∗, 5).

Generalization of d

We have seen a solution for the case when we use the numbers 2 through n; that is, $d = 2$. Now we consider what happens when we change the value of d.

Theorem 2 If a looped Langford sequence of order n and defect d exists, $d \geq 2$, then the restrictions on n are given as follows:

1. if $d \equiv 0$ or $3 \pmod 4$, then $n \equiv 2$ or $3 \pmod 4$;
2. if $d \equiv 1$ or $2 \pmod 4$, then $n \equiv 0$ or $1 \pmod 4$.

Again, we require the following equality, should a looped sequence exist:

$$\sum_{r=d}^{n}(a_r + a_r + r) \;=\; \sum_{i=1}^{2n-2d+1} i + 2n - 2d + 4\,.$$

From this, using the formula for the sum of the first n integers, we obtain

$$\sum_{r=d}^{n} a_r \;=\; \frac{3n^2 - 8nd + 9n + 5d^2 - 11d + 10}{4}\,.$$

So we have shown that it is necessary that $4|(3n^2 - 8nd + 3d^2 + 9n - 9d + 10)$, or equivalently, $3n^2 - 8nd + 9n + 5d^2 - 11d + 10 \equiv 0 \pmod 4$. Thus $-n^2 + n + d^2 + d - 2 \equiv 0 \pmod 4$, or $-n^2 + n - 2 \equiv d^2 + d \pmod 4$. We determine what values n can possess by considering the cases of d congruent to 0, 1, 2, or $3 \pmod 4$.

Case 1: $d \equiv 0$ or $3 \pmod 4$.

If $d \equiv 0 \pmod 4$ then $d^2 + d \equiv 0 \pmod 4$. Similarly, if $d \equiv 3 \pmod 4$, then $d^2 + d \equiv 3^2 + 3 \pmod 4 \equiv 12 \pmod 4 \equiv 0 \pmod 4$. So in either case, we have that $n^2 - n - 2 \equiv 0 \pmod 4$. Thus $(n - 2)(n + 1) \equiv 0 \pmod 4$. Since 2 cannot divide both of these factors, 4 must divide one of them. Now, $(n - 2) \equiv 0 \pmod 4$ gives us that $n \equiv 2 \pmod 4$, and $(n + 1) \equiv 0 \pmod 4$ implies $n \equiv 3 \pmod 4$.

Case 2: $d \equiv 1$ or $2 \pmod 4$.

If $d \equiv 1 \pmod 4$, then $d^2 + d \equiv 2 \pmod 4$, and if $d \equiv 2 \pmod 4$, we obtain that $d^2 + d \equiv 2^2 + 2 \pmod 4$, which is again congruent to $2 \pmod 4$. Thus $n^2 - n - 2 \equiv 2 \pmod 4$, and so $n(n - 1) \equiv 0 \pmod 4$. Thus $n \equiv 0 \pmod 4$, or $n \equiv 1 \pmod 4$.

So when $d \equiv 0$ or $3 \pmod 4$, we have that n is congruent to 2 or $3 \pmod 4$, and if $d \equiv 1$ or $2 \pmod 4$, we have that n is congruent to 0 or $1 \pmod 4$. We note that this is consistent with the case of $d = 2$, where we showed the necessary conditions to be $n \equiv 0$ or $1 \pmod 4$.

Conclusions

We have shown that a looped Langford sequence of order n and defect 2 exists if and only if $n \equiv 0$ or $1 \pmod 4$ [Theorem 1] . As well, we have shown the necessary conditions on n for the existence of a looped sequence of order $d \geq 2$ [Theorem 2]. We conjecture that the necessary conditions are also sufficient, but this is an open problem, and seems to require the use of tools beyond the scope of this work.

Addendum

Tara Stuckless won the prize for the best undergraduate paper when she presented this paper at the APICS conference 1998 (St Mary's University). It was brought to our attention that the result of this paper (Theorem 1) is included in a more general result, published in the Journal of Combinatorial Theory [5]. However, the proof given here is independent of the other proof and accessible to a larger mathematical readership.

Acknowledgements

The authors would like to say a special thank-you to Dr. Donald Rideout for his valuable comments, especially regarding the proof for Theorem 2. Thanks is given also to Dr. John Grant McLoughlin for his helpful criticism of the paper, and to Mr. Shannon Sullivan for his careful proof-reading.

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Nabil Shalaby
Department of Mathematics and Statistics
Memorial University of Newfoundland
St. John's, Newfoundland
Canada, A1C 5S7
email: nshalaby@math.mun.ca

Tara Stuckless
Department of Mathematics and Statistics
Memorial University of Newfoundland
St. John's, Newfoundland
Canada, A1C 5S7
email: taras@math.mun.ca
(Tara is now a graduate student
at Simon Fraser University).

THE SKOLIAD CORNER

No. 44

R.E. Woodrow

This number we give the problems of the 1999 Maritimes Mathematics Competition, written March 11, 1999. The contest is 3 hours long, and no calculators are allowed. It is emphasized that complete solutions with explanation are required. My thanks go to John Grant McLoughlin, Memorial University of Newfoundland, for sending me the problems.

1999 MARITIMES MATHEMATICS COMPETITION
CONCOURS DE MATHÉMATIQUES DES MARITIMES 1999
March 11, 1999

1. Let natural numbers be assigned to the letters of the alphabet as follows: $A = 1$, $B = 2$, $C = 3$, ..., $Z = 26$. The value of a word is defined to be the product of the numbers assigned to the letters in that word. For example, the value of $MATH$ is $13 \times 1 \times 20 \times 8 = 2080$. Find a word whose value is 285.

Faisons correspondre à chaque lettre de l'alphabet une valeur numérique comme suit : $A = 1$, $B = 2$, $C = 3$, ..., $Z = 26$. *La valeur d'un mot sera le produit de la valeur de ses lettres. Par exemple, la valeur du mot* $MATH$ *est* $13 \times 1 \times 20 \times 8 = 2080$. *Trouver un mot dont la valeur est* 285.

2. A rhombus is a parallelogram with all four sides having the same length. If one of the interior angles of a rhombus is 60°, find the ratio of the area of the rhombus to the area of the inscribed circle.

Un losange est un parallélogramme dont les côtés sont égaux. Si un des angles internes d'un losange mesure 60°, *trouver le rapport de l'aire du losange à l'aire du cercle inscrit.*

3. A straight line cuts the asymptotes of a hyperbola in points A and B and cuts the curve at points P and Q. Prove that $AP = BQ$. (Hint: Use the fact that every hyperbola can be rotated, translated and scaled so that it is given by the equation $xy = 1$, and the asymptotes in this case are just the x–axis and the y–axis.)

Une droite coupe les asymptotes d'une hyperbole aux points A *et* B *et coupe l'hyperbole en* P *et* Q. *Montrer que* $AP = BQ$. *(Indice : chaque hyperbole peut être convertit, par rotation, translation et contraction/dilation, en l'hyperbole, d'équation* $xy = 1$, *et dans ce cas les asymptotes sont tout simplement l'axe des* x *et l'axe des* y.*)*

4. Find the largest number n with the property that the sum of the cubes of its digits (in base 10) is greater than n.

Trouver le plus grand nombre n dont la somme des cubes des chiffres (en base 10) est plus grande que n.

5. Find all non-negative numbers x, y and z such that

$$\begin{aligned} z^x &= y^{2x}, \\ 2^z &= 2 \cdot 4^x, \\ x + y + z &= 16. \end{aligned}$$

Trouver tous les nombres non-négatifs x, y et z satisfaisant les équations ci-dessous

$$\begin{aligned} z^x &= y^{2x}, \\ 2^z &= 2 \cdot 4^x, \\ x + y + z &= 16. \end{aligned}$$

6. The following symmetric table is known as Sundaram's sieve. The first row and column is the arithmetic progression 4, 7, 10, 13, ... Successive rows are also arithmetic progressions, the common differences, respectively being the odd integers 3, 5, 7, 9, ... Show that for every positive integer n, $2n + 1$ is prime if and only if n is not in the table.

La matrice symmétrique ci-dessous s'appelle le crible de Sundaram. La première rangée et la première colonne contiennent la suite arithmétique 4, 7, 10, 13, ... Les rangées et colonnes successives du crible contiennent des suites arithmétiques dont les raisons sont les nombres impairs 3, 5, 7, 9, ... Montrer que pour tout nombre naturel n, le nombre $2n + 1$ est premier si et seulement si n n'apparaît pas dans le crible.

4	7	10	13	16	19	22	⋯
7	12	17	22	27	32	37	⋯
10	17	24	31	38	45	52	⋯
13	22	31	40	49	58	67	⋯
16	27	38	49	60	71	82	⋯
19	32	45	58	71	84	97	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Last issue we gave the problems from Part I of the Alberta High School Mathematics Competition written in November, 1999. Thanks go to Ted Lewis, University of Alberta, for forwarding the contest and the solutions to use in the *Corner*. Here are the solutions.

THE ALBERTA HIGH SCHOOL MATHEMATICS COMPETITION
PART I
November 16, 1999

1. Subtracting 99% of 19 from 19% of 99, the difference d satisfies

Solution. **(c)** Since multiplication is commutative, $d = 0$.

2. Suppose you multiply three different positive prime numbers together and get a product which is greater than 1999. The smallest possible size of the largest of your primes is

Solution. **(c)** We have $7 \cdot 11 \cdot 13 = 1001$ and $11 \cdot 13 \cdot 17 = 2431$.

3. Suppose you multiply three different positive prime numbers together and get a product which is greater than 1999. The largest possible size of the smallest of your primes is

Solution. **(e)** Since there is no end to the primes, there is no largest possible size of the smallest of the three primes.

4. The number of two-digit positive integers such that the difference between the integer and the product of its digits is 12 is

Solution. **(b)** Let the two digits be x and y respectively. From $10x + y - xy = 12$, we have $(x - 1)(10 - y) = 2$. This leads to 28 or 39.

5. The non-zero slope of a certain straight line is equal to its y–intercept if and only if the x–intercept a satisfies

Solution. **(b)** Let the y–intercept be b. Suppose the slope m is equal to b. Then the equation of the line is $y = bx + b$. Hence $0 = ba + b$ and $a = -1$. Conversely, suppose $a = -1$. Then the equation of the line is $\frac{y}{x+1} = m$. Hence $\frac{b}{0+1} = m$ and we do have $b = m$.

6. A and B are positive integers. The sum of the digits of A is 19. The sum of the digits of B is 99. The smallest possible sum of the digits of the number $A + B$ is

Solution. **(a)** Take $A = 9900000000001$ and $B = 99999999999$.

7. O is the origin of the coordinate plane. A, B and C are points on the x–axis such that $OA = AB = BC = 1$. D, E and F are points on the y–axis such that $OD = DE = EF \geq 1$. If $CD \cdot AF = BE^2$, then OD is

Solution. (**b**) Let $OD = x$. Then $(x^2 + 9)(9x^2 + 1) = CD^2AF^2 = BE^4 = (4x^2 + 4)^2$. This simplifies to $(x^2 - 7)(7x^2 - 1) = 0$.

8. The integer closest to $100(12 - \sqrt{143})$ is

Solution. (**c**) We have $12 - \sqrt{143} = \frac{1}{12+\sqrt{143}}$.
Note that $11 < \sqrt{143} < 12$.
It follows that we have $4 < \frac{100}{24} < \frac{100}{12+\sqrt{143}} < \frac{100}{23} < 4.5$.

9. A bag contains four balls numbered -2, -1, 1 and 2. Two balls are drawn at random from the bag, and the numbers on them are multiplied together. The probability that this product is either odd or negative (or both) is

Solution. (**d**) The six possible products are 2, 2, -1, -2, -2 and -4. Only two of them are positive and even.

10. The number of positive perfect cubes which divide 9^9 is

Solution. (**e**) The divisors of $9^9 = 3^{18}$ which are cubes are 3^0, 3^3, 3^6, 3^9, 3^{12}, 3^{15} and 3^{18}.

11. In the quadratic equation $x^2 - 14x + k = 0$, k is a positive integer. The roots of the equation are two different prime numbers p and q. The value of $\frac{p}{q} + \frac{q}{p}$ is

Solution. (**c**) We have $p + q = 14$, so that one of them is 3 and the other is 11.

12. In the quadrilateral $ABCD$, AB is parallel to CD, $AB = 4$ and $BC = CD = 9$. X is on BC and Y is on DA such that XY is parallel to AB. If the quadrilaterals $ABXY$ and $YXCD$ are similar, distance BX is

Solution. (**b**) We have $\frac{AB}{XY} = \frac{XY}{CD}$ so that $XY = 6$. We also have $\frac{AB}{XY} = \frac{BX}{CX} = \frac{BX}{BC-BX}$ so that $4(9 - BX) = 6BX$.

13. The country of Magyaria has three kinds of coins, each worth a different integral number of dollars. Matthew collected four Magyarian coins with a total worth of 28 dollars, while Daniel collected five with a total worth of 21 dollars. Each had at least one Magyarian coin of each kind. In dollars, the total worth of the three kinds of Magyarian coins is

Solution. (**b**) Let the worth of the three kinds of coins be a, b and c respectively. From Matthew's collection, we have $2a + b + c = 28$. From Daniel's, we have either $a + 2b + 2c = 21$ or $a + b + 3c = 21$. In the former case, addition yields $3a + 3b + 3c = 49$, but 49 is not a multiple of 3. Subtracting $a + b + 3c = 21$ from $2a + b + c = 28$, we have $a - 2c = 7$.

If $c = 1$, then $a = 9$ and $b = 9$, which must be rejected. If $c \geq 3$, then $a \geq 13$ and $b < 0$. Hence $c = 2$, $a = 11$ and $b = 4$.

14. Colin wants a function f which satisfies $f(f(x)) = f(x + 2) - 3$ for all integers x. If he chooses $f(1)$ to be 4 and $f(4)$ to be 3, then he must choose $f(5)$ to be

Solution. **(d)** We have $3 = f(4) = f(f(1)) = f(3) - 3$ so that $f(3) = 6$. It follows that we have $6 = f(3) = f(f(4)) = f(6) - 3$ so that $f(6) = 9$. Hence $9 = f(6) = f(f(3)) = f(5) - 3$ so that $f(5) = 12$.

15. Lindsay summed all the integers from a to b, including a and b. She chose these numbers so that $1 \leq a \leq 10$ and $11 \leq b \leq 20$. This sum cannot be equal to

Solution. **(d)** The sum is $S = \frac{b(b+1)}{2} - \frac{a(a-1)}{2} = \frac{(b+a)(b-a+1)}{2}$. Note that $12 \leq a + b \leq 30$. If $S = 91$, then $2S = 2 \times 7 \times 13$. We may have $a + b = 14$, $a = 1$ and $b = 13$. If $S = 92$, then $2S = 2 \times 2 \times 2 \times 23$. We must have $a + b = 23$, $a = 8$ and $b = 15$. If $S = 95$, then $2S = 2 \times 5 \times 19$. We must have $a + b = 19$, $a = 5$ and $b = 14$. For $S = 99$, we can take $a = 4$ and $b = 14$. However, if $S = 98$, then $2S = 2 \times 2 \times 7 \times 7$. One of $a + b$ and $b - a + 1$ is odd and the other is even. Hence $2a - 1 = (b + a) - (b - a + 1) \geq 28 - 7 = 21$, and $a \geq 11$. This contradicts $a \leq 10$.

16. A set of points in the plane is such that each of the numbers 1, 2, 4, 8, 16 and 32 is a distance between two of the points in the set. The minimum number of points in this set is

Solution. **(b)** First, note that we may have one point joined to six other points by segments of lengths 1, 2, 4, 8, 16 and 32 respectively. Suppose the task can be accomplished with six points. We draw the segments one at a time. At any point during this process, two points are said to be in the same group if we can travel from one to the other along one or more segments already drawn. At the start, each point is in a group by itself. When a segment is drawn, it joins two points either in different groups or already in the same group. The latter is impossible since this means that some subset of the segments of lengths 1, 2, 4, 8, 16 and 32 forms a polygon. However, the longest of them will be longer than the total length of the others. It follows that whenever a segment is drawn, it always joins two points in different groups. However, after five segments have been drawn, the six points would have merged into a single group, and it is no longer possible to draw the last segment.

That completes the *Skoliad Corner* for this issue. Please send me contest suitable materials, as well as your suggestions and comments about the Corner.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to Mathematical Mayhem, Department of Mathematics, University of Toronto, 100 St. George St., Toronto, Ontario, Canada. M5S 3G3. The electronic address is still

mayhem@math.toronto.edu

The Assistant Mayhem Editor is Cyrus Hsia (University of Western Ontario). The rest of the staff consists of Adrian Chan (Harvard University), Jimmy Chui (University of Toronto), and David Savitt (Harvard University)

Shreds and Slices

A Calculus Proof of a Trigonometric Identity

Michael Lambrou points out an easy calculus proof of the identity $\sin^2 x + \cos^2 x = 1$, in addition to Grant's combinatorial proof [1999 : 413]. In his own words,

> *It is mentioned there that " ...* [such an approach] would make it a challenge even to prove that $\sin^2 x + \cos^2 x = 1$". *Here is a proof:*
>
> *Differentiating the left hand side we see that the derivative is zero, so it is constant. Put $x = 0$ to find its value.*
>
> (**Note**: *that the derivative of $\sin x$ is trivially $\cos x$ etc., is one of the big advantages of the power series approach*).

Mayhem Problems

The Mayhem Problems editors are:

Adrian Chan	*Mayhem High School Problems Editor,*
Donny Cheung	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only solutions — the next issue will feature only problems.

We warmly welcome proposals for problems and solutions. With the schedule of eight issues per year, we request that solutions from the previous issue be submitted in time for issue 2 of 2001.

High School Solutions

Editor: Adrian Chan, 1195 Harvard Yard Mail Center, Cambridge, MA, USA 02138-7501 <ahchan@fas.harvard.edu>

H249. For a certain positive composite integer x, when the fraction $(60 - x)/120$ is reduced to lowest terms, the sum of the numerator and denominator exceeds 120. Determine x.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

If $x \geq 60$, then $60 - x \leq 0$. So if $(60 - x)/120 = a/b$ (in lowest terms), then $a + b \leq 180 - x \leq 120$. Hence, $x \leq 60$.

Now, if $2|x$, then we have $x = 2y$ for some integer y, so that $(60 - x)/120 = (30 - y)/60$, which implies that $a + b \leq 90 - y < 120$.

If $3|x$, then $x = 3y$ for some integer y, so $(60 - x)/120 = (20 - y)/40$, which implies that $a + b \leq 60 - y < 120$.

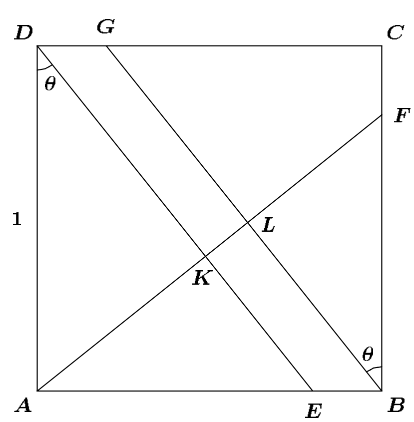
Finally, if $5|x$, then $x = 5y$ for some integer y, so $(60 - x)/120 = (12 - y)/24$, which implies that $a + b \leq 36 - y < 120$. We conclude that x is not divisible by 2, 3, or 5. The only such positive composite integer less than 60 is 49. We check $x = 49$: $(60 - x)/120 = 11/120$, and $11 + 120 > 120$. Therefore, the answer is $x = 49$.

H250. Let $ABCD$ be a unit square, and let E, F, G, and H be points on AB, BC, CD, and DA respectively such that $AE = BF = CG = DH = 1999/2000$. Construct the triangles AGB, BHC, CED, and DFA, and let S be the area of the region that is common to all four triangles. Show that

$$S = \frac{1}{1999^2 + 2000^2}.$$

Solution by D.J. Smeenk, Zaltbommel, the Netherlands.

Let us look at the following diagram: [we do not need H]



It is clear that $DE \parallel GB$ and $AF \perp DE$. Let $\theta = \angle ADE$. Then $\angle CBG = \theta$, so $DE = GB = AF = 1/\cos\theta$, and $AE = BF = \tan\theta$. Also, $AK = \sin\theta$, which implies that $LF = \sin^2\theta/\cos\theta$.

Now,

$$\begin{aligned} KL &= AF - AK - LF \\ &= \frac{1}{\cos\theta} - \sin\theta - \frac{\sin^2\theta}{\cos\theta} \\ &= \frac{1-\sin^2\theta}{\cos\theta} - \sin\theta \\ &= \cos\theta - \sin\theta\,. \end{aligned}$$

So, the square with side length KL has area

$$(\cos\theta - \sin\theta)^2 \;=\; 1 - \sin(2\theta) \;=\; 1 - \frac{2\tan\theta}{1+\tan^2\theta} \;=\; \frac{(1-\tan\theta)^2}{1+\tan^2\theta}\,.$$

Now, since $\tan\theta = 1999/2000$, the area is $1/(1999^2 + 2000^2)$ as required.

H251. We say that an arithmetic sequence is *astonishing* if it satisfies the following conditions:

(a) Every term in the sequence is an integer.

(b) No term in the sequence is greater than 10000.

(c) There are at least three terms in the sequence.

(d) The sum of the terms is 1999.

For example, the arithmetic sequence $-998, -997, \ldots, 999, 1000$ is astonishing. How many astonishing arithmetic sequences are there?

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Assume that we have an astonishing arithmetic sequence with first term a, common difference d, and n terms. Without loss of generality, we can assume that $d \geq 0$, since if $d < 0$, then we can "reverse" the sequence so that $d \geq 0$. Then by part (d),

$$\frac{n}{2}\left[2a + (n-1)d\right] \;=\; 1999\,.$$

Notice that 1999 is prime.

Case 1: n is odd.

If n is odd, then n divides 1999, so $n = 1999$ or $n = 1$ (rejected since $n \geq 3$). So, $a + 999d = 1$. Now, by part (b), $a + 1998d \leq 10000$, so $999d \leq 9999$, which implies that $d \leq 10$.

If $d = 0$, then $a = 1$ and our sequence is $\{1, 1, \dots, 1\}$.

If $d \neq 0$, then there exists a unique value for a. But since we can "reverse" the sequences, this gives a total of 20 sequences (since $d \le 10$).

Case 2: n is even.

If n is even, then $n/2$ divides 1999, so $n = 3998$ or $n = 2$ (rejected since $n \ge 3$). So, $2a + 3997d = 1$, and by part (b), $a + 3997d \le 10000$, so $3997d \le 19999$, which implies that $d \le 5$. But since $2a + 3997d = 1$, d must be odd, so d must be 1, 3, or 5. Again, each value of d yields a unique value for a, and by taking negative values of d, we obtain a total of 6 sequences.

Hence, there is a total of 27 astonishing sequences.

H252. Find a solution (a, b) in *rational* numbers to the following system:

$$\begin{aligned} 9a^2 + 16b^2 &= 25\,, \\ a^2 + b^2 &< \frac{25}{16} + \frac{1}{10}\,. \end{aligned}$$

(Query: Can you determine an infinite set of rational solutions to this system?)

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let $a = 5x/3$ and $b = 5y/4$, where x and y are rational. Then

$$\begin{aligned} x^2 + y^2 &= 1\,, \\ \frac{x^2}{9} + \frac{y^2}{16} &< \frac{1}{16} + \frac{1}{250}\,. \end{aligned}$$

Clearly, $(x, y) = (0, \pm 1)$ are solutions, or $(a, b) = (0, \pm 5/4)$.

Now let

$$x = \frac{2n}{n^2+1}\,, \quad y = \frac{n^2-1}{n^2+1}\,,$$

where n is an integer. Since $x \to 0$ as $n \to \infty$, there exists a K such that for all $n > K$, $x < \sqrt{72/875}$. Then,

$$x^2 + y^2 = \left(\frac{2n}{n^2+1}\right)^2 + \left(\frac{n^2-1}{n^2+1}\right)^2 = 1\,,$$

and

$$\frac{x^2}{9} + \frac{y^2}{16} = \frac{1}{16} + \frac{7x^2}{144} < \frac{1}{16} + \frac{1}{250}\,.$$

This generates an infinite number of rational solutions.

Also solved by Luyun Zhong-Qiao, Columbia International College, Hamilton, Ontario.

Advanced Problems

Editor: Donny Cheung, c/o Conrad Grebel College, University of Waterloo, Waterloo, Ontario, Canada. N2L 3G6 <dccheung@uwaterloo.ca>

A225. In an acute angled triangle, ABC, label its orthocentre H and its circumcentre O. Line BO is extended to meet the circumcircle at D. Show that $ADCH$ is a parallelogram.

Solution by Luyun Zhong-Qiao, Columbia International College, Hamilton, Ontario.

Extend AH and CH to meet BC and AB at E and F, respectively. Since BD is a diameter of cyclic quadrilateral $ABCD$, $BC \perp CD$. And since $BC \perp AE$, we have $AE \| CD$. Similarly, $DA \parallel CF$, so $ADCH$ is indeed a parallelogram.

Also solved by Andrei Simion, Brooklyn Technical HS, NY, USA.

A226. *Proposed by Naoki Sato.*

Let n be a positive integer. A $2 \times n$ array is filled with the entries 1, 2, $\ldots$, $2n$, using each exactly once, such that the entries increase reading left to right in each row, and top to bottom in each column.

For example, for $n = 5$, we could have the following array:

1	2	4	5	9
3	6	7	8	10

Find the number of possible such arrays in terms of n.

Solution by Miguel Carrión Álvarez, Universidad Complutense de Madrid, Spain.

This problem furnishes a nice application of the reflection principle. We will prove that the problem is equivalent to counting the number of paths joining two points on a directed lattice and satisfying a restriction, and then count the paths with the aid of the *reflection principle*.

Call a (possibly incomplete) arrangement of numbers in the $2 \times n$ array "admissible" if each number is greater than all those to its left or above it. Now consider constructing a complete admissible arrangement by adding the numbers 1, 2, $\ldots$, $2n$ one by one to the empty array in increasing order. If the numbers 1, 2, $\ldots$, m have been admissibly arranged, then there can be no empty cells above or to the left of any occupied cells, for in that case putting any number from $m + 1$, $\ldots$, $2n$ in the empty cells would make the arrangement inadmissible. Therefore, at each step the lowest unassigned number can only be assigned to the leftmost empty cell in either row, or only in the top row if both rows contain the same number of assigned cells. In the following examples, ? denotes one of 1, 2, 3, 4 (admissibly arranged), and $*$ a possible assignment for number 5:

?	?	?	*	
?	*			

?	?	?	*	
?	*			

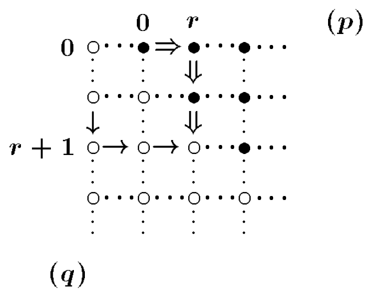
The process of adding one number at a time in the leftmost empty cell of a row can be represented as a directed lattice, in which $\rightarrow$ denotes adding a number to the top row and $\downarrow$ denotes adding a number to the bottom row, $\bullet$ denotes an admissible arrangement, and $\circ$, an inadmissible arrangement:

$$\begin{array}{ccccccccccc}
 & 0 & & 1 & & 2 & & 3 & & & (p) \\
0 & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \cdots & & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \\
1 & \circ & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \cdots & & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \\
2 & \circ & \rightarrow & \circ & \rightarrow & \bullet & \rightarrow & \bullet & \cdots & & \\
 & \vdots & & \vdots & & \vdots & & \vdots & & & \\
(q) & & & & & & & & & &
\end{array}$$

The dot at (p, q) represents an array in which the numbers $1, 2, \ldots, p+q$ have been assigned to the p leftmost cells of the top row and the q leftmost cells of the bottom row. Each possible arrangement of numbers is represented uniquely by a path joining $(0, 0)$ and (p, q). The arrangement is admissible if, and only if, the corresponding path does not pass any white dots. The problem is therefore equivalent to counting the number of admissible paths joining $(0, 0)$ with (n, n). It turns out that inadmissible paths are more easily counted, so we will evaluate this by subtracting the number of inadmissible paths from the total number of paths.

A path (admissible or inadmissible) joining $(0, 0)$ and (p, q) consists of p horizontal and q vertical links, in any order, so there are $\binom{p+q}{p} = \binom{p+q}{q}$ such paths. The evaluation of the number of inadmissible paths is more involved, and requires the use of the reflection principle.

There is a one-to-one correspondence between inadmissible paths from $(0, 0)$ to (p, q) and all paths from $(-1, 1)$ to (p, q), because $(-1, 1)$ is the result of "reflecting" $(0, 0)$ about the $(r, r+1)$ diagonal. To see this, consider the following argument. An inadmissible path from $(0, 0)$ to (p, q) begins and ends at black dots, but it must have at least one white intermediate dot. The first white dot in the path lies necessarily immediately below the (r, r) main diagonal, so it is of the form $(r, r + 1)$. Reflecting the section of the path from $(0, 0)$ to $(r, r + 1)$ about the $(r, r + 1)$ diagonal, we obtain a path from $(-1, 1)$ to (p, q) passing through $(r, r + 1)$. Similarly, a path from $(-1, 1)$ to (p, q) can be transformed into an inadmissible path from $(0, 0)$ to (p, q).



The number of paths from $(-1, 1)$ to (p, q) is equal to the number of paths from $(0, 0)$ to $(p+1, q-1)$; that is, $\binom{p+q}{p+1} = \binom{p+q}{q-1}$. As we have argued, this equals the number of inadmissible paths from $(0, 0)$ to (p, q). Therefore, the number of admissible paths from $(0, 0)$ to (p, q) is $\binom{p+q}{p} - \binom{p+q}{p-1}$.

The number of complete admissible arrangements in a $2 \times n$ array is equal to the number of admissible paths from $(0, 0)$ to (n, n), or

$$\binom{2n}{n} - \binom{2n}{n-1} = \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n+1)!(n-1)!} = \frac{(2n)!}{n!(n+1)!}\,.$$

[Ed. This is the n^{th} Catalan number. Does this suggest another approach?]

A227. *Proposed by Mohammed Aassila, Strasbourg, France.*

Let f, $g : \mathbb{N} \to \mathbb{N}$ be functions such that f is surjective, g is injective, and $f(n) \geq g(n)$ for all $n \in \mathbb{N}$. Prove that $f = g$.

Solution by Miguel Carrión Álvarez, Universidad Complutense de Madrid, Spain.

For all $k \in \mathbb{N}$, let $m_k = \min\{f^{-1}(k)\}$. This always exists because $\{f^{-1}(k)\}$ is a non-empty set, since f is surjective, and every non-empty subset of $\mathbb{N}$ has a minimal element. By definition, $f(m_k) = k$. We now prove by induction that $g(m_k) = k$ as well.

The case $k = 1$ is trivially true, since $g(m_1) \leq f(m_1) = 1$ implies $g(m_1) = 1$.

Assume now that $g(m_k) = k$ for all $k < K$. By hypothesis $g(m_K) \leq f(m_K) = K$, but since g is injective and $g(m_k) = k$ for all $k < K$, the only possibility is $g(m_K) = K$, as required.

Also solved by Lee Ho-Joo, Kwangwoon University, Seoul, South Korea.

A228. Given a sequence $a_1, a_2, a_3, \ldots$ of positive integers in which every positive integer occurs exactly once, prove that there exist integers $k < \ell < m$, such that $a_k + a_m = 2a_\ell$. (1997 Baltic Way)

Solution.

Let $k = 1$. Let ℓ be the smallest positive integer such that $a_\ell > a_k$. Then $2a_\ell - a_k$ is a positive integer, so there is a unique m so that $a_m = 2a_\ell - a_k$. Then $a_m > a_\ell$, so if $m < \ell$, then we would contradict the definition of ℓ. Clearly, we cannot have $k = \ell$ or $\ell = m$, so we have $1 = k < \ell < m$ with $a_k + a_m = 2a_\ell$.

Challenge Board Solutions

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

C83. *Proposed by Dima Arinkin, graduate student, Harvard University.*

Let f be a function from the plane $\mathbb{R}^2$ to the reals. Given a polygon $\mathcal{P}$ in the plane, let $f(\mathcal{P})$ denote the sum of the values of f at each of the vertices of $\mathcal{P}$. Suppose there exists a convex polygon $\mathcal{Q}$ in the plane such that for every polygon $\mathcal{P}$ similar to $\mathcal{Q}$, we find that $f(\mathcal{P}) = 0$. Show that f is identically zero.

Solution by the proposer.

Let $v_1, \ldots, v_k$ denote the vertices of $\mathcal{Q}$. Given any $x \in \mathbb{R}^2$ and integer n, the polygon with vertices $x + n \cdot v_1, \ldots, x + n \cdot v_k$ is similar to $\mathcal{Q}$, and so by hypothesis we have

$$\sum_{i=1}^{k} f(x + n \cdot v_i) = 0 .$$

Let T_i be the translation-by-v_i operator on functions on the plane; that is, given a function g on the plane, we have $(T_i g)(x) = g(x + v_i)$. Another way of writing that $\sum_{i=1}^{k} f(x + n \cdot v_i) = 0$, then, is to say that the function

$$\sum_{i=1}^{k} T_i^n f$$

is identically zero. Thus, f is in the kernel of each of the operators $P_n = \sum_{i=1}^{k} T_i^n$. (Recall that the kernel of an operator is just another name for the things which are mapped to zero by the operator.)

Define $S_1 = T_1 + \cdots + T_k$. Next, set $S_2 = T_1 \cdot T_2 + T_1 \cdot T_3 + \cdots + T_{k-1} \cdot T_k$, so that S_2 is the sum of the composition of pairs of distinct T_is. (Note that the T_i's all commute with one another, so this definition makes sense.) Continuing, for $n \leq k$, let S_n be the operator which is the sum of the composition of all n–tuples of distinct T_is; in other words,

$$S_n = T_1 \cdots T_{n-1} \cdot T_n + T_1 \cdots T_{n-1} \cdot T_{n+1} + \cdots + T_{k-n+1} \cdots T_k ,$$

and, in particular,

$$S_k = T_1 \cdots T_k .$$

It is a standard result that the S_n's may all be written as polynomials in terms of the P_n's. For example, $S_1 = P_1$, and $2S_2 = P_1^2 - P_2$. To prove this result, for example, observe that the result follows immediately by induction from the identity

$$(n+1)S_{n+1} = P_1 S_n - P_2 S_{n-1} + P_3 S_{n-2} + \cdots + (-1)^n P_{n+1} S_0 .$$

(Here, S_0 denotes the identity operator.) The identity follows by careful bookkeeping after substituting in the definitions, in terms of the T_i's, for the P_i's and S_i's.

Since our function f is in the kernel of each P_n, it is in the kernel of any polynomial in the P_n's, and specifically f is mapped to 0 by the S_n's. In particular, for any x,

$$0 = (S_k f)(x) = (T_1 \cdots T_k f)(x) = f(x + v_1 + \cdots + v_k) .$$

It follows immediately that f is identically zero.

C84. *Proposed by Christopher Long, graduate student, Rutgers University.*

Let $A(x) = \sum_{m=0}^{\infty} a_m x^m$ be a formal power series, with each a_m either 0 or 1 and with infinitely many of the a_m non-zero. Give a necessary and sufficient condition on the a_m for there to exist a formal power series $B(x) = \sum_{n=0}^{\infty} b_n x^{-n}$ with each b_n either 0 or 1, with infinitely many of the b_n non-zero, and such that the formal product $A(x)B(x)$ exists.

Solution by the proposer.

Let m_i be the i^{th} value of m such that $a_m = 1$. We claim that a necessary and sufficient condition for the existence of such a series is that $\limsup_i (m_{i+1} - m_i) = \infty$.

First, assume $\limsup_i (m_{i+1} - m_i) = \infty$, and let $n_0 = 0$. Since $\limsup_i (m_{i+1} - m_i) = \infty$, for all $k \geq 1$ there exists i_k such that $i_k > i_{k-1}$ and $m_{i_k+1} - m_{i_k} \geq 2k$. Define $n_k = m_{i_k} + k$; we claim that $B(x) = \sum_{n=0}^{\infty} x^{-n_k}$ is then a series of the desired form. This follows since only finitely many of the products $A(x)x^{-n_k}$ can yield x^i for any fixed i. Indeed, by construction $A(x)x^{-n_k}$ contains only x^j with $|j| \geq k$.

Next, if $\limsup_i (m_{i+1} - m_i) = N < \infty$, then for every sufficiently large n there would exist some m_i such that $|m_i - n| \leq N$, whence by the Pigeonhole Principle necessarily some x^k with $|k| \leq N$ would be represented by an infinite sum in the product $A(x)B(x)$ for any such $B(x)$.

Problem of the Month

Jimmy Chui, student, University of Toronto

Problem. Let $a_1, a_2, \ldots$ be a sequence of real numbers satisfying $a_{i+j} \leq a_i + a_j$ for all $i, j = 1, 2, 3, \ldots$. Prove that

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n} \;\geq\; a_n$$

for each positive integer n.

(1999 APMO, Problem 2)

Solution 1.

Lemma 1. For each positive integer $n \geq 2$,

$$a_n \;\leq\; \frac{2}{n-1} \sum_{i=1}^{n-1} a_i \,.$$

Proof. From the given condition,

$$\begin{aligned} a_n &\leq a_1 + a_{n-1} \,, \\ a_n &\leq a_2 + a_{n-2} \,, \\ &\cdots \,, \\ a_n &\leq a_{n-1} + a_1 \,. \end{aligned}$$

Summing these inequalities, we get

$$\begin{aligned} (n-1)a_n &\leq 2(a_1 + a_2 + \cdots + a_{n-1}) \;=\; 2\sum_{i=1}^{n-1} a_i \\ \Longrightarrow\; a_n &\leq \frac{2}{n-1} \sum_{i=1}^{n-1} a_i \,. \end{aligned}$$

Lemma 2. For each positive integer $n \geq 2$,

$$\frac{2}{n+1} \sum_{i=1}^{n} a_i \;\leq\; \frac{1}{n}\, a_n + \frac{2}{n} \sum_{i=1}^{n-1} a_i \,.$$

Proof. We have that

$$\begin{aligned}
\frac{2}{n+1}\sum_{i=1}^{n} a_i &= \frac{2}{n+1}\,a_n + \frac{2}{n+1}\sum_{i=1}^{n-1} a_i \\
&= \frac{1}{n}\,a_n + \frac{n-1}{n(n+1)}\,a_n + \frac{2}{n+1}\sum_{i=1}^{n-1} a_i \\
&\le \frac{1}{n}\,a_n + \frac{n-1}{n(n+1)}\cdot\frac{2}{n-1}\sum_{i=1}^{n-1} a_i + \frac{2}{n+1}\sum_{i=1}^{n-1} a_i \\
&\qquad\qquad\qquad\qquad \text{(By Lemma 1)} \\
&= \frac{1}{n}\,a_n + \frac{2}{n(n+1)}\sum_{i=1}^{n-1} a_i + \frac{2}{n+1}\sum_{i=1}^{n-1} a_i \\
&= \frac{1}{n}\,a_n + \frac{2}{n}\sum_{i=1}^{n-1} a_i\,.
\end{aligned}$$

Now,

$$a_n = \frac{1}{n}\,a_n + \frac{n-1}{n}\,a_n \le \frac{1}{n}\,a_n + \frac{2}{n}\sum_{i=1}^{n-1} a_i$$

by Lemma 1. With repeated application of Lemma 2, we obtain

$$\begin{aligned}
a_n &\le \frac{1}{n}\,a_n + \frac{1}{n-1}\,a_{n-1} + \frac{2}{n-1}\sum_{i=1}^{n-2} a_i \\
&\le \frac{1}{n}\,a_n + \frac{1}{n-1}\,a_{n-1} + \frac{1}{n-2}\,a_{n-2} + \frac{2}{n-2}\sum_{i=1}^{n-3} a_i \\
&\le \cdots \\
&\le \frac{1}{n}\,a_n + \frac{1}{n-1}\,a_{n-1} + \frac{1}{n-2}\,a_{n-2} + \cdots + \frac{2}{3}\sum_{i=1}^{2} a_i \\
&\le \frac{1}{n}\,a_n + \frac{1}{n-1}\,a_{n-1} + \frac{1}{n-2}\,a_{n-2} + \cdots + \frac{1}{2}\,a_2 + \frac{2}{2}\sum_{i=1}^{1} a_i \\
&\le \frac{1}{n}\,a_n + \frac{1}{n-1}\,a_{n-1} + \frac{1}{n-2}\,a_{n-2} + \cdots + \frac{1}{2}\,a_2 + a_1\,.
\end{aligned}$$

■

Solution 2. We proceed by strong mathematical induction. The result is clearly true for $n = 1$. Assume that it is true for $n = 1, 2, \ldots, k-1$ for some positive integer k. In other words, let

$$b_i \;=\; a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_i}{i}\,;$$

then $b_i \geq a_i$ for $i = 1, 2, \ldots, k-1$. Summing,

$$\begin{aligned} & b_1 + b_2 + b_3 + \cdots + b_{k-1} \\ = &\; (k-1)a_1 + \frac{k-2}{2}\,a_2 + \frac{k-3}{3}\,a_3 + \cdots + \frac{k-(k-1)}{k-1}\,a_{k-1} \\ \geq &\; a_1 + a_2 + a_3 + \cdots + a_{k-1}\,. \end{aligned}$$

Adding $a_1 + a_2 + a_3 + \cdots + a_k$ to both sides, we obtain

$$\begin{aligned} & k\left(a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_k}{k}\right) \\ \geq &\; 2(a_1 + a_2 + a_3 + \cdots + a_{k-1}) + a_k \\ \geq &\; k a_k \end{aligned}$$

by Lemma 1 of solution 1 above. Hence, the result is true for $n = k$, and by strong induction, for all n.

This question is a very difficult one. In fact, all the questions on the 1999 APMO were very challenging. With a problem such as this, if an approach does not work, try another! One approach to this question is to find a relationship between a_n and the other a_is, and hopefully/hopelessly grind that result into what we want to prove. Too bad I never got this question during the contest!

In Volume 25, Issue 6, three solutions are given to the Problem of the Month [1999 : 358]. Gordon Hookings of the University of Auckland points out that the problem also follows from Ptolemy's Theorem.

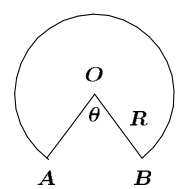
J.I.R. McKnight Problems Contest 1994

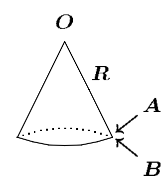
1. In a rectangular coordinate system we have $A(4,4)$, $B(3,5)$, and $C(1,1)$. Let $[X]$ denote the area of X. Determine the coordinates of point D such that

$$[DAB] : [DBC] : [DCA] \;=\; 1 : 2 : 3\,.$$

2. You are given a regular tetrahedron $ABCD$ with centroid G. Find the angle AGB to the nearest degree.

3. A sector containing an angle θ is cut off from a circle with radius R. The remaining part is then folded to form a circular cone. Find the angle θ such that the cone thus formed has the largest volume, and find the volume in terms of R.





4. Solve for x: $0 \le x \le \pi/2$,

$$8 \sin x \cos x - \sqrt{6} \sin x - \sqrt{6} \cos x + 1 = 0 .$$

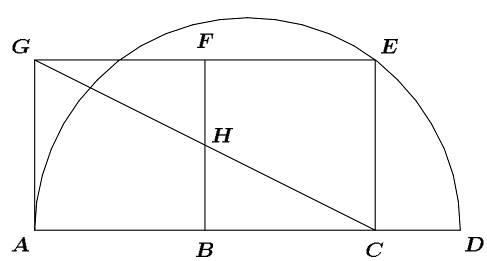
5. John, Larry, Peter, Terry and Vince each brought a gift to the Christmas party. These packages are placed in 5 identical boxes numbered randomly from 1 to 5. Each person will then draw one of the numbers from 1 to 5 from a hat and then pick up the corresponding box. What is the probability that ...

   (a) everybody will get back one's own gift?
   (b) 4 people will get back their own gifts?
   (c) 3 people will get back their own gifts?
   (d) 2 people will get back their own gifts?
   (e) one person will get back his own gift?
   (f) everybody will get a gift brought in by another person?

6. For all positive integers n, prove that

$$\sum_{r=1}^{n} r \binom{n}{r}^2 = \frac{(2n-1)!}{(n-1)!^2} .$$

7. A semicircle is drawn with diameter AD. Point E is chosen on the semicircle and rectangle $ACEG$ drawn along with square $ABFG$. Now CG is drawn cutting BF at H. Prove that $FH = CD$.



8. A point P is chosen inside equilateral triangle ABC such that $PA = 5$, $PB = 4$, and $PC = 3$. Calculate AB.

Dividing Points Equally, II

Cyrus C. Hsia
student, University of Toronto

In part I [1998 : 499], we started off with a simple problem of dividing $2n$ points in the plane with a straight line, and followed it through with some related problems that can be reduced to that problem. We looked at similar problems in three dimensional space and on the surface of a sphere. Because of its usefulness in the problems discussed and in the ones that follow, we will refer to it as the "Dividing Points Theorem".

Dividing Points Theorem
For every set of distinct $2n$ points in the plane, there exists a straight line so that there are an equal number of points on either side of the line.

For a solution, see Part I. Now we take another look at problems that reduce to this Dividing Points Theorem. For the reader's enjoyment, this article is about points with colours, so all of the following problems will deal with just that. The first problem comes from Mohammed Aassila that is very similar to a problem proposed by the Netherlands for the 1988 IMO.

Problem 1. For each set of $2n$ blue points and $2n$ red points in the plane, is it possible to divide the plane into two halves by a straight line, so that there are n blue points and n red points on either side? We are assuming that no three points are collinear.

Problem 1A. We are given 1988 points in the plane, no four of which are collinear. The points of a subset with 1788 points are coloured blue, and the remaining 200 are coloured red. Prove that there exists a line in the plane, such that each of the two parts into which the line divides the plane contains 894 blue points and 100 red points.

(1988 IMO Proposal by the Netherlands)

Here is a problem that immediately falls apart by using the Dividing Points Theorem, although at first glance, it seems unrelated.

Problem 2. We are given a set of $2n$ points in the plane. Exactly n points are coloured red, and the remaining n points are coloured blue. Is it always possible to draw n line segments, each connecting 2 different coloured points so that we may draw a straight line in the plane that intersects all these n line segments?

Solution. The answer is yes. By the Dividing Points Theorem, it is possible to draw a straight line that divides the $2n$ points in half. Now consider one side of the line. Suppose it has k points that are red. The remaining $n - k$ points on that side must be blue. But there are n red points and n blue points in total, so on the other side of the line, there must be $n - k$ red points and k blue points. This is just what the doctor ordered.

Connecting each of the k red points on the first side with one of the k blue points on the other side means that each of these line segments intersects the dividing straight line. Likewise, the $n - k$ blue points on the first side can be paired with the $n - k$ red points on the other side with given line segments that intersect the dividing straight line. Thus, the dividing straight line from the Dividing Points Theorem is the desired one, and we have also found our n line segments.

Problem 2A. Of course, we need not consider just two different colours and line segments. Consider 3 (or m resp.) colours with n points of each colour in the plane. A triangle (m–gon resp.) can be formed with 3 (m resp.) different colours. Is it possible to draw n triangles (m–gons resp.) with these points so that we may be able to draw a straight line that passes through all of these n triangles (m–gons resp.)?

Problem 3. A finite number of points are given in the plane, no three of which are collinear. Prove that it is possible to colour the points with two colours, so that it is impossible to divide the plane into two half-planes by a line, with exactly three points of the same colour on one side of the line.

(1992/3 Kürschák Competition)

Exercises

1. Send in your nice solutions to the problems above.

2. In Problem 1, show why it is necessary to have the condition, "no three points are collinear." That is, give an example where the result fails if we have collinear points.

3. Can problem 1 be extended to three or more points?

4. There are $9n$ smarties sprinkled on the top of a cake. There are $3n$ smarties of each of three colours, red, blue, and yellow. Can the cake necessarily be sliced into 3 sectors with n smarties of each colour on each sector?

More Exercises Without Colours

1. For a set of $2n$ points in the plane, is it always possible to draw a circle in the plane so that an equal number of points are inside and outside the circle?

2. Write a computer algorithm to find a straight line that divides $2n$ points into two equal halves, given their coordinates.

3. For $2n$ random line segments in the plane, is it always possible to draw a straight line that divides the plane into two halves, with an equal number of line segments on both sides?

Acknowledgements

Thanks to Mohammed Aassila for introducing me to these types of problems with different coloured points.

21 Van Allan Road
Scarborough
Ontario
M1G 1C3
hsia@math.toronto.edu

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}$"×11" or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 October 2000**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in LaTeX). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

The Editors would like to thank Waldemar Pompe for drawing to our attention that a venerable ***CRUX with MAYHEM*** problem proposer and solver will be celebrating his 90$^{\text{th}}$ birthday this month. So, we have seven Geometry problems in honour of Professor Toshio Seimiya's birthday. Happy birthday, and congratulations!!

2513. *Proposed by Waldemar Pompe, University of Darmstadt, Darmstadt, Germany; dedicated to Prof. Toshio Seimiya on his 90$^{\text{th}}$ birthday.*

A circle is tangent to the sides BC, AD of convex quadrilateral $ABCD$ in points C, D, respectively. The same circle intersects the side AB in points K and L. The lines AC and BD meet in P. Let M be the mid-point of CD. Prove that if $CL = DL$, then the points K, P, M are collinear.

2514. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

In $\triangle ABC$, the internal bisectors of $\angle ABC$ and $\angle BCA$ meet CA and AB at D and E respectively. Suppose that $AE = BD$ and that $AD = CE$. Characterize $\triangle ABC$.

2515. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

In $\triangle ABC$, the internal bisectors of $\angle BAC$, $\angle ABC$ and $\angle BCA$ meet BC, AC and AB at D, E and F respectively. Let p be the perimeter of $\triangle ABC$. Suppose that $AF + BD + CE = \frac{1}{2}p$. Characterize $\triangle ABC$.

2516. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

In isosceles $\triangle ABC$ (with $AB = AC$), let D and E be points on sides AB and AC respectively such that $AD < AE$. Suppose that BE and CD meet at P. Prove that $AE + EP < AD + DP$.

2517. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Suppose that D, E, F are the mid-points of the sides BC, CA, AB, respectively, of $\triangle ABC$. Let P be any point in the plane of the triangle, distinct from A, B and C.

1. Show that the lines parallel to AP, BP, CP, through D, E, F, respectively, are concurrent (at Q, say).
2. If X, Y, Z are the symmetrics of P with respect to D, E, F, respectively, show that AX, BY, CZ are concurrent at Q.

2518. *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

If P is a point on the altitude AN of $\triangle ABC$, if $\angle PBA = 20°$, if $\angle PBC = 40°$ and if $\angle PCB = 30°$, without using trigonometry, find $\angle PCA$.

2519. *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

In $\triangle ABC$, $\angle ACB = 40°$, $AB \perp BC$, P and Q are points on AB and BC respectively with $\angle PQB = 20°$. Without using trigonometry, prove that $AQ = 2BQ$ if and only if $PQ = CQ$.

2520. *Proposed by Paul Bracken, CRM, Université de Montréal, Montréal, Québec.*

Let a, b be real numbers such that $b \leq 0$ and $1 + ax + bx^2 \geq 0$ for every $x \in [0,1]$. Define

$$F_n(a,b) \;=\; \int_0^1 \left(a + ax + bx^2\right)^n \, dx\,.$$

Show that the following asymptotic expressions are valid for $F_n(a,b)$ as $n \to \infty$:

1. If $a < 0$ and $b \leq 0$, then
$$F_n(a,b) \;=\; -\frac{1}{an} + \frac{1}{n^2 a}\left(1 - \frac{2b}{a^2}\right) + \mathrm{O}\left(n^{-3}\right)\,.$$

2. If $a \geq 0$ and $b < 0$, then
$$F_n(a,b) \;\sim\; \sqrt{\frac{\pi}{n|b|}}\left(1 - \frac{a^2}{4|b|}\right)^{n+\frac{1}{2}}\,.$$

2521. *Proposed by Eric Postpschil, Nashua, New Hampshire, USA.*

Given a permutation τ, determine all pairs of permutations α and β, such that $\tau = \beta \circ \alpha$ and $\alpha^2 = \beta^2 = \imath$ (the identity permutation). That is, determine all factorizations of τ into two permutations, each composed of **disjoint** transpositions.

2522*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Suppose that a, b and c are positive real numbers. Prove that

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}\right) \;\geq\; \frac{9}{1+abc}\,.$$

2523. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Prove that, if $t \geq 1$, then

$$\ln t \;\leq\; \frac{t-1}{2(t+1)}\left(1+\sqrt{\frac{2t^2+5t+2}{t}}\right)\,.$$

Also, prove that, if $0 < t \leq 1$, then

$$\ln t \;\geq\; \frac{t-1}{2(t+1)}\left(1+\sqrt{\frac{2t^2+5t+2}{t}}\right)\,.$$

2524. *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

What conditions must the real numbers x, y and z satisfy so that

$$\cot x \cot y \cot z \;=\; \cot x + \cot y + \cot z\,,$$

where x, y, $z \neq n\pi$, with n being an integer?

2525. *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

Consider the recursions: $x_{n+1} = 2x_n + 3y_n$, $y_{n+1} = x_n + 2y_n$, with $x_1 = 2$, $y_1 = 1$. Show that, for each integer $n \geq 1$, there is a positive integer K_n such that

$$x_{2n+1} \;=\; 2\left(K_n^2 + (K_n+1)^2\right)\,.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

The name of Nikolaos Dergiades, Thessaloniki, Greece was inadvertantly omitted from the list of solvers of **2404** in the last issue.

2413. [1999: 50] *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

A deck of six cards consists of three black cards and three red cards, in some order. The top four cards are picked up, shuffled randomly, and then put on the *bottom* of the deck. This procedure is repeated n times.

Let $p(n)$ be the probability that after n such "shuffles", the deck alternates between red and black cards, either colour being the top card. (So $p(n)$ will, in general, depend on the initial ordering of the deck.)

Find $\lim\limits_{n\to\infty} p(n)$ — that is, the "long term" probability that the deck "tends to be alternating".

Solution by Kathleen E. Lewis, SUNY Oswego, Oswego, NY, USA.

If the bottom two cards at the n^{th} stage are the same colour, then the cards cannot be alternating at the $(n+1)^{\text{st}}$ stage. So we will first find the limit of d_n, the probability that the bottom two cards at the n^{th} stage are of different colours. This probability satisfies the recurrence relation

$$d_{n+1} = \tfrac{2}{3}d_n + \tfrac{1}{2}(1-d_n)\,.$$

[*Editorial remark*. Here is a little explanation. If the bottom two cards at the n^{th} stage are different colours, then there are $\binom{4}{2} = 6$ ways to arrange the other four cards on the bottom of the deck at the next stage, of which 4 ways have the bottom two cards coloured differently; whereas if the bottom two cards at the n^{th} stage are the same colour, then the corresponding numbers are $\binom{4}{1} = 4$ and 2.]

Letting $d = \lim_{n\to\infty} d_n$, we get the equation $d = \tfrac{2}{3}d + \tfrac{1}{2}(1-d)$.

Therefore $d = \tfrac{3}{5}$. Since $p(n+1) = \tfrac{d_n}{6}$, this shows that $\lim\limits_{n\to\infty} p(n) = \tfrac{1}{10}$.

Also solved by KEITH EKBLAW, Walla Walla, WA, USA; and GERRY LEVERSHA, St. Paul's School, London, England.

The proposer's solution was more complicated, and apparently incorrect! He obtained the answer

$$\lim_{n\to\infty} p_n = \frac{\sqrt{5}-2}{2} \approx 0.118034\,.$$

He follows this with the comment: "Note that the probability that a randomly chosen permutation of three 0*'s and three* 1*'s is alternating is*

$$\frac{2}{\binom{6}{3}} = \frac{1}{10}\,,$$

which is less than this. The reason for the difference is that permutations ending in 10 *or* 01 *are more likely than those ending* 00 *or* 11*, and also produce more such permutations when the deck is "shuffled" as in the problem, so an alternating string is more likely to occur than would happen by chance." In light of this comment, the proposer still wonders why the answer inconveniently comes out to be* $\frac{1}{10}$.

2414. [1999: 110] *Proposed by Wu Wei Chao, Guang Zhou Normal University, Guang Zhou City, Guang Dong Province, China, and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

For $1 < x \le e \le y$ or $e \le x < y$, prove that $x^x y^{x^y} > x^{y^x} y^x$.

Solution by Kee-Wai Lau, Hong Kong, China.

For $1 < x < e \le y$ or $e \le x < y$, let

$$f(x,y) = x\ln x + x^y \ln y - y^x \ln x - x \ln y$$

and

$$g(x,y) = (y-1)\ln x + \ln\ln y - (x-1)\ln y\,.$$

To prove the inequality, we need only to show that $f(x,y) > 0$.

For $y \ge e$, let $h(y) = y - 1 + \ln\ln y - (e-1)\ln y$. Since $h(e) = 0$ and

$$\frac{dh}{dy} = 1 - \frac{1}{y\ln y} - \frac{e-1}{y} \ge 1 - \frac{1}{e} - \frac{e-1}{e} = 0\,,$$

it follows that $h(y) \ge 0$.

Note that $\dfrac{\partial^2 g}{\partial x^2} = \dfrac{1-y}{x^2} < 0$. Since

$$\lim_{x\to 1^+} g(x,y) = \lim_{x\to 1^-} g(x,y) = \ln\ln y \ge 0$$

and $g(e,y) = h(y) \ge 0$, we see that $g(x,y) \ge 0$, or $x^{y-1}\ln y \ge y^{x-1}$.

Hence, $\dfrac{\partial f}{\partial y} = \dfrac{x^y - x}{y} + (x\ln x)\left(x^{y-1}\ln y - y^{x-1}\right) > 0$, and it follows that $f(x,y) > f(x,x) = 0$. This completes the proof.

Also solved by MICHAEL LAMBROU, University of Crete, Crete, Greece; and the proposers.

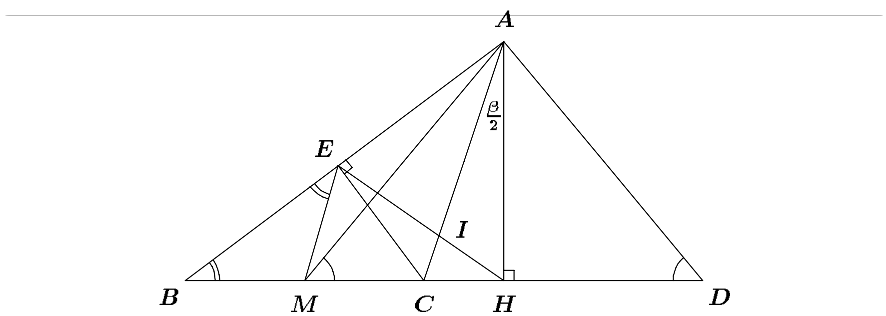
2416. [1999: 110] *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

Given $\triangle ABC$, where C is an obtuse angle, suppose that M is the mid-point of BC and that the circle with centre A and radius AM meets BC again at D. Assume also that $MD = AB$. The circle, Γ, with centre M and radius MB meets AB at E. Let H be the foot of the perpendicular from A to BC (extended). Suppose that AC and EH intersect at I.

Find the angles $\angle IAH$ and $\angle AHI$ as function of $\angle ABC$.

[This proposal was inspired by problem **2316**.]

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.



Let $\beta = \angle ABC$. We shall prove first that $\angle IAH = \frac{\beta}{2}$. Let AB be 2 units long, so that $MH = \dfrac{MD}{2} = \dfrac{AB}{2} = 1$.

Then $BM = 2\cos\beta - 1$, and $BC = 4\cos\beta - 2$. Therefore $\tan(\angle IAH) = \dfrac{CH}{AH} = \dfrac{2\cos\beta - (4\cos\beta - 2)}{2\sin\beta} = \dfrac{1-\cos\beta}{\sin\beta} = \tan\frac{\beta}{2}$.

Hence $\angle IAH = \frac{\beta}{2}$.

Consequently, $\angle ACB = 90^\circ + \frac{\beta}{2}$, and $\angle BAC = 90^\circ - \frac{3\beta}{2}$.

Next, since $MB = ME = MC$, we have $CE \perp AB$, so that $AHCE$ is a cyclic quadrilateral. Hence, $\angle IHA = \angle ACE = 90^\circ - \angle BAC = \frac{3\beta}{2}$.

Also solved by NIKOLAOS DERGIADES, Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; TOSHIO SEIMIYA, Kawasaki, Japan; PARAYIOU THEOKLITOS, Limassol, Cyprus, Greece; and the proposer.

2417. [1999: 111] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

In $\triangle ABC$, with $AB \neq AC$, the internal and external bisectors of $\angle BAC$ meet the circumcircle of $\triangle ABC$ again in L and M respectively. The points L' and M' lie on the extensions of AL and AM respectively, and satisfy $AL = LL'$ and $AM = MM'$. The circles ALM' and $AL'M$ meet again at P.

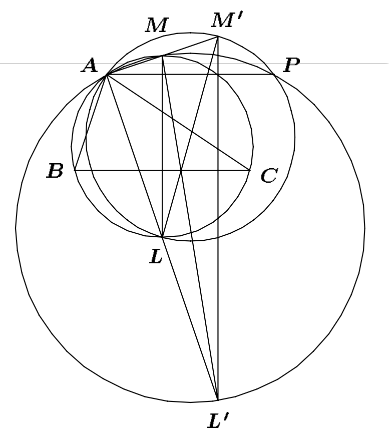
Prove that $AP \parallel BC$.

Solution by Nikolaos Dergiades, Thessaloniki, Greece.

The quadrilateral $MLL'M'$ is a trapezoid [trapezium in Europe] since M and L are the mid-points of AM' and AL'.

Since $\angle MAL = 90°$, it follows that LM' and ML' are the diameters of circles ALM' and $AL'M$.

Hence the line joining the centres of these circles, which is perpendicular to AP, is the line on the mid-points of the diagonals of the trapezoid. Therefore, $AP \perp ML$, yielding $AP \parallel BC$.



Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PARAYIOU THEOKLITOS, Limassol, Cyprus, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

2418. [1999: 111] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

In $\triangle ABC$, the lengths of the sides BC, CA, AB are 1998, 2000, 2002 respectively.

Prove that there exists exactly one point P (distinct from A and B) on the minor arc AB of the circumcircle of $\triangle ABC$ such that PA, PB, PC are all of integral length.

Solution by Nikolaos Dergiades, Thessaloniki, Greece.

Let $a = |BC| = 1998$, $b = |CA| = 2000$ and $c = |AB| = 2002$. Let $x = |PA|$, $y = |PB|$ and $z = |PC|$. Then Ptolemy's theorems give

$$cz = ax + by \tag{1}$$

and

$$\frac{z}{c} = \frac{ab + xy}{ay + bx}. \tag{2}$$

From (1) we have $2002z = 1998x + 2000y$, or

$$z = x + y - \frac{2x + y}{1001}.$$

Since z is an integer, we conclude that

$$y = 1001k - 2x \qquad \text{and} \qquad z = 1000k - x\,, \tag{3}$$

where k is an integer. Now, $x + z > b$, so $1000k > 2000$, and therefore, $k > 2$. Let s be the semi-perimeter of $\triangle ABC$. The diameter of the circumcircle of $\triangle ABC$ is

$$\frac{abc}{2\sqrt{s(s-a)(s-b)(s-c)}} = \frac{666666}{\sqrt{83333}} < 2310\,.$$

Also, $x < c$ (because P lies on the minor arc AB), and therefore $1000k = z + x < 2310 + 2002$. Hence $k < 5$. From (1) and (2),

$$z^2 = \frac{(ax+by)(ab+xy)}{ay+bx}\,. \tag{4}$$

If $k = 3$ then (3) and (4) give the equation

$$\frac{(x-3000)(x^2-3000x+1666665)}{998x-2999997} = 0\,,$$

which has no integer solution $x < 2002$.

If $k = 4$ then (3) and (4) give the equation

$$\frac{(x-4000)(x-1998)(x-2002)}{499x-1999998} = 0\,,$$

which has only one solution less than 2002, $x = 1998$. Then $y = 8$ and $z = 2002$ and the quadrilateral $PABC$ is an isosceles trapezium.

Also solved by COLIN DIXON, Newcastle upon Tyne, England; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong, China; D.J. SMEENK, Zaltbommel, the Netherlands; KENNETH M. WILKE, Topeka, KS, USA; and the proposer. There were also two incorrect solutions submitted

2419. [1999: 111] *Proposed by K.R.S. Sastry, Dodballapur, India.*
Find all solutions to the alphametic:

```
     M I X . E
  +  D B A . S
  ------------
     E S U . M
```

1. The letters before the decimal points represent base ten digits, and addition is done in that base.

2. The letters after the decimal points represent base six digits, and addition is done in that base.

3. The same letter stands for the same digit, distinct letters stand for distinct digits, and initial digits are non-zero.

Readers familiar with cricket will realize that this is a real world problem! [*Ed.* Readers not familiar with cricket may be interested to learn that an 'over' consists of six 'deliveries'!!]

Solution by Nikolaos Dergiades, Thessaloniki, Greece.

$\mathtt{M}$, $\mathtt{D}$, and $\mathtt{E}$ are initial digits, so $\mathtt{M}$, $\mathtt{D}$, and $\mathtt{E}$ are positive and $\mathtt{D} < \mathtt{E}$, $\mathtt{M} < \mathtt{E}$. Also, $\mathtt{S} > 0$, because, otherwise we would have $\mathtt{E} = \mathtt{M}$. Since $\mathtt{M} < \mathtt{E}$ then $\mathtt{E} + \mathtt{S} = 6 + \mathtt{M}$. On the other hand, $\mathtt{E}$, $\mathtt{S}$ and $\mathtt{M}$ are base six digits, so each of the digits $\mathtt{D}$, $\mathtt{E}$, $\mathtt{S}$ and $\mathtt{M}$ is less than 6. We have two cases for the initial digits: $\mathtt{M} + \mathtt{D} = \mathtt{E}$ or $\mathtt{M} + \mathtt{D} + 1 = \mathtt{E}$. If

$$\left\{ \begin{array}{l} \mathtt{E}+\mathtt{S}=6+\mathtt{M} \\ \mathtt{M}+\mathtt{D}=\mathtt{E}\,, \end{array} \right. \tag{1}$$

then

$$\left\{ \begin{array}{l} \mathtt{I}+\mathtt{B}=\mathtt{S} \\ \mathtt{X}+\mathtt{A}+1=\mathtt{U}\,. \end{array} \right. \tag{2}$$

or

$$\left\{ \begin{array}{l} \mathtt{I}+\mathtt{B}=\mathtt{S} \\ \mathtt{X}+\mathtt{A}+1=10+\mathtt{U}\,. \end{array} \right. \tag{3}$$

System (1) has five solutions for $(\mathtt{E},\mathtt{S},\mathtt{M},\mathtt{D})$: $(5,4,3,2)$, $(5,2,1,4)$, $(4,5,3,1)$, $(3,5,2,1)$, and $(3,4,1,2)$. None of these produces a solution to (2). However, $(\mathtt{E},\mathtt{S},\mathtt{M},\mathtt{D}) = (3,5,2,1)$ gives rise to four solutions to (3) and therefore to the alphametic:

$$(\mathtt{I},\mathtt{B},\mathtt{X},\mathtt{A},\mathtt{U}) \;=\; (0,4,8,7,6),(0,4,7,8,6),(4,0,8,7,6),(4,0,7,8,6)\,.$$

If

$$\left\{ \begin{array}{l} \mathtt{E}+\mathtt{S}=6+\mathtt{M} \\ \mathtt{M}+\mathtt{D}+1=\mathtt{E}\,, \end{array} \right. \tag{4}$$

then

$$\left\{ \begin{array}{l} \mathtt{I}+\mathtt{B}=10+\mathtt{S} \\ \mathtt{X}+\mathtt{A}+1=\mathtt{U} \end{array} \right. \tag{5}$$

or

$$\left\{ \begin{array}{l} \mathtt{I}+\mathtt{B}=10+\mathtt{S} \\ \mathtt{X}+\mathtt{A}+1=10+\mathtt{U}\,. \end{array} \right. \tag{6}$$

System (4) has three solutions for $(\mathtt{E},\mathtt{S},\mathtt{M},\mathtt{D})$: $(5,4,3,1)$, $(5,2,1,3)$, and $(4,3,1,2)$. None of these produces a solution to (6). However, the last two yield solutions to (5) and therefore to the alphametic. If $(\mathtt{E}, \mathtt{S}, \mathtt{M}, \mathtt{D}) = (5, 2, 1, 3)$, then system (5) gives the solutions

$$(\mathtt{I}, \mathtt{B}, \mathtt{X}, \mathtt{A}, \mathtt{U}) = (8,4,6,0,7), (8,4,0,6,7), (4,8,6,0,7), (4,8,0,6,7).$$

If $(\mathtt{E}, \mathtt{S}, \mathtt{M}, \mathtt{D}) = (4, 3, 1, 2)$, then system (5) produces the solutions

$$\begin{aligned}(\mathtt{I}, \mathtt{B}, \mathtt{X}, \mathtt{A}, \mathtt{U}) &= (8,5,6,0,7), (8,5,0,6,7), (5,8,6,0,7), (5,8,0,6,7), \\ &\quad (7,6,8,0,9), (7,6,0,8,9), (6,7,8,0,9), (6,7,0,8,9).\end{aligned}$$

Thus we have 16 solutions to the alphametic.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and GERRY LEVERSHA, St. Paul's School, London, England. There were also five incomplete solutions submitted.

2420. [1999: 111] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Suppose that x, y and z are integers. Solve the equation:

$$x^2 + y^2 = 2420z^2 .$$

Solution by Michael Lambrou, University of Crete, Crete, Greece.

First, note that if (x, y, z) is a solution, then (y, x, z) and $(\pm x, \pm y, \pm z)$ are also solutions. Since $x^2 + y^2$ is divisible by 4, both x and y must be even. Hence $x = 2x_1$, $y = 2y_1$, and the equation becomes $x_1^2 + y_1^2 = 5(11^2)z^2$. If n is an integer, then $n^2 \equiv 0,\ 1,\ 3,\ 4,\ 5,\ 9 \pmod{11}$. Therefore, $x_1^2 + y_1^2 \equiv 0 \pmod{11}$ if and only if x_1 and y_1 are divisible by 11. Thus $x_1 = 11x_2$, $y_1 = 11y_2$, and the equation becomes

$$x_2^2 + y_2^2 = 5z^2 .$$

There are a number of standard approaches for solving this equation. The one taught in Diophantos's *Arithmetica* is the following. One solution is $z = x_2 = y_2 = 0$. If $z \neq 0$, then the equation transforms to

$$\left(\frac{x_2}{z}\right)^2 + \left(\frac{y_2}{z}\right)^2 = 5 .$$

Hence we need to find the rational points on the circle

$$u^2 + v^2 = 5 , \qquad (1)$$

where $\frac{x_2}{z} = u$ and $\frac{y_2}{z} = v$. One such point is $(1, 2)$. Consider the line l through $(1, 2)$ and any other rational point (u, v) on the circle, $(u, v) \neq (1, \pm 2)$. The slope of l is the rational number $\frac{v-2}{u-1}$. Let $\frac{v-2}{u-1} = \frac{m}{n}$, where $\gcd(m,n) = 1$. Then $v - 2 = km$ and $u - 1 = kn$, where $k \neq 0$ is an integer. Substitute $v = km + 2$ and $u = kn + 1$ in the equation (1) and solve for k to obtain

$$k = -\frac{2(2m+n)}{m^2+n^2}\,.$$

Consequently,

$$u = \frac{m^2 - n^2 - 4mn}{m^2+n^2} \qquad \text{and} \qquad v = \frac{n^2 - m^2 - mn}{m^2+n^2}\,.$$

Back substitution gives

$$(x,y,z) = \big(22k(m^2 - n^2 - 4mn),\ 44k(n^2 - m^2 - mn),\ k(m^2+n^2)\big)\,.$$

Also solved by MICHEL BATAILLE, Rouen, France; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; GERRY LEVERSHA, St. Paul's School, London, England; PANOS E. TSAOUSSOGLOU, Athens, Greece; KENNETH M. WILKE, Topeka, KS, USA; and the proposer. There were also two incomplete solutions submitted.

2421. [1999: 111] *Proposed by Ice B. Risteski, Skopje, Macedonia.*

What is the probability that the k numbers in the Las Vegas lottery on a given payout day do not include two consecutive integers? (The winning numbers are an unordered random choice of k distinct integers from 1 to n, where $n > k$.)

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Let the $n - k$ numbers not chosen be represented by $n - k$ dots on a line. Then the k chosen numbers (represented by bars, |) can only be placed in the spaces between adjacent dots, including the space to the far left and the space to the far right. [Ed.: the diagram below depicts a possible scenario when $n = 7$ and $k = 3$.]

| • • | • | •

Since there are $n - k + 1$ such spaces to accommodate the |'s, the total number of admissible selections is $\binom{n-k+1}{k}$. Hence the required probability is $\dfrac{\binom{n-k+1}{k}}{\binom{n}{k}}$.

Also solved by KEITH EKBLAW, Walla Walla, WA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece (two solutions); GERRY LEVERSHA, St. Paul's School, London, England; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; MARK LYON, student, University of Arizona, Tuscon, AZ, USA; SKIDMORE

COLLEGE PROBLEM GROUP, Saratoga Springs, NY, USA; DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer. There was one incorrect solution.

Wang pointed out that the answer $\binom{n-k+1}{k}$ *obtained by Janous and other solvers is a well-known result in elementary combinatorics, and can be found in many books on combinatorics; for example, Ex. #40 on p. 51 of* Basic Techniques of Combinatorial Theory *by Daniel I.A. Cohen. The argument given by Janous is the usual one, and, to the best of this editor's knowledge, is the simplest and most elegant one. It is also easy to see that the probability is zero if* $k > \left\lfloor \frac{n+1}{2} \right\rfloor$. *This was noted by Hess, Lambrou, Leversha and Wang.*

2422★. [1999: 112] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let A, B, C be the angles of an arbitrary triangle. Prove or disprove that

$$\frac{1}{A}+\frac{1}{B}+\frac{1}{C} \;\geq\; \frac{9\sqrt{3}}{2\pi\left(\sin A\sin B\sin C\right)^{1/3}}\,.$$

Comment by the Editor.

This problem was already solved, by the proposer, on [1998: 306–307], as a remark following his solution of ***CRUX*** 2015. This was pointed out by VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and ECKARD SPECHT, Otto-von-Guericke Universität, Magdeburg, Germany. A solution was also sent in by CATHERINE SHEVLIN, Wallsend, England.

2423. [1999: 112] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $x_1, x_2, \ldots, x_n > 0$ be real numbers such that $x_1+x_2+\ldots+x_n = 1$, where $n > 2$ is a natural number. Prove that

$$\prod_{k=1}^{n}\left(1+\frac{1}{x_k}\right) \;\geq\; \prod_{k=1}^{n}\left(\frac{n-x_k}{1-x_k}\right)\,.$$

Determine the cases of equality.

Solution by the proposer.

We start from the following observation, valid for all convex functions $f(x)$:

$$\sum_{i=1}^{n} f(x_i) \;\geq\; \sum_{i=1}^{n} f\left(\frac{1-x_i}{n-1}\right)\,. \qquad (1)$$

To prove this, use Jensen's inequality:

$$\begin{aligned}\sum_{i=1}^{n} f(x_i) &= \sum_{i=1}^{n} \frac{\sum_{j\neq i} f(x_j)}{n-1} \\ &\geq \sum_{i=1}^{n} f\left(\frac{\sum_{j\neq i} x_j}{n-1}\right) = \sum_{i=1}^{n} f\left(\frac{1-x_i}{n-1}\right).\end{aligned}$$

[This is a slightly simplified version of the proposer's argument.—*Ed.*]

Now let $$f(x) = \ln\left(1+\frac{1}{x}\right) = \ln(1+x) - \ln x.$$

Then

$$f'(x) = \frac{1}{1+x} - \frac{1}{x} \quad\text{and}\quad f''(x) = \frac{1}{x^2} - \frac{1}{(1+x)^2} = \frac{1+2x}{x^2(1+x)^2} > 0$$

for $x > 0$, so that $f(x)$ is convex. Thus (1) yields

$$\sum_{i=1}^{n} \ln\left(1+\frac{1}{x_i}\right) \geq \sum_{i=1}^{n} \ln\left(1+\frac{n-1}{1-x_i}\right) = \sum_{i=1}^{n} \ln\left(\frac{n-x_i}{1-x_i}\right),$$

which is equivalent to the claimed inequality.

Also solved by KEE-WAI LAU, Hong Kong, China; and VEDULA N. MURTY, Visakhapatnam, India.

Lau and Murty both note that equality holds if and only if $x_1 = \cdots = x_n = 1/n$*, which can also be seen from the above proof.*

Murty's solution uses Lagrange multipliers. Lau first uses the AM–GM inequality to reduce the problem to proving the inequality

$$\prod_{i=1}^{n} \left(\frac{1-x_i^2}{x_i}\right) \geq \left(\frac{n^2-1}{n}\right)^n,$$

but his proof of this inequality uses calculus and is a little involved. Can someone finish the proof in a nice way without calculus?

2424. [1999: 112] *Proposed by K.R.S. Sastry, Dodballapur, India.*

In $\triangle ABC$, suppose that I is the incentre and BE is the bisector of $\angle ABC$, with E on AC. Suppose that P is on AB and Q on AC such that PIQ is parallel to BC. Prove that $BE = PQ$ if and only if $\angle ABC = 2\angle ACB$.

Solution by Nikolaos Dergiades, Thessaloniki, Greece (augmented by the editor).

We use the theorem:

If two triangles PQR and $P'Q'R'$ satisfy $PQ = P'Q'$ and $\angle R = \angle R'$, while the internal bisectors of $\angle R$ and $\angle R'$ have the same length, then the triangles are congruent.

Editor's comment. Books that deal with Euclidean constructions (for example, Alfred S. Posamentier and William Wernick, *Geometric Constructions*, J. Weston Walch, Publ. Portland, ME 1973, pp. 53-54, #105) show how to construct a unique triangle (when it exists) from one side, the opposite angle, and that angle's bisector. Jeremy Young included with his solution an argument that makes the existence and uniqueness both clear:

All triangles PQR with fixed side PQ and fixed angle magnitude at R have a common circumcircle with R on one arc $\overparen{PQ}$ and its internal angle bisector extended to meet the opposite arc at a fixed point Y (that lies halfway between P and Q). Call X the point where the segment PQ intersects RY, and consider how the length of the angle bisector RX varies as R moves along the circumference from P to the point diametrically opposite Y. RY increases (from the length of PX to the circle's diameter) while XY decreases. Therefore $RX = RY - XY$ increases, and each length of the angle bisector (within the possible range) is taken exactly once. Let us now return to Dergiades' argument.

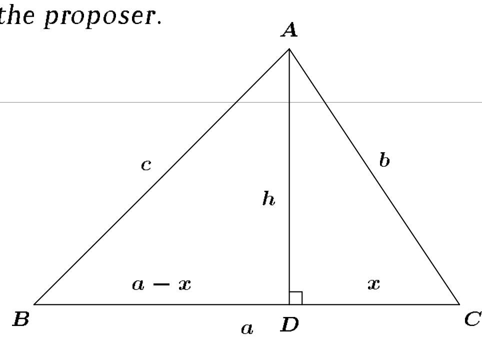
The theorem says that $BE = PQ$ if and only if $\triangle ABE \cong \triangle AQP$. By symmetry about the angle bisector AI, these two triangles are congruent if and only if $\angle ABI = \angle AQI$. This is equivalent to $\angle ABC = 2\angle ACB$ as required since [on the left] BI is the bisector of $\angle ABC$ so that $\angle ABC = 2\angle ABI$, while [on the right] $\angle AQI = \angle ACB$ [because $PQ||BC$].

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece (a second solution); WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer (2 solutions).

2425. [1999: 112] *Proposed by K.R.S. Sastry, Dodballapur, India.*

Suppose that D is the foot of the altitude from vertex A of an acute-angled Heronian triangle ABC (that is, one having integer sides and area). Suppose that the greatest common divisor of the side lengths is 1. Find the smallest possible value of the side length BC, given that $BD - DC = 6$.

Solution by the proposer.



Let x be the length of the line segment DC. Then $BD - DC = 6$ if and only if $a - 2x = 6$. From the Theorem of Pythagoras we have

$$c^2 = (a-x)^2 + h^2, \qquad b^2 = x^2 + h^2,$$

so that $c^2 - b^2 = 6a$ is even. Also $c - b$ and $c + b = (c - b) + (2b)$ have the same parity, namely even. Since $c + b > a$ we must have $c - b < 6$. Therefore $c - b = 2$ or 4.

(i) $c - b = 2$. This implies $c + b = 3a$, $a = 2k$ must be even, and $c, b = 3k \pm 1$. By Heron's triangle formula

$$\Delta^2 = s(s-a)(s-b)(s-c) = 4k(2k)(k^2-1)$$

which implies that $k^2 - 1 = 2d^2$, where $d \in \mathbb{N}$. Thus k is odd. However, then a, b, c are all even, so $\gcd(a, b, c) > 1$, a contradiction. Hence $c - b \neq 2$.

(ii) $c - b = 4$. Then $c + b = 3a/2$ must be even. Hence $a = 4k$, for some $k \in \mathbb{N}$. This gives $c, b = 3k \pm 2$. Now $\Delta^2 = 5k(k)4(k^2 - 1)$, which implies $k^2 - 1 = 5d^2$, for some $d \in \mathbb{N}$. This Fermat-Pell equation has the smallest solution $k = 9$, $d = 4$. Then $(a, b, c) = (36, 25, 29)$. Hence $BC = 36$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and D.J. SMEENK, Zaltbommel, the Netherlands. There were two incorrect solutions.