THE ACADEMY CORNER

No. 26

Bruce Shawyer

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Once again, we are pleased to present the Bernoulli Trials. Many thanks to Christopher Small for sending them to us.

The Bernoulli Trials 1999

Christopher G. Small & Ravindra Maharaj
University of Waterloo

The Bernoulli Trials is an undergraduate mathematics competition held annually at the University of Waterloo. Students competing in the Bernoulli Trials proceed through a sequence of rounds, with a new mathematical problem for each round. The mathematical problems are presented as propositions, and the students are given ten minutes to decide upon the correctness of each proposition. The double-knockout format means that students may make one mistake, but are eliminated after two mistakes.

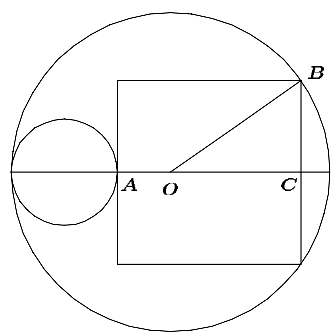
For the second year in a row, the competition was won by **Frédéric Latour**, who survived eleven rounds with only one mistake. **Dennis The** lasted ten rounds for a second place showing. Students ranked third through fifth were eliminated in the tenth round, and entered a playoff round to break the tie. After the tiebreaker, **Byung Kyu Chun** and **Derek Kisman** were still tied for third and fourth, while **Sabin Cautis** was ranked fifth.

In keeping with the spirit of the competition, Frédéric received 100 toonies (Can $200) in laundry money, a prize awarded by the Dean of Mathematics. The Dean also awarded 100 loonies (Can $100) to Dennis, 80 quarters each to Byung and Derek, and 40 quarters to Sabin. A total of 30 students participated.

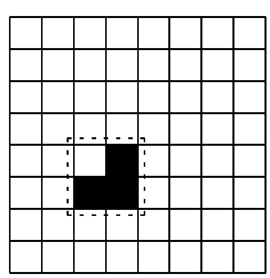
The contest organiser this year was Christopher Small. He was ably assisted by Ravindra Maharaj and Ken Davidson. Kristin Lord was the contest photographer.

The Problems

1. Let x and y be any positive integers satisfying $x\,y = 1999\,x + 1999\,y$.
   TRUE OR FALSE? It follows that $x \leq 3998000$.
2. TRUE OR FALSE? The equation $\sin(\cos x) = \cos(\sin x)$ has no solutions in real values x.
3. In the equation
   $$131079331213119595187 48AB = CD \times (5083^3 - 5083)^2$$
   four of the digits are omitted as shown. (The digits are not necessarily distinct.)
   TRUE OR FALSE? $B = 8$.
4. Circles with radii 12 and 4 are tangent as shown. A square is drawn inside the larger circle touching it and the smaller one as shown.

   The length of a side of the square is expressed as $\dfrac{a+\sqrt{b}}{c}$, where a, b and c are integers, and the ratio is expressed in lowest form.
   TRUE OR FALSE? $b = 2734$.
5. Suppose the 64 squares of an 8×8 chessboard of unit squares are to be coloured either black or white.

   TRUE OR FALSE? There are exactly 2^{15} ways of colouring the squares so that every 2×2 square of adjacent unit squares has exactly 3 of the 4 squares the same colour.

6. Consider the equation $f(x)\,f(x+1) \;=\; f\left(f(x)+x\right)$.

   TRUE OR FALSE? There exists a non-constant differentiable function $f:\mathbb{R}\to\mathbb{R}$ satisfying this equation for all real x.

7. Suppose $f:\mathbb{R}\to\mathbb{R}$ has derivatives of all orders and that the doubly infinite series

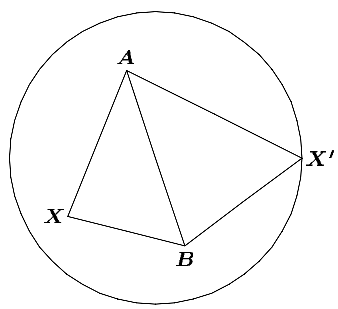
$$\cdots + f''(x) + f'(x) + f(x) + \int_0^x f(t)\,dt + \int_0^x \int_0^y f(t)\,dt\,dy + \cdots$$

   converges absolutely and uniformly to some function $\phi(x)$ for all $x \in [-1999, +1999]$. Suppose also that $\phi(0) = 1999$.

   TRUE OR FALSE? $\phi(-1999) < 0$.

8. TRUE or FALSE? $\dfrac{5}{4} \;<\; \displaystyle\int_0^1 x^{-x}\,dx \;<\; \dfrac{3}{2}$.

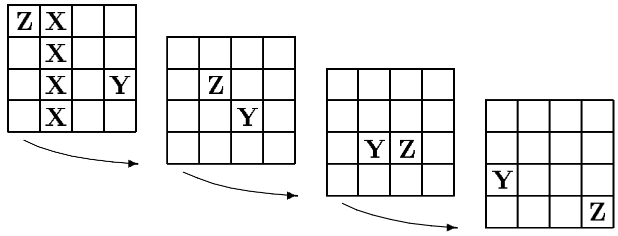
9. Three points A, B and X are randomly chosen from the interior of a circle. An additional point X' is randomly chosen on the boundary of the circle.



   Let Π be the probability that $\triangle ABX$ is acute, and let Π' be the probability that $\triangle ABX'$ is acute.

   TRUE OR FALSE? $\Pi < \Pi'$.

10. In 3–dimensional tic-tac-toe on a $4\times4\times4$ board, a winning line is a sequence of four counters in a straight line. In the figure, the $\mathbf{X}$'s form a winning line, as do the $\mathbf{Y}$'s and the $\mathbf{Z}$'s.



Note that the **X**'s form a line parallel to an edge of the cube, the **Y**'s form a line parallel to a face but not to any edge, and the **Z**'s are not parallel to any face or edge.

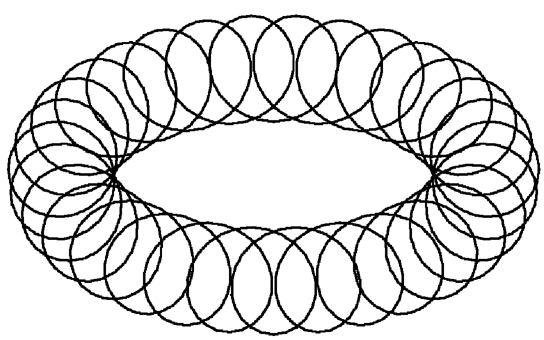
TRUE OR FALSE? It is possible to distribute 16 counters among the 64 cells so that every winning line of the opponent parallel to a face or edge is blocked.

11. Four men and four women decide to play a mixed-doubles tennis tournament on two adjacent courts at a tennis club so that each person plays in one mixed-doubles match per day.

    The tournament is arranged so that

    - a man and a woman always play against a man and a woman;
    - no person ever plays *with* anyone else more than once;
    - no person ever plays *against* anyone else more than once.

    TRUE OR FALSE? Such a tournament cannot be played on the two courts on three days.



THE OLYMPIAD CORNER

No. 200

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

As a set of problems for this issue, we give problems of the XL Mathematical Olympiad of the Republic of Moldova. My thanks go to Ravi Vakil for collecting them when he was Team Leader for Canada at the International Mathematical Olympiad at Mumbai.

REPUBLIC OF MOLDOVA
XL MATHEMATICAL OLYMPIAD
Chişinău, 17–20 April, 1996
First Day (Time: 4 hours)

10 Form

1. Let $n = 2^{13} \cdot 3^{11} \cdot 5^7$. Find the number of divisors of n^2 which are less than n and are not divisors of n.

2. Distinct square trinomials $f(x)$ and $g(x)$ have leading coefficient 1. It is known that $f(-12)+f(2000)+f(4000) = g(-12)+g(2000)+g(4000)$. Find all the real values of x which satisfy the equation $f(x) = g(x)$.

3. Through the vertices of a triangle tangents to the circumcircle are constructed. The distances of an arbitrary point of the circle to the straight lines containing the sides of the triangle are equal to a, b and c and to the tangents are equal to x, y and z. Prove that $a^2 + b^2 + c^2 = xy + xz + yz$.

4. Two brothers sold n lambs at a price n dollars. They divide the money as follows: the elder brother took 10 dollars, the younger one took 10 dollars, the elder one took again 10 dollars and so on. At the end it turned out that the sum for the younger brother was less than 10. He took the remainder and the pen-knife of his brother. The brothers agreed that the division was correct. What is the cost of the pen-knife?

11–12 Form

1. Prove the equality

$$\frac{1}{666}+\frac{1}{607}+\cdots+\frac{1}{1996} = 1+\frac{2}{2\cdot 3\cdot 4}+\frac{2}{5\cdot 6\cdot 7}+\cdots+\frac{2}{1994\cdot 1995\cdot 1996}.$$

2. Prove that the product of the roots of the equation

$$\sqrt{1996} \cdot x^{\log_{1996} x} = x^6$$

is an integer and find the last four digits of this integer.

3. Two disjoint circles C_1 and C_2 with centres O_1 and O_2 are given. A common exterior tangent touches C_1 and C_2 at points A and B respectively. The segment O_1O_2 cuts C_1 and C_2 at points C and D respectively. Prove that:

(a) the points A, B, C and D are concyclic;
(b) the straight lines (AC) and (BD) are perpendicular.

4. Among n coins, identical by form, less than half are false and differ by weight from the true coins. Prove that with the help of scales (without weights) using no more than $n - 1$ weighings one can find at least one true coin.

Second Day (Time: 4 hours)

10 Form

5. Prove that for all natural numbers $m \geq 2$ and $n \geq 2$ the smallest among the numbers $\sqrt[n]{m}$ and $\sqrt[m]{n}$ does not exceed the number $\sqrt[3]{3}$.

6. Prove the inequality $2^{a_1}+2^{a_2}+\cdots+2^{a_{1996}} \leq 1995+2^{a_1+a_2+\cdots+a_{1996}}$ for any real non-positive numbers a_1, a_2, $\ldots$, a_{1996}.

7. The perpendicular bisector to the side $[BC]$ of a triangle ABC intersects the straight line (AC) at a point M and the perpendicular bisector to the side $[AC]$ intersects the straight line (BC) at a point N. Let O be the centre of the circumcircle to the triangle ABC. Prove that:

(a) points A, B, M, N and O lie on a circle S;
(b) the radius of S equals the radius of the circumcircle to the triangle MNC.

8. Among 1996 coins, identical by form, two are false. One is heavier and the second is lighter than a true coin. With the help of scales (without weights) using four weighings, what is the way to show if the combined weight of these two coins is equal, greater than or less than, the combined weight of two true coins?

11–12 Form

5. Let p be the number of functions defined on the set $\{1, 2, \ldots, m\}$, $m \in N^*$, with values in the set $\{1, 2, \ldots, 35, 36\}$ and q be the number of functions defined on the set $\{1, 2, \ldots, n\}$, $n \in N^*$, with values in the set $\{1, 2, 3, 4, 5\}$. Find the least possible value for the expression $|p - q|$.

6. Solve in real numbers the equation

$$2x^2 - 3x = 1 + 2x\sqrt{x^2 - 3x}\,.$$

7. On a sphere distinct points A, B, C and D are chosen, so that segments $[AB]$ and $[CD]$ cut each other at point F, and points A, C and F are equidistant to a point E. Prove that the straight lines (BD) and (EF) are perpendicular.

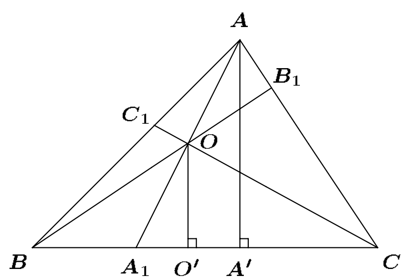
8. 20 children attend a rural elementary school. Every two children have a grandfather in common. Prove that some grandfather has not less than 14 grandchildren in this school.

Next, we give solutions by our readers to problems of the VIII Nordic Mathematical Contest [1998: 133].

1. Let O be a point in the interior of an equilateral triangle ABC with side length a. The lines AO, BO and CO intersect the sides of the triangle at the points A_1, B_1 and C_1 respectively. Prove that

$$|OA_1| + |OB_1| + |OC_1| < a\,.$$

Solutions by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec; by Pierre Bornsztein, Courdimanche, France; and by Toshio Seimiya, Kawasaki, Japan. We give the solution by Seimiya.



Since $AB = AC$ we have, in $\triangle AA_1C$

$$\angle ABC \;=\; \angle ACB \;=\; \angle ACA_1\,,$$

so that $AA_1 < AC = a$.

Similarly we have that $BB_1 < a$ and $CC_1 < a$. We denote the area of $\triangle PQR$ by $[PQR]$. Let A', O' be the feet of the perpendiculars from A, O to BC respectively. There, we have

$$\frac{OA_1}{AA_1} \;=\; \frac{OO'}{AA'} \;=\; \frac{[OBC]}{[ABC]}\,;$$

$$\frac{OB_1}{BB_1} \;=\; \frac{[OCA]}{[ABC]}\,; \qquad \frac{OC_1}{CC_1} \;=\; \frac{[OAB]}{[ABC]}\,.$$

Thus we get

$$\frac{OA_1}{AA_1} + \frac{OB_1}{BB_1} + \frac{OC_1}{CC_1} = \frac{[OBC]+[OCA]+[OAB]}{[ABC]} = 1\,.$$

Since $AA_1 < a$, $BB_1 < a$, $CC_1 < a$ we have

$$\frac{OA_1}{AA_1} + \frac{OB_1}{BB_1} + \frac{OC_1}{CC_1} > \frac{OA_1}{a} + \frac{OB_1}{a} + \frac{OC_1}{a}\,.$$

Thus we have $\quad 1 > \dfrac{OA_1}{a} + \dfrac{OB_1}{a} + \dfrac{OC_1}{a}\,.$

This implies that $\quad OA_1 + OB_1 + OC_1 < a\,.$

2. A finite set S of points in the plane with integer coordinates is called a *two-neighbour set*, if for each (p,q) in S exactly two of the points $(p+1,q)$, $(p,q+1)$, $(p-1,q)$, $(p,q-1)$ are in S. For which n does there exist a two-neighbour set which contains exactly n points?

Solution by Pierre Bornsztein, Courdimanche, France.

On va prouver que les valeurs de n cherchées sont tous les entiers pairs supérieurs ou éqaux à 4 sauf 6. On vérifie facilement que $n \geq 4$ est nécessaire.

Soit S un tel ensemble avec card $S = n$. Pour $M(p,q) \in S$, on dit que le point P est voisin de M lorsque $P \in S$ et $P \in \{(p+1,q), (p,q+1), (p-1,q), (p,q-1)\}$. Alors *tout point de S admet exactement deux voisins*.

Soit $M_1(p,q) \in S$.

Soit M_2 un voisin de M_1.

On pose M_3 est *le* voisin de M_2 avec $M_3 \neq M_1$, etc.

Si M_k et M_{k+1} sont construits $(k \geq 1)$ on pose M_{k+2} est *le* voisin de M_{k+1} avec $M_{k+2} \neq M_k$.

On construit ainsi, à partir de M_1, une suite de points de S. Mais S est fini. Donc il existe $i, j \in \mathbb{N}^*$ avec $i < j$ tels que $M_i = M_j$. Soit alors $E = \{k \in \mathbb{N}^* \mid \exists\, i \in \mathbb{N}^*, i < k \text{ et } M_i = M_k\}$. On a $E \neq \phi$, $E \subset \mathbb{N}^*$, donc E admet un plus petit élément. On pose $p = \min E$.

Soit alors $i < p$, $i \in \mathbb{N}^*$ tel que $M_i = M_p$. On a alors $i = 1$: en effet, si on suppose $i > 1$ alors M_{i-1}, M_{i+1}, M_{p-1} sont des voisins de $M_i = M_p$ avec

$$\begin{cases} M_{i-1} \neq M_{i+1} & \text{par construction} \\ M_{i-1} \neq M_{p-1} & \text{car } p = \min E \end{cases}$$

d'où $M_{i+1} = M_{p-1}$, et
comme $p = \min E$, on a $i+1 = p-1$ (on a $p-1 \leq p$ et $i+1 \leq p$) d'où $p = i+2$, et $M_i = M_{i+2}$, ce qui est impossible par construction.

Définition : *Une chaîne* C, de longueur a ($a \in \mathbb{N}^*, a \geq 2$) est un ensemble $\{M_1, \ldots, M_a\}$ de points de S tels que pour tous i, j de $\{1, \ldots, a\}$

- $M_i \in S$,
- M_{i+1} et M_i sont voisins,
- M_a et M_1 sont voisins,
- si $i \neq j$, alors $M_i \neq M_j$.

Sans perte de généralité on peut toujours supposer que pour tout $i \in \{1, \ldots, a\}$, $M_i(p_i, q_i)$ avec

- $p_i \geq 1$
- si $i > 1$ et $p_i = p_1$ alors $q_i > q_1$
- $M_2(p_1 + 1, q_1)$

Cela entraîne $M_a(p_1, q_1 + 1)$ et permet d'éviter de considérer deux foix la même chaîne si l'on change de “sens de parcours” ou de “point de départ”.

On verifie aisément que $a \geq 4$.

Propriété : Si $C = \{M_1, \ldots, M_a\}$ et $C' = \{P_1, \ldots, P_b\}$ sont deux chaînes alors $C \cap C' = \phi$ ou $C = C'$.

Preuve : Par l'absurde. Si $C \cap C' \neq \phi$ et $C \neq C'$, alors il existe $M \in C \cap C'$ tel que :

- il existe $i \in \{1, \ldots, a\}$, $j \in \{1, \ldots, b\}$, $M = M_i = P_j$
- $\mathbf{card}(M_{i-1}, M_{i+1}, P_{j-1}, P_{j+1}) \geq 3$ (On pose $M_{-1} = M_a$, $P_{-1} = P_b$).

Mais alors $M \in S$ possède au moins trois voisins, qui est un contradiction.

D'après ce qui précède, $\forall M \in S \; \exists$ chaine C tel que $M \in C$, donc $S \subset \bigcup_{\text{chaîne}} C$.

Par définition, toute chaîne est contenue dans S, d'où $S = \bigcup_{\text{chaîne}} C$.

Toutes ces chaînes étant deux à deux disjointes, alors

$$n \; = \; \mathbf{card}\; S \; = \; \sum_{\text{chaîne}} \mathbf{card}\; C\,. \qquad (1)$$

Soit une chaîne $C = \{M_1, \ldots, M_a\}$.

On pose h = nombre de “sauts” vers le haut (c.à.d. le nombre d'indices i tel que $p_{i+1} = p_i$ et $q_{i+1} = q_i + 1$)

b = nombre de “sauts” vers le bas (idem avec $p_{i+1} = p_i$, $q_{i+1} = q_i - 1$),

g = nombre de “sauts” vers la gauche (idem avec $p_{i+1} = p_i - 1$, $q_{i+1} = q_i$),

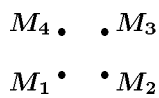
d = nombre de “sauts” vers la droite (idem avec $p_{i+1} = p_i + 1$, $q_{i+1} = q_i$).

Le nombre de sauts vers la gauche est égal au nombre de sauts vers la droite, et le nombre de sauts vers le haut est égal au nombre de sauts vers le bas plus un (rappel $q_a = q_1 + 1$).

C.à.d. $h = b + 1$, $g = d$;
or a = nombre de points = nombre de sauts plus un = $d + g + h + b + 1$.
Donc $a = 2(g + b + 1)$, *pair*.

Toute chaîne est donc de longeur paire.

D'après (1), on en déduit que n *est pair*.

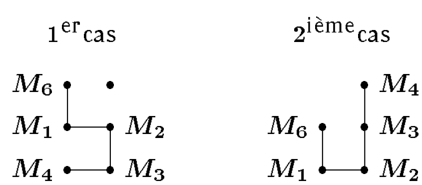
Pour $n = 4$: 

convient.

Pour $n = 6$: Comme $a \geq 4$, un tel ensemble S, s'il existe, n'utilise q'une seule chaîne, et $g + b = 3$, avec $g, b \in \mathbb{N}^*$, d'où $g = b = 1$ et $d = 1$, $h = 2$.

Si M_1 est choisis, M_2 et M_6 sont imposé par construction.

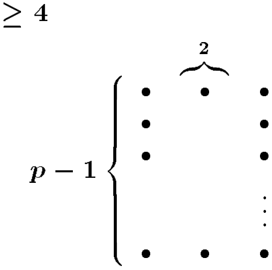
Car $d = 1$, il y a deux places possibles pour M_3 :



Dans les deux cas, la place pour M_4 est imposé car $d = 1 = b$. Dans le 1er cas, M_1 a donc 3 voisins, une contradiction. Dans le 2ième cas, M_3 a trois voisins, contradiction.

Donc $n = 6$ est impossible.

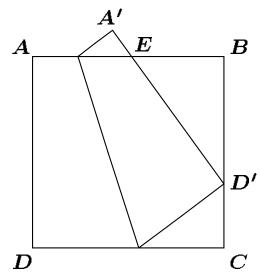
Pour $n = 2p$, $p \geq 4$



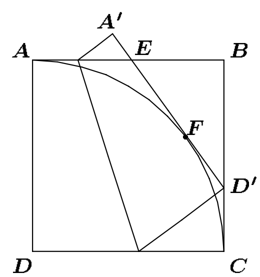
convient .

D'où le résultat annoncé.

3. A square piece of paper $ABCD$ is folded by placing the corner D at some point D' on BC (see figure). Suppose AD is carried into $A'D'$, crossing AB at E. Prove that the perimeter of triangle EBD' is half as long as the perimeter of the square.



Solutions by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec; by Pierre Bornsztein, Courdimanche, France; and by Toshio Seimiya, Kawasaki, Japan. We give the solution by Aassila.



$$\begin{aligned} AB + BC &= AE + EB + BD' + D'C \\ &= EF + EB + BD' + FD' = ED' + EB + BD' . \end{aligned}$$

[Editor's comment: it is easy to show that ED' is tangent to the circle, centre D, radius DC; see *More Mathematical Morsels* by Ross Honsberger, MAA Doliciani Mathematical Expositions, 1991, p. 11.]

4. Determine all positive integers $n < 200$ such that $n^2 + (n+1)^2$ is a perfect square.

Solutions by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec; by Pierre Bornsztein, Courdimanche, France; by Panos E. Tsaoussoglou, Athens, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution and remark.

There are exactly three solutions, $n = 3$, 20, 119. Suppose that $n^2 + (n+1)^2 = k^2$ for some natural number k. Then $(n, n+1, k)$ is a Pythagorean triple. It is in fact a primitive Pythagorean triple since $(n, n+1) = 1$. By a well known result, there exist natural numbers s and t with opposite parities and $s > t$ such that either

(i) $n = 2st$, $n + 1 = s^2 - t^2$, $k = s^2 + t^2$; or

(ii) $n = s^2 - t^2$, $n + 1 = 2st$, $k = s^2 + t^2$.

In case (i) we have $s^2 - 2ts - (t^2 + 1) = 0$. Solving for s we find $s = t + \sqrt{2t^2 + 1}$. Clearly $s \in \mathbb{N}$ if and only if $2t^2 + 1$ is a perfect square. Since $n < 200$ implies $st < 100$, we have $t^2 < 100$ or $t < 10$. Substituting $t = 1, 2, \ldots, 9$ reveals that $2t^2 + 1$ is a square only when $t = 2$, in which case $s = 5$, $n = 20$, $n + 1 = 21$ and $k = 29$.

In case (ii) we have $s^2 - 2ts - (t^2 - 1) = 0$. Solving for s, we find $s = t + \sqrt{2t^2 - 1}$. (Note that $t - \sqrt{2t^2 - 1} < 0$ for all $t \geq 1$). As in case (i), $n + 1 < 201$ implies $st \leq 100$ and thus $t^2 < 100$ or $t < 10$. Substituting $t = 1, 2, \ldots, 9$ reveals that $2t^2 - 1$ is a perfect square exactly when $t = 1$ and $t = 5$. When $t = 1$, we get $s = 2$, $n = 3$, $n + 1 = 4$, $k = 5$. When $t = 5$, we get $s = 12$, $n = 119$, $k = 169$.

Remark. In fact, the characterization of *all* (primitive) Pythagorean triples of the form $(n, n + 1, k)$ is known; for example, in Chapter 2.4 of *Elementary Theory of Numbers* by W. Sierpinski, 2nd ed., 1985, it is proved that if the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are defined recursively by $x_{n+1} = 3x_n + 2z_n + 1$, $y_{n+1} = x_{n+1} + 1$, $z_{n+1} = 4x_n + 3z_n + 2$ for all $n \geq 1$, with initial values $x_1 = 3$, $y_1 = 4$ and $z_1 = 5$, then (x_n, y_n, z_n), $n = 1, 2, 3, \ldots$ would be all the (primitive) Pythagorean triples for which $y_n = x_n + 1$. Using these iterations one can easily verify that the triples we obtained in the solution do indeed give the first three such triples. The next four values of n are 696, 4059, and 23660.

We finish this number of the Corner by giving readers' solutions to problems of the 44$^{\text{th}}$ Lithuanian Mathematical Olympiad [1998: 196–197].

GRADE XI

1. You are given a set of 10 positive integers. Summing nine of them in ten possible ways we get only nine different sums: 86, 87, 88, 89, 90, 91, 93, 94, 95. Find those numbers.

Solutions by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec; by Pierre Bornsztein, Courdimanche, France; by Christopher J. Bradley, Clifton College, Bristol, UK; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bradley's write-up.

Let S be the sum of all ten positive integers and suppose x is the repeated sum. Call the elements $a_1, a_2, \ldots, a_{10}$. Then we have

$$S - a_1 = 86\,, \quad S - a_2 = 87\,, \quad \ldots\,, \quad S - a_9 = 95\,, \quad S - a_{10} = x\,.$$

Adding, $9S = 813 + x$. The only value of x from 86, 87, 88, ..., 95 which makes $813 + x$ divisible by 9 is $x = 87$ and then $S = 100$. It follows that the ten numbers are respectively 14, 13, 12, 11, 10, 9, 7, 6, 5 and 13.

2. What is the least possible number of positive integers such that the sum of their squares equals 1995?

Solutions by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec; by Pierre Bornsztein, Courdimanche, France; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; by Bob Prielipp, University of Wisconsin–Oshkosh, WI, USA; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Klamkin's solution.

First note that $1995 = 3 \cdot 5 \cdot 7 \cdot 19$. We now use some known theorems [1] on representations of a number as sums of squares.

1. A natural number n is the sum of two squares if and only if the factorization of n does not contain any prime of the form $4k + 3$ that has an odd exponent.

2. A natural number n can be the sum of three squares if and only if it is not of the form $4^l(8k + 7)$, where k, l are non-negative integers.

3. Each odd natural number is the sum of the squares of four integers, two of which are consecutive numbers.

In view of the above theorems, the minimum number is three and a representation is given by $1995 = 1^2 + 25^2 + 37^2$.

Comment. A number for which the minimum number of squares is four is given by $1992 = 3 \cdot 8 \cdot 83 = 10^2 + 18^2 + 28^2 + 28^2 = 2^2 + 4^2 + 6^2 + 44^2$. Also $1995 = 13^2 + 24^2 + 25^2 + 25^2$.

Reference

[1] W. Sierpinski, Elementary Theory of Numbers, Hafner, NY, 1964, pp. 351, 363, 367.

3. Replace the asterisks in the "equilateral triangle"

```
* * * * * * * * *
 * * * * * * * *
  * * * * * * *
   * * * * * *
    * * * * *
     * * * *
      * * *
       * *
        *
```

by the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 so that, starting from the second line, each number is equal to the absolute value of the difference of the nearest two numbers in the line above.

Is it always possible to inscribe the numbers $1, 2, \ldots, n$, in the way required, into the equilateral triangle with the sides having n asterisks?

Solutions by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec; and by Christopher J. Bradley, Clifton College, Bristol, UK. We give Bradley's solution.

The case $n = 9$ can be done as follows:

$$\begin{array}{ccccccccccccccccc}
1&&7&&8&&1&&9&&8&&1&&7&&6\\
&6&&1&&7&&8&&1&&7&&6&&1&\\
&&5&&6&&1&&7&&6&&1&&5&&\\
&&&1&&5&&6&&1&&5&&4&&&\\
&&&&4&&1&&5&&4&&1&&&&\\
&&&&&3&&4&&1&&3&&&&&\\
&&&&&&1&&3&&2&&&&&&\\
&&&&&&&2&&1&&&&&&&\\
&&&&&&&&1&&&&&&&&
\end{array}$$

For the case $n = 6m + 3$ one starts with the row

$$1\;4m{+}3\;\ldots\;1\;6m{+}1\;6m{+}2\;1\;\;\overbrace{6m+3}^{\text{centre square}}\;\;6m{+}2\;1\;6m{+}1\;6m\;1\;\ldots\;4m{+}3\;4m{+}2\,.$$

Then if one works down six rows you get

$$1\;4m{-}1\;\ldots\;1\;6m{-}5\;6m{-}4\;1\;6m{-}3\;6m{-}4\;1\;6m{-}5\;6m{-}6\;1\;\ldots\;4m{-}1\;4m{-}2$$

and the result then follows by induction on m, with the pattern up to $m = 1$ as in the first triangle.

4. A function $f : \mathbb{N} \to \mathbb{N}$ is such that $f(f(m) + f(n)) = m + n$ for all $m, n \in \mathbb{N}$ ($\mathbb{N} = \{1, 2, \ldots\}$ denotes the set of all positive integers). Find all such functions.

Solutions by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec; by Pierre Bornsztein, Courdimanche, France; and by Christopher J. Bradley, Clifton College, Bristol, UK. We give the solution by Bornsztein.

Pour $m, n \in \mathbb{N}^*$, d'après

$$f(f(m) + f(n)) \;=\; m + n \qquad (1)$$

$$f(f(m) + f(n)) + f(f(m) + f(n)) \;=\; 2(m + n)$$

et donc

$$\begin{aligned} f[f(f(m) + f(n)) + f(f(m) + f(n))] &= f(m) + f(n) + f(m) + f(n) \\ &= 2(f(m) + f(n)) \end{aligned}$$

et

$$f[f(f(m) + f(n)) + f(f(m) + f(n))] \;=\; f(2m + 2n)\,.$$

Ainsi

$$2f(m) + 2f(n) \;=\; f(2m + 2n)\,. \qquad (2)$$

Pour $m = n$, $\;4f(n) = f(4n)$.

Pour $m = 2p + 1$, $n = 2p - 1$, $2f(2p + 1) + 2f(2p - 1) = f(8p) = 4f(2p)$ d'où pour $p \geq 1$,

$$f(2p + 1) \;=\; 2f(2p) - f(2p - 1)\,. \qquad (3)$$

De même pour $m = 2p + 2$, $n = 2p - 2$, $(p \geq 2)$ on obtient

$$f(2p + 2) \;=\; 2f(2p) - f(2p - 2)\,. \qquad (4)$$

On pose $f(1) = a$, $f(2) = b$.

En utilisant (3) et (4) on trouve

$$\begin{aligned} f(3) &= 2b - a\,, \\ f(4) &= f(4 \times 1) = 4f(1) = 4a\,, \\ f(5) &= 9a - 2b\,, \\ f(6) &= 8a - b\,. \end{aligned}$$

Mais pour $m = 2$, $n = 1$, (2) conduit à $f(6) = 2f(2) + 2f(1) = 2a + 2b$. Donc $8a - b = 2a + 2b$; c.à.d. $b = 2a$. On en déduit que pour $n \in \{1, \ldots, 6\}$, $f(n) = an$. Une récurrence immédiate en utilisant (3) et (4) permet d'obtenir :

$$f(n) \;=\; an, \quad \text{pour tout} \quad n \in \mathbb{N}^*\,.$$

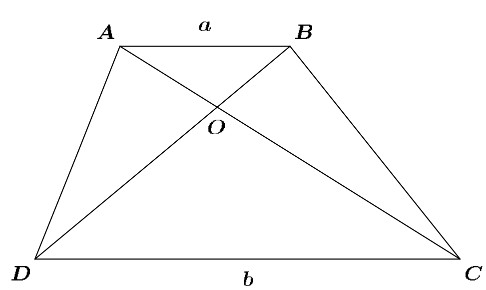
Alors dans (1), pour tous $m, n \in \mathbb{N}^*$

$$m + n \;=\; f(f(m) + f(n)) \;=\; f(a(m + n)) \;=\; a^2(m + n)$$

d'où $a = 1$ et $f = id_{\mathbb{N}^*}$. Réciproquent, $f = id_{\mathbb{N}^*}$ convient.

5. In the trapezium $ABCD$, the bases are $AB = a$, $CD = b$, and the diagonals meet at the point O. Find the ratio of the areas of the triangle ABO and trapezium.

Solutions by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec; by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Pierre Bornsztein, Courdimanche, France; and by Christopher J. Bradley, Clifton College, Bristol, UK. We give the solution by Amengual.



Since triangles with the same height have areas in proportion to their bases and since $\triangle AOB \sim \triangle COD$, we have

$$\frac{[DOA]}{[AOB]} = \frac{OD}{OB} = \frac{b}{a} \quad \text{or} \quad [DOA] = \frac{b}{a}[AOB],$$

where $[\mathcal{P}]$ denotes the area of polygon $[\mathcal{P}]$.

Also, since the areas of similar triangles are proportional to the squares on corresponding sides,

$$\frac{[COD]}{[AOB]} = \frac{b^2}{a^2} \quad \text{or} \quad [COD] = \frac{b^2}{a^2}[AOB].$$

Finally, since $\triangle ABD$ and $\triangle ABC$ have the same base and equal altitude $[ABD] = [ABC]$, and since

$$[ABD] = [AOB] + [DOA], \quad [ABC] = [AOB] + [BOC],$$

it follows that $[DOA] = [BOC]$.

Consequently,

$$\begin{aligned}[ABCD] &= [AOB] + [BOC] + [COD] + [DOA] \\ &= [AOB] + 2[DOA] + [COD];\end{aligned}$$

that is,

$$[ABCD] = \left(1 + 2 \cdot \frac{b}{a} + \frac{b^2}{a^2}\right)[AOB],$$

giving

$$[ABCD] = \left(\frac{a+b}{a}\right)^2 [AOB],$$

and

$$\frac{[AOB]}{[ABCD]} = \left(\frac{a}{a+b}\right)^2.$$

Remark. The following related problem appears in *Solving Problems in Geometry* by V. Gusev, V. Litvinenko and A. Mordkovich, Mir Publishers, Moscow 1988, page 60: The diagonals of a trapezoid $ABCD$ $(AD \| BC)$ intersect at the point O. Find the area of the trapezoid if it is known that the area of the triangle AOD is equal to a^2, and the area of the triangle BOC is equal to b^2.

GRADE XII

1. Consider all pairs of real numbers satisfying the inequalities

$$-1 \le x + y \le 1, \qquad -1 \le xy + x + y \le 1.$$

Let M denote the largest possible value of x.

(a) Prove that $M \leq 3$.

(b) Prove that $M \leq 2$.

(c) Find M.

Solutions by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec; by Pierre Bornsztein, Courdimanche, France; by Christopher J. Bradley, Clifton College, Bristol, UK; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We use the solution by Klamkin.

Letting $u = x + 1$ and $v = y + 1$, the given inequalities become

$$0 \leq uv \leq 2, \qquad 1 \leq u + v \leq 3.$$

Clearly u and v must each be non-negative and so the maximum value of u is 3. Finally, $x_{\max} = 2$.

2. A positive integer n is called an *ambitious* number if it possesses the following property: writing it down (in decimal representation) on the right of any positive integer gives a number that is divisible by n. Find:

(a) the first 10 ambitious numbers;

(b) all the ambitious numbers.

Solutions by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec; by Christopher J. Bradley, Clifton College, Bristol, UK; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bradley's solution.

We assume in this solution that n cannot begin with a zero (so that 04 is not acceptable). For n to be ambitious it must be a k–digit number dividing $10^k \times m$ for all m, which is so if and only if $n \mid 10^k$. The ambitious numbers are thus $1, 2, 5, 10, 20, 25, 50, 100, 125, 200, \ldots$.

3. The area of a trapezium equals 2; the sum of its diagonals equals 4. Prove that the diagonals are orthogonal.

Solutions by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec; by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Pierre Bornsztein, Courdimanche, France; by Christopher J. Bradley, Clifton College, Bristol, UK; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give the solution of Klamkin.

If a and b are the diagonal lengths and θ the angle between them, then the area $2 = \frac{1}{2}ab\sin\theta$. Since $4 = a + b \geq 2\sqrt{ab}$, the maximum value of ab is 4. Hence $\sin\theta = 1$, so that $\theta = \pi/2$.

4. 100 numbers are written around a circle. Their sum equals 100. The sum of any 6 neighbouring numbers does not exceed 6. The first number is 6. Find the remaining numbers.

Solutions by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec; and by Pierre Bornsztein, Courdimanche, France. We give the solution of Bornsztein.

On appelle $x_1, x_2, \ldots, x_{100}$ ces nombres dans le sens horaire, et on pose $x_{100+i} = x_i$. On a, pour $i = 1, 2, \ldots, 100$

$$x_i + x_{i+1} + \cdots + x_{100} + x_1 + \ldots x_{i-1} + x_i + x_{i+1} \;=\; 100 + x_i + x_{i+1}$$

et comme il y a $102 = 6 \times 17$ termes dans cette somme, en les groupant par 6 consécutifs, on a $100 + x_i + x_{i+1} \le 102$

$$\text{c.à.d.}\quad x_i + x_{i+1} \;\le\; 2\,.$$

Alors

$$100 \;=\; (x_1 + x_2) + (x_3 + x_4) + \cdots + (x_{99} + x_{100}) \;\le\; 2 \times 50$$

avec égalité ssi pour tout $i \ge 1$, $x_{2i-1} + x_{2i} = 2$. Donc pour $i \ge 1$, $x_{2i-1} + x_{2i} = 2$.

De même

$$100 \;=\; (x_2 + x_3) + (x_4 + x_5) + \cdots + (x_{100} + x_1) \;\le\; 2 \times 50$$

avec éqalité ssi $x_{2i} + x_{2i+1} = 2$ pour $i \ge 1$.

Finalement, pour tout $i \ge 1$, $x_i + x_{i+1} = 2$. Or $x_1 = 6$ donc $x_2 = -4$ et par récurrence immédiate pour $i = 1, 2, \ldots, 50$, $x_{2i-1} = 6$, $x_{2i} = -4$. Cette famille vérifie bien les conditions de l'énoncé.

5. Show that, at any time, moving both the hour-hand and the minute-hand of the clock symmetrically with respect to the vertical $(6 - 12)$ axis results in a possible position of the clock-hands. How many straight lines containing the centre of the clock-face possess the same property?

Solution by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.

$$\text{Eleven,}\quad \frac{360^\circ n}{11} \qquad (n \;=\; 0, 1, 2, \ldots, 10)\,.$$

That completes the Corner for this issue. Send me your nice solutions as well as contest materials.

BOOK REVIEWS

ALAN LAW

Problem-Solving Strategies for Efficient and Elegant Solutions by Alfred S. Posamentier and Stephen Krulik,
published by Corwin Press, 1998.
ISBN # 0-8039-6698-9, softcover, 249+ pages, $29.95 (U.S.).
Reviewed by **Ian VanderBurgh**, *University of Waterloo, Waterloo, Ontario.*

This new book from well-known authors Alfred Posamentier and Stephen Krulik is an excellent introduction to problem solving and the thought processes behind problem solving.

"Problem Solving Strategies ... " devotes each of ten chapters to a technique useful in solving problems. These techniques range from "Working Backwards" to "Finding a Pattern" to "Making a Drawing". Each chapter shows first how each technique can be used in everyday life, and then shows its use in a mathematical example. Posamentier and Krulik then present roughly 20 problems with solutions in each chapter.

There are definite advantages to their presentation. The layout of each chapter is consistent, which makes their book very easy to read. I liked the idea of showing clear applications in everyday life — this gives the reader something to relate to if he or she has never seen a particular technique previously in a mathematical setting. Also, with each problem, two solutions are usually given, one "brute force" and the other using the technique of the chapter, giving a clear contrast between a potentially ugly solution and an elegant one. The exposition does, however, tend to get a little wordy, which can be either a plus or a minus.

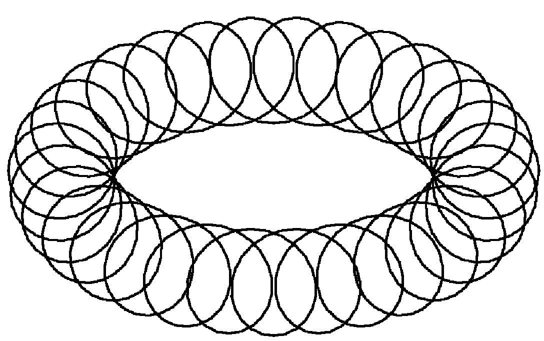
There are some difficulties that I encountered in working through this book. A couple of the problems and solutions are incomprehensible, and other problems are really overly simplistic. My major concern arose when some of the problems said "Show that ... " and the authors in their solution found a pattern or considered a couple of extreme cases, thus considering the problem finished without making any attempt to prove the required result. This is a fine way to demonstrate the use of their techniques, but could be dangerous for a student reader unfamiliar with the concept of a proof. My other concern is that the book is lacking a conclusion to tie together all of the ideas presented.

That being said, this book can serve as a good resource for teachers who want to brush up on their problem solving skills or teach them to younger high school students. The vast majority of the problems would be accessible to students in Grades 9–11. The techniques presented are good as general problem solving skills, but are also of excellent value for use on multiple-choice or answer-only contests, as they generally lead one to the correct result fairly quickly. If nothing else, this book is a good collection of problems, and thus could serve as a good resource.

The art of problem solving, Volume 2, by S. Lehoczky and R. Rusczyk. Published by Greater Testing Concepts, P.O. Box A-D, Stanford, CA 94309, 1994. Paperback, 390+ pages, solution manual 212 pages, without ISBN number, US $27 without solution manual or $35 with. (May be ordered in combination with Volume 1, which has been reviewed in *Crux* [1994: 135-136], US $47 without solution manuals or $60 with.)
Reviewed by **Andy Liu**, *University of Alberta, Edmonton, Alberta*

This volume consists of 27 chapters, 7 on algebra, 4 on geometry, 2 on analytic geometry, 2 on trigonometry, 2 on vectors, 5 on combinatorics, 3 on number theory, an introductory chapter on proof techniques and a final chapter of further problems. In all, there are 237 examples, 412 exercises and 509 problems. The companion manual contains solutions to all the exercises and problems.

Much that was said enthusiastically about Volume 1 applies here also. The only slight disappointment is that while the topics covered are more advanced, the treatment is at essentially the same level as that in Volume 1. Nevertheless, this set provides excellent preparation for introductory level mathematics competitions, and important stepping stones towards further study in solving problems of Olympiad calibre.



THE SKOLIAD CORNER

No. 40

R.E. Woodrow

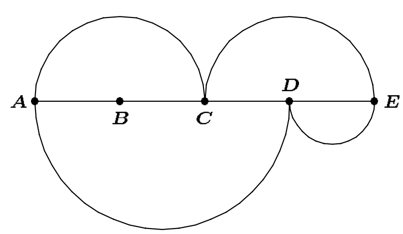
In this issue, we give the Final Round Parts A and B of the 1998 British Columbia Colleges Junior High School Mathematics Contest. My thanks go to Jim Totten, University College of the Cariboo, one of the organizers, for forwarding the materials for use in the *Corner*.

BRITISH COLUMBIA COLLEGES JR. HIGH SCHOOL MATHEMATICS CONTEST
Final Round 1998
Part A

1. Each edge of a cube is coloured either red or black. If every face of the cube has at least one black edge, the smallest possible number of black edges is:

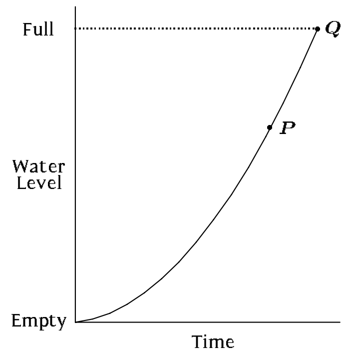
(a) 6 (b) 5 (c) 4 (d) 3 (e) 2

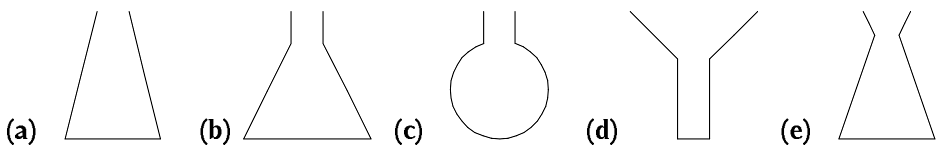
2. Line AE is divided into four equal parts by the points B, C and D. Semicircles are drawn on segments AC, CE, AD and DE creating semicircular regions as shown. The ratio of the area enclosed above the line AE to the area enclosed below the line is:



(a) $4:5$ (b) $5:4$ (c) $1:1$ (d) $8:9$ (e) $9:8$

3. A container is completely filled from a tap running at a uniform rate. The accompanying graph shows the level of the water in the container at any time while the container is being filled. The segment PQ is a straight line. The shape of the container which corresponds with the graph is:





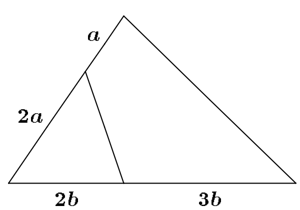
4. The digits 1, 9, 9, and 8 are placed on four cards. Two of the cards are selected at random. The probability that the sum of the numbers on the cards selected is a multiple of 3 is:

(a) $\frac{1}{4}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{2}{3}$ (e) $\frac{3}{4}$

5. The surface areas of the six faces of a rectangular solid are 4, 4, 8, 8, 18 and 18 square centimetres. The volume of the solid, in cubic centimetres is:

(a) 24 (b) 48 (c) 60 (d) 324 (e) 576

6. The area of the small triangle in the diagram is 8 square units. The area of the large triangle, in square units, is:



(a) 18 (b) 20 (c) 24 (d) 28 (e) 30

7. At 6:15 the hands of the clock form two positive angles with a sum of 360°. The difference of the degree measures of these two angles is:

(a) 165 (b) 170 (c) 175 (d) 180 (e) 185

8. The last digit of the number 8^{26} is:

(a) 0 (b) 2 (c) 4 (d) 6 (e) 8

9. For the equation $\frac{A}{x+3} + \frac{B}{x-3} = \frac{-x+9}{x^2-9}$ to be true for all values of x for which the expressions in the equation make sense, the value of AB is:

(a) 2 (b) -1 (c) -2 (d) -3 (e) -6

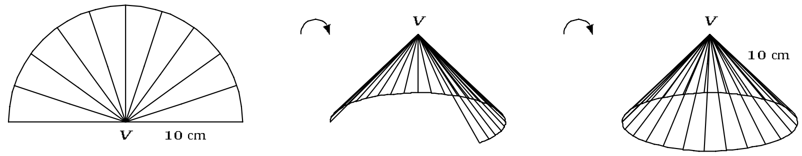
10. A hungry hunter came upon two shepherds, Joe and Frank. Joe had three small loaves of bread and Frank five loaves of the same size. The loaves were divided equally among the three people, and the hunter paid \$8 for his share. If the shepherds divide the money so that each gets an equitable share based on the amount of bread given to the hunter, the amount of money that Joe receives is:

(a) \$1 (b) \$1.50 (c) \$2 (d) \$2.50 (e) \$3

Part B

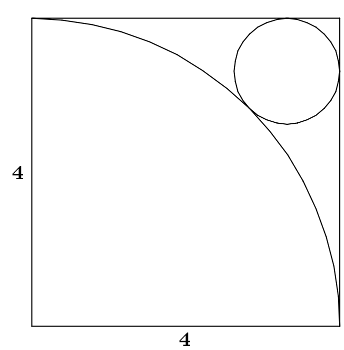
1. Four positive integers sum to 125. If the first of these numbers is increased by 4, the second is decreased by 4, the third is multiplied by 4 and the fourth is divided by 4, you produce four equal numbers. What are the four original numbers?

2. A semi-circular piece of paper of radius 10 cm is formed into a conical paper cup as shown (the cup is inverted in the diagram):



Find the height of the paper cup; that is, the depth of water in the cup when it is full.

3. In the diagram a quarter circle is inscribed in a square with side length 4, as shown. Find the radius of the small circle that is tangent to the quarter circle and two sides of the square.



4. Using the digits 1, 9, 9 and 8 *in that order* create expressions equal to 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10. You may use any of the four basic operations ($+$, $-$, $\times$, $\div$), the square root symbol ($\sqrt{}$) and parentheses, as necessary. For example, valid expressions for 25 and 36 would be

$$\begin{aligned} 25 &= -1+9+9+8 \\ 36 &= 1+9\times\sqrt{9}+8 \end{aligned}$$

Note: You may place a negative sign in front of 1 to create -1 if you wish.

5. At 6 am, one Saturday, you and a friend begin a recreational climb of Mt. Mystic. Two hours into your climb, you are overtaken by some scouts. As they pass, they inform you that they are attempting to set a record for ascending and descending the mountain. At 10 am, they pass you again on their way down, crowing that they had not stopped once to rest, not even at the top.

You finally reach the summit at noon. Assuming that both you and the scouts travelled at a constant vertical rate, both climbing and descending, when did the scouts reach the top of Mt. Mystic?

In the last number, we gave the problems of the Florida Mathematics Olympiad, Team Competition for 1998. Next we give the solutions. My thanks go to John Grant McLoughlin, Memorial University of Newfoundland, for forwarding the contest for our use.

FLORIDA MATHEMATICS OLYMPIAD
TEAM COMPETITION
May 14, 1998

1. Find all integers x, if any, such that $9 < x < 15$ and the sequence

$$1\,,\,2\,,\,6\,,\,7\,,\,9\,,\,x\,,\,15\,,\,18\,,\,20$$

does not have three terms in arithmetic progression. If there are no such integers, write "NONE."

Answer: $x = 14$.

The following chart allows us to eliminate 10, 11, 12, and 13:

x	Possible arithmetic progression(s) if x is in the sequence
10	2, 6, 10; 2, 10, 18; 10, 15, 20.
11	2, 11, 20; 7, 9, 11; 7, 11, 15.
12	2, 7, 12; 6, 9, 12; 6, 12, 18; 9, 12, 15.
13	1, 7, 13; 6, 13, 20.

As can be verified, the sequence

$$1, 2, 6, 7, 9, 14, 15, 18, 20$$

does not have three terms in arithmetic progression.

2. A sequence a_1, a_2, a_3, ... is said to satisfy a *linear recurrence relation of order two* if and only if there are numbers p and q such that, for all positive integers n,

$$a_{n+2} = pa_{n+1} + qa_n .$$

Find the next two terms of the sequence

$$2, 5, 14, 41, \ldots$$

assuming that this sequence satisfies a linear recurrence relation of order two.

Answer: The next two terms are 122 and 365.
Let a_1, a_2, a_3, a_4, ... be the sequence

$$2, 5, 14, 41, \ldots .$$

Since this sequence satisfies a linear recurrence of order two, we know that, in particular, there are numbers p and q such that

$$a_3 = pa_2 + qa_1 \quad \text{and} \quad a_4 = pa_3 + qa_2 .$$

That is,

$$14 = 5p + 2q \quad \text{and} \quad 41 = 14p + 5q .$$

Solving this system of equations in the usual way, we see that

$$p = 4 \quad \text{and} \quad q = -3 .$$

Thus, for all positive integers n,

$$a_{n+2} = 4a_{n+1} - 3a_n .$$

Hence the next two terms are

$$a_5 = 4a_4 - 3a_3 = 4(41) - 3(14) = 122 ,$$

and

$$a_6 = 4a_5 - 3a_4 = 4(122) - 3(41) = 365 .$$

3. Seven tests are given and on each test no ties are possible. Each person who is the top scorer on at least one of the tests or who is in the top six on at least four of these tests is given an award, but each person can receive at most one award. Find the maximum number of people who could be given awards if 100 students take these tests.

Answer: 15.

The maximum number of people who can receive an award occurs when

(i) no person is the top scorer more than once, and
(ii) the 35 positions second through sixth in the seven competitions are filled by each of 8 non-top-scorers at least four times.

This can happen as follows:

	Test						
Position	1	2	3	4	5	6	7
1	T	U	V	W	X	Y	Z
2	A	B	C	D	E	F	G
3	H	A	B	C	D	E	F
4	G	H	A	B	C	D	E
5	F	G	H	A	B	C	D
6	E	F	G	H	–	–	–

4. Some primes can be written as a sum of two squares. We have, for example, that

$$5 = 1^2 + 2^2, \quad 13 = 2^2 + 3^2, \quad 17 = 1^2 + 4^2,$$

$$29 = 2^2 + 5^2, \quad 37 = 1^2 + 6^2, \quad \text{and} \quad 41 = 4^2 + 5^2.$$

The odd primes less than 108 are listed below; the ones that can be written as a sum of two squares are boxed in.

$$3, \boxed{5}, 7, 11, \boxed{13}, \boxed{17}, 19, 23, \boxed{29}, 31,$$

$$\boxed{37}, \boxed{41}, 43, 47, \boxed{53}, 59, \boxed{61}, 67, 71,$$

$$\boxed{73}, 79, 83, \boxed{89}, \boxed{97}, \boxed{101}, 103, 107.$$

The primes that can be written as a sum of two squares follow a simple pattern. See if you can correctly find this pattern. If you can, use this pattern to determine which of the primes between 1000 and 1050 can be written as a sum of two squares; there are five of them. The primes between 1000 and 1050 are

$$1009, 1013, 1019, 1021, 1031, 1033, 1039, 1049.$$

No credit unless the correct five primes are listed.

Answer: 1009, 1013, 1021, 1033, 1049.

Remark: An odd prime can be written as a sum of two squares if and only if when the prime is divided by 4 the remainder is 1.

5. The sides of a triangle are 4, 13, and 15. Find the radius of the inscribed circle.

Answer: 1.5.

Let A, B, C be the vertices of the triangle. The area of $\triangle ABC$ is, by Hero's Formula,

$$\sqrt{16(12)3(1)} \;=\; 24\,.$$

Let O be the centre of the inscribed circle and let r be its radius. Thus

$$\begin{aligned} 24 &= \text{ area of } \triangle AOB + \text{ area of } \triangle BOC + \text{ area of } \triangle AOC \\ &= 0.5r(\overline{AB}) + 0.5r(\overline{BC}) + 0.5r(\overline{AC}) \\ &= 0.5r(4+13+15) \;=\; 16r\,, \end{aligned}$$

we see that $r = 1.5$.

6. In Athenian criminal proceedings, ordinary citizens presented the charges, and the 500-man juries voted twice: first on guilt or innocence, and then (if the verdict was guilty) on the penalty. In 399 BCE, Socrates (c. 469–399) was charged with dishonouring the gods and corrupting the youth of Athens. He was found guilty; the penalty was death. According to I.F. Stone's calculations on how the jurors voted:

(i) There were no abstentions;

(ii) There were 80 more votes for the death penalty than there were for the guilty verdict;

(iii) The sum of the number of votes for an innocent verdict and the number of votes against the death penalty equalled the number of votes in favour of the death penalty.

a) How many of the 500 jurors voted for an innocent verdict?
b) How many of the 500 jurors voted in favour of the death penalty?

Answer: a) 220 voted for an innocent verdict. b) 360 voted in favour of the death penalty.

Let x be the number of jurors who voted innocent and let y be the number of jurors who voted in favour of the death penalty. Since there were no abstentions,

$$500 - x \text{ jurors voted guilty}$$

and

$$500 - y \text{ jurors voted against the death penalty.}$$

From (ii) we see that

$$y - (500 - x) \;=\; 80$$

and from (iii) we see that

$$x + (500 - y) \;=\; y\,.$$

Simplifying we see that

$$x + y \;=\; 580 \quad \text{and} \quad x - 2y \;=\; -500\,.$$

Solving this system of equations we see that

$$\begin{aligned} 3y &= 1080 \\ y &= 360\,. \end{aligned}$$

Hence $x = 220$.

7. Find all x such that $0 \le x \le \pi$ and

$$\tan^3 x - 1 + \frac{1}{\cos^2 x} - 3\cot\left(\frac{\pi}{2} - x\right) \;=\; 3\,.$$

Your answer should be in radian measure.

Answer: $\frac{\pi}{3}$, $\frac{2\pi}{3}$, and $\frac{3\pi}{4}$.
Since

$$-1 + \frac{1}{\cos^2 x} \;=\; -1 + \sec^2 x \;=\; \tan^2 x$$

and

$$\cot\left(\frac{\pi}{2} - x\right) \;=\; \tan x\,,$$

we have the following chain of equations:

$$\begin{aligned} \tan^3 x - 1 + \frac{1}{\cos^2 x} - 3\cot\left(\frac{\pi}{2} - x\right) &= 3\,, \\ \tan^3 x + \tan^2 x - 3\tan x - 3 &= 0\,, \\ (\tan^2 x)(\tan x + 1) - 3(\tan x + 1) &= 0\,, \\ (\tan^2 x - 3)(\tan x + 1) &= 0\,, \end{aligned}$$

$$\tan x \;=\; \sqrt{3}\,, \quad \tan x \;=\; -\sqrt{3}\,, \quad \text{or} \quad \tan x \;=\; -1\,,$$

$$x \;=\; \frac{\pi}{3}\,,\ \frac{2\pi}{3}\,,\ \text{or}\ \frac{3\pi}{4}\,.$$

That completes the *Skoliad Corner* for this number. I need your suitable level contests and welcome your comments for future directions of the *Corner*.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to Mathematical Mayhem, Department of Mathematics, University of Toronto, 100 St. George St., Toronto, Ontario, Canada. M5S 3G3. The electronic address is still

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The Assistant Mayhem Editor is Cyrus Hsia (University of Western Ontario). The rest of the staff consists of Adrian Chan (Upper Canada College), Jimmy Chui (University of Toronto), David Savitt (Harvard University) and Wai Ling Yee (University of Waterloo).

Mayhem Problems

The Mayhem Problems editors are:

Adrian Chan	*Mayhem High School Problems Editor,*
Donny Cheung	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only solutions — the next issue will feature only problems.

We warmly welcome proposals for problems and solutions. With the schedule of eight issues per year, we request that solutions from the last issue be submitted in time for issue 6 of 2000.

High School Solutions

Editor: Adrian Chan, 229 Old Yonge Street, Toronto, Ontario, Canada. M2P 1R5 <all238@sprint.com>

H241. Find the integer n that satisfies the equation

$$1 \cdot 1998 + 2 \cdot 1997 + 3 \cdot 1996 + \cdots + 1997 \cdot 2 + 1998 \cdot 1 \;=\; \binom{n}{3}\,.$$

Solution by Shawn Godin, Cairine Wilson S.S., Orleans, Ontario; Penny Nom, University of Regina, Regina, Saskatchewan; and Edward T.H. Wang, *Wilfrid Laurier University, Waterloo, Ontario*. (All three had virtually identical solutions.)

Consider a set of three distinct numbers $\{a, b, c\}$ chosen from the set $\{1, 2, \ldots, n\}$ such that $a < b < c$. Clearly, $b = k$ for some $k = 2, 3, \ldots, n-1$. Now, having chosen b, there are $k-1$ ways to choose a, and $n-k$ ways to choose c. Hence,

$$\binom{n}{3} = \sum_{k=2}^{n-1}(k-1)(n-k) .$$

Our original equation is the case $n = 2000$ of this equation, so the required value of n is 2000.

Also solved by Masoud Kamgarpour, Carson Graham Secondary School, North Vancouver, BC; and D.J. Smeenk, Zaltbommel, the Netherlands.

Remark. Three other solutions by Godin were received.

H242. Let a and b be real numbers that satisfy $a^2 + b^2 = 1$. Prove the inequality

$$|a^2b + ab^2| \leq \frac{\sqrt{2}}{2} .$$

Determine the values of a and b for which equality occurs. (See how many ways you can solve this!)

Solution I by Masoud Kamgarpour, Carson Graham Secondary School, North Vancouver, BC.

Since $a^2+b^2-2ab = (a-b)^2 \geq 0$ for all real a and b, $2ab \leq a^2+b^2 = 1$, and so $ab \leq 1/2$. Since $a^2 + b^2 + 2ab = (a+b)^2 \geq 0$ for all real a and b, $2ab \geq -(a^2+b^2) = -1$, and so $ab \geq -1/2$. Combining these two inequalities, we have $|ab| \leq 1/2$.

Furthermore, $(a+b)^2 = a^2+2ab+b^2 = (a^2+b^2)+2ab \leq 1+1 = 2$, and so $|a+b| \leq \sqrt{2}$. Multiplying these together, we get

$$|a^2b + ab^2| = |ab(a+b)| = |ab||a+b| \leq \frac{\sqrt{2}}{2} ,$$

which leads to the desired result.

Equality occurs if and only if $|ab| = 1/2$ and $|a+b| = \sqrt{2}$, which implies $(a-b)^2 = a^2 - 2ab + b^2 = 1 - 2ab = 0$, so $a = b = \pm\sqrt{2}/2$.

Solution II. By the QM-AM-GM inequality, we have

$$\sqrt{\frac{1}{2}} = \sqrt{\frac{a^2+b^2}{2}} \geq \frac{|a|+|b|}{2} \geq \sqrt{|ab|} .$$

Thus, $|ab| \leq 1/2$ and $|a| + |b| \leq \sqrt{2}$, with equality if and only if $a = b$. But $a^2 + b^2 = 1$, so this implies that equality occurs if and only if $a = b = \pm\sqrt{2}/2$.

By the Triangle Inequality, $|a+b| \le |a|+|b| \le \sqrt{2}$, so $|a+b| \le \sqrt{2}$. Hence, $|a^2b+ab^2| = |ab(a+b)| = |ab||a+b| \le \sqrt{2}/2$, as required.

Solution III. Since a and b are real numbers with $a^2+b^2=1$, there exists a θ such that $a=\sin\theta$ and $b=\cos\theta$, $0 \le \theta < 2\pi$. Then we have that

$$\begin{aligned} \left|a^2b+ab^2\right| &= \left|\sin^2\theta\cos\theta+\cos^2\theta\sin\theta\right| \\ &= \left|\sin\theta\cos\theta(\sin\theta+\cos\theta)\right| = \frac{1}{2}\left|\sin 2\theta(\sin\theta+\cos\theta)\right| \\ &= \frac{\sqrt{2}}{2}\left|\sin 2\theta(\sin\theta\cdot\cos\tfrac{\pi}{4}+\cos\theta\cdot\sin\tfrac{\pi}{4})\right| \\ &= \frac{\sqrt{2}}{2}\left|\sin 2\theta\sin(\theta+\tfrac{\pi}{4})\right| . \end{aligned}$$

Since $|\sin 2\theta| \le 1$ and $|\sin(\theta+\frac{\pi}{4})| \le 1$, the desired inequality follows immediately. Equality occurs if and only if $|\sin 2\theta| = 1$ and $|\sin(\theta+\frac{\pi}{4})| = 1$, which occurs if and only if $\theta=\pi/4$ or $5\pi/4$, so $a=b=\pm\sqrt{2}/2$.

Also solved by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

H243. For a positive integer n, let $f(n)$ denote the remainder of n^2+2 when divided by 4. For example, $f(3)=3$ and $f(4)=2$. Prove that the equation

$$x^2+(-1)^y f(z) = 10y$$

has no integer solutions in x, y, and z.

Solution by Masoud Kamgarpour, Carson Graham Secondary School, North Vancouver, BC; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Since $(2k)^2 \equiv 0 \pmod 4$ and $(2k+1)^2 = 4k^2+4k+1 \equiv 1 \pmod 4$, every perfect square leaves a remainder of 0 or 1 when divided by 4. Therefore, when n^2+2 is divided by 4, the remainder must be 2 or 3. Hence, $f(z)=2$ or 3 for all z. So, $(-1)^y f(z) \equiv 2$, 3, 7, or 8 modulo 10.

But, x^2 must be 0, 1, 4, 5, 6, or 9 modulo 10. Therefore, $x^2+(-1)^y f(z)$ can never be 0 modulo 10, and the proof is complete.

H244. For a positive integer n, let $P(n)$ denote the sum of the digits of n. For example, $P(123) = 1+2+3 = 6$. Find all positive integers n satisfying the equation $P(n) = n/74$.

Solution by Keon Choi, student, A.Y. Jackson Secondary School, North York, Ontario.

Let n be a k-digit number, so $10^{k-1} \le n < 10^k$. Then $P(n) \le 9k$, since each of the k digits is at most 9, so

$$n - 74P(n) \ \ge\ 10^{k-1} - 74\cdot 9k \ =\ 10^{k-1} - 666k\,.$$

If $k = 5$, then $n - 74P(n) \geq 10^4 - 666 \cdot 5 > 0$, so $P(n)$ cannot equal $n/74$. If $k > 5$, then $n - 74P(n) > 0$, since 10^{k-1} increases much faster than $666k$ (a simple result by induction). So, in order for $P(n) = n/74$, we require $k \leq 4$.

By the rule of divisibility by 9, $P(n) \equiv n \pmod 9$. Hence, $74P(n) = n \equiv P(n) \implies 73P(n) \equiv 0 \pmod 9$. Since 9 is relatively prime to 73, $P(n) \equiv 0 \pmod 9$, and $P(n) = n/74$ must be a multiple of 9. Also, since n has at most four digits, $P(n) \leq 9 \cdot 4 = 36$.

If $n/74 = 9$, then $n = 666$, but $P(666) = 18 \neq 9$.
If $n/74 = 18$, then $n = 1332$, but $P(1332) = 9 \neq 18$.
If $n/74 = 27$, then $n = 1998$, and indeed $P(1998) = 27$.
If $n/74 = 36$, then $n = 2664$, but $P(2664) = 18 \neq 36$.

Hence, the only solution is $n = 1998$.

Also solved by Mara Apostol, student, A.Y. Jackson Secondary School, North York, Ontario; Nick Harland, student, Vincent Massey C.I., Winnipeg, Manitoba; Kenneth Ho, student, Don Mills C.I., Don Mills, Ontario; Masoud Kamgarpour, Carson Graham Secondary School, North Vancouver, BC; Condon Lau, student, David Thompson S.S., Vancouver, BC; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario; Dale Whitmore, student, Prince of Wales C.I., St. John's, Newfoundland; Wendy Yu, student, Danforth College and Tech. Inst., Toronto, Ontario.

Advanced Solutions

Editor: Donny Cheung, c/o Conrad Grebel College, University of Waterloo, Waterloo, Ontario, Canada. N2L 3G6 <dccheung@uwaterloo.ca>

A217. *Proposed by Alexandre Trichtchenko, OAC student, Brookfield High School, Ottawa.*

Show that for any odd prime p, there exists a positive integer n such that n, n^n, n^{n^n}, ... all leave the same remainder upon division by p where n does not leave a remainder of 0 or 1 upon division by p.

Solution. We claim that $n = 2p - 1$ will satisfy the given conditions. First, $n = 2p - 1$ is odd and $n = 2p - 1 \equiv -1 \pmod p$. So for any tower of ns, we have

$$n^{n^{\cdots^n}} \equiv (-1)^{n^{\cdots^n}} \equiv -1 \pmod p .$$

Therefore, all the numbers n, n^n, n^{n^n}, ... leave a remainder of $p - 1$ when divided by p. Finally, since p is an odd prime, $2p - 1$ cannot be 0 or 1.

A218. *Proposed by Mohammed Aassila, Strasbourg, France.*

(a) Suppose $f(x) = x^n + qx^{n-1} + t$, where q and t are integers, and suppose there is some prime p such that p divides t but p^2 does not divide t. Show, by imitating the proof of Eisenstein's Theorem, that either f is irreducible or f can be reduced into two factors, one of which is linear and the other irreducible.

(b) Deduce that if both q and t are odd then f is irreducible.

(Generalization of Question 1, IMO 1993)

[Ed.: Here "irreducible" means "irreducible over the integers".]

Solution by the proposer. (a) Suppose on the contrary that $f(x) = g(x)h(x)$ with

$$\begin{aligned} g(x) &= x^r + c_{r-1}x^{r-1} + \cdots + c_0\,, \\ h(x) &= x^s + b_{s-1}x^{s-1} + \cdots + b_0\,, \end{aligned}$$

$0 < r$, $s < n$ and $r + s = n$. Since p divides t and p^2 does not divide t, we have p dividing one of c_0 or b_0, but not both. Say p divides c_0, but it does not divide b_0. Let m be the largest i such that p divides the i^{th} coefficient c_i. Then $0 \le m < r$ and p does not divide c_{m+1}. The coefficient of x^{m+1} in $f(x)$ is

$$c_{m+1}b_0 + c_m b_1 + \cdots \;\equiv\; c_{m+1}b_0 \pmod{p}\,.$$

Further, this is not congruent to 0 modulo p. So $n > r \ge m+1 \ge n-1 \ge r$. Hence $n - 1 = r$, $s = 1$, and so $h(x)$ is a linear factor. It remains to show that $g(x)$ is irreducible, but since $m = r - 1$, we have that p divides c_0, c_1, ... , c_{r-1}, and p does not divide c_r. So it follows that $g(x)$ is irreducible by Eisenstein's criterion.

(b) If q and t are odd, then $f(x)$ is odd for any integer x, and so it has no integer roots. Thus it cannot have a linear factor and so by part (a) is irreducible.

A219. *Proposed by Mohammed Aassila, Strasbourg, France.*

Solve the following system:

$$3\left(x + \frac{1}{x}\right) \;=\; 4\left(y + \frac{1}{y}\right) \;=\; 5\left(z + \frac{1}{z}\right)\;; \qquad xy + yz + zx \;=\; 1\,.$$

Solution by D.J. Smeenk, Zaltbommel, the Netherlands.

The values x, y, and z are all positive or all negative. If (x_0, y_0, z_0) is a solution of the equations, then $(-x_0, -y_0, -z_0)$ must also be a solution.

In any triangle ABC, with angles A, B, and C, we have

$$\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} \;=\; 1\,.$$

We denote $x = \tan(A/2)$, $y = \tan(B/2)$, and $z = \tan(C/2)$. Then we have $x + 1/x = 2/\sin A$, $y + 1/y = 2/\sin B$, and $z + 1/z = 2/\sin C$. So the equation to be solved can be rewritten as

$$\frac{3}{\sin A} = \frac{4}{\sin B} = \frac{5}{\sin C}\,.$$

So triangle ABC has a right angle at C. We find that $\sin A = 3/5$, $\cos A = 4/5$, $\sin B = 4/5$, $\cos B = 3/5$, and $C = \pi/2$, and

$$\begin{aligned} x &= \tan\frac{A}{2} = \frac{1-\cos A}{\sin A} = \frac{1}{3}\,, \\ y &= \tan\frac{B}{2} = \frac{1-\cos B}{\sin B} = \frac{1}{2}\,, \\ z &= \tan\frac{C}{2} = 1\,. \end{aligned}$$

The solutions (x, y, z) are $(1/3, 1/2, 1)$ and $(-1/3, -1/2, -1)$.

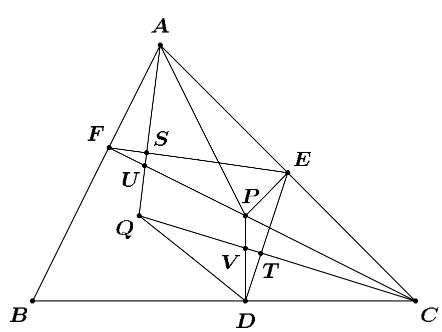
A220. *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

P is an interior point of triangle ABC. D, E, and F are the feet of the perpendiculars from P to the lines BC, CA, and AB, respectively. Let Q be the interior point of triangle ABC such that

$$\angle ACP = \angle BCQ \quad\text{and}\quad \angle BAQ = \angle CAP\,.$$

Prove that $\angle DEF = 90^\circ$ if and only if Q is the orthocentre of triangle BDF.

Solution. First for some notation. Let $x = \angle BAQ = \angle CAP$, $y = \angle ACP = \angle BCQ$. Let AQ intersect FP at U, CQ and DP at V, AQ and FE at S, and CQ and DE at T.



Notice that AQ and CQ are perpendicular to EF and ED respectively. To see this, consider triangle FAU. Then $\angle UFS = \angle PFE = \angle PAE = x$ since $PFAE$ is a cyclic quadrilateral. Now $\angle AUF = 90^\circ - x$, so that $\angle FSU = 90^\circ$. Thus EF is perpendicular to AQ. Likewise, in triangle DCV, we see that $\angle VDT = \angle PDE = \angle PCE = y$. Thus ED is perpendicular to CQ.

Assume that $\angle DEF = 90^\circ$. We show that DQ is parallel to PF. Since PF is perpendicular to AB, DQ extended to AB will be the altitude of triangle BDF from vertex D. Likewise, it can be shown that FQ is parallel to PD and it extended is the altitude of triangle BDF from vertex F. This will mean that Q, the intersection of these two altitudes, is the orthocentre of triangle BDF.

First, consider the quadrilateral $QSET$. Then $\angle QSE = \angle QTE = 90^\circ$ since we showed above that CQ is perpendicular to ED. Recall $\angle SET = \angle FED = 90^\circ$ by assumption. Thus $\angle SQT = 90^\circ$.

Now triangles DPC and QAC are similar since they both have a right angle and $\angle QCA = \angle DCP$. Thus $DC/PC = QC/AC$. Using this fact and knowing $\angle DCQ = \angle PCA$ implies that triangles QDC and APC are similar. Thus $\angle DQC = \angle PAC = x$.

Lines FE and QC are parallel since they are extended line segments of opposite sides of the rectangle $QSET$. Line FP makes an angle of x with FE since $\angle PFE = \angle PAE$. Line QD makes an angle of x also with QC and in the same configuration as shown in the diagram. Thus FP and QD are parallel.

Challenge Board Problems

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

C79. *Proposed by Mohammed Aassila, Strasbourg, France.*

Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing function, and assume that there are two constants $p > 0$ and $T > 0$ such that

$$\int_t^\infty f(s)^{p+1}\,ds \;\le\; Tf(0)^p f(t)$$

for all $t \in \mathbb{R}^+$. Prove that

$$f(t) \;\le\; f(0)\left(\frac{T+pt}{T+pT}\right)^{-1/p}$$

for all $t \ge T$.

Solution by the proposer. If $f(0) = 0$, then $f \equiv 0$ and there is nothing to prove. Otherwise, replacing $f(x)$ by $f(x)/f(0)$, we may assume that $f(0) = 1$, and our hypothesis becomes

$$\int_t^\infty f(s)^{p+1}\,ds \;\le\; Tf(t)$$

for all $t \in \mathbb{R}^+$. Introduce the auxiliary function $F : \mathbb{R}^+ \to \mathbb{R}^+$ defined as

$$F(t) \;=\; \int_t^\infty f(s)^{p+1}\,ds\,,$$

so that $F(t) \le Tf(t)$. Then F is non-increasing and locally absolutely continuous, and therefore

$$F'(t) \;=\; -f(t)^{p+1} \;\le\; -T^{-p-1}F(t)^{p+1}$$

almost everywhere in $(0, \infty)$.

(Readers who are unfamiliar with the terms 'locally absolutely continuous' and 'almost everywhere' may freely ignore them: they are technical conditions which allow us to solve the problem in greater generality. If instead we add the hypothesis that f is continuous, then we may replace 'locally absolutely continuous' with 'differentiable' and 'almost everywhere' with 'everywhere', and the proof carries through with no other changes.)

Letting B be the (possibly infinite) supremum of the set $\{t \mid F(t) > 0\}$, we find that

$$(F^{-p})' \;=\; -pF^{-p-1}F' \;\ge\; pT^{-p-1}$$

almost everywhere in $(0, B)$. It follows by integration that

$$F(t)^{-p} - F(0)^{-p} \;\ge\; pT^{-p-1}t$$

for all $t \in [0, B)$, whence

$$F(t) \;\le\; \left(F(0)^{-p} + pT^{-p-1}t\right)^{-1/p}$$

for all $t \in [0, B)$. Since $F(0) \le T$, it follows further that

$$F(t) \;\le\; (T^{-p} + pT^{-p-1}t)^{-1/p} \;=\; T^{(p+1)/p}(T + pt)^{-1/p}\,.$$

Since $t \ge B$ by definition implies $F(t) = 0$, the above inequality evidently holds for all $t \in \mathbb{R}^+$.

We can estimate the left-hand side of this inequality as follows:

$$\begin{aligned} F(t) \;&=\; \int_t^\infty f(s)^{p+1}\,ds \\ &\ge\; \int_t^{T+(p+1)t} f(s)^{p+1}\,ds \\ &\ge\; (T + pt)f(T + (p+1)t)^{p+1}\,, \end{aligned}$$

where the right-most inequality uses the fact that f is non-increasing. Consequently,

$$(T + pt)f(T + (p+1)t)^{p+1} \;\le\; T^{(p+1)/p}(T + pt)^{-1/p}\,,$$

which reduces immediately to

$$f(T+(p+1)t) \;\leq\; T^{1/p}(T+pt)^{-1/p}\,.$$

Writing $t = T + (p+1)t'$ for $t \geq T$, we find

$$f(t) \;\leq\; \left(\frac{T+pt'}{T}\right)^{-1/p} \;=\; \left(\frac{T+pt}{T+pT}\right)^{-1/p}\,,$$

as desired.

C80. Suppose $a_1, a_2, \ldots, a_m$ are transpositions in S_n (the symmetric group on n elements) such that $a_1 a_2 \cdots a_m = 1$. Show that if the a_i generate S_n, then $m \geq 2n-2$.

Solution. We take our S_n to be the symmetric group on the n integers $1, 2, \ldots, n$. Recall that when we say that the permutation a_i fixes the integer j, we mean that $a_i(j) = j$ or, equivalently, that j does not appear in the cycle decomposition of a_i. (We will use cycle-decomposition notation for these transpositions; that is, $a_i = (b\ c)$ is the permutation which swaps b and c and fixes everything else. Henceforth, we will also denote the identity permutation by e, so as not to confuse it with the integer 1 which is being permuted.)

To begin with, we will argue that unless none of the transpositions a_i fix 1 (that is, unless all the a_i are of the form $(1\ b_i)$ already), it is possible to replace the product $a_1 a_2 \cdots a_m = e$ with a new product $a_1' a_2' \cdots a_m' = e$ of the same length, such that the a_i' still generate S_n and such that fewer of the a_i' fix 1 than do the a_i. To show this, observe that either none of the a_i fix 1, or else there must exist some pair of transpositions of the form $a_j = (1\ b)$ and $a_k = (b\ c)$ with 1, b, and c all different: if not, this would certainly contradict the assumption that the a_i generate S_n. Choose such an a_j and a_k with $|k-j|$ as small as possible, and without loss of generality assume that $k > j$. Then by the minimality of $|k-j|$, all of the transpositions $a_{j+1}, \ldots, a_{k-1}$ must fix b and c, so the product $a_{j+1} \cdots a_{k-1}(b\ c) = (b\ c)a_{j+1} \cdots a_{k-1}$. Thus

$$\begin{aligned} a_j \cdots a_k &= (1\ b)a_{j+1} \cdots a_{k-1}(bc) \\ &= (1\ b)(b\ c)a_{j+1} \cdots a_{k-1} \\ &= (1\ c)(1\ b)a_{j+1} \cdots a_{k-1}\,. \end{aligned}$$

So,

$$\begin{aligned} e &= a_1 \cdots a_m \\ &= a_1 \cdots a_{j-1}(1\ c)(1\ b)a_{j+1} \cdots a_{k-1}a_{k+1} \cdots a_m\,, \end{aligned}$$

and the latter product is certainly still of the same length, its transpositions still generate S_n (because $(b\ c) = (1\ c)(1\ b)(1\ c)$), and fewer of the transpositions fix 1, as desired.

Evidently, by repeated application of the above procedure, we eventually arrive at a product $a'_1 \cdots a'_m = e$ of the same length m, and such that *none* of the a'_i fix 1; that is, all of the a'_i are of the form $(1\ b_i)$ for some integer b_i between 2 and n. Since the a'_i generate S_n, each of $(1\ 2), \ldots, (1\ n)$ must appear in the product at least once. However, if $(1\ b)$ were to appear in the product $a'_1 \cdots a'_m = e$ exactly once, then certainly the product could not fix b; so, each of $(1\ 2), \ldots, (1\ n)$ must appear in the product at least twice, and therefore $m \geq 2(n-1)$, as desired.

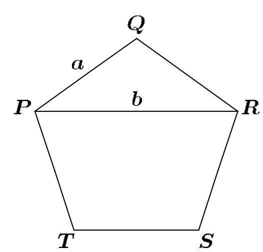
Problem of the Month

Jimmy Chui, student, University of Toronto

Problem. Let a be the length of a side and b be the length of a diagonal in the regular pentagon $PQRST$. Prove that

$$\frac{b}{a} - \frac{a}{b} = 1 .$$

(1998 Descartes, D1)



Solution I. The internal angle at each vertex of a regular pentagon is $108°$. Note that a and b are positive, and by the Cosine Law, we have

$$\begin{aligned} b^2 &= a^2 + a^2 - 2 \cdot a \cdot a \cdot \cos 108° \\ &= 2a^2(1 + \cos 72°) \;=\; 4a^2 \cos^2 36° , \end{aligned}$$

and so $b = 2a \cos 36°$.

Let $x = \cos 18°$. We know that $0 < x < 1$ and that $\cos(5 \cdot 18°) = 0$. Setting $\theta = 18°$ and using De Moivre's Theorem, we have $\text{cis}\, 5\theta = (\text{cis}\, \theta)^5$. Comparing real terms on both sides, we get

$$\begin{aligned} \cos 5\theta &= \binom{5}{0}(\cos\theta)^5 - \binom{5}{2}(\cos\theta)^3(\sin\theta)^2 + \binom{5}{4}(\cos\theta)(\sin\theta)^4 \\ &= x^5 - 10x^3(1-x^2) + 5x(1-x^2)^2 \;=\; 16x^5 - 20x^3 + 5x , \end{aligned}$$

so $16x^4 - 20x^2 + 5 = 0$, since $x \neq 0$.

By the quadratic formula, $x^2 = \dfrac{20 \pm \sqrt{400 - 320}}{32} = \dfrac{5 \pm \sqrt{5}}{8}$, with x positive. Since $\sqrt{\dfrac{5-\sqrt{5}}{8}} < \sqrt{\dfrac{5-1}{8}} = \sqrt{\dfrac{1}{2}} = \cos 45°$, we must have

$$\cos 18^\circ \;=\; x \;=\; \sqrt{\frac{5+\sqrt{5}}{8}}\,.$$

Then,

$$\cos 36^\circ \;=\; 2x^2 - 1 \;=\; \frac{1+\sqrt{5}}{4}\,.$$

Evaluating b, we obtain

$$b \;=\; \frac{1+\sqrt{5}}{2}\cdot a\,.$$

Finally,

$$\frac{b}{a}-\frac{a}{b} \;=\; \frac{1+\sqrt{5}}{2}-\frac{2}{1+\sqrt{5}} \;=\; \frac{(1+\sqrt{5})^2-4}{2(1+\sqrt{5})} \;=\; \frac{2+2\sqrt{5}}{2(1+\sqrt{5})} \;=\; 1\,.$$

Solution II. Draw the line PS. This has a length of b. By symmetry, $\angle QPR = \angle TPS = 36^\circ$, and so $\angle RPS = 36^\circ$. From triangle PQR, we get

$$\cos 36^\circ \;=\; \frac{a^2+b^2-a^2}{2ab} \;=\; \frac{b}{2a}\,.$$

From triangle PRS, we get

$$\cos 36^\circ \;=\; \frac{b^2+b^2-a^2}{2b^2} \;=\; \frac{2b^2-a^2}{2b^2}\,.$$

Now, a and b are both positive and cannot be equal, and so we have

$$\begin{aligned}
\frac{b}{2a} &= \frac{2b^2-a^2}{2b^2}\\
\Longrightarrow\quad b^3 &= 2ab^2-a^3\\
\Longrightarrow\quad 0 &= a^3-2ab^2+b^3\\
&= (a-b)(a^2+ab-b^2)\\
\Longrightarrow\quad 0 &= a^2+ab-b^2\\
\Longrightarrow\quad 1 &= \frac{b}{a}-\frac{a}{b}\,.
\end{aligned}$$

Solution III. Construct a point A on PR such that SA bisects the angle PSR. Chasing angles, we observe that triangle SRA is isosceles, with $RS = AS = a$, and that triangle SPA is isosceles, with $SA = PA = a$. We can also see that triangle SAR is similar to triangle PRS, and from this, we have

$$\frac{AR}{AS} \;=\; \frac{RS}{RP} \quad\Longrightarrow\quad \frac{b-a}{a} \;=\; \frac{a}{b} \quad\Longrightarrow\quad \frac{b}{a}-\frac{a}{b} \;=\; 1\,.$$

J.I.R. McKnight Problems Contest 1990

1. A certain number N consists of three digits which are consecutive terms of an arithmetic sequence. If N is divided by the sum of its digits the quotient is 48. Also, if 198 is subtracted from N, the resulting number comprises the digits of N in reversed order. Find N.

2. Two chords on a circle AB and CD intersect at right angles at E such that $AE = 2$, $EB = 6$ and $DE = 3$. Find the area of the circle.

3. Solve for x:
$$\log_{10}\left(\sqrt{x} + \frac{2}{\sqrt{x}}\right) = \frac{1}{2} + \log_{10} 2\,.$$
Leave your answer in simplest radical form.

4. A point P is taken on the curve $y = x^3$. The tangent at P intersects the curve at Q. Prove that the slope of the curve at Q is four times the slope at P.

5. The sum of the length of the hypotenuse and another side of a right angled triangle is 12. Prove that the area of the triangle is maximum when the angle between the two sides is $60°$.

6. A curve of intersection of a sphere with a plane through the centre of the sphere is called a great circle. One great circle divides a sphere's surface into 2 regions; 2 great circles divide the sphere into 4 regions. If no 3 great circles intersect in a common point, find the number of regions into which the surface of a sphere is divided by: 3, 4, 5, n great circles.

7. The sum of the first three terms of a geometric sequence is 37 and the sum of their squares is 481. Find the first three terms of all such sequences.

8. The point D is on the side BC of an equilateral triangle ABC such that $BD = \frac{1}{3}BC$ and E is a point on AB equidistant from A and D. Prove that $CE = EB + BD$.

9. Given that
$$\begin{aligned} \tan A + \tan B &= a\,, \\ \cot A + \cot B &= b\,, \\ \tan(A+B) &= c\,, \end{aligned}$$
find c in terms of a and b.

10. Determine the number of factors of 2 in the first positive integer greater than $(4 + 2\sqrt{3})^{1990}$.

Waiting in Wonderland

Cyrus Hsia
student, University of Toronto

Jack Wang, a student at Cedarbrae Collegiate Institute in Toronto, sent me the following fabulous problem. It was originally a problem in the 1998 J.I.R. McKnight Problem Solving Competition held annually in Scarborough and whose problems we are currently publishing in ***CRUX with MAYHEM***. The problem stated here is modified from the original problem in the paper.

Jack's Meeting Problem

A teacher at Cedarbrae, Mr. Iacobucc, and his girlfriend want to meet in Canada's Wonderland between 1:00 pm and 2:00 pm. [Ed: Whether or not his girlfriend's name is Alice is unconfirmed at the time this article was written.] Each of them agrees to wait for each other for 15 minutes. Assume each of them arrives between the designated times randomly. What is the probability that they will meet each other?

The reader is encouraged to find the answer to this problem on his/her own before reading the solution. The answer turns out to be a relatively nice rational number. If you solve it, you may wish to skip the next section which deals with some simple properties of probability. Otherwise, sit back and absorb some motivating facts that may be useful in solving the above problem.

An Aside on Simple Probability

Finite Outcomes

Readers will be familiar with the examples of tossing coins or picking coloured marbles from a bag. In any experiments similar to these, there are certain outcomes (tossing 2 heads or picking a red marble) whose likelihood of occurring we want to know. The desired outcome is called an *event*. The set of all possible outcomes in a given experiment is called a *sample space*.

Now to find the probability, we need to know the number of ways that the desired outcome (event) occurs compared to the number of ways of all possible outcomes (size of the sample space). In fact, the probability of an event, E, in a given sample space, S, is given by

$$P(E) = \frac{N(E)}{N(S)},$$

where $N(A)$ is the number of ways of event A occurring.

For example, suppose you wish to know the probability of tossing 2 heads in 2 coin tosses. Then N(tossing 2 heads) $= 1$ since there is only one way of doing this, when both tosses are heads. But N(all possible outcomes) equals 4, since we could have (tail, tail), (tail, head), (head, tail) or (head, head). The probability is then $1/4$.

Infinite Outcomes

Often the number of ways for the desired outcome or the sample space is infinite. In this case, the formula for probability given above is no longer valid. For example, suppose the real line from 0 to 4 is drawn on a piece of paper in black ink with total length 4 centimetres. Now arbitrarily highlight a line segment of length 1 centimetre. What is the probability that a randomly selected real number from 0 to 4 lies within the highlighted segment?

Now if we used the formula, we have to check the number of all possible outcomes. The number of ways of selecting a real number from 0 to 4 is infinite. In fact, the number of ways of selecting a real number from any arbitrary line segment of positive length is infinite.

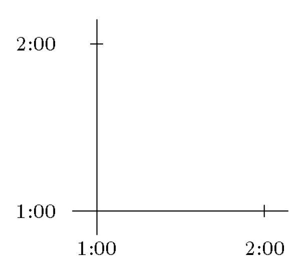
However, it is intuitive that a randomly selected real number will lie on the non-highlighted sections three times as likely than it would on the highlighted section, since the length of non-highlighted to highlighted regions is three to one. The probability should then be $1/4$.

We could apply similar reasoning to selecting points in higher dimensions. Suppose a circular archery target has radius 2 feet in length and the bull's-eye region is a circle of radius 1 foot in its centre. What is the probability of an arrow landing in the bull's-eye if the arrow lands randomly within the target? By calculating the areas, we have a π square feet region of the bull's-eye and a 4π square feet region of the whole target. The probability would then be $1/4$. [Ed: In case the reader is wondering, not every probability is $1/4$. The reader is strongly urged to verify this.]

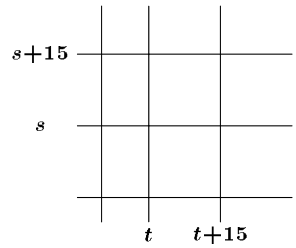
So what does all of this have to do with the problem? Let us go on to the solution.

Solution to Jack's Meeting Problem

Consider Mr. Iacobucc's time line on the x–axis and his girlfriend's on the y–axis. Say the origin is at 1:00 for both people and time progresses in the positive direction as shown in the following graph.

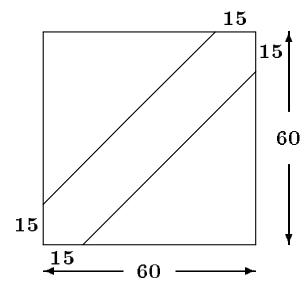


Now if Mr. Iacobucc is there from $[t, t+15]$, $0 \leq t \leq 60$ minutes after 1:00, draw in the two lines $x = t$ and $x = t + 15$. Similarly, his girlfriend gets there for $[s, s+15]$ and so draw in the two lines $y = s$ and $y = s+15$. The graph is now as follows.



So, for what cases of s and t do they meet? If they meet, then there is some point in time when they are both there. In other words, there is some time k in which $t \leq k \leq t+15$ and $s \leq k \leq s+15$. This time k exists if and only if the square formed by the four lines above intersect the line $y = x$. (Why?)

The question then becomes: What is the probability of a square of sides 15 units (minutes) intersecting the line $y = x$ in the square of sides 60 units (minutes)? Another way to look at this is to figure out all possible positions to place the lower left corner of the small square in the large square and the positions that would make the square intersect the line $y = x$.



It turns out that if we choose the lower left corner of the small square anywhere in region shown, then intersection occurs. Note: We allow for the smaller square to extend beyond the bounds of the larger square since we allow for either of the couple to come within the last 15 minutes.

By simply calculating the area of the region to the area of the whole square, we find that the probability of meeting is then

$$\frac{60^2 - 45^2}{60^2} = \frac{1575}{3600} = \frac{7}{16}.$$

In general, Jack pointed out the result for two people meeting in a common interval m and waiting for a time interval n. It simplifies to the following: The probability of meeting is

$$\frac{2n}{m} - \left(\frac{n}{m}\right)^2,$$

with of course $n \leq m$ being real numbers.

Another Aside on Probability

Jack also notes the following interesting similarity: Suppose A and B are two events. The probability of either one or the other happening is given by the equation

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$

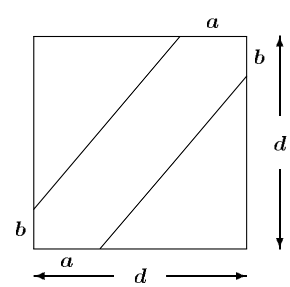
Suppose A and B are two independent events and that $P(A) = P(B) = n/m$. For example, suppose A is the event of throwing a heads and B is the event of getting an even number in a die roll. Now in one toss of a coin and a die roll, what is $P(A \text{ or } B)$, the probability of getting heads or an even number? To get a heads or an even number, we can get a heads and roll any number, roll an even number and toss heads or tails, then remove the case were we counted tossing heads and rolling an even number twice. So $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$. Since tossing a heads and rolling an even number are independent events, $P(A \text{ and } B) = P(A) \times P(B)$. The result is that $P(A \text{ or } B) = 2n/m - (n/m)^2$.

This answer makes us very suspicious that Jack's problem could be solved another way. The question to ask then, is whether or not we can apply this method to solving Jack's problem more elegantly.

Some Generalizations

Two People with Different Amounts of Patience

Unfortunately, not everyone has the same amount of patience. Suppose Mr. Iacobucc is only willing to wait for a minutes while his girlfriend will wait b minutes. What is the probability of meeting within a d minute interval?



Three's Company

Now suppose that Jack was also to meet the couple at Wonderland. Again, all three people are willing to wait 15 minutes and it is within the time from 1:00 pm to 2:00 pm. Now what is the probability that all three will meet?

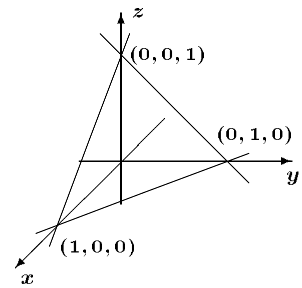
Here, using the same technique as above, we would be trying to find the ratio of a certain volume to the volume of a cube. The group would all meet at a certain point in time if a cube of sides 15 intersects the line $x = y = z$ in space.

Another Probability Problem

Problem.
Take a line segment of length 1. What is the probability that making two random cuts along the line will produce three line segments that are the sides of a non-degenerate triangle?

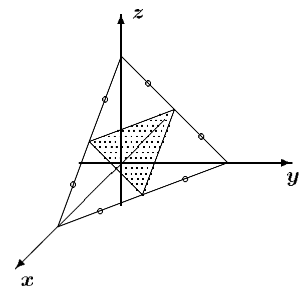
Solution.

First let us set up the problem. Suppose the three line segments have lengths x, y, and z. The constraints on them are $x + y + z = 1$, and x, y, $z \geq 0$. This defines the portion of a plane in the first quadrant of the three dimensional cartesian space as shown:



Now for the three lengths to form a non-degenerate triangle the following triangle inequalities must hold: $x + y > z$, $x + z > y$, and $y + z > x$. Each of these inequalities defines the space on one side of a given plane. For example, $x + y > z$ defines the space below the plane given by $x + y - z = 0$.

Now where do the three space domains intersect on the plane $x + y + z = 1$? The plane $x + y + z = 1$ intersects the first quadrant to form a triangle. The three space domains intersect in a triangle whose vertices are the mid-points of the sides of the first triangle.



The probability is then $1/4$.

Exercises

1. What is the answer to the probability of three people meeting as given above?
2. Exploration: Investigate a similar problem for four or more people.
3. Again consider three people. We assumed that each person will remain exactly 15 minutes. Usually, however, when one person meets another they will both wait and leave together so that either the first person urges the second to both leave or else the second will force the first to continue another 15 minutes. In either case, what is the probability of all three meeting?

Acknowledgements

Jack Wang, grade 12 student at Cedarbrae Collegiate Institute in Toronto, Ontario, Canada, for his interesting account of the problem.

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PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk (⋆) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}$"×11" or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 April 2000**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in LaTeX). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

2463⋆. *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

Suppose that $k \in \mathbb{Z}$. Prove or disprove that

$$\left(\tan\left(\frac{3n\pi}{11}\right)+4\sin\left(\frac{2n\pi}{11}\right)\right)^2 = \left(\tan\left(\frac{5n\pi}{11}\right)-4\sin\left(\frac{4n\pi}{11}\right)\right)^2 = \begin{cases} 11 & \text{for} \quad n \neq 11k\,, \\ 0 & \text{for} \quad n = 11k\,. \end{cases}$$

2464. *Proposed by Michael Lambrou, University of Crete, Crete, Greece.*

Given triangle ABC with circumcircle Γ, the circle Γ_A touches AB and AC at D_1 and D_2, and touches Γ internally at L. Define E_1, E_2, M, and F_1, F_2, N in a corresponding way. Prove that

(a) AL, BM, CN are concurrent;

(b) D_1D_2, E_1E_2, F_1F_2 are concurrent, and that the point of concurrency is the incentre of $\triangle ABC$.

2465*. *Proposed by Albert White, St. Bonaventure University, St. Bonaventure, NY, USA.*

For $n \geq 1$, prove that

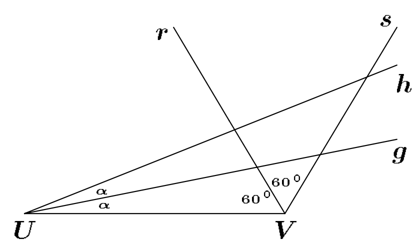
$$\sum_{i=0}^{n-1} \binom{n}{i} \sum_{j=0}^{n-1-i} \binom{n-1}{j} \;=\; 4^{n-1}\,.$$

2466. *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

Given a circle (but not its centre) and two of its arcs, AB and CD, and their mid-points M and N (which do not coincide and are not the end points of a diameter), prove that all the unmarked straightedge and compass constructions that can be carried out in the plane of the circle can also be done with an unmarked straightedge alone.

2467. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Given is a line segment UV and two rays, r and s, emanating from V such that $\angle(UV, r) = \angle(r, s) = 60°$, and two lines, g and h, on U such that $\angle(UV, g) = \angle(g, h) = \alpha$, where $0 < \alpha < 60°$.



The quadrilateral $ABCD$ is determined by g, h, r and s. Let P be the point of intersection of AB and CD.

Determine the locus of P as α varies in $(0, 60°)$.

2468. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For $c > 0$, let x, y, $z > 0$ satisfy

$$xy + yz + zx + xyz \;=\; c\,. \tag{1}$$

Determine the set of all $c > 0$ such that whenever (1) holds, then we have

$$x + y + z \;\geq\; xy + yz + zx\,.$$

2469. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Given a triangle ABC, consider the altitude and the angle bisector at each vertex. Let P_A be the intersection of the altitude from B and the bisector at C, and Q_A the intersection of the bisector at B and the altitude at C. These determine a line P_AQ_A. The lines P_BQ_B and P_CQ_C are analogously defined. Show that these three lines are concurrent at a point on the line joining the circumcentre and the incentre of triangle ABC. Characterize this point more precisely.

2470. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Given a triangle ABC, consider the median and the angle bisector at each vertex. Let P_A be the intersection of the median from B and the bisector at C, and Q_A the intersection of the bisector at B and the median at C. These determine a line P_AQ_A. The lines P_BQ_B and P_CQ_C are analogously defined. Show that these three lines are concurrent. Characterize this intersection more precisely.

2471. *Proposed by Vedula N. Murty, Dover, PA, USA.*

For all integers $n \geq 1$, determine the value of $\displaystyle\sum_{k=1}^{n} \frac{(-1)^{k-1}k}{k+1}\binom{n+1}{k}$.

2472. *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

If A, B, C are the angles of a triangle, prove that

$$\cos^2\left(\frac{A-B}{2}\right)\cos^2\left(\frac{B-C}{2}\right)\cos^2\left(\frac{C-A}{2}\right) \geq \left(8\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\right)^3 .$$

2473. *Proposed by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.*

Given a point S on the side AC of triangle ABC, construct a line through S which cuts lines BC and AB at P and Q respectively, such that $PQ = PA$.

2474. *Proposed by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.*

Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a decreasing continuous function satisfying, for all x, $y \in \mathbb{R}^+$:

$$f(x+y) + f\left(f(x)+f(y)\right) = f\Big(f\left(x+f(y)\right) + f\left(y+f(x)\right)\Big) .$$

Obviously $f(x) = c/x$ $(c > 0)$ is a solution. Determine all other solutions.

2475. *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

Prove that

$$\sum_{j=0}^{n}\sum_{k=1}^{n}(-1)^{j+k}\binom{2n}{2j}\binom{2n}{2k-1} = 0 .$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

The names of MICHEL BATAILLE, Rouen, France (2324), WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria (2324) and ARAM TANGBOONDOUANGJIT, Carnegie Mellon University, Pittsburgh, PA, USA (2352) were inadvertently omitted from the lists of solvers in the last issue.

2356. [1998: 303] *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

Five points, A, B, C, K, L, with whole number coordinates, are given. The points A, B, C do not lie on a line.

Prove that it is possible to find two points, M, N, with whole number coordinates, such that M lies on the line KL and $\triangle KMN$ is similar to $\triangle ABC$.

Solution by Jeremy Young, Nottingham High School, Nottingham, England.

Take A as the origin of the coordinate system, such that the points A, B, C have position vectors $\binom{0}{0}$, $\binom{x_1}{y_1}$, $\binom{x_2}{y_2}$ respectively. Let α be the angle through which we must rotate AB counterclockwise to make it parallel to KL. Since AB and KL each pass through two points with whole number coordinates, both have rational slopes. Therefore, $\tan\alpha = \frac{p}{q}$, some $p, q \in \mathbf{Z}$, if $\alpha \neq 90°$.

The images of B and C under rotation through $\alpha \neq 90°$ are

$$B' = \begin{pmatrix} x_1\cos\alpha - y_1\sin\alpha \\ x_1\sin\alpha + y_1\cos\alpha \end{pmatrix} = \frac{\cos\alpha}{q}\begin{pmatrix} qx_1 - py_1 \\ px_1 + qy_1 \end{pmatrix}$$

and

$$C' = \frac{\cos\alpha}{q}\begin{pmatrix} qx_2 - py_2 \\ px_2 + qy_2 \end{pmatrix}.$$

Define B'' and C'' by enlargement with scalar factor $\frac{q}{\cos\alpha}$, giving

$$B'' = \begin{pmatrix} qx_1 - py_1 \\ px_1 + qy_1 \end{pmatrix} \quad\text{and}\quad C'' = \begin{pmatrix} qx_2 - py_2 \\ px_2 + qy_2 \end{pmatrix}.$$

If $\alpha = 90°$ take $B'' = B' = \begin{pmatrix} -y_1 \\ x_1 \end{pmatrix}$ and $C'' = C' = \begin{pmatrix} -y_2 \\ x_2 \end{pmatrix}$. Since p, q, x_1, x_2, y_1 and y_2 are whole numbers, these are points with whole number coordinates and $\triangle KMN$ is simply a translation of $\triangle AB''C''$ with A going to K.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; and the proposer. There was one incomplete solution.

2357. [1998: 303] *Proposed by Gerry Leversha, St. Paul's School, London, England.*

An unsteady man leaves a place to commence a one-dimensional random walk. At each step he is equally likely to stagger one step to the east or one step to the west. Let his expected **absolute** distance from the starting point after $2n$ steps be a. Now consider $2n$ unsteady men each engaging in independent random walks of this type. Let the expected number of men at the starting point after $2n$ steps be b. Show that $a = b$.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

We label the points that it is possible to be stepped upon by $\ldots, -2, -1, 0, 1, 2, \ldots$; that is, $\mathbb{Z}$. We declare 0 to be the starting point of all walks.

Since there are $2n$ random steps, the only eligible final points are $2j$, where $-n \le j \le n$.

Let $a_j^{(2n)}$ be the number of all possible $2n$–walks ending in $2j$. Then, clearly, $a_0^{(0)} = 1$, $a_{-1}^{(2)} = a_1^{(2)} = 1$, $a_0^{(2)} = 2$, and $a_j^{(0)} = a_j^{(2)} = 0$ for all other values of j. Furthermore, $a_k^{(2n+2)} = a_{k-1}^{(2n)} + 2a_k^{(2n)} + a_{k+1}^{(2n)}$, where $n \ge 1$ and $k \in \mathbb{Z}$.

This relation is just the "doubled" Pascal Triangle Iteration, having $a_0^{(2n)} = \binom{2n}{n}$ as its central element.

Therefore, $a_k^{(2n)} = \binom{2n}{n+k}$, $-k \le n \le k$, and further,

$$P(X{=}2k) \;=\; \begin{cases} \dfrac{2\binom{2n}{n+k}}{2^{2n}}\,, & k > 0\,, \\[2ex] \dfrac{\binom{2n}{n}}{2^{2n}}\,, & k = 0\,. \end{cases}$$

Here, X denotes the absolute distance from the starting point after $2n$ steps.

Now,

$$E(X) \;=\; \sum_{k=0}^{n} 2kP(X{=}2k) \;=\; \frac{1}{2^{2n-1}}\sum_{k=1}^{n} 2k\binom{2n}{n+k} \;=\; \frac{s_n}{2^{2n-2}}\,,$$

where

$$\begin{aligned} s_n &= \sum_{k=1}^{n} k\binom{2n}{n+k} \;=\; \sum_{k=1}^{n}(n+k)\binom{2n}{n+k} - n\sum_{k=1}^{n}\binom{2n}{n+k} \\ &= 2n\sum_{k=1}^{n}\binom{2n-1}{n+k-1} - n\left(\frac{1}{2}\right)\left(2^{2n} - \binom{2n}{n}\right) \\ &= 2n\left(\frac{1}{2}\right)2^{2n-1} - n\,2^{2n-1} + \frac{n}{2}\binom{2n}{n} \;=\; \frac{n}{2}\binom{2n}{n}\,. \end{aligned}$$

Therefore, $E(X) = \frac{n}{2^{2n-1}}\binom{2n}{n} = a$.

Let Y be the number of men (out of $2n$ possibilities) being at the starting point after $2n$ steps.

Then, Y is binomial p–distributed with $p = P(X{=}0)$, yielding, at once, that

$$E(Y) = 2np = \frac{n}{2^{2n-1}}\binom{2n}{n} = b\,.$$

Thus, $E(X) = E(Y)$; that is $a = b$.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

Janous comments that this is one of his favourite problems *from* ***CRUX with MAYHEM*** *in 1998! He also calculated the variances, and found that*

$$V(X) \sim 2n\left(1 - \frac{2}{\pi}\right) \qquad \textit{and} \qquad V(Y) \sim \frac{2}{\pi}\left(\sqrt{n\pi} - 1\right)\,.$$

The proposer comments: This problem arose out of an English A level statistics course in which one pupil mistakenly answered the wrong question but got the 'right' answer. It is not too difficult to prove, by combinatorial arguments, that the two expectations are equal, but the real challenge is to show why they are. Can one problem be reduced to the other? Kathleen Lewis comes closest to achieving this by showing that the two satisfy parallel recurrence relations, but even so, I do not feel really satisfied that we are dealing with two aspects of the same thing. I await some marvellous illumination which will dispose of the matter in a nutshell so that I can say: There! It was obvious all the way along! [Ed. *Can any reader help?*]

2358. [1998: 303] *Proposed by Gerry Leversha, St. Paul's School, London, England.*

In triangle ABC, let the mid-points of BC, CA, AB be L, M, N, respectively, and let the feet of the altitudes from A, B, C be D, E, F, respectively. Let X be the intersection of LE and MD, let Y be the intersection of MF and NE, and let Z be the intersection of ND and LF. Show that X, Y, Z are collinear.

Editor's comment. Most solvers noted that points L, M, N, D, E and F all lie on the nine-point circle (a.k.a. Euler's circle or Feuerbach's circle), so by Pascal's Theorem (see, for example: Roger A. Johnson, *Modern Geometry*, Houghton-Mifflen Co., 1929, Theorem 385) X, Y, Z are collinear.

Bellot Rosado, Bataille, Smeenk and the proposer all note that X, Y and Z lie on the Euler line of $\triangle ABC$.

Solved by CHETAN BALWE, student, University of Michigan, Ann Arbor, MI, USA; MICHEL BATAILLE, Rouen, France (2 solutions); FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK (2 solutions); NIKOLAOS DERGIADES, Thessaloniki, Greece; C. FESTRAETS-HAMOIR, Brussels, Belgium; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands(2 solutions);

PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA(2 solutions); JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; NO NAME on the submission; and the proposer.

2359. [1998: 303] *Proposed by Vedula N. Murty, Visakhapatnam, India.*

Let $PQRS$ be a parallelogram. Let Z divide PQ internally in the ratio $k : l$. The line through Z parallel to PS meets the diagonal SQ at X. The line ZR meets SQ at Y.

Find the ratio $XY : SQ$.

I. *Solution by Michael Lambrou, University of Crete, Crete, Greece.*

We clearly have

$$\frac{SX}{XY} = \frac{SX}{SQ}\cdot\frac{SQ}{XY} = \frac{PZ}{PQ}\cdot\frac{SQ}{XY} = \frac{k}{k+l}\cdot\frac{SQ}{XY}.$$

Also from the similarity of $\triangle XYZ$ and $\triangle QYR$, we have

$$\frac{YQ}{XY} = \frac{QR}{XZ} = \frac{PS}{XZ} = \frac{PQ}{ZQ} = \frac{k+l}{l}.$$

Hence

$$\frac{SQ}{XY} = \frac{SX+XY+YQ}{XY} = \frac{SX}{XY}+1+\frac{YQ}{XY} = \frac{k}{k+l}\cdot\frac{SQ}{XY}+1+\frac{k+l}{l}$$

so that

$$\frac{l}{k+l}\cdot\frac{SQ}{XY} = \frac{k+2l}{l},$$

from which $XY : SQ = \dfrac{l^2}{(k+l)(k+2l)}$, giving the required ratio.

II. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

We shall use the method of "closed vector-chains". Let $\overrightarrow{PQ} = \overrightarrow{a}$ and $\overrightarrow{PS} = \overrightarrow{b}$. Then $\overrightarrow{a}$ and $\overrightarrow{b}$ are linearly independent. Set $\lambda = l/(k+l)$. Then $\overrightarrow{ZR} = \lambda\overrightarrow{a} + \overrightarrow{b}$, whence it follows that $\overrightarrow{ZY} = \alpha\overrightarrow{ZR}$ for a certain $\alpha \in \mathbb{R}$. Furthermore $\overrightarrow{QS} = \overrightarrow{b} - \overrightarrow{a}$ and $\overrightarrow{YX} = \beta\overrightarrow{QS}$ for a certain $\beta \in \mathbb{R}$. (We must find the value of β.) Finally $\overrightarrow{ZX} = \lambda\overrightarrow{b}$ (because $\triangle ZQX$ and $\triangle PQS$ are similar). Therefore, the "chain-relation" $\overrightarrow{ZX} = \overrightarrow{ZY} + \overrightarrow{YX}$ reads:

$$\lambda\overrightarrow{b} = \alpha(\lambda\overrightarrow{a} + \overrightarrow{b}) + \beta(\overrightarrow{b} - \overrightarrow{a});$$

that is,

$$\overrightarrow{b}(\lambda - \alpha - \beta) = \overrightarrow{a}(\alpha\lambda - \beta)$$

whence (due to linear independence)

$$\lambda - \alpha - \beta \;=\; 0 \qquad \text{and} \qquad \alpha\lambda - \beta \;=\; 0\,,$$

implying

$$\beta \;=\; \frac{\lambda^2}{\lambda+1} \;=\; \frac{l^2}{(k+l)(k+2l)}\,.$$

III. *Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.*

We will give a solution by way of coordinates. Suppose the points are coordinatized as $S(0,0)$, $P(1,v)$, $Z(1+k,v)$, $Q(1+k+l,v)$, $R(k+l,0)$, and $K(k,0)$. The equations of the concerned lines are:

$$\begin{aligned} SQ \quad &: \quad y = \frac{v}{1+k+l}x\,; \\ ZR \quad &: \quad \frac{x-k-l}{1-l} = \frac{y}{v}\,; \\ ZX \;=\; ZK \quad &: \quad y = v(x-k)\,. \end{aligned}$$

Then we have

$$SQ^2 \;=\; v^2 + (1+k+l)^2\,.$$

The coordinates of the points X and Y are:

$$\begin{aligned} X \;=\; ZK\cap SQ \;&=\; \left(\frac{k(1+k+l)}{k+l}, \frac{vk}{k+l}\right)\,, \\ Y \;=\; ZR\cap SQ \;&=\; \left(\frac{(k+l)(1+k+l)}{k+2l}, \frac{v(k+l)}{k+2l}\right)\,. \end{aligned}$$

A straightforward computation gives us

$$\frac{XY}{SQ} \;=\; \frac{l^2}{(k+l)(k+2l)}\,.$$

Also solved by SAM BAETHGE, Nordheim, TX, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; MASOUD KAMGARPOUR, student, Carson Graham Secondary School, North Vancouver, British Columbia; GEOFFREY A. KANDALL, Hamden, CT, USA; MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; GERRY LEVERSHA, St. Paul's School, London, England; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; MAX SHKARAYEV, Tucson, AZ, USA; D.J. SMEENK, Zaltbommel, the Netherlands; J. SUCK, Essen, Germany; PARAYIOU THEOKLITES, Limassol, Cyprus; UNIVERSITY OF ARIZONA PROBLEM SOLVING GROUP, Tucson, AZ, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer. There was one incorrect solution submitted.

2360. [1998: 304] *Proposed by K.R.S. Sastry, Dodballapur, India.*

In triangle ABC, let BE and CF be internal angle bisectors, and let BQ and CR be altitudes, where F and R lie on AB, and Q and E lie on AC. Assume that E, Q, F and R lie on a circle that is tangent to BC.

Prove that triangle ABC is equilateral.

Solution by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.

The lengths of the various segments are

$$AR = b\cos A\,,\ AQ = c\cos A\,,\ AF = \frac{bc}{a+b}\,,\ AE = \frac{bc}{a+c}\,.$$

Since E, Q, F, and R lie on a circle, $AE \cdot AQ = AF \cdot AR$, so that $\frac{c}{a+c} = \frac{b}{a+b}$. From this we have $b = c$.

Note that $CQ = a\cos C$ and $CE = \frac{ab}{a+c} = \frac{ac}{a+c}$. Since the given circle is tangent to BC and (by symmetry) that point of tangency is the mid-point of BC, we have $CQ\cdot CE = \left(\frac{a}{2}\right)^2$. From this, $\cos C = \frac{a+c}{4c}$. Since $\cos C = \frac{a/2}{c}$ in an isosceles triangle, we deduce that $a = c$ so that the given triangle is equilateral.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

2361. [1998: 304] *Proposed by K.R.S. Sastry, Dodballapur, India.*

The lengths of the sides of triangle ABC are given by relatively prime natural numbers. Let F be the point of tangency of the incircle with side AB. Suppose that $\angle ABC = 60°$ and $AC = CF$. Determine the lengths of the sides of triangle ABC.

Almost identical solutions by Sam Baethge, Nordheim, TX, USA, and by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.

Let D and E be the points of tangency of the incircle with sides AC and BC respectively. Let $BF = BE = x$, $CD = EC = y$, and $AD = AF = z$. Applying the Cosine Law to angle B of triangles BCF and BCA yields

$$FC^2 = (y+z)^2 = (x+y)^2 + x^2 - x(x+y) \qquad (1)$$

$$AC^2 = (y+z)^2 = (x+y)^2 + (x+z)^2 - (x+y)(x+z)\,. \qquad (2)$$

Equating the right members of (1) and (2) produces $z(z + x - y) = 0$. Since $z > 0$, $y = x + z$. Substituting in (2) yields $(2x - 3z)(x + z) = 0$, which implies that $x = \dfrac{3z}{2}$. Because the triangle's sides are relatively prime, $z = 2$, $x = 3$, and $y = 5$. Consequently, $AB = 5$, $BC = 8$, and $AC = CF = 7$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; MAX SHUARAYEV, Tuscon, AZ, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer (who provided two solutions).

Only Seimiya took the trouble to verify that these necessary conditions are also sufficient; in particular, $8 < 7 + 5$ so that the desired 5–7–8 triangle does, in fact, exist.

2362. [1998: 304] *Proposed by Mohammed Aassila, Université Louis Pasteur, Strasbourg, France.*

Suppose that a, b, $c > 0$. Prove that

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \;\geq\; \frac{3}{1+abc}\,.$$

Solution by Jun-hua Huang, the Middle School Attached To Hunan Normal University, Changsha, China.

We use the well-known inequality $t + \frac{1}{t} \geq 2$ for $t > 0$. Equality occurs if and only if $t = 1$. Note that

$$\frac{1+abc}{a(1+b)} \;=\; \frac{1+a}{a(1+b)} + \frac{b(1+c)}{1+b} - 1\,,$$

$$\frac{1+abc}{b(1+c)} \;=\; \frac{1+b}{b(1+c)} + \frac{c(1+a)}{1+c} - 1\,,$$

and

$$\frac{1+abc}{c(1+a)} \;=\; \frac{1+c}{c(1+a)} + \frac{a(1+b)}{1+a} - 1\,.$$

Then

$$\frac{1+abc}{a(1+b)} + \frac{1+abc}{b(1+c)} + \frac{1+abc}{c(1+a)} \;\geq\; 2+2+2-3 \;=\; 3\,,$$

by the above inequality. Equality holds when

$$\frac{1+a}{a(1+b)} \;=\; \frac{1+b}{b(1+c)} \;=\; \frac{1+c}{c(1+a)} \;=\; 1\,;$$

that is, when $a = b = c = 1$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong, China; PHIL McCARTNEY, Northern Kentucky University, KY, USA; VEDULA N. MURTY, Dover, PA, USA; M. PAROS ALEXANDROS, Paphos, Cyprus, and PARAYIOU THEOKLITOS, Limassol, Cyprus; PANOS E. TSAOUSSOGLOU, Athens, Greece; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer. There was also one incomplete solution submitted.

2363. [1998: 304] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For natural numbers $a, b, c > 0$, let

$$q(a,b,c) \;:=\; a + \cfrac{a + \cfrac{a+\cfrac{a+\frac{a+\cdots}{b+\cdots}}{b+\frac{c+\cdots}{a+\cdots}}}{b+\cfrac{c+\frac{a+\cdots}{b+\cdots}}{a+\frac{c+\cdots}{a+\cdots}}}}{b + \cfrac{c+\cfrac{a+\frac{a+\cdots}{b+\cdots}}{b+\frac{c+\cdots}{a+\cdots}}}{a+\cfrac{c+\frac{a+\cdots}{b+\cdots}}{a+\frac{c+\cdots}{a+\cdots}}}}$$

(in the n^{th} "column" above, from the third one onwards, we have, from top to bottom, the sequence a, b, c, a repeated 2^{n-3} times), where it is assumed that the right side (understood as an infinite process) yields a well-defined positive real number.

The original, a *Talent Search Problem*, asked to determine $q(1,3,5)$. The value is $\sqrt[3]{2}$ (see *Mathematics and Informatics Quarterly*, 7 (1997), No. 1, p. 53).

Determine whether or not there exist infinitely many triples (a, b, c) such that $q(a, b, c)$ is the cube root of a natural number.

Solution by Nikolaos Dergiades, Thessaloniki, Greece.

Let $x_1 = a/b$, $y_1 = c/a$, and

$$x_n = \frac{a + x_{n-1}}{b + y_{n-1}}\,, \qquad y_n = \frac{c + x_{n-1}}{a + y_{n-1}}\,; \tag{1}$$

then $q(a,b,c) = \lim_{n\to\infty}(a + x_{n-1})$, and from this we conclude that the sequence x_n converges to a positive real number, so we can put

$$x \;=\; \lim_{n\to\infty} x_n \;=\; \lim_{n\to\infty} x_{n-1}\,,$$

and from (1) we conclude that the limit of y_n exists and we can put

$$y \;=\; \lim_{n\to\infty} y_n \;=\; \lim_{n\to\infty} y_{n-1}\,.$$

So $q(a,b,c) = a + x$, and from (1) we get

$$x \;=\; \frac{a+x}{b+y} \quad \text{and} \quad y \;=\; \frac{c+x}{a+y}\,. \tag{2}$$

From (2) we have

$$y \;=\; \frac{a}{x} + 1 - b$$

and thus

$$\left(\frac{a}{x} + 1 - b\right)\left(a + \frac{a}{x} + 1 - b\right) \;=\; c + x\,.$$

or

$$x^3 + cx^2 \;=\; (a + x - bx)(ax + a + x - bx)\,.$$

or [after some rearrangement]

$$(a+x)^3 + (c - 4a + 2b + ba - b^2 - 1)x^2 + (2ba - 4a^2 - 2a)x \;=\; a^3 + a^2\,. \tag{3}$$

Taking $b = 2a + 1$ [to make the coefficient of x equal to zero] and then $c = a(2a+3)$ [to make the coefficient of x^2 equal to zero], we get from (3) that $(a+x)^3 = a^3 + a^2$ or

$$q(a,b,c) \;=\; \sqrt[3]{a^3 + a^2}\,.$$

where a can be any of an infinity of natural numbers.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; STAN WAGON, Macalester College, St. Paul, MN, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

All answers were the same. In fact, Lambrou notes that the above solutions are the only ones possible.

2364. [1998: 363] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

A sequence $\{x_n\}$ is given by the recursion: $x_0 = p$, $x_{n+1} = qx_n + q - 1$ $(n \geq 0)$, where p is a prime and $q \geq 2$ is an integer.

(1) Suppose that p and q are relatively prime. Prove that the sequence $\{x_n\}$ does not consist of only primes.

(2)* Suppose that $p|q$. Prove that the sequence $\{x_n\}$ does not consist of only primes.

I. *Solution to both parts by Manuel Benito and Emilio Fernandez, I.B. Praxedes Mateo Sagasta, Logroño, Spain.*

The sequence is

$$p\,,\quad q(p+1)-1\,,\quad q^2(p+1)-1\,,\quad \ldots\,,\quad q^n(p+1)-1\,,\quad \ldots$$

[as can easily be proved by induction — *Ed.*] If $x_1 = q(p+1)-1$ is prime then by the Fermat Theorem, $q^{x_1} \equiv q \pmod{x_1}$ and so

$$x_{x_1} \;=\; q^{x_1}(p+1)-1 \;\equiv\; q(p+1)-1 \;\equiv\; 0 \pmod{x_1}\,.$$

Hence x_{x_1} is not a prime number. [And if x_1 is not prime, we are done. Note that this proof does not use that p is prime. — *Ed.*]

II. *Solution to both parts by Heinz-Jürgen Seiffert, Berlin, Germany.*

More generally, we consider the sequence $\{x_n\}$ defined by the recursion

$$x_0 = p\,,\qquad x_{n+1} = qx_n + r\,,\quad n \ge 0\,,$$

where p, q and r are *any* positive integers. We shall prove that this sequence contains infinitely many composite natural numbers.

A simple induction argument shows that

$$x_n \;=\; pq^n + r\sum_{j=0}^{n-1} q^j\,,\quad n \ge 0\,. \tag{1}$$

First of all we note that $\{x_n\}$ is a strictly increasing sequence of positive integers. Let k be a positive integer such that $x_k = s$ is a prime; if such k does not exist, then we are done.

[*Editorial note.* Since $k \ge 1$, it is clear that $s = x_k > q$. This slightly simplifies the proposer's original proof.]

Case 1: $q = 1$. Equation (1) with $n = k$ gives $s = x_k = p + rk$. It follows that for all $u \ge 0$,

$$x_{us+k} \;=\; p + r(us+k) \;\equiv\; p + rk \;\equiv\; 0 \pmod{s}\,.$$

Case 2: $q > 1$. Since $q^{s-1} \equiv 1 \pmod{s}$ by Fermat's Little Theorem, and since $\gcd(q-1, s) = 1$, we then have

$$\frac{q^{u(s-1)}-1}{q-1} \;\equiv\; 0 \pmod{s} \quad \text{for all integers } u \ge 0\,.$$

Hence, for all $u \geq 0$,

$$\begin{aligned} x_{u(s-1)+k} &= pq^{u(s-1)}q^k + r\left(\sum_{j=0}^{k-1} q^j \;+\; q^k \sum_{j=0}^{u(s-1)-1} q^j\right) \\ &\equiv pq^k + r\sum_{j=0}^{k-1} q^j \;+\; rq^k \sum_{j=0}^{u(s-1)-1} q^j \pmod{s} \\ &= x_k + rq^k\left(\frac{q^{u(s-1)}-1}{q-1}\right) \equiv x_k = s \equiv 0 \pmod{s}\,, \end{aligned}$$

where we have used (1) twice and the closed form expression for finite geometric sums. This completes the solution, which is an extension of the proof of the following theorem found in [1]: *Let a, b and q be integers such that $a \geq 1$, $q \geq 2$, and $\gcd(aq, b) = 1$. Then the sequence $aq^n + b$, $n \geq 0$, does not consist of only primes.*

The sequence considered in the proposal is of the form $x_n = (p+1)q^n - 1$, $n \geq 0$, so that the statements of both parts of the proposal are obtained from the above theorem with $a = p + 1$ and $b = -1$.

Reference:

[1] A. Aigner, *Zahlentheorie*, de Gruyter Verlag, 1975, S. 190, Satz 105.

Both parts also solved by THEODORE CHRONIS, Athens, Greece; NIKOLAOS DERGIADES, Thessaloniki, Greece; MICHAEL LAMBROU, University of Crete, Crete, Greece; and JEREMY YOUNG, student, Nottingham High School, Nottingham, UK. Part (1) only solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GERRY LEVERSHA, St. Paul's School, London, England; and the proposer.

Lambrou also gave a generalization, which included the observation that the given sequence contains infinitely many composites.

2365. [1998: 363] *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

Triangle DAC is equilateral. B is on the line DC so that $\angle BAC = 70°$. E is on the line AB so that $\angle ECA = 55°$. K is the mid-point of ED. Without the use of a computer, calculator or protractor, show that $60° > \angle AKC > 57.5°$.

Solution by Manuel Benito and Emilio Fernandez, I.B. Praxedes Mateo Sagasta, Logroño, Spain [slightly modified by the editor].

Let us consider the circle with centre A and radius AC. From the given information, $\angle AEC = 55°$ and $\triangle AEC$ is isosceles. It follows that this circle passes through points D and E. Since K is the mid-point of the chord ED of that circle, we have $\angle KAC = 65°$. The radial line AK meets the circle at the point H, say, and the chord CE at the point L, say, and we obviously have that

$$\angle ALC > \angle AKC > \angle AHC.$$

But $\angle ALC = 60^\circ$ because $\angle ACL = 55^\circ$, and $\angle AHC = 57.5^\circ$ because $\triangle AHC$ is isosceles. This completes the proof.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; HIDETOSHI FUKAGAWA, Gifu, Japan; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong, China; GERRY LEVERSHA, St. Paul's School, London, England; GOTTFRIED PERZ, Pestalozzi-gymnasium, Graz, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Magdeburg, Germany; and the proposer.

2366. [1998: 364] *Proposed by Catherine Shevlin, Wallsend-upon-Tyne, England.*

Triangle ABC has area p, where $p \in \mathbb{N}$. Let

$$\Sigma = \min\left(AB^2 + BC^2 + CA^2\right),$$

where the minimum is taken over all possible triangles ABC with area p, and where $\Sigma \in \mathbb{N}$.

Find the least value of p such that $\Sigma = p^2$.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

We have $p = \frac{1}{2}ab\sin(C)$; that is $ab = \dfrac{2p}{\sin(C)}$. Furthermore,

$$\begin{aligned} a^2+b^2+c^2 &= 2\left(a^2+b^2-ab\cos(C)\right) \\ &\geq 2\left(2ab-ab\cos(C)\right) \\ &= 2\cdot 2p\left(\frac{2}{\sin(C)}-\cot(C)\right). \end{aligned}$$

Define $f(C) = \dfrac{2}{\sin(C)} - \cot(C)$. Hence $f'(C) = \dfrac{1-2\cos(C)}{\sin^2(C)}$.

It follows that $f(C)$ attains its minimum at $C = \frac{\pi}{3}$, yielding $\min\{a^2+b^2+c^2 \in \mathbb{R}\} = 4pf\left(\frac{\pi}{3}\right) = 4p\sqrt{3}$.

Hence, $\min\{a^2+b^2+c^2 \in \mathbb{N}\} = p^2 \geq 4p\sqrt{3}$, implying that $p \geq 4\sqrt{3}$; that is, $p_{\min} = 7$.

Also solved by MANUEL BENITO and EMILIO FERNANDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; C. FESTRAETS-HAMOIR, Brussels, Belgium; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong, China; and the proposer. There were two incorrect solutions.

All the solutions were quite different in approach. Benito and Fernández also show that the solution is, in fact, unique.

The proposer comments that this is based on a problem in the New Scientist, *but that the answer published there was wrong!*

2367. [1998: 364] *Proposed by K.R.S. Sastry, Dodballapur, India.*
In triangle ABC, the Cevians AD, BE intersect at P. Prove that

$$[ABC] \times [DPE] = [APB] \times [CDE].$$

(Here, $[ABC]$ denotes the area of $\triangle ABC$, etc.)

Solution by Toshio Seimiya, Kawasaki, Japan.

Since $\frac{PD}{AD} = \frac{[DPE]}{[DEA]}$ and $\frac{AE}{CE} = \frac{[DEA]}{[CDE]}$, we have

$$\frac{[DPE]}{[CDE]} = \frac{[DPE]}{[DEA]} \cdot \frac{[DEA]}{[CDE]} = \frac{PD}{AD} \cdot \frac{AE}{CE}. \qquad (1)$$

Since $\frac{PD}{AD} = \frac{[PBC]}{[ABC]}$ and $\frac{AE}{CE} = \frac{[APB]}{[PBC]}$, we have

$$\frac{[APB]}{[ABC]} = \frac{[APB]}{[PBC]} \cdot \frac{[PBC]}{[ABC]} = \frac{AE}{CE} \cdot \frac{PD}{AD}. \qquad (2)$$

From (1) and (2) we have

$$\frac{[DPE]}{[CDE]} = \frac{[APB]}{[ABC]}.$$

This implies $[ABC] \times [DPE] = [APB] \times [CDE]$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; SAM BAETHGE, Nordheim, TX, USA (2 solutions); MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; MANUEL BENITO and EMILIO FERNANDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; C. FESTRAETS-HAMOIR, Brussels, Belgium; HIDETOSHI FUKAGAWA, Gifu, Japan; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; VICTOR OXMAN, University of Haifa, Haifa, Israel; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; PARAGIOU THEOKLITOS, Limassol, Cyprus; PANOS E. TSAOUSSOGLOU, Athens, Greece; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

The proposer implied (and all solvers assumed) both (a) that P is inside $\triangle ABC$, and (b) that all areas and lengths are positive. Bradley points out that had signed areas and segments been employed, then $[ABC] \times [PDE] = [ABP] \times [CED]$ holds for any P in the plane of $\triangle ABC$.

Three years ago Kandall deduced Sastry's equality as the first step of his featured solution to problem 1469 in Mathematics Magazine, 69:2 (April, 1996) 144-146. *His solution there invokes Menelaus' Theorem (which is implicit in Seimiya's solution above). Indeed, many of the submitted solutions used Menelaus' Theorem; the use of either affine or areal coordinates also leads to nice solutions.*

2368. [1998: 364] *Proposed by Iliya Bluskov, Simon Fraser University, Burnaby, BC.*

Let $(a_1, a_2, \ldots, a_n)$ be a permutation of the integers from 1 to n with the property that $a_k + a_{k+1} + \ldots + a_{k+s}$ is not divisible by $(n+1)$ for any choice of k and s where $k \geq 1$ and $0 \leq s \leq n-k-1$. Find such a permutation

(a) for $n = 12$;

(b) for $n = 22$.

Solution by Jeremy Young, student, Nottingham High School, Nottingham, UK (modified slightly by the editor).

(a) Set $a_k \equiv 2^k \pmod{13}$, $1 \leq k \leq 12$, $1 \leq a_k \leq 12$. Then we obtain the following permutation of $1, 2, \ldots, 12$;

k	1	2	3	4	5	6	7	8	9	10	11	12
a_k	2	4	8	3	6	12	11	9	5	10	7	1

For any integers k and s with $k \geq 1$, $0 \leq s \leq 11-k$, we have

$$a_k + a_{k+1} + \cdots + a_{k+s} \equiv 2^k + 2^{k+1} + \cdots + 2^{k+s} = 2^k\left(2^{s+1}-1\right).$$

Clearly, $2^k \not\equiv 0 \pmod{13}$. Furthermore, since $s+1 \leq 12-k \leq 11$, we see from the table above that $2^{s+1} \not\equiv 1 \pmod{13}$. Hence, $a_k + a_{k+1} + \cdots + a_{k+s}$ is not divisible by 13.

(b) Set $a_k \equiv 5^k \pmod{23}$, $1 \leq k \leq 22$, $1 \leq a_k \leq 22$. Then we obtain the following permutation of $1, 2, \ldots, 22$;

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
a_k	5	2	10	4	20	8	17	16	11	9	22	18	21	13	19	3	15	6	7	12	14	1

For any integers k and s with $k \geq 1$, $0 \leq s \leq 21-k$, we have

$$\begin{aligned} a_k + a_{k+1} + \cdots + a_{k+s} &\equiv 5^k + 5^{k+1} + \cdots + 5^{k+s} \\ &= \frac{5^k\left(5^{s+1}-1\right)}{4}. \end{aligned}$$

Clearly, $5^k \not\equiv 0 \pmod{23}$. Furthermore, since $s+1 \leq 22-k \leq 21$, the table above reveals that $5^{s+1} \not\equiv 1 \pmod{23}$.

Thus $\dfrac{5^k\left(5^{s+1}-1\right)}{4} \not\equiv 0 \pmod{23}$. Hence, $a_k + a_{k+1} + \cdots + a_{k+s}$ is not divisible by 23.

Also solved by MANUEL BENITO MUÑOZ and EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; and the proposer. Part (a) *only was solved by CHARLES ASHBACHER, Cedar Rapids, IA, USA.*

The solutions given by Hess and the proposer were essentially the same as the one above. Ashbacher, Benito and Fernández, and Lewis all gave the same permutation $(1,2,3,4,5,6,10,7,12,11,8,9)$ *for* $n = 12$. *On the other hand, their permutations for* $n = 22$ *were all different.*

The proposer actually showed in general that if p *is a prime and if* q *is a primitive root* $\bmod p$ *(that is,* $(q,p) = 1$ *and* $p-1$ *is the smallest positive integer such that* $q^{p-1} \equiv 1 \pmod p$) — Ed.), *then* $(q^0, q^1, q^2, \ldots, q^{p-2})$ *has the desired property. The proposed problem is the special case when* $p = 13$ *and* 23. *The proof of this general fact is quite easy and is essentially contained in Young's solution.*

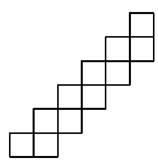
2369*. [1998: 364] *Proposed by Federico Arboleda, student, Bogotá, Colombia* (age 11).

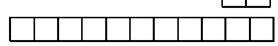
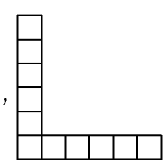
Prove or disprove that for every $n \in \mathbb{N}$, there exists a $2n$–omino such that every n–omino can be placed entirely on top of it.

(An n–omino is defined as a collection of n squares of equal size arranged with coincident sides.)

Solution by Michael Lambrou, University of Crete, Crete, Greece, modified by the editors.

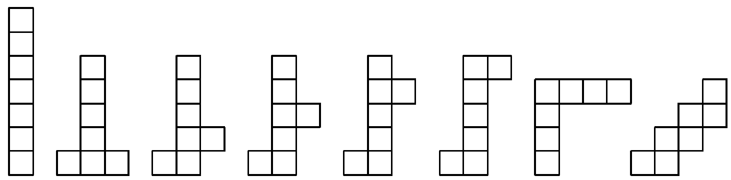
We show that the answer is negative for each large enough n. For simplicity we demonstrate this for $n = 11$, but the technique easily generalizes (*Ed.* for n odd).

In order to cover the "staircase",  , we need 11 squares.

To cover the "ladder",  , we need a further 9 squares since the staircase has only 2 squares in each row and column. Note that the staircase has 6 rows and 6 columns, and the ladder can overlap at most one row if it is placed horizontally and at most one column if it is placed vertically. So either 5 rows or 5 columns of the staircase are not covered by the ladder. Consider now the "el" shape,  , which has a row of 6 and a column of 6. There will be at least 3 squares on it uncovered by either the staircase or the ladder, since it can overlap at most 2 squares of the staircase which are not on the ladder if it has a maximum overlap with the ladder (if it does not have a maximum overlap with the ladder then there will be at least 6 squares on it uncovered by the ladder and the staircase). To sum up we need at least $11 + 9 + 3 = 23 > 22 = 2(11)$ squares to cover just these three 11–ominoes, completing the demonstration. (Generally, for $n = 2k+1$, the numbers 11, 9, 3 are respectively $2k+1$, $2k-1$, $k-2$ and $(2k+1) + (2k-1) + (k-2) > 2(2k+1)$ if $k \geq 5$.)

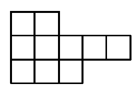
Also solved by MANUEL BENITO and EMILIO FERNANDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain.

Benito and Fernández give a set of eight 7–ominoes which require at least 15 squares to cover:

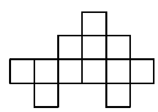


Editor's note: *It is interesting that, although the above example, as well as Lambrou's solution, considers only odd n, it certainly generalizes to even n as well.*

The proposer observes that the statement to be proved or disproved "is trivially true for $n = 1$, 2, 3 and 4. For $n = 5$ the following decamino shows that it is also true:



We believe the following dodecomino shows it is true for $n = 6$:



(end of quote). If we do not distinguish between reflections of 6–ominoes, then the above 12–omino seems to work; but it fails if we distinguish among the reflections; for example, the following cannot be covered without reflecting it: