THE ACADEMY CORNER

No. 25

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Abstracts • Résumés

Canadian Undergraduate Mathematics Conference 1998 — Part 3

The Brachistochrone Problem

Nils Johnson
The University of British Columbia

The brachistochrone problem is to find the curve between two points down which a bead will slide in the shortest amount of time, neglecting friction and assuming conservation of energy. To solve the problem, an integral is derived that computes the amount of time it would take a bead to slide down a given curve $y(x)$. This integral is minimized over all possible curves and yields the differential equation $y(1 + (y')^2) = k^2$ as a constraint for the minimizing function $y(x)$. Solving this differential equation shows that a cycloid (the path traced out by a point on the rim of a rolling wheel) is the solution to the brachistochrone problem. First proposed in 1696 by Johann Bernoulli, this problem is credited with having led to the development of the calculus of variations. The solution presented assumes knowledge of one-dimensional calculus and elementary differential equations.

The Theory of Error-Correcting Codes

Dennis Hill
University of Ottawa

Coding theory is concerned with the transfer of data. There are two issues of fundamental importance. First, the data must be transferred accurately. But equally important is that the transfer be done in an efficient manner. It is the interplay of these two issues which is the core of the theory of error-correcting codes.

Typically, the data is represented as a string of zeros and ones. Then a *code* consists of a set of such strings, each of the same length. The most fruitful approach to the subject is to consider the set $\{0, 1\}$ as a two-element field. We will then only

consider codes which are vector spaces over this field. Such codes will be called *linear codes*.

Since there is no way to always ensure that the message has been transferred accurately, it is important to add redundant structure so that, except for extreme cases, the receiver will realize that there has been an error. This is called *error-detection*. It would be even better if the receiver were able to not only detect that an error has occurred, but to determine the location of the incorrect digit or digits, and thus determine the original message. This is the notion of *error correction*.

This theory uses in a fundamental way ideas from linear algebra over finite fields, such abstract algebra as principal ideal domains, and metric space theory. One of the reasons this is such an exciting subject is that it combines principles of abstract algebra with very concrete and important applications. Among the many applications are the minimization of noise from compact disc recordings, transmission of data from satellites, and transmission of financial data.

I will introduce the basic concepts of coding theory, focusing on linear codes. I will discuss several more specialized topics, such as the most important methods of error correction. I will also discuss a number of specific examples. I will conclude with a brief discussion of *convolutional codes*, one of the more recent advances in the field.

Fixed Points and Diagonalization: An Abstraction of Gödel's Theorem

Todd A. Kemp
University of Calgary

Gödel's Incompleteness Theorem is one of the most provocative results in Mathematical Logic. It may be summarized "In any *'sufficiently powerful'* consistent theory, there must always be theorems (true sentences) which are *unprovable*."

In this paper, I address the question of what this heuristic phrase *'sufficiently powerful'* means. I will appeal to abstract conditions put forward by Smullyan, and show that any system meeting them also satisfies Gödel's Theorem. Further, after outlining some of the basic terminology and major results in the area, I will demonstrate that Robinson's Arithmetic—a well-known Gödel system—indeed meets Smullyan's conditions, hence lending some credibility to the claim that these conditions are not only sufficient but necessary.

Smullyan also goes on to show that his abstract form of Gödel's Theorem is actually a special case of a general Fixed Point Theorem—one which is also the primitive framework behind a major result in Recursive Function Theory, and Smullyan's favourite puzzle—the Mockingburd Puzzle. I will discuss this Fixed Point Theorem, and how it relates to Gödel's theorem.

Misuse of Statistics

Theodoro Koulis
McGill University

Statistics play a big role in the sciences. Statistics provide scientists with tools that help them verify the significance of their data. However, since these tools are widely distributed in popular software packages such as Microsoft Excel, they are often misused. Using some test statistics, I will show how some basic procedures can be misdirected and how conclusions can be misinterpreted. These will be:

1. The sign test and hypothesis testing:
   Foundations of hypothesis testing (the likelihood ratio principle)
2. Goodness of fit test statistics:
   Good and bad measures: Pearson's χ^2 and Kolmogorov-Smirnov tests
3. Correlation and rank correlation (if time permits)

An Introduction to Random Walks from Pólya to Self-Avoidance

Michael Kozdron
University of British Columbia

This paper provides an introduction to random walks. We begin with some basic definitions and culminate with the classical theorem of Pólya that a simple random walk in $\mathbb{Z}^d$, $d \geq 3$ is transient and recurrent otherwise. We then discuss the more contemporary topic of self-avoiding random walks and survey some currently open problems. This paper assumes only a minimal background in probability including the notion of a random variable and an expectation.

Of Graphs and Coffi Grounds: Decompiling Java

Patrick Lam
McGill University

Java programmers write applications and applets in plain English-like text, and then apply a java compiler to the text to obtain *class files*. Class files, which are typically transmitted across the Web, are a low-level representation of the original text; they are not human-readable. Consider a compiler as a function from text to class files. My goal is to compute the inverse function: given the compiled class file, I wish to find the best approximation to the original text possible. This is called decompilation.

Given a class file, one can build an unstructured graph representation of the program. The main goal of my work is to develop graph algorithms to convert these unstructured graphs into structured graphs corresponding to high-level program constructs, and I will discuss these in detail. I shall also mention some results concerning possibility and impossibility which show that decompilation is always possible if the program may be lengthened.

THE OLYMPIAD CORNER

No. 198

R.E. Woodrow

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We begin this number with the problems of the XXXIX Republic Competition of Mathematics in Macedonia and the problems of the third Macedonian Mathematical Olympiad. My thanks go to Ravi Vakil who collected them for me while he was Canadian Team Leader to the IMO at Mumbai.

XXXIX REPUBLIC COMPETITION OF MATHEMATICS IN MACEDONIA

Class I

1. The sum of three integers a, b and c is 0. Prove that $2a^4+2b^4+2c^4$ is the square of an integer.

2. Prove that if

$$a_0^{a_1} = a_1^{a_2} = \cdots = a_{1995}^{a_{1996}} = a_{1996}^{a_0}, \quad a_1 \in \mathbb{R}^*,$$

then

$$a_0 = a_1 = \cdots = a_{1996}.$$

3. Let h_a, h_b and h_c be the altitudes of the triangle with edges a, b and c, and r be the radius of the inscribed circle in the triangle. Prove that the triangle is equilateral if and only if $h_a+h_b+h_c = 9r$.

4. Prove that each square can be cut into n ($n \geq 6$) squares.

Class II

1. Prove that for positive real numbers a and b

$$2\cdot\sqrt{a}+3\cdot\sqrt[3]{b} \geq 5\cdot\sqrt[5]{ab}.$$

2. The point M is the mid-point on the side $\overline{BC} = a$ of a triangle ABC. Let r_1, r_2, r_3 be the radii of the inscribed circles in the triangles ABC, ABM and ACM respectively. Prove that

$$\frac{1}{r_1}+\frac{1}{r_2} \geq 2\left(\frac{1}{4}+\frac{2}{a}\right).$$

3. Let $A = \{z_1, z_2, \ldots, z_{1996}\}$ be a set of complex numbers and for each $i \in \{1, 2, \ldots, 1996\}$ suppose $\{z_1 z_1, z_1 z_2, \ldots, z_1 z_{1996}\} = A$.

(a) Prove that $|z_i| = 1$ for each i.

(b) Prove that $z \in A$ implies $\overline{z} \in A$.

4. Find the biggest value of the difference $x - y$ if $2(x^2 + y^2) = x + y$.

Class III

1. Solve the equation $x^{1996} - 1996x^{1995} + \cdots + 1 = 0$ (the coefficients in front of $x, \ldots, x^{1994}$ are unknown), if it is known that its roots are positive real numbers.

2. Let AH, BK and CL be the altitudes of an arbitrary triangle ABC. Prove that

$$\overline{AK} \cdot \overline{BL} \cdot \overline{CH} = \overline{AL} \cdot \overline{BH} \cdot \overline{CK} = \overline{HK} \cdot \overline{KL} \cdot \overline{LH}.$$

3. An initial triple of numbers $2, \sqrt{2}, \frac{1}{\sqrt{2}}$ is given. A new triple may be obtained from an old one as follows: two numbers a and b of the triple are changed to $\frac{(a+b)}{\sqrt{2}}$ and $\frac{(a-b)}{\sqrt{2}}$ and the third number is unchanged. Is it possible after a finite number of such steps to obtain the triple $1, \sqrt{2}, 1 + \sqrt{2}$?

4. A finite number of points in the plane are given such that not all of them are collinear. A real number is assigned to each point. The sum of the numbers for each line containing at least two of the given points is zero. Prove that all numbers are zeros.

Class IV

1. Let $a_1, a_2, \ldots, a_n$ be real numbers which satisfy:

There exists a real number M such that $|a_i| \leq M$ for each $i \in \{1, \ldots, n\}$.

Prove that $a_1 + 2a_2 + \cdots + na_n \leq \dfrac{Mn^2}{4}$.

2. Two circles with radii R and r touch from inside. Find the side of an equilateral triangle having one vertex at the common point of the circles and the other two vertices lying on the two circles.

3. The same problem as problem 3 given for Class III.

4. The same problem as problem 4 given for Class III.

PROBLEMS ON THE THIRD MACEDONIAN MATHEMATICAL OLYMPIAD

1. Let $ABCD$ be a parallelogram which is not a rectangle and E be a point in its plane, such that $AE \perp AB$ and $BC \perp EC$. Prove that $\angle DAE = \angle CEB$. [Ed. We know this is incorrect — can any reader supply the correct version?]

2. Let $\mathcal{P}$ be the set of all polygons in the plane and let $M : \mathcal{P} \to \mathbb{R}$ be a mapping which satisfies:

(i) $M(P) \geq 0$ for each polygon P;

(ii) $M(P) = x^2$ if P is an equilateral triangle of side x;

(iii) If P is a polygon separated into two polygons S and T, then $M(P) = M(S) + M(T)$; and

(iv) If P and T are congruent polygons, then $M(P) = M(T)$.

Find $M(P)$ if P is a rectangle with edges x and y.

3. Prove that if α, β and γ are angles of a triangle, then

$$\frac{1}{\sin\alpha} + \frac{1}{\sin\beta} \geq \frac{8}{3 + 2\cos\gamma}\,.$$

4. A polygon is called "good" if the following conditions are satisfied:

(i) all angles belong to $(0, \pi) \cup (\pi, 2\pi)$;

(ii) two non-neighbouring sides do not have any common point; and

(iii) for any three sides, at least two are parallel and equal.

Find all non-negative integers n such that there exists a "good" polygon with n sides.

5. Find the biggest number n such that there exist n straight lines in space, $\mathbb{R}^3$, which pass through one point and the angle between each two lines is the same. (The angle between two intersecting straight lines is defined to be the smaller one of the two angles between these two lines.)

Next we give the problems of the Ninth Irish Mathematical Olympiad, written Saturday, May 4, 1996. My thanks again go to Ravi Vakil for collecting the problems and sending them to me while he was Canadian Team Leader to the IMO at Mumbai.

NINTH IRISH MATHEMATICAL OLYMPIAD

First Paper — May 4, 1996

Time: 3 hours

1. For each positive integer n, let $f(n)$ denote the greatest common divisor of $n!+1$ and $(n+1)!$ (where ! denotes "factorial"). Find, with proof, a formula for $f(n)$ for each n.

2. For each positive integer n, let $S(n)$ denote the sum of the digits of n (when n is written in base 10). Prove that for every positive integer n,

$$S(2n) \le 2S(n) \le 10S(2n).$$

Prove also that there exists a positive integer n with $S(n) = 1996S(3n)$,

3. Let K be the set of all real numbers x with $0 \le x \le 1$. Let f be a function from K to the set of all real numbers $\mathbb{R}$ with the following properties:

(i) $f(1) = 1$;

(ii) $f(x) \ge 0$ for all $x \in K$;

(iii) if x, y and $x+y$ are all in K, then $f(x+y) \ge f(x)+f(y)$.

Prove that $f(x) \le 2x$ for all $x \in K$.

4. Let F be the mid-point of the side BC of the triangle ABC. Isosceles right-angled triangles ABD and ACE are constructed externally on the sides AB and AC with the right angles at D and E, respectively.

Prove that DEF is a right-angled isosceles triangle.

5. Show, with proof, how to dissect a square into at most five pieces in such a way that the pieces can be reassembled to form three squares no two of which have the same area.

Second Paper — May 4, 1996

Time: 3 hours

6. The Fibonacci sequence $F_0, F_1, F_2, \ldots$ is defined as follows: $F_0 = 0$, $F_1 = 1$ and for all $n \ge 0$, $F_{n+2} = F_n + F_{n+1}$. (So $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8, \ldots$.) Prove that

(i) The statement "$F_{n+k} - F_n$ is divisible by 10 for all positive integers n" is true if $k = 60$ but it is not true for any positive integer $k < 60$.

(ii) The statement "$F_{n+t} - F_n$ is divisible by 100 for all positive integers n" is true if $t = 300$ but it is not true for any positive integer $t < 300$.

7. Prove that the inequality $2^{1/2} \cdot 4^{1/4} \cdot 8^{1/8} \cdots (2^n)^{1/2^n} < 4$ holds for all positive integers n.

8. Let p be a prime number and a and n positive integers. Prove that if $2^p + 3^p = a^n$, then $n = 1$.

9. Let ABC be an acute-angled triangle and let D, E, F be the feet of the perpendiculars from A, B, C onto the sides BC, CA, AB, respectively. Let P, Q, R be the feet of the perpendiculars from A, B, C onto the lines EF, FD, DE respectively. Prove that the lines AP, BQ, CR (extended) are concurrent.

10. We are given a rectangular chessboard of size 5×9 (so there are five rows of squares, each row containing nine squares). The following game is played: Initially, a number of discs are randomly placed on some of the squares, no square being allowed to contain more than one disc. A complete move consists of moving every disc from the square containing it to another square subject to the following rules:

(i) each disc may be moved one square up or down, or left or right, of the square it occupies, to an adjoining square;

(ii) if a particular disc is moved up or down as part of a complete move, then it must be moved left or right in the next complete move;

(iii) if a particular disc is moved left or right as part of a complete move, then it must be moved up or down in the next complete move;

(iv) at the end of each complete move, no square can contain two or more discs.

The game stops if it becomes impossible to perform a complete move. Prove that if initially 33 discs are placed on the board then the game must eventually stop. Prove also that it is possible to place 32 discs on the board in such a way that the game could go on forever.

Now we turn to solutions by our readers to problems of the 30^{th} Spanish Mathematical Olympiad, Final Round, November 26–27, 1993 [1998: 69–70].

30th SPANISH MATHEMATICAL OLYMPIAD
First Round — November 26–27, 1993

1. Show that, for all $n \in \mathbb{N}$, the fractions

$$\frac{n-1}{n}, \frac{n}{2n+1}, \frac{2n+1}{2n^2+2n},$$

are irreducible.

Solution by Pierre Bornsztein, Courdimanche, France.

Soit $n \in \mathbb{N}^*$.

On a, pour $n \geq 2$, $(n, n-1) = 1$. En plus$(n, 2n+1) = 1$ et donc $\frac{n}{2n+1}$ est irréductible. Enfin $(2n+1, 2n) = 1$ et $(2n+1, n+1) = 1$, car si p divise $2n+1$ et $n+1$ alors p divise $(2n+1) - (n+1) = n$, donc p divise $(n, n+1) = 1$. D'où $(2n+1, 2n(n+1)) = 1$ et donc $\frac{2n+1}{2n^2+2n}$ est irréductible.

3. Solve the following system of equations:

$$x \cdot |x| + y \cdot |y| = 1, \qquad \lfloor x \rfloor + \lfloor y \rfloor = 1\,,$$

in which $|t|$ and $\lfloor t \rfloor$ represent the absolute value and the integer part of the real number t.

Solutions by Pierre Bornsztein, Courdimanche, France; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.

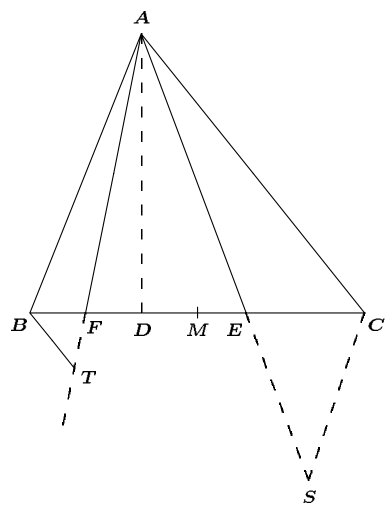
We show that there are only two solutions: $(x, y) = (1, 0)$ and $(0, 1)$. Clearly, x and y cannot both be negative. Hence by symmetry, there are two cases to be considered:

(i) If $x \geq 0$ and $y \geq 0$, then from $x^2 + y^2 = 1$ we get $0 \leq x$, $y \leq 1$. If $x < 1$ and $y < 1$, then $\lfloor x \rfloor + \lfloor y \rfloor = 0$, a contradiction. Hence $x = 1$ or $y = 1$. Then from $x^2 + y^2 = 1$ we obtain the two solutions $(1, 0)$ and $(0, 1)$.

(ii) If $x \geq 0$ and $y < 0$ then $x^2 - y^2 = 1$, and from $\lfloor x \rfloor \leq x$, $\lfloor y \rfloor \leq y$ we get $x + y \geq 1$. Since $x - y > x + y \geq 1$ we have $x^2 - y^2 = (x-y)(x+y) > 1$, a contradiction. Therefore, there are no solutions in this case.

4. Let AD be the internal bisector of the triangle ABC ($D \in BC$), E the point symmetric to D with respect to the mid-point of BC, and F the point of BC such that $\angle BAF = \angle EAC$. Show that $\frac{BF}{FC} = \frac{c^3}{b^3}$.

Solutions by Pierre Bornsztein, Courdimanche, France; by Toshio Seimiya, Kawasaki, Japan; and by D.J. Smeenk, Zaltbommel, the Netherlands. We give Seimiya's solution.



Let T be the point on AF such that $BT \parallel AC$ and let S be the point on AE such that $CS \parallel AB$. Then $\angle ABT + \angle BAC = 180^\circ$ and $\angle ACS + \angle BAC = 180^\circ$.

Thus we have $\angle ABT = \angle ACS$. Since $\angle BAT = \angle BAF = \angle CAE = \angle CAS$ we have $\triangle ABT \sim \triangle ACS$ so that

$$\frac{BT}{CS} = \frac{AB}{AC} = \frac{c}{b} . \tag{1}$$

As $BT \parallel AC$ we get

$$\frac{BF}{FC} = \frac{BT}{AC} = \frac{BT}{b} . \tag{2}$$

As $AB \parallel CS$ we have

$$\frac{BE}{EC} = \frac{AB}{CS} = \frac{c}{CS} . \tag{3}$$

Let M be the mid-point of BC.

Since E is the point symmetric to D with respect to M we have

$$EC = BD \quad \text{and} \quad BE = DC ,$$

so that $\dfrac{BE}{EC} = \dfrac{DC}{BD}$.

Since AD is the bisector of $\angle BAC$, we get $\dfrac{DC}{BD} = \dfrac{AC}{AB} = \dfrac{b}{c}$. Thus we have

$$\frac{BE}{EC} = \frac{b}{c} . \tag{4}$$

From (3) and (4) we have

$$\frac{c}{CS} = \frac{b}{c}, \quad \text{so that} \quad CS = \frac{c^2}{b}.$$

Hence, from (1), we get $BT = \frac{c^3}{b^2}$; whence from (2), $\frac{BF}{FC} = \frac{c^3}{b^3}$.

5. Find all the natural numbers n such that the number

$$n(n+1)(n+2)(n+3)$$

has exactly three prime divisors.

Solutions by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.

The only such n's are $n = 2$, 3 and 6.

Let $P(n) = n(n+1)(n+2)(n+3)$. Then $P(1) = 2^3 \times 3$ and thus $n = 1$ is not a solution. Hence we assume $n \geq 2$. Note first that for all $k \in \mathbb{N}$, $(k, k+1) = (2k-1, 2k+1) = 1$ and hence if n is odd, then each of n, $n+1$ and $n+2$ must be a distinct prime power. We are led to two cases:

(a) If n is odd, then since $n+1$ is even we must have $n = p^a$, $n+1 = 2^b$ and $n+2 = q^c$ where a, b, c, p, $q \in \mathbb{N}$ with p and q being distinct odd primes. Note that $n+3 = 2^b + 2 = 2(2^{b-1}+1)$ where $b \geq 2$. Since the only possible prime divisors of $n+3$ are 2, p or q, we have either

$$\text{(i) } 2^{b-1} + 1 = p^\alpha \quad \text{for some} \quad \alpha \in \mathbb{N}$$

or

$$\text{(ii) } 2^{b-1} + 1 = q^\beta \quad \text{for some} \quad \beta \in \mathbb{N}.$$

In case (i) we have $2p^\alpha = n + 3 = p^a + 3$. Clearly $\alpha \leq a$ and thus $p^\alpha \mid p^a$. Hence $p^\alpha \mid 3$ which implies $p = 3$, $\alpha = 1$. Thus $b = 2$ and $n = 3$. Indeed, $n = 3$ is a solution since $P(3) = 2^3 \times 3^2 \times 5$.

In case (ii) we have $2q^\beta = n + 3 = q^c + 1$, which is clearly impossible since $q \not| 1$.

(b) If n is even, then by the same argument, we have $n+1 = p^a$, $n+2 = 2^b$ and $n+3 = q^c$ where a, b, c, p, $q \in \mathbb{N}$ with p and q being distinct odd primes. Note that $n = 2^b - 2 = 2(2^{b-1} - 1)$ where $b \geq 2$. If $b = 2$, then $n = 2$, which is indeed a solution since $P(2) = 2^3 \times 3 \times 5$. If $b > 2$, then we must have either

$$\text{(iii) } 2^{b-1} - 1 = p^\alpha \quad \text{for some} \quad \alpha \in \mathbb{N}$$

or

$$\text{(iv) } 2^{b-1} - 1 = q^\beta \quad \text{for some} \quad \beta \in \mathbb{N}.$$

In case (iii) we have $2p^\alpha = n = p^a - 1$ which is clearly impossible since $p \not| 1$.

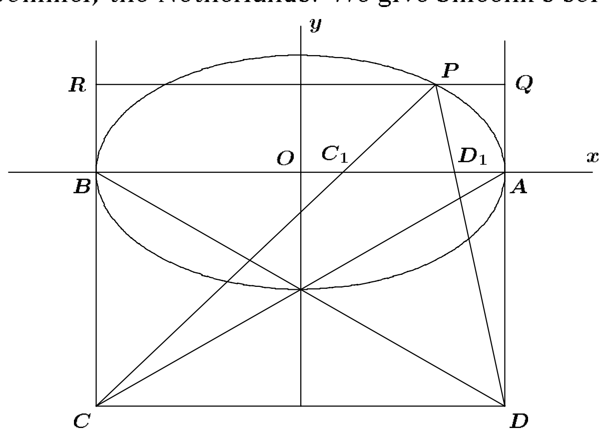
In case (iv) we have $2q^\beta = n = q^c - 3$. Clearly $\beta \le c$ and thus $q^\beta \mid 3$ which implies $q = 3$, $\beta = 1$. Thus $b = 3$ and $n = 6$, which is indeed a solution since $P(6) = 2^4 \times 3^3 \times 7$.

To summarize, $n(n+1)(n+2)(n+3)$ has exactly three prime divisors if and only if $n = 2$, 3 or 6.

6. An ellipse is drawn taking as major axis the biggest of the sides of a given rectangle, such that the ellipse passes through the intersection point of the diagonals of the rectangle.

Show that, if a point of the ellipse, external to the rectangle, is joined to the extreme points of the opposite side, then three segments in geometric progression are determined on the major axis.

Solutions by Pierre Bornsztein, Courdimanche, France; and by D.J. Smeenk, Zaltbommel, the Netherlands. We give Smeenk's solution.



An equation of the ellipse E is $b^2x^2 + a^2y^2 - a^2b^2 = 0$. The rectangle $ABCD$ has sides $AB = CD = 2a$, $AD = BC = 2b$. A point P of E is $P(a\cos\varphi, b\sin\varphi)$, with $\sin\varphi > 0$. Let the line through P parallel to AB meet BC and AD in R and Q, respectively. Let PD and PC meet the major axis at D_1, C_1 respectively.

From $\triangle DPQ$ and $\triangle DD_1A$, we have

$$D_1A : PQ = 2b : (2b+b\sin\varphi), \qquad D_1A : a(1-\cos\varphi) = 2 : (2+\sin\varphi);$$

$$\Longrightarrow \quad D_1A = \frac{2a(1-\cos\varphi)}{2+\sin\varphi}. \tag{1}$$

In the same way, from $\triangle PCD$ and $\triangle PC_1D_1$, we have

$$C_1D_1 = \frac{2a\sin\varphi}{2+\sin\varphi}, \tag{2}$$

and from $\triangle CPR$ and $\triangle CC_1B$, we have

$$BC_1 = \frac{2a(1+\cos\varphi)}{2+\sin\varphi}. \qquad (3)$$

From (1), (2) and (3) $D_1A : C_1D_1 : BC_1 = (1-\cos\varphi) : \sin\varphi : (1+\cos\varphi)$. And as $\sin^2\varphi = 1-\cos^2\varphi$ the three segments are in geometric progression indeed.

7. Let $a \in \mathbb{R}$ be given. Find the real numbers $x_1, \ldots, x_n$ which satisfy the system of equations

$$\begin{aligned} x_1^2 + ax_1 + \left(\tfrac{a-1}{2}\right)^2 &= x_2\,, \\ x_2^2 + ax_2 + \left(\tfrac{a-1}{2}\right)^2 &= x_3\,, \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ x_{n-1}^2 + ax_{n-1} + \left(\tfrac{a-1}{2}\right)^2 &= x_n\,, \\ x_n^2 + ax_n + \left(\tfrac{a-1}{2}\right)^2 &= x_1\,. \end{aligned}$$

Solutions by Pierre Bornsztein, Courdimanche, France; by Pavlos Maragoudakis, Pireas, Greece; and by Panos E. Tsaoussoglou, Athens, Greece. We give the write-up of Maragoudakis, although the methods were the same.

The first equation becomes

$$x_1^2 + (a-1)x_1 + \left(\frac{a-1}{2}\right)^2 = x_2 - x_1 \quad \text{or} \quad \left(x_1 + \frac{a-1}{2}\right)^2 = x_2 - x_1\,.$$

Similarly

$$\begin{aligned} \left(x_1 + \tfrac{a-1}{2}\right)^2 &= x_2 - x_1\,, \\ \left(x_2 + \tfrac{a-1}{2}\right)^2 &= x_3 - x_2\,, \\ \cdots\cdots\cdots\cdots & \\ \left(x_{n-1} + \tfrac{a-1}{2}\right)^2 &= x_n - x_{n-1}\,, \\ \left(x_n + \tfrac{a-1}{2}\right)^2 &= x_1 - x_n\,. \end{aligned}$$

Adding, we get:

$$\left(x_1 + \tfrac{a-1}{2}\right)^2 + \left(x_2 + \tfrac{a-1}{2}\right)^2 + \cdots + \left(x_{n-1} + \tfrac{a-1}{2}\right)^2 + \left(x_n + \tfrac{a-1}{2}\right)^2 = 0\,.$$

So $x_1 = x_2 = \cdots = x_{n-1} = x_n = \frac{1-a}{2}$.

We next give solutions to two of the problems from Peru's Selection Test for the XII IberoAmerican Olympiad [1998: 131].

1. Given an integer $a_0 > 2$, the sequence $a_0, a_1, a_2, \ldots$ is defined as follows:

$$a_{k+1} = a_k(1 + a_k), \quad \text{if } a_k \text{ is an odd number}$$
$$a_{k+1} = \frac{a_k}{2}, \quad \text{if } a_k \text{ is an even number.}$$

Prove that there is a non-negative integer p such that $a_p > a_{p+1} > a_{p+2}$.

Solution by Pierre Bornsztein, Courdimanche, France.

On montre facilement par récurrence sur k que pour tout $k \in \mathbb{N}$, $a_k \in \mathbb{N}$.

Alors pour tout $k \geq 0$, $a_{k+1} \neq a_k$ si a_k est impair alors $a_{k+1} = a_k + a_k^2 > a_k$. Cependant si a_k est pair alors $a_{k+1} < a_k$. Donc $a_p > a_{p+1} > a_{p+2}$ si et seulement si $a_p \equiv 0 \pmod 4$.

Lemme : Soit $n \in \mathbb{N}$, $n \geq 2$. S'il existe $k \geq 0$ tel que $a_k = 2 + 2^n q$ où q impair, $q \geq 1$, alors il existe $p \geq k$ tel que $a_p > a_{p+1} > a_{p+2}$.

Preuve du Lemme : Par récurrence sur n

Pour $n = 2$, si $a_k = 2 + 4q$ où q impair, $q \geq 1$.

Alors

$$\begin{aligned} a_{k+1} &= 1 + 2q \quad \text{impair, d'où} \\ a_{k+2} &= (1+2q)(2+2q) = 2(1+2q)(1+q) \end{aligned}$$

avec $1 + q$ pair, donc $a_{k+2} \equiv 0 \pmod 4$, et il suffit de choisir $p = k + 2$.

Soit $n \geq 2$ fixé. Supposons la propriété vraie pour ce n. Soit a_k tel que $a_k = 2 + 2^{n+1} q$, où q impair, $q \geq 1$. Alors

$$\begin{aligned} a_{k+1} &= 1 + 2^n q, \quad \text{impair et} \\ a_{k+2} &= (1 + 2^n q)(2 + 2^n q) = 2 + 2^n(q + 2q + 2^n q^2)\,, \end{aligned}$$

et comme $q+2q+2^n q^2$ est impair, d'après l'hypothèse de récurrence il existe $p \geq k + 2 \geq k$ tel que $a_p \equiv 0 \pmod 4$, d'où le résultat pour $n + 1$.

Ce qui achève la recurrence et prouve le Lemme.

On distingue quatre cas.

1. $a_0 \equiv 0 \pmod 4$: il suffit de choisir $p = 0$.

2. $a_0 \equiv 3 \pmod 4$: alors a_0 est impair, d'où $a_1 = a_0(1 + a_0)$ et $1 + a_0 \equiv 0 \pmod 4$. Donc $a_1 \equiv 0 \pmod 4$, et il suffit de choisir $p = 1$.

3. $a_0 \equiv 2 \pmod 4$: alors il existe $n \geq 2$, il existe q, impair, $q \geq 1$ (car $a_0 > 2$) tel que $a_0 = 2 + 2^n q$. Le Lemme permet de conclure.

4. $a_0 \equiv 1 \pmod 4$: alors $a_0 = 1 + 4k$, $k \geq 1$, car $a_0 > 2$. D'où, a_0 impair et

$$\begin{aligned} a_1 &= (1+4k)(2+4k) \\ &= 2 + 4(3k + 4k^2) \end{aligned}$$

donc $a_1 \equiv 2 \pmod 4$ et on est ramené au $3^{\text{ème}}$ cas.

Donc, dans tous les cas, il existe $p \geq 0$ tel que $a_p \equiv 0 \pmod 4$, c.à.d. tel que $a_p > a_{p+1} > a_{p+2}$.

2. A positive integer is called "almost-triangular" if the number is itself triangular or is the sum of different triangular numbers. How many almost-triangular numbers are there in the set $\{1, 2, 3, \ldots, 1997\}$?

Note: The triangular numbers are a_1, a_2, a_3, $\ldots$, a_k, $\ldots$, where $a_1 = 1$, and $a_k = k + a_{k-1}$, for all $k \geq 2$.

Solution by Pierre Bornsztein, Courdimanche, France.

On pose $T_i = i^{\text{ème}}$ nombre triangulaire $= \frac{i(i+1)}{2}$, $i \geq 1$. On a $T_1 = 1$, $T_2 = 3$, $T_3 = 6$, $T_4 = 10$, $T_5 = 15$, $T_6 = 21$, $T_7 = 28$, $T_8 = 36$, $T_9 = 45$, $T_{10} = 55$, $T_{11} = 66$.

On vérifie facilement que si $n \leq 33$ alors n est presque triangulaire sauf pour $n \in I = \{2, 5, 8, 12, 23, 33\}$.

Supposons $n \geq 34$.

On vérifie que : si $34 \leq n \leq 66$ alors n convient et que $n = \sum T_i$ avec $T_i < 66$.

On en déduit que si $n = 66 + a = T_{11} + a$, où $a \in \{1, \ldots, 66\}$ et $a \notin I$, alors n convient (on est sûr que l'on peut décomposer a sans utiliser T_{11}). On vérifie que si $n = 66 + a$ avec $a \in I$, alors n convient puisque :

$$\begin{array}{rcl@{\qquad}rcl} 68 & = & 55 + 10 + 3 & 78 & = & 36 + 28 + 10 + 3 + 1 \\ 71 & = & 55 + 10 + 6 & 89 & = & 55 + 28 + 6 \\ 74 & = & 55 + 10 + 6 + 3 & 99 & = & 55 + 28 + 10\,. \end{array}$$

Par conséquent si $n \leq 132$ alors n convient et $n = \sum T_i$ avec $T_i \leq 66$. Or $T_{13} = 91$ et donc tout $n = 91 + a$ convient pour $a \in \{34, \ldots, 132\}$. Donc si $n \leq 223$, alors n convient et $n = \sum T_i$ avec $T_i \leq 91$.

Or $T_{19} = 190$. On en déduit que tout $n = 190 + a$, où $a \in \{34, \ldots, 223\}$ convient. Donc si $n \leq 413$ alors n convient et $n = \sum T_i$ avec $T_i \leq 190$.

Or $T_{25} = 325$. Donc tout $n = 325 + a$ où $a \in \{34, \ldots, 413\}$ convient. Donc si $n \leq 738$ alors n convient et $n = \sum T_i$ avec $T_i \leq 325$.

Or $T_{37} = 703$. Donc tout $n = 703 + a$ où $a \in \{34, \ldots, 738\}$ convient, donc si $n \leq 1441$ alors n convient et $n = \sum T_i$ avec $T_i \leq 703$. Or $T_{52} = 1378$.

Donc tout $n = 1378 + a$ où $a \in \{34, \ldots, 1441\}$ convient, donc si $n \leq 2819$ alors n convient.

Finalement : les seuls $n \in \{1, \ldots, 1997\}$ qui ne sont pas presque triangulaires sont 2, 5, 8, 12, 23, 33. Il y a donc 1991 nombres presque triangulaires dans $\{1, \ldots, 1997\}$.

Remarque : "Tout nombre $n \geq 34$ est presque triangulaire". Ce résultat serait dû à R. Graham et P. Erdős, "L'enseignement mathématiques" 1980.

We now turn to solutions from the readers to problems for the Third Grade of the 38th Mathematics Competition of the Republic of Slovenia [1998: 132].

1. Let n be a natural number. Prove: if $2n+1$ and $3n+1$ are perfect squares, then n is divisible by 40.

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Pierre Bornsztein, Courdimanche, France; by Masoud Kamgarpour, Carson Graham Secondary School, North Vancouver, BC; by Pavlos Maragoudakis, Pireas, Greece; by Panos E. Tsaoussoglou, Athens, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution by Amengual Covas.

Since $40 = 2^3 \cdot 5$, it is sufficient to prove that n is divisible by 8 and 5. Set

$$2n+1 \;=\; x^2 \tag{1}$$

and

$$3n+1 \;=\; y^2 \tag{2}$$

where x, y are natural numbers.

We note that the number x^2 is odd, and thus also the number x is odd; consequently $x = 2a+1$, where a is a natural number.

Equation (1) implies the equality $2n+1 = (2a+1)^2$, whence $n = 2a^2+2a$.

The number n, as the sum of two even numbers, is even. It follows from equation (2) that the number y^2 is odd, and thus also the number y is odd; consequently $y = 2b+1$, where b is a natural number.

1° We subtract (1) from (2) and find that

$$n \;=\; y^2 - x^2 \;=\; (2b+1)^2 - (2a+1)^2 \;=\; 4(b+a+1)(b-a)\,.$$

Since both of $b+a$, $b-a$ are either even or odd, one of the numbers $b+a+1$, $b-a$ is even, whence the number n is divisible by 8.

2° We can eliminate n between (1) and (2) to get

$$3x^2 - 2y^2 \;=\; 1\,.$$

Since the square of an odd number ends in 1, 5, or 9, each of the numbers x^2 and y^2 ends in 1, 5, or 9. Therefore the number $3x^2$ ends in 3, 5, or 7 and $2y^2$ ends in 2, 0 or 8.

Since $3x^2 - 2y^2 = 1$, $3x^2$ must have ended in 3 and $2y^2$ must have ended in 2, whence both of the numbers x^2 and y^2 end in 1.

Hence $n = y^2 - x^2$ ends in 0, and consequently n is divisible by 5.

Comment: A related problem appears in Arthur Engel's *Problem-Solving Strategies*, Springer-Verlag 1998, page 131. If $2n+1$ and $3n+1$ are squares, then $5n+3$ is not a prime.

2. Show that $\cos(\sin x) > \sin(\cos x)$ holds for every real number x.

Solutions by Mohammed Aassila, Univérsité Louis Pasteur, Strasbourg, France; by Pierre Bornsztein, Courdimanche, France; by Pavlos Maragoudakis, Pireas, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the two solutions of Maragoudakis.

First Solution. $-\frac{\pi}{2} < -1 \leq \sin x \leq 1 < \frac{\pi}{2} \Longrightarrow \cos(\sin x) > 0$ for $x \in \mathbb{R}$.

If $\cos x \leq 0$, then $\cos x \in (-\frac{\pi}{2}, 0]$, so $\sin(\cos x) \leq 0 < \cos(\sin x)$.

If $\cos x > 0$, then $\cos x \in (0, \frac{\pi}{2})]$. It is known that $\sin y < y$ for $y \in (0, \frac{\pi}{2})$. So

$$\sin(\cos x) \ < \ \cos x \, . \tag{1}$$

Also $\cos y \geq 1 - \frac{y^2}{2}$ for $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Since $\sin x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ we have

$$\cos(\sin x) \ \geq \ 1 - \frac{\sin^2 x}{2} \ = \ \frac{1+\cos^2 x}{2} \, . \tag{2}$$

But, using (1) and (2), $\dfrac{1+\cos^2 x}{2} \geq \cos x \Longrightarrow \cos(\sin x) > \sin(\cos x)$.

Second Solution.

$$\begin{aligned}
&\cos(\sin x) - \sin(\cos x) \\
&\quad = \ \cos(\sin x) - \cos\left(\frac{\pi}{2} - \cos x\right) \\
&\quad = \ 2\sin\left(\frac{\sin x - \cos x + \frac{\pi}{2}}{2}\right) \sin\left(\frac{\frac{\pi}{2} - \sin x - \cos x}{2}\right) \\
&\quad = \ 2\sin\left(\frac{\sqrt{2}}{2}\sin\left(x - \frac{\pi}{4}\right) + \frac{\pi}{4}\right)\left(\sin\left(\frac{\pi}{4} - \frac{\sqrt{2}}{2}\sin\left(x + \frac{\pi}{4}\right)\right)\right) \, .
\end{aligned}$$

It is easy to prove that

$$0 \ < \ \frac{\pi}{4} - \frac{\sqrt{2}}{2} \ \leq \ \frac{\sqrt{2}}{2}\sin\left(x - \frac{\pi}{4}\right) + \frac{\pi}{4} \, ,$$

$$\frac{\pi}{4} - \frac{\sqrt{2}}{2}\sin\left(x + \frac{\pi}{4}\right) \ \leq \ \frac{\pi}{4} + \frac{\sqrt{2}}{2} < \frac{\pi}{2} \, ,$$

so that

$$\sin\left(\frac{\sqrt{2}}{2}\sin\left(x-\frac{\pi}{4}\right)+\frac{\pi}{4}\right)\,, \qquad \sin\left(\frac{\pi}{4}-\frac{\sqrt{2}}{2}\sin\left(x+\frac{\pi}{4}\right)\right) \;>\; 0\,.$$

3. The polynomial $p(x) = x^3 + ax^2 + bx + c$ has only real roots. Show that the polynomial $q(x) = x^3 - bx^2 + acx - c^2$ has at least one non-negative root.

Solutions by Pierre Bornsztein, Courdimanche, France; by Pavlos Maragoudakis, Pireas, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bornsztein's solution.

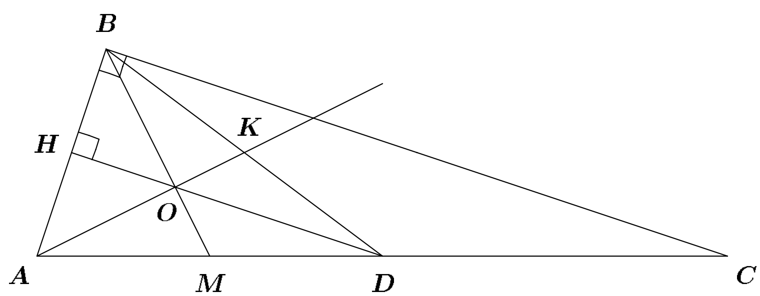
Les hypothèses sont inutiles car :

$$\lim_{x\to+\infty} q(x) \;=\; +\infty \quad \text{et} \quad q(0) \;=\; -c^2 \leq 0\,.$$

Donc, d'après le théorème des valeurs intermediaires (q est continue sur $\mathbb{R}^+$), il existe $\alpha \in \mathbb{R}^+$ tel que $q(\alpha) = 0$.

4. Let the point D on the hypotenuse AC of the right triangle ABC be such that $|AB| = |CD|$. Prove that the bisector of the angle $\angle A$, the median through B and the altitude through D of the triangle ABD have a common point.

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Pierre Bornsztein, Courdimanche, France; by Masoud Kamgarpour, Carson Graham Secondary School, North Vancouver, BC; and by Pavlos Maragoudakis, Pireas, Greece. We give the solution of Kamgarpour.



Let M be the foot of the median from B, K the point of intersection of the angle bisector with BD and H the foot of the altitude from D in $\triangle ABD$.

We use Ceva's Theorem to prove that AM, DH and AK have a common point.

$$\frac{MA}{MD} \;=\; 1 \qquad \text{(because } BM \text{ is a median)}$$

$$\frac{KD}{KB} \;=\; \frac{AD}{AB} \qquad \text{(Bisector Property)}$$

$$\triangle AHD \sim \triangle ABC \quad \Longrightarrow \quad \frac{HB}{HA} = \frac{DC}{DA} = \frac{AB}{AD}.$$

Thus

$$\frac{MA}{MD} \cdot \frac{KD}{KB} \cdot \frac{HB}{HA} = \frac{AD}{AB} \cdot \frac{AB}{AD} = 1.$$

That completes the *Corner* this issue. Send me your Olympiad contests, your nice solutions, and generalizations.

Challenge Answer

In the February 1999 issue [1999: 32], we issued the challenge:

What is the 10^{th} term in the following sequence, and why?

n	x_n
0	0
1	$\frac{1}{16}\left(\sqrt{30}+\sqrt{10}-\sqrt{6}-\sqrt{2}-2\left(\sqrt{3}-1\right)\sqrt{5+\sqrt{5}}\right)$
2	$\frac{1}{8}\left(\sqrt{30-6\sqrt{5}}-\sqrt{5}-1\right)$
3	$\frac{1}{8}\left(\sqrt{10}+\sqrt{2}-2\sqrt{5-\sqrt{5}}\right)$
4	$\frac{1}{8}\left(\sqrt{10+2\sqrt{5}}-\sqrt{15}+\sqrt{3}\right)$
5	$\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$
6	$\frac{1}{4}\left(\sqrt{5}-1\right)$
7	$\frac{1}{16}\left(2\left(\sqrt{3}+1\right)\sqrt{5-\sqrt{5}}-\sqrt{30}+\sqrt{10}-\sqrt{6}+\sqrt{2}\right)$
8	$\frac{1}{8}\left(\sqrt{15}+\sqrt{3}-\sqrt{10-2\sqrt{5}}\right)$
9	$\frac{1}{8}\left(2\sqrt{5+\sqrt{5}}-\sqrt{10}+\sqrt{2}\right)$
10	?

The answer, sent in by Luyun Zhong-Qiao, Columbia International College, Hamilton, Ontario, is $\frac{1}{2}$. He notes that $T_n = \sin(3n^\circ)$.
[Ed.: note there was a typo in T_7.]

BOOK REVIEWS

ALAN LAW

Elementary Mathematical Models by Dan Kalman,
published the The Mathematical Association of America, 1997
ISBN# 0-88385-707-3, soft cover, 340+ pages.
Reviewed by **Richard Charron**, *Memorial University of Newfoundland, St. John's, Newfoundland.*

Subtitled *Order Aplenty and a Glimpse of Chaos*, this book is intended for post-secondary students who are pursuing a course of study requiring no more mathematics than college algebra. The idea is to embed the rudiments of college algebra into a course whose focus is not algebra itself but rather the use of algebra in model problems common to the sciences, economics and business.

The book starts with basic notions of sequences, recursion, and difference equations to then introduce arithmetic growth models, quadratic growth, geometric growth and ultimately logistic growth models. The models are all presented in a particular context, going from problems dealing with pollution data, consumption of non-renewable resources, on to population growth. As the models are presented, a set of natural questions arises. It is in answering these questions that the usual algebraic techniques are intertwined, covering the usual range of questions from determining the equation of a line, computing slope and intercepts, to quadratic functions and their roots, polynomials, rational functions and their graphs, exponential and logarithmic functions. The book does end with a problem giving a glimpse into the interesting dynamics of the logistic equation.

As the focus of this book is not algebra but rather algebra to assist in answering modeling questions, the instructor/student who prefers a course emphasizing definitions, theorems and techniques will not enjoy this book. Students looking for a litany of worked-out examples will not be entirely satisfied either. Those who teach/learn algebra but prefer to do so in a broader scientific or social context will find the text to be quite interesting. The author does not shy away from the fact that life, data and models are never in agreement and makes a conscientious effort to convey to the reader the importance of understanding the limitations of one's models. On the minus side, the text is a bit verbose. The author admits his guilt in this respect in the introductory remarks indicating he preferred to err in this fashion.

Overall it stands as a recommended text for the subject.

Principles of Mathematical Problem Solving by Martin J. Erickson and Joe Flowers, published by Prentice-Hall, Inc., 1999.
ISBN # 0-13-096445-X, hardcover, 252+ pages.
Reviewed by **Christopher Small**, *University of Waterloo, Waterloo, Ontario.*

Teaching mathematical problem solving is a bit like trying to teach someone to find a light switch in a dark room in the middle of the night. Unlike the tidy presentation of theorem, corollary and lemma that makes up the standard course, training people to be good problem solvers is a very inexact science. Nevertheless, like finding the light switch in a dark room, mathematical problem solving has many rules of thumb to get you through the task.

Principles of Mathematical Problem Solving by Martin J. Erickson and Joe Flowers contains many of the standard rules of thumb, and is pitched at a level for the undergraduate university student possibly preparing for the William Lowell Putnam Competition. While the book requires some mathematical expertise beyond the high school level, the advanced high school student will find much that is useful as well. In many respects, it is similar in academic level and coverage of topics to Loren C. Larson's *Problem Solving Through Problems*. However the overall level of presentation is easier and more suitable for the novice who has little experience with mathematics competitions.

The book explains many standard methods with careful attention to the fundamental arguments. Many examples are well developed, and the problems at the end are judiciously chosen, although many are fairly well known.

It is good to note that the problems have been selected in part because of their elegance. The importance of this cannot be overestimated in providing students with an eye for mathematical beauty and an appreciation for the subject.

THE SKOLIAD CORNER

No. 38

R.E. Woodrow

In this issue we give an example of a rather different contest, the Newfoundland and Labrador Teachers' Association Senior Mathematics League, for 1998–1999. The League started in 1987 as a contest in the St. John's area and grew to a province-wide event, with schools competing in local leagues at several sites. The same contest occurs simultaneously at the sites, and the top schools in each district coming together for the provincial finals. The emphasis is on cooperative problem solving. A school team consists of four students who work together on problems. Individual work is rewarded, but to foster collaboration there are bonus marks for correct work as a team. A typical competition consists of ten questions and a relay. Unlike most contests, the problems are presented separately, answers collected, and solutions discussed before going on to the next problem. Individual student solutions may earn 1 mark each, while a correct team solution gains 5 marks. Incorrect answers score 0. The relay has four points, with the answer to each part being an input to the next. One point is awarded if only part 1 is correct, two for parts 1 and 2, three for parts 1, 2, and 3 and five marks for all four parts of the relay. Bonus points are awarded for each minute remaining in the first ten minutes allotted for the relay. (If an incorrect answer is submitted early, the team is told the answer is wrong but not why, and they may go back to rework it.) A tie-breaker may be required. This is based on speed, but to deter silly guesses, an incorrect answer means the team cannot answer for one minute. My thanks go to John Grant McLoughlin, Memorial University of Newfoundland, for forwarding me the contest and background information.

NLTA SENIOR MATH LEAGUE
GAME 1 — 1998–99

1. Find a two-digit number that equals twice the product of its digits.

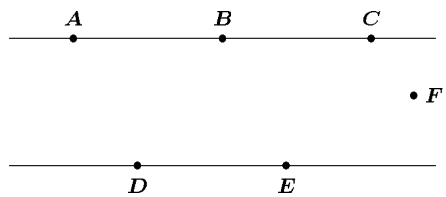
2. The degree measures of the interior angles of a triangle are A, B, C where $A \leq B \leq C$. If A, B, and C are multiples of 15, how many possible values of (A, B, C) exist?

3. Place an operation ($+$, $-$, $\times$, $\div$) in each square so that the expression using 1, 2, 3, ..., 9 equals 100.

$$1 \square 2 \square 3 \square 4 \square 5 \square 6 \square 7 \square 8 \square 9 \;=\; 100$$

You may also freely place brackets before/after any digits in the expression. Note that the squares must be filled in with operational symbols only.

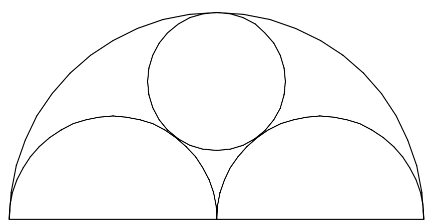
4. A, B and C are points on a line that is parallel to another line containing points D and E, as shown. Point F does not lie on either of these lines.



How many distinct triangles can be formed such that all three of their vertices are chosen from A, B, C, D, E, and F?

5. Michael, Jane and Bert enjoyed a picnic lunch. The three of them were to contribute an equal amount of money toward the cost of the food. Michael spent twice as much money as Jane did buying food for lunch. Bert did not spend any money on food. Instead, Bert brought \$6 which exactly covered his share. How much (in dollars) of Bert's contribution should be given to Michael?

6. Two semicircles of radius 3 are inscribed in a semicircle of radius 6. A circle of radius R is tangent to all three semicircles, as shown. Find R.



7. If $5^A = 3$ and $9^B = 125$, find the value of AB.

8. The legs of a right-angled triangle are 10 and 24 cm respectively.

Let	A	=	the length (cm) of the hypotenuse,
	B	=	the perimeter (cm) of the triangle,
	C	=	the area (cm^2) of the triangle.

Determine the lowest common multiple of A, B, and C.

9. A lattice point is a point (x, y) such that both x and y are integers. For example, $(2, -1)$ is a lattice point, whereas, $(3, \frac{1}{2})$ and $(-\frac{1}{3}, \frac{2}{3})$ are not.

How many lattice points lie inside the circle defined by $x^2 + y^2 = 20$? (Do NOT count lattice points that lie on the circumference of the circle.)

10. The quadratic equation $x^2 + bx + c = 0$ has roots, r_1 and r_2, that have a sum which equals 3 times their product. Suppose that $(r_1 + 5)$ and $(r_2 + 5)$ are the roots of another quadratic equation $x^2 + ex + f = 0$. Given that the ratio of $e : f = 1 : 23$, determine the values of b and c in the original quadratic equation.

RELAY

R1. Operations $*$ and $\diamond$ are defined as follows:

$$A * B \;=\; \frac{A^B + B^A}{A+B} \quad\text{and}\quad A \diamond B \;=\; \frac{A^B - B^A}{A-B}\,.$$

Simplify $N = (3 * 2) * (3 \diamond 2)$. Write the value of N in Box #1 of the relay answer sheet.

R2. A square has a perimeter of P cm and an area of Q cm^2? Given that $3NP = 2Q$, determine the value of P. Write the value of P in Box #2 of the relay answer sheet.

R3. List all two-digit numbers that have digits whose product is P. Call the sum of these two-digit numbers S. Write the value of S in Box #3 of the relay answer sheet.

R4. How many integers between 6 and 24 share no common factors with S that are greater than 1? Write the number in Box #4 of the relay answer sheet.

TIE-BREAKER

Find the maximum value of

$$f(x) \;=\; 14 - \sqrt{x^2 - 6x + 25}\,.$$

Last number we gave the problems of Part I of the Alberta High School Mathematics Competition written in November 1998. Next we give the official solutions. My thanks go to the organizing committee, chaired by Ted Lewis, University of Alberta, Edmonton, for supplying the contest problems and the official solutions.

THE ALBERTA HIGH SCHOOL MATHEMATICS COMPETITION
Part I — November 1998

1. A restaurant usually sells its bottles of wine for 100% more than it pays for them. Recently it managed to buy some bottles of its most popular wine for half of what it usually pays for them, but still charged its customers what it would normally charge. For these bottles of wine, the selling price was what percent more than the purchase price?

Solution. The answer is: **(c)** 300. Let the old buying price be $2x$. Then the new buying price is x and the selling price is $4x$. The latter exceeds the former by $3x$.

2. How many integer solutions n are there to the inequality $34n \geq n^2 + 289$?

Solution. The answer is: (b) 1. The only solution is $n = 17$ since the inequality can be rewritten as $0 \geq (n - 17)^2$.

3. A university evaluates five magazines. Last year, the rankings were MacLuck with a rating of 150, followed by MacLock with 120, MacLick with 100, MacLeck with 90, and MacLack with 80. This year, the ratings of these magazines are down 50%, 40%, 20%, 10% and 5% respectively. How does the ranking change for MacLeck?

Solution. The answer is: (a) up 3 places. The new ratings are 75 for MacLuck, 72 for MacLock, 80 for MacLick, 81 for MacLeck and 76 for MacLack. *This problem illustrates that quality and ranking are two different things*.

4. Parallel lines are drawn on a rectangular piece of paper. The paper is then cut along each of the lines, forming n identical rectangular strips. If the strips have the same length to width ratio as the original, what is this ratio?

Solution. The answer is: (a) $\sqrt{n} : 1$. Suppose the length to width ratio for the original rectangle is $a : b$ with $a > b$. Then the ratio for the strips is $b : \frac{a}{n} = a : b$. Hence $\frac{a}{b} = \sqrt{n}$.

5. "The smallest integer which is at least $a\%$ of 20 is 10." For how many integers a is this statement true?

Solution. The answer is: (e) 5. The largest possible value of a is 50 and the smallest is 46.

6. Let $S = 1 + 2 + 3 + \cdots + 10^n$. How many factors of 2 appear in the prime factorization of S?

Solution. The answer is: (c) $n - 1$. We have $S = \frac{1}{2}10^n(10^n + 1) = 2^{n-1}5^n(10^n + 1)$.

7. When $1 + x + x^2 + x^3 + x^4 + x^5$ is factored as far as possible into polynomials with integral coefficients, what is the number of such factors, not counting trivial factors consisting of the constant polynomial 1?

Solution. The answer is: (c) 3. We have $(1+x+x^2)+(x^3+x^4+x^5) = (1 + x + x^2)(1 + x^3) = (1 + x + x^2)(1 + x)(1 - x + x^2)$. The two quadratic factors are irreducible over polynomials with integral coefficients since the only possible linear factors are $1 + x$ and $1 - x$, but neither divides them.

8. In triangle ABC, $AB = AC$. The perpendicular bisector of AB passes through the midpoint of BC. If the length of AC is $10\sqrt{2}$ cm, what is the area of ABC in cm^2 ?

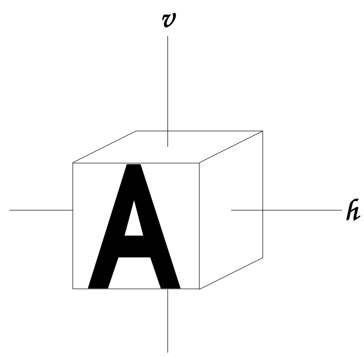
Solution. The answer is: (e) none of these. The line joining the midpoints of AB and BC is parallel to AC. Hence $\angle CAB = 90°$ and the area of triangle ABC is $\frac{1}{2}AB \cdot AC = 100$.

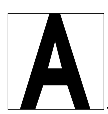
9. If $f(x) = x^x$, what is $f(f(x))$ equal to?

Solution. The answer is: **(d)** $x^{(x^{(x+1)})}$. We have

$$f(f(x)) = (x^x)^{(x^x)} = x^{xx^x} .$$

10. A certain TV station has a logo which is a rotating cube in which one face has an A on it and the other five faces are blank. Originally the A-face is at the front of the cube as shown on the right. Then you perform the following sequence of three moves over and over: rotate the cube $90°$ around the vertical axis v, so that the front face moves to the left; then rotate the cube $90°$ around a horizontal axis h, so that the new front face moves down; then rotate the cube $90°$ around the vertical axis again, so that the new front face moves to the left. Suppose you perform this sequence of three moves a total of 1998 times. What will the front face look like when you have finished?



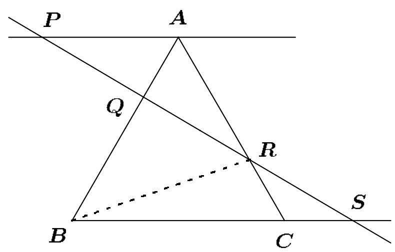
Solution. The answer is: **(a)** 

. When the sequence is performed, the A-face first goes to the left, pointing up, then stays at the left but pointing to the front, and finally goes to the back, pointing to the left. When the sequence is performed again, the A-face first goes to the right, pointing to the back, then stays at right, pointing up, and finally returns to the front, pointing up.

11. How many triples (x, y, z) of real numbers satisfy the simultaneous equations $x + y = 2$ and $xy - z^2 = 1$?

Solution. The answer is: **(a)** 1. We have $1 + z^2 = x(2 - x)$ which is equivalent to $(x - 1)^2 + z^2 = 0$. Hence $x = y = 1$ and $z = 0$.

12. In the diagram, ABC is an equilateral triangle of side length 3 and PA is parallel to BS. If $PQ = QR = RS$, what is the length of BR?



Solution. The answer is: **(d)** $\sqrt{7}$. Triangles PRA and SRC are similar. Since $PR = 2RS$ and $AC = 3$, we have $CR = 1$. Let the foot of perpendicular from R to BC be T. Since $\angle ACB = 60°$, we have $RT = \frac{1}{2}\sqrt{3}$ and $CT = \frac{1}{2}$, so that $BT = \frac{5}{2}$. By Pythagoras' Theorem, $BR^2 = RT^2 + BT^2$.

13. Let a, b, c and d be the roots of $x^4 - 8x^3 - 21x^2 + 148x - 160 = 0$. What is the value of $\frac{1}{abc} + \frac{1}{abd} + \frac{1}{acd} + \frac{1}{bcd}$?

Solution. The answer is: **(b)** $-\frac{1}{20}$. We have $a + b + c + d = 8$ while $abcd = -160$. The desired expression is equal to $\frac{a+b+c+d}{abcd}$.

14. Wei writes down, in order of size, all positive integers b with the property that b and 2^b end in the same digit when they are written in base 10. What is the 1998^{th} number in Wei's list?

Solution. The answer is: **(b)** 19976. Since 2^b is even, so is b. When b is even, 2^b ends alternately in 4 and 6. For $2 \le b \le 20$, the only matches are $b = 14$ and $b = 16$. Since everything repeats in a cycle of 20, Wei's list is 14, 16, 34, 36, $\ldots$. The $(2n-1)$-st number is $10(2n-1)+4$ and the $2n$-th number is $10(2n-1)+6$.

15. Suppose $x = 3^{1998}$. How many integers are between $\sqrt{x^2 + 2x + 4}$ and $\sqrt{4x^2 + 2x + 1}$?

Solution. The answer is: **(b)** $3^{1998} - 1$. Note that

$$x + 1 \ < \ \sqrt{x^2 + 2x + 4} \ < \ x + 2\,,$$

while

$$2x \ < \ \sqrt{4x^2 + 2x + 1} \ < \ 2x + 1\,.$$

Hence the number of integers between the two radicals is $x - 1$.

16. The lengths of all three sides of a right triangle are positive integers. The area of the triangle is 120. What is the length of the hypotenuse?

Solution. The answer is: **(c)** 26. Suppose that the three side lengths are $x \le y \le z$. Then $x^2 + y^2 = z^2$ since we have a right triangle. Hence $\frac{1}{2}xy = 120$ and $xy = 2^4 3 \cdot 5$.

Of the numbers 3^2+80^2, 5^2+48^2, 15^2+16^2, 6^2+40^2, 10^2+24^2 and 12^2+20^2, we only have an integral value for z when $x=10$ and $y=24$, namely, $z^2=2^2(5^2+12^2)=(2\cdot 13)^2$.

In the February 1999 number of the *Corner* we gave the solutions to the problems of the Old Mutual Mathematical Olympiad 1991, Final Paper 1. One of our readers proposes an alternate "variable free" solution to one of them.

2. [1998: 477; 1999: 28] What is the value of

$$\sqrt{17-12\sqrt{2}}+\sqrt{17+12\sqrt{2}}$$

in its simplest form?

Alternate solution by Luyun Zhong Qiho, Mathematics Teacher, Hamilton, Ontario.

$$\begin{aligned}
&\sqrt{17-12\sqrt{2}} \;+\; \sqrt{17+12\sqrt{2}}\\
&\quad= \sqrt{9-2(3)\left(2\sqrt{2}\right)+8} \;+\; \sqrt{9+2(3)\left(2\sqrt{2}\right)+8}\\
&\quad= \sqrt{\left(3-2\sqrt{2}\right)^2} \;+\; \sqrt{\left(3+2\sqrt{2}\right)^2}\\
&\quad= \left(3-2\sqrt{2}\right)+\left(3+2\sqrt{2}\right) \qquad \text{since } 3>2\sqrt{2}\\
&\quad= 3+3\\
&\quad= 6\,.
\end{aligned}$$

That completes the *Skoliad Corner* for this month. Send me suitable contest materials, novel solutions, and suggestions for future directions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to the Mayhem Editor, Naoki Sato, Department of Mathematics, Yale University, PO Box 208283 Yale Station, New Haven, CT 06520–8283 USA. The electronic address is still

mayhem@math.toronto.edu

The Assistant Mayhem Editor is Cyrus Hsia (University of Toronto). The rest of the staff consists of Adrian Chan (Upper Canada College), Donny Cheung (University of Waterloo), Jimmy Chui (Earl Haig Secondary School), David Savitt (Harvard University) and Wai Ling Yee (University of Waterloo).

Shreds and Slices

A Combinatorial Proof

In Issue 2, this volume, we issued the following challenge to our audience.

Problem. Find a combinatorial proof of the following identity:

$$(n-r)\binom{n+r-1}{r}\binom{n}{r} = n\binom{n+r-1}{2r}\binom{2r}{r}.$$

Dave Arthur, of Toronto, Ontario, was the first, and so far only person to give a combinatorial proof. His prize is a copy of "Riddles of the Sphinx", by Martin Gardner; we print his solution here. The proposer still seeks a combinatorial proof that proves the identity in *one* step, and so an additional prize will be offered for such a solution.

Solution by Dave Arthur.

Lemma 1. Let us consider the number of ways of choosing two distinct sets, A and B, each with r elements from a set of $n+r-1$ elements.

There are $\binom{n+r-1}{2r}$ ways to choose the elements that will belong to their union, and there are $\binom{2r}{r}$ ways to choose the elements of these that will belong to A. Therefore, the number of ways is

$$\binom{n+r-1}{2r}\binom{2r}{r}.$$

However, we also note there are $\binom{n+r-1}{r}$ ways to choose A, and $\binom{n-1}{r}$ ways to choose B from the remaining elements. It follows that

$$\binom{n+r-1}{2r}\binom{2r}{r} = \binom{n+r-1}{r}\binom{n-1}{r}.$$

Lemma 2. Let us consider the number of ways of choosing two distinct sets, C and D, such that C has 1 element and D has r elements, from a set of n elements.

There are n ways to choose C first, and $\binom{n-1}{r}$ ways to choose D from the remaining elements, so the number of ways is $n\binom{n-1}{r}$. Also, there are $\binom{n}{r}$ ways to choose D first, and $n-r$ ways to choose C from the remaining elements, so it follows that

$$(n-r)\binom{n}{r} = n\binom{n-1}{r}.$$

Therefore, by lemmas 1 and 2,

$$n\binom{n+r-1}{2r}\binom{2r}{r} = n\binom{n+r-1}{r}\binom{n-1}{r} = (n-r)\binom{n}{r}\binom{n+r-1}{r}.$$

Erratum

On page 169, Issue 3 of this Volume, in problem 6 of the Qualifying Round of the 1990 Swedish Mathematical Olympiad, the dimensions of rectangle $ABCD$ are described as 3000 metres by 500 metres. This is a typo: the dimensions should be 300 metres by 500 metres. Thanks to Jim Totten for the correction.

Mayhem Problems

The Mayhem Problems editors are:

Adrian Chan	*Mayhem High School Problems Editor,*
Donny Cheung	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only solutions — the next issue will feature only problems.

We warmly welcome proposals for problems and solutions. With the new schedule of eight issues per year, we request that solutions from the last issue be submitted in time for issue 4 of 2000.

High School Solutions

Editor: Adrian Chan, 229 Old Yonge Street, Toronto, Ontario, Canada. M2P 1R5 <all238@sprint.com>

H228. Verify that the following three inequalities hold for positive reals x, y, and z:

(i) $x(x-y)(x-z)+y(y-x)(y-z)+z(z-x)(z-y) \geq 0$. (This is known as Schur's Inequality.)

(ii) $x^4+y^4+z^4+xyz(x+y+z) \geq 2(x^2y^2+y^2z^2+z^2x^2)$.

(iii) $9xyz+1 \geq 4(xy+yz+zx)$, where $x+y+z=1$.

(Can you derive an ingenious method that allows you to solve the problem without having to prove all three inequalities directly?)

Additional solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

(i) As indicated, this is the special case $n=1$ of Schur's Inequality:

$$x^n(x-y)(x-z)+y^n(y-z)(y-x)+z^n(z-x)(z-y) \;\geq\; 0$$

for n real. For a simple proof, let $x \geq y \geq z$ without loss of generality. Then for $n \geq 0$,

$$x^n(x-y)(x-z) \;\geq\; y^n(y-z)(x-y) \qquad \text{and} \qquad z^n(x-z)(y-z) \;\geq\; 0\,.$$

For $n \leq 0$,

$$z^n(x-z)(y-z) \;\geq\; y^n(y-z)(x-y) \qquad \text{and} \qquad x^n(x-y)(x-z) \;\geq\; 0\,.$$

There is equality if and only if $x=y=z$. It also follows that if n is an even integer, then x, y and z need not be positive.

(ii) Since as known

$$\begin{aligned} &2(y^2z^2+z^2x^2+x^2y^2)-(x^4+y^4+z^4) \\ &\quad= \;(x+y+z)(y+z-x)(z+x-y)(x+y-z) \end{aligned}$$

(related to Heron's formula for the area of a triangle), the inequality reduces to

$$xyz \;\geq\; (y+z-x)(z+x-y)(x+y-z)\,. \qquad (1)$$

In terms of the elementary symmetric functions $T_1 = x+y+z$, $T_2 = yz+zx+xy$, and $T_3 = xyz$, (1) becomes $T_1^3 + 9T_3 \geq 4T_1T_2$, which is the same as (i).

(iii) In homogeneous form, the inequality is equivalent to

$$9xyz + (x+y+z)^3 \;\geq\; 4(x+y+z)(yz+zx+xy)\,,$$

which is the same as (1).

For a generalization of (1) to

$$\begin{aligned}&(ux+vy+wz)(vx+wy+uz)(wx+uy+vz)\\ &\geq\;\; (y+z-x)(z+x-y)(x+y-z)\,,\end{aligned}$$

where $u+v+w=1$ and $0 \leq u,v,w \leq 1$, see [1].

Reference

1. Klamkin, M.S., *Inequalities for a triangle associated with n given triangles*, Publ. Electrotechn. Fak. Ser. Mat. Fiz. Univ. Beograd, No. 330 (1970), pp. 3–7.

H237. The letters of the word MATHEMATICAL are arranged at random. What is the probability that the resulting arrangement contains no adjacent A's?

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

First, the total number of arrangements is

$$\frac{12!}{3!2!2!}\,.$$

Now, to count the number of arrangements with no adjacent A's, we first arrange the other 9 letters, for a total of

$$\frac{9!}{2!2!}$$

ways. For each such arrangement, we choose any 3 of the 10 "spaces" that are between consecutive letters, including those on the ends. This can be done in a total of $\binom{10}{3}$ ways.

Now, we insert the 3 A's into the 3 spaces that we have chosen. This corresponds to a unique arrangement with no adjacent A's. The total number is then

$$\frac{9!}{2!2!}\cdot\binom{10}{3}\,.$$

The required probability is then equal to

$$\frac{\frac{9!}{2!2!}\cdot\binom{10}{3}}{\frac{12!}{3!2!2!}} \;=\; \frac{6}{11}\,.$$

H238. Johnny is dazed and confused. Starting at $A(0,0)$ in the Cartesian grid, he moves 1 unit to the right, then r units up, r^2 units left, r^3 units down, r^4 units right, r^5 units up, and continues the same pattern indefinitely. If r is a positive number less than 1, he will be approaching a point $B(x,y)$. Show that the length of the line segment AB is greater than $\frac{7}{10}$.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let $P_n = (x_n, y_n)$ denote the point where Johnny is at after n moves, $n = 0, 1, 2, \ldots$. So, $P_0 = A = (0,0)$, $P_1 = (1,0)$, $P_2 = (1,r)$, $P_3 = (1-r^2, r)$, etc. A simple pattern reveals itself, such that for all $m \geq 2$,

$$\begin{aligned} x_{2m-1} = x_{2m} &= 1 - r^2 + r^4 - \cdots + (-1)^{m-1} r^{2m-2}\,, \\ y_{2m} = y_{2m+1} &= r - r^3 + r^5 - \cdots + (-1)^{m-1} r^{2m-1}\,. \end{aligned}$$

Hence,

$$\begin{aligned} x &= \frac{1}{1-(-r^2)} = \frac{1}{1+r^2}\,, \\ y &= \frac{r}{1-(-r^2)} = \frac{r}{1+r^2}\,. \end{aligned}$$

So,

$$AB^2 = x^2 + y^2 = \frac{1}{1+r^2}\,.$$

And since $r < 1$,

$$AB = \frac{1}{\sqrt{1+r^2}} > \frac{1}{\sqrt{2}} = \frac{5\sqrt{2}}{10} > \frac{7}{10}\,.$$

H239. Find all pairs of integers (x,y) which satisfy the equation $y^2(x^2+1) + x^2(y^2+16) = 448$.

Solution. We have

$$\begin{aligned} & y^2(x^2+1) + x^2(y^2+16) &=& \; 448 \\ \Longrightarrow\; & 2x^2y^2 + 16x^2 + y^2 - 448 &=& \; 0 \\ \Longrightarrow\; & 2x^2(y^2+8) + y^2 + 8 &=& \; (2x^2+1)(y^2+8) \\ & &=& \; 456 \;=\; 2^3 \times 3 \times 19\,. \end{aligned}$$

Since x and y are integers, both $2x^2+1$ and y^2+8 are positive integers. Since $2x^2+1$ is odd, it must equal one of the odd factors of 456, namely 1, 3, 19, and 57. Checking each of these cases, we find that $x = 0$, ± 1, and ± 3 are the only solutions.

If $x = 0$, then $y^2 + 8 = 456$, for which there is no solution.

If $x = \pm 1$, then $y^2 + 8 = 152$, from which we obtain $y = \pm 12$.

If $x = \pm 3$, then $y^2 + 8 = 24$, from which we obtain $y = \pm 4$.

Therefore, there are eight solutions, namely $(\pm 1, \pm 12)$ and $(\pm 3, \pm 4)$.

Also solved by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

H240. *Proposed by Alexandre Trichtchenko, Brookfield High School, Ottawa, ON.*

A Pythagorean triple (a, b, c) is a triple of integers satisfying the equation $a^2 + b^2 = c^2$. We say that such a triple is *primitive* if $\gcd(a, b, c) = 1$. Let p be an odd integer with exactly n prime divisors. Show that there exist exactly 2^{n-1} primitive Pythagorean triples where p is the first element of the triple. For example, if $p = 15$, then $(15, 8, 17)$ and $(15, 112, 113)$ are the primitive Pythagorean triples with first element 15.

Solution. We may write p uniquely in the form

$$p = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n} ,$$

where $p_1, p_2, \ldots , p_n$ are distinct odd primes, and $e_1, e_2, \ldots , e_n$ are positive integers. We seek all primitive Pythagorean triples (p, b, c). For such a triple, $p^2 + b^2 = c^2$, or $c^2 - b^2 = (c+b)(c-b) = p^2$. Suppose p_i divides both $c+b$ and $c - b$ for some i. Then p_i divides $(c + b) - (c - b) = 2b$, and since p_i is odd, it follows that p_i divides b. Similarly, p_i divides $(c + b) + (c - b) = 2c$, so p_i divides c. Then (p, b, c) fails to be a primitive Pythagorean triple, since p_i divides all three numbers, so for each i, p_i divides at most one of $c + b$ and $c - b$.

This implies that in the factorization $p^2 = (c+b)(c-b)$, for each i, all the factors of p_i must reside in $c + b$ or $c - b$. In other words, for each i, we can make one of two choices of where to place all the factors of p_i, for a total of 2^n factorizations. However, half of them must be discarded, since $c + b$ must be the greater number, and $c - b$ the lesser. (We cannot have $c + b$ and $c - b$ equal, since they have different prime factors.) Each of the other half, however, does lead to a unique solution: If $p^2 = xy$, where $x > y$ and x and y are relatively prime, then $b = (x - y)/2$ and $c = (x + y)/2$. Hence, there are 2^{n-1} such Pythagorean triples.

Also solved by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Advanced Solutions

Editor: Donny Cheung, c/o Conrad Grebel College, University of Waterloo, Waterloo, Ontario, Canada. N2L 3G6 <dccheung@uwaterloo.ca>

A212. Let A and B be real $n \times n$ matrices such that $A^2 + B^2 = AB$. Prove that if $AB - BA$ is an invertible matrix, then n is divisible by 3.

(International Competition in Mathematics for University Students)

Solution. Let $S = A + \omega B$, where ω is a primitive cube root of unity. Then we have that

$$\begin{aligned} S\overline{S} &= (A+\omega B)(\overline{A+\omega B}) \\ &= (A+\omega B)(A+\overline{\omega}B) \;=\; A^2+\omega BA+\overline{\omega}AB+B^2 \\ &= AB+\omega BA+\overline{\omega}AB \;=\; \omega(BA-AB)\,, \end{aligned}$$

since $\overline{\omega}+1 = -\omega$. Also, $\det(S\overline{S}) = \det S \cdot \det \overline{S}$ is a real number and $\det \omega(BA-AB) = \omega^n \det(BA-AB) \neq 0$, so ω^n must be a real number. Hence, n is divisible by 3.

A213. Show that the number of positive integer solutions to the equation $a+b+c+d = 98$, where $0 < a < b < c < d$, is equal to the number of positive integer solutions to the equation $p+2q+3r+4s = 98$, where $0 < p, q, r, s$.

I. *Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Suppose that a, b, c, and d are positive integers such that $a < b < c < d$ and satisfy $a+b+c+d = 98$. Then letting $p = d-c$, $q = c-b$, $r = b-a$, and $s = a$, we see that p, q, r, and s are positive integers satisfying

$$p+2q+3r+4s = (d-c)+2(c-b)+3(b-a)+4a = a+b+c+d \;=\; 98\,.$$

Conversely, suppose that p, q, r, and s are positive integers satisfying $p+2q+3r+4s = 98$. Then letting $a = s$, $b = r+s$, $c = q+r+s$, and $d = p+q+r+s$, we see that a, b, c and d are positive integers satisfying $a < b < c < d$, and

$$a+b+c+d \;=\; p+2q+3r+4s \;=\; 98\,.$$

This establishes a bijection between the solutions (a,b,c,d) and (p,q,r,s).

II. *Solution*. First count the number of ways of distributing 98 identical marbles into 4 distinct boxes, with the first box containing the least number of marbles, the second box containing the second least number, and so on. This value is equal to the number of solutions to the first equation with the appropriate conditions. Another way of counting this is to distribute the same number h to each of the 4 boxes. Then distribute g to each of the second, third and fourth boxes. This ensures that the first box has the least. Distribute f to each of boxes three and four. This ensures that the second box is the second least and so on. The number of ways to do this is equal to the number of solutions to the second equation.

A214. Show that any rational number can be written as the sum of a finite number of distinct unit fractions. A unit fraction is of the form $1/n$, where n is an integer.

Solution.
We solve the problem by step-climbing through several cases.

Case I: Positive rationals in $[0,1]$.

Let r be a positive rational number in $[0,1]$. Then we show that r can be written as a sum of distinct unit fractions by providing an algorithm that explicitly finds the unit fractions, namely the greedy algorithm.

Let $r = a/b$, in reduced terms. Let $n = \lceil b/a \rceil$. In other words, n is the integer b/a rounded up, so $b/a \le n < b/a + 1$. Then $1/n$ is the largest unit fraction which is less than or equal to a/b, and when we subtract $1/n$ from a/b, we obtain $r' = \dfrac{a}{b} - \dfrac{1}{n} = \dfrac{an-b}{bn}$.

Since $b/a \le n < b/a + 1$, $b \le an < b + a$, or $0 \le an - b < a$. So, the numerator in r' is less than the numerator in r. If r' is reducible, then we reduce, and the numerator decreases still. We now subtract the largest unit fraction from r', and so on. Since the numerator decreases by at least 1 each step, the algorithm must stop at some point, in fact after at most $a-1$ steps. Then r is the sum of the unit fractions produced by the algorithm.

For example, for $r = 6/7$, we have that

$$\frac{6}{7} - \frac{1}{2} = \frac{5}{14}, \qquad \frac{5}{14} - \frac{1}{3} = \frac{1}{42},$$

so that

$$\frac{6}{7} = \frac{1}{2} + \frac{1}{3} + \frac{1}{42}.$$

Case II: Positive rationals greater than 1.
Let r be a positive rational greater than 1. Since the harmonic series $\sum_{i=1}^{\infty} 1/i$ diverges, there exists a positive integer k such that

$$\sum_{i=1}^{k} \frac{1}{i} \le r < \sum_{i=1}^{k+1} \frac{1}{i}.$$

If $r = \sum_{i=1}^{k} 1/i$, then we have an expression of r as a sum of distinct unit fractions, so assume that $r > \sum_{i=1}^{k} 1/i$. Consider the rational

$$s = r - \sum_{i=1}^{k} \frac{1}{i}, \qquad \text{so that} \qquad 0 < s < \frac{1}{k+1}.$$

Then s is a positive rational in $[0,1]$, so by Case I, s is the sum of distinct unit fractions

$$s = \sum_{j=1}^{m} \frac{1}{n_j}.$$

Since $s < 1/(k+1)$, each n_j is at least $k+1$ (otherwise $n_j \leq k \implies 1/n_j \geq 1/k \implies s \geq 1/k$, contradiction), so that

$$r = \sum_{i=1}^{k} \frac{1}{i} + \sum_{j=1}^{m} \frac{1}{n_j}$$

is an expression of r as a sum of distinct unit fractions.

Case III: Negative rationals and 0.

If $r < 0$, then $-r$, by Cases I and II, can be written as the sum of distinct unit fractions. Negate each term to get such a sum for r. Finally, 0 can be written as $0 = \frac{1}{1} - \frac{1}{2} - \frac{1}{3} - \frac{1}{6}$.

A215. For a fixed integer $n \geq 2$, determine the maximum value of $k_1 + \cdots + k_n$, where $k_1, \ldots, k_n$ are positive integers with $k_1^3 + \cdots + k_n^3 \leq 7n$.

(Polish Mathematical Olympiad)

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let $T = \sum_{i=1}^{n} k_i^3$ and $S = \sum_{i=1}^{n} k_i$. We claim that

$$S \leq n + \left\lfloor \frac{6n}{7} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes, as usual, the greatest integer less than or equal to x. This maximum is attained if and only if $\lfloor 6n/7 \rfloor$ of the k_i equal 2 and the rest equal 1.

First note that if $k_i \geq 2$ for all i, then $T \geq 8n$, a contradiction. Hence $k_i = 1$ for at least one i, say $k_1 = 1$. If $k_j \geq 3$ for some $j > 1$, then we consider T' obtained from T as follows: Replace k_1 and k_j by $k_1' = 2$ and $k_j' = k_j - 1$ respectively, and leave all the other k_i unchanged. Then clearly the value of S is unchanged. On the other hand, $T' \leq T$, which means that $(k_j - 1)^3 + 8 \leq k_j^3 + 1$. Rewriting this, we obtain $(k_j + 1)(k_j - 2) \geq 0$. This is true as long as $k_j \geq 3$. Hence $T' \leq 7n$, and to obtain the maximum value of S, we may assume, without loss of generality, that $k_i = 1$ or 2 for all i.

Suppose among all the k_i, that there are m 2's and $n - m$ 1's. Then $T = 8m + (n - m) \leq 7n$, which implies that $m \leq \lfloor 6n/7 \rfloor$. Clearly, the maximum value of S is attained when $m = \lfloor 6n/7 \rfloor$, in which case we obtain $S = 2\lfloor 6n/7 \rfloor + n - \lfloor 6n/7 \rfloor = n + \lfloor 6n/7 \rfloor$.

A216. Given a continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying the conditions:

$$f(1000) = 999,$$

$$f(x) \cdot f(f(x)) = 1 \text{ for all } x \in \mathbb{R}.$$

Determine $f(500)$.

(Polish Mathematical Olympiad)

Solution. By the second condition, $f(1000)f(f(1000)) = 1$, so we have that $999f(999) = 1$ or $f(999) = 1/999$.

Since f is a continuous function, by the Intermediate Value Theorem, there exists an $a \in [999, 1000]$ such that $f(a) = 500$.

Then $f(a)f(f(a)) = 1$, giving $500f(500) = 1$, so $f(500) = 1/500$.

In fact, $f(x) = 1/x$ for all $x \in [1/999, 999]$. To complete the function, for any x outside this range, set $f(x)$ to any value, within the interval $[1/999, 999]$. Then for any x, $1/999 \leq f(x) \leq 999$, and so $f(f(x)) = 1/f(x)$.

Note: Because $f(1000) = 999 \neq 1/1000$, $f(x)$ can never equal 1000.

Challenge Board Solutions

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

C77. Let F_i denote the i^{th} Fibonacci number, with $F_0 = 1$ and $F_1 = 1$. (Then $F_2 = 2$, $F_3 = 3$, $F_4 = 5$, etc.)

(a) Prove that each positive integer is uniquely expressible in the form $F_{a_1} + \cdots + F_{a_k}$, where the subscripts form a strictly increasing sequence of positive integers no pair of which are consecutive.

(b) Let $\tau = \frac{1}{2}(1 + \sqrt{5})$, and for any positive integer n, let $f(n)$ equal the integer nearest to $n\tau$. Prove that if $n = F_{a_1} + \cdots + F_{a_k}$ is the expression for n from part (a) and if $a_2 \neq 3$, then $f(n) = F_{a_1+1} + \cdots + F_{a_k+1}$.

(c) Keeping the notation from part (b), if $a_2 = 3$ (so that $a_1 = 1$), it is not always true that the formula $f(n) = F_{a_1+1} + \cdots + F_{a_k+1}$ holds. For example, if $n = 4 = F_3 + F_1 = 1 + 3$, then the closest integer to $n\tau = 6.47\ldots$ is 6, not $F_2 + F_4 = 2 + 5 = 7$. Fortunately, in the cases where the formula fails, we can correct the problem by setting $a_1 = 0$ instead of $a_1 = 1$: For example, $4 = F_0 + F_3 = 1 + 3$ as well, and indeed $6 = F_1 + F_4 = 1 + 5$. Determine for which sequences of a_i this correction is necessary.

Solution.

(a) Suppose we know that every positive integer less than F_n is uniquely expressible in the desired form. (The base case $n = 1$ is vacuous.) If N satisfies $F_n \leq N < F_{n+1}$, then by our inductive assumption, $N - F_n$ is expressible in the desired form, say $N - F_n = F_{a_1} + \cdots + F_{a_k}$. Since

$N - F_n < F_{n+1} - F_n = F_{n-1}$, we must have that $a_k \le n-2$, and so the expression

$$N = F_{a_1} + \cdots + F_{a_k} + F_n$$

is of the desired form. It remains to show that this is the *unique* expression for N as such a sum; observe, by induction, that it suffices to show that any such sum for N must include F_n.

Given an expression

$$N = F_{a_1} + \ldots + F_{a_k}$$

of the necessary form, if we assume that $a_k < n$, then we know that a_k is at most $n-1$, so a_{k-1} is at most $n-3$, and so on. Thus, N is at most

$$F_1 + F_3 + F_5 + \cdots + F_{n-1}$$

if n is even and

$$F_2 + F_4 + \cdots + F_{n-1}$$

if n is odd. However, it is a straightforward induction argument to prove that in fact for any positive integer m,

$$F_1 + F_3 + F_5 + \cdots + F_{2m-1} = F_{2m} - 1$$

and

$$F_2 + F_4 + \cdots + F_{2m} = F_{2m+1} - 1\,,$$

so we obtain a contradiction.

(b) Since $f(n)$ is the unique integer N such that $|N - n\tau| < 1/2$, we need only show that

$$|F_{a_1+1} + \cdots + F_{a_k+1} - \tau(F_{a_1} + \cdots + F_{a_k})| < \frac{1}{2}\,.$$

Recall that $F_i = \frac{\tau^{i+1} - \sigma^{i+1}}{\sqrt{5}}$, where $\sigma = \frac{1}{2}(1 - \sqrt{5})$. Hence, we want to prove that

$$\left|\sum_{i=1}^{k} \left((\tau^{a_i+2} - \sigma^{a_i+2}) - \tau(\tau^{a_i+1} - \sigma^{a_i+1})\right)\right| < \frac{\sqrt{5}}{2}\,.$$

Using the fact that $\sigma\tau = -1$, this becomes the same as showing that

$$\left|\sum_{i=0}^{k} (\sigma^{a_i+2} + \sigma^{a_i})\right| < \frac{\sqrt{5}}{2}\,,$$

and since $\sigma^2 + 1 = -\sigma\sqrt{5}$, we are further reduced to showing that

$$\left|\sum_{i=0}^{k} \sigma^{a_i+1}\right| < \frac{1}{2}\,.$$

Let $\mathcal{P}$ be the sum of the positive terms in the sum $\sum \sigma^{a_i+1}$, and let $\mathcal{N}$ be the sum of the negative terms.

Then $\mathcal{P}$ is at most $\displaystyle\sum_{i=1}^{\infty} \sigma^{2i} = \frac{\sigma^2}{1-\sigma^2} < 0.62$,

and $\mathcal{N}$ is at least $\displaystyle\sum_{i=1}^{\infty} \sigma^{2i+1} = \frac{\sigma^3}{1-\sigma^2} > -0.39$. In particular, the only way that $|\mathcal{P}+\mathcal{N}| \geq 1/2$ is if $\mathcal{P} \geq \dfrac{1}{2} + |\mathcal{N}|$.

If $a_2 \neq 3$, then either σ^2 or σ^4 is not in the sum for $\mathcal{P}$. Since $\sigma^2 > 0.38$ and $\sigma^4 > 0.14$, it follows that $\mathcal{P} < 1/2$, so $\mathcal{P} < 1/2 + |\mathcal{N}|$ and the desired formula holds.

(c) Keeping our notation from (b), if $\mathcal{P}+\mathcal{N} > 1/2$, then $a_1 = 1$ and $a_2 = 3$. Setting $a_1' = 0$ and $a_i' = a_i$ for $i > 1$, we find that

$$\left|\sum_{i=0}^{k} \sigma^{a_i'+1}\right| = \left|(\mathcal{P}-\sigma^2)+(\mathcal{N}+\sigma)\right| = |\mathcal{P}+\mathcal{N}-1| < \frac{1}{2},$$

so the suggested correction does indeed work when necessary.

We would like to decide precisely when this correction is necessary; that is, when

$$\mathcal{S} = \sum_{i=0}^{k} \sigma^{a_i'+1} > -\frac{1}{2}.$$

Let us retain the assumption that $a_1 = 1$ (so that the definition of the a_i' makes sense) but drop the assumption that $a_2 = 3$. This will turn out to be more convenient, in the end.

Observing, by direct calculation, that $\displaystyle\sum_{i=0}^{\infty} \sigma^{3i+1} = -\frac{1}{2}$, let j be the smallest (non-negative) integer i such that $a_{i+2}' \neq 3(i+1)$.

(If $a_{i+2}' = 3(i+1)$ for all $i \leq k$, then put $j = k-1$.) Then

$$\mathcal{S}+\frac{1}{2} = \frac{1}{2}+\sigma+\cdots+\sigma^{3j+1}+\sum_{i=j+2}^{k} \sigma^{a_i'+1} = \frac{1}{2}\sigma^{3j+3}+\sum_{i=j+2}^{k} \sigma^{a_i'+1}.$$

We will attempt to show (in what will amount to a clumsy verification) that $\mathcal{S}+1/2$ has the same sign as $\frac{1}{2}\sigma^{3j+3}$, and so the correction is necessary precisely when j is odd.

Remembering the definition of j and the fact that the a_i are non-consecutive, using $a_{j+1}' = 3j$, we break into three cases: $a_{j+2}' = 3j+2$, $a_{j+2}' = 3j+4$, or $a_{j+2}' \geq 3j+5$.

To begin with, in the first case,

$$\left|\sum_{i=j+3}^{k} \sigma^{a'_i+1}\right| < \left|\sigma^{3j+5}\right|\sum_{i=0}^{\infty} \sigma^{2i} = \left|\sigma^{3j+4}\right| < \left|\frac{3}{2}\,\sigma^{3j+3}\right| = \left|\frac{1}{2}\,\sigma^{3j+3} + \sigma^{a'_{j+2}+1}\right| ,$$

as desired. Similarly, in the second case,

$$\left|\sum_{i=j+3}^{k} \sigma^{a'_i+1}\right| < \left|\sigma^{3j+7}\right|\sum_{i=0}^{\infty} \sigma^{2i} = \left|\sigma^{3j+6}\right| < \left|\frac{1}{2}\,\sigma^{3j+3} + \sigma^{3j+5}\right| = \left|\frac{1}{2}\,\sigma^{3j+3} + \sigma^{a'_{j+2}+1}\right| .$$

Finally, in the third case,

$$\left|\sum_{i=j+2}^{k} \sigma^{a'_i+1}\right| < \left|\sigma^{3j+6}\right|\sum_{i=0}^{\infty} \sigma^{2i} = \left|\sigma^{3j+5}\right| < \left|\frac{1}{2}\sigma^{3j+3}\right| ,$$

and we are done.

Thus, we have shown: *If* $a_1 = 1$*, then if* j *is the smallest* non-negative *integer* i *such that* $a_{i+2} \neq 3(i+1)$*, then the correction is necessary if and only if* j *is odd.*

Remark: In the solution of part (b), we never fully used the fact that we were using the representation from part (a). In particular, the proof of (b) actually showed that if $n = F_{a_1} + \cdots + F_{a_k}$ and the a_i are distinct positive integers, then as long as 1 and 3 are not both among the a_i, we can conclude that $f(n) = F_{a_1+1} + \cdots + F_{a_k+1}$. Unfortunately, as should not be a surprise, not every integer can be expressed in this way, $n = 4$ being the smallest example.

Problem of the Month

Jimmy Chui, student, Earl Haig S.S.

Problem. A rectangular wine rack, $PQRS$, holds five rows of identical bottles (Figure 1). The bottom row contains enough room for three bottles (A, B, and C) but not enough room for a fourth bottle. The second row, consisting of just two bottles (D and E), holds B in place somewhere between A and C, and pushes A and C to the sides of the rack. The third row, consisting of three bottles (F, G, and H), lies on top of those two bottles, and F and H rest against the sides of the rack. The fourth layer holds two bottles (I and J) and the fifth layer contains three bottles (K, L and M). Prove that the fifth row is perfectly horizontal regardless of how A, B and C are positioned.

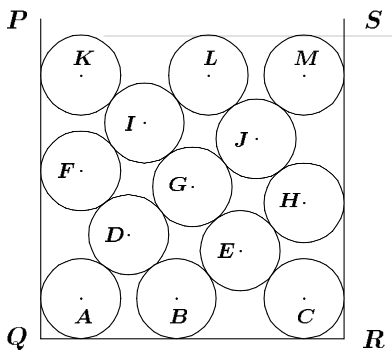


Figure 1.

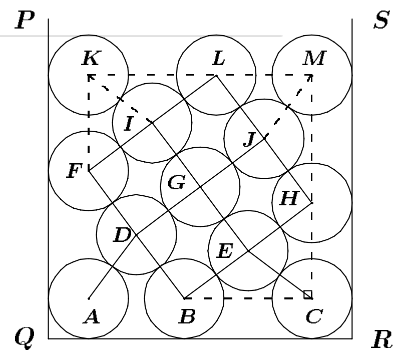


Figure 2.

Solution. All the bottles are identical, and so F and K are the same distance away from the wall (Figure 2). Since FK is vertical, we only need to show that $\angle FKL = 90°$.

The distance between the centres of touching bottles is constant; it is the diameter of a bottle. Therefore, IF, IK, and IL are all equal, and hence I is the circumcentre of triangle FKL. In order for $\angle FKL$ to be a right angle, then from properties of right angle triangles, the circumcentre I is also the mid-point of FL. Hence, we will show that I is the mid-point of FL.

Note that the four quadrilaterals $GDFI$, $GILJ$, $GJHE$, $GEBD$ are all rhombi (they have side length of a bottle diameter). So $\overrightarrow{FI} = \overrightarrow{BE}$ and $\overrightarrow{IL} = \overrightarrow{EH}$. Furthermore, since EB, EC, and EH are all equal, E is the circumcentre of triangle BCH. However, we know that BCH is a right triangle. Hence, E is the mid-point of BH. Thus, I is the mid-point of FL, and it follows that $\angle FKL = 90°$.

Similarly, $\angle HML$ is a right angle. Therefore, the top row is perfectly horizontal, QED.

Four Ways to Count

Jimmy Chui
student, Earl Haig Secondary School

Problem. Evaluate

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n},$$

where n is a positive integer.

Solution 1. Let the given sum be equal to S. Now,

$$\begin{aligned} S &= 0\binom{n}{0} + 1\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} \\ &= 0\binom{n}{n} + 1\binom{n}{n-1} + 2\binom{n}{n-2} + 3\binom{n}{n-3} + \cdots + n\binom{n}{0} \\ &= n\binom{n}{0} + (n-1)\binom{n}{1} + (n-2)\binom{n}{2} + (n-3)\binom{n}{3} + \cdots + 0\binom{n}{n}. \end{aligned}$$

Adding the first and third equations and dividing by 2, we obtain

$$\begin{aligned} S &= \frac{n}{2}\left(\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n}\right) \\ &= \frac{n}{2} \cdot 2^n \\ &= n2^{n-1}. \end{aligned}$$

Solution 2. We claim that

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}.$$

We will prove this by mathematical induction. The base case $n = 1$ is trivial. Now we assume, for some $n = k$, that

$$\binom{k}{1} + 2\binom{k}{2} + 3\binom{k}{3} + \cdots + k\binom{k}{k} = k2^{k-1}.$$

Then,

$$\begin{aligned}
&\binom{k+1}{1} + 2\binom{k+1}{2} + 3\binom{k+1}{3} + \cdots + k\binom{k+1}{k} + (k+1)\binom{k+1}{k+1} \\
&\quad = 1\left(\binom{k}{0} + \binom{k}{1}\right) + 2\left(\binom{k}{1} + \binom{k}{2}\right) + 3\left(\binom{k}{2} + \binom{k}{3}\right) \\
&\qquad + \cdots + k\left(\binom{k}{k-1} + \binom{k}{k}\right) + (k+1)\binom{k}{k} \\
&\quad = 1\binom{k}{0} + 3\binom{k}{1} + 5\binom{k}{2} + 7\binom{k}{3} + \cdots + (2k+1)\binom{k}{k} \\
&\quad = \binom{k}{0} + \binom{k}{1} + \binom{k}{2} + \binom{k}{3} + \cdots + \binom{k}{k} \\
&\qquad + 2\left(0\binom{k}{0} + 1\binom{k}{1} + 2\binom{k}{2} + 3\binom{k}{3} + \cdots + k\binom{k}{k}\right) \\
&\quad = 2^k + 2(k2^{k-1}) \\
&\quad = (k+1)2^k\,.
\end{aligned}$$

Hence, the claim is true for $n = k+1$, and by the principle of mathematical induction, for all positive integers n.

Solution 3. Let

$$f(x) = \sum_{i=0}^{n} \binom{n}{i} x^i = 1 + \sum_{i=1}^{n} \binom{n}{i} x^i\,.$$

Then

$$f'(x) = \sum_{i=1}^{n} i\binom{n}{i} x^{i-1}\,,$$

so that

$$f'(1) = \sum_{i=1}^{n} i\binom{n}{i}\,.$$

But

$$f(x) = (1+x)^n$$

by the Binomial Theorem. Then

$$f'(x) = n(1+x)^{n-1}\,,$$

so that

$$f'(1) = n2^{n-1}\,.$$

Hence,

$$\sum_{i=1}^{n} i\binom{n}{i} = n2^{n-1}\,.$$

Solution 4. Consider a set of n people. We wish to count the number of different teams that can be formed, with the condition that there is one and only one leader.

One way is to find the total number of team members first, and then select a leader from those chosen. Suppose you choose i members. (Note that $1 \le i \le n$.) There are $\binom{n}{i}$ such subsets. In each of these subsets, there are i possible leaders. Hence, the total number of teams that can be formed with i members is $i\binom{n}{i}$. Therefore, the total number of teams with any number of members is merely the sum

$$\sum_{i=1}^{n} i\binom{n}{i} .$$

Another way is to choose the leader first, and the rest of the members afterwards. The leader can be chosen in n ways. The members can be chosen out of the other $n-1$ people in any way, and there are 2^{n-1} ways of doing so. Hence the total number of teams is $n2^{n-1}$.

Thus,

$$\sum_{i=1}^{n} i\binom{n}{i} = n2^{n-1} .$$

Comment. This question appears in the strangest of places, and it is pleasant to see four very different, yet equally elegant, solutions.

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) *after a number indicates that a problem was submitted without a solution.*

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ *or* A4 *sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than* **1 December 1999**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in* LaTeX*). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

G.P. Henderson, Garden Hill, Campbellcroft, Ontario asks us to point out that problem **2405** is not the problem that he submitted. It is a modification made by the editors. It should be regarded as an unsolved problem proposed by the editors.

2439. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that $ABCD$ is a square with side a. Let P and Q be points on sides BC and CD respectively, such that $\angle PAQ = 45^\circ$. Let E and F be the intersections of PQ with AB and AD respectively. Prove that $AE + AF \geq 2\sqrt{2}a$.

2440. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given: triangle ABC with $\angle BAC = 90^\circ$. The incircle of triangle ABC touches BC at D. Let E and F be the feet of the perpendiculars from D to AB and AC respectively. Let H be the foot of the perpendicular from A to BC.

Prove that the area of the rectangle $AEDF$ is equal to $\dfrac{AH^2}{2}$.

2441. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, Florida, USA.*

Suppose that D, E, F are the mid-points of the sides BC, CA, AB of $\triangle ABC$. The incircle of $\triangle AEF$ touches EF at X, the incircle of $\triangle BFD$ touches FD at Y, and the incircle of $\triangle CDE$ touches DE at Z.

Show that DX, EY, FZ are collinear. What is the intersection point?

2442. *Proposed by Michael Lambrou, University of Crete, Crete, Greece.*

Let $\{a_n\}_1^\infty$, $\{x_n\}_1^\infty$, $\{y_n\}_1^\infty$, $\dots$, $\{z_n\}_1^\infty$, be a finite number of given sequences of non-negative numbers, where all $a_n > 0$. Suppose that $\sum a_n$ is divergent and all the other infinite series, $\sum x_n$, $\sum y_n$, $\dots$, $\sum z_n$, are convergent. Let $A_n = \sum_{k=1}^n a_k$.

(a) Show that, for every $\epsilon > 0$, there is an $n \in \mathbb{N}$ such that, simultaneously,

$$0 \le \frac{A_n x_n}{a_n} < \epsilon\,, \quad 0 \le \frac{A_n y_n}{a_n} < \epsilon\,, \quad \dots\,, \qquad 0 \le \frac{A_n z_n}{a_n} < \epsilon\,.$$

(b) From part (a), it is clear that if $\lim\limits_{n\to\infty} \dfrac{A_n x_n}{a_n}$ exists, it must have value zero. Construct an example of sequences as above such that the stated limit does not exist.

2443. *Proposed by Michael Lambrou, University of Crete, Crete, Greece.*

Without the use of any calculating device, find an explicit example of an integer, M, such that $\sin(M) > \sin(33)(\approx 0.99991)$. (Of course, M and 33 are in radians.)

2444. *Proposed by Michael Lambrou, University of Crete, Crete, Greece.*

Determine $\displaystyle\lim_{n\to\infty} \left(\frac{\ln(n!)}{n} - \frac{1}{n} \sum_{k=1}^{n} \ln(k) \left(\sum_{j=k}^{n} \frac{1}{j} \right) \right)$.

2445. *Proposed by Michael Lambrou, University of Crete, Crete, Greece.*

Let A, B be a partition of the set $C = \{q \in \mathbb{Q} : 0 < q < 1\}$ (so that A, B are disjoint sets whose union is C).

Show that there exist sequences $\{a_n\}$, $\{b_n\}$ of elements of A and B respectively such that $(a_n - b_n) \to 0$ as $n \to \infty$.

2446. *Proposed by Catherine Shevlin, Wallsend upon Tyne, England.*

A sequence of integers, $\{a_n\}$ with $a_1 > 0$, is defined by

$$a_{n+1} = \begin{cases} \frac{a_n}{2} & \text{if } n \equiv 0 \pmod 4, \\ 3a_n + 1 & \text{if } n \equiv 1 \pmod 4, \\ 2a_n - 1 & \text{if } n \equiv 2 \pmod 4, \\ \frac{a_n+1}{4} & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Prove that there is an integer m such that $a_m = 1$.

(Compare **OQ**.117 in *OCTOGON*, vol 5, No. 2, p. 108.)

2447. *Proposed by Gerry Leversha, St. Paul's School, London, England.*

Two circles intersect at P and Q. A variable line through P meets the circles again at A and B. Find the locus of the orthocentre of triangle ABQ.

2448. *Proposed by Gerry Leversha, St. Paul's School, London, England.*

Suppose that S is a circle, centre O, and P is a point outside S. The tangents from P to S meet the circle at A and B. Through any point Q on S, the line perpendicular to PQ intersects OA at T and OB at U. Prove that $OT \times OU = OP^2$.

2449. *Proposed by Gerry Leversha, St. Paul's School, London, England.*

Two circles intersect at D and E. They are tangent to the sides AB and AC of $\triangle ABC$ at B and C respectively.. If D is the mid-point of BC, prove that $DA \times DE = DC^2$.

2450. *Proposed by Gerry Leversha, St. Paul's School, London, England.*

Find the exact value of $$\frac{\displaystyle\sum_{k=0}^{\infty} \frac{(2k)!}{(k!)^3}}{\displaystyle\sum_{k=0}^{\infty} \frac{1}{(k!)^2}}.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2329*. [1998: 176, 301] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Suppose that p and $t > 0$ are real numbers. Define

$$\lambda_p(t) := t^p + t^{-p} + 2^p \quad \text{and} \quad \kappa_p(t) := \left(t + t^{-1}\right)^p + 2\,.$$

(a) Show that $\lambda_p(t) \le \kappa_p(t)$ for $p \ge 2$.

(b) Determine the sets of p: A, B and C, such that

1. $\lambda_p(t) \le \kappa_p(t)$,
2. $\lambda_p(t) = \kappa_p(t)$,
3. $\lambda_p(t) \ge \kappa_p(t)$.

Solution by Michael Lambrou, University of Crete, Crete, Greece.

(a) This part of the problem is included in (b), so we pass on to the latter.

(b) We show that

$$A := \{p|\lambda_p(t) \le \kappa_p(t) \quad \text{for all } t > 0\} = [0,1] \cup [2,\infty)\,,$$

$$B := \{p|\lambda_p(t) = \kappa_p(t) \quad \text{for all } t > 0\} = \{0,\,1,\,2\}\,,$$

and

$$C := \{p|\lambda_p(t) \ge \kappa_p(t) \quad \text{for all } t > 0\} = (-\infty,0] \cup [1,2]\,.$$

The case for B will be settled once we observe $B = A \cap C$, so we determine A and C.

We shall make repeated use of the inequalities: for any $x > 0$,

$$\begin{aligned} x^q - 1 &\ge q(x-1) \quad \text{if } q \le 0 \ \text{ or } \ q \ge 1\,, && (1)\\ x^q - 1 &\le q(x-1) \quad \text{if } 0 \le q \le 1 && (2)\end{aligned}$$

(see for example Hardy, Littlewood and Pólya, *Inequalities*, Theorem 42, page 40).

Set $f_p(t) = \kappa_p(t) - \lambda_p(t)$. First suppose $p \ge 2$. For $0 < t < 1$, and by using (1) twice with $q = p - 1 \ge 1$ [and with first $x = t^2 + 1$ and then

$x = t^2$], we have (since $1 - t^2 \geq 0$)

$$\begin{aligned} \frac{df_p(t)}{dt} &= p(t + t^{-1})^{p-1}\left(1 - \frac{1}{t^2}\right) - pt^{p-1} + pt^{-p-1} \\ &= \frac{-p}{t^{p+1}}\left[(1-t^2)(t^2+1)^{p-1} + t^2(t^2)^{p-1} - 1\right] \\ &\leq \frac{-p}{t^{p+1}}\left[(1-t^2)\Big(1 + (p-1)t^2\Big) + t^2\Big(1 + (p-1)(t^2-1)\Big) - 1\right] \\ &= 0\,. \end{aligned} \qquad (3)$$

Thus $f_p(t)$ is decreasing in $(0, 1]$. Note that $f_p(1/t) = f_p(t)$ so $f_p(t)$ is increasing in $[1, \infty)$, so $t = 1$ gives an absolute minimum. Thus $f_p(t) \geq f_p(1) = 0$ for all $t > 0$, showing that $[2, \infty) \subseteq A$.

If now $1 \leq p \leq 2$ then $q = p - 1 \in [0, 1]$, so we use (2), showing that inequality (3) is reversed and hence that $[1, 2] \subseteq C$. If $0 \leq p \leq 1$ then $q = p - 1 \leq 0$ so we use (1) again to show $[0, 1] \subseteq A$. Finally, if $p \leq 0$ then $q = p - 1 \leq 0$ and $-p \geq 0$, so by use of (1) we get $df_p(t)/dt \geq 0$ instead of (3), so $(-\infty, 0] \subseteq C$, completing the proof.

Remark. It is easy to see that for $0 < p < 1$ and $1 < p < 2$ the limit $\lim_{t\to\infty} f_p(t)$ exists and equals $2 - 2^p$. Thus by the monotonicity of f_p shown above we have the best possible inequalities

$$2 - 2^p \;<\; \kappa_p(t) - \lambda_p(t) \;\leq\; 0 \qquad \text{for} \quad 1 < p < 2\,,$$

$$0 \;\leq\; \kappa_p(t) - \lambda_p(t) \;<\; 2 - 2^p \qquad \text{for} \quad 0 < p < 1\,.$$

For the rest of p it is easy to see that $\lim_{t\to\infty} f_p(t) = \pm\infty$ so no corresponding inequality exists.

Also solved by NIKOLAOS DERGIADES, Thessaloniki, Greece; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; KEE-WAI LAU, Hong Kong, China; and the proposer. One other reader sent in a counterexample to the incorrect version of this problem published on [1998: 176].

2337. [1998: 177] *Proposed by Iliya Bluskov, Simon Fraser University, Burnaby, BC.*

Let $F(1) = \left\lceil \dfrac{n^2 + 2n + 2}{n^2 + n + 1} \right\rceil$, and, for each $i > 1$, let

$$F(i) = \left\lceil \frac{n^2 + 2n + i + 1}{n^2 + n + i}\, F(i-1) \right\rceil.$$

Find $F(n)$.

Solution by Michael Lambrou, University of Crete, Crete, Greece.

If the given n is $n = 1$, then $F(n) = F(1) = \left\lceil \dfrac{1^2 + 2 + 2}{1^2 + 1 + 1} \right\rceil = \left\lceil \dfrac{5}{3} \right\rceil = 2$.
Let us then do the more interesting case when the given n is ≥ 2.

We show that for each i with $1 \leq i \leq n$ we have $F(i) = i + 1$, so that in particular $F(n) = n + 1$. We use induction on i.

For $i = 1$ we have

$$F(i) \;=\; F(1) \;=\; \left\lceil \frac{n^2 + 2n + 2}{n^2 + n + 1} \right\rceil \;=\; \left\lceil 1 + \frac{n+1}{n^2 + n + 1} \right\rceil .$$

Thus

$$1 \;<\; F(1) \;=\; \left\lceil 1 + \frac{n+1}{n^2 + n + 1} \right\rceil \leq \left\lceil 1 + \frac{n+1}{n^2 + n} \right\rceil \;=\; \left\lceil 1 + \frac{1}{n} \right\rceil \;=\; 2 ,$$

from which we see that $F(1) = 2$, as required.

If we assume validity for $i = m - 1$ (that is, $F(m - 1) = m$, where $2 \leq m \leq n$) then

$$\begin{aligned} F(m) \;&=\; \left\lceil \frac{n^2 + 2n + m + 1}{n^2 + n + m} \cdot m \right\rceil \;=\; \left\lceil m + \frac{(n+1)m}{n^2 + n + m} \right\rceil \\ &\leq\; \left\lceil m + \frac{(n+1)m}{n^2 + n} \right\rceil \;=\; \left\lceil m + \frac{m}{n} \right\rceil \leq \lceil m + 1 \rceil \;=\; m + 1 , \end{aligned}$$

but as $\dfrac{(n+1)m}{n^2 + n + m} \geq 0$, we have $m < \left\lceil m + \dfrac{(n+1)m}{n^2 + n + m} \right\rceil$.

Thus we have $m < F(m) \leq m + 1$, showing that $F(m) = m + 1$, completing the induction step and proving the claim.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; FLORIAN HERZIG, student, Cambridge, UK; RICHARD I. HESS, Rancho Palos Verdes, California, USA; JOEL SCHLOSBERG, student, Bayside, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer.

2338. [1998: 234] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose $ABCD$ is a convex cyclic quadrilateral, and P is the intersection of the diagonals AC and BD. Let I_1, I_2, I_3 and I_4 be the incentres of triangles PAB, PBC, PCD and PDA respectively. Suppose that I_1, I_2, I_3 and I_4 are concyclic.

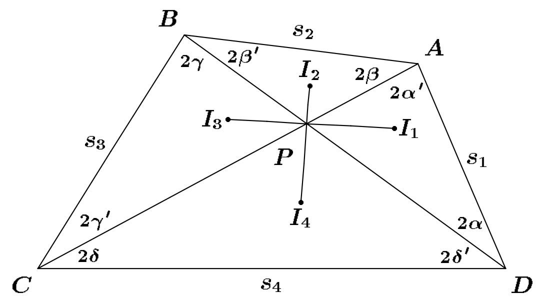
Prove that $ABCD$ has an incircle.

Solution by Peter Y. Woo, Biola University, La Miranda, California, USA.

$ABCD$ does not have to be cyclic. More precisely,

When a convex quadrilateral is subdivided into 4 triangles by its two diagonals, then the incentres of the 4 triangles are concyclic if and only if the quadrilateral has an incircle.

Notation. Let P be the point where the diagonals intersect and let the triangles be T_1, T_2, T_3, T_4 (labelled counterclockwise as in the figure), with the respective incentres I_1, I_2, I_3, I_4. Denote the 8 angles formed by the diagonals with the four sides by 2α, $2\alpha'$, 2β, $2\beta'$, 2γ, $2\gamma'$, 2δ, $2\delta'$ (counterclockwise with 2α, $2\alpha'$ in T_1, etc.).



Step 1. In the usual notation (used only here in step 1) for $\triangle ABC$ with sides a, b, c, incentre I, and semiperimeter $s = \frac{a+b+c}{2}$, AI satisfies

$$AI^2 \;=\; bc\tan\frac{B}{2}\tan\frac{C}{2}\,.$$

The proof follows from familiar formulas. In E.W. Hobson's *Treatise on Plane and Advanced Trigonometry*, for example, in section 123 the author shows that

$$AI \;=\; \frac{r}{\sin\frac{A}{2}}\,,\;\; \tan\frac{B}{2} \;=\; \frac{r}{s-b}\,,\;\; \tan\frac{C}{2} \;=\; \frac{r}{s-c}\,,$$

$$\text{and}\quad \sin^2\frac{A}{2} \;=\; \frac{(s-b)(s-c)}{bc}\,,$$

which, when combined, proves the claim.

Step 2. I_1, I_2, I_3, I_4 are concyclic if and only if

$$\tan\alpha\tan\alpha'\tan\gamma\tan\gamma' \;=\; \tan\beta\tan\beta'\tan\delta\tan\delta'\,.$$

Proof. Since PI_1 and PI_3 bisect vertically opposite angles, as do PI_2 and PI_4, the Intersecting Chords Theorem says that I_1, I_2, I_3, I_4 are concyclic if and only if $PI_1 \cdot PI_3 = PI_2 \cdot PI_4$. The desired equality then follows from step 1.

Step 3. A quadrilateral has an incircle if and only if the sum of one pair of opposite sides equals the sum of the other. This is a standard result of elementary geometry. See, for example, Nathan Altshiller Court's *College Geometry*, p. 135.

Step 4. Let $ABCD$ be any quadrilateral (that is, any four points, no three collinear), *I, I' be incentres of $\triangle ABC$ and $\triangle ADC$, and IN, $I'N'$ be* *perpendiculars to the diagonal AC from I and I'. Then $CN \geq CN'$ if and only if $AD + BC \geq AB + CD$, with equality for both or for neither.*

Proof. $CB - AB = CN - AN$ [because $CN = s - c$ and $AN = s - a$ in the notation of step 1] and $AD - CD = AN' - CN'$. Add these two equalities [noting that $AN = AC - CN$ and $AN' = AC - CN'$].

Step 5. $AD + BC \geq AB + CD$ if and only if

$$\tan \angle BAI \tan \angle DCI' \;\geq\; \tan \angle BCI \tan \angle DAI' ,$$

with equality for both or for neither.

Proof. This follows from step 4 since we have $\tan \angle BAI = \frac{IN}{AC-CN}$, $\tan \angle DCI' = \frac{I'N'}{CN'}$, $\tan \angle BCI = \frac{IN}{CN}$, and $\tan \angle DAI' = \frac{I'N'}{AC-CN'}$. [Note that the incentres here generally do not coincide with those of the main result. The key observation is that IA (for example) bisects the angle between a diagonal and side of the quadrilateral, while the angles α, α', etc. in step 2 each are equal to half the angle between a diagonal and side.]

Proof of the main result. If $ABCD$ has an incircle then $AD + BC = AB + CD$ (step 3), so that $\tan\alpha\tan\gamma = \tan\beta'\tan\delta'$ and $\tan\alpha'\tan\gamma' = \tan\beta\tan\delta$ (step 5). By step 2, I_1, I_2, I_3, I_4 are therefore concyclic. On the other hand, if $ABCD$ does not circumscribe some circle, then let s_i be the side of T_i opposite P (for $i = 1, 2, 3, 4$). Without loss of generality, assume $s_2 + s_4 > s_1 + s_3$. Then by step 5, $\tan\alpha\tan\gamma > \tan\beta'\tan\delta'$ and $\tan\alpha'\tan\gamma' > \tan\beta\tan\delta$, so that by step 2, I_1, I_2, I_3, I_4 are not concyclic.

Also solved by NIELS BEJLEGAARD, Stavanger, Norway; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VJECKOSLAV KOVAČ, student, Univ. Zagreb, Croatia; D.J. SMEENK, Zaltbommel, the Netherlands; JEREMY YOUNG, student, Nottingham, England; and the proposer.

All solvers except Woo and Janous proved the theorem as stated (with $ABCD$ cyclic). Janous remembers having seen the stronger version before, but he did not recall the reference. Can any reader provide a reference?

2340. [1998: 235] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $\lambda > 0$ be a real number and a, b, c be the sides of a triangle. Prove that

$$\prod_{\text{cyclic}} \frac{s + \lambda a}{s - a} \geq (2\lambda + 3)^3.$$

[As usual s denotes the semiperimeter.]

I. *Solution by Jeremy Young, student, Nottingham High School, Nottingham, England.*

By the common substitution that a, b, c are the sides of a triangle if and only if $a = y + z$, $b = z + x$, $c = x + y$ for x, y, $z > 0$, the inequality is equivalent to

$$\frac{1}{(2\lambda+3)^3} \prod_{\text{cyclic}} \Big(x + (\lambda+1)y + (\lambda+1)z \Big) \;\geq\; xyz\,. \tag{1}$$

Now by the weighted AM–GM inequality,

$$\left(\frac{1}{2\lambda+3}\right)x + \left(\frac{\lambda+1}{2\lambda+3}\right)y + \left(\frac{\lambda+1}{2\lambda+3}\right)z \;\geq\; x^{\frac{1}{2\lambda+3}} y^{\frac{\lambda+1}{2\lambda+3}} z^{\frac{\lambda+1}{2\lambda+3}}\,,$$

$$\left(\frac{\lambda+1}{2\lambda+3}\right)x + \left(\frac{1}{2\lambda+3}\right)y + \left(\frac{\lambda+1}{2\lambda+3}\right)z \;\geq\; x^{\frac{\lambda+1}{2\lambda+3}} y^{\frac{1}{2\lambda+3}} z^{\frac{\lambda+1}{2\lambda+3}}\,,$$

and

$$\left(\frac{\lambda+1}{2\lambda+3}\right)x + \left(\frac{\lambda+1}{2\lambda+3}\right)y + \left(\frac{1}{2\lambda+3}\right)z \;\geq\; x^{\frac{\lambda+1}{2\lambda+3}} y^{\frac{\lambda+1}{2\lambda+3}} z^{\frac{1}{2\lambda+3}}\,.$$

Multiplying these three lines together gives (1), the required result.

II. *Solution by Michael Lambrou, University of Crete, Crete, Greece.*

Set

$$f(\lambda) \;=\; s(s+\lambda a)(s+\lambda b)(s+\lambda c) - (2\lambda+3)^3\Delta^2$$

where $\lambda \geq 0$ and Δ is the area of the triangle. Then

$$\begin{aligned} f'(\lambda) \;&=\; s \sum_{\text{cyclic}} a(s+\lambda b)(s+\lambda c) - 6(2\lambda+3)^2\Delta^2 \\ &=\; s\left(s^2(a+b+c) + 2\lambda s(ab+bc+ca) + 3\lambda^2 abc\right) - 6(2\lambda+3)^2\Delta^2\,. \end{aligned}$$

Using $a+b+c = 2s$ and

$$s^2 \geq 3\Delta\sqrt{3}\,, \qquad ab+bc+ca \geq 4\Delta\sqrt{3}\,, \qquad 9abc \geq 8s\Delta\sqrt{3}$$

(see for example Bottema et al, *Geometric Inequalities*, items 4.2, 4.5, 4.13) we have

$$\begin{aligned} f'(\lambda) \;&\geq\; \frac{2}{3}s^2\Delta\sqrt{3}(9+12\lambda+4\lambda^2) - 6(2\lambda+3)^2\Delta^2 \\ &\geq\; 6\Delta^2(2\lambda+3)^2 - 6(2\lambda+3)^2\Delta^2 \;=\; 0\,. \end{aligned}$$

Thus, f is increasing in $[0,\infty)$, and so

$$f(\lambda) \;\geq\; f(0) \;=\; s^4 - 27\Delta^2 \;\geq\; 0\,.$$

Hence by Heron's formula,

$$f(\lambda) \;=\; s(s+\lambda a)(s+\lambda b)(s+\lambda c) - (2\lambda+3)^3 s(s-a)(s-b)(s-c) \;\geq\; 0\,,$$

which reduces to the given inequality.

Also solved by ED BARBEAU, University of Toronto, Toronto, Ontario; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, Athens, Greece; NIKOLAOS DERGIADES, Thessaloniki, Greece; FLORIAN HERZIG, student, Cambridge, UK; RICHARD I. HESS, Rancho Palos Verdes, California, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; VJEKOSLAV KOVAČ, student, University of Zagreb, Croatia; KEE-WAI LAU, Hong Kong, China; JESSIE LEI, student, Vincent Massey Secondary School, Windsor, Ontario; VEDULA N. MURTY, Dover, PA, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

Herzig's solution is the same as Young's. Herzig also points out that the inequality is true for all $\lambda \geq -1$, and that equality holds if and only if $\lambda = -1$ or $x = y = z$; that is, $a = b = c$.

2341. [1998: 235] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let a, b, c be the sides of a triangle. For real $\lambda > 0$, put

$$s(\lambda) := \left|\sum_{\text{cyclic}} \left[\left(\frac{a}{b}\right)^{\lambda} - \left(\frac{b}{a}\right)^{\lambda}\right]\right|$$

and let $\Delta(\lambda)$ be the supremum of $s(\lambda)$ over all triangles.

1. Show that $\Delta(\lambda)$ is finite if $\lambda \in (0,1]$ and $\Delta(\lambda)$ is infinite for $\lambda > 1$.

2.★ What is the exact value of $\Delta(\lambda)$ for $\lambda \in (0,1)$?

I. *Solution to part* 1 *by Florian Herzig, student, Cambridge, UK.*

Note that

$$s(\lambda) \;=\; \left|\frac{(c^\lambda - b^\lambda)(c^\lambda - a^\lambda)(b^\lambda - a^\lambda)}{(abc)^\lambda}\right|\,.$$

First, I show that, for positive reals a and b, the inequality $a^x + b^x \geq (a+b)^x$ holds for all $0 \leq x \leq 1$. Without loss of generality, assume that $a + b = 1$. Then $0 < a, b < 1$ and hence $a^x \geq a$ and $b^x \geq b$. Therefore $a^x + b^x \geq a + b = (a+b)^x$ as claimed.

Suppose $a \leq b \leq c$ without loss of generality. By the above and the triangle inequality, $a^x + b^x \geq (a+b)^x > c^x$. Hence $a^x > c^x - b^x \geq 0$ and $b^x > c^x - a^x \geq 0$. Trivially also $c^x > b^x - a^x \geq 0$. Thus for all $0 < \lambda \leq 1$,

$$s(\lambda) \;\leq\; \frac{a^\lambda b^\lambda c^\lambda}{(abc)^\lambda} \;=\; 1\,,$$

and so $\Delta(\lambda) \leq 1$.

If $\lambda > 1$ then consider the triangle with sides $1 + \frac{1}{n}$, $n + \frac{1}{n}$, $n + 1$ for a positive integer n. Then in the expression for $s(\lambda)$ [that is,

$$s(\lambda) \;=\; \left|\left(\frac{n+1}{n^2+1}\right)^{\lambda}+\left(\frac{n^2+1}{n^2+n}\right)^{\lambda}+n^{\lambda}-\left(\frac{n^2+1}{n+1}\right)^{\lambda}-\left(\frac{n^2+n}{n^2+1}\right)^{\lambda}-\left(\frac{1}{n}\right)^{\lambda}\right|$$

— *Ed.*] two terms tend to 0 and two tend to 1 as $n \to \infty$. Consider the remaining two terms:

$$n^{\lambda} - \left(\frac{n^2+1}{n+1}\right)^{\lambda} \;\geq\; n^{\lambda} - \left(n - \frac{1}{2}\right)^{\lambda}. \tag{1}$$

where the inequality holds for all $n \geq 3$. Since

$$n^{\lambda} \;=\; \left(n - \frac{1}{2} + \frac{1}{2}\right)^{\lambda} \geq \left(n - \frac{1}{2}\right)^{\lambda} + \frac{\lambda}{2}\left(n - \frac{1}{2}\right)^{\lambda-1}$$

[by the Binomial Theorem], the left-hand side of (1) and hence $s(\lambda)$ tends to infinity as $n \to \infty$. It follows that $\Delta(\lambda)$ is infinite.

II. *Partial solution to part 2 by Nikolaos Dergiades, Thessaloniki, Greece (with editorial comments).*

All readers who attempted part 2 agreed that there will be no explicit formula for $\Delta(\lambda)$ for arbitrary $0 < \lambda < 1$. They all gave $\lambda = 1/2$ as a special case (and so did the proposer, in fact), and further agreed that in this case $\Delta(1/2) \approx 0.0740033$.

Dergiades, however, managed to find the exact value of $\Delta(1/2)$. First he puts $a \leq b \leq c$ without loss of generality, in which case (as in Solution I)

$$\begin{aligned} s(\lambda) \;&=\; \frac{(c^{\lambda}-b^{\lambda})(c^{\lambda}-a^{\lambda})(b^{\lambda}-a^{\lambda})}{(abc)^{\lambda}} \\ &=\; \left(\frac{a}{b}\right)^{\lambda}+\left(\frac{b}{c}\right)^{\lambda}+\left(\frac{c}{a}\right)^{\lambda}-\left(\frac{b}{a}\right)^{\lambda}-\left(\frac{c}{b}\right)^{\lambda}-\left(\frac{a}{c}\right)^{\lambda}. \end{aligned}$$

Considering a, b and λ as constant and calling this function $F(c)$, he gets

$$F'(c) \;=\; \frac{\lambda\left[c^{2\lambda}-(ab)^{\lambda}\right](b^{\lambda}-a^{\lambda})}{c(abc)^{\lambda}} \;\geq\; 0,$$

so F is increasing. Thus to maximize s he puts $c = a + b$, $b = x$ and without loss of generality $a = 1$, which for $\lambda = 1/2$ means that s can be rewritten as the function

$$f(x) \;=\; \frac{1}{\sqrt{x}}+\frac{\sqrt{x}}{\sqrt{1+x}}+\sqrt{1+x}-\sqrt{x}-\frac{\sqrt{1+x}}{\sqrt{x}}-\frac{1}{\sqrt{1+x}},$$

which now must be maximized over $x \geq 1$. Setting the derivative of this to zero results in a sixth degree polynomial equation

$$x^6 - 4x^5 - 32x^4 - 58x^3 - 32x^2 - 4x + 1 \;=\; 0\,,$$

whose root $\rho \approx 8.57318922$ can be substituted into $f(x)$ to obtain the above maximum value of $\Delta(1/2)$. This is where the other solvers stopped. However, Dergiades then transforms the equation for $f(x)$ by putting

$$x \;=\; \tan^2 y \qquad \text{and} \qquad \sin 2y \;=\; z\,; \qquad \text{that is,} \quad z \;=\; \frac{2\sqrt{x}}{1+x}\,,$$

getting the function

$$h(z) \;=\; \frac{\sqrt{1-z}\cdot(2+z-2\sqrt{1+z})}{z}\,.$$

Solving $h'(z) = 0$ gives a **cubic** equation

$$z^3 - 3z^2 + 8z - 4 \;=\; 0$$

which can be solved. He gets the real root

$$r \;=\; 1 - \left(1 + \frac{2}{9}\sqrt{114}\right)^{1/3} + \frac{5}{3}\left(1 + \frac{2}{9}\sqrt{114}\right)^{-1/3}\,,$$

so the exact value of $\Delta(1/2)$ is $h(r)$ (which, by my calculator, is indeed the previously found 0.0740033).

Part 1 also solved by VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer. The approximate value of $\Delta(1/2)$ was found by Konečný, Lambrou and the proposer.

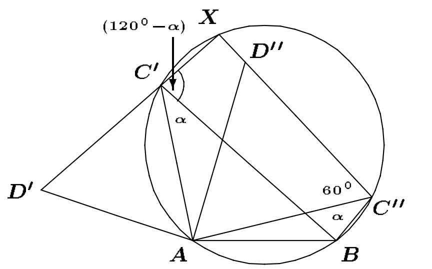
2342. [1998: 235] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Given A and B are fixed points of circle Γ. The point C moves on Γ, on one side of AB. D and E are points outside $\triangle ABC$ such that $\triangle ACD$ and $\triangle BCE$ are both equilateral.

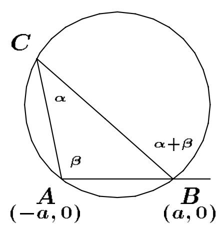
(a) Show that CD and CE each pass through a fixed point of Γ when C moves on Γ.

(b) Determine the locus of the mid-point of DE.

Solution by Jeremy Young, student, Nottingham High School, Nottingham, England.



Let X be the point subtending angles $120°$ and $60°$ with A. By "angles in the same segment", CD always passes through X. Two possible positions of C are shown: C' and C''. Similarly, two corresponding positions of D are shown: D' and D''.



Similarly, define Y relative to B.

Introduce a Cartesian coordinate system with A, B as $(-a,0)$, $(a,0)$ respectively. Let C be a point with positive y-coordinate such that $\angle ACB = \alpha$.

By the Sine Rule, we have $AC = 2a\dfrac{\sin(\alpha+\beta)}{\sin\alpha}$ and $BC = 2a\dfrac{\sin\beta}{\sin\alpha}$.

Therefore D has position vector

$$\begin{pmatrix} -a \\ 0 \end{pmatrix} + 2a\frac{\sin(\alpha+\beta)}{\sin\alpha}\begin{pmatrix} \cos(\beta+60°) \\ \sin(\beta+60°) \end{pmatrix}.$$

Similarly, E has position vector

$$\begin{pmatrix} a \\ 0 \end{pmatrix} + 2a\frac{\sin\beta}{\sin\alpha}\begin{pmatrix} \cos(\alpha+\beta-60°) \\ \sin(\alpha+\beta-60°) \end{pmatrix}.$$

Hence, the mid-point has position vector

$$\begin{aligned} &\frac{a}{\sin\alpha}\begin{pmatrix} \sin(\alpha+\beta)\cos(\beta+60°)+\sin\beta\cos(\alpha+\beta-60°) \\ \sin(\alpha+\beta)\sin(\beta+60°)+\sin\beta\sin(\alpha+\beta-60°) \end{pmatrix} \\ &= \frac{a}{2\sin\alpha}\begin{pmatrix} \sin(\alpha+2\beta) \\ 2\cos(\alpha-60°)-\cos(\alpha+2\beta) \end{pmatrix} \\ &= \frac{a}{2\sin\alpha}\begin{pmatrix} \sin(\alpha+2\beta) \\ -\cos(a+2\beta) \end{pmatrix} + \frac{a}{\sin\alpha}\begin{pmatrix} 0 \\ \cos(\alpha-60°) \end{pmatrix}, \end{aligned}$$

where $0 \le \beta \le 18° - \alpha$. Thus the required locus is an arc of a circle, radius $R/2$ (where $R = a/2\sin\alpha$ is the circumradius of $\triangle ABC$) and centre $(0, 2R\cos(\alpha-60°))$, which is exterior to the equilateral triangle with AB as base.

Also solved by NIELS BEJLEGAARD, Stavanger, Norway; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, *UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; VJEKOSLAV KOVAČ, student, University of Zagreb, Croatia; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer.*

Most solvers used pure geometric methods. Bellot Rosado and López Chamorro solved the problem entirely with the use of complex numbers.

2343. [1998: 235] *Proposed by Doru Popescu Anastasiu, Liceul "Radu Greceanu", Slatina, Olt, Romania.*

For positive numbers sequences $\{x_n\}_{n\geq 1}$, $\{y_n\}_{n\geq 1}$, $\{z_n\}_{n\geq 1}$ with conditions: for $n \geq 1$, we have

$$\begin{aligned}(n+1)x_n^2 + (n^2+1)y_n^2 + (n^2+n)z_n^2 & \\ = \; 2\sqrt{n}\left(nx_ny_n + \sqrt{n}x_nz_n + y_nz_n\right)\,, & \qquad (1)\end{aligned}$$

and for $n \geq 2$, we have

$$x_n + \sqrt{n}y_n - nz_n \;=\; x_{n-1} + y_{n-1} - \sqrt{n-1}z_{n-1}\,. \qquad (2)$$

Find $\lim\limits_{n\to\infty} x_n$, $\lim\limits_{n\to\infty} y_n$ and $\lim\limits_{n\to\infty} z_n$.

Solution by Christo Saragiotis, Thessaloniki, Greece.

Equation (1) can be re-written as

$$\left(\sqrt{n}x_n - ny_n\right)^2 + (x_n - nz_n)^2 + \left(y_n - \sqrt{n}z_n\right)^2 \;=\; 0\,.$$

Thus, $y_n = \dfrac{x_n}{\sqrt{n}}$, $z_n = \dfrac{x_n}{n}$ and $y_n = \sqrt{n}z_n$ for all $n \geq 1$. Substituting these into equation (2) yields $x_n = x_{n-1}$ for all $n \geq 2$.

Therefore, $\lim\limits_{n\to\infty} x_n = x_1$ and $\lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} z_n = 0$.

Also solved by THEODORE CHRONIS, Athens, Greece; OSCAR CIAURRI, Logroño, Spain; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VJEKOSLAV KOVAČ, student, University of Zagreb, Croatia; MICHAEL LAMBROU, University of Crete, Crete, Greece; LAURENT LESSARD, student, Le Collège français, Toronto, Ontario; GERRY LEVERSHA, St. Paul's School, London, England; and the proposer. There were two partially incorrect solutions.

The solutions of almost all the solvers were virtually identical to the one given above.

2344. [1998: 235] *Proposed by Murali Vajapeyam, student, Campina Grande, Brazil and Florian Herzig, student, Perchtoldsdorf, Austria.*

Find all positive integers N that are quadratic residues modulo all primes greater than N.

Solution by Mansur Boase, student, Cambridge, England.

Clearly any square is a quadratic residue modulo any prime. Suppose N is not a square. Then we can write N as $m^2 p_1 p_2 \cdots p_r$ where $p_i \neq p_j$ for any pair (i, j) with $i \neq j$, and where r is some positive integer. Without loss of generality let us assume $p_1 < p_2 < \cdots < p_r$. We shall show that N cannot be a perfect square modulo all primes congruent to $1 \pmod 4$. Let q be a prime congruent to $1 \pmod 4$. Then, introducing Legendre symbols, we have

$$\begin{aligned}\left(\frac{N}{q}\right) = \left(\frac{m^2 p_1 p_2 \cdots p_r}{q}\right) &= \left(\frac{m^2}{q}\right)\left(\frac{p_1}{q}\right)\left(\frac{p_2}{q}\right)\cdots\left(\frac{p_r}{q}\right) \\ &= \left(\frac{p_1}{q}\right)\left(\frac{p_2}{q}\right)\cdots\left(\frac{p_r}{q}\right) \\ &= \left(\frac{q}{p_1}\right)\left(\frac{q}{p_2}\right)\cdots\left(\frac{q}{p_r}\right) .\end{aligned}$$

The latter equality follows from the Law of Quadratic Reciprocity since $q \equiv 1 \pmod 4$.

Suppose $p_1 = 2$. Then if $q \equiv 1 + 4p_2 p_3 \cdots p_r \pmod{8p_2 p_3 \cdots p_r}$, we have $q \equiv 1 \pmod{p_i}$ for $2 \leq i \leq r$, whence

$$\left(\frac{q}{p_i}\right) = \left(\frac{1}{p_i}\right) = 1 \qquad \text{for } 2 \leq i \leq r .$$

Also, since all the p_2, p_3, $\ldots$, p_r are odd, $q \equiv 5 \pmod 8$, and therefore, $\left(\frac{2}{q}\right) = (-1)^{(q^2-1)/8} = -1$. Thus $\left(\frac{N}{q}\right) = -1$, which would be a contradiction. It thus remains to show that a prime greater than N exists satisfying $q \equiv 1 + 4p_2 p_3 \cdots p_r \pmod{8p_2 p_3 \cdots p_r}$. But by Dirichlet's Theorem there are infinitely many such primes. Thus we cannot have $p_1 = 2$, and all the primes are odd.

There are $(p_i - 1)/2$ quadratic residues modulo the prime p_i, $(p_i \geq 3)$ and this is a positive integer. Suppose n is a quadratic non-residue modulo p_1. Then by the Chinese Remainder Theorem, there exists a solution modulo $4p_1 p_2 \cdots p_r$ to the set of congruences:

$$x \equiv 1 \pmod 4 , \quad x \equiv n \pmod{p_1} , \quad x \equiv 1 \pmod{p_i} \quad \text{for } 2 \leq i \leq r .$$

as $(2, p_i) = 1$ for all i, $1 \leq i \leq r$, and $(p_i, p_j) = 1$ for all $i \neq j$. Suppose the prime q satisfies $q \equiv x \pmod{4p_1 p_2 \cdots p_r}$ and hence q satisfies the set of congruences above. Then

$$\begin{aligned}\left(\frac{N}{q}\right) &= \left(\frac{q}{p_1}\right)\left(\frac{q}{p_2}\right)\cdots\left(\frac{q}{p_r}\right) \\ &= \left(\frac{n}{p_1}\right)\left(\frac{1}{p_2}\right)\cdots\left(\frac{1}{p_r}\right) \\ &= -1 ,\end{aligned}$$

which would be a contradiction. Thus we must prove that there can be no primes greater than N congruent to $x \pmod{4p_1p_2\cdots p_r}$, as all such numbers satisfy the requirement of being congruent to $1 \pmod 4$. But by Dirichlet's Theorem, as $(x, 4p_1p_2\cdots p_r) = 1$ there must be infinitely many primes in this arithmetic progression. [$n \neq 0$, so the set of congruences above shows that x is relatively prime to each of 4, p_1, p_2, $\ldots$, p_r, and hence also to the product.]

Thus N can satisfy the conditions of the problem only if N is a perfect square.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; and the proposer. There was one incorrect solution submitted.

Lau comments that Theorem 3 of Chapter 5 of Ireland and Rosen's "A Classical Introduction to Modern Number Theory" (Springer-Verlag, 1982) proves that if N is a non-square integer, then there are infinitely many primes p for which N is a quadratic nonresidue. Combining this with the observation that the squares are quadratic residues modulo all primes greater than them solves the problem.

2345. [1998: 236] *Proposed by Vedula N. Murty, Visakhapatnam, India.*

Suppose that $x > 1$.

(a) Show that $\ln(x) > \dfrac{3(x^2-1)}{x^2+4x+1}$.

(b) Show that $\dfrac{a-b}{\ln(a)-\ln(b)} < \dfrac{1}{3}\left(2\sqrt{ab}+\dfrac{a+b}{2}\right)$,

where $a > 0$, $b > 0$ and $a \neq b$.

Solution by Kee-Wai Lau, Hong Kong, China.

(a) For $x \geq 1$, let $f(x) = \ln(x) - \dfrac{3(x^2-1)}{x^2+4x+1}$. Then straightforward computations show that $f'(x) = \dfrac{(x-1)^4}{x\left(x^2+4x+1\right)^2}$, and so $f'(x) > 0$ for all $x > 1$. Since $f(1) = 0$, we have $f(x) > 0$ for $x > 1$, and the inequality follows.

(b) Clearly, we can assume that $a > b$. Then the desired inequality follows readily by substituting $x = \sqrt{a/b}$ into the inequality in (a).

Also solved by THEODORE CHRONIS, Athens, Greece; OSCAR CIAURRI; Logroño, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; RICHARD B. EDEN, student, Ateneo de Manila University, Quezon City, Philippines; FLORIAN HERZIG, student, Cambridge, UK; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; LAURENT LESSARD, student, Le Collège Français, Toronto, Ontario; GERRY LEVERSHA, St. Paul's School, London, England; PHILIP McCARTNEY, Northern Kentucky University, Highland Heights, Kentucky, USA; BOB PRIELIPP, University of Wisconsin–Oshkosh, Wisconsin, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DIGBY SMITH, Mount Royal College, Calgary, Alberta;

PANOS E. TSAOUSSOGLOU, Athens, Greece; JEREMY YOUNG, student, Nottingham High School, Nottingham, England; and the proposer. There was one incorrect solution.

Solutions virtually identical to the one given above were submitted by about half of the solvers.

Herzig remarked that the right hand side of the inequality in (a) *is a Padé approximation to* $\ln x$ *at* $x = 1$. *Janous commented that this inequality is "quite sharp" in positive neighbourhoods of* $x = 1$. *Lambrou remarked that part* (b) *is a repetition of problem* **2206** [1997: 46]. *He and Konečný both pointed out that the inequality in* (a) *is reversed for* $0 < x < 1$. (Ed: *This is obvious from the solution given above.*) *Leversha considered some generalizations of* (b) *by setting* $x = \left(\frac{a}{b}\right)^{1/n}$ *in the inequality of* (a); *for example, putting* $n = 4$ *would give*

$$\frac{a-b}{\ln a - \ln b} < \frac{1}{12}\left(\sqrt{a}+\sqrt{b}+4\sqrt[4]{ab}\right)\left(\sqrt{a}+\sqrt{b}\right),$$

which is sharper than the inequality in (b). *Although a few solvers commented that the inequality in* (b) *is known as* Pólya's Inequality, *Sieffert (who was the proposer of problem* **2206** *mentioned above) stated that it should really be called the* Pólya–Szegö Inequality, *since it first appeared in a joint paper by them in 1951.*

2347. [1998: 236] *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Prove that the equation $x^2 + y^2 = z^{1998}$ has infinitely many solutions in positive integers, x, y and z.

Solution by Richard I. Hess, Rancho Palos Verdes, California, USA. Let k be any positive integer. Multiply both sides of the equality $3^2 + 4^2 = 5^2$ by $5^{1996}k^{1998}$. The result is

$$(3 \cdot 5^{998}k^{999})^2 + (4 \cdot 5^{998}k^{999})^2 = (5k)^{1998}.$$

Hence $(3 \cdot 5^{998}k^{999}, 4 \cdot 5^{998}k^{999}, 5k)$ is an infinite family of solutions to the given equation.

Also solved by CHARLES ASHBACHER, Cedar Rapids, Iowa, USA; NIELS BEJLEGAARD, Stavanger, Norway; M. BENITO and E. FERNÁNDEZ, Logroño, Spain (two solutions); MANSUR BOASE, student, Cambridge, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, Athens, Greece; GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; KEITH EKBLAW, Walla Walla, Washington, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, Connecticut, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA (two solutions); MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong, China; GERRY LEVERSHA, St. Paul's School, London, England; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria (three solutions); BOB PRIELIPP, University of Wisconsin–Oshkosh, Wisconsin, USA; CHRISTOS SARAGIOTIS, Thessaloniki, Greece; DIGBY SMITH, Mount Royal College, Calgary, Alberta; PANOS E. TSAOUSSOGLOU, Athens, Greece; STAN WAGON, Macalester College, St. Paul, Minnesota, USA (two solutions); JEREMY YOUNG, student, Nottingham High School, Nottingham, England; and the proposer.

2349. [1998: 236] *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*

Suppose that $\triangle ABC$ has acute angles such that $A < B < C$. Prove that

$$\sin^2 B \sin\left(\tfrac{A}{2}\right) \sin\left(A + \tfrac{B}{2}\right) \;>\; \sin^2 A \sin\left(\tfrac{B}{2}\right) \sin\left(B + \tfrac{A}{2}\right) \;.$$

Solution by Florian Herzig, student, Cambridge, UK.

Let AD, BE be the angle bisectors in the triangle. By the angle bisector property, $EC = \frac{ab}{a+c}$. Hence in triangle BEC,

$$\frac{\sin\frac{B}{2}}{\sin(A+\frac{B}{2})} \;=\; \frac{EC}{BC} \;=\; \frac{b}{a+c}\,.$$

A similar result holds in triangle ADC. Since also $\sin B : \sin A = b : a$, the given inequality is equivalent to $\frac{b^2}{a^2} > \frac{b+c}{a} \cdot \frac{b}{a+c}$, or $b(a+c) > a(b+c)$; that is, $b > a$, and this is, of course, true.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, Athens, Greece; GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; NIKOLAOS DERGIADES, Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong, China; GERRY LEVERSHA, St. Paul's School, London, England; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; PANOS E. TSAOUSSOGLOU, Athens, Greece; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, England; and the proposer.

Most of the solutions were along the same lines as the featured one, which was chosen for its brevity.

2350. [1998: 236] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Suppose that the centroid of $\triangle ABC$ is G, and that M and N are the mid-points of AC and AB respectively. Suppose that circles ANC and AMB meet at (A and) P, and that circle AMN meets AP again at T.

1. Determine $AT : AP$.

2. Prove that $\angle BAG = \angle CAT$.

Identical independent solutions by Florian Herzig, student, Cambridge, UK and by Vjekoslav Kovač, student, U. of Zagreb, Croatia.

Invert in a circle with centre A, and denote the image of a point X by X'. Then C' is the mid-point of AM' since $AC : AM = AM' : AC'$. Similarly B' is the mid-point of AN'. P' is the intersection of lines $M'B'$ and $N'C'$, so that it is the centroid of $\triangle AM'N'$. T' is then the point of intersection of line AP' and line $M'N'$ (that is, the mid-point of segment $M'N'$). By the property of medians,

$$AT : AP \;=\; AP' : AT' \;=\; 2 : 3\,.$$

Note that $\triangle ABC \sim \triangle ANM \sim \triangle AM'N'$ and hence

$$\angle CAT \;=\; \angle C'AT' \;=\; \angle M'AP' \;=\; \angle BAG$$

as we wanted to show.

Also solved by NIELS BEJLEGAARD, Stavanger, Norway; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer.