THE ACADEMY CORNER

No. 22

Bruce Shawyer

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APICS Mathematics Competition 1998

held at **Saint Mary's University**, Halifax, Nova Scotia,

Rules:

- Teams of two are to work in cooperation and to submit *one* set of answers each.
- No notes, calculators, or other such aids are permitted.
- You may not communicate with noncontestants (except invigilators) or other teams.
- There are nine questions.

1. Fred and Cathy play the following game. They are given the polynomial $f(x) = ax^3 + bx^2 + cx + d$. They take turns, Cathy first, in replacing a, then b, then c and finally d with positive integers. Fred wins if the resulting polynomial has at least two distinct roots. Who should win and what is the winning strategy?

2. Define the integer sequence $\{T_n\}$ by $T_0 = 0$, $T_1 = 1$, $T_2 = 2$ and $T_{n+1} = T_n + T_{n-1} + T_{n-2}$ ($n \geq 2$). Compute
$$S := \sum_{n=0}^{\infty} \frac{T_n}{2^n}\,.$$

3. Let $X_1, X_2, \ldots, X_n$ be independent, integer-valued random variables with p =probability$\{X_k$ is even$\}$. Form the sum S_n of the random variables. Show that the probability that the sum is even is
$$[1 + (2p-1)^n]/2\,.$$

4. Show that there do not exist four points in the Euclidean plane such that the pairwise distances between them are all odd integers.

5. If $\{a_n\}$ is a sequence of positive integers such that
$$\lim_{n\to\infty} \frac{a_n}{a_1 + a_2 + \cdots + a_n} = 0\,,$$
show that there is a sequence $\{b_n\}$ of positive integers such that for every positive integer $n \geq 2$
$$\frac{b_n}{b_1 + b_2 + \cdots + b_n} \leq \frac{1}{3}\,,$$
and for some positive integer N we have $a_n = b_n$ for all $n \geq N$.

6. For $a > 1$ evaluate
$$\int_0^a x a^{(-\lfloor \log_a x \rfloor)} dx\,,$$
where $\lfloor t \rfloor$ denotes the greatest integer less than or equal to t.

7. Let $ABCD$ be a cyclic quadrilateral, inscribed in a circle ω. Let A', B', C', D' be the points where the tangents at A and B, at B and C, at C and D and at D and A, respectively, intersect. Prove that the lines AC, BD, $A'C'$ and $B'D'$ are concurrent; that is, they intersect at one point.

8. The expression
$$\underbrace{(\ldots(((x-2)^2-2)^2-2)^2-\ldots-2)^2}_{n\,-\,\text{times}}$$
is multiplied out and coefficients of equal powers are collected. Find the coefficient of x^2.

9. Let $f(n) = 2n^2 + 14n + 25$. We see that $f(0) = 25 = 5^2$. Find two positive integers n such that $f(n)$ is a perfect square.

The winning teams were:

1. Ian Caines and Alex Fraser — Dalhousie University;
2. Dave Morgan and Shannon Sullivan — Memorial University;
3. Tara Small and Kit Yan Wong — University of New Brunswick.

All six students received a free subscription to ***CRUX with MAYHEM***, in addition to some other prizes.

We will publish solutions later this year. Will your name be attached to a solution? Send them to me as soon as you can!

Thanks to Karl Dilcher, Dalhousie University, Halifax, Nova Scotia, for sending me the LaTeX file.

THE OLYMPIAD CORNER

No. 195

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

Here it is, the start of another year. How time flies. It is time to recognize all the individual contributions which go to make up the *Olympiad Corner*. First I would like to thank Joanne Longworth who somehow manages to translate my offerings into LaTeX, and to keep me nearly on schedule too. My thanks, too, for the excellent figures prepared by the Editorial Office.

The *Corner* could not exist without its readers who supply me with good Olympiad materials as well as interesting solutions, generalizations, and corrections. I hope that we have missed no one in the following list of contributors for 1998:

Mohammed Aassila
Miguel Amengual Covas
David Arthur
Jamie Batuwantudawe
Francisco Bellot Rosado
Adrian Birka
Mansur Boase
Pierre Bornsztein
Adrian Chan
Sonny Chan
Keon Choi
Jimmy Chui
Filip Crnogorac
F.J. Flanigan
Chen He
Walther Janous
Murray Klamkin
Marcin Kuczma
Michael Lebedinsky
Yin Lei
Pavlos Maragoudakis
Vasiliou Meletis
Richard Nowakowski
Bob Prielipp
Bill Sands
Sree Sanyal
Heinz-Jürgen Seiffert
Toshio Seimiya
Michael Selby
Zun Shan
D.J. Smeenk
Daryl Tingley
Jim Totten
Panos E. Tsaoussoglou
Enrique Valeriano Cuba
Aliya Walji
Edward T.H. Wang

Thanks, and all the best for good problem solving in 1999.

As a first sample of Olympiad materials we give the Second Round 1995 and First Round 1996 Problems of the Bundeswettbewerb Mathematik (the Federal Contest in Mathematics in Germany). My thanks go to Ravi Vakil, Canadian Team Leader to the 37th IMO in Mumbai, India, for collecting them.

BUNDESWETTBEWERB MATHEMATIK
Federal Contest in Mathematics (Germany)
Second Round 1995

1. Starting in $(1/1)$, a stone is moved according to the following rules:

(i) From any point (a/b), it can be moved to $(2a/b)$ or $(a/2b)$.

(ii) From any point (a/b), if $a > b$ it can be moved to $(a - b/b)$, and if $a < b$ it can be moved to $(a/b - a)$.

Determine a necessary and sufficient relation between x and y so that the stone can reach (x/y) after some moves.

2. In a segment of unit length, a finite number of mutually disjoint subsegments are coloured such that no two points with distance 0.1 are both coloured. Prove that the total length of the coloured subsegments is not greater than 0.5.

3. Every diagonal of a given pentagon is parallel to one side of the pentagon. Prove that the ratio of the lengths of a diagonal and its corresponding side is the same for each of the five pairs. Determine the value of this ratio.

4. Prove that every integer k, $(k > 1)$ has a multiple which is less than k^4 and which can be written in decimal representation with at most four different digits.

First Round 1996

1. Is it possible to cover a square of length 5 completely with three squares of length 4?

2. The cells of an $n \times n$-board are numbered according to the example shown for $n = 5$. You may choose n cells, not more than one from each row and each column, and add the numbers in the cells chosen. Which are the possible values of this sum?

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

3. There are four straight lines in the plane, each three of them determining a triangle. One of these straight lines is parallel to one of the medians of the triangle formed by the other three lines. Prove that each of the other straight lines has the same property.

4. Determine the set of all positive integers n for which $n \cdot 2^{n-1}$ is a perfect square.

As a second set this issue, we give the problems of the Grade XII level of the XLV Lithuanian Mathematical Olympiad (1996). My thanks again go to Ravi Vakil for collecting them when he was Canadian Team Leader at the 37$^{\text{th}}$ IMO at Mumbai, India.

XLV LITHUANIAN MATHEMATICAL OLYMPIAD
1996
XII Grade

1. Solve the following equation in positive integers:

$$x^3 - y^3 \;=\; xy + 61\,.$$

2. Sequences $a_1, \ldots, a_n, \ldots$ and $b_1, \ldots, b_n, \ldots$ are such that $a_1 > 0$, $b_1 > 0$, and

$$a_{n+1} \;=\; a_n + \frac{1}{b_n}\,, \qquad b_{n+1} \;=\; b_n + \frac{1}{a_n}\,, \qquad n \in \mathbb{N}\,.$$

Prove that

$$a_{25} + b_{25} \;>\; 10\sqrt{2}\,.$$

3. Two pupils are playing the following game. In the system

$$\begin{cases} *x + *y + *z = 0\,, \\ *x + *y + *z = 0\,, \\ *x + *y + *z = 0\,, \end{cases}$$

they alternately replace the asterisks by any numbers. The first player wins if the final system has a non-zero solution. Can the first player always win?

4. How many sides has the polygon inscribed in a given circle and such that the sum of the squares of its sides is the largest one?

5. Given ten numbers 2, 3, 5, 6, 7, 8, 10, 11, 12 ,13, one must cross out several of them so that the total of any of the remaining numbers would not be an exact square (that is, the sum of any two, three, four,. . . , and of all the remaining numbers would not be an exact square). At most, how many numbers can remain?

And now, shifting to the other side of the globe, we give the problems of Category A and Category B of the Vietnamese Mathematical Olympiad 3/1996. Again our thanks go to Ravi Vakil, Canadian Team Leader at the 37^{th} IMO in Mumbai, India, for collecting the contest materials.

VIETNAMESE MATHEMATICAL OLYMPIAD 3/1996

Category A

First Day — Time: 3 hours

1. Solve the system of equations:

$$\left\{\begin{array}{rcl} \sqrt{3x}\left(1+\frac{1}{x+y}\right) & = & 2\,, \\ \sqrt{7y}\left(1-\frac{1}{x+y}\right) & = & 4\sqrt{2}\,. \end{array}\right.$$

2. Let $Sxyz$ be a trihedron (a figure determined by the intersection of three planes). A plane (P), not passing through S, cuts the rays Sx, Sy, Sz respectively at A, B, C. In the plane (P), construct three triangles DAB, EBC, FCA such that each has no interior point of triangle ABC and $\triangle DAB = \triangle SAB$, $\triangle EBC = \triangle SBC$, $\triangle FCA = \triangle SCA$. Consider the sphere (T) satisfying simultaneously two conditions:

(i) (T) touches the planes (SAB), (SBC), (SCA), (ABC);

(ii) (T) is inside the trihedron $Sxyz$ and is outside the tetrahedron $SABC$.

Prove that the circumcentre of triangle DEF is the point where (T) touches (P).

3. Given two positive integers k and n, $0 < k \leq n$, find the number of the k-arrangements $(a_1, a_2, \ldots, a_k)$ of the first n positive integers $1, 2, \ldots, n$ such that each k-arrangement $(a_1, a_2, \ldots, a_k)$ satisfies at least one of the two conditions:

(i) there exist $s, t \in \{1, 2, \ldots, k\}$ such that $s < t$ and $a_s > a_t$;

(ii) there exists $s \in \{1, 2, \ldots, k\}$ such that $(a_s - s)$ is not divisible by 2.

Second Day — Time: 3 hours

4. Determine all functions $f : N^* \to N^*$ satisfying:

$$f(n) + f(n+1) = f(n+2)f(n+3) - 1996$$

for every $n \in N^*$, (N^* is the set of positive integers).

5. Consider the triangle ABC, the measure of side BC (which is 1) and the measure of angle BAC (which is a given number α $(\alpha > \frac{\pi}{3})$). For triangle ABC, find the distance from the incentre to the centre of gravity of ABC which attains the least value. Calculate this least value in terms of α.

Let $f(\alpha)$ be the least value. When α varies in the interval $(\frac{\pi}{3}, \pi)$, at which value of α does the function $f(\alpha)$ attain its greatest value?

6. We are given four non-negative real numbers a, b, c, d satisfying the condition:

$$2(ab + ac + ad + bc + bd + cd) + abc + abd + acd + bcd \;=\; 16\,.$$

Prove that:

$$a + b + c + d \;\geq\; \frac{2}{3}(ab + ac + ad + bc + bd + cd)\,.$$

When does equality occur?

Category B

First Day — Time: 3 hours

1. Determine the number of real solutions of the system of equations with unknowns x, y:

$$\begin{cases} x^3y - y^4 &= a^2\,, \\ x^2y + 2xy^2 + y^3 &= b^2\,, \end{cases}$$

where a, b are real parameters.

2. Let $ABCD$ be a tetrahedron with $AB = AC = AD$, inscribed in a sphere with centre O. Let G be the centre of gravity of triangle ACD, E be the mid-point of BG and F be the mid-point of AE. Prove that OF is perpendicular to BG if and only if OD is perpendicular to AC.

3. Let us be given n $(n \geq 4)$ numbers a_1, a_2, $\ldots$, a_n, distinct from one another. Determine the number of permutations of these n numbers such that in each permutation, no three of the four numbers a_1, a_2, a_3, a_4 lie in three consecutive positions.

Second Day — Time: 3hours

4. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying simultaneously two conditions:

(i) $f(1995) = 1996$

(ii) for every $n \in Z$, if $f(n) = m$ then $f(m) = n$ and $f(m + 3) = n - 3$, ($\mathbb{Z}$ is the set of integers).

5. Consider triangles ABC, with side BC of length 1, and angle BAC measuring α $(\alpha > \frac{\pi}{3})$. For triangles ABC, does the distance from the incentre to the centre of gravity of ABC attain the least value? Calculate this least value in terms of α.

6. Let x, y, z be three non-negative real numbers satisfying the condition:

$$xy + yz + zx + xyz = 4 .$$

Prove that:

$$x + y + z \geq xy + yz + zx .$$

When does equality occur?

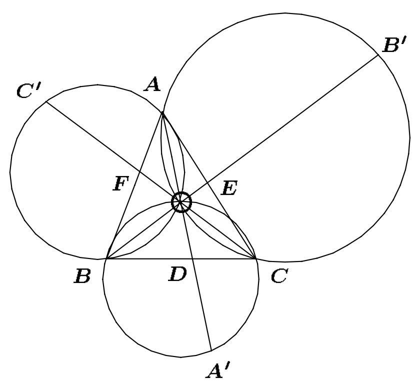
Last number we began giving readers' solutions to problems proposed to the jury but not used at the 37th IMO at Mumbai, India [1997: 450]. We now continue with readers' solutions.

13. Let ABC be an acute-angled triangle with circumcentre O and circumradius R. Let AO meet the circle BOC again in A', let BO meet the circle COA again in B' and let CO meet the circle AOB again in C'. Prove that

$$OA' \cdot OB' \cdot OC' \geq 8R^3 .$$

When does equality hold?

Solution by Toshio Seimiya, Kawasaki, Japan.



Let D, E, F be the intersections of AO, BO, CO with BC, CA, AB, respectively. We denote the area of $\triangle PQR$ by $[PQR]$. Since $\frac{AD}{OD} = \frac{[ABC]}{[OBC]}$, we get

$$\frac{OA}{OD} = \frac{AD}{OD} - 1 = \frac{[ABC] - [OBC]}{[OBC]} = \frac{[OCA] + [OAB]}{[OBC]} .$$

We put $[OBC] = x$, $[OCA] = y$ and $[OAB] = z$. Then we have $\frac{OA}{OD} = \frac{y+z}{x}$; that is $\frac{R}{OD} = \frac{y+z}{x}$.

Similarly we have $\dfrac{R}{OE} = \dfrac{z+x}{y}$ and $\dfrac{R}{OF} = \dfrac{x+y}{z}$. Multiplying these three equations, we have

$$\frac{R^3}{OD \cdot DE \cdot OF} = \frac{(y+z)(z+x)(x+y)}{xyz}. \qquad (1)$$

Since $y+z \geq 2\sqrt{yz}$, $z+x \geq 2\sqrt{zx}$ and $x+y \geq 2\sqrt{xy}$, we have $(y+z)(z+x)(x+y) \geq 8xyz$, with equality holding when $x = y = z$.

Hence from (1),

$$\frac{R^3}{OD \cdot OE \cdot OF} \geq 8. \qquad (2)$$

Since O, B, A', C are concyclic we get

$$\angle OA'C = \angle OBC = \angle OCB = \angle OCD,$$

so that we get $OD \cdot OA' = OC^2 = R^2$. Thus we have $OA' = \frac{R^2}{OD}$.

Similarly, we have $OB' = \frac{R^2}{OE}$, and $OC' = \frac{R^2}{OF}$. Multiplying these three equations we get

$$OA' \cdot OB' \cdot OC' = \frac{R^6}{OD \cdot DE \cdot OF}. \qquad (3)$$

Therefore, we obtain from (2) and (3)

$$OA' \cdot OB' \cdot OC' \geq 8R^3.$$

Equality holds when $[OBC] = [OCA] = [OAB]$; that is when $\triangle ABC$ is equilateral.

16. Four integers are marked on a circle. On each step we simultaneously replace each number by the difference between this number and the next number on the circle, moving in a clockwise direction; that is, the numbers a, b, c, d are replaced by $a-b$, $b-c$, $c-d$, $d-a$. Is it possible after 1996 such steps to have numbers a, b, c, d such that the numbers $|bc-ad|$, $|ac-bd|$, $|ab-cd|$ are primes?

Solution by Pierre Bornsztein, Courdimanche, France.

La réponse est non, après 1996 étapes tout comme après n'importe quel nombre d'étapes.

En effet, si après k étapes ($k \in \mathbb{N}^*$) on obtient a, b, c, d; on appelle m, n, p, q, les entiers précédents avec $a = m-n$, $b = n-p$, $c = p-q$, $d = q-m$, et donc $-d = a+b+c$.

Mais alors,

$$\begin{aligned} bc - ad &= bc + a(a+b+c) &= (a+b)(a+c) \\ ac - bd &= ac + b(a+b+c) &= (a+b)(b+c) \\ ab - cd &= ab + c(a+b+c) &= (a+c)(b+c), \end{aligned}$$

d'où

$$|bc - ad|\ |ac - bd|\ |ab - cd| \ = \ [(a+b)(b+c)(c+a)]^2 \,.$$

Or, le produit de trois nombres premiers ne peut pas être un carré, d'où l'affirmation du départ.

18. Find all positive integers a and b for which

$$\left[\frac{a^2}{b}\right] + \left[\frac{b^2}{a}\right] \ = \ \left[\frac{a^2+b^2}{ab}\right] + ab \,,$$

where, as usual, $[t]$ refers to the greatest integer which is $\leq t$.

Solution by Pierre Bornsztein, Courdimanche, France.
Soient a, b dans $\mathbb{N}^*$ qui vérifient

$$\left[\frac{a^2}{b}\right] + \left[\frac{b^2}{a}\right] \ = \ \left[\frac{a^2+b^2}{ab}\right] + ab \,. \tag{1}$$

Par symétrie des rôles on peut imposer $a \geq b$.

Dans le cas où $a = b$, (1) s'écrit $a + a = 2 + a^2$; c.à.d. $(a-1)^2 + 1 = 0$, qui est impossible.

Si $a > b$, on pose $x = \left[\frac{a^2}{b}\right]$ et $y = \left[\frac{b^2}{a}\right]$, $x, y \in \mathbb{N}$. Remarquons que $x > 1$ car $\frac{a^2}{b} > a > b \geq 1$, a et b entiers. De plus $x \leq \frac{a^2}{b}$ et $y \leq \frac{b^2}{a}$, d'où $xy \leq ab$.

Et alors, d'après (1)

$$\begin{aligned} x + y = \left[\frac{a}{b} + \frac{b}{a}\right] + ab \ &\geq \ 2 + ab \quad \left(\text{car } t + \frac{1}{t} \geq 2\right) \\ &\geq \ 2 + xy \,, \end{aligned}$$

d'où $(x-1)(y-1) \leq -1$ avec $x > 1$ et $y \geq 0$, y entier, et donc $y = 0$. C'est à dire $a > b^2$.

On pose $a = b^2 + c$ avec $c \in \mathbb{N}^*$.

Conséquemment, (1) s'écrit

$$b^3 + 2bc + \left[\frac{c^2}{b}\right] \ = \ b + \left[\frac{c}{b} + \frac{b}{b^2+c}\right] + b^3 + bc \,;$$

c.à.d.

$$b(c-1) + \left[\frac{c^2}{b}\right] \ = \ \left[\frac{c}{b} + \frac{b}{b^2+c}\right] \,. \tag{2}$$

Puisque

$$\left[\frac{c^2}{b}\right] \ > \ \frac{c^2}{b} - 1 \ > \ \frac{c}{b} - 1$$

et

$$\left[\frac{c}{b}+\frac{b}{b^2+c}\right] \leq \frac{c}{b}+\frac{b}{b^2+c} \leq \frac{c}{b}+1\,,$$

nous avons à partir de (2) $b(c-1)+\frac{c}{b}-1<\frac{c}{b}+1$, donc $b(c-1)<2$ avec $b\geq 1$ et $c\geq 1$ et alors $c\leq 2$.

Lorsque $c=2$, $b=1$ et donc $a=b^2+c=3$. Or pour $a=3$, $b=1$, $\left[\frac{a^2}{b}\right]+\left[\frac{b^2}{a}\right]=9$ et $\left[\frac{a}{b}+\frac{b}{a}\right]+ab=3+3=6$, donc (1) n'est pas vérifiée.

Enfin si $c=1$, $a=b^2+1$, alors

$$\begin{aligned}\left[\frac{a^2}{b}\right]+\left[\frac{b^2}{a}\right] &= \left[\frac{a^2}{b}\right] = b^3+2b+\left[\frac{1}{b}\right]\,, \qquad \text{et donc}\\ \left[\frac{a}{b}+\frac{b}{a}\right]+ab &= b+\left[\frac{1}{b}+\frac{b}{b^2+1}\right]+b^3+b\,.\end{aligned}$$

Pour $b=1$ $\left[\frac{1}{b}\right]=\left[\frac{1}{b}+\frac{b}{b^2+1}\right]=1$, d'où (1) est vraie.

Pour $b>1$ $\frac{1}{b}\leq\frac{1}{2}$ et $\frac{b}{b^2+1}<\frac{1}{2}$, donc $\left[\frac{1}{b}\right]=0=\left[\frac{1}{b}+\frac{b}{b^2+1}\right]$, et donc (1) est vérifiée.

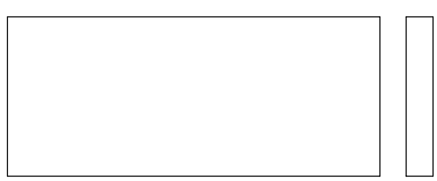
Par suite, les couples-solutions sont les couples de la forme (b^2+1,b) ou (b,b^2+1) avec $b\in\mathbb{N}^*$.

Another solution has arrived to a problem of the Irish Mathematical Olympiad 1994 given in the November 1997 *Corner* [1997: 388] and for which some solutions appeared last issue [1998: 456].

4. Consider the set of $m\times n$ matrices with every entry either 0 or 1. Determine the number of such matrices with the property that the number of "1"s in each row and in each column is even.

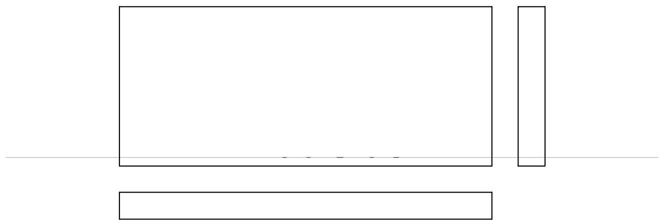
Solution by Felipe Gago, Universidade de Santiago de Compostela, Spain.

Given any $m\times n$ matrix whose entries are all 0 or 1, we can construct an $m\times(n+1)$ whose entries are also 0 or 1 and whose rows have an even number of 1's.



Indeed, in each row we must enter a 1 if the number of 1's was odd, and a 0 otherwise. If we choose the number of the new column, say the last one, this process is unique.

Similarly we can construct an $(m+1)\times n$ matrix whose entries are also 0 or 1 and whose columns have each an even number of 1's.



If we combine both processes, given an $m \times n$ matrix, we obtain a configuration like the one depicted above in which all complete rows and columns have an even number of 1's. Therefore, the $m \times (n+1)$ matrix and the $(m+1)\times n$ matrix contain both an even number of 1's, and so the number of 1's in the new column and in the new row have the same parity. By adding a 1, if they are odd, or a 0, if they are even, we obtain an $(m+1) \times (n+1)$ matrix with the required property. The uniqueness of the processes tells us that the number of such $(m+1) \times (n+1)$ matrices equals the number of $m \times n$ matrices whose entries are 0 or 1; that is, equals $2^{m\cdot n}$, and so the solution to our problem is $2^{(m-1)\cdot(n-1)}$.

We next turn to solutions by our readers to problems of the Croatian National Mathematics Competition (4th Class), May 13, 1994 [1997: 454].

1. One member of an infinite arithmetic sequence in the set of natural numbers is a perfect square. Show that there are infinitely many members of this sequence having this property.

Solutions by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; by Bob Prielipp, University of Wisconsin–Oshkosh, Wisconsin, USA; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution and comment.

Let the arithmetic sequence be $\{a + nd : n = 0, 1, 2, \ldots, \}$ where $a, d \in \mathbb{Z}$, $a, d > 0$. Suppose $a + kd = q^2$, for some $k, q \in \mathbb{Z}$, $k \geq 0$, $q > 0$. Then

$$\begin{aligned} a + (k + 2q^2 + dq^2)d &= a + kd + (d^2 + 2d)q^2 \\ &= q^2 + (d^2 + 2d)q^2 \\ &= ((d+1)q)^2\,, \end{aligned}$$

showing that $a + (k + 2q^2 + dq^2)d$ is also a perfect square. The conclusion then follows by iteration.

Remark: The iteration described above by no means exhausts all the perfect squares in the sequence. For example, if $d \geq 2$ then by the same method, we see that if $a+kd = q^2$, then $a+(k-2q^2+dq^2)d = ((d-1)q)^2$. For example, if the arithmetic sequence is $\{1+3n : n = 0, 1, 2, \ldots, \}$, then starting with 1, the first perfect square, we would get, using the iteration given in our solution, the sequence of perfect squares 1^2, 4^2, 16^2, 64^2, ... If we use the second iteration mentioned above, then we would get the "refinement" 1^2, 2^2, 4^2, 8^2, ... Note, however, that neither of these two sequences includes the perfect square 49 which is a member of the given arithmetic sequence.

Solution and Generalization of Klamkin.

If the square term is a^2 and the common difference is d, then the terms of the progression $a^2 + d(2an + n^2 d)$ for all natural n are perfect squares $(a+nd)^2$.

More generally, if there is a term of the form $P(a)$ in the arithmetic progression, where P is an integral polynomial, (e.g. $2a^7 + 3a^6 - 1$), then the terms $P(a) + d\left(\frac{P(a+nd)-P(a)}{d}\right)$ are of the same form for all natural n.

2. For a complex number z let $w = f(z) = \dfrac{2}{3-z}$.

(a) Determine the set $\{w : z = 2 + iy, y \in \mathbb{R}\}$ in the complex plane.

(b) Show that the function w can be written in the form

$$\frac{w-1}{w-2} = \lambda\frac{z-1}{z-2}.$$

(c) Let $z_0 = \frac{1}{2}$ and the sequence (z_n) be defined recursively by

$$z_n = \frac{2}{3 - z_{n-1}}, \qquad n \geq 1.$$

Using the property (b) calculate the limit of the sequence (z_n).

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

(a) Let $w = u + iv$, where $u, v \in R$. Then from $w = \frac{2}{1-iy} = \frac{2(1+iy)}{1+y^2}$ we get $u = \frac{2}{1+y^2}$ and $v = \frac{2y}{1+y^2}$. Eliminating y, we get $u^2 + v^2 = \frac{4}{1+y^2} = 2u$, or $(u-1)^2 + v^2 = 1$. Hence $\{w : z = 2 - iy, y \in \mathbb{R}\}$ is the set of all points on the circle centred at $(1, 0)$ with radius 1.

(b) Straightforward computations show that

$$\frac{w-1}{w-2} = \left(\frac{z-1}{3-z}\right) \div \left(\frac{2z-4}{3-z}\right) = \frac{1}{2}\left(\frac{z-1}{z-2}\right);$$

that is, $\lambda = \frac{1}{2}$.

(c) Using the equation in (b), we find, by iteration, that

$$\frac{z_n - 1}{z_n - 2} = \frac{1}{2}\left(\frac{z_{n-1}-1}{z_{n-1}-2}\right) = \left(\frac{1}{2}\right)^n \left(\frac{z_0 - 1}{z_0 - 2}\right) = \frac{1}{3}\left(\frac{1}{2}\right)^n.$$

Since $(\frac{1}{2})^n \to 0$ as $n \to \infty$, we find $\lim_{n\to\infty} z_n = 1$.

Remark: One could also solve for z_n from the equation in (b) and obtain $1 + \frac{1}{z_n - 2} = \frac{1}{3}(\frac{1}{2})^n$ or $z_n = 2 + (\frac{1}{3}(\frac{1}{2})^n - 1)^{-1}$. This explicit expression for z_n can also be proved directly by induction.

3. Determine all polynomials $P(x)$ with real coefficients such that for some $n \in \mathbb{N}$ we have $xP(x-n) = (x-1)P(x)$, for all $x \in \mathbb{R}$.

Solutions by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution of Wang.

The only such polynomials are $P(x) = cx$, where c is an arbitrary constant. Clearly $P(x) \equiv 0$ satisfies the given equation. So assume $P(x) \not\equiv 0$. Setting $x = 0$ in $xP(x-n) = (x-1)P(x)$, we get $P(0) = 0$ and thus $(n-1)P(n) = 0$. If $n \neq 1$, then $P(n) = 0$ and a straightforward induction shows that $P(kn) = 0$ for all $k \in \mathbb{N}$, which is impossible since $P(x) \not\equiv 0$. Hence $n = 1$ and we have $xP(x-1) = (x-1)P(x)$ which implies $P(2) = 2P(1)$. Suppose that $P(m) = mP(1)$ for some $m \geq 2$. Then from $(m+1)P(m) = mP(m+1)$, we get $P(m+1) = (m+1)P(1)$. Hence $P(m) = mP(1) = mc$ for all $m \in N$, where $c = P(1)$. Let $Q(x) = P(x) - cx$. Then $Q(m) = P(m) - cm = 0$ for all $m \in N$ and so $Q(x) \equiv 0$ from which $P(x) \equiv cx$. Noting that $c = 0$ if and only if $P(x) \equiv 0$, the conclusion follows.

4. In the plane five points P_1, P_2, P_3, P_4, P_5 are chosen having integer coordinates. Show that there is at least one pair (P_i, P_j), for $i \neq j$ such that the line P_iP_j contains a point Q, with integer coordinates, and is strictly between P_i and P_j.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

With respect to parity, there are only 4 kinds of points (x, y). They are (E, E), (E, O), (O, E) and (O, O), where E denotes an even integer, and O denotes an odd integer. Since there are 5 points, there must be at least two (x_1, y_1) and (x_2, y_2) such that x_1 and x_2 have the same parity and also y_1 and y_2. Hence Q can be taken as the mid-point of these two points.

We now give a solution to one of the problems of the *Additional Competition for the Olympiad* of the Croatian National Mathematical Competition, May 14, 1994 [1997: 454].

2. Construct a triangle ABC if the lengths $|AO|$, $|AU|$ and radius r of incircle are given, where O is orthocentre and U the centre of the incircle.

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give the solution by Amengual Covas.

Denote the orthocentre, the incentre and the circumcentre of triangle ABC by H, I, and O, respectively.

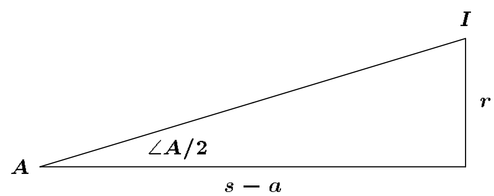


Figure 1

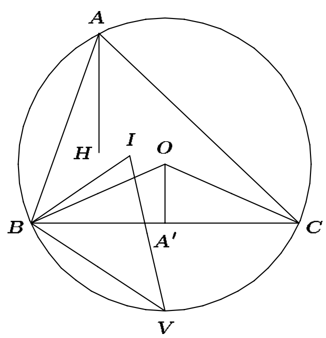


Figure 2

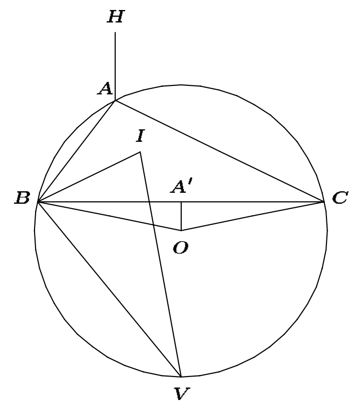


Figure 3

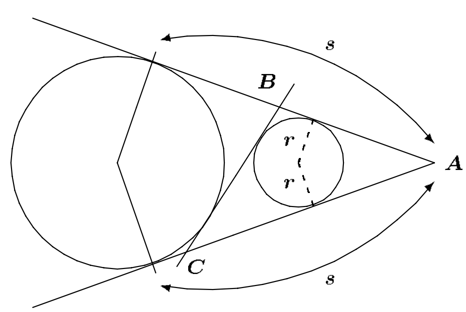


Figure 4

Analysis. Imagine the construction completed (Figures 2 and 3), with A' the mid-point of side BC and V the midpoint of arc BC. It is a standard exercise (for example #1.8 and #3.4 in Trajan Lalesco, La Géometrie du Triangle) to show that

$$OA' = \frac{1}{2}AH \qquad \text{and} \qquad BV = VI\,.$$

Geometrical Construction. Since AI and r are given, $\angle A$ and $s-a$ are known (Figure 1).

If $\angle A = 90°$ (or equivalently, $AI = r\sqrt{2}$), then A and H coincide and there are an infinite number of triangles circumscribing the r circle.

Suppose $\angle A \neq 90^\circ$. In *isosceles* triangle BOC we then know

$$\angle BOC = \begin{cases} 2\cdot\angle A & \text{if } \angle A < 90^\circ \\ & \text{(or, equivalently } AI > r\sqrt{2}) \\ 2\cdot(180^\circ - \angle A), & \text{if } \angle A > 90^\circ \\ & \text{(or, equivalently } AI < r\sqrt{2}), \end{cases}$$

and altitude OA' $(= \frac{1}{2}AH)$, whence $\triangle BOC$ and the circumcircle of $\triangle ABC$ can be constructed. Then,

Solution 1. Triangle BOC and the circumcircle of $\triangle ABC$ being constructed, draw XY parallel to BC and a distance r from it (on the same side as O if $\angle A < 90^\circ$ and on the opposite side if $\angle A > 90^\circ$). Bisect the arc BC at V and let the circle, centre V, radius VB, cut XY at I and I'. These points will be the incentres for the two (symmetrical) solutions. Let VI meet the circumcircle again at A. Then ABC is the required triangle.

Solution 2. Triangle BOC being constructed, we then know $\angle A$, r and s $(= a + (s - a))$ where s is the semiperimeter of $\triangle ABC$ and $a = \overline{BC}$.

These elements suffice to determine the incircle and the excircle lying in $\angle A$, which are readily constructed.

The common internal tangents of these circles intersect the sides of the angle A at B and C to form the required triangle (Figure 4).

The details of the construction may be omitted since they are obvious.

We finish this number of the *Corner* with a more direct solution than the one published in the [1997: 460]. There are also some generalizations.

18. Show that in a non-obtuse triangle the perimeter of the triangle is always greater than two times the diameter of the circumcircle.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

Here we give a more direct solution and some generalizations.

If we also allow degenerate triangles, then the inequality here is

$$a + b + c \geq 4R,$$

and since $a = 2R\sin A$, etc., we equivalently have the known inequality

$$\sin A + \sin B + \sin C \geq 2 \tag{1}$$

for non-obtuse triangles. Also, by using the pedal angle transformation, $A = (\pi - A')/2$, etc., (1) becomes

$$\cos A'/2 + \cos B'2 + \cos C'/2 \geq 2 \tag{1'}$$

for all triangles $A'B'C'$.

The following generalization will include (1). We consider the same problem where the maximum angle of the triangle is given as θ and we have two cases:

(i) $\theta > \pi/2$,

(ii) $\theta \le \pi/2$.

To establish our inequalities, we use the powerful majorization inequality. Here we say a vector (x, y, z) majorizes another vector (x', y', z') if $x \ge y \ge z$, $x' \ge y' \ge z'$ and $x \ge x'$, $x+y \ge x'+y'$, $x+y+z = x'+y'+z'$ and we write this as $(x, y, z) \succ (x', y', z')$. Then for any convex function F, we have the inequality

$$F(x) + F(y) + F(z) \;\ge\; F(x') + F(y') + F(z')\,,$$

and for a concave function like $\sin t$, the inequality is reversed.

Case (i). Here $(\theta, \pi - \theta, 0) \succ (A, B, C) \succ (\pi/3, \pi/3, \pi/3)$, so that

$$3\sin\pi/3 \;\ge\; \sin A + \sin B + \sin C \ge 2\sin\theta\,.$$

Case (ii). Here $(\theta, \theta, \pi - 2\theta) \succ (A, B, C) \succ (\pi/3, \pi/3, \pi/3)$, so that

$$3\sin\pi/3 \;\ge\; \sin A + \sin B + \sin C \ge 2\sin\theta + \sin 2\theta\,.$$

If we now let $\theta = \pi/2$, we recapture (1) and there is equality only for the degenerate triangle of angles $\pi/2$, $\pi/2$, 0.

We now consider the same original problem where the minimum angle of the triangle is given as φ. Then since

$$(\pi - 2\varphi, \varphi, \varphi) \;\succ\; (A, B, C) \;\succ\; (\{\pi - \varphi\}/2, \{\pi - \varphi\}/2, \varphi)\,,$$

$$2\cos\varphi/2 + \sin\varphi \;\ge\; \sin A + \sin B + \sin C \;\ge\; \sin 2\varphi + 2\sin\varphi\,.$$

What is done above in restricting one of the angles can be applied to generalize other triangle inequalities; for example,

$$\begin{aligned}
&3/2 \;\ge\; \sin A/2 + \sin B/2 + \sin C/2 \;\ge\; 1\,,\\
&3\sqrt{3}/8 \;\ge\; \sin A \sin B \sin C \;\ge\; 0\,,\\
&2\sqrt{3}/9 \;\ge\; \sin A \sin B \sin C/2 \;\ge\; 0\,,\\
&3\sqrt{3}/8 \;\ge\; \cos A/2 \cos B/2 \cos C/2 \;\ge\; 0\,,\\
&\tan A/2 + \tan B/2 + \tan C/2 \;\ge\; \sqrt{3}\,,\\
&\cot A/2 + \cot B/2 + \cot C \;\ge\; 3\sqrt{3}\,.
\end{aligned}$$

That concludes the *Olympiad Corner* for this issue. Send me your Olympiad Contests and your nice solutions.

BOOK REVIEWS

ALAN LAW

First, we must introduce our new Book Reviews Editor, Alan Law.

Alan is a Professor in the Department of Computer Science, Faculty of Mathematics, University of Waterloo. He was previously Dean of Science at Memorial University, St. John's, Newfoundland, and before that, Head of the Department of Computer Science at the University of Regina, Regina, Saskatchewan. He has also served on the Canadian Mathematical Olympiad Committee.

During his 37 years as a full-time academic, he has authored or co-authored over 45 papers in refereed journals and over 30 refereed conference proceedings papers, as well as having edited or co-edited three books. He has given lectures all over the world at universities and research institutes. He was also instrumental in persuading the present Editor-in-Chief to take on the position.

Since Andy Liu has, as usual, been very efficient in getting books reviewed, the next few issues will, in fact, be from him.

Over and Over Again, by Genghe Chang and Thomas Sederberg,
published by the Mathematical Associatlon of America, 1997,
P.O. Box 91112, Washington DC 20090-1112.
ISBN # 0-88385-641-7, softcover, 309+ pages, $31.50.
Reviewed by **Murray S. Klamkin**, *University of Alberta, Edmonton, Alberta.*

This book is #39 in the excellent well edited New Mathematical Library series of the MAA. It consists of 32 chapters dealing mainly with varied repeated transformations (maps) over and over again; that is, iterations of various functions.

The first eighteen chapters require secondary school mathematics with a few exceptions. Also included here are geometry, complex numbers, vectors, and inequalities. Most of the problems at the end of these chapters come from mathematical olympiads from various countries, and there are hints and solutions at the end of the book for them. This part of the book should be interesting and very suitable for students studying for mathematical competitions. The following is a small sample of some of these problems.

1. We start a list of numbers with 1 and 2. The operation **O** consists of taking all pairs a and b of different numbers already listed and adding the numbers $a + b + ab$ to the list. Give a simple description of the numbers which will be in the list if we repeat **O** infinitely often.

2. Let $d(n)$ be the number of 1's in the base 2 representation of n. Show that the number of odd binomial coefficients $\binom{n}{i}$ equals $d(n)$.

3. Five points A, B, C, U and V lie in the same plane. If triangles AUV, UBV and UVC are directly similar to each other, show that triangle ABC is directly similar to each of them.

4. Show that the equation $z^{n+1} - z^n - 1 = 0$ (n a positive integer) has a solution on the unit circle in the complex plane if and only if $6|(n+2)$.

5. Show that a circular disk D can never be covered by two smaller circular disks D_1 and D_2.

The last fourteen chapters are more advanced and so are more suitable for college students. They deal with functional iteration, Chebyshev polynomials, chaos, Bernstein polynomials, Bezier curves, surfaces and splines. As before, there are also problems at the end of these chapters with hints and solutions at the end of the book. This part of the book is a good supplement for study in calculus, numerical analysis, and computer-aided geometric design. The following is a small sample of the problems from these chapters.

6. Determine $\lim_{n\to\infty} a_n$ where $a_{n+1} = \frac{a_n^2}{a_n - 1}$, $n = 0, 1, \ldots$ and $a_0 = 2$. (The solution given indicates how to determine a_n explicitly.)

7. Let $f_{n+1} = f(f_n)$, where $f_0 = x$ and $f(x) = 4\left(x - \frac{1}{2}\right)^2$. Show that for any positive integer p, there is an x in $[0,1]$ such that the sequence f_n is periodic of least period p.

8. Show that if $\{a_i\}$, $i = 0, 1, \ldots$ is an arithmetic progression, then $\sum_{i=1}^{n} a_i \binom{n}{i} x^i (1-x)^{n-i}$ is either a constant or a polynomial of degree 1 for any positive integer n.

9. Show that the parametric equations $x = P_1(t)$ and $y = P_2(t)$, where P_1 and P_2 are polynomials, cannot represent the arc of a circle exactly. Find a pair of rational functions which does represent an arc of a circle.

10. Show that a polynomial $P(x)$ of degree at most m can be represented uniquely in the form $P(x) = a_0 x^n + a_1(x+1)^n + \cdots + a_n(x+n)^n$, where $n \le m$.

Generalisations of a Four-Square Theorem

Hiroshi Okumura and John F. Rigby

In the 17th–19th centuries, Japanese people often wrote their mathematical results on a wooden board and dedicated it to a shrine or a temple. Then the board was hung under the roof there. Such a board is called a **sangaku**. In this paper, we shall discuss generalisations of a sangaku problem involving four squares, which we state as the following theorem (see Figure 1.). [*Ed.* see ***CRUX*** problem **1496** [1989: 298; 1998: 56].]

Theorem 1. [1, p.47] *Let $CAEA'$, $CBGB'$, $ADFB$, $A'D'F'B'$ be squares as in Figure 1. Then D, E, D' are collinear if and only if $CAEA'$ is half the size of $CBGB'$.*

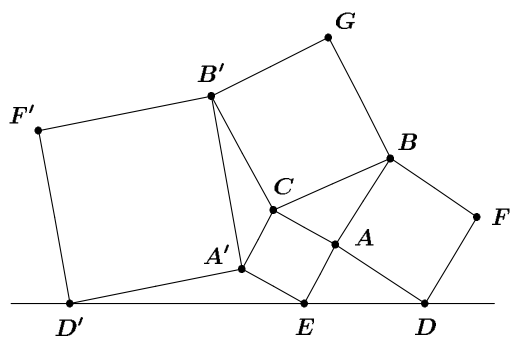


Figure 1.

The angle CBA, denoted by $\angle CBA$, is the angle between $\overrightarrow{BC}$ and $\overrightarrow{BA}$; it is positive if the direction of rotation from $\overrightarrow{BC}$ to $\overrightarrow{BA}$ is anticlockwise, and negative otherwise. The following lemma is the key of our generalizations (see Figure 2).

Lemma 1. *If the vertices of two triangles ABC and AED lie anticlockwise in this order, and if they satisfy*

$$\frac{BC - CA}{AB} = \frac{AE}{DA}, \qquad \angle BAC + \angle EAD = \pi, \tag{1}$$

then $2\angle DEA = \pi - \angle ACB$.

Proof. Let f be a dilative rotation with centre A mapping B into D. If we denote the ratio of magnification by λ, then

$$\lambda = \frac{DA}{AB} = \frac{AE}{BC - CA}$$

Research supported in part by NFS grant DMS-9300657.

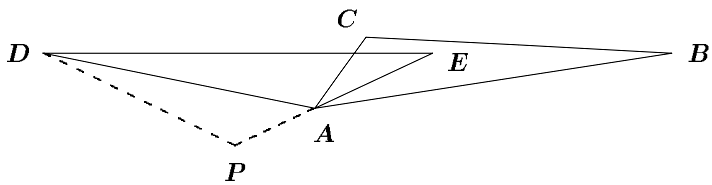


Figure 2.

by the first part of (1). Let $P = f(C)$. Then A lies on the segment PE since $\angle DAP + \angle EAD = \pi$ by the second part of (1), while

$$\begin{aligned} PE &= PA + AE = \lambda CA + AE = \frac{CA \cdot AE}{BC - CA} + AE \\ &= \frac{BC \cdot AE}{BC - CA} = \lambda BC = DP\,. \end{aligned}$$

Hence EPD is an isosceles triangle with summit P. Therefore, we get $2\angle DEA = \pi - \angle APD = \pi - \angle ACB$. ∎

Reflecting Figure 2 in a line, we get a similar lemma with opposite orientations: *If the vertices of two triangles ABC and AED lie clockwise in this order, and if they satisfy*

$$\frac{BC - CA}{AB} = \frac{AE}{DA}, \qquad \angle BAC + \angle EAD = -\pi\,,$$

then $2\angle DEA = -\pi - \angle ACB$.

Lemma 2. *If* $\angle EAC = \pm\pi/2$ *in* Lemma 1, *then the angle bisector of* $\angle BCA$ *is parallel to* DE.

Proof. In this situation, the triangle ADP is obtained from the triangle ABC by a dilative rotation with centre A and angle of rotation $\pm\pi/2$. ∎

In the following examples, we shall agree that $ACA'E$, $CBGB'$, $ADFB$ and $A'B'F'D'$ have anti-clockwise orientations. Complete proofs will be given later.

Example 1. (Figure 3) The ratio of the sides of the two squares $ACA'E$ and $CBGB'$ is $n : n{+}1$ (n is an integer). The two rectangles consist of n squares as in the figure. Let $AE = na$; then $BC = (n+1)a$. Hence $(BC - CA)/AB = a/AB = (na)/(nAB) = AE/DA$. And we can show that D, E, D' are collinear and that this line is parallel to the angle bisectors of $\angle ACB$ and $\angle A'CB'$.

Example 2. (Figure 4) Four similar rectangles where the ratio of the sides of $CAEA'$ and $CBGB'$ is $1 : 2$. Then D, E, D' are collinear, and this line is parallel to the angle bisectors of $\angle ACB$ and $\angle A'CB'$ (this is a special case of Example 4 below).

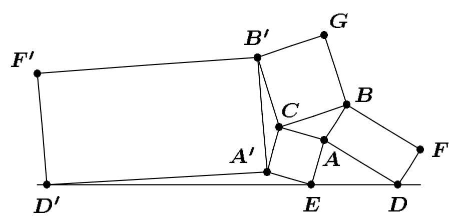


Figure 3. $CA : CB = n : (n{+}1)$ $(n = 2)$.

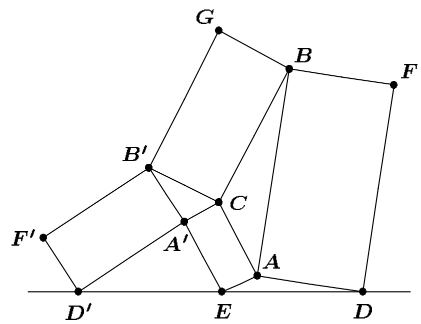


Figure 4. $CA : CB = 1 : 2$

Example 3. (Figure 5) Four similar rectangles where $CA : AE = s : a$ and $CA : BC = s^2 : s^2{+}1$. We get $BC = (s^2+1)CA/s^2 = (s^2+1)CA'/s$, and

$$\frac{BC - CA}{AB} = \frac{(s^2+1)CA/s^2 - CA}{DA/s} = \frac{CA/s}{DA} = \frac{AE}{DA},$$

as well as

$$\frac{B'C - CA'}{A'B'} = \frac{sBC - CA'}{A'B'} = \frac{(s^2+1)CA' - CA'}{sD'A'} = \frac{sCA'}{D'A'} = \frac{A'E}{D'A'}.$$

And we can show that D, E, D' are collinear, and that this line is parallel to the angle bisectors of $\angle ACB$ and $\angle A'CB'$.

Example 4. (Figure 6) Four similar cyclic quadrilaterals. The cyclic quadrilaterals $CAEA'$, $CBGB'$, $BADF$, $B'F'D'A'$ are similar, with $\angle CAE = \pi/2$ and $BC = 2CA$. In the figure, we get $(BC - CA)/AB = CA/AB = AE/DA$. Further, we can show that D, E, D' are collinear, and this line is parallel to the angle bisectors of $\angle ACB$ and $\angle A'CB'$.

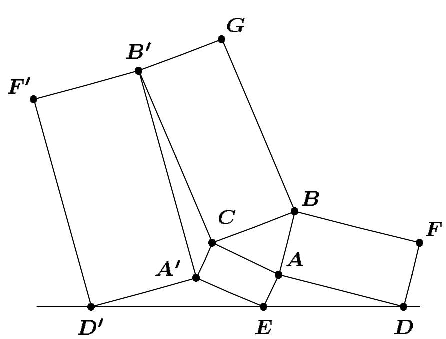


Figure 5.

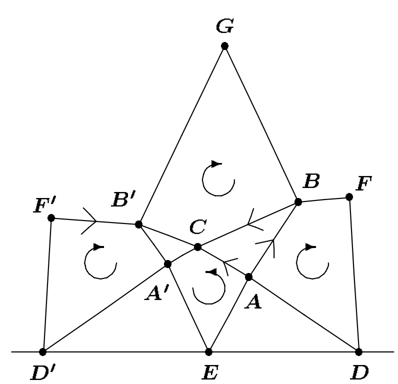


Figure 6.

Example 5. (Figure 7) Four similar harmonic cyclic quadrilaterals: $CAEA'$ is a cyclic quadrilateral with the property $CA{\cdot}EA' = AE{\cdot}A'C$. The quadrilaterals $CBGB'$ $BFDA$, $B'A'D'F'$ are similar to $CAEA'$, and

$BC = 2CA$. In the figure,

$$\frac{BC - CA}{AB} = \frac{CA}{AB} = \frac{AE \cdot A'C}{EA'} \frac{FD}{DA \cdot BF} = \frac{AE}{DA} \frac{(A'C/EA')}{(BF/FD)}$$
$$= \frac{AE}{DA} \frac{(CA/AE)}{(BF/FD)} = \frac{AE}{DA},$$

and we can show that D, E, D' are collinear.

In example 4, the quadrilaterals $CBGB'$, $BADF$, $B'F'D'A'$ are directly similar, and $CAEA'$ is oppositely similar to the other three, whereas, in example 5, it is $CBGB'$ that is oppositely similar to the other three quadrilaterals.

In all the above examples, the triangles ABC and AED satisfy the hypotheses of Lemma 1 (see Figure 8). Also, the triangles $A'B'C$ and $A'ED'$ satisfy the hypotheses of the reflected lemma. Thus we get $\angle DEA = \pi/2 - \angle ACB/2$, and $\angle D'EA' = -\pi/2 - \angle A'CB'/2$. Hence we get

$$\begin{aligned} \angle DED' &= \angle DEA + \angle AEA' + \angle A'ED' \\ &= \tfrac{1}{2}(\pi - \angle ACB) + (\pi - \angle A'CA) + \tfrac{1}{2}(\pi + \angle A'CB') \\ &= 2\pi - \tfrac{1}{2}(\angle ACB + 2\angle A'CA + \angle B'CA'). \end{aligned}$$

But since $\angle A'CA = \angle BCB'$, we have

$$\begin{aligned} &\tfrac{1}{2}(\angle ACB + 2\angle A'CA + \angle B'CA') \\ &\quad = \tfrac{1}{2}(\angle ACB + \angle BCB' + \angle B'CA' + \angle A'CA) \equiv 0 \pmod{\pi}. \end{aligned}$$

Hence $\angle DED' \equiv 0 \pmod{\pi}$. This implies that D, E, D' are collinear. Also, examples 1, 2, 3, and 4 satisfy the hypotheses of Lemma 2, and DD' is parallel to the angle bisector of $\angle ACB$ in all four examples.

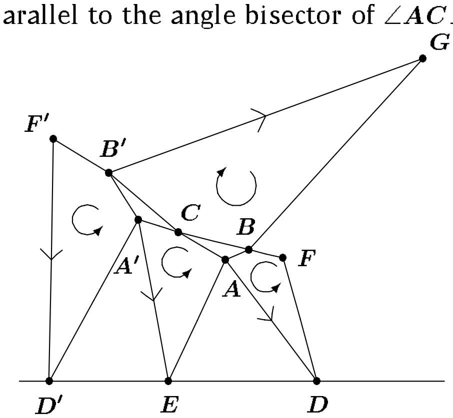


Figure 7.

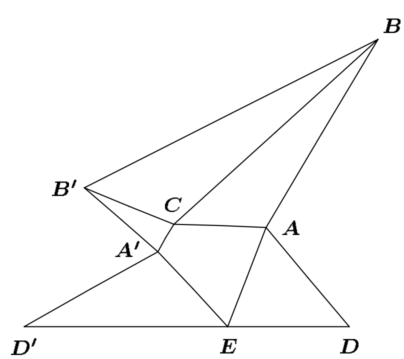


Figure 8.

Example 6. (Figures 9A and 9B) Four regular $2n$–gons. The smallest one is half the size of the one above. The collinearity described in the figure is obtained from example 4.

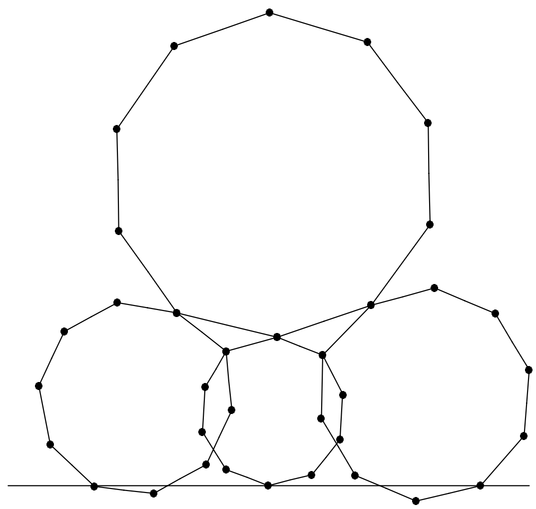


Figure 9A ($n = 5$).

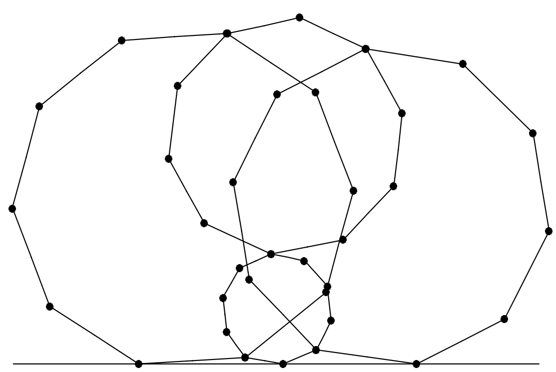


Figure 9B ($n = 5$).

Reference

[1.] H. Fukagawa and Pedoe, D., *Japanese Temple Geometry Problems*, Charles Babbage Research Centre, Winnipeg, Canada, 1989.

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THE SKOLIAD CORNER

No. 35

R.E. Woodrow

We begin this issue with the problems of the first round of the Old Mutual Mathematical Olympiad 1992. Thanks go to John Grant McLoughlin of the Faculty of Education, Memorial University of Newfoundland, for collecting this contest and forwarding it for use in the *Corner*.

OLD MUTUAL MATHEMATICAL OLYMPIAD 1992

Time: 1 hour

1. $(0.4)^2 - (0.1)^2$ equals:

(a) 0.04 (b) 1.5 (c) 0.15 (d) 0.6 (e) 0.06

2. The angles of a triangle are in the ratio $2 : 3 : 4$. The size of the largest angle in degrees is:

(a) $40°$ (b) $80°$ (c) $45°$ (d) $90°$ (e) $72°$

3. Reduced to the lowest terms $\frac{a^2-b^2}{ab} - \frac{ab-b^2}{ab-a^2}$ is equal to:

(a) $\frac{a^2-2b^2}{ab}$ (b) $a - 2b$ (c) a^2 (d) $\frac{a}{b}$ (e) none of these

4. If the radius of a circle is increased by 100%, the area is increased by:

(a) 100% (b) 200% (c) 300% (d) 400% (e) 10000%

5. The area of the largest triangle that can be inscribed in a semicircle of radius r is:

(a) r^2 (b) r^3 (c) $2r^2$ (d) $2r^3$ (e) $\frac{1}{2}r^2$

6. A manufacturer built a machine which will address 500 envelopes in 8 minutes. He wishes to build another machine so that when both are operating together they will address 500 envelopes in 2 minutes. The equation used to find how many minutes x, it would require the second machine to address 500 envelopes alone is:

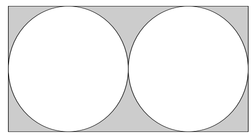
(a) $8 - x = 2$ (b) $\frac{500}{8} + \frac{500}{x} = 500$ (c) $\frac{1}{8} + \frac{1}{x} = \frac{1}{2}$

(d) $\frac{x}{2} + \frac{x}{8} = 1$ (e) none of these.

7. The expression $\sqrt{\frac{4}{3}} - \sqrt{\frac{3}{4}}$ is equal to:

(a) $\sqrt{\frac{7}{12}}$ (b) $\sqrt{-1}$ (c) $\frac{1}{2\sqrt{3}}$ (d) $\sqrt{\frac{1}{7}}$ (e) -1

8. Two circles of equal size are contained in a rectangle as shown.



If the radius of each circle is 1 cm, then the area of the shaded portion in cm^2 is:

(a) $\pi - 4$ (b) $4 - 2\pi$ (c) $8 - \pi$ (d) $8 - 2\pi$ (e) 4

9. The yearly changes in the population of a town for four consecutive years are respectively 10% increase, 10% increase, 10% decrease, 10% decrease. The net change over four years to the nearest percent is:

(a) -2 (b) -1 (c) 0 (d) 1 (e) 12

10. If $\log_{10} 2 = a$ and $\log_{10} 3 = b$, then $\log_{10} 12$ equals:

(a) $a^2 + b$ (b) $2a + b$ (c) $4b$ (d) $2ab$ (e) a^2b

11. Of the following, which is the best approximation to the positive square root of $\frac{1992}{10000}$:

(a) 0.0045 (b) 0.014 (c) 0.0446 (d) 0.1411 (e) 0.4463

12. A circle and a square have the same perimeter. Then:

(a) their areas are equal
(b) the area of the circle is $\frac{4}{\pi}$ times that of the square
(c) the area of the circle is $\frac{2}{\pi}$ times that of the square
(d) the area of the circle is π times the area of the square
(e) none of these

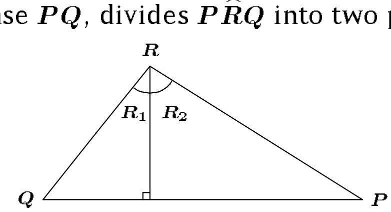
13. The square of an integer is called a *perfect square*; for example, 4, 9, 16, 25 are all perfect squares. If n is a perfect square, then the next perfect square greater than n is:

(a) $n + 1$ (b) $n^2 + 1$ (c) $n^2 + 2n + 1$ (d) $n^2 + n$ (e) $n + 2\sqrt{n} + 1$

14. The last digit of 7^{1992} is:

(a) 1 (b) 2 (c) 6 (d) 7 (e) 9

15. The figure below shows a triangle PQR, where $P\widehat{Q}R > Q\widehat{P}R$. The altitude to the base PQ, divides $P\widehat{R}Q$ into two parts R_1 and R_2.



Then:

(a) $R_1 + R_2 = \widehat{P} + \widehat{Q}$ (b) $R_1 - R_2 = \widehat{Q} - \widehat{P}$ (c) $R_1 - R_2 = \widehat{P} - \widehat{Q}$
(d) $R_1 + R_2 = \widehat{Q} - \widehat{P}$ (e) $R_1 - R_2 = \widehat{Q} + \widehat{P}$

16. A cylinder is such that the area of its curved surface is twice its volume. Then its radius is:

(a) $\frac{1}{2}$ (b) 2 (c) $\frac{\pi}{2}$ (d) $\sqrt{\frac{2}{\pi}}$ (e) 1

17. $\left(-\frac{1}{8}\right)^{-1/3}$ is equal to:

(a) -2 (b) 2 (c) $-\frac{1}{2}$ (d) 8 (e) 83

18. A man wishes to travel 1992 kilometres at an average speed of 100 kmh^{-1}. He travels the first half of this distance at 50 kmh^{-1}. How fast must he go over the remaining half?

(a) 150 kmh^{-1} (b) 200 kmh^{-1} (c) 400 kmh^{-1}
(d) 496 kmh^{-1} (e) none of these

19. Six numbers are in arithmetic progression. The sum of the first and last is 5. Then the sum of the third and fourth is:

(a) 5 (b) 6 (c) 7 (d) 12 (e) impossible to determine

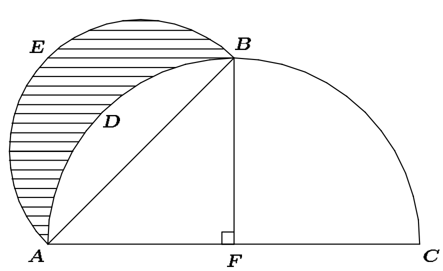
20. If $\boxed{n}$ means n^n, so that $\boxed{3} = 3^3 = 27$, then $\boxed{\boxed{2}}$ is:

(a) 2 (b) 4 (c) 16 (d) 64 (e) 256

Last issue we gave the problems of the Final Round of the Old Mutual Mathematical Olympiad 1991. My thanks go to John Grant McLoughlin of the Faculty of Education, Memorial University of Newfoundland for forwarding the questions and "official" solutions to the questions of paper 2, and problem 4 of paper 1.

OLD MUTUAL MATHEMATICAL OLYMPIAD 1991
Final Paper 1
Solutions (Time: 2 hours)

1. In the figure shown ABC and AEB are semicircles and F is the mid-point of AC and $AF = 1$ cm. Find the area of the shaded region.



Solution. AB is the diameter of the semicircle AEB and the hypotenuse of isosceles right triangle AFB, so $AB = \sqrt{2}$, and the area of semicircle AEB is $\frac{1}{2}\pi(\frac{\sqrt{2}}{2})^2 = \frac{\pi}{4}$. Now semicircle ABC has area $\frac{1}{2}\pi$, while right triangle AFB has area $\frac{1}{2}$. Hence the area of the circular sector ADB and the triangle AFB is is $\frac{\pi}{4} - \frac{1}{2}$, and the shaded area is $\frac{\pi}{4} - (\frac{\pi}{4} - \frac{1}{2}) = \frac{1}{2}$.

2. What is the value of $\sqrt{17 - 12\sqrt{2}} + \sqrt{17 + 12\sqrt{2}}$ in its simplest form?

Solution. Let $x = \sqrt{17 - 12\sqrt{2}} + \sqrt{17 + 12\sqrt{2}}$. Now

$$\begin{aligned} x^2 &= 17 - 12\sqrt{2} + 17 + 12\sqrt{2} + 2\sqrt{17 - 12\sqrt{2}}\,\sqrt{17 + 12\sqrt{2}} \\ &= 34 + 2\sqrt{(17 - 12\sqrt{2})(17 + 12\sqrt{2})} \\ &= 34 + 2\sqrt{(17)^2 - (12)^2(\sqrt{2})^2} \\ &= 34 + 2\sqrt{289 - 288} = 34 + 2 = 36\,. \end{aligned}$$

So $x = -6$, or $x = 6$. Since $x > 0$, $x = 6$.

3. In a certain mathematics examination, the average grade of the students passing was $x\%$, while the average of those failing was $y\%$. The average of all students taking the examination was $z\%$. Find, in terms of x, y and z, the percentage who fail.

Solution. The total of the scores for those passing the exam is $\frac{x}{100} \cdot p$, where p is the number of students who pass. Similarly, with f the number who fail $\frac{y}{100}f$ is the total of the failing scores. The total of all students scores is $\frac{z(p+f)}{100} = \frac{xp}{100} + \frac{yf}{100}$ so $zp + zf = xp + yf$,

$$(z - x)p = (y - z)f \qquad \text{and} \qquad p = \frac{y - z}{z - x}f\,.$$

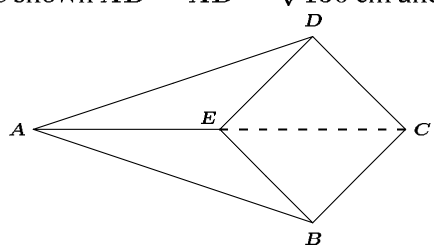
Now

$$\begin{aligned} p + f = \frac{y - z}{z - x}f + f &= \left(\frac{y - z}{z - x} + 1\right) f \\ &= \frac{y - x}{z - x}f\,. \end{aligned}$$

The percentage who fail is

$$\frac{f}{p + f} \times 100 = \frac{f}{\frac{y - x}{z - x}f} \times 100 = \frac{z - x}{y - x} \times 100\,.$$

4. In the figure shown $\overline{AB} = \overline{AD} = \sqrt{130}$ cm and $BEDC$ is a square.



Also the area of $\triangle AEB$ = area of square $BEDC$.
Find the area of $BEDC$.

Solution. Let BOD be a diagonal of the square, with O its centre. Now triangles $\triangle BEA$ and BCE have the same altitude, from B, namely BO. Since the area of square $BEDC$ equals the area of triangle $\triangle AEB$, and the area of the square is twice the area of $\triangle BCA$, we have $\overline{AE} = 2\overline{EC}$. Clearly $\overline{BO} = \overline{EO} = \frac{1}{2}\overline{EC}$, so $\overline{AE} = 4\overline{EO}$.

By Pythagoras, $\overline{AO}^2 + \overline{BO}^2 = \overline{AB}^2 = 130$. Thus,

$$(\overline{AE} + \overline{EO})^2 + \overline{EO}^2 = 130$$

or

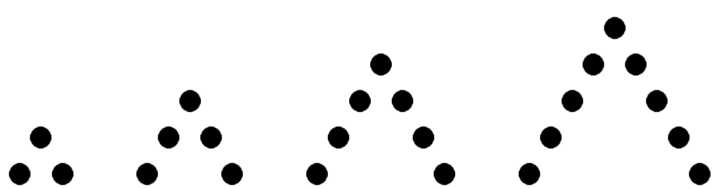
$$\begin{aligned} (4\overline{EO} + \overline{EO})^2 + \overline{EO}^2 &= 130 \\ (25+1)\overline{EO}^2 &= 130 \\ \overline{EO}^2 &= \frac{130}{26} = 5\,. \end{aligned}$$

But the area of the square is just twice the area of the square with side length EO, so the area is $2 \times 5 = 10$.

Final Paper 2

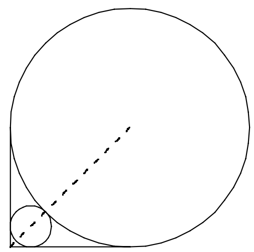
Solutions (Time: 2 hours)

1. If the pattern below of dot-figures is continued, how many dots will there be in the 100^{th} figure?



Solution. By inspection the number of dots in the n^{th} figure is $2n + 1$. So the 100^{th} has 201 dots.

2. It is required to place a small circle in the space left by a large circle as shown. If the radius of the large one is a and that of the small one is b, find the ratio a/b.



Solution. Let O be the centre of the larger circle and let O_1 be the centre of the small circle. The distance of O from P, the point of intersection of the two tangents shown, is $\sqrt{2}\,a$, by Pythagoras, and similarly the distance from O_1 to P is $\sqrt{2}\,b$. But $OO_1 = a + b$. So $\sqrt{2}\,b + a + b = \sqrt{2}\,a$, giving $\frac{a}{b} = \frac{\sqrt{2}+1}{\sqrt{2}-1} = 3 + 2\sqrt{2}$.

3. Find **all** solutions to the simultaneous equations

$$\begin{aligned} x + y &= 2\,, \\ xy - z^2 &= 1\,, \end{aligned}$$

and **prove** that there are no other solutions.

Solution. We find all *real* solutions. The first observation is that

$$xy \geq 1 \qquad (1)$$

in order that $xy - z^2 = 1$. Thus x and y are both positive or both negative. But for $x + y = 2$, x and y must both be positive. Now for any $x, y > 0$, $\sqrt{xy} \leq \frac{x+y}{2}$ (this is just the Arithmetic-Geometric Mean Inequality). Thus $\sqrt{xy} \leq \frac{2}{2} = 1$, and $xy \leq 1$. From (1) we have $xy = 1$ and $z = 0$. But the AM/GM inequality is strict unless $x = y$ so $x = y = 1$ and $z = 0$ is the only solution.

4. If a, b, c and d are numbers such that

$$\begin{aligned} a + b &< c + d\,, \\ b + c &< d + e\,, \\ c + d &< e + a\,, \\ \text{and} \quad d + e &< a + b\,. \end{aligned}$$

Prove that the largest number is a and the smallest is b.

Solution.

From $d+e<a+b$ and $a+b<c+d$ we see $d+e<c+d$ so $e<c$. (1)

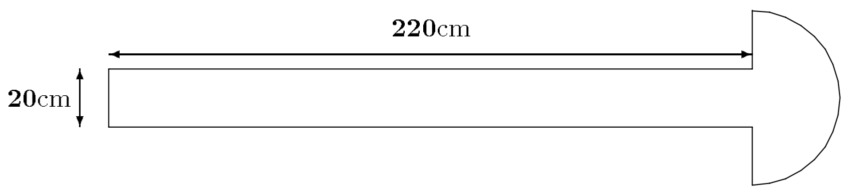
From $a+b<c+d$ and $c+d<e+a$ we get $b<e$. (2)

From $b+c<d+e$ and $d+e<a+b$ we get $c<a$. (3)

Now from $d+e<a+b$ we read $d-a<b-e<0$ by (2), so $d<a$. From (1), (2) and (3) $b<e<c<a$, so with $d<a$, a is the largest number.

Similarly from $a+b<c+d$ we see $0<a-c<d-b$ from (3) and $b<d$. This makes b the smallest number.

5. The diagram below [rotated through 90°] shows a container whose lower part is a hemisphere and whose upper part is a cylinder.

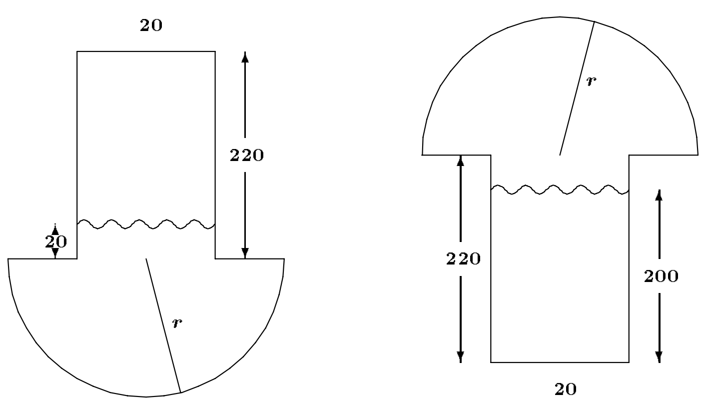


The cylindrical part has internal diameter of 20 cm and is 220 cm long. Water is poured into it and rises to a height of 20 cm in the cylindrical part. The top is then sealed with a flat cover and the container is turned upside down. The water is now 200 cm high in the cylindrical part.

(i) Calculate the volume of the hemisphere in terms of π.
(ii) Find the total height of the container.

[Note: The volume of a sphere of radius R is $\frac{4}{3}\pi R^3$.]

Solution. (Diagrams are not to scale!)



The volume V of water is $V=\pi\cdot 10^2\cdot 200$, from the information about the upside down figure.

But $V=\frac{1}{2}\,\frac{4}{3}\pi r^3+\pi\cdot 10^2\cdot 20$ from the initial configuration.

So the volume of the hemisphere V_H is

$$\begin{aligned} V_H = \frac{1}{2}\,\frac{4}{3}\pi r^3 &= \pi \cdot 10^2 \cdot 200 - \pi \cdot 20^2 \cdot 20 \\ &= \pi \times 18000\,. \end{aligned}$$

Now

$$\begin{aligned} \frac{2}{3}\pi r^3 &= \pi \times 18000 \\ r^3 &= 27000 \\ r &= 30\,. \end{aligned}$$

The total height is $220 + 30 = 250$ cm

That completes the *Skoliad Corner* for this issue. Please send me contest materials and suggestions for other features of the *Corner*.

Challenge

What is the 10^{th} term in the following sequence, and why?

n	x_n
0	0
1	$\frac{1}{16}\left(\sqrt{30}+\sqrt{10}-\sqrt{6}-\sqrt{2}-2\left(\sqrt{3}-1\right)\sqrt{5+\sqrt{5}}\right)$
2	$\frac{1}{8}\left(\sqrt{30-6\sqrt{5}}-\sqrt{5}-1\right)$
3	$\frac{1}{8}\left(\sqrt{10}+\sqrt{2}-2\sqrt{5-\sqrt{5}}\right)$
4	$\frac{1}{8}\left(\sqrt{10+2\sqrt{5}}-\sqrt{15}+\sqrt{3}\right)$
5	$\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$
6	$\frac{1}{4}\left(\sqrt{5}-1\right)$
7	$\frac{1}{16}\left(2\left(\sqrt{3}-1\right)\sqrt{5-\sqrt{5}}-\sqrt{30}+\sqrt{10}-\sqrt{6}+\sqrt{2}\right)$
8	$\frac{1}{8}\left(\sqrt{15}+\sqrt{3}-\sqrt{10-2\sqrt{5}}\right)$
9	$\frac{1}{8}\left(2\sqrt{5+\sqrt{5}}-\sqrt{10}+\sqrt{2}\right)$
10	?

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to the Mayhem Editor, Naoki Sato, Department of Mathematics, Yale University, PO Box 208283 Yale Station, New Haven, CT 06520–8283 USA. The electronic address is still

mayhem@math.toronto.edu

The Assistant Mayhem Editor is Cyrus Hsia (University of Toronto). The rest of the staff consists of Adrian Chan (Upper Canada College), Donny Cheung (University of Waterloo), Jimmy Chui (Earl Haig Secondary School), David Savitt (Harvard University) and Wai Ling Yee (University of Waterloo).

A new year brings change, and this is especially true for 1999. We are pleased to announce two new appointments. Adrian Chan, a high school student in his graduating year at Upper Canada College, is the new High School Problems Editor, and Donny Cheung, a 3^{rd} year undergraduate student at the University of Waterloo, is the new Advanced Problems Editor. Both bring their knowledge and experience to MAYHEM, including the IMO. We also introduce a new column, "Problem of the Month", which will hopefully appeal to young problem solvers. We hope these changes better serve you, our reader, which brings me to my second point.

MAYHEM was founded on the principle that it be a journal produced *for* students, *by* students (people still sending mail to **Professor** Naoki Sato may take particular note), and it is a principle that we have been committed to, for we take our audience as seriously as we take our content. We encourage our readership, in particular students, to take an active role in what goes into MAYHEM, and there are many ways one can do this. First, you can comment on our material. Do you think it is too easy or too hard? Are there any particular topics you would like to see? We are a small organization, so this kind of feedback is important to us. So, write to us, and remember, it does not hurt to be specific. Second, you can have a stronger say by writing an article yourself. It does not have to be deep or profound (that is for CRUX), just something that appeals to students. Finally, if you think you have what it takes, then consider joining our "illustrious" staff. (By contract, we must always use the word "illustrious" when describing our staff.) Candidates must be enthusiastic about mathematics, dedicated to their work, knowledgeable with word processing, and naturally, students. We cannot promise prestige or glamour. We cannot promise large sums of money, or even small sums of money. But, we do offer a chance to be a part of a unique Canadian mathematics journal. I am very serious about these offers, because MAYHEM has always been strongly driven by its readership, and I want to make this absolutely clear. That being said, I feel very optimistic about the coming year, and am confident that our cause will prosper. I would like to thank all the staff who have worked hard to make MAYHEM what it is today, and the CMS, and in particular Bruce Shawyer for his patience and guidance. I will see you all in 1999!

Your Editor, Naoki Sato

Mayhem Problems

The Mayhem Problems editors are:

Adrian Chan	*Mayhem High School Problems Editor,*
Donny Cheung	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only problems — the next issue will feature only solutions.

We warmly welcome proposals for problems and solutions. With the new schedule of eight issues per year, we request that solutions from this issue be submitted by 1 June 1999, for publication in issue 2 of volume 26.

High School Problems

Editor: Adrian Chan, 229 Old Yonge Street, Toronto, Ontario, Canada. M2P 1R5 <all238@ipoline.com>

H249. For a certain positive composite integer x, when the fraction $(60 - x)/120$ is reduced to lowest terms, the sum of the numerator and denominator exceeds 120. Determine x.

H250. Let $ABCD$ be a unit square, and let E, F, G, and H be points on AB, BC, CD, and DA respectively such that $AE = BF = CG = DH = 1999/2000$. Construct the triangles AGB, BHC, CED, and DFA, and let S be the area of the region that is common to all four triangles. Show that

$$S = \frac{1}{1999^2 + 2000^2}.$$

H251. We say that an arithmetic sequence is *astonishing* if it satisfies the following conditions:

(a) Every term in the sequence is an integer.

(b) No term in the sequence is greater than 10000.

(c) There are at least three terms in the sequence.

(d) The sum of the terms is 1999.

For example, the arithmetic sequence $-998, -997, \ldots, 999, 1000$ is astonishing. How many astonishing arithmetic sequences are there?

H252. Find a solution (a, b) in *rational* numbers to the following system:

$$\begin{aligned} 9a^2 + 16b^2 &= 25\,, \\ a^2 + b^2 &< \frac{25}{16} + \frac{1}{10}\,. \end{aligned}$$

(Query: Can you determine an infinite set of rational solutions to this system?)

Advanced Problems

Editor: Donny Cheung, c/o Conrad Grebel College, University of Waterloo, Waterloo, Ontario, Canada. N2L 3G6 <dccheung@uwaterloo.ca>

A225. In an acute-angled triangle, ABC, label its orthocentre H and its circumcentre O. Line BO is extended to meet the circumcircle at D. Show that $ADCH$ is a parallelogram.

A226. *Proposed by Naoki Sato.*

Let n be a positive integer. A $2 \times n$ array is filled with the entries 1, 2, ..., $2n$, using each exactly once, such that the entries increase reading left to right in each row, and top to bottom in each column.

For example, for $n = 5$, we could have the following array:

1	2	4	5	9
3	6	7	8	10

Find the number of possible such arrays in terms of n.

A227. *Proposed by Mohammed Aassila, Université Louis Pasteur, Strasbourg, France.*

Let f, $g : \mathbb{N} \to \mathbb{N}$ be functions such that f is surjective, g is injective, and $f(n) \geq g(n)$ for all $n \in \mathbb{N}$. Prove that $f = g$.

A228. Given a sequence a_1, a_2, a_3, ... of positive integers in which every positive integer occurs exactly once. Prove that there exist integers $k < \ell < m$, such that $a_k + a_m = 2a_\ell$.

(1997 Baltic Way)

Challenge Board Problems

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

C83. *Proposed by Dima Arinkin, graduate student, Harvard University.*

Let f be a function from the plane $\mathbb{R}^2$ to the reals. Given a polygon $\mathcal{P}$ in the plane, let $f(\mathcal{P})$ denote the sum of the values of f at each of the vertices of $\mathcal{P}$. Suppose there exists a convex polygon $\mathcal{Q}$ in the plane such that for every polygon $\mathcal{P}$ similar to $\mathcal{Q}$, we find that $f(\mathcal{P}) = 0$. Show that f is identically zero.

C84. *Proposed by Christopher Long, graduate student, Rutgers University.*

Let $A(x) = \sum_{m=0}^{\infty} a_m x^m$ be a formal power series, with each a_m either 0 or 1 and with infinitely many of the a_m non-zero. Give a necessary and sufficient condition on the a_m for there to exist a formal power series $B(x) = \sum_{n=0}^{\infty} b_n x^{-n}$ with each b_n either 0 or 1, with infinitely many of the b_n non-zero, and such that the formal product $A(x)B(x)$ exists.

Problem of the Month

Jimmy Chui, student, Earl Haig S.S.

We introduce a new section, originally titled "Problem of the Month" (yes, this is the most creative name we could think of). As you might guess, we plan to exhibit a problem and solution each issue, and not just for its elegance, but for its simplicity. For example, the last problems on a contest or olympiad always seem difficult, but the official solutions are inevitably a few lines long, and usually easy to understand. We would like to illustrate the same insight here. We begin with a problem from the 1997 Asian Pacific Mathematical Olympiad.

Problem. (1997 APMO)

Let ABC be a triangle inscribed in a circle and let

$$l_a = \frac{m_a}{M_a}, \qquad l_b = \frac{m_b}{M_b}, \qquad l_c = \frac{m_c}{M_c},$$

where m_a, m_b, m_c are the lengths of the angle bisectors (internal to the triangle) and M_a, M_b, M_c are the lengths of the angle bisectors extended until they meet the circle. Prove that

$$\frac{l_a}{\sin^2 A} + \frac{l_b}{\sin^2 B} + \frac{l_c}{\sin^2 C} \geq 3,$$

and that equality holds if and only if ABC is an equilateral triangle.

Solution.

Let the altitudes from A to BC, B to CA, and C to AB be h_a, h_b, and h_c respectively. Let the circumradius of triangle ABC be R. Note that $h_a = b\sin C$, $h_b = c\sin A$, and $h_c = a\sin B$, so that $h_ah_bh_c = abc\sin A\sin B\sin C$. We can see that $m_a \ge h_a$, $m_b \ge h_b$, $m_c \ge h_c$, $M_a \le 2R$, $M_b \le 2R$, and $M_c \le 2R$. Hence,

$$\begin{aligned}
&\frac{l_a}{\sin^2 A} + \frac{l_b}{\sin^2 B} + \frac{l_c}{\sin^2 C} \\
&= \frac{m_a}{M_a \sin^2 A} + \frac{m_b}{M_b \sin^2 B} + \frac{m_c}{M_c \sin^2 C} \\
&\ge \frac{h_a}{2R\sin^2 A} + \frac{h_b}{2R\sin^2 B} + \frac{h_c}{2R\sin^2 C} \\
&= \frac{h_a}{a\sin A} + \frac{h_b}{b\sin B} + \frac{h_c}{c\sin C} \qquad \text{(Extended Sine Law)} \\
&\ge 3\sqrt[3]{\frac{h_a}{a\sin A}\cdot\frac{h_b}{b\sin B}\cdot\frac{h_c}{c\sin C}} \qquad \text{(AM-GM)} \\
&= 3\,.
\end{aligned}$$

J.I.R. McKnight Problems Contest 1986

1. A bookshelf whose width lies between 30 and 50 cm holds exactly 5 poetry books each p cm thick, 3 history books each h cm thick, and 5 dictionaries each d cm thick (p, h, and d are unequal integers). It could instead hold exactly 4 poetry books, 5 history books and 4 dictionaries. If instead 7 poetry and 4 history books are placed on it, how many dictionaries must then be added to fill the shelf exactly? Also find the thickness of each book and the width of the shelf.

2. $ABCDEF$ is a regular hexagon, G is the mid-point of AB, and GE meets AC at P. Find the ratio into which P divides AC.

3. Prove that the sum of the squares of the first n even natural numbers exceeds the sum of the squares of the first n odd natural numbers by $n(2n+1)$. Hence, or otherwise, find the sum of the squares of the first n odd natural numbers.

4. (a) Andre, Barbara, Carole and Donato, whose ages were respectively 19, 17, 15 and 13 years, inherited $13\,750$ dollars which was so divided that their respective shares, at 10 percent per annum simple interest, amounted to equal sums when they arrived at the age of 21 years. What was the share that each received when they reached the age of 21?

(b) Prove that in any acute triangle ABC,

$$\cot A + \cot B + \cot C \;=\; \frac{a^2 + b^2 + c^2}{4K},$$

where K is the area of triangle ABC.

5. The point $P(t^2, t^3)$ lies on the curve $y^2 = x^3$. Q is another point on the curve such that PQ subtends a right angle at the origin O. Prove that the tangents at P and Q intersect on the curve $4y^2 = 3x - 1$.

[*Ed*. Unfortunately, we have lost the last page with questions 6, 7 and 8. Anyone with a spare copy, please send to the editor. Thank you.]

Contest Dates

Here are some upcoming contest dates to mark on your calendar.

Contest	**Grade**	**Date**
Gauss	Grades 7 & 8	May 14, 1999
Pascal	Grade 9	
Cayley	Grade 10	February 24, 1999
Fermat	Grade 11	
Euclid	Grade 12	April 20, 1999
Descartes	Grades 12 & 13	April 21, 1999
AHSME	High School	February 9, 1999
AIME	High School	March 16, 1999
USAMO	High School	April 27, 1999
OPEN	High School	November 24, 1999
CMO	High School	March 31, 1999
IMO	High School	July 10–22, 1999

More information can be found at the following sites:

`<http://camel.math.ca/CMS/Competitions>`
Canadian Mathematical Society Competitions

`<http://math.uwaterloo.ca/CMC/CMCHome.html>`
Canadian Mathematics Competitions

`<http://www.unl.edu/amc/>`
American Mathematics Competitions

Counting Snakes, Differentiating the Tangent Function, and Investigating the Bernoulli-Euler Triangle

Harold Reiter

In this paper we will examine three apparently unrelated mathematical objects. One is a family of permutations of finite sets of integers, called *snakes*. The second is the tangent and secant functions from trigonometry, and their higher derivatives. The third is a Pascal-like triangular array of numbers called the Bernoulli-Euler triangle.

Snakes and Up-Down Permutations.

What are "snakes" and what are "up-down" permutations? Before we can answer these questions, we must discuss permutations of finite sets and establish some notation. We denote by $[n]$ the set $\{1, 2, 3, 4, \ldots, n\}$.

Definition. A *permutation* π of the set $[n]$ is a one-to-one function from $[n]$ to itself:

$$\begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ \pi(1) & \pi(2) & \pi(3) & \ldots & \pi(n) \end{pmatrix},$$

or simply $(\pi(1), \pi(2), \pi(3), \ldots, \pi(n))$. For example, the permutation π of $[4]$ given by $\pi(1) = 3, \pi(2) = 1, \pi(3) = 2$ and $\pi(4) = 4$ is denoted $(3, 1, 2, 4)$.

Definition. A permutation π is called

A. an *up-down* permutation if $\pi(1) < \pi(2), \pi(2) > \pi(3), \pi(3) < \pi(4), \ldots,$

and

B. a *down-up* permutation if $\pi(1) > \pi(2), \pi(2) < \pi(3), \pi(3) > \pi(4), \ldots$.

A permutation which is either up-down or down-up is called a *snake*. Our main goal here is to find a formula for the number of snakes on $[n]$. Let us list the up-down and the down-up permutations of $[n]$ for small values of n.

1. The up-down permutations of [1]: (1)
   The down-up permutations of [1]: (1)

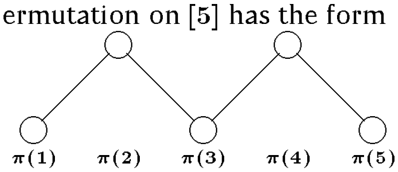
2. The up-down permutations of [2]: $(1, 2)$
   The down-up permutations of [2]: $(2, 1)$

3. The up-down permutations of [3]: $(1,3,2)$, $(2,3,1)$
   The down-up permutations of [3]: $(2,1,3)$, $(3,1,2)$

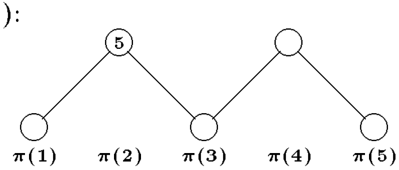
4. The up-down permutations of [4]: $(1,3,2,4)$, $(2,3,1,4)$, $(1,4,2,3)$, $(2,4,1,3)$, $(3,4,1,2)$
   The down-up permutations of [4]: $(4,2,3,1)$, $(4,1,3,2)$, $(3,2,4,1)$, $(3,1,4,2)$, $(2,1,4,3)$

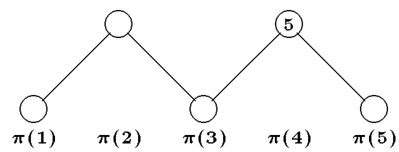
Let A_n and B_n denote the number of up-down and down-up permutations of $[n]$, $n \geq 1$. For convenience, let $A_0 = 1$ and $B_0 = 1$. Note from the above that $A_1 = B_1 = 1$, $A_2 = B_2 = 1$, $A_3 = B_3 = 2$ and $A_4 = B_4 = 5$. Let us find A_5.

Each up-down permutation on [5] has the form



Clearly there are just two places for the 5. That is 5 must be $\pi(2)$ or $\pi(4)$:

A: 

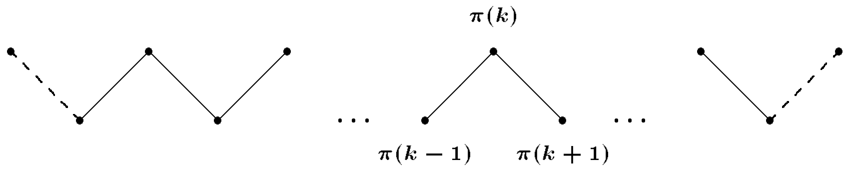
and B: 

Do you see a one-to-one correspondence between those of type A and those of type B? Because of this correspondence, we will concentrate only on those of type A. Notice that $\pi(1)$ can be any of 1, 2, 3, or 4 and the other three can become $\pi(3)$, $\pi(4)$, $\pi(5)$ in two ways ($\pi(4)$ is the largest of the three, and $\pi(3)$ could be either of the other two). Thus, there are $4 \times 2 = 8$ of type A, and so there are 16 up-down perms of [5]. Our next step is to find a general formula for A_n. Before we do this, let us consider the relationship between the A_n and the B_n.

Theorem. For all n, $A_n = B_n$.

Proof. Let $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$ be an up-down permutation of $[n]$. Consider the function $\overline{\pi} = (n+1-\pi(1), n+1-\pi(2), \ldots, n+1-\pi(n))$. For example if $\pi = (1,3,2,5,4)$, then $\overline{\pi} = (6-1, 6-3, 6-2, 6-5, 6-4) = (5,3,4,1,2)$. Note that $\overline{\pi}$ is a function from $[n]$ to $[n]$, that $\overline{\pi}$ is one-to-one (hence a permutation), and that $\overline{\pi}$ is a down-up permutation. Notice also that if $\pi_1 \neq \pi_2$, then $\overline{\pi_1} \neq \overline{\pi_2}$. This means that the mapping $\pi \rightarrow \overline{\pi}$ is one-to-one. It is also onto; that is, if σ is any down-up permutation, there is an up-down permutation σ' such that the mapping above takes σ to σ'. Try working out σ' from σ. Thus $A_n = B_n$ for all n. ∎

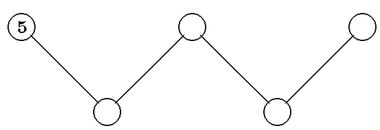
In what follows, we will find a formula for the numbers $A_n + B_n = 2A_n$. Fix n. Consider the possible positions for n in a down-up or an up-down permutation. Of course, n must appear in an up position, which we denote by k.



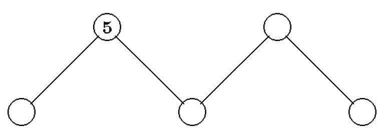
Notice that π is an up-down permutation if and only if k is even. In other words, the even positions are up positions if π is an up-down permutation and down positions if π is a down-up permutation. The numbers in positions 1 through $k-1$ are in up-down or down-up order, and those in positions $k+1$ through n are in up-down order. There are $\binom{n-1}{k-1}$ ways to select the numbers for positions 1 through k. Having made that selection, there are A_{k-1} ways of arranging them and A_{n-k} ways to arrange the rest of the numbers. Thus, there are $\binom{n-1}{k-1} A_{k-1}A_{n-k}$ up-down or down-up permutations with the number n in the k^{th} position. (Our choice of $A_0 = 1$ comes into play here.) Since n can appear in any of the n positions, we have

$$2A_n = A_n + B_n = \sum_{k=1}^{n} \binom{n-1}{k-1} A_{k-1}A_{n-k}\,. \qquad (2)$$

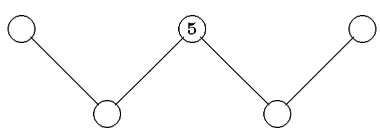
To illustrate for $n = 5$:



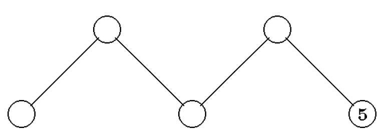
there are $\binom{4}{0} A_0A_4 = 1 \cdot 1 \cdot 5 = 5$;



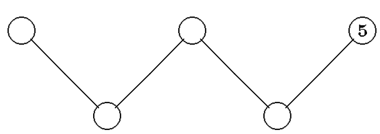
there are $\binom{4}{1} A_1A_3 = 4 \cdot 1 \cdot 2 = 8$;



there are $\binom{4}{2} A_2A_2 = 6 \cdot 1 \cdot 1 = 6$;



there are $\binom{4}{3} A_3A_1 = 4 \cdot 2 \cdot 1 = 8$;



there are $\binom{4}{4} A_4A_0 = 1 \cdot 5 \cdot 1 = 5$.

Hence, $A_5 + B_5 = 5 + 8 + 6 + 8 + 5 = 32$ and $A_5 = 16$.

Let $a_k = A_k/k!$. Thus $a_0 = 1/0! = 1$, $a_1 = 1/1! = 1$, $a_2 = 1/2! = 1/2$, etc. Writing out (2) and using the factorial form of $\binom{n-1}{k-1}$, we get

$$\begin{aligned}
2A_n &= \sum_{k=1}^{n} \binom{n-1}{k-1} A_{k-1}A_{n-k} = \sum_{k=1}^{n} \frac{(n-1)!A_{k-1}A_{n-k}}{(k-1)!(n-k)!} \\
&= (n-1)! \sum_{k=1}^{n} \frac{A_{k-1}}{(k-1)!} \frac{A_{n-k}}{(n-k)!} \\
&= (n-1)! \sum_{k=1}^{n} a_{k-1}a_{n-k} .
\end{aligned}$$

We may conclude from this that

$$\frac{2nA_n}{n(n-1)!} = \sum_{k=1}^{n} a_{k-1}a_{n-k} ,$$

which is equivalent to

$$2na_n = \sum_{k=1}^{n} a_{k-1}a_{n-k} . \tag{3}$$

Let f be the ordinary generating function of the a_n. That is,

$$f(t) = \sum_{k=1}^{\infty} a_k t^k .$$

When expressed in terms of the A_n, f is the exponential generating function. That is

$$f(t) = \sum_{k=1}^{\infty} \frac{A_k}{k!} t^k .$$

Next, compute $f^2(t)$ using equation (3).

$$\begin{aligned}
f^2(t) &= (a_0 + a_1 t + a_2 t^2 + \dots)(a_0 + a_1 t + a_2 t^2 + \dots) \\
&= a_0^2 + (a_0 a_1 + a_1 a_0)t + (a_0 a_2 + a_1 a_1 + a_2 a_0)t^2 + \dots \\
&\quad + \sum_{k=1}^{n} (a_{k-1}a_{n-k}) t^{n-1} \dots \\
&= a_0^2 + 2 \cdot 2 \cdot a_2 t + 2 \cdot 3 \cdot a_3 t^2 + \dots + 2na_n t^{n-1} + \dots \\
&= a_0^2 + 2\left[\sum_{k=2}^{\infty} k a_k t^{k-1}\right] = a_0^2 + 2\left[\sum_{k=1}^{\infty} k a_k t^{k-1} - a_1\right] \\
&= a_0^2 - 2a_1 + 2f'(t) = 1 - 2 + 2f'(t) .
\end{aligned}$$

Hence f satisfies the differential equation

$$y^2 + 1 \;=\; 2y' .$$

To solve this, write $1/2 = y'/(y^2+1)$ and anti-differentiate both sides with respect to t to get

$$\frac{t}{2} + C \;=\; \arctan y .$$

Take the tangent of both sides to get

$$\tan\left(\frac{t}{2} + C\right) \;=\; \tan(\arctan(f(t))) \;=\; f(t)$$

for some constant C. At $t = 0$, we have $a_0 = f(0) = 1 = \tan(C)$, so $C = \pi/4$ will do. Then

$$f(t) \;=\; \tan\left(\frac{t}{2} + \frac{\pi}{4}\right) \;=\; a_0 + a_1 t + a_2 t^2 + \dots ,$$

and

$$f(-t) \;=\; \tan\left(-\frac{t}{2} + \frac{\pi}{4}\right) \;=\; a_0 - a_1 t + a_2 t^2 - \dots .$$

It can be shown that

$$\tan\left(\frac{t}{2} + \frac{\pi}{4}\right) + \tan\left(-\frac{t}{2} + \frac{\pi}{4}\right) \;=\; 2/\cos t$$

and that

$$\tan\left(\frac{t}{2} + \frac{\pi}{4}\right) - \tan\left(-\frac{t}{2} + \frac{\pi}{4}\right) \;=\; 2\tan t$$

for all $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Thus $f(t) + f(-t) = 2/\cos t$, or

$$\sec t \;=\; a_0 + a_2 t^2 + a_4 t^4 + \dots \;=\; A_0 + \frac{A_2}{2!} t^2 + \frac{A_4}{4!} t^4 + \dots ,$$

and similarly

$$\tan t \;=\; a_1 + a_3 t^3 + a_5 t^5 + \dots \;=\; A_1 + \frac{A_3}{3!} t^3 + \frac{A_5}{5!} t^5 + \dots .$$

Therefore, to find A_n, for odd n, we find the n^{th} derivative of $\tan t$, and evaluate at $t = 0$. That is,

$$A_n \;=\; \frac{d^n}{dt^n} \tan t \Big]_{t=0}$$

and for even n, we find the n^{th} derivative of $\sec t$, and evaluate at $t = 0$. That is,

$$A_n \;=\; \frac{d^n}{dt^n} \sec t \Big]_{t=0} .$$

Differentiating the Tangent Function.

We seek to differentiate the tangent function repeatedly at 0. Since $\frac{d}{dx}\tan x = \sec^2 x$ and $\frac{d}{dx}\sec^2 x = 2\tan x \sec^2 x$, it is convenient to set $y = \tan x$ and $z = \sec^2 x$. We use D in place of $\frac{d}{dx}$ for convenience. Then $Dy = D\tan x = \tan' x = z$ and $D^2 y = Dz = 2yz$.

1. $D^0 y = y$.
2. $Dy = z$.
3. $D^2 y = Dz = 2yz$.
4. $D^3 y = D2yz = 2z \cdot z + 2y \cdot 2yz = 2z^2 + 4y^2 z$
5. $D^4 y = D(2z^2 + 4y^2 z) = 2 \cdot 2z \cdot 2yz + 4(2y \cdot z \cdot z + y^2 \cdot 2yz)$
   $= 16yz^2 + 8y^3 z$.
6. $D^5 y = D(16yz^2 + 8y^3 z) = 16z \cdot z^2 + 16y \cdot 2z \cdot 2yz + 24y^2 z + 8y^3 \cdot 2yz$
   $= 16z^3 + 64y^2 z^2 + 24y^2 z + 16y^4 z$.
7. $D^6 y = D(16z^3 + 64y^2 z^2 + 24y^2 z + 16y^4 z)$
   $= 48z^2 \cdot 2yz + 64(2yz \cdot z^2 + y^2 \cdot 2z \cdot 2yz) + 24(2yz^2 + y^2 \cdot 2yz)$
   $+16(4y^3 z \cdot z + y^4 \cdot 2yz)96yz^3 + 128yz^3 + 256y^3 z^2 + 48yz^2 + 48y^3 z$
   $+64y^3 z^2 + 32y^5 z = 224yz^3 + 320y^3 z^2 + 48y^3 z + 32y^5 z + 48yz^2$.
8. $D^7 y = D(224yz^3 + 320y^3 z^2 + 48y^3 z + 32y^5 z + 48yz^2)$
   $= 224(z^4 + y \cdot 3z^2 \cdot 2yz) + 320(3y^2 z \cdot z^2 + y^3 \cdot 2z \cdot 2yz)$
   $+48(3y^2 \cdot z \cdot z + y^3 \cdot 2yz) + 32(5y^4 \cdot z \cdot z + y^5 \cdot 2yz) + 48(z \cdot z^2 + y \cdot 2z \cdot 2yz)$.

Evaluating each of these at 0, noting that $\tan 0 = 0$ and $\sec 0 = 1$, we get:

1.	$\tan 0 = 0$,	2.	$\tan' 0 = 1$,
3.	$\tan'' 0 = 0$,	4.	$\tan^{(3)} 0 = 2$,
5.	$\tan^{(4)} 0 = 0$,	6.	$\tan^{(5)} 0 = 16$,
7.	$\tan^{(6)} 0 = 0$,	8.	$\tan^{(7)} 0 = 272$.

The Bernoulli-Euler Triangle.

This third and final section provides a straightforward and fast method for counting the up-down permutations on $[n]$. The Bernoulli-Euler (BE) triangle, like Pascal's triangle is a triangular array of numbers each row of which is obtainable from the previous row. The BE triangle:

1
1 0
0 1 1
2 2 1 0
0 2 4 5 5
16 16 14 10 5 0
0 16 32 46 56 61 61
272 272 256 224 178 122 61 0

The triangle can be filled in as follows. In line 1, write 1. Every even line is filled in from right to left by writing in each position the sum of all the numbers of the previous row to the right of the position. Odd lines are filled in from left to right similarly, summing those numbers in the row above which lie to the left of the position. For example, the number 46 above is in the sixth row, and it is obtained by adding the two 16's and the 14 which lie to its left in the previous row. We will see below that the entries at the ends of the rows are just the numbers we seek.

What do the numbers in the triangle mean? What do the $S(n,h)$ count? Consider the triangle

$$
\begin{array}{ccccccccc}
 & & & & S(1,1) & & & & \\
 & & & S(2,1) & \longleftarrow & S(2,2) & & & \\
 & & S(3,3) & \longrightarrow & S(3,2) & \longrightarrow & S(3,1) & & \\
 & S(4,1) & \longleftarrow & S(4,2) & \longleftarrow & S(4,3) & \longleftarrow & S(4,4) & \\
S(5,5) & \longrightarrow & S(5,4) & \longrightarrow & S(5,3) & \longrightarrow & S(5,2) & \longrightarrow & S(5,1)
\end{array}
$$

Here is the answer. If n is odd, $S(n,h)$ counts the number of up-down permutations of $[n]$ whose last entry is h, and if n is even, $S(n,h)$ counts the number of down-up permutations of $[n]$ whose last entry is h. For example,

$$(S(5,5), S(5,4), S(5,3), S(5,2), S(5,1)) = (0,2,4,5,5),$$

which means that

- there are no up-down permutations of [5] with last value 5;
- there are two up-down permutations with last value 4 — they are $(1,3,2,5,4)$ and $(2,3,1,5,4)$;
- there are four up-down permutations with last value 3 — they are $(1,4,2,5,3)$, $(2,4,1,5,3)$, $(1,5,2,4,3)$ and $(2,5,1,4,3)$;
- there are five up-down permutations with last value 2 — they are $(1,4,3,5,2)$, $(3,4,1,5,2)$, $(1,5,3,4,2)$, $(3,5,1,4,2)$ and $(4,5,1,3,2)$; and
- there are five up-down permutations with last value 1 — they are $(2,4,3,5,1)$, $(3,4,2,5,1)$, $(2,5,3,4,1)$, $(3,5,2,4,1)$ and $(4,5,2,3,1)$.

The next formula is a recursive construction of the triangle. Using the notation $S(n,i)$ as we did earlier for the i^{th} entry of even numbered rows and the $n-i+1^{\text{st}}$ entry of odd numbered rows, $S(1,1) = 1$, and for $n > 1$,

$$S(n,i) = \begin{cases} 0 & \text{if } i = n\,, \\ S(n,i+1) + S(n-1,n-i) & \text{otherwise.} \end{cases}$$

To illustrate why the recursive formulas works, consider the example obtained for $n = 6$ and $i = 4$:

$$S(6,5) + S(5,2) \;=\; S(6,4)\,.$$

$S(6,5)$ counts the five down-up permutations of [6] which end in 5: $(2,1,4,3,6,5)$, $(3,1,4,2,6,5)$, $(3,2,4,1,6,5)$, $(4,1,3,2,6,5)$, and $(4,2,3,1,6,5)$. These give rise to five up-down permutations ending in 4 by interchanging the 4 and the 5:

$$\begin{aligned}(2,1,4,3,6,5) &\rightarrow (2,1,5,3,6,4),\\ (3,1,4,2,6,5) &\rightarrow (3,1,5,2,6,4),\\ (3,2,4,1,6,5) &\rightarrow (3,2,5,1,6,4),\\ (4,1,3,2,6,5) &\rightarrow (5,1,3,2,6,4),\\ (4,2,3,1,6,5) &\rightarrow (5,2,3,1,6,4).\end{aligned}$$

Note that $S(5,2)$ counts the five up-down permutations of [5] which end with 2: $(1,4,3,5,2)$, $(3,4,1,5,2)$, $(1,5,3,4,2)$, $(3,5,1,4,2)$, and $(4,5,1,3,2)$. To see how these are transformed into new (other than those counted above) permutations counted by $S(6,4)$, first replace (x,y,u,v,w) with $(6-x,6-y,6-u,6-v,6-w)$, obtaining a down-up permutation of [5]. Next, replace the 5 in $(6-x,6-y,6-u,6-v,6-w)$ with 6, replace the last position $6-w$ (which must be $6-2=4$, because of where (x,y,u,v,w) came from) with 5, and finally, append the number 4 at the end. Thus, for example,

$$(1,4,3,5,2) \rightarrow (5,2,3,1,4) \rightarrow (6,2,3,1,4) \rightarrow$$

$$(6,2,3,1,5) \rightarrow (6,2,3,1,5,4).$$

All permutations obtained in this way are different because they end with the pair $4\ 5$, whereas the ones obtained above all end with $6\ 4$.

References:

1. *Combinatorik*, E. Netto, 1930.

2. Why Study Mathematics, Vladimir Arnold, *Quantum*, Sept–Oct, 1994, pp. 25-31.

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PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 September 1998**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in LaTeX). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

2401. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

In triangle ABC, CD is the altitude from C to AB. E and F are the mid-points of AB and CD respectively. P and Q are points on line segments BC and AC respectively, and are such that $PQ \| BA$. The projection of Q onto AB is R. PR and EF intersect at S.

Prove that

(a) S is the mid-point of line segment PR,

(b) $\dfrac{1}{PR^2} \leq \dfrac{1}{AB^2} + \dfrac{1}{CD^2}$.

2402. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Find all sets of positive integers a, b, c, d, such that

$$bd > ad+bc \quad \text{and} \quad (9ac+bd)(ad+bc) = a^2d^2+10abcd+b^2c^2 .$$

2403. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

The positive integers a, b, c have the following properties:

1. a is odd;
2. the greatest common denominator of a, b, c is 1;
3. they satisfy the Diophantine equation:

$$\frac{2}{a} + \frac{1}{b} = \frac{1}{c}.$$

Prove that abc is a perfect square.

2404*. *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*

Let $f(x) = \dfrac{(1-2x)^p + 2(x+1)^p}{2\left((x+1)(1-2x)\right)^p + (x+1)^{2p}}$, where $x \in \left(0, \frac{1}{2}\right)$ and $p \geq 2$ is an integer.

The relative maximum of $f(x)$ on $\left(0, \frac{1}{2}\right)$ has been found by computer to be at the point x_p for several values of p as shown:

p	2	3	4	5	6	7
x_p	0.2538	0.1133	0.0633	0.0403	0.0279	0.0205

Explain why x_p is "near" $\frac{1}{p^2}$.

2405. *Proposed by G.P. Henderson, Garden Hill, Campbellcroft, Ontario, (adapted by the editors).*

Given two n-sided dice, one with a_k sides with k dots $(1 \leq k \leq n)$ such that $\sum_{k=1}^{n} a_k = n$ and the other with b_k sides with k dots $(1 \leq k \leq n)$ such that $\sum_{k=1}^{n} b_k = n$. Both are rolled. Let r_k be the probability that the sum of the two faces showing is k.

How should the a_k and the b_k be chosen to minimize $\displaystyle\sum_{k=1}^{n} \left(r_k - \frac{1}{n-1}\right)^2$

(a) with $n = 6$?

(b)* with general n?

2406. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, Florida, USA.*

Suppose that an integer-sided triangle contains a 120° angle, with the two containing arms differing by 1. Prove that the length of the longest side is the sum of two consecutive squares.

2407. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Triangle ABC is given with $\angle BAC = 72^\circ$. The perpendicular from B to CA meets the internal bisector of $\angle BCA$ at P. The perpendicular from C to AB meets the internal bisector of $\angle ABC$ at Q.

If A, P and Q are collinear, determine $\angle ABC$ and $\angle BCA$.

2408. *Proposed by Mansur Boase, student, St. Paul's School, London, England.*

Perpendiculars are dropped from a point P inside an acute-angled triangle ABC to the sides BC, CA, AB, meeting them at D, E, F respectively.

(a) Prove that the perpendiculars from A to EF, B to FD, C to DE are concurrent (at K, say).

(b) A point, P, is called "Cevic" with respect to $\triangle ABC$ if AD, BE and CF are concurrent. Prove that K is Cevic with respect to $\triangle DEF$ if and only if P is Cevic with respect to $\triangle ABC$.

2409. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Triangle ABC with sides a, b, c, has O as the centre of its circumcircle (with radius R) and H as its orthocentre. Suppose that OH intersects CB and CA at P and Q respectively.

(a) Prove that quadrilateral $ABPQ$ is cyclic if and only if $a^2 + b^2 = 6R^2$.

(b) If quadrilateral $ABPQ$ is cyclic, find a formula for the length of PQ in terms of a, b and c alone.

2410. *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*

For $n \geq 1$, define $v_n = [1, 2, 3, \ldots, n-1, n]$, where the square bracket denotes the least common multiple. Let $p_1 < p_2$ be twin primes.

Prove or disprove that $v_{p_2} = p_2\, v_{p_1}$ for $p_1 > 3$.

2411. *Proposed by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

It is well known and easy to show that the product of four consecutive positive integers plus one is always a perfect square. It is also easy to show that the product of any two consecutive positive integers plus one is never a perfect square. Also, note that

$$2 \times 3 \times 4 + 1 = 5^2 \qquad \text{and} \qquad 4 \times 5 \times 6 + 1 = 11^2\,.$$

(a) Find another natural number n such that $n(n+1)(n+2) + 1$ is a perfect square.

(b)* Are there further examples?

2412. *Proposed by Darko Veljan, University of Zagreb, Zagreb, Croatia.*

Suppose that A_1, A_2, A_3, A_4 are the vertices of a tetrahedron $\mathcal{T}$. On the faces opposite A_1, A_2, A_3, construct tetrahedra outside $\mathcal{T}$ with apexes A_1', A_2', A_3', and volumes V_1, V_2, V_3, respectively.

Let A_4' be the point such that $\overrightarrow{A_1A_4'} = \overrightarrow{BA_4}$, where B is the point of intersection of the planes through A_i' parallel to the respective bases $(i = 1, 2, 3)$.

Let V_4 be the volume of tetrahedron $A_1A_2A_3A_4'$.

Prove that $V_4 = V_1 + V_2 + V_3$.

2413. *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

A deck of six cards consists of three black cards and three red cards, in some order. The top four cards are picked up, shuffled randomly, and then put on the *bottom* of the deck. This procedure is repeated n times.

Let $p(n)$ be the probability that after n such "shuffles", the deck alternates between red and black cards, either colour being the top card. (So $p(n)$ will, in general, depend on the initial ordering of the deck.)

Find $\lim\limits_{n\to\infty} p(n)$ — that is, the "long term" probability that the deck "tends to be alternating".

Correction

2324. [1998: 109] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

Find the exact value of $\sum\limits_{n=1}^{\infty} \dfrac{1}{u_n}$, where u_n is given by the recurrence

$$u_n = n! + \left(\frac{n-1}{n}\right) u_{n-1},$$

with the initial condition $u_1 = 2$.

Notice

Proposers are asked to note that we have now published all of the proposals, submitted in 1997, that the Editorial Board has accepted for use.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2299. [1997: 503] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let x, y, $z > 0$ be real numbers such that $x + y + z = 1$. Show that

$$\prod_{\text{cyclic}} \left(\frac{(1-y)(1-z)}{x}\right)^{(1-y)(1-z)/x} \geq \frac{256}{81}\,.$$

Determine the cases of equality.

Composite solution by Theodore Chronis, Athens, Greece and Kee-Wai Lau, Hong Kong.

The given inequality is clearly equivalent to

$$A = \sum_{\text{cyclic}} \left(1 + \frac{xy}{z}\right) \ln\left(1 + \frac{xy}{z}\right) \geq 4 \ln\left(\frac{4}{3}\right)\,. \qquad (1)$$

Since the function $x\ln(x)$ is convex on $(0, \infty)$, we have, by Jensen's Inequality, that

$$\frac{A}{3} \geq \frac{1}{3}\left(3 + \frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y}\right) \ln\left(\frac{1}{3}\left(3 + \frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y}\right)\right)\,. \qquad (2)$$

Since $(xy)^2 + (yz)^2 + (zx)^2 \geq (xy)(yz) + (yz)(zx) + (zx)(xy) = xyz$, we get

$$\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} \geq 1\,. \qquad (3)$$

From (2) and (3), we obtain (1) immediately.

It is clear that equality holds if and only if $x = y = z = \frac{1}{3}$.

Also solved by NIKOLAOS DERGIADES, Thessaloniki, Greece; FLORIAN HERZIG, student, Cambridge, UK; RICHARD I. HESS, Rancho Palos Verdes, California, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; MICHAEL LAMBROU, University of Crete, Crete, Greece; VEDULA N. MURTY, Visakhapatnam, India; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer.

2300. [1997: 503] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Suppose that ABC is a triangle with circumradius R. The circle passing through A and touching BC at its mid-point has radius R_1. Define R_2 and R_3 similarly. Prove that

$$R_1^2 + R_2^2 + R_3^2 \geq \frac{27}{16} R^2\,.$$

Solution by G. Tsintsifas, Thessaloniki, Greece.

We use m_i for the medians and h_i for the altitudes. Let M be the mid-point of BC and O_1 the centre of the circle through A touching BC at M. From triangle O_1MA we have: $\frac{1}{2}m_1 = \frac{1}{2}AM = R_1\cos\omega = R_1\frac{h_1}{m_1}$, where $\omega = \widehat{O_1MA}$ or $\frac{1}{2}m_1^2 = R_1h_1$. Similarly, $\frac{1}{2}m_2^2 = R_2h_2$, $\frac{1}{2}m_3^2 = R_3h_3$. Therefore,

$$\frac{3}{8}(a^2+b^2+c^2) \;=\; \frac{m_1^2+m_2^2+m_3^2}{2} \;=\; R_1h_1+R_2h_2+R_3h_3\,. \qquad (1)$$

From the Cauchy-Schwarz inequality, we have

$$R_1h_1+R_2h_2+R_3h_3 \;\le\; \left[\left(R_1^2+R_2^2+R_3^2\right)\left(h_1^2+h_2^2+h_3^2\right)\right]^{1/2}\,. \qquad (2)$$

But, it is well known that:

$$h_1^2+h_2^2+h_3^2 \;=\; \frac{a^2b^2+b^2c^2+c^2a^2}{4R^2}\,. \qquad (3)$$

Using (1), (2), (3) we obtain

$$\frac{9R^2}{16}\frac{\left(a^2+b^2+c^2\right)^2}{a^2b^2+b^2c^2+c^2a^2} \;\le\; R_1^2+R_2^2+R_3^2\,. \qquad (4)$$

Obviously we have:

$$3 \;\le\; \frac{\left(a^2+b^2+c^2\right)^2}{a^2b^2+b^2c^2+c^2a^2}\,. \qquad (5)$$

So, from (4), (5), it follows that $\frac{27}{16}R^2 \le R_1^2+R_2^2+R_3^2$.

Also solved by NIELS BEJLEGAARD, Stavanger, Norway; THEODORE CHRONIS, Athens, Greece; NIKOLAOS DERGIADES, Thessaloniki, Greece; FLORIAN HERZIG, student, Cambridge, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

2301. [1998: 45] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Suppose that ABC is a triangle with sides a, b, c, that P is a point in the interior of $\triangle ABC$, and that AP meets the circle BPC again at A'. Define B' and C' similarly.

Prove that the perimeter $\mathcal{P}$ of the hexagon $AB'CA'BC'$ satisfies

$$\mathcal{P} \;\ge\; 2\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)\,.$$

Solution by Florian Herzig, student, Cambridge, UK.

Let $x = \angle BPC$, $y = \angle CPA$, and $z = \angle APB$.

Then, as $A'CPB$ is cyclic, the angles in $\triangle A'CB$ are $\pi - x$, $\pi - z$, $\pi - y$ and so by the Sine Law:

$$A'B = \frac{a \sin z}{\sin x} \quad \text{and} \quad A'C = \frac{a \sin y}{\sin x}\,.$$

Thus the perimeter of $\triangle A'CB$ equals

$$\frac{a(\sin x + \sin y + \sin z)}{\sin x}\,.$$

Analogously we obtain the perimeters of triangles $B'AC$ and $C'BA$. Summing these yields

$$(\sin x + \sin y + \sin z)\left(\frac{a}{\sin x} + \frac{b}{\sin y} + \frac{c}{\sin z}\right)\,.$$

By the Cauchy-Schwarz inequality,

$$P + a + b + c \;\geq\; \left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right)^2\,,$$

or equivalently,

$$P \;\geq\; 2\left(\sqrt{bc} + \sqrt{ca} + \sqrt{ab}\right)\,.$$

Equality holds if

$$\frac{a}{\sin^2 x} = \frac{b}{\sin^2 y} = \frac{c}{\sin^2 z}\,.$$

Also solved by MANSUR BOASE, student, Cambridge, England; GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; JOEL SCHLOSBERG, student, Bayside, NY, USA; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; G. TSINTSIFAS, Thessaloniki, Greece; and the proposer.

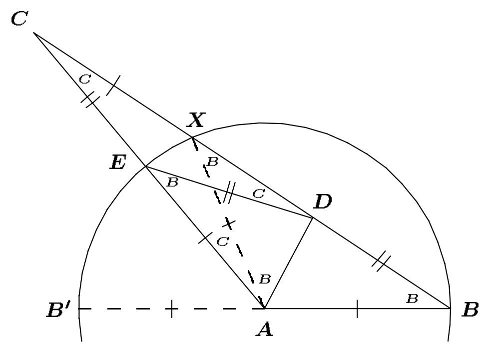
Most of the submitted solutions are similar to the one given above.

2302. [1998: 45] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that the bisector of angle A of triangle ABC intersects BC at D. Suppose that $AB + AD = CD$ and $AC + AD = BC$.

Determine the angles B and C.

I. *Solution by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA (somewhat modified by the editor).*



The figure shows the circle with centre A and radius AB intersecting CB at X, CA at E and AB (extended) again at B'. Subtract the first of the given equalities from the second to get $AC - AB = BC - CD$, which, because $AB = AE$, implies that $CE = BD$ (as the figure shows). Using the symmetry of AB and AE about the angle bisector AD, we see that BD also equals ED, and that B equals the angle at E ($\angle AED$). This latter angle is an exterior angle of the isosceles triangle EDC, so that $B = \angle AED = 2C$. The exterior angle at A of triangle ABC (namely $\angle B'AE$) is therefore $3C$, while $\angle B'AX = 4C$ (since the angle at the centre is twice the angle at the circumference). This implies that

$$\angle XAC = C, \qquad (1)$$

so that triangle XAC is isosceles, with XC equal to the radius AX. The figure now recalls Archimedes' trisection of $\angle B'AE$ by compass and marked ruler (as in David R. Davis, *Modern College Geometry*, Addison-Wesley, 1958, section 10–8): using $CX = AB$, we see that the given condition $AB + AD = CD$ yields $XD = AD$, so that triangle DXA is isosceles, and (since $\angle DXA$ is an exterior angle of triangle AXC) $\angle DAX = 2C$. This, together with (1) says that half of angle A, namely $\angle DAC$, equals $3C$, so that $A = 6C$. Thus $180^\circ = A + B + C = 6C + 2C + C = 9C$. We conclude that $A = 120^\circ$, $B = 40^\circ$ and $C = 20^\circ$.

II. *Solution by Nikolaos Dergiades, Thessaloniki, Greece.*

Let A, B, C be the angles and a, b, c be the sides of $\triangle ABC$. Then, from $c + AD = CD$ and $b + AD = a$, we get $a - CD = b - c$, so that $BD = b - c$ and $AD = a - b$.

If E is the point between A and C with $AE = AB$, then $EC = AC - AE = b - c = BD = DE$, so that $\triangle DEC$ is isosceles, and $B = \angle DEA = 2C$.

[*Editor's comment.* Note that the argument up to here has repeated the first step of solution I, avoiding reference to the figure. Several other solvers derived $B = 2C$ by first showing that $ca + c^2 = b^2$, noting that these equivalent conditions are familiar to ***CRUX with MAYHEM*** readers, having appeared in [1976: 74], [1984: 278], [1996: 265], etc.] [It follows that

$\sin A = \sin(C+B) = \sin(3C)$, and $\sin\left(\frac{A}{2}\right) = \sin\left(\frac{\pi}{2} - \frac{(B+C)}{2}\right) = \cos\left(\frac{3C}{2}\right)$.]
The Sine Law applied to $\triangle ABD$, and then to $\triangle ABC$, gives

$$\begin{aligned}
\frac{AD}{\sin B} = \frac{BD}{\sin(A/2)} \quad &\Longrightarrow \quad \frac{a-b}{\sin B} = \frac{b-c}{\sin(A/2)} \\
&\Longrightarrow \quad \frac{\sin A - \sin B}{\sin B} = \frac{\sin B - \sin C}{\sin(A/2)} \\
&\Longrightarrow \quad \frac{\sin(3C) - \sin(2C)}{\sin(2C)} = \frac{\sin(2C) - \sin C}{\cos(3C/2)} \\
&\Longrightarrow \quad \frac{2\sin(C/2)\cos(5C/2)}{\sin(2C)} = \frac{2\sin(C/2)\cos(3C/2)}{\cos(3C/2)} \\
&\Longrightarrow \quad \cos(5C/2) = \sin(2C) \Longrightarrow \frac{5C}{2} + 2C = 90^\circ\,.
\end{aligned}$$

Thus, $C = 20^\circ$, $B = 40^\circ$ and $A = 120^\circ$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines and GIOVANNI MAZZARELLO, Ferrovie dello Stato, Florence, Italy; FLORIAN HERZIG, student, Cambridge, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; ECKARD SPECHT, Otto-von-Guericke-Universität, Magdeburg, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; PARAYIOU THEOKLITOS, Limassol, Cyprus; PANOS E. TSAOUSSOGLOU, Athens, Greece; JOHN VLACHAKIS, Athens, Greece; and the proposer. There was one incomplete solution.

2303. [1998: 45] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that ABC is a triangle with angles B and C satisfying $C = 90^\circ + \frac{1}{2}B$, that the exterior bisector of angle A intersects BC at D, and that the side AB touches the incircle of $\triangle ABC$ at E.

Prove that $CD = 2AE$.

Solution by Nikolaos Dergiades, Thessaloniki, Greece.

We know that $2AE = b + c - a$ (see, for example, **290**, Roger A. Johnson, *Modern Geometry*, Houghton-Mifflin, 1929). In the extension of BA we take $AF = b = AC$.

Since AD is the external bisector of the angle A of $\triangle ABC$, F is the symmetric point of C with axis of symmetry AD. Then

$$\angle AFD \;=\; \angle ACD \;=\; 180 - \angle C \;=\; 90 - \frac{\angle B}{2}\,,$$

so $\triangle BFD$ is isosceles; that is, $BD = BF$ or

$$a + CD \;=\; b + c \qquad \text{or} \qquad CD \;=\; b + c - a = 2AE\,.$$

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain (2 solutions); SAM BAETHGE, Nordheim, Texas, USA; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; MANSUR BOASE, student, Cambridge, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; FLORIAN HERZIG, student, Cambridge, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; JOEL SCHLOSBERG, student, Bayside, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke Universität, Magdeburg, Germany; PARAYIOU THEOKLITOS, Limassol, Cyprus; JOHN VLACHAKIS, Athens, Greece; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA; and the proposer.

2304. [1998: 46] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

An acute angled triangle ABC is given, and equilateral triangles ABD and ACE are drawn outwardly on the sides AB and AC. Suppose that CD and BE meet AB and AC at F and G respectively, and that CD and BE intersect at P.

Suppose that the area of the quadrilateral $AFPG$ is equal to the area of the triangle PBC. Determine angle BAC.

Solution by Michael Lambrou, University of Crete, Crete, Greece.

If $[AFPG] = [PBC]$, then $[ABG] = [BFC]$ and so $\frac{1}{2}\overline{AG}c\sin A = \frac{1}{2}\overline{BF}a\sin B$; hence, by the Sine Rule,

$$\frac{AG}{BF} = \frac{a\sin B}{c\sin A} = \frac{b\sin A}{c\sin A} = \frac{b}{c}\,.$$

Similarly, $\dfrac{AF}{CG} = \dfrac{c}{b}$, so that $\dfrac{AG}{CG} = \dfrac{BF}{AF}$.

Writing $\angle AEG = \varphi$, $\angle BDF = \omega$, we have from the equilateral triangles

$$\frac{\sin\varphi}{\sin(60-\varphi)} = \frac{\sin\angle AEG}{\sin\angle GEC} = \frac{AG}{CG} = \frac{BF}{AF} = \frac{\sin\angle BDF}{\sin\angle FDA} = \frac{\sin\omega}{\sin(60-\omega)}\,.$$

Hence, $\sin\varphi\left(\dfrac{\sqrt{3}}{2}\cos\omega - \dfrac{1}{2}\sin\omega\right) = \left(\dfrac{\sqrt{3}}{2}\cos\varphi - \dfrac{1}{2}\sin\varphi\right)\sin\omega$ and $\sin(\varphi-\omega) = 0$. This shows that $\varphi = \omega$ (as both are acute).

Recall that P (Fermat or Napoleon point) satisfies $\angle FPG = 120^\circ$. Now summing the angles of quadrilateral $DAEP$, we have

$$\begin{aligned} 360^\circ &= \angle ADP + \angle DPE + \angle PEA + (60^\circ + \angle BAC + 60^\circ) \\ &= 60^\circ - \omega + 120^\circ + \varphi + 120^\circ + \angle BAC\,, \end{aligned}$$

and hence $\angle BAC = 60^\circ$.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; NIKOLAOS DERGIADES, Thessaloniki, Greece; FLORIAN HERZIG, student, Cambridge, UK; GERRY LEVERSHA, St. Paul's School, London, England; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke Universität, Magdeburg, Germany; PARAYIOU THEOKLITOS, Limassol, Cyprus; and the proposer.

2305. [1998: 46] *Proposed by Richard I. Hess, Rancho Palos Verdes, California, USA.*

An integer-sided triangle has angles $p\theta$ and $q\theta$, where p and q are relatively prime integers. Prove that $\cos\theta$ is rational.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria (slightly modified by the editor).

Let $\mathbb{Q}$ denote the set of all rationals. By the Cosine Law, we see that $\cos(p\theta) \in \mathbb{Q}$ and $\cos(q\theta) \in \mathbb{Q}$.

Furthermore, since $\cos((p+q)\theta) = -\cos(\pi - (p+q)\theta)$, we have as well that $\cos((p+q)\theta) \in \mathbb{Q}$.

From $\cos(p\theta)\cos(q\theta) = \frac{1}{2}\left(\cos((p+q)\theta) + \cos((p-q)\theta)\right)$, we infer that $\cos((p-q)\theta) \in \mathbb{Q}$.

Similarly, from $\cos(p\theta)\cos((p\pm q)\theta) = \frac{1}{2}\left(\cos(2p \pm q)\theta + \cos(q\theta)\right)$, we infer that $\cos((2p\pm q)\theta) \in \mathbb{Q}$.

Analogously, we have $\cos((p \pm 2q)\theta) \in \mathbb{Q}$.

A simple induction then yields that $\cos((lp+kq)\theta) \in \mathbb{Q}$ for all integers l and k. The result now follows, since there exist integers l and k such that $lp + kq = 1$.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, Athens, Greece; DIANE and RAY DOWLING, University of Manitoba, Winnipeg, Manitoba; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; FLORIAN HERZIG, student, Cambridge, UK; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; PAVLOS MARAGOUDAKIS, Pireas, Greece; JOEL SCHLOSBERG, student, Bayside, NY, USA; and the proposer.

Besides Janous, Godin also obtained the stronger result. Klamkin showed that $\cos((lp-kq)\theta) \in \mathbb{Q}$ for all non-negative integers l and k. Their arguments, which are very similar, all used induction. The Dowlings gave a proof based on de Moivre's Theorem and the Binomial Theorem. Bradley gave the following example to show that such triangles do exist: $a = 135$, $b = 144$, $c = 31$. In this case, the angles of the triangle are 2θ, 3θ and $\pi - 5\theta$, where $\cos\theta = \frac{5}{6}$. (Thus, $\theta = \cos^{-1}\left(\frac{5}{6}\right) \approx 0.5857$ radians, or 33.5573° — Ed.)

2306. [1998: 46, 175] *Proposed by Vedula N. Murty, Visakhapatnam, India.*

(a) Give an elementary proof of the inequality:

$$\left(\sin\left(\frac{\pi x}{2}\right)\right)^2 > \frac{2x^2}{1+x^2}\,; \qquad (0 < x < 1)\,. \tag{1}$$

(b) Hence (or otherwise) show that

$$\tan \pi x \begin{cases} < \frac{\pi x(1-x)}{1-2x}\,; & \left(0 < x < \frac{1}{2}\right)\,, \\ > \frac{\pi x(1-x)}{1-2x}\,; & \left(\frac{1}{2} < x < 1\right)\,. \end{cases} \tag{2}$$

(c) Find the maximum value of $f(x) = \dfrac{\sin(\pi x)}{x(1-x)}$ on the interval $(0,1)$.

Comments on (a) *and* (c) *by the editor.*

As pointed out by several solvers (indicated by a dagger † before their names), part (a) is identical to the first half of problem **2296** [1997: 503] with solution on [1998: 533]. Part (c) follows immediately from the second half of that problem since $f\left(\frac{1}{2}\right) = 4$, where $f(x) = \frac{\sin(\pi x)}{x(1-x)}$. This, together with the (corrected) inequality $f(x) \le 4$, shows that $\max f(x) = 4$.

Solution to (b) *by* †*Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Consider the function $f(x) = \dfrac{\pi x(1-x)}{1-2x} - \tan(\pi x), \quad 0 < x < 1.$ Then $f(0) = 0$. We claim that $f'(x) > 0$ for all $x \in (0, \frac{1}{2})$.

Since $f'(x) = \pi\left(\dfrac{(1-2x)^2 + 2x(1-x)}{(1-2x)^2} - \dfrac{1}{\cos^2(\pi x)}\right)$, it suffices to show that

$$\cos^2(\pi x) > \frac{(1-2x)^2}{(1-2x)^2 + 2x(1-x)} \tag{3}$$

Letting $y = 1 - 2x$, we have $y \in (0,1)$ and $x = \frac{1}{2}(1-y)$. Simple manipulations show that (3) is equivalent to

$$\sin^2\left(\frac{\pi y}{2}\right) > \frac{y^2}{y^2 + (1-y)(1+y)/2} \qquad \text{or} \qquad \sin^2\left(\frac{\pi y}{2}\right) > \frac{2y^2}{1+y^2}\,,$$

which is valid by (a). Hence $f(x) > 0$ for all $x \in (0, \frac{1}{2})$.

If $\frac{1}{2} < x < 1$, then $0 < 1 - x < \frac{1}{2}$, and hence, by the result just established, we have $f(1-x) > 0$; that is,

$$\frac{\pi x(1-x)}{1-2(1-x)} > \tan(\pi(1-x))\,, \qquad \text{or} \qquad \frac{\pi x(1-x)}{-(1-2x)} > -\tan(\pi x)\,,$$

from which we obtain that $\tan(\pi x) > \dfrac{\pi x(1-x)}{1-2x}$.

Also solved by GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; RICHARD I. HESS, Rancho Palos Verdes, California, USA; †VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA; †HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer. Part (a) *only was solved by AISSA GUESMIA, Institut de recherche mathématique avancée, Université Louis Pasteur et CNRS, Strasbourg, France. There was also one incorrect solution.*

Seiffert also cited the following known inequalities (V.D. Mascioni und K. Schütte, Aufgabe 979, Elem. Math. 44, 1989, 20):

$$\frac{8x(1-x)}{\pi(1-2x)} < \tan(\pi x) < \frac{\pi x}{1-4x^2}, \qquad 0 < x < \tfrac{1}{2},$$

and noted that the right inequality of this refines the right inequality in (b). *Replacing x by $x - \frac{1}{2}$, he derived the following inequalities:*

$$\frac{\pi(1-x)}{(1-2x)(3-2x)} < \tan(\pi x) < \frac{8x(1-x)}{\pi(1-2x)}, \qquad \tfrac{1}{2} < x < 1,$$

and remarked that the left inequality of this is sharper than the left inequality in (b). *He also pointed out that this problem and problem* **2296** *are both simple consequences of problem* **519** [1980: 44; 1981: 65] *and problem 144* [Matyc J. 14, 1980, 72 *and* 15, 1981, 156]. *All of these problems were posed by this proposer.*

2307. [1998: 46] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

It is known that every regular $2n$-gon can be dissected into $\binom{n}{2}$ rhombuses with the same side length.

(a) How many different classes of rhombuses are there?

(b) How many rhombuses are there in each class?

Solution by Michael Lambrou, University of Crete, Crete, Greece.

It is stated on page 10 of *Disections, Plane and Fancy* by G.N. Frederickson, Cambridge University Press, 1977, that a regular $2n$-gon can be dissected into rhombi with the same side length according to the following rules:

(a) If n is odd with $n = 2m + 1$, then the $2n$-gon can be dissected into n such rhombi, each of angle $j\pi/n$, for each $j = 1, 2, \ldots, m$. In other words, we have $m = \frac{n-1}{2}$ classes of rhombi with n rhombi in each class, making a total of $\frac{1}{2}n(n-1) = \binom{n}{2}$ pieces.

(b) If n is even with $n = 2m$, then the $2n$-gon can be dissected into n such rhombi, each of angle $j\pi/n$, for each $j = 1, 2, \ldots, m-1$, plus m squares. In other words, we have $m = \frac{n}{2}$ classes of rhombi with a total of $n(m-1) + m = 2m(m-1) + m = m(2m-1) = \frac{1}{2}n(n-1) = \binom{n}{2}$ pieces.

Although the proof is not given in detail in Frederickson, the accompanying figures for the cases $n = 2, 3, 4, 5$ and 6 make it obvious how to do the general case, so there is no need to repeat the details here.

Also solved by the proposer.

2308. [1998: 46] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

A sequence $\{v_n\}$ has initial value $v_0 = 1$ and, for $n \geq 0$, satisfies the recurrence relation

$$v_{n+1} = 2^{n+1} - \sum_{k=0}^{n} v_k\, v_{n-k}\,.$$

Find a formula for v_n in terms of n.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

We use the method of generating functions and show

$$v_n \;=\; \binom{n}{\lfloor n/2 \rfloor}\,.$$

where $\lfloor x \rfloor$ is the greatest integer not exceeding x. Let

$$f(x) \;=\; \sum_{n=0}^{\infty} v_n x^n\,.$$

Then using $f^2(x)$ to mean $[f(x)]^2$ we get

$$f^2(x) \;=\; \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} v_k v_{n-k}\right) x^n \;=\; \sum_{n=0}^{\infty}(-v_{n+1} + 2^{n+1})x^n\,.$$

Hence (via $-v_0 + 1 = 0$) we get

$$\begin{aligned} xf^2(x) &= \sum_{n=0}^{\infty}(-v_{n+1} + 2^{n+1})x^{n+1} \;=\; \sum_{m=1}^{\infty}(-v_m + 2^m)x^m \\ &= \sum_{m=0}^{\infty}(-v_m + 2^m)x^m \;=\; -f(x) + \frac{1}{1-2x}\,; \\ \text{that is,}\quad 0 &= xf^2(x) + f(x) - \frac{1}{1-2x}\,, \end{aligned}$$

from which it follows that

$$f(x) \;=\; \frac{1}{2x}\left(-1 \pm \sqrt{\frac{1+2x}{1-2x}}\right)\,.$$

Because of $f(0) = 1$ we have to use the positive root. Thus

$$f(x) \;=\; \frac{1}{2x}\left(-1 + (1+2x)(1-4x^2)^{-1/2}\right)\,.$$

It is known that

$$(1-4z)^{-1/2} \;=\; \sum_{k=0}^{\infty}\binom{2k}{k} z^k\,.$$

Hence

$$f(x) = \frac{1}{2x}\left(-1+(1+2x)\sum_{k=0}^{\infty}\binom{2k}{k}x^{2k}\right)$$
$$= \sum_{k=0}^{\infty}\binom{2k}{k}x^{2k} + \sum_{k=1}^{\infty}\frac{1}{2}\binom{2k}{k}x^{2k-1}\,;$$

that is

$$v_{2k} = \binom{2k}{k} \qquad \text{and} \qquad v_{2k-1} = \frac{1}{2}\binom{2k}{k} = \binom{2k-1}{k-1}\,,$$

showing $v_n = \binom{n}{\lfloor n/2\rfloor}$, as claimed.

Also solved by MICHEL BATAILLE, Rouen, France; PAUL BRACKEN, CRM, Université de Montréal, Montréal, Québec; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; FLORIAN HERZIG, student, Cambridge, UK; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer. There was one incomplete solution submitted.

Dimminie remarks that if $\{C_n\}$ *is the sequence of Catalan numbers defined by* $C_0 = 1$ *and*

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$$

for $n \geq 0$, *then the solution may also be expressed as*

$$v_{2n} = (n+1)C_n \qquad \text{for } n \geq 0$$

and

$$v_{2n-1} = (2n-1)C_{n-1} \qquad \text{for } n \geq 1\,.$$

Janous also observes that by using

$$(1-4z)^{1/2} = -\sum_{k=0}^{\infty}\binom{2k}{k}\frac{z^k}{2k-1}$$

we get

$$f(x) = \frac{1}{2x}\left(-1+\frac{(1-4x^2)^{1/2}}{1-2x}\right)$$
$$= -\frac{1}{2x}\left(1+\sum_{k=0}^{\infty}\binom{2k}{k}\frac{x^{2k}}{2k-1}\cdot\sum_{\ell=0}^{\infty}2^{\ell}x^{\ell}\right)$$
$$= -\frac{1}{2x}\left[1+\sum_{m=0}^{\infty}\left(\sum_{2k+\ell=m}\binom{2k}{k}\frac{2^{\ell}}{2k-1}\right)x^m\right]$$
$$= -\sum_{m=1}^{\infty}\left(\sum_{2k+\ell=m}\binom{2k}{k}\frac{2^{\ell-1}}{2k-1}\right)x^{m-1}$$
$$= -\sum_{n=0}^{\infty}\left(\sum_{2k+\ell=n+1}\binom{2k}{k}\frac{2^{\ell-1}}{2k-1}\right)x^{n}\,.$$

Hence we get the curious (and apparently new) identity [compare coefficients and replace ℓ by $n+1-2k$]:

$$\sum_{k=0}^{\lfloor (n+1)/2 \rfloor} \binom{2k}{k} \frac{2^{n+1-2k-1}}{2k-1} \;=\; -\binom{n}{\lfloor n/2 \rfloor}\,,$$

which simplifies to

$$\sum_{k=0}^{\lfloor (n+1)/2 \rfloor} \binom{2k}{k} \frac{1}{4^k(2k-1)} \;=\; -\frac{1}{2^n}\binom{n}{\lfloor n/2 \rfloor}\,.$$

2310. [1998: 47] *Proposed by K.R.S. Sastry, Dodballapur, India.*

Let $n \in \mathbb{N}$. I call a positive integral divisor of n, say d, a *unitary* divisor if $\gcd(d, n/d) = 1$.

Let $\Upsilon(n)$ denote the sum of the unitary divisors of n.

Find a characterization of n so that $\Upsilon(n) \equiv 2 (\bmod 4)$.

Solution by Mansur Boase, student, Cambridge, England.

Suppose $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$, where p_i are distinct primes and $\alpha_i > 0$ for all i, $1 \le i \le r$. Then

$$\Upsilon(n) \;=\; (1+p_1^{\alpha_1})(1+p_2^{\alpha_2})\ldots(1+p_r^{\alpha_r})\,,$$

as each term when this is multiplied out is a unitary divisor of n and every unitary divisor of n occurs exactly once in the expansion.

If $\Upsilon(n) \equiv 2 \pmod 4$, then it must be even, but not divisible by 4. For odd p_i, the term $(1 + p_i^{\alpha_i})$ is always even, so at most one odd prime can divide n and n must be of the form $2^m p^k$.

Now $\Upsilon(n) \equiv 2 \pmod 4$ if and only if $(1 + p^k) \equiv 2 \pmod 4$. If $p \equiv 1 \pmod 4$, this will be the case for any k; if $p \equiv 3 \pmod 4$, then k must be even.

Therefore, $n = 2^m p^k$ with $m \ge 0$, p an odd prime, with k an arbitrary positive integer if $p \equiv 1 \pmod 4$ and k an arbitrary even integer if $p \equiv 3 \pmod 4$.

Also solved by GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; NIKOLAOS DERGIADES, Thessaloniki, Greece; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; KENNETH M. WILKE, Topeka, Kansas, USA; and the proposer. There were two incorrect solutions submitted.

2311. [1998: 47] *Proposed by K.R.S. Sastry, Dodballapur, India.*

Let $\Upsilon_e(n)$ denote the sum of the even unitary divisors, and $\Upsilon_o(n)$, the sum of the odd unitary divisors, of n. Assume that $\Upsilon_e(n) - \Upsilon_o(n) = n$.

(a) If n is composed of powers of exactly two distinct primes, show that n must be the product of two consecutive integers, one of which is a Mersenne prime.

(b) Give an example of a natural number n that is composed of powers of more than two distinct primes.

Solution by Kenneth M. Wilke, Topeka, Kansas, USA.

For $n = 2^d p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k}$ where $d \geq 1$ and each p_i denotes an odd prime which divides n, we have

$$\Upsilon_o(n) = \prod_{i=1}^{k} (p_i^{e_i} + 1) \qquad \text{and} \qquad \Upsilon_e(n) = 2^d \prod_{i=1}^{k} (p_i^{e_i} + 1)\,,$$

so

$$n = \Upsilon_e(n) - \Upsilon_o(n) = (2^d - 1) \prod_{i=1}^{k} (p_i^{e_i} + 1)\,. \tag{1}$$

(a) Here $i = 1$ so that [writing p for p_1 and e for e_1]

$$n = 2^d p^e = (2^d - 1)(p^e + 1)$$

and hence

$$2^d = p^e + 1 \tag{2}$$

where d and e are positive integers. Since $p \geq 3$, we must have $e \geq 1$ and $d \geq 2$. Suppose e is even, say $e = 2l$. Then $p^{2l} \equiv 1 \bmod 4$ regardless of the choice of the odd prime p and the integer l. Then since $2^d \equiv 0 \bmod 4$ for all integers $d \geq 2$, we have that

$$p^e + 1 = p^{2l} + 1 \equiv 2 \not\equiv 0 \equiv 2^d \bmod 4\,,$$

which is impossible from (2). Hence e must be odd, say $e = 2l + 1$. Then from (2) we have

$$2^d = (p + 1)(p^{2l} - p^{2l-1} + \cdots - p + 1)\,.$$

Since the second factor on the right hand side is odd, equality can be maintained only if $l = 0$ and $p = 2^d - 1$, a Mersenne prime.

(b) From (1) and our choice of n,

$$1 = \left(\frac{2^d - 1}{2^d}\right) \left(\frac{p_1^{e_1} + 1}{p_1^{e_1}}\right) \left(\frac{p_2^{e_2} + 1}{p_2^{e_2}}\right) \cdots \left(\frac{p_k^{e_k} + 1}{p_k^{e_k}}\right)\,.$$

We can use the values $d = 3$, $(p_1, e_1) = (5, 2)$, $(p_2, e_2) = (7, 2)$ and $(p_3, e_3) = (13, 1)$ to find

$$n = 2^3 \cdot 5^2 \cdot 7^2 \cdot 13 = 127400$$

as a solution for part (b).

Editorial remark. The above solution was also found by solver Hess and by the proposer. The smallest solution to part (b), and the one most solvers gave, is $n = 180$. Other solutions found are

$$n = 441000 \qquad \text{(found only by the proposer)}$$

and

$$n = 2646000 \qquad \text{(found by Hess and Leversha).}$$

Are there any others?

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; NIKOLAOS DERGIADES, Thessaloniki, Greece; FLORIAN HERZIG, student, Cambridge, UK; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; JOEL SCHLOSBERG, student, Bayside, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; and the proposer.

In part (a), *one solver obtained equation* (2) *and then concluded rather hastily that* $e = 1$ *and so* p *is a Mersenne prime. Another solver believed* (2) *to imply that* $e = 1$ *but admitted he had no proof. Both have been given the benefit of the doubt (in the spirit of the season — this is being written during the Christmas holidays). Other solvers merely said that this is a known result, or proved it themselves (as Wilke did), or gave a reference. For example, Lambrou mentioned the article "A note on Mersenne numbers", by Ligh and Neal, on pp.* 231–233 *of* Math. Magazine **47** (1974), *and Manes quoted problem E1221 of the* Amer. Math. Monthly, *solution in Volume* **64** (1957), *p.* 110. *The solution of the* Monthly *problem also contains some much earlier references to this result.*