THE OLYMPIAD CORNER

No. 193

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As a first Olympiad to give you puzzling pleasure, we give the 18th Austrian-Polish Mathematics Competition written in Austria June 28–30, 1995. My thanks go to Bill Sands, University of Calgary, who collected this contest while assisting at the International Olympiad in Toronto in 1995, as well as to Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

18th AUSTRIAN-POLISH MATHEMATICS COMPETITION

Problems of the Individual Contest

June 28-29, 1995 (Time: 4.5 hours)

1. For a given integer $n \geq 3$ find all solutions $(a_1, \dots, a_n)$ of the system of equations

$$a_3 = a_2 + a_1, \quad a_4 = a_3 + a_2, \dots, a_n = a_{n-1} + a_{n-2}$$

$$a_1 = a_n + a_{n-1}, \quad a_2 = a_1 + a_n$$

in real numbers.

2. Let A_1, A_2, A_3, A_4 be four distinct points in the plane and let $X = \{A_1, A_2, A_3, A_4\}$. Show that there exists a subset Y of the set X with the following property: there is no disc K such that $K \cap X = Y$. *Note:* All points of the circle limiting a disc are considered to belong to the disc.

3. Let $P(x) = x^4 + x^3 + x^2 + x + 1$. Show that there exist polynomials $Q(y)$ and $R(y)$ of positive degrees, with integer coefficients, such that $Q(y) \cdot R(y) = P(5y^2)$ for all y.

4. Determine all polynomials $P(x)$ with real coefficients, such that

$$(P(x))^2 + (P(1/x))^2 = P(x^2)P(1/x^2) \quad \text{for all} \quad x \neq 0.$$

5. An equilateral triangle ABC is given. Denote the mid-points of sides BC, CA, AB respectively by A_1, B_1, C_1. Three distinct parallel lines p, q, r are drawn through A_1, B_1, C_1, respectively. Line p cuts B_1C_1 at A_2;

line q cuts C_1A_1 at B_2; line r cuts A_1B_1 at C_2. Prove that the lines AA_2, BB_2, CC_2 concur at a point D lying on the circumcircle of triangle ABC.

6. The Alpine Club consisting of n members organizes four high-mountain expeditions for its members. Let E_1, E_2, E_3, E_4 be the four teams participating in these expeditions. How many ways are there to compose those teams, given the condition that $E_1 \cap E_2 \neq \emptyset$, $E_2 \cap E_3 \neq \emptyset$, $E_3 \cap E_4 \neq \emptyset$?

Problems of the Team Contest
June 30, 1995 (Time: 4 hours)

7. For every integer c consider the equation $3y^4 + 4cy^3 + 2xy + 48 = 0$, with integer unknowns x and y. Determine all integers c for which the number of solutions (x, y) in pairs of integers satisfying the additional conditions (A) and (B) is a maximum:

(A) the number $|x|$ is the square of an integer;

(B) the number y is square-free (that is, there is no prime p with p^2 dividing y).

8. Consider the cube with vertices $\{\pm 1, \pm 1, \pm 1)$; that is, the set $\{(x, y, z) : |x| \leq 1, |y| \leq 1, |z| \leq 1\}$. Let $V_1, \dots, V_{95}$ be points of that cube. Denote by v_i the vector from $(0, 0, 0)$ to V_i. Consider the 2^{95} vectors of the form $s_1v_1 + s_2v_2 + \cdots + s_{95}v_{95}$, where $s_i = 1$ or $s_i = -1$.

(a) Let $d = 48$. Show that among all such vectors one can find a vector $w = (a, b, c)$ with $a^2 + b^2 + c^2 \leq d$.

(b) Find a number $d < 48$ with the same property.

Note: The smaller d, the better mark will be attracted by the solution.

9. Prove that the following inequality holds for all integers $n, m \geq 1$ and all positive real numbers x, y:

$$(n-1)(m-1)(x^{n+m} + y^{n+m}) + (n+m-1)(x^n y^m + x^m y^n) \\ \geq nm(x^{n+m-1}y + xy^{n+m-1}).$$

The next contest we give was also collected by Bill Sands while he was assisting at the IMO in Toronto. These are the problems of the 9th Iberoamerican Mathematical Olympiad held September 20, 21 in Fortaleza, Brazil. Students were given $4\frac{1}{2}$ hours each day.

9th IBEROAMERICAN MATHEMATICAL OLYMPIAD
Fortaleza, Brazil, September 20–21, 1994
First Day — Time: 4.5 hours

1. (Mexico): A natural number n is called *brazilian* if there exists an integer r, with $1 < r < n - 1$, such that the representation of the number

n in base r has all the digits equal. For example, 62 and 15 are brazilian, because 62 is written 222 in base 5 and 15 is 33 in base 4. Prove that 1993 is **not** brazilian, but 1994 is brazilian.

2. (Brazil): Let $ABCD$ be a cyclic quadrilateral. We suppose that there exists a circle with centre in AB, tangent to the other sides of the quadrilateral.

(i) Show that $AB = AD + BC$.

(ii) Calculate, in terms of $x = AB$ and $y = CD$, the maximal area that such a quadrilateral can reach.

3. (Brazil): In each cell of an $n \times n$ chessboard is a lamp. When a lamp is touched, the state of this lamp, and also the state of all the lamps in its row and in its column, is changed (switched from OFF to ON and vice versa). At the beginning, all the lamps are OFF. Show that it is always possible, with suitable sequence of touches, to turn ON all the lamps of the chessboard, and find, in terms of n, the minimal number of touches in order that all the lamps of the chessboard are ON.

Second Day — Time: 4.5 hours

4. (Brazil): The triangle ABC is acute, with circumcircle k. Let P be an internal point to k. The lines AP, BP, CP meet k again at X, Y, Z. Determine the point P for which triangle XYZ is equilateral.

5. (Brazil): Let n and r be two positive integers. We wish to construct r subsets of $\{0, 1, \ldots, n-1\}$, called $A_1, \ldots, A_r$, with $\text{card}(A_i) = k$ and such that, for each integer x, $0 \leq x \leq n-1$, there exist $x_1 \in A_1$, $x_2 \in A_2, \ldots, x_r \in A_r$ (an element in each subset), with

$$x = x_1 + x_2 + \cdots + x_r.$$

Find, in terms of n and r, the minimal value of k.

6. (Brazil): Show that all natural numbers $n \leq 2^{1000000}$ can be obtained beginning at 1 with less than 1100000 sums; that is, there exists a finite sequence of natural numbers $x_0, x_1, \ldots, x_k$, with $k < 1100000$, $x_0 = 1$, $x_k = n$, such that for each $i = 1, 2, \ldots, k$, there exists r, s, with $0 \leq r < i$, $0 \leq s < i$, and $x_i = x_r + x_s$.

As a final problem set to challenge you, we present the problems of the IX, X and XI Grade of the Georgian Mathematical Olympiad, Final Round for 1995. It is interesting that 60% of the Grade XI problems come from the Grade IX paper. My thanks again go to Bill Sands, University of Calgary, for collecting these problems while he assisted with the IMO in Toronto in 1995.

GEORGIAN MATHEMATICAL OLYMPIAD 1995
Final Round
GRADE IX

1. A three-digit number was decreased by the sum of its digits. Then the result was decreased by the sum of *its* digits and so on. Show that on the 100^{th} step of this procedure the result will be zero, whatever the initial three-digit number is chosen. How many repetitions are enough to get zero?

2. Two circles of the same size are given. Seven arcs, each of them of 3° measure, are taken on the first circle and 10 arcs, each of them of 2° measure, are taken on the second one. Prove that it is possible to place one circle on the other so that these arcs do not intersect. Is it or is it not possible to prove the same if the number of arcs with measure 2° is 11?

3. Prove that if the product of three positive numbers is 1 and their sum is more than the sum of their reciprocals, then only one of these numbers can be more than 1.

4. Prove that in any convex hexagon there exists a diagonal which cuts from the hexagon a triangle with area less than $\frac{1}{6}$ of the area of the hexagon.

5. The set M of integers has the following property: if the numbers a and b are in M, then $a + 2b$ also belongs to M. It is known that the set contains positive as well as negative numbers. Prove that if the numbers a, b and c are in M, then $a + b - c$ is also in M.

GRADE X

1. (a) Five different numbers are written in one line. Is it always possible to choose three of them placed in increasing or decreasing order?

(b) Is it always possible to do the same, if we have to choose four numbers from nine?

2. (Same as IX.2)

3. Prove that for any natural number n, the average of all its factors lies between the numbers $\sqrt{n}$ and $\frac{n+1}{2}$.

4. The incircle of a triangle divides one of its medians into three equal parts. Find the ratio of the sides of the triangle.

5. The function f is given on the segment $[0,1]$. It is known that $f(x) \geq 0$ and $f(1) = 1$. Besides that, for any two numbers x_1 and x_2, if $x_1 \geq 0$, $x_2 \geq 0$ and $x_1 + x_2 \leq 1$, then $f(x_1 + x_2) \geq f(x_1) + f(x_2)$.

(a) Prove that $f(x) \leq 2x$ for any x.

(b) Does the inequality $f(x) \leq 1.9x$ hold for every x?

GRADE XI

1. (Same as IX.3)

2. (Same as IX.1)

3. How many solutions has the equation $x = 1995 \sin x + 199$?

4. (Same as IX.4)

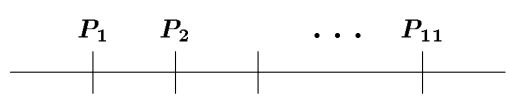
5. A natural number is written in each square of an $m \times n$ rectangular table. By one move, it is allowed to double all numbers of any row or subtract 1 from all numbers of any column. Prove that by repeating these moves several times, all numbers in the table become zeros.

Next a bit of housekeeping. After the columns were set, and before they appeared in print form, I received solutions from Pavlos Maragoudakis to the six problems of the Swedish contest for which we published the solutions last issue [1997: 196; 1998: 328–329]. He also submitted solutions to problems 1, 2, 3 and 5 of the Dutch Mathematical Olympiad, Second Round, 1993 [1997: 197; 1998: 329–332]. Last issue we gave solutions to all but the last problem. It was rather unfortunate timing in terms of acknowledging his contribution, but we are able to close out the file by having a complete set of solutions from the readers.

5. $P_1, P_2, \ldots, P_{11}$ are eleven distinct points on a line. $P_iP_j \leq 1$ for every pair P_i, P_j. Prove that the sum of all (55) distances P_iP_j, $1 \leq i < j \leq 11$ is smaller than 30.

Solution by Pavlos Maragoudakis, Pireas, Greece.

Without loss of generality we suppose that $P_1, P_2, \ldots, P_{11}$ are adjacent.



Now if $1 \leq i < j \leq 11$, then $P_iP_j = P_jP_1 - P_iP_1$. So

$$\begin{aligned}
\sum_{1 \leq i < j \leq 11} P_iP_j &= \sum_{1 \leq i < j \leq 11} (P_jP_1 - P_iP_1) \\
&= 10P_{11}P_1 + 9P_{10}P_1 - P_{10}P_1 + 8P_9P_1 - 2P_9P_1 \\
&\quad +7P_8P_1 - 3P_8P_1 + 6P_7P_1 - 4P_7P_1 + 5P_6P_1 \\
&\quad -5P_6P_1 + 4P_5P_1 - 6P_5P_1 + 3P_4P_1 - 7P_4P_1 \\
&\quad +2P_3P_1 - 8P_3P_1 + P_2P_1 - 9P_2P_1
\end{aligned}$$

$$\begin{aligned} &= 10P_{11}P_1 + 8P_{10}P_1 + 6P_9P_1 + 4P_8P_1 + 2P_7P_1 \\ &\quad -2P_5P_1 - 4P_4P_1 - 6P_3P_1 - 8P_2P_1 \\ &= 10P_{11}P_1 + 8(P_{10}P_1 - P_2P_1) + 6(P_9P_1 - P_3P_1) \\ &\quad +4(P_8P_1 - P_4P_1) + 2(P_7P_1 - P_5P_1) \\ &= 10P_{11}P_1 + 8P_{10}P_2 + 6P_9P_3 + 4P_8P_4 + 2P_7P_5 \\ &< 10 \cdot 1 + 8 \cdot 1 + 6 \cdot 1 + 4 \cdot 1 + 2 \cdot 1 = 30 \end{aligned}$$

Also setting the record straight, I found amongst the solutions for another contest, the solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain to problem 2 of the Dutch Mathematical Olympiad, Second Round, for which we published a solution last issue [1997: 197, 1998: 330–331]. My apologies.

While we do not normally give solutions to problems of the USAMO, I am giving two comments/solutions from our readers to problems of the USAMO 1997 [1997: 261, 262].

2. Let ABC be a triangle, and draw isosceles triangles BCD, CAE, ABF externally to ABC, with BC, CA, AB as their respective bases. Prove that the lines through A, B, C perpendicular to the lines $\overleftrightarrow{EF}, \overleftrightarrow{FD}, \overleftrightarrow{DE}$, respectively, are concurrent.

Comment by Mansur Boase, student, St. Paul's School, London, England.

The result is immediate from Steiner's Theorem:

If the perpendiculars from the vertices A, B, C of a triangle ABC to the sides B_1C_1, C_1A_1, and A_1B_1, respectively, of a second triangle A_1, B_1, C_1 are concurrent, then the perpendiculars from the vertices A_1, B_1, C_1 of the triangle $A_1B_1C_1$ to the sides BC, CA, AB are also concurrent.

5. Prove that for positive real numbers a, b, c,

$$(a^3 + b^3 + abc)^{-1} + (b^3 + c^3 + abc)^{-1} + (c^3 + a^3 + abc)^{-1} \le (abc)^{-1}.$$

Solutions by Mansur Boase, student, St. Paul's School, London, England and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give Klamkin's presentation.

Since the inequality is homogeneous, we can assume $abc = 1$. Then if we let $x = a^3$, $y = b^3$, $z = c^3$, the inequality becomes

$$\frac{1}{1+x+y} + \frac{1}{1+y+z} + \frac{1}{1+z+x} \le 1 \qquad (1)$$

where $xyz = 1$ and x, y, z are positive. On expanding, (1) is equivalent to

$$(x+y+z)(xy+yz+zx-2) \ge 3 .$$

This follows from the known elementary inequalities

$$\frac{x+y+z}{3} \geq \left(\frac{yz+zx+xy}{3}\right)^{1/2} \geq (xyz)^{1/3}.$$

There is equality if and only if $x = y = z = 1$.

Comment: The inequality in the form (1) was also given in the Spring 1997, Senior A-Level Tournament Of The Towns competition. A generalization to

$$\frac{1}{1+S-x_1} + \frac{1}{1+S-x_2} + \cdots + \frac{1}{1+S-x_n} \leq 1,$$

where $S = x_1 + x_2 + \cdots + x_n$, $x_1 x_2 \ldots x_n = 1$, and $x_i > 0$ is due to Dragos Hrimiuc, University of Alberta, and will probably appear as a problem in Math. Magazine.

Next we give two solutions by our readers to two problems of the 3rd Ukrainian Mathematical Olympiad, March 26–27, 1994 given in [1997: 262].

2. (9–10) A convex polygon and point O inside it are given. Prove that for any $n > 1$ there exist points $A_1, A_2, \ldots, A_n$ on the sides of the polygon such that $\overrightarrow{OA_1} + \overrightarrow{OA_2} + \ldots + \overrightarrow{OA_n} = \overrightarrow{0}$.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

It follows by continuity that there always exists a chord A_1OA_1' such that $A_1O = A_1'O$ and hence $\overrightarrow{OA_1} + \overrightarrow{OA_1'} = \overrightarrow{0}$. Similarly, there exists a chord A_2A_3' which is bisected by the midpoint O_1 of OA_1'. It follows by the parallelogram law that $\overrightarrow{OA_2} + \overrightarrow{OA_3'} = \overrightarrow{OA_1'}$ and hence $\overrightarrow{OA_1} + \overrightarrow{OA_2} + \overrightarrow{OA_3'} = \overrightarrow{0}$. Again similarly there exists a chord A_3A_4' which is bisected by the midpoint of OA_3' so that $\overrightarrow{OA_1} + \overrightarrow{OA_2} + \overrightarrow{OA_3} + \overrightarrow{OA_4'} = \overrightarrow{0}$, and so on for any number of vectors $n > 1$.

3. (10) A sequence of natural numbers a_k, $k \geq 1$, such that for each k, $a_k < a_{k+1} < a_k + 1993$ is given. Let all prime divisors of a_k be written for every k. Prove that we receive an infinite number of different prime numbers.

Solution by Pavlos Maragoudakis, Pireas, Greece.

We suppose that there is a sequence of natural numbers such that $a_k < a_{k+1} < a_k + 1993$, $k \geq 1$, and the set of all prime divisors of all a_k is finite. Let $p_1, p_2, \ldots, p_r$ list all the prime divisors of all a_k. Now every a_k has the form $p_1^{a_1} \ldots p_r^{a_r}$, $a_i = 0, 1, 2, \ldots$, $i = 1, \ldots, r$.

Let $S = \{p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r} \mid a_i = 0, 1, 2, \ldots,\ i = 1, 2, \ldots, r\}$.

Define (x_n) with $x_1 < x_2 < \ldots$ such that $S = \{x_n \mid n \in \mathbb{N}^*\}$. We have that $a_k < a_{k+1}$ and $a_k \in S$, $k \geq 1$. Thus (a_k) is a subsequence of

(x_k). But $a_k < a_{k-1} + 1993 < a_{k-2} + 2 \cdot 1993 < \cdots < a_1 + (k-1)1993 < k(a_1 + 1993), k \geq 1$.

Hence $a_k < k(a_1 + 1993)$, $k \geq 1$. Therefore

$$\sum_{n=1}^{\infty} \frac{1}{a_n} > \sum_{n=1}^{\infty} \frac{1}{n(a_1+1993)} = \frac{1}{a_1+1993} \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$$

and
$$\sum_{n=1}^{\infty} \frac{1}{a_n} \leq \sum_{n=1}^{\infty} \frac{1}{x_n} = \sum_{a_1,\dots,a_r \geq 0} \frac{1}{p_1^{\alpha_1} p_2^{a_2} \cdots p_r^{a_4}}$$
$$= \left(\sum_{a_1=0}^{\infty} \frac{1}{p_1^{\alpha_1}}\right)\left(\sum_{a_2=0}^{\infty} \frac{1}{p_2^{a_2}}\right) \cdots \left(\sum_{\alpha_r=0}^{\infty} \frac{1}{p_r^{a_r}}\right)$$
$$= \frac{p_1}{p_1-1} \cdot \frac{p_2}{p_2-1} \cdots \frac{p_4}{p_4-1} < +\infty,$$

which is a contradiction.

We now turn our attention to the solutions by readers to problems of the Mock Test of the Hong Kong Committee for the IMO 1994 [1997: 322–323].

INTERNATIONAL MATHEMATICAL OLYMPIAD 1994

Hong Kong Committee — Mock Test, Part I

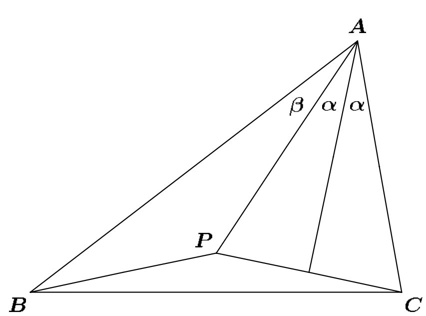
Time: 4.5 hours

1. In a triangle $\triangle ABC$, $\angle C = 2\angle B$. P is a point in the interior of $\triangle ABC$ satisfying that $AP = AC$ and $PB = PC$. Show that AP trisects the angle $\angle A$.

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and by D.J. Smeenk, Zaltbommel, the Netherlands. We give the solution by Amengual Covas.

Let $\angle PAC$ and $\angle BAP$ be 2α and β respectively. Then, since $\angle C = 2\angle B$, we deduce from $A + B + C = 180°$ that

$$2\alpha + \beta + 3B = 180°. \tag{1}$$



The angles at the base of the isosceles triangle PAC are each $90° - \alpha$. Also $\triangle BPC$ is isosceles, having base angles

$$C - (90° - \alpha) = 2B + \alpha - 90°,$$

and so

$$\begin{aligned}\angle BPA &= 180° - (\angle PBA + \angle BAP) \\ &= 180° - [B - (2B + \alpha - 90°) + 180° - 2\alpha - 3B] \\ &= 4B + 3\alpha - 90°.\end{aligned}$$

As usual, let a, b and c denote the lengths of the sides BC, AC and AB. By the Law of Cosines, applied to $\triangle BPA$, where $\overline{PA} = b$ and $\overline{PB} = \overline{PC} = 2b\sin\alpha$,

$$c^2 = b^2 + (2b\sin\alpha)^2 - 2 \cdot b \cdot 2b\sin\alpha \cdot \cos(4B + 3\alpha - 90°),$$

so that

$$c^2 = b^2[1 + 4\sin^2\alpha - 4\sin\alpha\sin(4B + 3\alpha)]. \qquad (2)$$

We now use the fact that $\angle C = 2\angle B$ is equivalent to the condition $c^2 = b(b + a)$, which has appeared before in ***CRUX*** [1976: 74], [1984: 278] and [1996: 265–267]. Since $a = 2 \cdot \overline{PC} \cdot \cos(2B + \alpha - 90°) = 4b\sin\alpha\sin(2B + \alpha)$, we have

$$c^2 = b^2[1 + 4\sin\alpha\sin(2B + \alpha)]. \qquad (3)$$

Therefore, from (2) and (3), we get

$$b^2[1 + 4\sin^2\alpha - 4\sin\alpha\sin(4B + 3\alpha)] = b^2[1 + 4\sin\alpha\sin(2B + \alpha)],$$

which simplifies to

$$\sin\alpha - \sin(4B + 3\alpha) = \sin(2B + \alpha).$$

Since $\sin\alpha - \sin(4B + 3\alpha) = -2\cos(2B + 2\alpha)\sin(2B + \alpha)$, this equation may be rewritten as

$$\sin(2B + \alpha) \cdot [1 + 2\cos(2B + 2\alpha)] = 0.$$

Since, from (1), $2B + \alpha < 180°$, we must have $1 + 2\cos(2B + 2\alpha) = 0$, giving $\cos(2B + 2\alpha) = -1/2$; that is,

$$2B + 2\alpha = 120° \qquad (4)$$

since, again from (1), $2B + 2\alpha < 180°$.

Finally, we may eliminate B between (1) and (4) to obtain $\alpha = \beta$. The result follows.

Mock Test, Part II

Time: 4.5 hours

1. Suppose that $yz + zx + xy = 1$ and x, y, and $z \geq 0$. Prove that

$$x(1-y^2)(1-z^2) + y(1-z^2)(1-x^2) + z(1-x^2)(1-y^2) \;\leq\; \frac{4\sqrt{3}}{9}.$$

Solutions by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and Pavlos Maragoudakis, Pireas, Greece. We give the solution by Klamkin.

We first convert the inequality to the following equivalent homogeneous one:

$$\begin{aligned} x(T_2-y^2)(T_2-z^2) + y(T_2-z^2)(T_2-x^2) + z(T_2-x^2)(T_2-y^2) \\ \leq\; (4\sqrt{3}/9)(T_2)^{5/2} \end{aligned}$$

where $T_2 = yz+zx+xy$, and for subsequent use $T_1 = x+y+z$, $T_3 = xyz$. Expanding out, we get

$$T_1T_2^2 - T_2\sum x(y^2+z^2) + T_2T_3 \;\leq\; (4\sqrt{3}/9)(T_2)^{5/2},$$

or

$$T_1T_2^2 - T_2(T_1T_2 - 3T_3) + T_2T_3 = 4T_2T_3 \;\leq\; (4\sqrt{3}/9)(T_2)^{5/2}.$$

Squaring, we get one of the known Maclaurin inequalities for symmetric functions:

$$\sqrt[3]{T_3} \;\leq\; \sqrt[2]{T_2/3}\,.$$

There is equality if and only if $x = y = z$.

To finish this number of the *Corner* we give two solutions to problems of the 45$^{\text{th}}$ Mathematical Olympiad in Poland, Final Round [1997: 323–324].

1. Determine all triples of positive rational numbers (x, y, z) such that $x+y+z$, $x^{-1}+y^{-1}+z^{-1}$ and xyz are integers.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

Let $x+y+z = n_1$, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = n_2$, and $xyz = n_3$, where n_1, n_2, n_3 are integers. Then $yz+zx+xy = n_2n_3$ and x, y, z are roots of the cubic

$$t^3 - n_1t^2 + n_2n_3t - n_3 \;=\; 0\,.$$

As known, the only rational roots of the latter are factors of n_3, and consequently x, y, z are integers.

The only triples of integers (x, y, z), aside from permutations, which satisfy $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = n_2$ are

$$(1,1,1),\ (1,2,2),\ (2,3,6),\ (2,4,4), \quad \text{and} \quad (3,3,3).$$

5. Let $A_1, A_2, \ldots, A_8$ be the vertices of a parallelepiped and let O be its centre. Show that

$$4(OA_1^2 + OA_2^2 + \cdots + OA_8^2) \ \leq\ (OA_1 + OA_2 + \cdots + OA_8)^2 .$$

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

Let one of the vertices be the origin and let the vectors $B + C$, $C + A$, $A + B$ denote the three coterminal edges emanating from this origin. Then the vectors to the remaining four vertices are $S + A$, $S + B$, $S + C$, and $2S$ where $S = A + B + C$ and which is also the vector to the centre. The inequality now becomes

$$2(S^2 + A^2 + B^2 + C^2) \ \leq\ (|S| + |A| + |B| + |C|)^2 ,$$

or

$$S^2 + A^2 + B^2 + C^2 \ \leq\ 2|S|\{|A| + |B| + |C|\} + 2\{|B|\,|C| + |C|\,|A| + |A|\,|B|\}.$$

Since

$$S^2 \ =\ A^2 + B^2 + C^2 + 2B \cdot C + 2C \cdot A + 2A \cdot B,$$

the inequality now becomes

$$S^2 - B\cdot C - C\cdot A - A\cdot B \ \leq\ |S|\{|A| + |B| + |C|\} + \{|B|\,|C| + |C|\,|A| + |A|\,|B|\}.$$

Clearly,

$$S^2 \ \leq\ |S|\{|A| + |B| + |C|\}$$

and

$$-B \cdot C - C \cdot A - A \cdot B \ \leq\ |B|\,|C|\,|A| + |A|\,|B| .$$

There is equality if and only if the parellelepiped is degenerate, for example, $B = C = O$.

That completes this number of the *Olympiad Corner*. Send me your nice solutions and Olympiad contests.

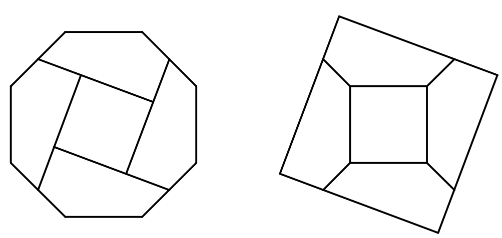
BOOK REVIEWS

Edited by ANDY LIU

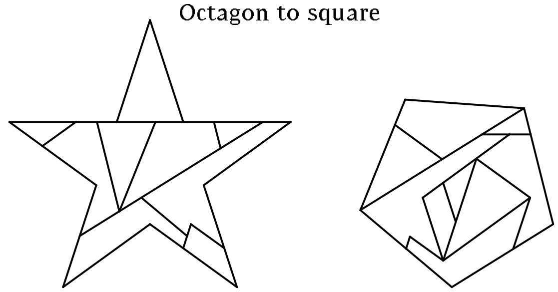
Dissections : Plane & Fancy
by Greg N. Frederickson,
published by Cambridge University Press, 1997,
ISBN# 0-521-57197-9, hardcover, 310+ pages, $34.95.
Reviewed by ***Andy Liu****, University of Alberta.*

This is a much awaited sequel to Harry Lindgren's 1964 classic work, **Geometric Dissections**, which the author (G.N.F.) revised and augmented in the 1972 Dover edition. Actually, the current volume is much more than just a sequel. It is the most comprehensive treatise on the subject of geometric dissections. It may be enjoyed on at least three levels.

First and foremost, this book is a collection of interesting dissection puzzles, old and new. Only some background in high school geometry is needed to fully enjoy these problems. *Can you cut an octagon into five pieces and rearrange them into a square? How about turning a star into a pentagon?* The solutions, which are both appealing but for somewhat opposing reasons, are shown below.

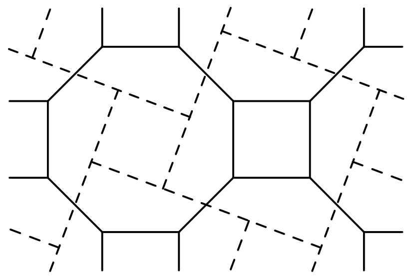


Octagon to square

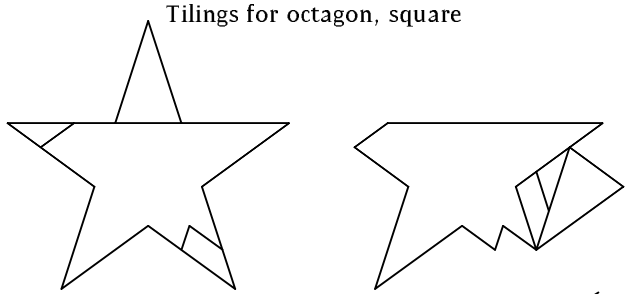


Star to pentagon

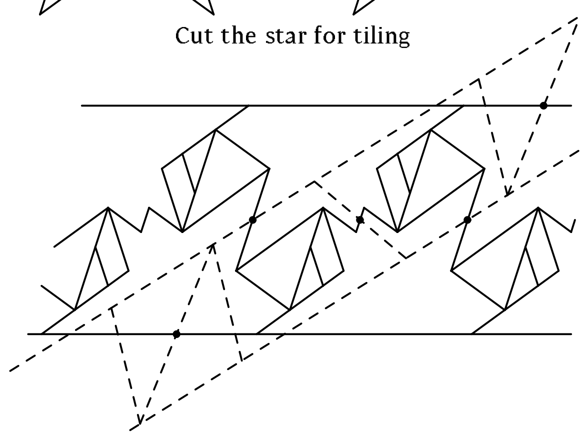
This book is also an instructive manual on the art and science of geometric dissections. While one may admire the ingenuity which produced the spectacular solutions, the author probes into the underlying fabric which might have led to such incisive insight. Many techniques are discussed, too many to enumerate here. A favourite is that of tessellation. Below are two tilings which might have suggested the dissections above.



Tilings for octagon, square



Cut the star for tiling



Tilings for star, pentagon

Finally, this book is an important historical document, detailing the inter-cultural development of the subject. *Travel from the palace school of tenth-century Baghdad to the mathematical puzzle columns in turn-of-the-century newspapers, from the 1900 Paris Congress of Mathematicians to the night sky of Canberra*. Readers puzzled by this quote need look no further than the illustrious names of Abū'l Wafā, Henry Dudeney/Sam Loyd, David Hilbert/Max Dehn and Harry Lindgren. Biographical sketches of Wafā, Dudeney, Loyd and Lindgren are provided, along with those of over forty other people who have made significant contributions to geometric dissections. The writing style is very engaging, and the book is good reading even if one skips over some of the more complicated technical details.

In conclusion, the reviewer echoes Martin Gardner that this book will be a classic. It comes with the highest recommendation.

THE SKOLIAD CORNER

No. 33

R.E. Woodrow

As a contest this issue we give the Senior High School Mathematics Contest, Preliminary Round 1998 of the British Columbia Colleges. My thanks go to the organizer, Jim Totten, The University College of the Cariboo, for forwarding me the contest materials. Time allowed is 45 minutes!

BRITISH COLUMBIA COLLEGES
Senior High School Mathematics Contest
Preliminary Round 1998

Time: 45 minutes

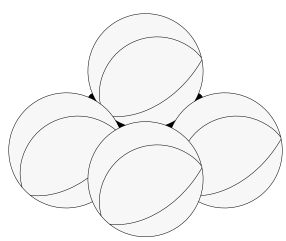
1. The integer $1998 = (n-1)n^n(10n + c)$ where n and c are positive integers. It follows that c equals:

(a) 2 (b) 5 (c) 6 (d) 7 (e) 8

2. The value of the sum $\log\frac{1}{2} + \log\frac{2}{3} + \log\frac{3}{4} + \cdots + \log\frac{9}{10}$ is:

(a) -1 (b) 0 (c) 1 (d) 2 (e) 3

3. Four basketballs are placed on the gym floor in the form of a square with each basketball touching two others. A fifth basketball is placed on top of the other four so that it touches all four of the other balls, as shown. If the diameter of a basketball is 25 cm, the height, in centimetres, of the centre of the fifth basketball above the gym floor is:



(a) $25\sqrt{2}$ (b) $\frac{25}{2}\sqrt{2}$ (c) 20 (d) $\frac{25}{2}(1+\sqrt{2})$ (e) $25(1+\sqrt{2})$

4. Last summer, I planted two trees in my yard. The first tree came in a fairly small pot, and the hole that I dug to plant it in filled one wheelbarrow load of dirt. The second tree came in a pot, the same shape as that of the first tree, that was one-and-a-third times as deep as the first pot and one-and-a-half times as big around. Let us make the following assumptions:

i) The hole for the second tree was the same shape as for the first tree.

ii) The ratios of the dimensions of the second hole to those of the first hole are the same as the ratios of the dimensions of the pots.

Based on these assumptions, the number of wheelbarrows of dirt that I filled when I dug the hole for the second tree was:

(a) 2 (b) 2.5 (c) 3 (d) 3.5 (e) none of these

5. You have an unlimited supply of 5-gram and 8-gram weights that may be used in a pan balance. If you use only these weights and place them only in one pan, the largest number of grams that you cannot weigh is:

(a) 22 (b) 27 (c) 36 (d) 41 (e) there is no largest number of grams

6. If all the whole numbers from 1 to 1,000,000 are printed, the number of times that the digit 5 appears is:

(a) 100,000 (b) 500,000 (c) 600,000 (d) 1,000,000 (e) 2,000,000

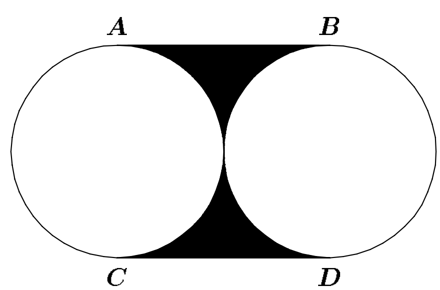
7. The perimeter of a rectangle is x centimetres. If the ratio of two adjacent sides is $a : b$, with $a > b$, then the length of the shorter side, in centimetres, is:

(a) $\frac{bx}{a+b}$ (b) $\frac{x}{2} - b$ (c) $\frac{2bx}{a+b}$ (d) $\frac{ax}{2(a+b)}$ (e) $\frac{bx}{2(a+b)}$

8. The sum of the positive solutions to the equation $x^{x\sqrt{x}} = (x\sqrt{x})^x$ is:

(a) 1 (b) $1\frac{1}{2}$ (c) $2\frac{1}{4}$ (d) $2\frac{1}{2}$ (e) $3\frac{1}{4}$

9. Two circles, each with a radius of one unit, touch as shown. AB and CD are tangent to each circle. The area, in square units, of the shaded region is:



(a) π (b) $\frac{\pi}{4}$ (c) $2 - \frac{\pi}{2}$ (d) $4 - \pi$ (e) none of these

10. A parabola with a vertical axis of symmetry has its vertex at $(0, 8)$ and an x–intercept of 2. If the parabola goes through $(1, a)$, then a is:

(a) 5 (b) 5.5 (c) 6 (d) 6.5 (e) 7

11. A five litre container is filled with pure orange juice. Two litres of juice are removed and the container is filled up with pure water and mixed thoroughly. Then two litres of the mixture are removed and again the container is filled up with pure water. The percentage of the final mixture that is orange juice is:

(a) 27 (b) 25 (c) 30 (d) 36 (e) 24

12. The lengths of the sides of a triangle are $b + 1$, $7 - b$ and $4b - 2$. The number of values of b for which the triangle is isosceles is:

(a) 0 (b) 1 (c) 2 (d) 3 (e) none of these

13. The number of times in one day when the hands of a clock form a right angle is:

(a) 46 (b) 22 (c) 24 (d) 44 (e) 48

14. In my town some of the animals are really strange. Ten percent of the dogs think they are cats and ten percent of the cats think they are dogs. All the other animals are perfectly normal. One day, I tested all the cats and dogs in the town and found that 20% of them thought that they were cats. The percentage of the dogs and cats in the town that really are cats is:

(a) 12.5 (b) 18 (c) 20 (d) 22 (e) 22.5

15. A short hallway in a junior high school contains a bank of lockers numbered one to ten. On the last day of school the lockers are emptied and the doors are left open. The next day, a malicious math student walks down the hallway and closes the door of every locker that has an even number. The following day, the same student again walks down the hallway and, for every locker whose number is a multiple of three, closes the door if it is open and opens it if it is closed. On the next day, the student does the same thing with every locker whose number is divisible by four. If the student continues this procedure for a total of nine days, the number of lockers that are closed after the ninth day is:

(a) 4 (b) 5 (c) 6 (d) 7 (e) 8

Last issue we gave the Junior High School Mathematics Contest, Preliminary Round 1998 of the British Columbia Colleges. Here are the "official solutions", which come our way from the organizer, Jim Totten, The University College of the Cariboo.

BRITISH COLUMBIA COLLEGES
Junior High School Mathematics Contest
Preliminary Round 1998

Time: 45 minutes

1. A number is prime if it is greater than one and divisible only by one and itself. The sum of the prime divisors of 1998 is: **(c)**

Solution. We can factor the number 1998 as follows: $1998 = 2 \times 999 = 2 \times 3^2 \times 111 = 2 \times 3^3 \times 37$. Hence, the sum of its prime divisors is $2 + 3 + 37 = 42$.

2. Successive discounts of 10% and 20% are equivalent to a single discount of: **(c)**

Solution. If P denotes the initial price then the new price after deducting the two consecutive discounts is $P(1 - 0.1)(1 - 0.2) = 0.72P$. This gives the total discount of $(1 - 0.72) \times 100\% = 28\%$.

3. Suppose that $\text{(}A\text{)} = A^2$ and $A \square B = A - 2B$. Then the value of (7) $\square$ (3) is: **(e)**

Solution. According to our definitions, $(7) \square (3) = 7^2 \square 3^2 = 49 \square 9 = 49 - 2 \times 9 = 31$.

4. The expression that is not equal to the value of the four other expressions listed is: **(d)**

Solution. The values of the five expressions are:

(a) $1^{\sqrt{9}} + 9 - 8 = 2$,
(b) $(1 + 9) \div (-\sqrt{9} + 8) = 2$,
(c) $-1 \times 9 + \sqrt{9} + 8 = 2$,
(d) $(1 - \sqrt{9}) \times (9 - 8) = -2$,
(e) $19 - 9 - 8 = 2$.

Thus,the expression (d) has a different value than the other four expressions.

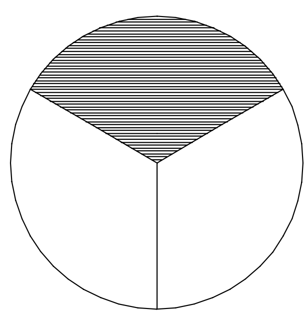
5. The sum of all of the digits of the number $10^{75} - 75$ is: **(c)**

Solution. Consider the procedure for subtracting 75 from 10^{75} "by hand":

$$\begin{array}{r} 100\ldots000 \\ -75 \\ \hline 99\ldots925 \end{array}$$

Hence, the decimal digits of $10^{75} - 75$ are: 5, 2 and seventy-three copies of 9. The sum of the digits is $5 + 2 + 73 \times 9 = 664$.

6. A circle is divided into three equal parts and one part is shaded as in the accompanying diagram. The ratio of the perimeter of the shaded region, including the two radii, to the circumference of the circle is: **(d)**



Solution. The ratio is given by

$$\frac{\frac{2\pi r}{3}+r+r}{2\pi r}=\frac{\frac{2\pi r+6r}{3}}{2\pi r}=\frac{2\pi r+6r}{6\pi r}=\frac{\pi+3}{3\pi}.$$

7. The value of

$$\frac{1}{2-\frac{1}{2-\frac{1}{2-\frac{1}{2}}}}$$

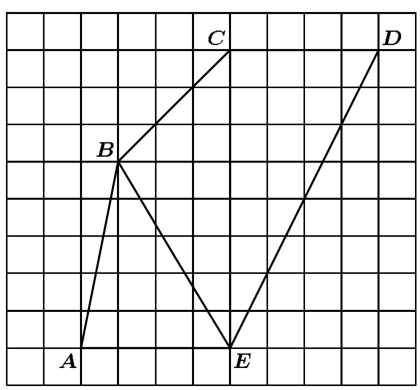
is: **(b)**

Solution. We can simplify this compound fraction by working successively from the bottom to the top of the expression:

$$\frac{1}{2-\frac{1}{2-\frac{1}{2-\frac{1}{2}}}}=\frac{1}{2-\frac{1}{2-\frac{1}{\left(\frac{3}{2}\right)}}}=\frac{1}{2-\frac{1}{2-\frac{2}{3}}}=\frac{1}{2-\frac{1}{\left(\frac{4}{3}\right)}}=\frac{1}{2-\frac{3}{4}}=\frac{1}{\frac{5}{4}}=\frac{4}{5}.$$

8. If each small square in the accompanying grid is one square centimetre, then the area in square centimetres of the polygon $ABCDE$ is: **(a)**

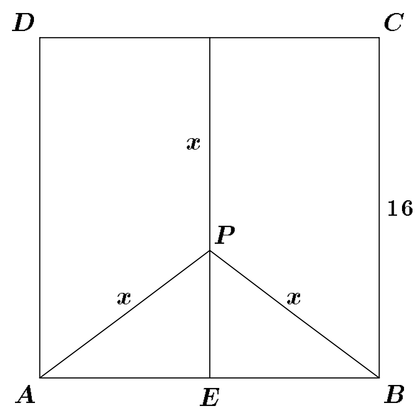
Solution. We can find the area by decomposing the polygon $ABCDE$ into simpler figures, for example, into three triangles: ABE, BCE, and CDE.



If we choose AE, CE, and CD as bases of the triangles then the lengths of the corresponding perpendicular heights are 5, 3, and 8 cm. Hence, the area of the polygon is $\frac{1}{2}\times 4\times 5+\frac{1}{2}\times 8\times 3+\frac{1}{2}\times 4\times 8=38$.

9. A point P is inside a square $ABCD$ whose side length is 16. P is equidistant from two adjacent vertices, A and B, and the side CD opposite these vertices. The distance PA equals: **(e)**

Solution. The situation is illustrated by the following diagram, where x denotes the distance PA.



The Pythagorean Theorem applied to triangle PEB gives $(16-x)^2+8^2 = x^2$, so that $16^2 - 32x + x^2 + 8^2 = x^2$, and $x = \frac{16^2+8^2}{32} = 10$.

10. A group of 20 students has an average mass of 86 kg per person. It is known that 9 people from this group have an average mass of 75 kg per person. The average mass in kilograms per person of the remaining 11 people is: **(b)**

Solution. If $m_1, m_2, \ldots, m_{20}$ denote the masses of the students then $\frac{m_1+m_2+\cdots+m_{20}}{20} = 86$. Hence, $m_1 + m_2 + \cdots + m_{20} = 20 \times 86 = 1720$. We can assume, without loss of generality, that the average mass of the first nine students is 75; that is, $\frac{m_1+m_2+\cdots+m_9}{9} = 75$. Hence, $m_1 + m_2 + \cdots + m_9 = 9 \times 75 = 675$. The total mass of the remaining 11 people is $1720 - 675 = 1045$. This gives the average of $\frac{1045}{11} = 95$.

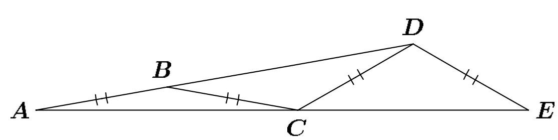
11. In the following display each letter represents a digit:

3	B	C	D	E	8	G	H	I

The sum of any three successive digits is 18. The value of H is: **(a)**

Solution. We have $3 + B + C = 18$. Consequently, $B + C = 15$. By subtracting this equation from $B + C + D = 18$ we get $D = 3$. Now, $D+E+8 = 18$ gives $E = 10-D = 10-3 = 7$. Finally, $E+8+G = 18$ gives $G = 10 - E = 10 - 7 = 3$, and $8 + G + H = 18$ gives $H = 18 - 8 - 3 = 7$.

12. In the accompanying diagram $\angle ADE = 140°$. The sides are congruent as indicated. The measure of $\angle EAD$ is: **(e)**

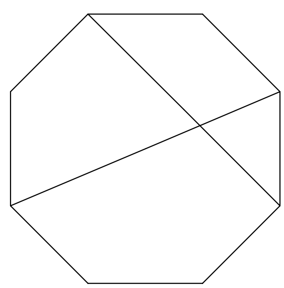


Solution. If $\angle EAD = \alpha$, then also $\angle ACB = \alpha$, since triangle ABC is isosceles. Hence, $\angle CBD = 2\alpha$ as an external angle of triangle ABC. Consequently, $\angle ADC = 2\alpha$, since triangle BCD is isosceles. Further, $\angle ECD = \angle ADC + \angle CAD$ as an external angle of triangle ADC. Hence, $\angle ECD = 2\alpha + \alpha = 3\alpha$. Now, $\angle AED = \angle ECD = 3\alpha$, because triangle CDE is isosceles. This implies that $\angle CDE = 180° - 6\alpha$. Finally, $\angle ADE = \angle ADC + \angle CDE = 2\alpha + 180° - 6\alpha = 180° - 4\alpha$. Thus, $180° - 4\alpha = 140°$. This yields $\alpha = 10°$.

13. The area (in square units) of the triangle bounded by the x–axis and the lines with equations $y = 2x + 4$ and $y = -\frac{2}{3}x + 4$ is: **(e)**

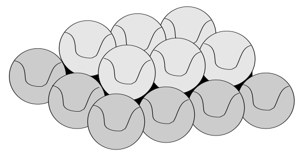
Solution. Two vertices of the triangle lie on the x–axis, so they are the x–intercepts of the lines. The x–intercept of the first line, determined by the equation $2x + 4 = 0$, is -2. Similarly, the second x–intercept, determined by $-\frac{2}{3}x_2 + 4 = 0$, is 6. Consequently, the length of the base of the triangle is $6 - (-2) = 8$. Since both lines have the same y–intercept 4, they intersect each other and the y–axis at level 4 and, consequently, the length of the height of the triangle is 4. Therefore, the area of the triangle is $A = \frac{1}{2}(4 \times 8) = 16$.

14. Two diagonals of a regular octagon are shown in the accompanying diagram. The total number of diagonals possible in a regular octagon is: **(d)**



Solution. Let d_i, for $i = 1, 2, \ldots, 8$, denote the number of diagonals connected to the i^{th} vertex. Then $d_1 = d_2 = \ldots = d_8 = 5$, since each vertex of the octagon is connected to five diagonals. On the other hand, each diagonal joins two vertices. Therefore, in the sum $d_1 + d_2 + \cdots + d_8 = 8 \times 5 = 40$, each diagonal is counted twice. Hence, the number of diagonals in the octagon is 20.

15. A local baseball league is running a contest to raise money to send a team to the provincial championship. To win the contest it is necessary to determine the number of baseballs stacked in the form of a rectangular pyramid. The fifth and sixth levels from the base of the stack of baseballs are shown. If the stack contains a total of seven levels, the number of baseballs in the stack is: **(d)**



Solution. The fifth level has $3 \times 4 = 12$ balls, the sixth $2 \times 3 = 6$ balls, and the seventh $1 \times 2 = 2$ balls. We notice that the number of balls in both sides of the rectangle they form increases by one each time we move one level down. Thus, the total number of balls is $1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + 5 \times 6 + 6 \times 7 + 7 \times 8 = 168$.

That completes the *Skoliad Corner* for this issue. We need suitable contests and solutions. I welcome any comments, criticisms, or suggestions for the future direction of this feature.

Advance Announcement

The 1999 Summer Meeting of the Canadian Mathematical Society will take place at Memorial University in St. John's, Newfoundland, from Saturday, 29 May 1999 to Tuesday, 1 June 1999.

The Special Session on Mathematics Education will feature the topic

What Mathematics Competitions do for Mathematics.

The invited speakers are

Ed Barbeau (University of Toronto),
Ron Dunkley (University of Waterloo),
Tony Gardiner (University of Birmingham, UK),
Rita Janes (Newfoundland and Labrador Senior Mathematics League), and
Shannon Sullivan (student, Memorial University).

Requests for further information, or to speak in this session, as well as suggestions for further speakers, should be sent to the session organizers:

Bruce Shawyer and Ed Williams
CMS Summer 1999 Meeting, Education Session
Department of Mathematics and Statistics, Memorial University
St. John's, Newfoundland, Canada A1C 5S7

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to the Mayhem Editor, Naoki Sato, Department of Mathematics, Yale University, PO Box 208283 Yale Station, New Haven, CT 06520–8283 USA. The electronic address is still

mayhem@math.toronto.edu

The Assistant Mayhem Editor is Cyrus Hsia (University of Toronto). The rest of the staff consists of Adrian Chan (Upper Canada College), Jimmy Chui (Earl Haig Secondary School), Richard Hoshino (University of Waterloo), David Savitt (Harvard University) and Wai Ling Yee (University of Waterloo).

Shreds and Slices

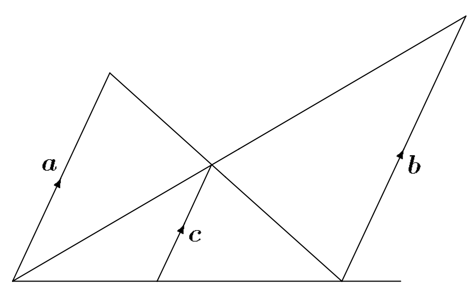
An Algebraic Relation with a Geometric Twist

Cyrus Hsia

Consider the following algebraic relationship between the positive real numbers a, b, and c:

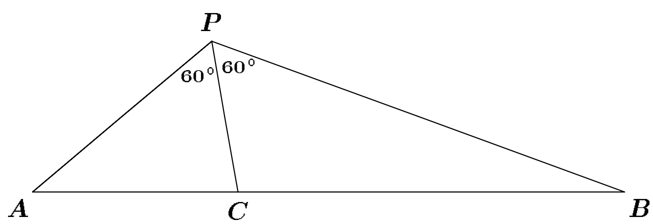
$$\frac{1}{c} = \frac{1}{a} + \frac{1}{b}.$$

If we consider line segments with lengths of a, b, and c, then they are related to each other as shown in the following diagram.



The diagram shows the three line segments parallel to each other and emanating from a common line. This figure and relationship between the line segments appear a lot. The reader is encouraged to prove this.

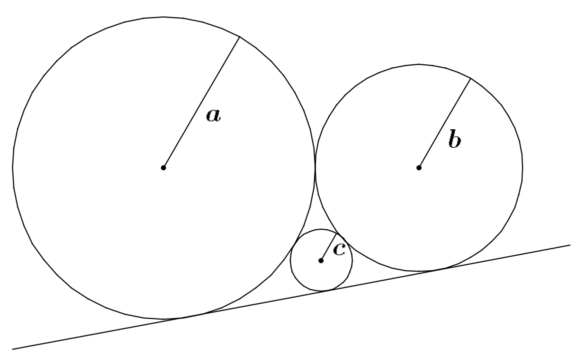
As a corollary of this fact, here is another geometric example. Let P, A, B, and C be points in the plane such that $\angle APC = \angle CPB = 60°$ and A, C, and B are collinear. Show that $1/PC = 1/PA + 1/PB$.



Another algebraic relation between three positive numbers with an interesting geometric interpretation is the following:

$$\frac{1}{\sqrt{c}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}.$$

The reader is encouraged to find a geometric interpretation for the above relation before looking at the diagram below. Use a, b, and c as the length of three line segments, and determine a geometric figure that relates the three.



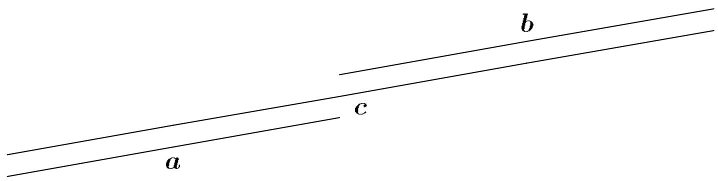
It turns out that if a, b, and c are considered to be the lengths of the radii of three circles, then the circles may all be tangent to a common line and to each other as shown. Again readers are encouraged to prove this themselves.

Now what about a generalization? Consider the following algebraic relation between positive reals a, b, c, and a real number x:

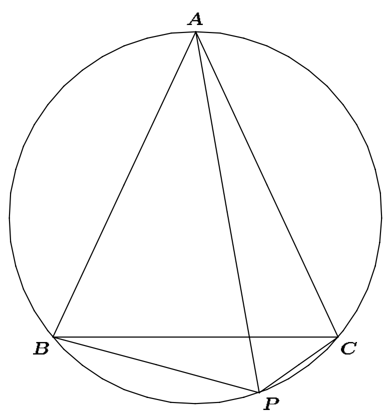
$$c^x = a^x + b^x.$$

The first case then corresponds to the value $x = -1$ and the second case to $x = -\frac{1}{2}$.

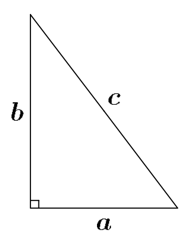
The case $x = 1$ is trivial, as we could interpret it geometrically as a line segment of length c is made up of the sum of its parts of lengths a and b.



If we wanted to get fancy, we could give the following geometric example instead. Consider an equilateral triangle ABC inscribed in a circle, as shown. P is a point on arc BC. Prove that $PA = PB + PC$.



The reader is probably already familiar with the famous case $x = 2$ known as the Pythagorean Theorem: A triangle with sides a, b, and c is a right-angled triangle if and only if the lengths satisfy $a^2 + b^2 = c^2$.



Of course, no discussion about algebraic relations in the above form is complete without mentioning the notorious Fermat's Last Theorem and the recent announcement that it has finally been laid to rest. If x is an integer with $x > 2$, then the claim is that no solution in the natural numbers exists for a, b, and c. However, in our general case, the values are real, so we are not limited by the above result to finding wild and wacky geometric or other interpretations for it.

If the reader is curious, as are we, then try the following exercises to find geometric interpretations for special cases of the above relation. The exercises are explorational and may not have nice solutions, if any. Readers are welcome to submit any interesting results they find.

Exercises

1. Let a, b, and c be the lengths of three line segments. Determine how these three line segments are related geometrically if they satisfy the relation

   (a) $c^3 = a^3 + b^3$,

   (b) $\sqrt{c} = \sqrt{a} + \sqrt{b}$,

   (c) $\frac{1}{c^2} = \frac{1}{a^2} + \frac{1}{b^2}$.

2. It is clear that algebraically, all the relations are similar. However, the geometric interpretations do not appear to be related. Is there a general geometric description where each of the above geometric figures is a special case?

3. The algebraic relation clearly does not work for the case $x = 0$. Is there a way to define the relation so that it would be consistent with everything else mentioned so far?

We extend congratulations to Ravi Vakil and Alice Staveley, who were married at St. John's, NF on Monday, 12 October 1998.

Mayhem Problems

The Mayhem Problems editors are:

Richard Hoshino	*Mayhem High School Problems Editor,*
Cyrus Hsia	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only problems — the next issue will feature only solutions.

We warmly welcome proposals for problems and solutions. We request that solutions from this issue be submitted by 1 September 1999, for publication in issue 8 of 1999.

High School Problems

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H245. Determine how many distinct integers there are in the set

$$\left\{\left\lfloor\frac{1^2}{1998}\right\rfloor, \left\lfloor\frac{2^2}{1998}\right\rfloor, \left\lfloor\frac{3^2}{1998}\right\rfloor, \ldots, \left\lfloor\frac{1998^2}{1998}\right\rfloor\right\}.$$

H246. Let $S(n)$ denote the sum of the first n positive integers. We say that an integer n is *fantastic* if both n and $S(n)$ are perfect squares. For example, 49 is fantastic, because $49 = 7^2$ and $S(49) = 1+2+3+\cdots+49 = 1225 = 35^2$ are both perfect squares. Find another integer $n > 49$ that is fantastic.

H247. Say that the integers a, b, c, d, p, and r form a cyclic set (a, b, c, d, p, r) if there exists a cyclic quadrilateral with circumradius r, sides a, b, c, and d, and diagonals p and $2r$.

(a) Show that if $r < 25$, no cyclic set exists.

(b) Find a cyclic set (a, b, c, d, p, r) for $r = 25$.

H248. Consider a tetrahedral die that has the four integers 1, 2, 3, and 4 written on its faces. Roll the die 2000 times. For each i, $1 \le i \le 4$, let $f(i)$ represent the number of times that i turned up. (So, $f(1) + f(2) + f(3) + f(4) = 2000$.) Also, let S denote the total sum of the 2000 rolls.

If $S^4 = 6144 \cdot f(1)f(2)f(3)f(4)$, determine the values of $f(1)$, $f(2)$, $f(3)$, and $f(4)$.

Advanced Problems

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A221. Construct, using straightedge and compass only, the common tangents of two non-intersecting circles.

A222. Does there exist a set of n consecutive positive integers such that for every positive integer $k < n$, it is possible to pick k of these numbers whose mean is still in the set?

A223. *Proposed by Mohammed Aassila, Université Louis Pasteur, Strasbourg, France.*

Suppose p is a prime with $p \equiv 3 \pmod 4$. Show that for any set of $p-1$ consecutive integers, the set cannot be divided into two subsets so that the product of the members of the one set is equal to the product of the members of the other set.

(Generalization of Question 4, IMO 1970)

A224. *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

Let P be an interior point of triangle ABC such that $\angle PBA = \angle PCA = (\angle ABC + \angle ACB)/3$. Prove that

$$\frac{AC}{AB+PC} = \frac{AB}{AC+PB}.$$

Challenge Board Problems

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C81. Let $\{a_n\}$ be the sequence defined as follows: $a_0 = 0$, $a_1 = 1$, and $a_{n+1} = 4a_n - a_{n-1}$ for $n = 1, 2, 3, \ldots$.

(a) Prove that $a_n^2 - a_{n-1}a_{n+1} = 1$ for all $n \geq 1$.

(b) Evaluate $\displaystyle\sum_{k=1}^{\infty} \arctan\left(\frac{1}{4a_k^2}\right)$.

C82. Find the smallest multiple of 1998 which appears as a partial sum of the increasing sequence

$$1,\ 1,\ 2,\ 2,\ 2,\ 4,\ 4,\ 4,\ 4,\ 8,\ \ldots,$$

in which the number 2^k appears $k+2$ times (for k a non-negative integer).

IMO Report

Adrian Chan, student, UCC, Toronto

This year's Canadian IMO team began with a week of training at the University of Calgary with lavish meal tickets. Then they were off to beautiful and rocky Kananaskis where we all had an "adventurous" time. Once the team stepped off the air-conditioned plane and into hot and muggy Taipei, Taiwan, it marked the team's official arrival to the 39$^{\text{th}}$ International Mathematical Olympiad.

The team consisted of the following members: Adrian "Oops I dropped my..." Birka, Adrian "If You Will" Chan, Jimmy "Nuclear Aerial Strike" Chui, Mihaela "Baia" Enachescu, Jessie "So Cute" Lei, and Adrian "Nailing Radar" Tang. Team leader Dr. Christopher "Focus" Small was driven to the edge, while deputy leader J.P. "It's So Easy" Grossman calmly polished off old competitions one by one. Special thanks to leader observer Arthur "Rubik's Cube" Baragar and deputy observer Dorette "Dutch" Pronk for their coaching and experience. Also, thanks must go to Dr. Bill Sands of the University of Calgary for organizing such a fun training session.

The contest itself seemed to continue the trend of difficult IMO's and low medal cutoffs. With 76 countries competing, Canada fared extremely well, bringing back 1 gold, 1 silver, 2 bronze and an honourable mention. The scores were as follows:

CAN 1	Adrian Birka	10	
CAN 2	Adrian Chan	31	Gold Medal
CAN 3	Jimmy Chui	14	Bronze Medal
CAN 4	Mihaela Enachescu	30	Silver Medal
CAN 5	Jessie Lei	13	Honourable Mention
CAN 6	Adrian Tang	15	Bronze Medal

In this year's contest, Canada placed 20$^{\text{th}}$ out of 76 countries, up from last year's 29$^{\text{th}}$ ranking. Best of luck to CAN 1, 4, and 6 as they continue university studies at MIT, Harvard, and Waterloo respectively. CAN 2, 3, and 5 are all eligible for next year's team. Hopefully there won't be as much moaning of "Where did I go wrong?" next year around!

Special thanks must also go to Dr. Graham Wright of the Canadian Mathematical Society for again taking care of the tab, and Professor Ed Barbeau for his hard work and dedication to the training of potential IMO candidates through his year-long correspondence program.

Although sometimes things didn't make sense, and the IMO flag somehow disappeared, the 39$^{\text{th}}$ International Mathematical Olympiad ran smoothly and was definitely a success. The new experience of a place halfway around the world with a stimulating culture was new to most of us. Best of luck to all IMO hopefuls for the 1999 team, as yet another Canadian IMO journey begins next July in Bucharest, Romania.

Bogus Arguments and Arcane Identities

Ravi Vakil
Princeton University

In Euler's time, mathematics was faster and looser than today, and niceties such as limits were blatantly ignored. Here is an argument of Euler's that seems to have no right to work, but does nonetheless. We conclude with some avenues for exploration and an open question.

If $r_1, \ldots, r_n$ are the zeros of a polynomial $a_0 + a_1 x + \cdots + a_n x^n$, and none of the r_i are zero (or $a_0 \neq 0$), then the negative of the linear term over the constant term is the sum of the reciprocals of the roots:

$$-\frac{a_1}{a_0} = \frac{1}{r_1} + \cdots + \frac{1}{r_n}.$$

What about power series? For example, $\cos x = 1 - x^2/2 + x^4/24 - \cdots$, so that

$$\cos\sqrt{x} = 1 - \frac{x}{2} + \frac{x^2}{24} - \cdots.$$

The zeroes of this function are the squares of the odd multiples of $\pi/2$: $((2n+1)\pi/2)^2$, $n = 0, 1, 2, \ldots$. One might hope that the principle for polynomials given above still holds:

$$\frac{1}{2} = \sum_{n=0}^{\infty} \frac{1}{((2n+1)\pi/2)^2}$$

which can be rewritten as

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}. \tag{1}$$

This is actually true!

Another possibility is to use $\sin x$, which has power series expansion

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots.$$

Can you use the power series for $(\sin\sqrt{x})/x$ to "prove" the identity

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\ ? \tag{2}$$

Can you relate (2) to (1) by arguing that (2) minus a quarter of (2) is (1)?

If you write a short computer program to compute $\sum_{n=1}^{1000} \frac{1}{n^2}$, and compare it to $\pi^2/6$, you'll see that they are indeed very close. In fact, they differ by almost exactly $1/1000$. Can you explain why this might be? Can you guesstimate the difference between $\sum_{n=1}^{1000} \frac{1}{(2n+1)^2}$ and $\pi^2/8$?

And finally, can you conjure up other examples of this sort of argument, to "prove" other arcane identities? If so, please let us know!

Acknowledgements. This note was inspired by the example of $\cos\sqrt{x}$ given in [A].

References

[A] S. Abhyankar, *Historical ramblings in algebraic geometry and related algebra*, Amer. Math. Monthly **83** (1976), no. 6, 409–448.

[E] L. Euler, Introductio in Analysin Infinitorum, Berlin Academy, 1748.

The Fibonacci Sequence

Wai Ling Yee
student, University of Waterloo

The sequence defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ is called the *Fibonacci sequence*. Named after Leonardo of Pisa, who is also known as Fibonacci (unsurprisingly), it is one of the most widely studied sequences of all time. The Fibonacci sequence is an excellent topic with which to begin learning some basic number theory and various techniques for working with recurrence relations.

Basic Results

Theorem 1. For all $n \geq 1$,

$$F_n^2 - F_{n-1}F_{n+1} \;=\; (-1)^{n-1}\,.$$

Proof by Induction. When $n = 1$, $F_1^2 - F_0F_2 = 1^2 - 0 \cdot 1 = 1 = (-1)^{1-1}$, so the formula holds for $n = 1$. Assume that the formula holds for some $n = k$, $k \geq 1$. For $n = k + 1$,

$$\begin{aligned}
F_{k+1}^2 - F_kF_{k+2} &= F_{k+1}^2 - F_k(F_{k+1} + F_k) \\
&= (F_{k+1} - F_k)F_{k+1} - F_k^2 \\
&= F_{k-1}F_{k+1} - F_k^2 \\
&= -(-1)^{k-1} \quad \text{by the induction hypothesis} \\
&= (-1)^k,
\end{aligned}$$

so the formula holds for $n = k + 1$. Therefore, by mathematical induction, the formula holds for all $n \geq 1$.

Theorem 2. For all $n \geq 1$,

$$F_1 + F_3 + \cdots + F_{2n-1} \;=\; F_{2n}\,.$$

Proof. Using the recurrence relation n times, we have

$$\begin{aligned}
F_{2n} &= F_{2n-1} + F_{2n-2} \\
&= F_{2n-1} + F_{2n-3} + F_{2n-4} \\
&= \cdots \\
&= F_{2n-1} + F_{2n-3} + \cdots + F_3 + F_1 + F_0 \\
&= F_{2n-1} + F_{2n-3} + \cdots + F_3 + F_1\,.
\end{aligned}$$

Exercise 1. Prove

$$\sum_{j=1}^{n} F_j^2 = F_n F_{n+1} .$$

Divisibility

Theorem 3. For all m, $n \geq 1$,

$$F_{n+m} = F_n F_{m+1} + F_{n-1} F_m .$$

Proof. We will prove this by induction on m. When $m = 1$,

$$F_{n+1} = F_n \cdot 1 + F_{n-1} \cdot 1 = F_n F_2 + F_{n-1} F_1,$$

so the formula holds for all n when $m = 1$. Assume that the formula holds for all n when $m = M$. For $m = M + 1$,

$$\begin{aligned} F_{n+M+1} &= F_{n+M} + F_{(n-1)+M} \\ &= F_n F_{M+1} + F_{n-1} F_M + F_{n-1} F_{M+1} + F_{n-2} F_M \\ &\qquad \text{by the induction hypothesis} \\ &= F_n F_{M+1} + (F_{n-1} + F_{n-2}) F_M + F_{n-1} F_{M+1} \\ &= F_n (F_M + F_{M+1}) + F_{n-1} F_{M+1} \\ &= F_n F_{M+2} + F_{n-1} F_{M+1}, \end{aligned}$$

so the formula holds for all n for $m = M + 1$. By mathematical induction, the formula holds for all m, $n \geq 1$.

Corollary 4. For all m, $n \geq 1$, $F_n | F_{nm}$.

Proof. We will prove this by induction on m. When $m = 1$, F_n certainly divides itself for every positive integer n. Suppose the statement holds for all n when $m = M$. For $m = M + 1$,

$$F_{n(M+1)} = F_{nM+n} = F_{nM} F_{n+1} + F_{nM-1} F_n$$

by Theorem 3. Since F_{nM} is divisible by F_n by the induction hypothesis, $F_{nM} F_{n+1} + F_{nM-1} F_n$ is also divisible by F_n. This is equivalent to $F_n | F_{n(M+1)}$, so the result holds for $m = M + 1$. By mathematical induction, the formula holds for all m, $n \geq 1$.

Exercise 2. Prove that for every positive integer n, there exist n consecutive, composite Fibonacci numbers.

Number Theory 101

We will now define a few terms in the interests of formality. For integers a and b, we say that a *divides* b if there exists an integer q such that $b = aq$, and a is called a *divisor* of b. Given two non-zero integers a and b, the largest number which divides both of them, denoted $\gcd(a, b)$, is called their *greatest common divisor*. If $\gcd(a, b) = 1$, then a and b are said to be *relatively prime*.

Theorem 5. (The Division Algorithm) Given a positive integer a and an integer b, there exist unique integers q and r such that $b = aq + r$ and $0 \le r < a$. Then q is called the *quotient*, and r is called the *remainder* upon division of b by a.

Proof. Consider the set

$$S = \{s : s = b - aq \ge 0, q \in \mathbb{Z}\}.$$

S cannot be empty. If $b \ge 0$, then select $q = 0$ to give $b \in S$. Otherwise, if $b < 0$, select $q = b$ so that $b - ab = b(1 - a) \ge 0$, which means that $b - ab \in S$. Since S is non-empty and contains only non-negative integers, we can find the smallest element in S. Call it r.

Suppose $r \ge a$. Then $0 \le r - a = b - aq - a = b - a(q + 1)$ for some q, so $r - a \in S$ and it is smaller than r, contradiction. Thus $0 \le r < a$. Suppose that we can find $0 \le r_1 < r_2 < a$ in S and corresponding q_1 and q_2. Then $b = aq_1 + r_1 = aq_2 + r_2$, which implies that $a(q_1 - q_2) = r_2 - r_1$. Thus a divides $r_2 - r_1$. However, we also know that $0 < r_2 - r_1 < a$; that is, $r_2 - r_1$ lies between two consecutive multiples of a and thus cannot be divisible by a, contradiction. We have shown the existence and uniqueness of q and r.

Exercise 3. Prove that $\gcd(a, b) = \gcd(a, b - aq)$ for any non-zero integers a and b and any integer q.

Exercise 4. Prove that if a and q are relatively prime, then $\gcd(a, qb) = \gcd(a, b)$, where a, b, q are non-zero integers.

Exercise 5. Prove that F_n and F_{n+1} are relatively prime.

Number Theory 102

The Euclidean Algorithm. The Euclidean Algorithm is an algorithm used to determine the greatest common divisor of two numbers. Suppose we have two distinct positive integers a and b where, without loss of generality, $a < b$. By the Division Algorithm, $b = aq_1 + r_1$ where $0 \le r_1 < a$ for unique integers q_1 and r_1. If $r_1 = 0$, then a divides b so our greatest common divisor is a. Otherwise, by Exercise 3, $\gcd(a, b) = \gcd(a, b - aq_1) = \gcd(a, r_1) = \gcd(r_1, a)$. In this case, we then repeat the same argument using r_1 and a where we used a and b before, respectively. We have $a = r_1q_2 + r_2$ where $0 \le r_2 < r_1$ for unique integers q_2 and r_2 by the Division Algorithm, and $\gcd(r_1, a) = \gcd(r_1, a - q_2r_1) = \gcd(r_2, r_1)$. Continue applying this argument. Since the r_is are strictly decreasing and non-negative, there must be a last remainder, say r_n, that is bigger than 0. So we have

$$\begin{array}{rcll}
b & = & aq_1 + r_1, & 0 \le r_1 < a, \\
a & = & r_1 q_2 + r_2, & 0 \le r_2 < r_1, \\
r_1 & = & r_2 q_3 + r_3, & 0 \le r_3 < r_2, \\
\cdots & & & \\
r_{n-2} & = & r_{n-1} q_n + r_n, & 0 \le r_n < r_{n-1}, \\
r_{n-1} & = & r_n q_{n+1}, &
\end{array}
\qquad
\begin{array}{rcl}
\gcd(a,b) & = & \gcd(r_1,a), \\
\gcd(r_1,a) & = & \gcd(r_2,r_1), \\
\gcd(r_2,r_1) & = & \gcd(r_3,r_2), \\
 & \cdots & \\
\gcd(r_{n-1},r_{n-2}) & = & \gcd(r_n,r_{n-1}), \\
\gcd(r_n,r_{n-1}) & = & r_n.
\end{array}$$

We have found that $\gcd(a, b) = r_n$.

Theorem 6. For all a, $b \ge 1$,

$$\gcd(F_a, F_b) = F_{\gcd(a,b)}.$$

Proof. If a and b are equal, the result is immediate, so assume that $a < b$. Apply the Euclidean Algorithm to obtain

$$\begin{array}{rcll}
b & = & aq_1 + r_1, & 0 \le r_1 < a, \\
a & = & r_1 q_2 + r_2, & 0 \le r_2 < r_1, \\
r_1 & = & r_2 q_3 + r_3, & 0 \le r_3 < r_2, \\
 & \cdots & & \\
r_{n-2} & = & r_{n-1} q_n + r_n, & 0 \le r_n < r_{n-1}, \\
r_{n-1} & = & r_n q_{n+1}. &
\end{array}$$

We have

$$\gcd(F_a, F_b) \;=\; \gcd(F_a, F_{aq_1+r_1}) \;=\; \gcd(F_a, F_{aq_1-1}F_{r_1} + F_{aq_1}F_{r_1+1})$$

by Theorem 3. Since $F_{aq_1}F_{r_1+1}$ is a multiple of F_a by Corollary 4,

$$\gcd(F_a, F_b) \;=\; \gcd(F_a, F_{aq_1-1}F_{r_1} + F_{aq_1}F_{r_1+1}) \;=\; \gcd(F_a, F_{aq_1-1}F_{r_1})$$

by Exercise 3. By Exercise 5, $\gcd(F_{aq_1}, F_{aq_1-1}) = 1$. Since F_a divides F_{aq_1}, $\gcd(F_a, F_{aq_1-1}) = 1$ also. By Exercise 4, since F_a and F_{aq_1-1} are relatively prime,

$$\gcd(F_a, F_{aq_1-1}F_{r_1}) \;=\; \gcd(F_a, F_{r_1}).$$

We conclude that $\gcd(F_a, F_b) = \gcd(F_{r_1}, F_a)$. Repeating this argument, we obtain

$$\gcd(F_{r_1}, F_a) \;=\; \gcd(F_{r_2}, F_{r_1}) \;=\; \cdots \;=\; \gcd(F_{r_n}, F_{r_{n-1}}).$$

Since r_n divides r_{n-1}, F_{r_n} divides $F_{r_{n-1}}$, which implies that $\gcd(F_{r_n}, F_{r_{n-1}}) = F_{r_n}$. Thus,

$$\gcd(F_a, F_b) \;=\; F_{r_n} \;=\; F_{\gcd(a,b)}.$$

Finding F_n Explicitly

The monic quadratic with roots α and β is

$$(x-\alpha)(x-\beta) \;=\; x^2 - (\alpha+\beta)x + \alpha\beta.$$

Let α and β be the roots of $x^2 - x - 1$ in particular. Then, comparing coefficients, $\alpha + \beta = 1$ and $\alpha\beta = -1$. Using this, we can rewrite the recurrence relation $F_n = F_{n-1} + F_{n-2}$ as $F_n = (\alpha + \beta)F_{n-1} - \alpha\beta F_{n-2}$. From this equation, we obtain

$$F_n - \alpha F_{n-1} = \beta(F_{n-1} - \alpha F_{n-2}).$$

Let $s_{n-1} = F_n - \alpha F_{n-1}$ for all $n \geq 2$. Rewriting the above equation in terms of the s_i, we obtain $s_{n-1} = \beta s_{n-2}$. In other words, the sequence $\{s_n\}$ is a geometric sequence with common ration β. We conclude that $s_n = \beta^{n-1}s_1$.

Similarly, $F_n - \beta F_{n-1} = \alpha(F_{n-1} - \beta F_{n-2})$, and if we let $t_{n-1} = F_n - \beta F_{n-1}$ for $n \geq 2$, then $t_n = \alpha^{n-1}t_1$ for $n \geq 1$. Hence,

$$\begin{aligned} F_n &= \frac{\alpha - \beta}{\alpha - \beta} F_n + \frac{\alpha\beta}{\alpha - \beta} F_{n-1} - \frac{\alpha\beta}{\alpha - \beta} F_{n-1} \\ &= \frac{\alpha(F_n - \beta F_{n-1}) - \beta(F_n - \alpha F_{n-1})}{\alpha - \beta} \\ &= \frac{\alpha t_{n-1} - \beta s_{n-1}}{\alpha - \beta} \\ &= \frac{\alpha^{n-1}t_1 - \beta^{n-1}s_1}{\alpha - \beta} \\ &= \frac{\alpha^{n-1}(F_2 - \beta F_1) - \beta^{n-1}(F_2 - \alpha F_1)}{\alpha - \beta} \\ &= \frac{\alpha^{n-1}(1 - \beta) - \beta^{n-1}(1 - \alpha)}{\alpha - \beta}. \end{aligned}$$

Recall that $\alpha + \beta = 1$, so the equation above is, in fact,

$$\frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

Solving $x^2 - x - 1 = 0$ for the values of α and β, we have, without loss of generality,

$$\alpha = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1 - \sqrt{5}}{2}.$$

Substituting these values, we conclude that

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

This is called *Binet's Formula* for the Fibonacci sequence.

Exercise 5. Let $\tau = (1 + \sqrt{5})/2$. Prove that F_n is the integer closest to $\dfrac{\tau^n}{\sqrt{5}}$.

Problems

1. Prove that the product of every four consecutive Fibonacci numbers is the area of a Pythagorean triangle.

2. Prove that every positive integer can be written as a sum of distinct Fibonacci numbers.

3. Prove
$$F_n = \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-k}{k-1}.$$

4. Prove that if F_n is prime and $n \geq 5$, then n is prime.

5. Prove that $F_n + 1$ is always composite for $n \geq 4$.

6. Show that for any positive integer n, among the first n^2 Fibonacci numbers, there exists at least one that is divisible by n.

7. Define a Fibonacci prime to be a Fibonacci number that is prime. Prove or disprove: There are infinitely many Fibonacci primes. (Note that this is an open problem.)

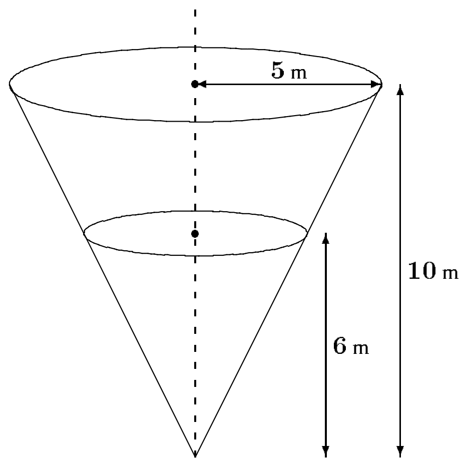
J.I.R. McKnight Problems Contest 1984

1. Find the real roots of the equation:

$$\sqrt{x+3-4\sqrt{x-1}}+\sqrt{x+8-6\sqrt{x-1}} \;=\; 1\,.$$

(Note: All square roots are to be taken as positive.)

2. Consider a reservoir in the shape of an inverted cone as shown in the diagram below. Water runs into the reservoir at the constant rate of $2\ \mathrm{m}^3$ per minute. How fast is the water level rising when it is 6 metres deep?



3. Three forces of magnitudes 10 N, 15 N, and 10 N act at angles of $30°$, $70°$, and $120°$ respectively, to the real axis O_x. Using the complex numbers and the imaginary axis O_y find the magnitude and direction of the resultant force.

4. Normals are drawn from the point $(\frac{15}{4}, -\frac{3}{4})$ to the parabola whose equation is $y^2 = 4x$. Find the coordinates of the points where the normals meet the parabola.

5. The horizontal base of a triangular pyramid is an equilateral triangle QRS, each of whose sides is 20 cm long. The sloping edges of the pyramid $PQRS$ are respectively 20 cm, 20 cm, and 12 cm long.

   (a) Calculate the perpendicular height of the pyramid to the nearest millimetre.

   (b) Calculate the angle of inclination of each of the three edges with the base to the nearest tenth of a degree.

6. Prove that if $\tan A = \tan^3 B$ and $\tan 2B = 2\tan C$, then $A+B-C = n\pi$ for some $n \in \mathbb{Z}$.

Swedish Mathematics Olympiad

1988 Qualifying Round

1. Show that the function

$$f(x) = \sqrt{x - 4\sqrt{x-1} + 3} + \sqrt{x - 6\sqrt{x-1} + 8}$$

is constant on the closed interval $5 \le x \le 10$.

2. Find the rational root of the equation

$$(2x)^{\log 2} = (3x)^{\log 3}, \quad x > 0$$

in the form $\frac{p}{q}$, where p and q are integers.

3. We will call two squares on a chessboard "neighbours" if they have a side or corner in common. The numbers 1 to 64 are arbitrarily placed on the 64 squares of a chessboard. Show that there are always two "neighbours" whose numbers have positive difference at least 9.

4. A car's tires wear proportionally with the distance driven. Furthermore, front tires last a km and back tires b km, where $a < b$. If, after an appropriate distance is driven, the tires are rotated (that is, back tires placed on front wheels and front tires on back wheels), the distance which can be driven without needing to replace any of the tires can be increased. What is the longest distance which can be driven with a set of tires, before any new tires must be bought?

5. P, Q, and R are points on the circumference of a circle such that PQR is an equilateral triangle. S is an arbitrary point on the circumference of the circle. Consider the lengths of the line segments PS, QS, and RS. Show that one of them is the sum of the other two.

6. Show that for every positive integer n, there exist positive integers x and y such that

$$\sqrt{x^2 + nxy + y^2}$$

is an integer.

1988 Final Round

1. The sides of a triangle have lengths $a > b > c$, and the corresponding perpendiculars have lengths h_a, h_b, and h_c. Show that

$$a + h_a \;>\; b + h_b \;>\; c + h_c \,.$$

2. Six ducklings swim on the surface of a pond, which is in the shape of a circle with radius 5 m. Show that, at every instant, two of the ducklings swim at a distance of at most 5 m from each other.

3. Show that for aribtrary real numbers x_1, x_2, and x_3,

   if $x_1 + x_2 + x_3 = 0$, then $x_1x_2 + x_2x_3 + x_3x_1 \;\leq\; 0$.

   Find all $n \geq 4$ for which the statement

   if $x_1 + x_2 + \cdots + x_n = 0$, then $x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1 \leq 0$

   is true. (Both sums have n terms.)

4. Let $P(x)$ be a polynomial of degree 3 with exactly three distinct real roots. Find the number of real roots of the equation

$$(P'(x))^2 - 2P(x)P''(x) \;=\; 0\,.$$

5. Let m and n be positive integers. Show that there exists a constant $\alpha > 1$, independent of m and n, such that

$$\frac{m}{n} < \sqrt{7} \quad \text{implies that} \quad 7 - \frac{m^2}{n^2} \geq \frac{\alpha}{n^2}\,.$$

6. The sequence a_1, a_2, ... , is defined by the recursion formula

$$a_{n+1} \;=\; \sqrt{a_n^2 + \frac{1}{a_n}} \quad n \geq 1\,,$$

   and $a_1 = 1$. Show that one can choose α such that

$$\frac{1}{2} \leq \frac{a_n}{n^\alpha} \leq 2 \qquad \text{for all } n \geq 1.$$

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) *after a number indicates that a problem was submitted without a solution.*

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ *or* A4 *sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than* **1 April 1999**. *They may also be sent by email to* **crux-editors@cms.math.ca**. (*It would be appreciated if email proposals and solutions were written in* LaTeX). *Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

2374. [1998: 365] **(Correction)** *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given triangle ABC with $\angle BAC > 60^\circ$. Let M be the mid-point of BC. Let P be any point in the plane of $\triangle ABC$.

Prove that $AP + BP + CP \geq 2AM$.

2376. *Proposed by Albert White, St. Bonaventure University, St. Bonaventure, NY, USA.*

Suppose that ABC is a right-angled triangle with the right angle at C. Let D be a point on hypotenuse AB, and let M be the mid-point of CD. Suppose that $\angle AMD = \angle BMD$. Prove that

1. $\overline{AC}^2\,\overline{MC}^2 + 4[ABC]\,[BCD] \;=\; \overline{AC}^2\,\overline{MB}^2$;
2. $4\overline{AC}^2\,\overline{MC}^2 - \overline{AC}^2\overline{BD}^2 \;=\; 4[ACD]^2 - 4[BCD]^2$,

where $[XYZ]$ denotes the area of $\triangle XYZ$.

(This is a continuation of problem **1812**, [1993: 48].)

2377. *Proposed by Nikolaos Dergiades, Thessaloniki, Greece.*

Let ABC be a triangle and P a point inside it. Let $BC = a$, $CA = b$, $AB = c$, $PA = x$, $PB = y$, $PC = z$, $\angle BPC = \alpha$, $\angle CPA = \beta$ and $\angle APB = \gamma$.

Prove that $ax = by = cz$ if and only if $\alpha - A = \beta - B = \gamma - C = \frac{\pi}{3}$.

2378. *Proposed by David Doster, Choate Rosemary Hall, Wallingford, Connecticut, USA.*

Find the exact value of: $\cot\left(\dfrac{\pi}{22}\right) - 4\cos\left(\dfrac{3\pi}{22}\right)$.

2379. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Suppose that M_1, M_2 and M_3 are the mid-points of the altitudes from A to BC, from B to CA and from C to AB in $\triangle ABC$. Suppose that T_1, T_2 and T_3 are the points where the excircles to $\triangle ABC$ opposite A, B and C, touch BC, CA and AB.

Prove that M_1T_1, M_2T_2 and M_3T_3 are concurrent.

Determine the point of concurrency.

2380. *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

When the price of a certain book in a store is reduced by $1/3$ and rounded to the nearest cent, the cents and dollars are switched. For example, if the original price was \$43.21, the new price would be \$21.43 (this does not satisfy the "reduced by $1/3$" condition, of course). What was the original price of the book? [For the benefit of readers unfamiliar with North American currency, there are 100 cents in one dollar.]

2381. *Proposed by Angel Dorito, Geld, Ontario.*

Solve the equation $\log_2 x = \log_4(x+1)$.

2382. *Proposed by Mohammed Aassila, Université Louis Pasteur, Strasbourg, France.*

If $\triangle ABC$ has inradius r and circumradius R, show that

$$\cos^2\left(\frac{B-C}{2}\right) \geq \frac{2r}{R}.$$

2383. *Proposed by Mohammed Aassila, Université Louis Pasteur, Strasbourg, France.*

Suppose that three circles, each of radius 1, pass through the same point in the plane. Let A be the set of points which lie inside at least two of the circles. What is the least area that A can have?

2384. *Proposed by Paul Bracken, CRM, Université de Montréal, Québec.*

Prove that $2(3n-1)^n \geq (3n+1)^n$ for all $n \in \mathbb{N}$.

2385. *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

A die is thrown $n \geq 3$ consecutive times. Find the probability that the sum of its n outcomes is greater than or equal to $n + 6$ and less than or equal to $6n - 6$.

2386*. *Proposed by Clark Kimberling, University of Evansville, Evansville, IN, USA.*

Write

$$\begin{matrix} 1 \end{matrix} \rightarrow \begin{matrix} 1 \\ 1 \end{matrix} \rightarrow \begin{matrix} 3 \\ 1 \end{matrix} \rightarrow \begin{matrix} 4 & 1 \\ 1 & 3 \end{matrix} \rightarrow \begin{matrix} 6 & 2 & 1 \\ 1 & 3 & 4 \end{matrix} \rightarrow \begin{matrix} 8 & 1 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 & 6 \end{matrix} \rightarrow$$

(The last ten numbers shown indicate that up to this point, eight 1's, one 2, three 3's, two 4's and one 6 have been written.)

(a) If this is continued indefinitely, will 5 eventually appear?

(b) Will every positive integer eventually be written?

Note: 11 is a number and not two 1's.

2387. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For fixed $p \in \mathbb{N}$, consider the power sums

$$S_p(n) := \sum_{k=1}^{n} (2k-1)^p, \qquad \text{where } n \geq 1,$$

so that $S_p(n)$ is a polynomial in n of degree $p + 1$ with rational coefficients. Prove that

(a) If all coefficients of $S_p(n)$ are integers, then $p = 2^m - 1$ for some $m \in \mathbb{N}$.

(b)* The only values of p yielding such polynomials are $p = 1$ and $p = 3$ (with $S_1(n) = n^2$ and $S_3(n) = 2n^4 - n^2$).

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

1637. [1991: 114; 1992: 125; 1994: 165] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Prove that
$$\sum \frac{\sin B + \sin C}{A} > \frac{12}{\pi}$$
where the sum is cyclic over the angles A, B, C (measured in radians) of a nonobtuse triangle.

Comment by Waldemar Pompe, student, University of Warsaw, Poland.

We can use ***Crux* 2015** [1998: 305] to derive the strengthening of ***Crux* 1637** given on [1994: 165], namely that

$$\sum \frac{\sin B + \sin C}{A} \geq \frac{9\sqrt{3}}{\pi}$$

for *all* triangles ABC.

Without loss of generality, we can assume that $A \geq B \geq C$. Then

$$\frac{1}{A} \leq \frac{1}{B} \leq \frac{1}{C}$$

and
$$\sin B + \sin C \leq \sin A + \sin C \leq \sin A + \sin B \,.$$
By Chebyshev's Inequality and ***Crux* 2015**, we have

$$3\sum \frac{\sin B + \sin C}{A} \geq \left(2\sum \sin A\right) \cdot \left(\sum \frac{1}{A}\right) \geq \frac{27\sqrt{3}}{\pi} \,,$$

and the result follows.

2257. [1997: 300] *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

The diagonals AC and BD of a convex quadrilateral $ABCD$ intersect at the point O. Let OK, OL, OM, ON, be the altitudes of triangles $\triangle ABO$, $\triangle BCO$, $\triangle CDO$, $\triangle DAO$, respectively.

Prove that if $OK = OM$ and $OL = ON$, then $ABCD$ is a parallelogram.

I. *Solution by Con Amore Problem Group, Royal Danish School of Educational Studies, Copenhagen, Denmark.*

Let $OK = OM = h$, $OA = a$, $OB = b$, $OC = c$, $OD = d$, and $\angle AOB = v$. Expressing in two ways the area of $\triangle AOB$, we get

$$\tfrac{1}{2}AB \cdot h = \tfrac{1}{2}ab\sin v,$$

and so

$$\frac{1}{h} = \frac{\sqrt{a^2+b^2-2ab\cos v}}{ab\sin v} = \frac{1}{\sin v}\sqrt{\frac{1}{b^2}+\frac{1}{a^2}-2\cdot\frac{1}{b}\cdot\frac{1}{a}\cdot\cos v}\,.$$

Similarly,

$$\frac{1}{h} = \frac{1}{\sin v}\sqrt{\frac{1}{d^2}+\frac{1}{c^2}-2\cdot\frac{1}{d}\cdot\frac{1}{c}\cdot\cos v}\,,$$

so that

$$\sqrt{\frac{1}{b^2}+\frac{1}{a^2}-2\cdot\frac{1}{b}\cdot\frac{1}{a}\cdot\cos v} = \sqrt{\frac{1}{d^2}+\frac{1}{c^2}-2\cdot\frac{1}{d}\cdot\frac{1}{c}\cdot\cos v}\,, \qquad (1)$$

and similarly,

$$\sqrt{\frac{1}{a^2}+\frac{1}{d^2}+2\cdot\frac{1}{a}\cdot\frac{1}{d}\cdot\cos v} = \sqrt{\frac{1}{b^2}+\frac{1}{c^2}+2\cdot\frac{1}{b}\cdot\frac{1}{c}\cdot\cos v}\,. \qquad (2)$$

Now, consider another convex quadrilateral $A_1B_1C_1D_1$, with diagonals intersecting in O_1, and such that $\angle A_1O_1B_1 = v$, $O_1A_1 = \frac{1}{a}$, $O_1B_1 = \frac{1}{b}$, $O_1C_1 = \frac{1}{c}$, and $O_1D_1 = \frac{1}{d}$. The equalities (1) and (2) imply that the opposite sides of $A_1B_1C_1D_1$ are equal in length, which means that $A_1B_1C_1D_1$ is a parallelogram. So $\frac{1}{a} = \frac{1}{c}$, and $\frac{1}{b} = \frac{1}{d}$, implying $a = c$ and $b = d$. This proves that $ABCD$ is a parallelogram.

II. *Solution by the proposer* (*slightly edited*).

If $ABCD$ is a trapezoid with $AB \parallel CD$, then

$$\frac{AB}{CD} = \frac{OK}{OM} = 1\,,$$

which means that $ABCD$ is a parallelogram. Hence, assume that $ABCD$ is not a trapezoid and set $P = AB\cap CD$, $Q = AD\cap BC$. Indeed, $P \neq Q$. The assumption on the given quadrilateral says exactly that PO bisects $\angle BPC$ and QO bisects $\angle AQB$. Thus

$$\frac{AP}{PC} = \frac{AO}{OC} = \frac{AQ}{QC}\,,$$

implying that P, O, and Q lie on the Apollonius circle with centre on the line AC. Similarly, since

$$\frac{BP}{PD} = \frac{BO}{OD} = \frac{BQ}{QD}\,,$$

P, O, and Q lie on the Apollonius circle with centre on the line BD. This implies that O is the circumcentre of $\triangle POQ$; that is, points P and Q coincide, a contradiction.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MICHAEL LAMBROU, University of Crete, Crete, Greece; TOSHIO SEIMIYA, Kawasaki, Japan; and D.J. SMEENK, Zaltbommel, the Netherlands. There were also five incorrect solutions submitted.

Most of the submitted solutions are similar to the proposer's solution.

2260. [1997: 301] *Proposed by Vedula N. Murty, Visakhapatnam, India.*

Let n be a positive integer and $x > 0$. Prove that

$$(1+x)^{n+1} \geq \frac{(n+1)^{n+1}}{n^n}\,x\,.$$

Solution by Florian Herzig, student, Cambridge, UK; Gerry Leversha, St. Paul's School, London, England; Nick Lord, Tonbridge School, Tonbridge, Kent, England; and Panos E. Tsaoussoglou, Athens, Greece.

By the AM–GM Inequality applied to the $n+1$ positive numbers $x, \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}$, we have $\left(\frac{x+1}{n+1}\right)^{n+1} \geq \frac{x}{n^n}$, with equality if and only if $x = \frac{1}{n}$. This is clearly equivalent to the given inequality.

Also solved by PAUL BRACKEN, CRM, Université de Montréal, Montréal, Québec; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; KEITH EKBLAW, Walla Walla, Washington, USA; RUSSELL EULER and JAWAD SADEK, NW Missouri State University, Maryville, Missouri, USA; RICHARD I. HESS, Rancho Palos Verdes, California, USA; JOE HOWARD, New Mexico Highlands University, Las Vegas, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; PAVLOS MARAGOUDAKIS, Hatzikiriakio, Pireas, Greece; PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA; VICTOR OXMAN, University of Haifa, Haifa, Israel; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; JOHN VLACHAKIS, Athens, Greece; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; ROGER ZARNOWSKI, TREY SMITH, CHARLES DIMINNIE and GERALD ALLEN (jointly), Angelo State University, San Angelo, TX, USA; and the proposer. There were also three incomplete solutions.

The majority of solvers used a standard calculus approach to establish the given inequality. The only exceptions are the five listed in the solutions above plus Lambrou and Maragoudakis, who used Bernoulli's Inequality. Both Janous and Lambrou noted that the given inequality holds for all positive real n.

Janous also generalized the problem by showing that if $a > 0$, and $\alpha > \beta > 0$ are given real numbers, then the largest constant $C = C(a, \alpha, \beta)$ such that $(a + x)^\alpha \geq Cx^\beta$ holds for all $x > 0$ is given by $C = \left(\dfrac{a}{\alpha - \beta}\right)^{\alpha-\beta} \dfrac{\alpha^\alpha}{\beta^\beta}$. The given problem is the special case when $a = \beta = 1$ and $\alpha = n + 1$.

By applying the AM–GM Inequality to the $n + 1$ positive numbers: $1, \dfrac{x}{n}, \dfrac{x}{n}, \dots, \dfrac{x}{n}$, Lord and the proposer obtained $(1 + x)^{n+1} \geq \dfrac{(n+1)^{n+1}}{n^n}\, x^n$, which is stronger than the proposed inequality for $x > 1$.

Zarnowski et al. *commented that when n is odd, the inequality is true for all real x, while if n is even, there is a number $x_n \leq -2$ such that the inequality holds for all $x \geq x_n$.*

2261. [1997: 301] *Proposed by Angel Dorito, Geld, Ontario.*

Assuming that the limit exists, find

$$\lim_{N\to\infty}\left(1 + \frac{2 + \frac{N+\cdots}{1+\cdots}}{N + \frac{1+\cdots}{2+\cdots}}\right),$$

where every fraction in this expression has the form

$$\frac{a + \frac{b+\cdots}{c+\cdots}}{b + \frac{c+\cdots}{a+\cdots}}$$

for some cyclic permutation a, b, c of 1, 2, N.

[*Proposer's comment*: this problem was suggested by Problem 4 of Round 21 of the International Mathematical Talent Search, *Mathematics and Informatics Quarterly*, Vol. 6, No. 2, p. 113.]

Solution by Keith Ekblaw, Walla Walla, Washington, USA.

It will be shown that

$$1 + \frac{2 + \frac{N+\cdots}{1+\cdots}}{N + \frac{1+\cdots}{2+\cdots}} \longrightarrow \frac{1+\sqrt{5}}{2} \qquad \text{as} \quad N \to \infty.$$

First, consider

$$J_N = N + \frac{1 + \frac{2+\cdots}{N+\cdots}}{2 + \frac{N+\cdots}{1+\cdots}}.$$

Note that

$$\frac{1 + \frac{2+\cdots}{N+\cdots}}{2 + \frac{N+\cdots}{1+\cdots}} > 0.$$

Thus $J_N > N$ and hence $J_N \to \infty$ as $N \to \infty$. Now let

$$K_N = 1 + \frac{2 + \frac{N+\cdots}{1+\cdots}}{N + \frac{1+\cdots}{2+\cdots}} = 1 + \frac{2 + \frac{J_N}{K_N}}{J_N} = 1 + \frac{2}{J_N} + \frac{1}{K_N}.$$

Thus as $N \to \infty$ (and hence $J_N \to \infty$), $K_N \to 1 + 1/K_N$. Letting $K = \lim_{N\to\infty} K_N$, we have $K = 1 + 1/K$ or $K^2 - K - 1 = 0$ and so the required limit is

$$K = \frac{1+\sqrt{5}}{2}\,.$$

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; FLORIAN HERZIG, student, Cambridge, UK; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; and the proposer.

Hess calculates that if N is replaced by 3.5, then the expression inside the limit (which is itself a limit, actually) is equal to 2. Readers may like to find other "nice" triples of numbers a, b, c so that the expression

$$a + \frac{b + \frac{c+\cdots}{a+\cdots}}{c + \frac{a+\cdots}{b+\cdots}}$$

is rational, say.

The proposer notes (as can be seen from the above proof) that the answer is still the golden ratio if the 2's in the given expression are replaced by any constant real number.

2262. [1997: 301] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Consider two triangles $\triangle ABC$ and $\triangle A'B'C'$ such that $\angle A \geq 90^\circ$ and $\angle A' \geq 90^\circ$ and whose sides satisfy $a > b \geq c$ and $a' > b' \geq c'$. Denote the altitudes to sides a and a' by h_a and h'_a.

Prove that (a) $\dfrac{1}{h_a h'_a} \geq \dfrac{1}{bb'} + \dfrac{1}{cc'}$, (b) $\dfrac{1}{h_a h'_a} \geq \dfrac{1}{bc'} + \dfrac{1}{b'c}$.

Solution by Christopher J. Bradley, Clifton College, Bristol, UK.

(a) By the Cauchy-Schwarz inequality on the vectors $\left(\dfrac{1}{b}, \dfrac{1}{c}\right)$ and $\left(\dfrac{1}{b'}, \dfrac{1}{c'}\right)$, we have

$$\frac{1}{bb'} + \frac{1}{cc'} \;\leq\; \left(\frac{1}{b^2} + \frac{1}{c^2}\right)^{\frac{1}{2}} \left(\frac{1}{b'^2} + \frac{1}{c'^2}\right)^{\frac{1}{2}}.$$

Now $\dfrac{1}{b^2} + \dfrac{1}{c^2} = \dfrac{b^2+c^2}{b^2c^2} \leq \dfrac{a^2}{b^2c^2}$, since $\angle A \geq 90^\circ$.

Also $\dfrac{a^2}{b^2c^2} \leq \dfrac{a^2}{b^2c^2\sin^2 A} = \dfrac{1}{h_a^2}$, since $\frac{1}{2}ah_a$ and $\frac{1}{2}bc\sin A$ are both formulae for the area of $\triangle ABC$. Similarly for $\triangle A'B'C'$.

Hence $\dfrac{1}{bb'} + \dfrac{1}{cc'} \leq \dfrac{1}{h_a h'_a}$, as required.

The proof of part (b) is the same, except that it starts with the vectors $\left(\frac{1}{b}, \frac{1}{c}\right)$ and $\left(\frac{1}{c'}, \frac{1}{b'}\right)$.

Also solved by CLAUDIO ARCONCHER, Jundiaí, Brazil; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; FLORIAN HERZIG, student, Cambridge, UK; RICHARD I. HESS, Rancho Palos Verdes, California, USA; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; VICTOR OXMAN, University of Haifa, Haifa, Israel; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; GEORGE TSAPAKIDIS, Agrinio, Greece; JOHN VLACHAKIS, Athens, Greece; and the proposer.

Most of the submitted solutions are similar to the one given above. Several solvers pointed out that the restrictions on the sides are unnecessary, and that equality in (a) *occurs if and only if* $\angle A = \angle A' = 90^\circ$ *and* $b/c = b'/c'$ *and in* (b) *if and only if* $\angle A = \angle A' = 90^\circ$ *and* $b/c = c'/b'$.

Janous proved more generally that for any real number $p \geq 1$

$$\frac{1}{(h_a h'_a)^p} \geq \frac{1}{(bb')^p} + \frac{1}{(cc')^p}, \quad \text{and} \quad \frac{1}{(h_a h'_a)^p} \geq \frac{1}{(bc')^p} + \frac{1}{(b'c)^p}.$$

2263. [1997: 364] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle, and the internal bisectors of $\angle B$, $\angle C$, meet AC, AB at D, E, respectively. Suppose that $\angle BDE = 30^\circ$. Characterize $\triangle ABC$.

Solution by the proposer.

Let F be the reflection of E across BD. Since $\angle EBD = \angle CBD$, it follows that F lies on BC. So $\angle BDF = \angle BDE = 30^\circ$, and $DF = DE$. Then $\triangle DEF$ is equilateral, so $EF = ED$ and $\angle FED = 60^\circ$. By the Law of Sines for $\triangle EFC$ and $\triangle EDC$ we obtain that

$$\frac{EC}{\sin\angle EFC} = \frac{EF}{\sin\angle ECF} = \frac{ED}{\sin\angle ECD} = \frac{EC}{\sin\angle EDC},$$

which gives $\sin\angle EFC = \sin\angle EDC$. It follows that we have either that $\angle EFC = \angle EDC$, or that $\angle EFC + \angle EDC = 180^\circ$.

Case 1. $\angle EFC = \angle EDC$. Then

$$\angle FEC = \angle DEC = \tfrac{1}{2}\angle FED = 30^\circ.$$

Let I be the intersection of BD and CE. Then

$$\angle DIC = \angle IED + \angle IDE = 30^\circ + 30^\circ = 60^\circ.$$

Since $\angle DIC = 90^\circ - \frac{1}{2}\angle A$, we obtain $90^\circ - \frac{1}{2}\angle A = 60^\circ$, so $\angle A = 60^\circ$.

Case 2. $\angle EFC + \angle EDC = 180^\circ$. Then
$\angle FED + \angle FCD = 180^\circ$, so that $60^\circ + \angle FCD = 180^\circ$.
Thus $\angle FCD = 120^\circ$; that is, $\angle ACB = 120^\circ$.

Therefore, ABC is a triangle with either $\angle A = 60°$ or $\angle C = 120°$.

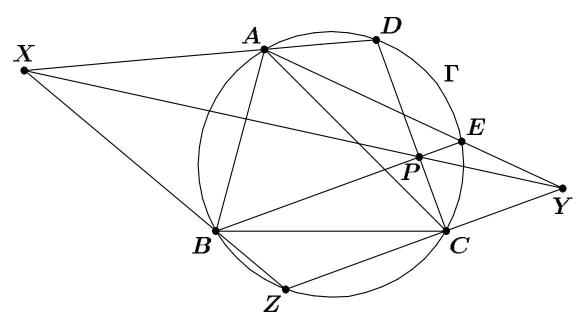
Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; FLORIAN HERZIG, student, Cambridge, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; VICTOR OXMAN, University of Haifa, Haifa, Israel; D.J. SMEENK, Zaltbommel, the Netherlands. There were also two incomplete solutions submitted.

Herzig and Lambrou have also shown that the characteristic condition is sufficient, that is, $\angle A = 60°$ or $\angle C = 120°$ implies $\angle BDE = 30°$.

2265. [1997: 364] *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

Given triangle ABC, let ABX and ACY be two variable triangles constructed outwardly on sides AB and AC of $\triangle ABC$, such that the angles $\angle XAB$ and $\angle YAC$ are fixed and $\angle XBA + \angle YCA = 180°$. Prove that all the lines XY pass through a common point.

Solution by Toshio Seimiya, Kawasaki, Japan.



We denote the circumcircle of $\triangle ABC$ by Γ. Let BX and CY meet at Z. Since $\angle XBA + \angle YCA = 180°$, we get $\angle XBA = 180° - \angle YCA = \angle ACZ$, so that A, B, Z, C are concyclic, that is, Z lies on Γ. Let D, E be the second intersections of AX, AY respectively with Γ. Since AX and AY are fixed lines, D and E are fixed points. Let P be the intersection of BE and CD. Since hexagon $ADCZBE$ is inscribed in Γ, by Pascal's Theorem the intersections of AD and BZ, of DC and BE, and of CZ and EA are collinear. Therefore variable line XY always passes through the fixed point P. [*Editorial note*: if the diagram differs from the one shown, for example if Z lies between X and B, the proof still works with minor changes.]

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; FLORIAN HERZIG, student, Cambridge, UK; MICHAEL LAMBROU, University of Crete, Crete, Greece; MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, *Valladolid, Spain; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer. One incorrect solution and one comment were sent in.*

Seimiya and the proposer had similar solutions.

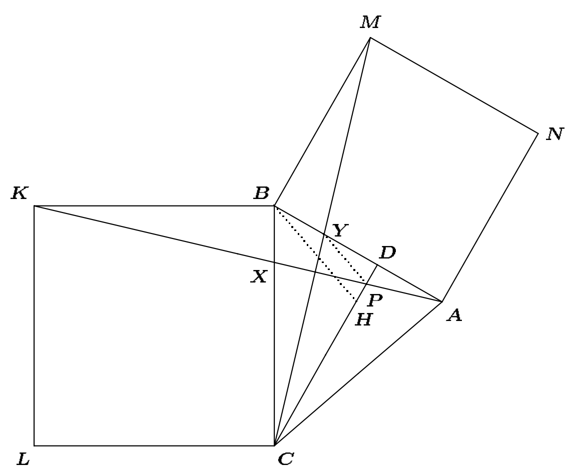
Herzig notes that, if the fixed angles are chosen to be AXB and AYC instead, then the lines XY still pass through a fixed point. Readers may like to show this themselves.

2266. [1997: 364] *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

$BCLK$ is the square constructed outwardly on side BC of an acute triangle ABC. Let CD be the altitude of $\triangle ABC$ (with D on AB), and let H be the orthocentre of $\triangle ABC$. If the lines AK and CD meet at P, show that

$$\frac{HP}{PD} = \frac{AB}{CD}.$$

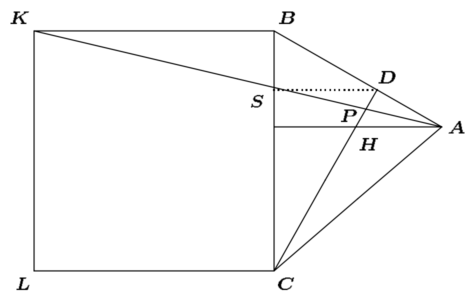
I. *Solution by Florian Herzig, student, Cambridge, UK.*



Construct a square $ABMN$ outwardly on $\triangle ABC$. Let X and Y be the points of intersection of AK, BC and AB, CM respectively. The rotation through a right angle about B maps $\triangle MBC$ onto $\triangle ABK$. Hence AK and CM are perpendicular and it follows that AP and CD are altitudes in $\triangle AYC$. Therefore P is the orthocentre in that triangle, and as a consequence $YP \perp AC$ or $YP \parallel BH$. Thus

$$\frac{HP}{PD} = \frac{BY}{YD} = \frac{BM}{CD} = \frac{AB}{CD}.$$

II. *Solution by Toshio Seimiya, Kawasaki, Japan.*



Let S be a point on AK such that $DS \perp BC$ and so $DS \parallel AH \parallel BK$. Since $AH \parallel DS$ we get

$$\frac{HP}{PD} = \frac{AH}{DS}. \tag{1}$$

Since $DS \parallel BK$ and $BK = BC$, we have

$$\frac{AD}{AB} = \frac{DS}{BK} = \frac{DS}{BC}. \tag{2}$$

Since $AH \perp BC$ and $CD \perp AB$ we get $\angle HAD = \angle BCD$. Moreover we have $\angle HDA = \angle BDC (= 90°)$, so that $\triangle HAD \sim \triangle BCD$. Thus

$$\frac{AH}{BC} = \frac{AD}{CD}. \tag{3}$$

From (2) and (3) we have

$$\frac{AH}{DS} = \frac{AH}{BC} \cdot \frac{BC}{DS} = \frac{AD}{CD} \cdot \frac{AB}{AD} = \frac{AB}{CD},$$

so that we obtain from (1) that $\dfrac{HP}{PD} = \dfrac{AB}{CD}$.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; D.J. SMEENK, Zaltbommel, the Netherlands; JOHN VLACHAKIS, Athens, Greece; and the proposer.

The proposer's solution was the same as Herzig's. Most other solvers used either similar triangles (as in II*), trigonometry, or coordinates.*

2268. [1997: 364] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let x, y be real. Find all solutions of the equation

$$\frac{2xy}{x+y}+\sqrt{\frac{x^2+y^2}{2}} \;=\; \sqrt{xy}+\frac{x+y}{2}.$$

Solution by Nikolaos Dergiades, Thessaloniki, Greece.

Let $A=\sqrt{\dfrac{x^2+y^2}{2}}$ and $B=\sqrt{xy}$. Then

$$2(A^2+B^2)=(x+y)^2 \qquad \text{and} \qquad 2(A^2-B^2)=(x-y)^2$$

and the given equation yields

$$\begin{aligned} A-B &= \frac{x+y}{2}-\frac{2xy}{x+y} \\ \text{or} \qquad \frac{2(A^2-B^2)}{A+B} &= \frac{(x-y)^2}{x+y} \\ \frac{(x-y)^2}{A+B} &= \frac{(x-y)^2}{x+y}. \end{aligned}$$

Therefore, we have either $(x-y)^2=0$ (which implies that $x=y$) or $A+B=x+y$. Let us consider $A+B=x+y$:

$$\begin{aligned} A+B=x+y \quad &\iff \quad (A+B)^2=2(A^2+B^2) \\ &\iff \quad (A-B)^2=0 \quad \iff \quad A=B \\ &\iff \quad \sqrt{\frac{x^2+y^2}{2}}=\sqrt{xy} \\ &\iff \quad (x-y)^2=0 \quad \iff \quad x=y \end{aligned}$$

In conclusion, all solutions have $x=y\neq 0$, because $x+y\neq 0$.

Also solved by HAYO AHLBURG, Benidorm, Spain; PAUL BRACKEN, CRM, Université de Montréal; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark (2 solutions); CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; RUSSELL EULER and JAWAD SADEK, NW Missouri State University, Maryville, Missouri, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; VICTOR OXMAN, University of Haifa, Haifa, Israel; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; ROGER

ZARNOWSKI, Angelo State University, San Angelo, Texas, USA; and the proposer. There were 14 incorrect solutions submitted, 11 of which simply did NOT exclude the origin from the solution set.

2270. [1997: 365] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Given $\triangle ABC$ with sides a, b, c, a circle, centre P and radius ρ intersects sides BC, CA, AB in A_1 and A_2, B_1 and B_2, C_1 and C_2 respectively, so that

$$\frac{\overline{A_1A_2}}{a} = \frac{\overline{B_1B_2}}{b} = \frac{\overline{C_1C_2}}{c} = \lambda \geq 0.$$

Determine the locus of P.

Solution by the proposer.

[Assume that no two sides of the triangle are equal.] The distance from

$$P \text{ to } BC \text{ is } \quad x = \sqrt{\rho^2 - \frac{\lambda^2 a^2}{4}},$$

$$P \text{ to } CA \text{ is } \quad y = \sqrt{\rho^2 - \frac{\lambda^2 b^2}{4}},$$

$$P \text{ to } AB \text{ is } \quad z = \sqrt{\rho^2 - \frac{\lambda^2 c^2}{4}}.$$

It follows that $\dfrac{x^2 - y^2}{y^2 - z^2} = \dfrac{b^2 - a^2}{c^2 - b^2}$, or

$$x^2(c^2 - b^2) + y^2(a^2 - c^2) + z^2(b^2 - a^2) = 0. \tag{1}$$

Considering x, y, z to be the triangular coordinates of P with respect to $\triangle ABC$, we conclude that (1) represents a conic K. Note that K passes through the incentre $I(1,1,1)$ and the excentres $I_a(-1,1,1), I_b(1,-1,1)$, and $I_c(1,1,-1)$, [and also the circumcentre $O(\cos A, \cos B, \cos C)$]. So K is the conic through O of the pencil determined by I, I_a, I_b, I_c. Since the degenerate conics of the pencil are degenerate orthogonal hyperbolas (that is, pairs of perpendicular lines), K must be an orthogonal hyperbola.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK.

Neither solver mentioned the obvious special cases:

The locus is a pair of perpendicular lines when $\triangle ABC$ is isosceles, and just the four points I, I_a, I_b, I_c when equilateral.

Bradley points out that it is clear from the statement of the problem (with no need for coordinates) that the locus includes I, I_a, I_b, I_c (when $\lambda = 0$) and O (when $\lambda = 1$).

2271. [1997: 365] *Proposed by F.R. Baudert, Waterkloof Ridge, South Africa.*

A municipality charges householders per month for electricity used according to the following scale:

first 400 units — 4.5¢ per unit;

next 1100 units — 6.1¢ per unit;

thereafter — 5.9¢ per unit.

If E is the total amount owing (in dollars) for n units of electricity used, find a closed form expression, $E(n)$.

Solution by Michael Lambrou, University of Crete, Greece.

We may view the charges as consisting of

(i) 4.5¢ per unit and, additionally,

(ii) a surcharge of $6.1 - 4.5 = 1.6$¢ per unit, but with

(iii) a refund of $6.1 - 5.9 = 0.2$¢ per unit for units consumed in excess of $400 + 1100 = 1500$.

So (in dollars) the amount owing for n units is:

$$\frac{45}{1000}n + \frac{16}{1000}\max\{0, n-400\} - \frac{2}{1000}\max\{0, n-1500\}.$$

Now writing $\max\{a,b\} = \frac{1}{2}\left(|a-b| + a + b\right)$, this simplifies to:

$$\frac{1}{1000}\left\{52n + 8|n-400| - |n-1500| - 1700\right\}.$$

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; FLORIAN HERZIG, student, Cambridge, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; KEE-WAI LAU, Hong Kong; and the proposer. There were two incorrect solutions submitted. Some solvers used unit step functions instead of absolute value to express the answer.

2272*. [1997: 365] *Proposer unknown (please identify yourself!)*

Write $r \precsim s$ if there is an integer k satisfying $r < k < s$. Find, as a function of n $(n \geq 2)$, the least positive integer satisfying

$$\frac{k}{n} \precsim \frac{k}{n-1} \precsim \frac{k}{n-2} \precsim \cdots \precsim \frac{k}{2} \precsim k.$$

Solution by Florian Herzig, student, Cambridge, UK (modified by the editor).

Let k_n denote the least positive integer k satisfying

$$\frac{k}{n} \precsim \frac{k}{n-1} \precsim \frac{k}{n-2} \precsim \cdots \preceq \frac{k}{2} \precsim k \tag{1}$$

We claim that $k_n = \left\lfloor \left(\frac{n+x_n}{2}\right)^2 + 1 \right\rfloor$, where $x_n = \lfloor n+1-2\sqrt{n-1} \rfloor$. We first show that k_n satisfies (1). To this end, let $u = \lfloor \sqrt{k_n} \rfloor$. Then for all integers $a = 1, 2, \cdots, u-1$ we have, from $a \leq \sqrt{k_n} - 1$ that $a(a+1) \leq \sqrt{k_n}(\sqrt{k_n}-1) < k_n$. Hence $\frac{k_n}{a} - \frac{k_n}{a+1} > 1$, which implies $\frac{k_n}{a+1} \precsim \frac{k_n}{a}$. Therefore we have

$$\frac{k_n}{u} \precsim \frac{k_n}{u-1} \precsim \cdots \precsim \frac{k_n}{2} \preceq k_n. \tag{2}$$

Next we show that for all integers $a = u+1, u+2, \cdots, n$

$$n + x_n - a < \frac{k_n}{a} < n + x_n - a + 1. \tag{3}$$

In fact, the left inequality in (3) holds for all $a = 1, 2, \cdots, n$. To see this, note that from $k_n > \left(\frac{n+x_n}{2}\right)^2$ we get $a^2 - ax_n - an + k_n = \left(a - \frac{n+x_n}{2}\right)^2 + k_n - \left(\frac{n+x_n}{2}\right)^2 > 0$, and thus $n + x_n - a < \frac{k_n}{a}$.

On the other hand, note that the right inequality in (3) is equivalent to

$$\left(a - \frac{n+x_n+1}{2}\right)^2 < \left(\frac{n+x_n+1}{2}\right)^2 - k_n. \tag{4}$$

Since $k_n > \left(\frac{n+x_n}{2}\right)^2$, we have

$$a \geq u+1 = \left\lfloor \sqrt{k_n} \right\rfloor + 1 \geq \left\lfloor \frac{n+x_n}{2} \right\rfloor + 1 \geq \frac{n+x_n+1}{2}.$$

[*Ed: The last inequality holds since $n + x_n$ is an integer.*]

Hence it suffices to establish (4) for $a = n$.

Substituting $a = n$ into the right inequality of (3), we need to show that $k_n < nx_n + n$. From $n - 2\sqrt{n-1} < x_n < n$ we get $(n-x_n)^2 < 4(n-1)$ or $(n+x_n)^2 + 4 < 4(nx_n+n)$. Hence $k_n \leq \left(\frac{n+x_n}{2}\right)^2 + 1 < nx_n + n$. Therefore (3) holds, and by setting $a = u+1, u+2, \cdots, n$ we get

$$\frac{k_n}{n} < x_n + 1 < \frac{k_n}{n-1} < \cdots < x_n + n - u - 1 < \frac{k_n}{u+1} < x_n + n - u < \frac{k_n}{u}.$$

Hence

$$\frac{k_n}{n} \precsim \frac{k_n}{n-1} \cdots \precsim \frac{k_n}{u+1} \precsim \frac{k_n}{u}. \tag{5}$$

From (2) and (5) we conclude that k_n satisfies (1).

Now we show that if k is any integer satisfying (1), then $k \geq k_n$. To this end, let $x = \left\lfloor \frac{k}{n} \right\rfloor$.We first show that $x \geq x_n$. Since there exists an integer z such that $\frac{k}{n} < z < \frac{k}{n-1}$ and $x = \left\lfloor \frac{k}{n} \right\rfloor < z$, we have $z - x \geq 1$ and hence $\frac{k}{n-1} > x+1$. Similarly, $\frac{k}{n-2} > x+2$, $\frac{k}{n-3} > x+3, \cdots$, $\frac{k}{1} > x+n-1$. That is, for all $a = 1, 2, \cdots, n$ we have $k > (n-a)(x+a)$

$$\text{or} \quad k \geq (n-a)(x+a)+1 = -a^2 + a(n-x) + nx + 1. \tag{6}$$

Hence

$$k \geq \left(\frac{n-x}{2}\right)^2 + nx + 1 - \left(a - \frac{n-x}{2}\right)^2. \tag{7}$$

Note that $x \geq 1$. [*Ed: If* $k \leq n-1$, *then* $0 < \frac{k}{n} < \frac{k}{n-1} \leq 1$, *contradicting* $\frac{k}{n} \precsim \frac{k}{n-1}$. *Hence* $k \geq n$].

On the other hand, it is clear that $x_n \leq n-1$. Suppose, contrary to what we claim, that $x < x_n$. Then we have $1 \leq x < x_n \leq n-1$ and so $2 \leq n-x < n$ or $1 \leq \frac{n-x}{2} < \frac{n}{2}$. (Here we must assume that $n \geq 3$. The case when $n = 2$ can be treated separately, since it is easy to verify that $k_2 = 3$.) Hence we may let $a = \left\lfloor \frac{n-x}{2} \right\rfloor$ in (6) and (7) and obtain

$$k \geq \left(\frac{n-x}{2}\right)^2 + nx + 1 - \left(\left\lfloor \frac{n-x}{2} \right\rfloor - \left(\frac{n-x}{2}\right)\right)^2. \tag{8}$$

Since the right side of (7) is an integer and since the last squared term in (8) is either 0 or $\frac{1}{4}$ we get

$$k \geq \left\lfloor \left(\frac{n-x}{2}\right)^2 + nx + 1 \right\rfloor = \left\lfloor \left(\frac{n+x}{2}\right)^2 + 1 \right\rfloor. \tag{9}$$

Thus $x = \left\lfloor \frac{k}{n} \right\rfloor \geq \left\lfloor \frac{1}{n} \left\lfloor \left(\frac{n+x}{2}\right)^2 + 1 \right\rfloor \right\rfloor = \left\lfloor \frac{1}{n} \left(\frac{n+x}{2}\right)^2 + \frac{1}{n} \right\rfloor$.

[*Ed: It is known and easy to show that* $\left\lfloor \frac{\lfloor z \rfloor}{n} \right\rfloor = \left\lfloor \frac{z}{n} \right\rfloor$ *for all real numbers* z *and positive integers* n.]

Hence $0 \geq \left\lfloor \frac{1}{n}\left(\frac{n+x}{2}\right)^2 + \frac{1}{n} \right\rfloor - x = \left\lfloor \frac{1}{n}\left(\frac{n-x}{2}\right)^2 + \frac{1}{n} \right\rfloor$, which implies that $\frac{1}{n}\left(\left(\frac{n-x}{2}\right)^2 + 1\right) \leq 1$ or $\left(\frac{n-x}{2}\right)^2 \leq n-1$.

Thus $n - x < 2\sqrt{n-1}$ or $x > n - 2\sqrt{n-1}$, from which we get $x \geq \lfloor n+1-2\sqrt{n-1} \rfloor = x_n$, a contradiction. Hence $x \geq x_n$. Therefore we may replace x by x_n in (6), (7), and (9) and conclude that $k \geq \left\lfloor \left(\frac{n+x_n}{2}\right)^2 + 1 \right\rfloor = k_n$. This completes the proof.

Also solved by PETER TINGLEY, student, University of Waterloo, Waterloo, Ontario. There was one incorrect solution. Tingley gave the answer

$$k_n = n\lfloor n - 2\sqrt{n-1} + 1\rfloor + \left\lfloor \left(\frac{n - \lfloor n - 2\sqrt{n-1} + 1\rfloor}{2}\right)^2 + 1 \right\rfloor$$

which is readily seen to be the same as the one obtained by Herzig. The proposer had conjectured that

$$k_n = \begin{cases} 1 + (n-m)^2 & \text{if } m^2 \leq n-2 \\ 1 + (n-m)^2 + (n-m) & \text{otherwise} \end{cases}$$

where $m = \left\lfloor \frac{1+\sqrt{4n-7}}{2} \right\rfloor$ *and had verified it for* $2 \leq n \leq 600$ *using a computer. In a private communication Tingley has actually proved that this conjectured formula is equivalent to the answer given by Herzig and himself. Interested readers may find the proof of this fact quite challenging.*

2273. [1997: 366] *Proposed by Tim Cross, King Edward's School, Birmingham, England.*

Consider the sequence of positive integers: $\{1, 12, 123, 1\,234, 12\,345, \ldots\}$, where the next term is constructed by lengthening the previous term at its right-hand end by appending the next positive integer. Note that this next integer occupies only one place, with "carrying" occurring as in addition: thus the ninth and tenth terms of the sequence are $123\,456\,789$ and $1\,234\,567\,900$ respectively.

Determine which terms of the sequence are divisible by 7.

Solution by Heinz-Jürgen Seiffert, Berlin, Germany.

The sequence $\{a_n\}$ under consideration satisfies the recurrence

$$a_1 = 1 \quad \text{and} \quad a_n = 10a_{n-1} + n \quad \text{for } n \geq 2.$$

A simple induction argument shows that

$$81a_n = 10^{n+1} - 9n - 10, \quad n \in \mathbb{N}.$$

Let $n \in \mathbb{N}$. Applying the Euclidean Algorithm twice, we see that there exist non-negative integers j, k, r such that $0 \le k \le 6$, $0 \le r \le 5$, and $n = 42j + 6k + r$. Since $3^{42j} \equiv 3^{6k} \equiv 1 \pmod 7$ by Fermat's Little Theorem, it follows that

$$4a_n \equiv 3^{r+1} + 2k - 2r - 3 \pmod 7.$$

The following table gives the remainder when the expression on the right hand side of the above congruence is divided by 7:

$r \backslash k$	0	1	2	3	4	5	6
0	0	2	4	6	1	3	5
1	4	6	1	3	5	0	2
2	6	1	3	5	0	2	4
3	2	4	6	1	3	5	0
4	1	3	5	0	2	4	6
5	2	4	6	1	3	5	0

Inspecting this table and using the above congruence, we see that a_n is divisible by 7 if and only if $n \equiv 0, 22, 26, 31, 39$, or $41 \pmod{42}$.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; FLORIAN HERZIG, student, Cambridge, UK; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; J.A. MCCALLUM, Medicine Hat, Alberta; JOEL SCHLOSBERG, student, Robert Louis Stevenson School, New York, NY, USA; TREY SMITH, GERALD ALLEN, NOEL EVANS, CHARLES DIMINNIE, AND ROGER ZARNOWSKI (jointly), Angelo State University, San Angelo, Texas; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA; and the proposer. There was one incorrect solution submitted.

2274. [1997: 366] *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*

A. Let m be a non-negative integer. Find a closed form for $\displaystyle\sum_{k=1}^{n}\prod_{j=0}^{m}(k+j)$.

B. Let $m \in \{1, 2, 3, 4\}$. Find a closed form for $\displaystyle\sum_{k=1}^{n}\prod_{j=0}^{m}(k+j)^2$.

C$\star$. Let m and α_j ($j = 0, 1, \ldots, m$) be non-negative integers. Prove or disprove that $\sum_{k=1}^{n} \prod_{j=0}^{m} (k+j)^{\alpha_j}$ is divisible by $\prod_{j=0}^{m+1} (n+j)$.

I. *Solution by Florian Herzig, student, Cambridge, UK.*

A. Note that

$$\begin{aligned}\sum_{k=1}^{n} \prod_{j=0}^{m} (k+j) &= \sum_{k=1}^{n} \frac{(m+k)!}{(k-1)!} = (m+1)! \sum_{k=1}^{n} \binom{m+k}{k-1} \\ &= (m+1)! \left[\binom{m+1}{0} + \binom{m+2}{1} + \binom{m+3}{2} + \cdots + \binom{m+n}{n-1} \right].\end{aligned}$$

This combinatorial sum [inside the square brackets] is well-known and can be evaluated as follows. Note that the first two terms add to $\binom{m+3}{1}$ which in turn adds with the third term to $\binom{m+4}{2}$, and so on until we obtain the desired closed form $\binom{m+n+1}{n-1}$ in the end. Thus

$$\begin{aligned}\sum_{k=1}^{n} \prod_{j=0}^{m} (k+j) &= (m+1)! \binom{m+n+1}{n-1} \\ &= \frac{(m+n+1)!}{(m+2)(n-1)!} = \frac{n(n+1)\ldots(n+m+1)}{m+2}.\end{aligned}$$

B. The expressions reduce to sums of the form $\sum_{k=1}^{n} k^m$. With the help of a calculator I got

$$\binom{n+2}{3} \frac{2(3n^2+6n+1)}{5} \quad \text{for } m = 1,$$

$$\binom{n+3}{4} \frac{12(2n+3)(5n^2+15n+1)}{35} \quad \text{for } m = 2,$$

$$\binom{n+4}{5} \frac{8(35n^4+280n^3+685n^2+500n+12)}{21} \quad \text{for } m = 3,$$

$$\binom{n+5}{6} \frac{40(126n^5+1575n^4+6860n^3+12075n^2+7024n+60)}{77} \quad \text{for } m = 4.$$

C. I assume that "is divisible by" means *polynomial* division.

Clearly the α_j should be *positive* integers, and in this case I will prove that the claim is true. Define

$$P(n) = \sum_{k=1}^{n} \prod_{j=0}^{m} (k+j)^{\alpha_j},$$

a polynomial of degree $(\sum_{j=1}^{n}\alpha_j)+1$ (since each $\sum_{k=1}^{n}k^l$ is a polynomial of degree $l+1$). Then

$$P(n) \;=\; \sum_{k=-m}^{n}\prod_{j=0}^{m}(k+j)^{\alpha_j} \;=\; \sum_{k=1}^{m+n+1}\;\prod_{j=0}^{m}(k-m-1+j)^{\alpha_j}\,,$$

as in the first of these sums all the terms for $k\le 0$ vanish. Hence, for all integers a such that $-m\le a\le 0$ we get

$$P(a) = \sum_{k=1}^{m+a+1}(k-m-1)^{\alpha_0}(k-m)^{\alpha_1}\ldots(k-1)^{\alpha_m} = 0\,,$$

since each term is zero. [Thus $n-a$ must be a factor of $P(n)$ for each such a, so $P(n)$ must be divisible by each of $n, n+1, \ldots, n+m$. — *Ed.*] For $n=-m-1$ the above sum is empty and so $P(-m-1)=0$ as well (to avoid empty sums one can instead write $P(n)$ as $\sum_{k=0}^{m+n+1}p(k)-p(0)$ where $p(k)$ is the above product). Therefore $P(n)$ is divisible by $\prod_{j=0}^{m+1}(n+j)$ as claimed.

II. *Solution to part* A *by Michael Lambrou, University of Crete, Greece.*
A. From the identity

$$\prod_{j=0}^{m}(k+j) = \frac{1}{m+2}\left[\prod_{j=0}^{m+1}(k+j) - \prod_{j=0}^{m+1}(k-1+j)\right]$$

(easily verified by considering the common factor $\prod_{j=0}^{m}(k+j)$ of the two products on the right), we obtain telescopically

$$\sum_{k=1}^{n}\prod_{j=0}^{m}(k+j) = \frac{1}{m+2}\left[\prod_{j=0}^{m+1}(n+j) - \prod_{j=0}^{m+1}j\right] = \frac{1}{m+2}\prod_{j=0}^{m+1}(n+j).$$

[*Editorial note*: Lambrou also solved parts B and C.]

All three parts also solved by G. P. HENDERSON, Garden Hill, Ontario. Parts A *and* B *only solved by THEODORE N. CHRONIS, Athens, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.*

As Herzig mentions above, part A *at least is a fairly familiar result. (For example, see formula 2.50, page 50 of* Concrete Mathematics *by Graham, Knuth and Patashnik.) In fact it was proposed for publication in* ***CRUX*** *in 1991 by an Edmonton high school student, Jason Colwell, but was not accepted by the then editor.*

Chronis notes that, in the solution for part B, *when $m=4$, the fifth degree polynomial has $2n+5$ as a factor.*

2275. [1997: 366] *Proposed by M. Perisastry, Vizianagaram, Andhra Pradesh, India.*

Let $b > 0$ and $b^a \geq ba$ for all $a > 0$. Prove that $b = e$.

I. *Solution by Gerald Allen, Charles Diminnie, Trey Smith and Roger Zarnowski (jointly), Angelo State University, San Angelo, TX, USA; Russell Euler and Jawad Sadek (jointly), NW Missouri State University, Maryville, Missouri, USA; Michael Parmenter, Memorial University of Newfoundland, St. John's, Newfoundland; Reza Shahidi, student, University of Waterloo, Waterloo, Ontario; George Tsapakidis, Agrinio, Greece; and John Vlachakis, Athens, Greece.*

Let $f(x) = b^x - bx$ for $x > 0$. Since $f(x) \geq 0$ for all $x > 0$ and $f(1) = 0$, it follows that f has a relative (as well as an absolute) minimum at $x = 1$.

Since $f'(1)$ exists, we have $f'(1) = 0$; that is, $b \ln b - b = 0$. Since $b > 0$, we get $\ln b = 1$ or $b = e$.

II. *Solution by Theodore Chronis, student, Aristotle University of Thessaloniki, Greece; Florian Herzig, student, Cambridge, UK; Michael Lambrou, University of Crete, Crete, Greece; Gerry Leversha, St. Paul's School, London, England; Vedula N. Murty, Visakhapatnam, India; Heinz-Jürgen Seiffert, Berlin, Germany; and David R. Stone, Georgia Southern University, Statesboro, Georgia, USA.*

The given inequality is equivalent to $b^{a-1} \geq a$ for all $a > 0$. Letting $a = 1 + \dfrac{1}{n}$ where $n \in \mathbb{N}$, we get $b^{\frac{1}{n}} \geq 1 + \dfrac{1}{n}$, or $b \geq \left(1 + \dfrac{1}{n}\right)^n$. Hence $b \geq \lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n = e$.

On the other hand, letting $a = \left(1 + \dfrac{1}{n}\right)^{-1} = \dfrac{n}{n+1}$, where $n \in \mathbb{N}$, we get from $b^{1-a} \leq \dfrac{1}{a}$, that $b^{\frac{1}{n+1}} \leq 1 + \dfrac{1}{n}$, or $b \leq \left(1 + \dfrac{1}{n}\right)^{n+1}$. Hence $b \leq \lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n = e$. Therefore, $b = e$.

Also solved by FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, MA, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; LUZ M. DeALBA, Drake University, Des Moines, IA, USA; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; KEITH EKBLAW, Walla Walla, Washington, USA; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; F.J. FLANIGAN, San Jose State University, San Jose, California, USA; RICHARD I. HESS, Rancho Palos Verdes, California, USA; JOE HOWARD, New Mexico Highlands University, Las Vegas, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; KEE-WAI LAU, Hong Kong; D.J. SMEENK, Zaltbommel, the Netherlands; DIGBY SMITH, Mount Royal College, Calgary, Alberta; and the *proposer. There was one incorrect solution.*

Although the problem did not ask to show that the condition $b = e$ is both necessary and sufficient, a few solvers did provide a proof of the simple fact that $e^x \geq ex$ for all $x > 0$.

2277. [1997: 431] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

For $n \geq 1$, define

$$u_n = \left[\frac{1}{(1,n)}, \frac{2}{(2,n)}, \ldots, \frac{n-1}{(n-1,n)}, \frac{n}{(n,n)}\right],$$

where the square brackets [ ] and the parentheses ( ) denote the **least common multiple** and **greatest common divisor** respectively.

For what values of n does the identity $u_n = (n-1)u_{n-1}$ hold?

Solution by Florian Herzig, student, Cambridge, UK.

We first introduce some notation:

for any prime p, let $(n)_p = \max\{\alpha | p^\alpha \leq n\}$, $[n]_p = \max\{\alpha | p^\alpha \text{divides } n\}$. Thus $p^{[n]_p} || n$. For each $k = 1, 2, \cdots, n$ we determine $a = a(k)$ such that $p^a || k$. If $a \leq [n]_p$, then since $p^a | n$, we have $p^a || (k,n)$ and so $p \not| \dfrac{k}{(k,n)}$. If $a > [n]_p$, then since $p^{[n]_p} | k$ we have $p^{[n]_p} || (k,n)$ and so $p^{a-[n]_p} || \dfrac{k}{(k,n)}$. Thus the highest power of p in any $\dfrac{k}{(k,n)}$ arises when $a = (n)_p$. This shows that $u_n = \prod_p p^{(n)_p - [n]_p}$, where the product is over all primes. Note that $n - 1 = \prod_p p^{[n-1]_p}$ and hence $u_n = (n-1)u_{n-1}$ is equivalent to

$$(n)_p - [n]_p = (n-1)_p \text{ for all primes } p \qquad (1)$$

We distinguish two cases:

Case (i) Suppose n is a prime power, say $n = q^b$ where q is a prime and $b > 0$. For $p \neq q$, (1) is satisfied since $(n)_p = (n-1)_p$ and $[n]_p = 0$. For $p = q$ we have $n = p^b$ and so $(n)_p = [n]_p = b$ and $(n-1)_p = b - 1$. Hence (1) holds if and only if $b - 1 = 0$; that is, $b = 1$.

Case (ii) If n is not a prime power, then $(n)_p = (n-1)_p$ for all primes p. Hence (i) holds if and only if $[n]_p = 0$ for all primes p, and so $n = 1$. [*Ed: Clearly $n = 1$ is not a solution, since u_0 is undefined.*]

Therefore $u_n = (n-1)u_{n-1}$ if and only if n is a prime.

Also solved by ED BARBEAU, University of Toronto, Toronto; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; JOEL SCHLOSBERG, student, Robert Louis Stevenson School, New York, NY, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; and the proposer.

From the proof given above, it is not difficult to see that, in fact, we have $u_n = [1, 2, \cdots, n]/n$. *This was explicitly pointed out by Konečný, Lambrou, and the proposer, but only Lambrou and the proposer actually gave a proof.*

2278. [1997: 431] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

Determine the value of a_n, which is the number of ordered n-tuples $(k_2, k_3, \ldots, k_n, k_{n+1})$ of non-negative integers such that

$$2k_2 + 3k_3 + \ldots + nk_n + (n+1)k_{n+1} \;=\; n + 1.$$

I. *Solution by Michael Lambrou, University of Crete, Crete, Greece.*

We show that $a_n = p(n+1) - p(n)$ for $n \geq 1$ where $p(m)$ denotes the number of partitions of m into positive integral parts. Our argument is based on the well-known observation that to a partition of m where l_k k's appear ($k = 1, 2, \ldots, m$), so that

$$1l_1 + 2l_2 + \cdots + ml_m \;=\; m\,, \qquad (1)$$

corresponds the ordered m-tuple $(l_1, l_2, \ldots, l_m)$. Conversely, to any given ordered m-tuple $(l_1, l_2, \ldots, l_m)$ of positive integers satisfying (1), there corresponds a partition of m.

For fixed $n \geq 1$ consider the partitions of $n+1$ as above. They are of two types:

(a) those for which the number 1 is absent in the decomposition; or

(b) those for which the number 1 appears at least once.

The number of partitions of type (a) is clearly a_n. Moreover, for each partition in case (b), if we delete one 1, we get a partition of n. Conversely, every partition of n with an extra 1 added on gives a partition of $n+1$ of type (b). Clearly then $p(n+1) = a_n + p(n)$, as required.

II. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

We deal more generally with the equation:

$$k_1 + 2k_2 + \cdots + (j-1)k_{j-1} + (j+1)k_{j+1} + \cdots + (n+1)k_{n+1} \;=\; n+1$$

where $j \in \{1, 2, \ldots, n+1\}$ is fixed and determine the number $a_n(j)$ of its non-negative solutions in $\mathbb{Z}^n$.

For this we recall that (since Euler's days) such problems are dealt with best by generating functions, namely:

$$\begin{aligned} E(x) &= (1 + x^1 + x^{2\cdot 1} + \dots) \cdot (1 + x^2 + x^{2\cdot 2} + \dots) \cdot \dots \\ &= \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \dots = \sum_{k=0}^{\infty} p(k) x^k \end{aligned}$$

where $p(k)$ denotes the number of partitions of k; that is, the number of unordered representations of k as $k = s_1 + s_2 + \cdots + s_e$ with s_j a positive integer for $j = 1, 2, \dots, e$, or equivalently $k = 1n_1 + 2n_2 + \cdots + kn_k$, where $n_j \geq 0$ is the number of appearances of summand j. Therefore, all partitions with summand j forbidden are obtained via

$$(1-x^j)E(x) = (1-x^j) \sum_{k=0}^{\infty} p(k)x^k = \sum_{k=0}^{j-1} p(k)x^k + \sum_{k=j}^{\infty} (p(k) - p(k-j))x^k .$$

Hence the desired amount $a_n(j)$ equals:

$$a_n(j) = \begin{cases} p(n+1), & \text{if} \quad n+1 \leq j-1 \\ p(n+1) - p(n+1-j), & \text{if} \quad n+1 \geq j \end{cases}$$

Also solved by HEINZ-JÜRGEN SEIFFERT, Berlin, Germany. There was one incorrect solution submitted.

Janous remarks how his ideas above can be extended to include the case where the excluded summand can be a subset of the values from 1 *to* $n+1$.