THE ACADEMY CORNER

No. 18

Bruce Shawyer

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Memorial University Undergraduate Mathematics Competition

September 25, 1997

We present in this issue, two solutions from Bob Prielipp, University of Wisconsin–Oshkosh, Wisconsin, USA, to the above competition, which was printed in Academy Corner No. 15 [1997: 449].

2. The surface area of a closed cylinder is twice the volume. Determine the radius and height of the cylinder given that the radius and height are both integers.

Solution.

Let r be the radius and h be the height of the closed cylinder, where r and h are both positive integers.

If the surface area of the closed cylinder is twice the volume, then

$$2\pi rh + 2\pi r^2 = 2\pi r^2 h.$$

It follows that $h + r = rh$. Thus

$$(r \Leftrightarrow 1)(h \Leftrightarrow 1) = rh \Leftrightarrow r \Leftrightarrow h + 1 = 0 + 1 = 1.$$

Hence $r \Leftrightarrow 1 = 1$ and $h \Leftrightarrow 1 = 1$ (since r and h are both positive integers), so that $r = 2$ and $h = 2$.

It is easily checked that if $r = 2$ and $h = 2$, then the surface area of the closed cylinder is twice the volume.

3. Prove that

$$1+\frac{1}{4}+\frac{1}{9}+\ldots+\frac{1}{n^2} \;<\; 2.$$

Solution. [Slightly shortened by the editor.]
We use mathematical induction to prove, for each positive integer n,

$$1+\frac{1}{4}+\frac{1}{9}+\ldots+\frac{1}{n^2} \;<\; 2 \Leftrightarrow \frac{1}{n}\,.$$

The result clearly holds for $n = 1$.

Assume that the result holds for $n = k$. Then

$$\begin{aligned}
&1+\frac{1}{4}+\frac{1}{9}+\ldots+\frac{1}{k^2}+\frac{1}{(k+1)^2}\\
&< \left(2 \Leftrightarrow \frac{1}{k}\right)+\frac{1}{(k+1)^2}\\
&= 2 \Leftrightarrow \frac{(k+1)^2 \Leftrightarrow k}{k(k+1)^2} \;=\; 2 \Leftrightarrow \frac{k^2+k+1}{k(k+1)^2}\\
&< 2 \Leftrightarrow \frac{k(k+1)}{k(k+1)^2} \;=\; 2 \Leftrightarrow \frac{1}{k+1}\,.
\end{aligned}$$

Hence, by induction, the result holds.

Equality holds if and only if $n = 1$.

Advance Notice

At the summer 1999 meeting of the Canadian Mathematical Society, to be held in St. John's, Newfoundland, there will be a Mathematics Education Session on the topic "What Mathematics Competitions do for Mathematics".

Invited speakers include Edward Barbeau, Toronto; Tony Gardner, Birmingham, England; Ron Dunkley, Waterloo; and Rita Janes, St. John's. Anyone interested in giving a paper at this session should contact one of the organizers, Bruce Shawyer or Ed Williams, at the Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada.

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THE OLYMPIAD CORNER

No. 189

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We begin this issue with an exam set sent in by one of our newer correspondents. My thanks go to Enrique Valeriano, National University of Engineering, Lima, Peru.

PERU'S SELECTION TEST FOR
THE XII IBEROAMERICAN OLYMPIAD
August 1997 — 3.5 hours

1. Given an integer $a_0 > 2$, the sequence $a_0, a_1, a_2, \ldots$ is defined as follows:

$$a_{k+1} = a_k(1 + a_k), \quad \text{if } a_k \text{ is an odd number,}$$
$$a_{k+1} = \frac{a_k}{2}, \quad \text{if } a_k \text{ is an even number.}$$

Prove that there is a nonnegative integer p such that $a_p > a_{p+1} > a_{p+2}$.

2. A positive integer is called "almost-triangular" if the number is itself triangular or is the sum of different triangular numbers. How many almost-triangular numbers are there in the set $\{1, 2, 3, \ldots, 1997\}$?

Note: The triangular numbers are $a_1, a_2, a_3, \ldots, a_k, \ldots$, where $a_1 = 1$, and $a_k = k + a_{k-1}$, for all $k \geq 2$.

3. An $n \times n$ chessboard ($n \geq 2$) is numbered with n^2 non-zero numbers. This chessboard is called an "Incaican Board" if, for each square the number written on the square is the difference between two of the numbers written on two of the neighbouring squares (sharing a common edge). For which values of n can one obtain Incaican Boards?

4. Let ABC be a given acute triangle. Give a ruler and compass construction of an equilateral triangle DEF with D on BC, E on AC, and F on AB such that the perpendiculars to BC at D, to AC at E, and to AB at F, respectively, are concurrent.

Next we give the Third and Fourth Grade problems of the 38^{th} Mathematics Competition for Secondary School Students of the Republic of Slovenia. My thanks go to Richard Nowakowski, Canadian Team Leader to the IMO in Hong Kong, for collecting this contest and forwarding it to me.

REPUBLIC OF SLOVENIA
38^{th} Mathematics Competition for Secondary School Students
April, 1994

Third Grade

1. Let n be a natural number. Prove: if $2n + 1$ and $3n + 1$ are perfect squares, then n is divisible by 40.

2. Show that $\cos(\sin x) > \sin(\cos x)$ holds for every real number x.

3. The polynomial $p(x) = x^3 + ax^2 + bx + c$ has only real roots. Show that the polynomial $q(x) = x^3 \Leftrightarrow bx^2 + acx \Leftrightarrow c^2$ has at least one nonnegative root.

4. Let the point D on the hypotenuse AC of the right triangle ABC be such that $|AB| = |CD|$. Prove that the bisector of $\angle BAC$, the median through B, and the altitude through D, of the triangle ABD have a common point.

Fourth Grade

1. Prove that there does not exist a function $f : \mathbb{Z} \to \mathbb{Z}$, for which $f(f(x)) = x + 1$ for every $x \in \mathbb{Z}$.

2. Put a natural number in every empty field of the table so that you get an arithmetic sequence in every row and every column.

	74			
				186
		103		
0				

3. Prove that every number of the sequence

$$49, \quad 4489, \quad 444889, \quad 44448889, \quad \ldots$$

is a perfect square (in every number there are n fours, $n \Leftrightarrow 1$ eights and a nine).

4. Let Q be the midpoint of the side AB of an inscribed quadrilateral $ABCD$ and S the intersection of its diagonals. Denote by P and R the orthogonal projections of S on AD and BC respectively. Prove that $|PQ| = |QR|$.

As a final contest for your puzzling pleasure in this number, we give the VIII Nordic Mathematical Contest. Again my thanks go to Richard Nowakowski, Canadian Team leader to the IMO in Hong Kong, for collecting this contest and forwarding it to me.

VIII NORDIC MATHEMATICAL CONTEST

March 17^{th}, 1994

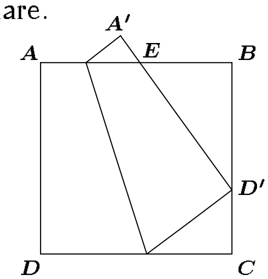
Time: 4 hours

1. Let O be a point in the interior of an equilateral triangle ABC with side length a. The lines AO, BO and CO intersect the sides of the triangle at the points A_1, B_1 and C_1 respectively. Prove that

$$|OA_1| + |OB_1| + |OC_1| < a.$$

2. A finite set S of points in the plane with integer coordinates is called a *two-neighbour set*, if for each (p, q) in S exactly two of the points $(p+1, q)$, $(p, q+1)$, $(p \Leftrightarrow 1, q)$, $(p, q \Leftrightarrow 1)$ are in S. For which n does there exist a two-neighbour set which contains exactly n points?

3. A square piece of paper $ABCD$ is folded by placing the corner D at some point D' on BC (see figure). Suppose AD is carried into $A'D'$, crossing AB at E. Prove that the perimeter of triangle EBD' is half as long as the perimeter of the square.



4. Determine all positive integers $n < 200$ such that $n^2 + (n+1)^2$ is a perfect square.

Turning now to comments and solutions related to the February 1997 number of the corner, we welcome two alternate solutions to problems of the Sixth Irish Mathematical Olympiad sent in by another new contributor whom we also welcome.

3. [1995: 151–152; 1997: 9–13] *Sixth Irish Mathematical Olympiad.*

For nonnegative integers n, r the binomial coefficient $\binom{n}{r}$ denotes the number of combinations of n objects chosen r at a time, with the convention that $\binom{n}{0} = 1$ and $\binom{n}{r} = 0$ if $n < r$. Prove the identity

$$\sum_{d=1}^{\infty} \binom{n \Leftrightarrow r + 1}{d} \binom{r \Leftrightarrow 1}{d \Leftrightarrow 1} = \binom{n}{r}$$

for all integers n, r with $1 \leq r \leq n$.

Alternate Solution by Mohammed Aassila, UFR de Mathématique et d'Informatique, L'Université Louis Pasteur, Strasbourg, France.

We have

$$\begin{aligned}(1+x)^n &= (1+x)^{n-r+1}(1+x)^{r-1} \\ &= \left[\sum_{i=0}^{m-r+1} \binom{n-r+1}{i} x^i\right]\left[\sum_{j=0}^{r-1}\binom{r-1}{j}x^j\right] \\ &= \sum_{i=0}^{m-r+1}\sum_{j=0}^{r-1}\binom{n-r+1}{i}\binom{r-1}{j}x^{i+j}\end{aligned}$$

The coefficient of x^r is

$$\sum_{d=0}^{r}\binom{n-r+1}{d}\binom{r-1}{r-d} = \sum_{d=1}^{r}\binom{n-r+1}{d}\binom{r-1}{d-1}$$

4. [1995:151–152; 1997: 9–13] *Sixth Irish Mathematical Olympiad*.

Let x be a real number with $0 < x < \pi$. Prove that, for all natural numbers n, the sum

$$\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots + \frac{\sin(2n-1)x}{2n-1}$$

is positive.

Alternate solution by Mohammed Aassila, UFR de Mathématique et d'Informatique, L'Université Louis Pasteur, Strasbourg, France.

We know that

$$2\sin x\sin(2k-1)x = \cos(2k-2)x - \cos 2kx.$$

Hence

$$\begin{aligned}&2\sin x\left(\sin x + \frac{\sin 3x}{3} + \cdots + \frac{\sin(2n-1)x}{2n-1}\right) \\ &= 1-\cos 2x + \frac{\cos 2x - \cos 4x}{3} + \cdots + \frac{\cos(2n-2)x - \cos 2nx}{2n-1} \\ &= 1-\cos 2x\left(1-\frac{1}{3}\right) - \cos 4x\left(\frac{1}{3}-\frac{1}{5}\right) - \cdots - \frac{\cos 2nx}{2n-1} \\ &\geq 1-\left[\left(1-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots+\frac{1}{2n-1}\right] \;=\; 0.\end{aligned}$$

Next we turn to solutions to problems of the Latvian 44 Mathematical Olympiad given in the February number of the Corner last year [1997: 78].

LATVIAN 44 MATHEMATICAL OLYMPIAD

Final Grade, 3rd Round

Riga, 1994

1. It is given that $\cos x = \cos y$ and $\sin x = \Leftrightarrow\sin y$. Prove that $\sin 1994x + \sin 1994y = 0$.

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Mansur Boase, student, St. Paul's School, London, England; by Panos E. Tsaoussoglou, Athens, Greece; and by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution of Covas.

More generally, we prove that $\sin mx + \sin my = 0$ where m is an integer.

Since $\sin mx + \sin my = 2\sin\left(m\frac{x+y}{2}\right)\cos\left(m\left(\frac{x-y}{2}\right)\right)$, it is sufficient to show that $\sin\left(\frac{x+y}{2}\right) = 0$ since $\sin\left(m\frac{x+y}{2}\right) = 0$ follows easily by induction from

$$\begin{aligned}\sin\left((m+1)\frac{x+y}{2}\right) &= \sin\left(m\frac{(x+y)}{2}\right)\cos\left(\frac{x+y}{2}\right)\\ &+ \cos\left(m\frac{(x+y)}{2}\right)\sin\left(\frac{x+y}{2}\right).\end{aligned}$$

Now,

$$\begin{aligned}\cos x = \cos y &\iff \cos x \Leftrightarrow\cos y = 0\\ &\iff \Leftrightarrow2\sin\frac{x+y}{2}\sin\frac{x\Leftrightarrow y}{2} = 0 \qquad (1)\end{aligned}$$

and

$$\begin{aligned}\sin x = \Leftrightarrow\sin y &\iff \sin x + \sin y = 0\\ &\iff 2\sin\frac{x+y}{2}\cos\frac{x\Leftrightarrow y}{2} = 0\,. \qquad (2)\end{aligned}$$

Squaring each of (1) and (2) and adding, we find

$$4\sin^2\left(\frac{x+y}{2}\right)\underbrace{\left(\sin^2\frac{(x\Leftrightarrow y)}{2} + \cos^2\frac{(x\Leftrightarrow y)}{2}\right)}_{=1} = 0.$$

Hence $\sin\frac{x+y}{2} = 0$.

3. It is given that $a > 0$, $b > 0$, $c > 0$, $a + b + c = abc$. Prove that at least one of the numbers a, b, c exceeds $17/10$.

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Heinz-Jürgen Seiffert, Berlin, Germany; by Panos E. Tsaoussoglou, Athens, Greece; and by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give two solutions of Shan and Wang.

First Solution.

We show, in general, that if $x_i > 0$ for $i = 1, 2, \ldots, n$, such that $\sum_{i=1}^n x_i = \prod_{i=1}^n x_i$, then $\max\{x_i : i = 1, 2, \ldots, n\} \geq n^{1/(n-1)}$. In particular, when $n = 3$ we get

$$\max\{x_1, x_2, x_3\} \geq \sqrt{3} > 1.7 = \frac{17}{10}.$$

By the arithmetic-geometric mean inequality we have

$$\left(\frac{1}{n}\sum_{i=1}^n x_i\right)^n \geq \prod_{i=1}^n x_i = \sum_{i=1}^n x_i,$$

and thus $\sum_{i=1}^n x_i \geq n^{n/(n-1)}$.

Without loss of generality, we may assume that $\max\{x_i : i = 1, 2, \ldots, n\} = x_n$. Then $nx_n \geq \sum_{i=1}^n x_i \geq n^{n/(n-1)}$ from which $x_n \geq n^{1/(n-1)}$ follows.

Second Solution (without the AM-GM inequality).

Suppose $x_i > 0$ $(i = 1, 2, \ldots, n)$ such that $\sum_{i=1}^n x_i = \prod_{i=1}^n x_i$. Then dividing both sides by $\prod_{i=1}^n x_i$ we get

$$\sum_{i=1}^n \frac{1}{x_1 x_2 \ldots \hat{x}_i \ldots x_n} = 1,$$

where $\hat{x}_i$ indicates that the factor x_i is missing. Hence for some j we have

$$\frac{1}{x_1 x_2 \ldots \hat{x}_j \ldots x_n} \leq \frac{1}{n} \quad \text{or} \quad n \leq x_1 x_2 \ldots \hat{x}_j \ldots x_n.$$

Without loss we may assume that $x_n = \max\{x_i : i = 1, 2, \ldots, n\}$. Then $x_n^{n-1} \geq x_1 x_2 \ldots \hat{x}_j \ldots x_n \geq n$, from which $x_n \geq n^{1/(n-1)}$ follows. In particular, for $n = 3$, we get $x_3 \geq \sqrt{3} > 1.7 = \frac{17}{10}$.

4. Solve the equation $1! + 2! + 3! + \cdots + n! = m^3$ in natural numbers.

Solution by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

We solve the more general problem of finding all solutions of the Diophantine equation $1! + 2! + 3! + \cdots + n! = m^k$ in natural numbers, n, m, and k. For convenience, let $S_n = 1! + 2! + 3! + \cdots + n!$.

When $k = 1$ clearly $m = S_n$ is the only solution for any n.

When $k = 2$ we claim that the equation $S_n = m^2$ has exactly two solutions: $n = m = 1$ and $n = m = 3$. Note first that $d! \equiv 0 \pmod{10}$

for all $d \geq 5$ and $S_4 = 1 + 2 + 6 + 24 = 33 \equiv 3 \pmod{10}$. Hence $S_n \equiv 3 \pmod{10}$ for all $n \geq 4$. However, it is easy to see that the last digit of a perfect square can never be 3 and so there are no solutions if $n \geq 4$. Checking the cases when $n = 1, 2, 3$ directly reveals that there are precisely two solutions, as given above.

When $k \geq 3$ we show that $n = m = 1$ is the only solution. If $n \geq 2$ then clearly $S_n \equiv 0 \pmod 3$. But $m^k \equiv 0 \pmod 3$ implies $m \equiv 0 \pmod 3$ and so $m^k \equiv 0 \pmod{27}$ as $k \geq 3$. Since $d! \equiv 0 \pmod{27}$ for all $d \geq 9$ and since

$$S_8 = 1 + 2 + 6 + 24 + 120 + 720 + 5040 + 40320 = 46233 \not\equiv 0 \pmod{27}$$

there are no solutions if $n \geq 8$. On the other hand, direct checking shows that for $n = 2, 3, 4, 5, 6, 7$, $S_n = 3, 9, 33, 153, 873$, and 5913, none of which is a perfect k^{th} power for any $k \geq 3$. Finally, it is trivial to see that $n = m = 1$ is a solution.

Remark: The special case of this problem when $k = 2$ was proposed by E.T.H. Wang as Quicky Q657 in the *Mathematics Magazine*, 52 (1979), p. 47. The general case was also proposed by him as problem #4203 in *Mathmedia* (in Chinese) 4(2), 1980; p. 64 with solution in 4(3), 1980, p. 49. This is a journal published by the Institute of Mathematics, Academia Sinica, Taipei, Taiwan.

5. There are 1994 employees in the office. Each of them knows 1600 others of them. Prove that we can find 6 employees, each of them knowing all 5 others.

Solution by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let E denote the set of these 1994 employees. For each $x \in E$, let $S(x)$ denote the set of all employees whom x does *not* know. Then by assumption, $|S(x)| = 393$ for all $x \in E$. Let a and b be any two employees who know each other. Since

$$|S(a) \cup S(b)| \leq 2 \times 393 = 786 < 1992,$$

$\exists c \in E$ such that a, b, and c form a triple of mutual acquaintances. Since

$$|S(a) \cup S(b) \cup S(c)| \leq 3 \times 393 = 1179 < 1991,$$

$\exists d \in E$ such that a, b, c, and d form a quadruple of mutual acquaintances. Since

$$|S(a) \cup S(b) \cup S(c) \cup S(d)| \leq 4 \times 393 = 1572 < 1990,$$

$\exists e \in E$ such that a, b, c, d, and e form a quintuple of mutual acquaintances. Finally, since

$$|S(a) \cup S(b) \cup S(c) \cup S(d) \cup S(e)| \leq 5 \times 393 = 1965 < 1989,$$

$\exists f \in E$ such that a, b, c, d, e, and f form a sextuple of mutual acquaintances.

1$^{\text{st}}$ SELECTION ROUND

1. It is given that x and y are positive integers and $3x^2 + x = 4y^2 + y$. Prove that $x \Leftrightarrow y$, $3x + 3y + 1$ and $4x + 4y + 1$ are squares of integers.

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Panos E. Tsaoussoglou, Athens, Greece; and by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution by Shan and Wang and their comment.

Note first that the given equation implies the following two equations:

$$(x \Leftrightarrow y)(3x + 3y + 1) = y^2 \qquad (1)$$
$$(x \Leftrightarrow y)(4x + 4y + 1) = x^2 \qquad (2)$$

(1) $\times$ (2) yields $(x \Leftrightarrow y)^2(3x + 3y + 1)(4x + 4y + 1) = (xy)^2$ which implies that $(3x + 3y + 1)(4x + 4y + 1)$ is a perfect square. But clearly $\gcd(3x+3y+1, 4x+4y+1) = 1$ since $4(3x+3y+1) \Leftrightarrow 3(4x+4y+1) = 1$. Therefore, $3x + 3y + 1$ and $4x + 4y + 1$ are both squares, which, together with (1) (or (2)), implies that $x \Leftrightarrow y$ is also a square.

Comment: This would be a much better problem had it asked to show only that $x \Leftrightarrow y$ is a square. This is an example of a case when asking to prove too many things actually gives the solution away, in some sense.

Shan and Wang also proposed the following problem inspired by this one.

The Diophantine equation $3x^2 + x = 4y^2 + y$ is satisfied when $x = 30$ and $y = 26$.

(a) Find another solution in positive integers.

(b) Are there infinitely many solutions in positive integers? Is so, describe all of them.

2$^{\text{nd}}$ SELECTION ROUND

1. It is given that $0 \leq x_i \leq 1$, $i = 1, 2, \ldots, n$. Find the maximum of the expression

$$\frac{x_1}{x_2x_3\ldots x_n + 1} + \frac{x_2}{x_1x_3x_4\ldots x_n + 1} + \cdots + \frac{x_n}{x_1x_2\ldots x_{n-1} + 1}.$$

Solutions by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We first give the solution of Shan and Wang.

Clearly $n \geq 2$ for the question to make sense. Let S_n denote the given sum. We prove that $S_n \leq n \Leftrightarrow 1$. For $n = 2$, equality holds if and only if $x_1 = 0$, $x_2 = 1$ or $x_1 = 1$, $x_2 = 0$ or $x_1 = x_2 = 1$. For $n \geq 3$, equality holds if and only if one of the x_i's is 0 and the others are all equal to 1. We first establish a lemma:

Lemma. Suppose that the x_i's are reals such that $0 \le x_i \le 1$ for all $i = 1, 2, \ldots, n$, where $n \ge 2$. Then $x_1 + x_2 + \cdots + x_n \le n \Leftrightarrow 1 + x_1 x_2 \cdots x_n$ with equality holding if and only if $x_i \ne 1$ for at most one i, $i = 1, 2, \ldots, n$.

Proof. For $n = 2$, $x_1 + x_2 \le x_1 x_2 + 1 \Leftrightarrow (1 \Leftrightarrow x_1)(1 \Leftrightarrow x_2) \ge 0$, which is clearly true. Equality holds if and only if $x_1 = 1$ or $x_2 = 1$. Suppose the assertion holds for some $n \ge 2$. Then

$$\begin{aligned} x_1 + x_2 + \cdots + x_n + x_{n+1} &\le n \Leftrightarrow 1 + x_1 x_2 \cdots x_n + x_{n+1} \\ &\le n \Leftrightarrow 1 + x_1 x_2 \cdots x_n x_{n+1} + 1 \\ &= n + x_1 x_2 \cdots x_{n+1}. \end{aligned}$$

(The 2^{nd} inequality is by the $n = 2$ case.)

If equality holds, then it must hold in both inequalities above. By the induction hypotheses, we then have

(i) at most one of $x_1, x_2, \ldots, x_n$ is different from 1 and

(ii) either $x_{n+1} = 1$ or $x_1 x_2 \cdots x_n = 1$.

Since $x_1 x_2 \cdots x_n = 1$ clearly implies $x_1 = x_2 = \cdots = x_n = 1$, our assertion about the equality follows.

Now we proceed to prove the claim about S_n. For $n = 2$,

$$S_2 \;=\; \frac{x_1}{x_2 + 1} + \frac{x_2}{x_1 + 1} \;\le\; \frac{x_1}{x_2 + x_1} + \frac{x_2}{x_1 + x_2} \;=\; 1.$$

It is readily seen that equality holds if and only if $x_1 = 0$, $x_2 = 1$ or $x_1 = 1$, $x_2 = 0$, or $x_1 = x_2 = 1$. For $n \ge 3$ we apply the lemma above and obtain

$$S_n \;\le\; \frac{x_1 + x_2 + \cdots + x_n}{x_1 x_2 \cdots x_n + 1} \;\le\; \frac{(n \Leftrightarrow 1) + x_1 x_2 \cdots x_n}{x_1 x_2 \cdots x_n + 1} \;\le\; n \Leftrightarrow 1.$$

If equality holds, then the 2^{nd} inequality implies $x_j \ne 1$ for at most one j and the 3^{rd} inequality implies $(n \Leftrightarrow 1) x_1 x_2 \cdots x_n = x_1 x_2 \cdots x_n$, which implies that $x_i = 0$ for at least one i. Hence $x_i = 0$ for some i and $x_j = 1$ for all $j \ne i$. It is obvious that this condition is also sufficient. This completes the proof.

And here is Klamkin's somewhat more abbreviated solution with generalization.

Since the expression is convex in each of the variables, the maximum value is achieved when each variable takes on 0 or 1. Clearly this occurs when one variable is 0 and the rest are 1 giving the maximum value of $n \Leftrightarrow 1$. The same maximum occurs if any of the numerators x_i are replaced by $x_i^{\alpha_i}$ where $\alpha_i \ge 1$.

A similar result, using convexity, that

$$\sum \frac{x_i^u}{1 + s \Leftrightarrow x_i} + \prod (1 \Leftrightarrow x_i)^v \;\le\; 1,$$

where $0 \leq x_i \leq 1$, $u, v \geq 1$, $s = \sum x_i$ and the sum and product are over $i = 1, 2, \ldots, n$, is given in [1].

Reference:

[1] M.S. Klamkin, *USA Mathematical Olympiads* 1972–1986, M.A.A., Washington D.C., 1988, p. 82.

3. A triangle ABC is given. From the vertex B, n rays are constructed intersecting the side AC. For each of the $n+1$ triangles obtained, an incircle with radius r_i and excircle (which touches the side AC) with radius R_i is constructed. Prove that the expression

$$\frac{r_1 r_2 \ldots r_{n+1}}{R_1 R_2 \ldots R_{n+1}}$$

depends on neither n nor on which rays are constructed.

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.

If A, B, C are the angles of a triangle, then

$$r = s \tan\frac{A}{2} \tan\frac{B}{2} \tan\frac{C}{2} \qquad \text{and} \qquad r_a = s \tan\frac{A}{2},$$

where r, r_a are the inradius and the radius of the excircle opposite A, and s is the semiperimeter.

It follows that

$$\frac{r}{r_a} = \tan\frac{B}{2} \tan\frac{C}{2}.$$

Next we apply this result to each of $n+1$ triangles obtained (see figure at the top of the next page). This yields

$$\begin{aligned}
\frac{r_1}{R_1} &= \tan\frac{A}{2} \tan\frac{\alpha_1}{2} \\
\frac{r_2}{R_2} &= \tan\frac{180^\circ - \alpha_1}{2} \tan\frac{\alpha_2}{2} \\
& \ldots\ldots\ldots \\
& \ldots\ldots\ldots \\
\frac{r_n}{R_n} &= \tan\frac{180^\circ - \alpha_{n-1}}{2} \tan\frac{\alpha_n}{2} \\
\frac{r_{n+1}}{R_{n+1}} &= \tan\frac{180^\circ - \alpha_n}{2} \tan\frac{C}{2}.
\end{aligned}$$

Multiplying these equalities, we observe that the product of all the right hand members is

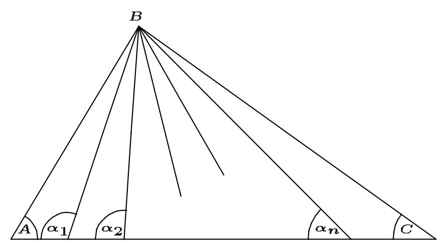
$$\tan\frac{A}{2} \cdot \left(\tan\frac{\alpha_1}{2} \cot\frac{\alpha_1}{2}\right) \cdot \left(\tan\frac{\alpha_2}{2} \cot\frac{\alpha_2}{2}\right) \cdots \left(\tan\frac{\alpha_n}{2} \cot\frac{\alpha_n}{2}\right) \cdot \tan\frac{C}{2}$$

$$= \tan\frac{A}{2} \cdot \tan\frac{C}{2},$$

and we get

$$\frac{r_1 r_2 \cdots r_{n-1}}{R_1 R_2 \cdots R_{n+1}} = \tan\frac{A}{2} \cdot \tan\frac{C}{2}$$

which depends on neither n nor on which rays are constructed.

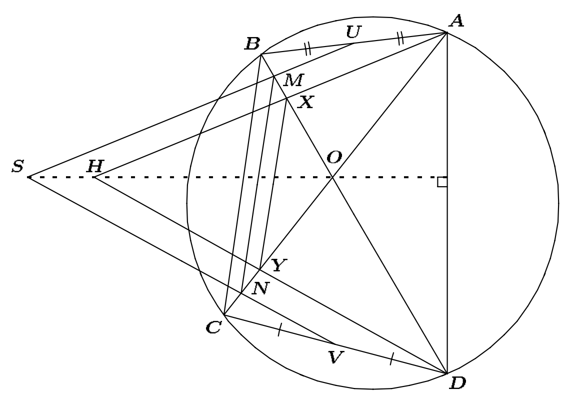


3^{rd} SELECTION ROUND

3. Let $ABCD$ be an inscribed quadrilateral. Its diagonals intersect at O. Let the midpoints of AB and CD be U and V. Prove that the lines through O, U and V, perpendicular to AD, BD and AC respectively, are concurrent.

Solution by Toshio Seimiya, Kawasaki, Japan.

Case 1. Neither is AC orthogonal to BD, nor is AD a diameter.



Let M, N be the feet of the perpendiculars from U, V to BD, AC respectively, and let S be the intersection of UM and VN. Let X, Y be the feet of the perpendiculars from A, D to BD, AC respectively, and let H be the intersection of AX and DY. Since U is the midpoint of AB, and $UM \| AX$, M is the midpoint of BX. Similarly N is the midpoint of CY.

Since $\angle AXD = \angle AYD = 90°$, A, X, Y, D are concyclic. Therefore $\angle YXD = \angle YAD = \angle CAD = \angle CBD$. Thus we have $XY \| BC$.

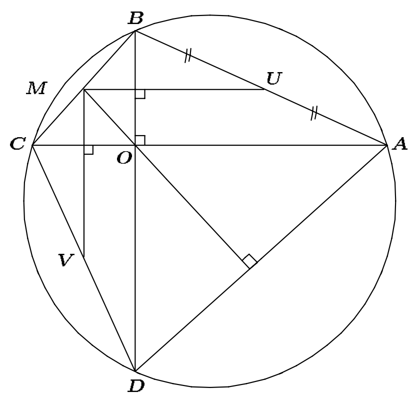
Since M, N are the midpoints of BX, CY respectively, we have $MN \| XY$.

Since $SM \| HX$, $XN \| HY$, and $MN \| XY$, MX, NY and SH are concurrent at O. Therefore S, H, O are collinear.

Since $AH \perp OD$ and $DH \perp OA$, H is the orthocentre of ΔOAD, so that $HO \perp AD$. Thus we have $SO \perp AD$.

Thus the lines through O, U and V, perpendicular to AD, BD and AC respectively, are concurrent at S.

Case 2. AC is orthogonal to BD.

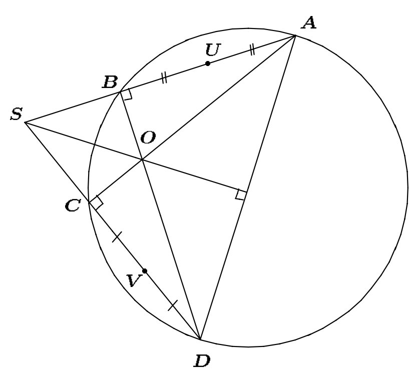


Let M be the midpoint of BC. Since U is the midpoint of AB, we get $UM \perp AC$, so that $UM \perp BD$. Similarly we have $VM \perp AC$.

Since $AC \perp BD$ and M is the midpoint of BC, by Brahmagupta's Theorem we have $MO \perp AD$. (See: H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited*, p. 59).

Thus the lines through O, U and V, perpendicular to AD, BD and AC respectively, are concurrent at M.

Case 3. AD is a diameter.



Let S be the intersection of AB and CD. Since $AB \perp BD$ and $CD \perp AC$, S is the orthocentre of ΔOAD. Thus $SO \perp AD$.

Hence the lines through O, U and V, perpendicular to AD, BD and AC respectively, are concurrent at S.

That completes this number of the Olympiad Corner. Send me your nice solutions and suggestions for future issues.

BOOK REVIEWS

Edited by ANDY LIU

Vita Mathematica
edited by Ronald Calinger,
published by the MAA, Notes and Reports Series, 1996,
ISBN# 0-88385-097-4, softcover, 350+ pages, $34.95.
Reviewed by **Maria de Losada**, *Bogotá, Colombia.*

This collection of papers read at the August 1992 Quadrennial Meeting of the International Study Group on the Relations between History and Pedagogy of Mathematics at the University of Toronto and the Seventh International Congress on Mathematical Education at Université Laval in Quebec, edited and refereed, begins with a somewhat ponderous reflection on the different tendencies in research in the history of mathematics, contrasting the approaches of cultural and mathematical historians.

Looking at the role of problems in the history and teaching of mathematics, Evelyne Barbin reminds us that "one of the perverse effects of education ... [is] that answers are given to questions that have not been asked. The history of mathematics shows us that questions must come first, and [that] it is through questions that we make sense of mathematical concepts."

In the historical studies (from antiquity to the Scientific Revolution) can be found intriguing material that is not of easy access. Swetz's paper on the enigmas of Chinese mathematics is informative as it strives to provide a balanced presentation and Katz's treatment of medieval Hebrew and Islamic mathematics provides useful and little known information in a concise and well ordered manner.

Noteworthy among the more recent historical studies is Judith Grabiner's paper which contrasts historical perspectives of the calculus, the geometric (McLaurin) and algebraic (Lagrange), linking these naturally with education and culture and suggesting that "progress in mathematics is made by those who sharpen their thinking by exercising the courage of their sometimes idiosyncratic convictions". On an entirely different note, the detailed history given by Ronald Calinger of the University of Berlin Mathematics Seminar examines every facet of that paradigmatically successful structuring of a research tradition, in which many of the recurrent themes of this book, such as starting from problems and learning from the masters, were put into practice.

The third section of the collection, devoted to the integration of history with mathematics teaching recounts many valuable experiences, two titles of note being **Mathematical Masterpieces: Teaching with Original Sources** and **A History of Mathematics Course for Teachers, Based on Great Quotations**.

Although they are presented as summaries and have catchy titles, these articles refer to teaching experiences that are both substantive and well structured. In the first of these, Reinhard Laubenbacher and David Pengelley reveal not only a polished list of original sources that can be used with undergraduates, but also aspects of their pedagogical approach. In the second, Israel Kleiner begins with quotations addressing the question "What is mathematics?" that are arranged in "antagonistic" pairs to bring across the underlying message of mathematics as an activity whose history is susceptible to chronological, thematical, topical and biographical study, as long as sight is not lost of Lakatos' remark: "The history of mathematics, lacking the guidance of philosophy [is] blind, while the philosophy of mathematics, turning its back on the most intriguing phenomena in the history of mathematics, [is] empty."

It is unfortunate that the book has so many typos! Not only are footnote numbers routinely formatted incorrectly, but there are also plenty of errors in the text.

In general, this is good formative reading for those with an appetite for historical material, but especially useful in its treatment of interrelations between history and pedagogy.

Sum of powers of a finite sequence: a geometric approach

William O.J. Moser

In this note we give a geometric-combinatoric derivation of a formula for the sum of r^{th} powers of a finite positive integer sequence:

$$a_1^r + a_2^r + \cdots + a_n^r = \sum_{\ell=1}^{r} \sum_{i=1}^{n} \binom{a_i}{\ell} \mu(r, \ell)\,, \tag{1}$$

where

$$\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!}, & \text{if } 0 \le k \le n\,, \\ 0, & \text{if } k > n \ge 1\,, \end{cases}$$

and the numbers $\mu(r, \ell)$ are determined by the recurrence

$$\begin{aligned} \mu(r,1) &= 1, \;\; r = 1,2,3,\cdots; \quad \mu(1,\ell) = 0, \;\; \ell = 2,3,\cdots; \\ \mu(r,\ell) &= \ell(\mu(r \Leftrightarrow 1, \ell \Leftrightarrow 1) + \mu(r \Leftrightarrow 1, \ell)), \; r \ge 2, \; \ell \ge 2\,. \end{aligned} \tag{2}$$

The derivation parallels, simplifies and generalizes the proof given in [1], and seems to be more elementary than proofs given in [2] and [3].

For fixed (but arbitrary) positive integers m, r, let $L_m^{(r)}$ denote the r-dimensional integer lattice points

$$\{(x_1, x_2, \cdots, x_r) \,|\, x_i \text{ integers, } 0 \le x_1, x_2, \cdots, x_r \le m\}\,,$$

and for $w \ge 1$ let $S_m^{(r)}(w)$ denote the set of r-dimensional "cubes" with faces parallel to the coordinate planes, vertices in $L_m^{(r)}$ and width w. Of course $S_m^{(r)}(w) = \emptyset$ if $w > m$. A cube in $S_m^{(r)}(w)$ is identified by its vertex $(\alpha) = (\alpha_1, \alpha_2, \cdots, \alpha_r)$ closest to $(0, \cdots, 0)$. Clearly

$$w + \alpha_1 \le m, \quad w + \alpha_2 \le m, \quad \cdots, \quad w + \alpha_r \le m, \qquad \alpha_1, \alpha_2, \cdots, \alpha_r \ge 0\,,$$

or

$$0 \le \alpha_1 \le m \Leftrightarrow w, \quad 0 \le \alpha_2 \le m \Leftrightarrow w, \quad \cdots,$$

$$\cdots \quad 0 \le \alpha_r \le m \Leftrightarrow w, \qquad 1 \le w \le m\,, \tag{3}$$

so the number of cubes in $S_m^{(r)}(w)$ is

$$|S_m^{(r)}(w)| = \begin{cases} (m \Leftrightarrow w + 1)^r\,, & \text{if } 1 \le w \le m\,, \\ 0\,, & \text{if } w > m \ge 1\,. \end{cases}$$

For given $1 \leq w \leq m$ we can also find the number of (α)'s satisfying (3) as follows. Any r-tuple (α) satisfying (3) determines the sequence (β) consisting of the distinct values, in increasing order, of the entries in (α):

$$(\beta) : 0 \leq \beta_1 < \beta_2 < \cdots < \beta_\ell \leq m \Leftrightarrow w, \quad \ell \leq r\,. \tag{4}$$

Given a sequence (β) satisfying (4) there are many r-tuples (α) satisfying (3) which have entries with values (β). How many? Let $\mu(r, \ell)$ denote this number. It depends on ℓ (and not on the particular numbers $\beta_1, \cdots, \beta_\ell$) and satisfies the recurrence (2). For, given (β), there are ℓ choices (namely $\beta_1, \beta_2, \cdots, \beta_\ell$) for the first entry α_1 of (α); (α) can then be completed in $\mu(r \Leftrightarrow 1, \ell)$ ways if the integer chosen for α_1 appears among the entries $\alpha_2, \alpha_3, \cdots, \alpha_r$, and in $\mu(r \Leftrightarrow 1, \ell \Leftrightarrow 1)$ ways otherwise; (2) then follows.

The numbers $\mu(r, \ell)$ are exhibited in the display below:

$r \backslash \ell$	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	2!	0	0	0	0
3	1	6	3!	0	0	0
4	1	14	36	4!	0	0
5	1	30	150	240	5!	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Since there are $\binom{m-w+1}{\ell}$ ways of choosing the sequence (β) satisfying (4) we find that

$$(m \Leftrightarrow w + 1)^r = \sum_{\ell=1}^{r} \binom{m \Leftrightarrow w + 1}{\ell} \mu(r, \ell), \qquad 1 \leq w \leq m, \qquad 1 \leq \ell \leq r,$$

and setting $a = m \Leftrightarrow w + 1 \geq 1$

$$a^r = \sum_{\ell=1}^{r} \binom{a}{\ell} \mu(r, \ell), \quad a \geq 1, \quad r \geq 1\,.$$

Summing over a set of a's we have (1). Taking $a_i = a + (i \Leftrightarrow 1)d$, $i = 1, 2, \cdots, n$ in (1), we have

$$a^r + (a + d)^r + \cdots + (a + (n \Leftrightarrow 1)d)^r = \sum_{\ell=1}^{r} \sum_{i=1}^{n} \binom{a + (i \Leftrightarrow 1)d}{\ell} \mu(r, \ell)\,,$$

the sum of the r^{th} powers of an arithmetic progression. When $a = d = 1$

$$s^{(r)}(n) = 1^r + 2^r + \cdots + n^r = \sum_{\ell=1}^{r} \binom{n+1}{\ell+1} \mu(r, \ell)\,,$$

since $\sum_{i=1}^{n} \binom{i}{\ell} = \binom{n+1}{\ell+1}$.

The display below exhibits $s^{(r)}(n)$ for $r = 1, 2, 3, 4, 5$.

$$
\begin{array}{lllllllllll}
s^{(1)}(n) & & & & & 1\binom{n+1}{2} & & & & & \\
s^{(2)}(n) & & & & 1\binom{n+1}{2} & + & 2!\binom{n+1}{3} & & & & \\
s^{(3)}(n) & & & 1\binom{n+1}{2} & + & 6\binom{n+1}{3} & + & 3!\binom{n+1}{4} & & & \\
s^{(4)}(n) & & 1\binom{n+1}{2} & + & 14\binom{n+1}{3} & + & 36\binom{n+1}{4} & + & 4!\binom{n+1}{5} & & \\
s^{(5)}(n) & 1\binom{n+1}{2} & + & 30\binom{n+1}{3} & + & 150\binom{n+1}{4} & + & 240\binom{n+1}{5} & + & 5!\binom{n+1}{6} &
\end{array}
$$

References

1. [1] Moser, W., Sums of d^{th} powers. *Math. Gazette* **75** (1991) 332.

2. [2] Paul, J.L., On the sum of the k^{th} powers of the first n integers. *Amer. Math. Monthly* **78** (1971) 271–273. MR 43 #4092.

3. [3] Wagner, C., Combinatorial proofs of formulas for power sums. *Arch. Math.* (Basel) **68** (1997), no. 6, 464–467.

THE SKOLIAD CORNER

No. 29

R.E. Woodrow

In this number we give the first round of the 1996–97 Alberta High School Mathematics Competition written in November of 1996. The top students from Round 1 are invited to take part in the second round competition written in February. There are book prizes for individuals and teams based on the Round 1 results, and scholarships and cash prizes for Round 2. (See [1997:410-411, 479-481] where we gave Round 2 of the 1996–97 contest and the answers.) More information about the contest can be found at the website: `www.math.ualberta.ca/ ahsmc/`. My thanks go to the organizers of the committee (chaired by T. Lewis, University of Alberta) for permission to use these materials.

ALBERTA HIGH SCHOOL MATHEMATICS COMPETITION
Part I — November, 1996

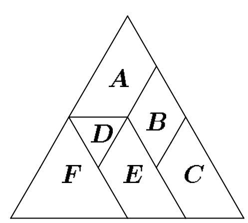
1. An eight-inch pizza is cut into 3 equal slices. A ten-inch pizza is cut into 4 equal slices. A twelve-inch pizza is cut into 6 equal slices. A fourteen inch pizza is cut into 8 equal slices. From which pizza would you take a slice if you want as much pizza as possible?

(a) 8-inch **(b)** 10-inch **(c)** 12-inch **(d)** 14-inch **(e)** does not matter

2. One store sold red plums at four for a dollar and yellow plums at three for a dollar. A second store sold red plums at four for a dollar and yellow plums at six for a dollar. You bought m red plums and n yellow plums from each store, spending a total of ten dollars. How many plums in all did you buy?

(a) 10 **(b)** 20 **(c)** 30 **(d)** 40 **(e)** not enough information

3.



Six identical cardboard pieces are piled on top of one another, and the result is shown in the diagram.

The third piece to be placed is:

(a) A **(b)** B **(c)** C **(d)** D **(e)** E

4. A store offered triple the GST in savings. A sales clerk calculated the selling price by first reducing the original price by 21% and then adding the 7% GST based on the reduced price. A customer protested, saying that the store should first add the 7% GST and then reduce that total by 21%. They agreed on a compromise: the clerk just reduced the original price by the 14% difference. How do the three ways compare with one another from the customer's point of view?

(a) The clerk's way is the best.
(b) The customer's way is the best.
(c) The compromise is the best.
(d) All three ways are the same.
(e) The compromise is the worst while the other two are the same.

5. If m and n are integers such that $2m \Leftrightarrow n = 3$, then what will $m \Leftrightarrow 2n$ equal?

(a) $\Leftrightarrow 3$ only **(b)** 0 only **(c)** only multiples of 3 **(d)** any integer **(e)** none of these

6. If x is $x\%$ of y, and y is $y\%$ of z, where x, y and z are positive real numbers, what is z?

(a) 100 **(b)** 200 **(c)** $10,000$ **(d)** does not exist **(e)** cannot be determined

7. About how many lines can one rotate a regular hexagon through some angle x, $0° < x < 360°$, so that the hexagon again occupies its original position?

(a) 1 **(b)** 3 **(c)** 4 **(d)** 6 **(e)** 7

8. AB is a diameter of a circle of radius 1 unit. CD is a chord perpendicular to AB that cuts AB at E. If the arc CAD is $2/3$ of the circumference of the circle, what is the length of the segment AE?

(a) $\frac{2}{3}$ **(b)** $\frac{3}{2}$ **(c)** $\frac{\pi}{2}$ **(d)** $\frac{\sqrt{3}}{2}$ **(e)** none of these

9. One of Kerry and Kelly lies on Mondays, Tuesdays and Wednesdays, and tells the truth on the other days of the week. The other lies on Thursdays, Fridays and Saturdays, and tells the truth on the other days of the week. At noon, the two had the following conversation:

Kerry : I lie on Saturdays.
Kelly : I will lie tomorrow.
Kerry : I lie on Sundays.

On which day of the week did this conversation take place?

(a) Monday **(b)** Wednesday **(c)** Thursday **(d)** Saturday **(e)** Sunday

10. How many integer pairs (m, n) satisfy the equation

$$m(m+1) = 2n \ ?$$

(a) 0 (b) 1 (c) 2 (d) 3 (e) more than 3

11. Of the following triangles given by the lengths of their sides, which one has the greatest area?

(a) $5, 12, 12$ (b) $5, 12, 13$ (c) $5, 12, 14$ (d) $5, 12, 15$ (e) $5, 12, 16$

12. If $x < y$ and $x < 0$, which of the following numbers is never greater than any of the others?

(a) $x + y$ (b) $x \Leftrightarrow y$ (c) $x + |y|$ (d) $x \Leftrightarrow |y|$ (e) $\Leftrightarrow |x + y|$

13. An x by y flag, with $x < y$, consists of two perpendicular white stripes of equal width and four congruent blue rectangles at the corners. If the total area of the blue rectangles is half that of the flag, what is the length of the shorter side of each blue rectangle?

(a) $\frac{x-y+\sqrt{x^2+y^2}}{4}$ (b) $\frac{x-y+\sqrt{x^2+y^2}}{2}$ (c) $\frac{3x+y+\sqrt{x^2+y^2}}{4}$ (d) $\frac{3x+y+\sqrt{x^2+y^2}}{2}$ (e) none of these

14. A game is played with a deck of ten cards numbered from 1 to 10. Shuffle the deck thoroughly.

i) Take the top card. If it is numbered 1, you win. If it is numbered k, where $k > 1$, go to (ii).

ii) If this is the third time you have taken a card, you lose. Otherwise, put the card back into the deck at the k^{th} position from the top and go to (i).

What is the probability of winning?

(a) $\frac{1}{5}$ (b) $\frac{5}{18}$ (c) $\frac{13}{45}$ (d) $\frac{3}{10}$ (e) none of these

15. Five of the angles of a convex polygon are each equal to $108°$. In which of the following five intervals does the maximum angle of all such polygons lie?

(a) $(105°, 120°)$ (b) $(120°, 135°)$ (c) $(135°, 150°)$ (d) $(150°, 165°)$ (e) $(165°, 180°)$

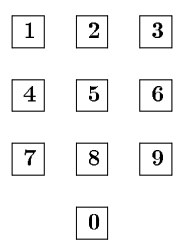
16. Which one of the following numbers cannot be expressed as the difference of the squares of two integers?

(a) 314159265 (b) 314159266 (c) 314159267 (d) 314150268 (e) 314159269

In the last number we gave the problems of the Final Round of the British Columbia Colleges Junior High School Mathematics Contest 1997. Here are the official solutions. My thanks for sending the materials to me go to John Grant McLoughlin, now of the Faculty of Education, Memorial University of Newfoundland, who participated in formulating the exams while he was at Okanagan College.

BRITISH COLUMBIA COLLEGES
Junior High School Mathematics Contest
Final Round 1997 — Part A

1. The buttons of a phone are arranged as shown at the right. If the buttons are one centimetre apart, centre-to-centre, when you dial the number 592-7018 the distance, in centimetres, travelled by your finger is:



Solution. Each of the distances between successive digits in the phone number is the hypotenuse of a right-angled triangle with integer sides. The 6 lengths can be easily computed as $\sqrt{2}$, $\sqrt{5}$, $\sqrt{5}$, $\sqrt{2}$, $\sqrt{10}$, and $\sqrt{5}$, for a total of $2\sqrt{2} + 3\sqrt{5} + \sqrt{2}\sqrt{5}$. This can be rearranged into $\sqrt{5}(3 + \sqrt{2}) + 2\sqrt{2}$. *Answer is* **(A)**

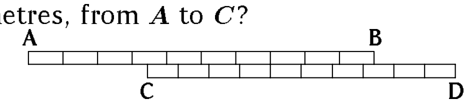
2. What is the total number of ones digits needed in order to write the integers from 1 to 100?

Solution. Clearly we need only one 1 for all the single digit numbers, and (since we are only interested in one 3-digit number, namely 100) we need only one 1 for all the 3-digit numbers under consideration. For the 2-digit numbers there are nine which have a 1 in the units position $(11, 21, \ldots, 91)$, and 10 which have a 1 in the tens position $(10, 11, \ldots, 19)$, for a grand total of 21 ones needed. *Answer is* **(D)**

3. The number of solutions (x, y, z) in positive integers for the equation $3x + y + z = 23$ is:

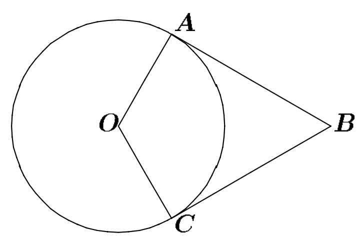
Solution. Clearly, there are only 7 possible values for x, namely 1 through 7. When $x = 7$, we have $y + z = 2$, which has a single solution $y = z = 1$; when $x = 6$, we have $y + z = 5$, which has 4 solutions: $(y, z) = (1, 4), (2, 3), (3, 2), (4, 1)$. In this way we see that by reducing the value of x by 1 we increase the number of solutions by 3. Thus the total number of solutions are $1 + 4 + 7 + 10 + 13 + 16 + 19 = 70$. *Answer is* **(D)**

4. In the diagram below the upper scale AB has ten 1 centimetre divisions. The lower scale CD also has ten divisions but it is only 9 centimetres long. If the right hand end of the fourth division of scale CD coincides exactly with the right hand end of the seventh division of scale AB, what is the distance, in centimetres, from A to C?



Solution. Let E be the right hand end of the fourth division of scale CD (which is also the right hand end of the seventh division of scale AB). The distance AE is 7 cm. The distance CE is $4 \times 0.9 = 3.6$ cm. Thus the distance $AC = AE \Leftrightarrow CE = 3.4$ cm. *Answer is* **(E)**

5. Triangle ABC is equilateral with sides tangent to the circle with centre at O and radius $\sqrt{3}$. The area of the quadrilateral $AOCB$, in square units, is:



Solution. First join the points B and O. Since an intersecting tangent and radius of a circle are perpendicular, this produces 2 congruent right angled triangles, namely BOA and BOC, each of which is a $30° \Leftrightarrow 60° \Leftrightarrow 90°$ triangle, which means that side BO is twice the length of side AO. Using the Theorem of Pythagoras we have

$$BA^2 = (2\sqrt{3})^2 \Leftrightarrow (\sqrt{3})^2 = 12 \Leftrightarrow 3 = 9.$$

Thus $BA = BC = 3$. The area of triangle BOA is now $\frac{1}{2} \cdot \sqrt{3} \cdot 3$. Since this is half the area we seek, the answer is $3\sqrt{3}$. *Answer is* **(A)**

6. Times such as 1:01, 1:11,... are called palindromic times because their digits read the same forwards and backwards. The number of palindromic times on a digital clock between 1:00 a.m. and 11:59 a.m. is:

Solution. Let us first consider single digit hours. The first and last digits must be the same, but the middle digit can be anything. Thus there are 6 such palindromic times each hour (since there are 6 possible middle digits), for a total of 54 such palindromic times between 1:00 am and 9:59 am. For two digit hours (that is, 10 and 11), in order to be a palindromic time the minutes are completely determined by the hour; there are only two such times, namely 10:01 and 11:11. So the grand total number of palindromic times between 1:00 a.m. and 11:59 a.m. is 56. *Answer is* **(D)**

7. Ted's television has channels 2 through 42. If Ted starts on channel 15 and surfs, pushing the channel up button 518 times, when he stops he will be on channel:

Solution. Ted's television set has a total of 41 channels. Thus every time he pushes the channel up button 41 times he returns to the channel he starts with. Since

$$518 = 12 \cdot 41 + 26$$

we see that he has effectively gone up 26 channels from his starting point at channel 15. He should thus be at channel 15+26=41. *Answer is* **(E)**

8. Consider a three-digit number with the following properties:

1. If its tens and ones digits are switched, its value would increase by 36.
2. Instead, if its hundreds and ones digits are switched its value would decrease by 198.

Suppose that only the hundreds and tens digits are switched. Its value would:

Solution. Let n be a three-digit number with the given properties. Let a, b, and c represent its hundreds digit, its tens digit and its ones digit, respectively. Then $n = 100a + 10b + c$. By property 1 we see that

$$\begin{aligned} 100a + 10c + b &= n + 36 = (100a + 10b + c) + 36, \\ \text{or} \quad 9(c \Leftrightarrow b) &= 36; \\ \text{that is,} \quad c \Leftrightarrow b &= 4. \end{aligned}$$

By property 2 we have

$$\begin{aligned} 100c + 10b + a &= n \Leftrightarrow 198 = (100a + 10b + c) \Leftrightarrow 198, \\ \text{or} \quad 99(a \Leftrightarrow c) &= 198; \\ \text{that is,} \quad a \Leftrightarrow c &= 2. \end{aligned}$$

Note that the two relations established above (when added) yield $a \Leftrightarrow b = 6$. Now let us consider the switch of the hundreds and the tens digits of n; the new number is

$$\begin{aligned} 100b + 10a + c &= 100a + 10b + c + 100(b \Leftrightarrow a) + 10(a \Leftrightarrow b) \\ &= n \Leftrightarrow 90(a \Leftrightarrow b) = n \Leftrightarrow 540. \end{aligned}$$

Answer is **(B)**

9. Speedy Sammy Seamstress sews seventy-seven stitches in sixty-six seconds. The time, in seconds, it takes Sammy to stitch fifty-five stitches is:

Solution. It takes Sammy $\frac{6}{7}$ seconds to sew a single stitch. For 55 stitches it will take $55 \cdot \frac{6}{7} = 47\frac{1}{7}$ seconds. *Answer is* **(E)**

10. How many positive integers less than or equal to 60 are divisible by 3, 4, or 5?

Solution. There are 20 integers between 1 and 60 which are divisible by 3, 15 divisible by 4, and 12 divisible by 5. If we simply add up these numbers to get $20 + 15 + 12 = 47$, we will have counted those integers divisible by at least two of 3, 4, or 5 more than once. Thus we must subtract from this total the number of integers between 1 and 60 divisible by both 3 and 4 (that is, divisible by 12), namely 5; the number divisible by both 3 and 5, namely 4; and the number divisible by both 4 and 5, namely 3. Thus our accumulated total is now $47 \Leftrightarrow (5 + 4 + 3) = 35$. However, the integer 60, which is the only positive integer in our range divisible by all of 3, 4, and 5 was at first counted 3 times, and now is not counted at all. Thus we must add 1 back into the total. So our total is 36.

Alternate method: Simply enumerate all the numbers between 1 and 60 and test each one. Those that are divisible by 3, 4, or 5 are: 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 21, 24, 25, 27, 28, 30, 32, 33, 35, 36, 39, 40, 42, 44, 45, 48, 50, 51, 52, 54, 55, 56, 57 and 60. There are 36 numbers in this list. *Answer is* **(C)**

Final Round 1997 — Part B

1. (a) The pages of a thick telephone directory are numbered from 1 to N. A total of 522 digits is required to print the pages. Find N.

Solution. From 1 to 9 inclusive there are 9 single digit numbers, which uses up 9 digits. From 10 to 99 inclusive there are 90 2-digit numbers, which uses up a further 180 digits. So far we have used up 189 digits before we get to 3-digit numbers leaving us with $522 \Leftrightarrow 189 = 333$ digits still to be accounted for. With this many digits we can make up 111 3-digit numbers. Starting at 100 and proceeding for 111 numbers brings us to page 210. Thus $N = 210$.

(b) There are 26 pages in the local newspaper. Suppose that you pull a sheet out and drop it on the floor. One of the pages facing you is numbered 19. What are the other page numbers on the sheet?

Solution. Since the newspaper has 26 pages, the outer sheet has pages 1, 2, 25, and 26. The next outermost sheet has pages 3, 4, 23, and 24. The next sheet has 5, 6, 21, and 22. Finally the sheet we are interested in has pages 7, 8, 19, and 20. So the other page numbers are 7, 8, and 20.

2. Create expressions for the numbers $1, 2, 3, \ldots, 10$ by using each of the digits 1, 9, 9, and 7. Note that the digits must appear separately; that is, numbers like 17 are not allowed. Only the basic operations $+$, $\Leftrightarrow$, $\times$, $\div$ and brackets (if necessary) may be used. Other mathematical symbols such as $\sqrt{}$ are *not* allowed. Every expression must include one 1, two 9's, and one 7, in any order.

Solution. There are several acceptable answers for this question. Here are some:

$$\begin{aligned}
1 &= 7 \times (9 \Leftrightarrow 9) + 1 \\
2 &= (9 + 7) \div (9 \Leftrightarrow 1) \\
3 &= 9 \div (9 + 1 \Leftrightarrow 7) \\
4 &= (9 \Leftrightarrow 1) \div (9 \Leftrightarrow 7) \\
5 &= (9 + 1) \div (9 \Leftrightarrow 7) = 7 \Leftrightarrow (9 \div 9) \Leftrightarrow 1 \\
6 &= 9 \Leftrightarrow 9 + 7 \Leftrightarrow 1 \\
7 &= 9 \Leftrightarrow 9 + 7 \div 1 = (9 \Leftrightarrow 9) \times 1 + 7 \\
8 &= 9 \Leftrightarrow 9 + 7 + 1 \\
9 &= 9 \div 9 + 7 + 1 \\
10 &= 9 + 9 \Leftrightarrow 7 \Leftrightarrow 1
\end{aligned}$$

3. (a) Decide which is greater: $\sqrt{6} + \sqrt{8}$ or $\sqrt{5} + \sqrt{9}$.
Solution. Let $x = \sqrt{6} + \sqrt{8}$ and $y = \sqrt{5} + \sqrt{9}$. Then

$$x^2 = 6 + 8 + 2\sqrt{48} = 14 + 2\sqrt{48}$$

$$y^2 = 5 + 9 + 2\sqrt{45} = 14 + 2\sqrt{45}$$

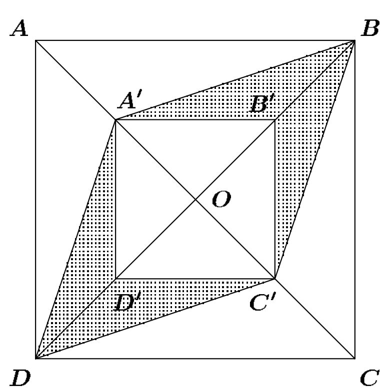
Since $\sqrt{48} > \sqrt{45}$, we see that $x^2 > y^2$ and thus it follows that $x > y$ since both are positive real numbers. Therefore, the larger value is $\sqrt{6} + \sqrt{8}$.

(b) Show that $\frac{x^2+1}{x} \geq 2$ for any real number $x > 0$.

Solution. Note that $(x \Leftrightarrow 1)^2 \geq 0$ for all real numbers x. This can be rewritten as $x^2 \Leftrightarrow 2x + 1 \geq 0$, and thus we have $x^2 + 1 \geq 2x$. Since we are given $x > 0$ we can divide both sides of the inequality by x and preserve the direction of the inequality to get

$$\frac{x^2 + 1}{x} \geq 2.$$

4. In the plane figure shown below, $ABCD$ is a square with $AB = 12$. If A', B', C', and D' are the midpoints of AO, BO, CO, and DO, respectively, then:



(a) Find the area of the square $A'B'C'D'$.

Solution. Triangles AOB and $A'OB'$ are similar since both are isosceles right angled triangles. Since $A'O$ is one half the length of AO we see that $A'B'$ is also one half the length of AB; that is, $A'B'$ is 6 units in length, which makes the square $A'B'C'D'$ have area 36 square units.

(b) Find the area of the shaded region.

Solution. Triangle $BA'B'$ has a base $A'B'$ of 6 units, and an altitude which is one half of the altitude of triangle AOB (from its base AB); that is, triangle $BA'B'$ has altitude 3 from its base $A'B'$. Thus the area of triangle $BA'B'$ is $\frac{1}{2} \cdot 6 \cdot 3 = 9$ square units. Since the shaded area comprises 4 such triangles, the total shaded area is 36 square units.

Alternate method for (b): Triangles $A'OB$ and $A'B'B$ have the same altitude from the base $OB'B$, and they have the same length for a base, since B' is the midpoint of OB. Thus their areas must be the same. But the area of triangle $A'OB'$ is clearly $\frac{1}{4}$ of the area of the square $A'B'C'D'$ computed in part (a); that is, 9 square units. Thus, triangle $BA'B'$ has area 9 and we finish as in the first method.

(c) Find the area of the trapezoid $AA'B'B$.

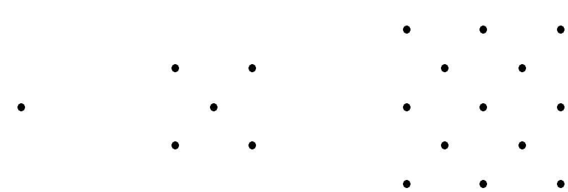
Solution. Since the parallel sides of the trapezoid have lengths 12 and 6, and since its altitude is 3 (as seen in part (b) above), we see that the area of the trapezoid is:

$$\frac{1}{2}3(6+12) = 27.$$

Alternate method for (c): The area we seek is $\frac{1}{4}$ of the difference between the areas of the squares $ABCD$ and $A'B'C'D'$, which is:

$$\frac{1}{4}(12^2 \Leftrightarrow 6^2) = \frac{1}{4}(144 \Leftrightarrow 36) = 27.$$

5. The figure below shows the first three in a sequence of square arrays of dots. The numbers of dots in the three arrays are 1, 5, and 13.



(a) Find the number of dots in the next array in the sequence.

Solution. There are several ways of looking at the arrays. One way is to notice that in going from one array to the next we build around the outside of the array a new set of dots. That is, we add 4 dots to the single dot to get array 2. We then add $4 \cdot 2 = 8$ dots around array 2 to get array 3. We would then add $4 \cdot 3 = 12$ dots around array 3 to get array 4. This leaves us with $13 + 12 = 25$ dots in the next array.

(b) Find the number of dots in the sixth array in the sequence.

Solution. To carry on to array 6 we need first to get array 5 from array 4. Using the process described in part (a) we would get $25 + 4 \cdot 4 = 41$ dots in array 5 and $41 + 4 \cdot 5 = 61$ dots in array 6.

(c) Find an expression for a_n in terms of n alone.

Solution. By observing the dot pattern, the number of dots seems to be in the order of n^2. If we subtract n^2 from a_n for the first few values of n we see that what remains is $(n \Leftrightarrow 1)^2$. Since $a_n = n^2 + (n \Leftrightarrow 1)^2$ satisfies the relation from part (d) above, and since it coincides with the first few values of a_n, this must be the solution for a_n.

(d) If a_n is the number of dots in the n^{th} array in the sequence, find a relation between a_{n+1}, a_n, and n.

Solution. From the above analysis, if we denote by $a_n$the number of dots in the n^{th} array, then

$$a_{n+1} = a_n + 4n.$$

That completes the Skoliad Corner for this number. I need contest materials suited to this feature, so please send me your materials, as well as comments and suggestions for the future of the Skoliad Corner.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to the Mayhem Editor, Naoki Sato, Department of Mathematics, Yale University, PO Box 208283 Yale Station, New Haven, CT 06520–8283 USA. The electronic address is still

mayhem@math.toronto.edu

The Assistant Mayhem Editor is Cyrus Hsia (University of Toronto). The rest of the staff consists of Adrian Chan (Upper Canada College), Jimmy Chui (Earl Haig Secondary School), Richard Hoshino (University of Waterloo), David Savitt (Harvard University) and Wai Ling Yee (University of Waterloo).

Mayhem Problems

The Mayhem Problems editors are:

Richard Hoshino	*Mayhem High School Problems Editor,*
Cyrus Hsia	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only problems — the next issue will feature only solutions.

We warmly welcome proposals for problems and solutions. We request that solutions from this issue be submitted by 1 February 1999, for publication in issue 4 of 1999. Also, starting with this issue, we would like to re-open the problems to all ***CRUX with MAYHEM*** readers, not just students, so now all solutions will be considered for publication.

Erratum

We regret to report that the High School porblems in volume 24, issue 1 [1998: 42], were mislabelled. Instead of **H223**, **H224**, **H225** and **H226**, they should have been labelled **H233**, **H234**, **H235** and **H236** respectively. We kindly ask readers to respect the new labelling when submitting solutions.

High School Problems

Editor: Richard Hoshino, 17 Norman Ross Drive, Markham, Ontario, Canada. L3S 3E8 <rhoshino@undergrad.math.uwaterloo.ca>

H237. The letters of the word MATHEMATICAL are arranged at random. What is the probability that the resulting arrangement contains no adjacent A's?

H238. Johnny is dazed and confused. Starting at $A(0,0)$ in the Cartesian grid, he moves 1 unit to the right, then r units up, r^2 units left, r^3 units down, r^4 units right, r^5 units up, and continues the same pattern indefinitely. If r is a positive number less than 1, he will be approaching a point $B(x,y)$. Show that the length of the line segment AB is greater than $\frac{7}{10}$.

H239. Find all pairs of integers (x,y) which satisfy the equation $y^2(x^2+1)+x^2(y^2+16)=448$.

H240. *Proposed by Alexandre Trichtchenko, Brookfield High School, Ottawa, Ontario.*

A Pythagorean triple (a,b,c) is a triple of integers satisfying the equation $a^2+b^2=c^2$. We say that such a triple is *primitive* if $\gcd(a,b,c)=1$. Let p be an odd integer with exactly n prime divisors. Show that there exist exactly 2^{n-1} primitive Pythagorean triples where p is the first element of the triple. For example, if $p=15$, then $(15,8,17)$ and $(15,112,113)$ are the primitive Pythagorean triples with first element 15.

Advanced Problems

Editor: Cyrus Hsia, 21 Van Allan Road, Scarborough, Ontario, Canada. M1G 1C3 <hsia@math.toronto.edu>

A213. Show that the number of non-negative integer solutions to the equation $a+b+c+d=98$, where $a\le b\le c\le d$, is equal to the number of non-negative integer solutions to the equation $p+2q+3r+4s=98$.

A214. Show that any rational number can be written as the sum of a finite number of distinct unit fractions. A unit fraction is of the form $\frac{1}{n}$, where n is an integer.

A215. For a fixed integer $n\ge 2$, determine the maximum value of $k_1+\cdots+k_n$, where $k_1,\ldots,k_n$ are positive integers with $k_1^3+\cdots+k_n^3\le 7n$.

(Polish Mathematical Olympiad)

A216. Given a continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying the conditions:

$$f(1000) = 999,$$

$$f(x) \cdot f(f(x)) = 1 \text{ for all } x \in \mathbb{R}.$$

Determine $f(500)$.

(Polish Mathematical Olympiad)

Challenge Board Problems

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

Editorial Notes: Part (b) is the interesting part of C77. Part (a) is a commonly asked problem, but I think it is better to ask it again than simply to state it for use in (b).

C77. Let F_n denote the n^{th} Fibonacci number, with $F_0 = 1$ and $F_1 = 1$. (Then $F_2 = 2$, $F_3 = 3$, $F_4 = 5$, etc.)

(a) Prove that each positive integer is uniquely expressible in the form $F_{a_1} + \cdots + F_{a_k}$, where the subscripts form a strictly increasing sequence of positive integers, no pair of which are consecutive.

(b) Let $\tau = \frac{1}{2}(1 + \sqrt{5})$, and for any positive integer n, let $f(n)$ equal the integer nearest to τn. If $n = F_{a_1} + \cdots + F_{a_k}$ is the expression for n from part (a), prove that $f(n) = F_{a_1+1} + \cdots + F_{a_k+1}$.

C78. Let n be a positive integer. An $n \times n$ matrix A is a *magic matrix* of order m if each entry is a non-negative integer and each row and column sum is m. (That is, for all i and j, $\sum_k A_{ik} = \sum_k A_{kj} = m$.)

Let A be a magic matrix of order m. Show that A can be expressed as the sum of m magic matrices of order 1.

Tips on Inequalities

Naoki Sato
graduate student, Yale University

Inequalities can be difficult to solve because there are few systematic methods for tackling even the most simple formulations. Indeed, solving usually involves a trial and error of different approaches, before one hits the right combination of estimations and manipulations. In this article, we expose some useful standard approaches and techniques. We recall two basic and fundamental inequalities:

AM-GM Inequality. For all $x_1, x_2, \ldots, x_n \geq 0$,

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n},$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Cauchy-Schwarz Inequality (CSB). For all real x_i, y_i, $i = 1, 2, \ldots, n$,

$$(x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)^2 \leq (x_1^2 + x_2^2 + \cdots + x_n^2)(y_1^2 + y_2^2 + \cdots + y_n^2),$$

with equality if and only if the vectors $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are proportional.

Manipulating the Expressions

A typical approach in proving an inequality of the form $A \geq B$ is to find intermediate expressions, so we have a chain

$$A \geq P_1 \geq P_2 \geq \cdots \geq P_n \geq B.$$

These are usually found through using the classic inequalities, and by manipulating terms until we get what we want. As anyone who has worked with inequalities knows, this takes great care; one constantly has to make sure that the estimates are not too crude, and that the inequality signs are going the right way. In this kind of approach, there are several things one should keep in mind:

1. Is the inequality sharp or strict?

An inequality is sharp if equality occurs at a point, and strict if equality never occurs. It is always a good idea to check which type it is, though it is usually given or obvious. A strict inequality may allow for generous estimates, but not always. A sharp inequality leaves no such allowance.

2. If equality does occur, when/where does it occur?

The points where equality occurs are points you must work around. In the chain above, each intermediate inequality must become an equality at these points. This is a good check of whether your intermediate expressions are the right ones.

Pairing and Grouping

In inequalities where several terms are involved, it might be possible to group terms together and prove "smaller" inequalities.

Problem 1.

(a) Prove that $x^2 + y^2 + z^2 \geq xy + yz + zx$ for all x, y, $z \in \mathbb{R}$.

(b) Prove that $x^3 + y^3 + z^3 \geq x^2y + y^2z + z^2x$ for all x, y, $z \geq 0$.

Solution. (a) No grouping is immediately obvious. We know $x^2 + y^2 \geq 2xy$, but how can we incorporate this? By adding $x^2 + y^2 \geq 2xy$, $y^2 + z^2 \geq 2yz$, and $z^2 + x^2 \geq 2zx$, and dividing by 2, we obtain the desired inequality.

(b) Here, we know $2x^3 + y^3 = x^3 + x^3 + y^3 \geq 3x^2y$ by AM-GM. We then add the other two corresponding inequalities, and then divide by 3.

Problem 2. For all a, b, c, $d > 0$, show that

$$\frac{a^3+b^3+c^3}{a+b+c}+\frac{a^3+b^3+d^3}{a+b+d}+\frac{a^3+c^3+d^3}{a+c+d}+\frac{b^3+c^3+d^3}{b+c+d} \;\geq\; a^2+b^2+c^2+d^2 .$$

Solution. We claim that

$$\frac{a^3+b^3+c^3}{a+b+c} \;\geq\; \frac{a^2+b^2+c^2}{3} .$$

Then, the problem follows by symmetry. Our inequality is equivalent to

$$3a^3 + 3b^3 + 3c^3 \;\geq\; (a^2+b^2+c^2)(a+b+c) ,$$

which, in turn, is equivalent to

$$\begin{aligned} &2a^3+2b^3+2c^3-a^2b-ab^2-a^2c-ac^2-b^2c-bc^2 \\ &\quad= (a^3-a^2b-ab^2+b^3)+(a^3-a^2c-ac^2+c^3)+(b^3-b^2c-bc^2+c^3) \\ &\quad= (a-b)^2(a+b)+(a-c)^2(a+c)+(b-c)^2(b+c) \;\geq\; 0 , \end{aligned}$$

which is true. Note that this inequality also follows from Chebyshev's inequality.

The Cauchy-Schwarz Inequality

The Cauchy-Schwarz inequality is a good way to deal with squares and especially fractions.

Problem 3. For $x_1, x_2, \ldots, x_n > 0$, show that

$$\frac{x_1^2}{x_1+x_2}+\frac{x_2^2}{x_2+x_3}+\cdots+\frac{x_n^2}{x_n+x_1} \;\geq\; \frac{x_1+x_2+\cdots+x_n}{2} .$$

Solution. By CSB,

$$\begin{aligned}&[(x_1+x_2)+(x_2+x_3)+\cdots+(x_n+x_1)]\\ &\times\left[\frac{x_1^2}{x_1+x_2}+\frac{x_2^2}{x_2+x_3}+\cdots+\frac{x_n^2}{x_n+x_1}\right]\\ &\geq\ (x_1+x_2+\cdots+x_n)^2\,.\end{aligned}$$

The result then follows by dividing each side by $2\,(x_1+x_2+\cdots+x_n)$.

Problem 4. Prove that for $a_1,\ a_2,\ \ldots\ ,\ a_n>0$,

$$\frac{(a_1+a_2+\cdots+a_n)^2}{2(a_1^2+a_2^2+\cdots+a_n^2)}\ \leq\ \frac{a_1}{a_2+a_3}+\frac{a_2}{a_3+a_4}+\cdots+\frac{a_n}{a_1+a_2}\,.$$

(1990–1991 IMO Correspondence)

Solution. Recall that CSB states that for all real $x_i,\ y_i,\ i=1,\ 2,\ \ldots\ ,\ n$,

$$(x_1y_1+x_2y_2+\cdots+x_ny_n)^2\ \leq\ \left(x_1^2+x_2^2+\cdots+x_n^2\right)\left(y_1^2+y_2^2+\cdots+y_n^2\right)\,.$$

Setting $x_i=\sqrt[4]{a_i},\ y_i=\sqrt[4]{a_i^3}$, we obtain

$$\begin{aligned}&(a_1+a_2+\cdots+a_n)^2\\ &\leq\ \left(\sqrt{a_1}+\sqrt{a_2}+\cdots+\sqrt{a_n}\right)\left(a_1\sqrt{a_1}+a_2\sqrt{a_2}+\cdots+a_n\sqrt{a_n}\right)\,.\end{aligned}$$

Setting $x_i=\sqrt{a_i},\ y_i=a_i$, we obtain

$$\begin{aligned}&\left(a_1\sqrt{a_1}+a_2\sqrt{a_2}+\cdots+a_n\sqrt{a_n}\right)^2\\ &\leq\ (a_1+a_2+\cdots+a_n)\left(a_1^2+a_2^2+\cdots+a_n^2\right)\,.\end{aligned}$$

Combining these two, we obtain

$$\begin{aligned}&(a_1+a_2+\cdots+a_n)^3\\ &\leq\ \left(\sqrt{a_1}+\sqrt{a_2}+\cdots+\sqrt{a_n}\right)^2\left(a_1^2+a_2^2+\cdots+a_n^2\right)\,.\end{aligned}$$

Finally, setting $x_i=\sqrt{a_{i+1}+a_{i+2}},\ y_i=\sqrt{\frac{a_i}{a_{i+1}+a_{i+2}}}$, we obtain

$$\begin{aligned}&\left(\sqrt{a_1}+\sqrt{a_2}+\cdots+\sqrt{a_n}\right)^2\\ &\leq\ 2\,(a_1+a_2+\cdots+a_n)\left(\frac{a_1}{a_2+a_3}+\frac{a_2}{a_3+a_4}+\cdots+\frac{a_n}{a_1+a_2}\right)\,.\end{aligned}$$

The last two inequalities give the desired result.

Elementary Symmetric Polynomials

Given a set of variables X, such as $X=\{x,y,z\}$, a polynomial is **symmetric** in X if it is invariant under any permutation of the variables under X, and **homogeneous of degree** k if every term in the polynomial has degree k.

For example, $x^2 + y^2 + z^2 \Leftrightarrow xyz$ is symmetric but not homogeneous in X, and $x^3 + x^2y \Leftrightarrow xyz$ is homogeneous of degree 3 but not symmetric in X.

The elementary symmetric polynomials in X are the polynomials obtained as the sum of the products of the variables in X, taken k at a time. For $X = \{x, y, z\}$, these would be $x + y + z$, $xy + xz + yz$, and xyz. Note that each of these is homogeneous. A theorem of Gauss states that any symmetric, homogeneous polynomial in X can be expressed as a polynomial in these elementary polynomials.

After all these tedious definitions, we finally get to the point that it can be useful to use these elementary polynomials.

Problem 5. For non-negative reals x, y, and z satisfying $x + y + z = 1$, show that

$$\left(\frac{1}{x} + 1\right)\left(\frac{1}{y} + 1\right)\left(\frac{1}{z} + 1\right) \;\geq\; 64\,.$$

Proof. Expanding, we first must show

$$1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{xy} + \frac{1}{xz} + \frac{1}{yz} + \frac{1}{xyz} \;\geq\; 64\,.$$

By AM-GM,

$$xyz \;\leq\; \left(\frac{x+y+z}{3}\right)^3 \;=\; \frac{1}{27} \qquad \text{so that} \qquad \frac{1}{xyz} \;\geq\; 27\,.$$

Therefore,

$$\begin{aligned} & 1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{xy} + \frac{1}{xz} + \frac{1}{yz} + \frac{1}{xyz} \\ \geq\; & 1 + \frac{3}{\sqrt[3]{xyz}} + \frac{3}{\sqrt[3]{x^2y^2z^2}} + \frac{1}{xyz} \\ =\; & \left(1 + \frac{1}{\sqrt[3]{xyz}}\right)^3 \\ \geq\; & 4^3 \;=\; 64\,. \end{aligned}$$

Problem 6. Prove that

$$0 \;\leq\; yz + zx + xy \Leftrightarrow 2xyz \;\leq\; \frac{7}{27},$$

where x, y, and z are non-negative real numbers for which $x + y + z = 1$.
(1984 IMO, #1)

Proof. Let $S = xy + xz + yz \Leftrightarrow 2xyz$, $P = (1 \Leftrightarrow 2x)(1 \Leftrightarrow 2y)(1 \Leftrightarrow 2z)$. Then

$$P \;=\; 1 \Leftrightarrow 2(x+y+z) + 4(xy+xz+yz) \Leftrightarrow 8xyz \;=\; 4S \Leftrightarrow 1\,.$$

We must show that $0 \le S \le \frac{7}{27}$, or equivalently, $\Leftrightarrow 1 \le P \le \frac{1}{27}$. Since $0 \le x, y, z \le 1$, $\Leftrightarrow 1 \le 2x \Leftrightarrow 1, 2y \Leftrightarrow 1, 2z \Leftrightarrow 1 \le 1$, so $\Leftrightarrow 1 \le P$. Now, if one of the variables, say x, was greater than $1/2$, then the other two would be less than $1/2$, and we would have $1 \Leftrightarrow 2x < 0$, $1 \Leftrightarrow 2y$, $1 \Leftrightarrow 2z > 0$, and $P < 0 < \frac{1}{27}$. Otherwise, x, y, and z are at most $1/2$, and all factors of P are non-negative, so by AM-GM,

$$P \le \left(\frac{1 \Leftrightarrow 2x + 1 \Leftrightarrow 2y + 1 \Leftrightarrow 2z}{3} \right)^3 = \frac{1}{27}.$$

Introducing and Removing Constraints

Inequalities often come with constraints on the variables. Removing these constraints can simplify the problem. Alternately, introducing them may help as well. The most common way of removing a constraint is to "homogenize" the given inequality. For example, suppose we are given the expression $x^3 + xy \Leftrightarrow 2$, where $xyz = 1$. Then

$$x^3 + xy \Leftrightarrow 2 = x^3 + xy\sqrt[3]{xyz} \Leftrightarrow 2xyz.$$

This new expression is homogeneous of degree 3. It's not pretty, but for the expressions given in problems, most of the time it will be nice and much easier to work with.

At this point, we introduce an inequality that is not well-known, but that seems to pop up from time to time.

Schur's Inequality. For all x, y, $z \ge 0$ and non-negative integers n,

$$x^n(x \Leftrightarrow y)(x \Leftrightarrow z) + y^n(y \Leftrightarrow z)(y \Leftrightarrow x) + z^n(z \Leftrightarrow x)(z \Leftrightarrow y) \ge 0.$$

For $n = 1$, this becomes

$$x^3 + y^3 + z^3 + 3xyz \ge x^2y + xy^2 + x^2z + xz^2 + y^2z + yz^2,$$

or in shorthand, $\sum x^3 + 3xyz \ge \sum x^2y$. This is a useful inequality to know, with which we present another solution.

Problem 6. Prove that

$$0 \le yz + zx + xy \Leftrightarrow 2xyz \le \frac{7}{27},$$

where x, y, z are non-negative real numbers for which $x + y + z = 1$.
(1984 IMO, #1)

Solution. This is equivalent to the following problem: For x, y, $z \ge 0$, prove that

$$0 \le (yz + zx + xy)(x + y + z) \Leftrightarrow 2xyz \le \frac{7}{27}(x + y + z)^3.$$

This is the homogeneous version of the original inequality. The expression in the middle expands to $\sum x^2y + xyz$, which is clearly non-negative. We focus on the right inequality, which becomes

$$\sum x^2y + xyz \;\le\; \frac{7}{27}\sum x^3 + \frac{7}{9}\sum x^2y + \frac{14}{9}\sum xyz\,,$$

which implies

$$6\sum x^2y \;\le\; 7\sum x^3 + 15xyz\,.$$

This is where we must think backwards. What results do we know that we can use to prove this? By Schur's, $5\sum x^2y \le 5\sum x^3 + 15xyz$ (we try to eliminate the xyz term). Hence, to prove the above inequality, we must show that $\sum x^2y \le 2\sum x^3$, which is left as an exercise for the reader (it has been virtually done already in this article).

We stated early in the solution that the modified problem was equivalent to the original problem. It is easy to see that the modified problem implies the original problem (which is the only direction we actually needed), but what about the converse? What if $x + y + z \neq 1$? In such a case, we can normalize.

A property of homogeneous polynomials, and an alternate definition, is the following: $p(x_1, x_2, \ldots, x_n)$ is homogeneous of degree k if

$$p(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) \;=\; \lambda^k p(x_1, x_2, \ldots, x_n)$$

for all $\lambda \in \mathbb{R}$.

Going back to the original problem,

$$p(x, y, z) \;=\; (yz + zx + xy)(x + y + z) \Leftrightarrow 2xyz\,.$$

If $x + y + z = 0$, then all three variables must be 0, and the inequality follows. Otherwise, we can set $\lambda = \frac{1}{x+y+z}$, and so we must show

$$0 \;\le\; p\left(\frac{x}{x+y+z}, \frac{y}{x+y+z}, \frac{z}{x+y+z}\right) \;\le\; \frac{7}{27}\,.$$

This is the original problem. Thus, we can set $x + y + z$ equal to 1, or indeed anything we want to (except 0). It can be vital to exploit this degree of freedom. We perform a similar setting in the next problem.

Problem 7. Given $0 < a \le b$, and $x_1, x_2, \ldots, x_n \ge 0$, show that

$$(x_1^a + x_2^a + \cdots + x_n^a)^{1/a} \;\ge\; (x_1^b + x_2^b + \cdots + x_n^b)^{1/b}.$$

Solution. If $x_1^b + x_2^b + \cdots + x_n^b = 0$, then the problem is solved. Otherwise, by the arguments above, we can assume $x_1^b + x_2^b + \cdots + x_n^b = 1$. Then

$$\begin{aligned} x_i^b \le 1 \;&\Longrightarrow\; x_i \le 1 \Longrightarrow x_i^{b-a} \le 1 \Longrightarrow x_i^b \le x_i^a \\ &\Longrightarrow\; x_1^a + x_2^a + \cdots + x_n^a \ge x_1^b + x_2^b + \cdots + x_n^b = 1 \\ &\Longrightarrow\; (x_1^a + x_2^a + \cdots + x_n^a)^{1/a} \ge 1 = (x_1^b + x_2^b + \cdots + x_n^b)^{1/b}. \end{aligned}$$

Problems

1. Prove that if x, y, and z are non-negative real numbers such that $x + y + z = 1$, then

$$2(x^2 + y^2 + z^2) + 9xyz \ \geq \ 1 \, .$$

2. For any real numbers a, b, and c, show that

$$\min \left[(a \Leftrightarrow b)^2, (b \Leftrightarrow c)^2, (c \Leftrightarrow a)^2 \right] \ \leq \ \frac{a^2 + b^2 + c^2}{2} \, .$$

3. Let a, b, and c be the sides of a triangle with perimeter 2. Prove that $a^2 + b^2 + c^2 + 2abc < 2$.

4. For non-negative reals x, y, and z satisfying $2xyz + xy + xz + yz = 1$, prove that $x + y + z \geq \frac{3}{2}$.

5. For all positive integers n, show that

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n \Leftrightarrow 1}{2n} \ \leq \ \frac{1}{\sqrt{3n + 1}} \, .$$

(Hint: The inequality is almost certainly not sharp, so there is some room for approximation. The RHS suggests squaring.)

6. Show that if x, y, and z are non-negative reals such that $x + y + z = 1$, then

$$\left(\frac{1}{x} \Leftrightarrow 1 \right) \left(\frac{1}{y} \Leftrightarrow 1 \right) \left(\frac{1}{z} \Leftrightarrow 1 \right) \ \geq \ 8 \, .$$

(Note: The solution in Problem 5 does not work!)

7. Given a, b, c, d, $e > 0$, $abcde = 1$, show that

$$a^4 + b^4 + c^4 + d^4 + e^4 \ \geq \ a + b + c + d + e \, .$$

(A favourite of Ravi Vakil's. Find as many different solutions as you can.)

8. Show that if non-negative reals a, b, and c satisfy

$$\frac{1}{1 + a} + \frac{1}{1 + b} + \frac{1}{1 + c} \ = \ 1 \, ,$$

then $abc \geq 8$.

Riveting Properties of Pascal's Triangle

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Consider the following table of integers, known as **Pascal's Triangle**:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & \vdots & & & & &
\end{array}$$

This table is named after the French mathematician Blaise Pascal, who conceived of it at the age of thirteen (although the Chinese discovered some properties of this triangle in the early 14^{th} century, centuries before Pascal was even born!).

To generate a row of Pascal's Triangle, look at the row immediately above it. Each element of the triangle is the sum of the two elements above it (for example, the second element of the fourth row is 4 because $4 = 1 + 3$). By convention, we denote the top row as the 0^{th} row, and we denote the left most entry of each row as the 0^{th} entry, even though it may seem a little awkward at first.

Let us show that this table is the same as the following table:

$$\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \\
 & & & & & \vdots & & & & &
\end{array}$$

In this table, $\binom{n}{k}$ is determined by the formula $\frac{n!}{k!(n-k)!}$. Thus, we will show that the k^{th} element of the n^{th} row of Pascal's Triangle equals $\binom{n}{k}$ for all integers n and k. Each of these elements in the triangle, namely $\binom{0}{0}$, $\binom{1}{0}$, $\binom{1}{1}$, $\binom{2}{0}$, $\binom{2}{1}$, ..., is called a **binomial coefficient**, since the coefficients of the expansion of the binomial $(1 + x)^n$ correspond to the entries of the n^{th} row of Pascal's Triangle (for example, $(1+x)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$). Let us prove these statements, as they will be fundamental to our analysis of the properties of Pascal's Triangle.

Now $\binom{n}{k}$ denotes the number of ways we may select k objects from a set of n objects. By convention, we let $\binom{n}{0} = 1$, since there is technically only "one" way we may select nothing from a set of n objects. Let us show that

$$\binom{n}{k} + \binom{n}{k \Leftrightarrow 1} = \binom{n+1}{k}$$

for all n and k. Note that the right side denotes the number of ways we may select a k-member committee from a class of n girls and one boy (we choose k members from a set of $(n+1)$ people). Now, if the boy is on the committee, then we have $\binom{n}{k-1}$ ways of selecting the remaining $k \Leftrightarrow 1$ members. And if he is not on the committee, then there are $\binom{n}{k}$ ways of selecting the committee. Hence,

$$\binom{n}{k} + \binom{n}{k \Leftrightarrow 1} = \binom{n+1}{k}.$$

This is known as **Pascal's Identity**.

Since $\binom{0}{0} = 1$, the table of binomial coefficients corresponds directly to Pascal's Triangle – since the initial element is the same and, like Pascal's Triangle, each element is the sum of the two directly above it. Thus, we can determine any element in Pascal's Triangle with this formula. For example, the thirty-fifth element in the seventy-ninth row of Pascal's Triangle is $\binom{79}{35}$.

We now prove that the entries in the n^{th} row of Pascal's Triangle are the coefficients in the expansion of $(1+x)^n$. We proceed by induction. The case is trivial for $n = 1$. Suppose that

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{k}x^k + \cdots + \binom{n}{n}x^n$$

for some n. Then

$$\begin{aligned}
(1+x)^{n+1} &= (1+x)^n(1+x) \\
&= \left[\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n\right](1+x) \\
&= \binom{n}{0} + \left[\binom{n}{1} + \binom{n}{0}\right]x + \cdots + \left[\binom{n}{n} + \binom{n}{n \Leftrightarrow 1}\right]x^n + \binom{n}{n}x^{n+1}.
\end{aligned}$$

Since $\binom{n}{0} = \binom{n+1}{0} = \binom{n}{n} = \binom{n+1}{n+1} = 1$, and using Pascal's Identity for all the other terms, we immediately arrive at the case for $n+1$. Hence, the coefficient of x^k in $(1+x)^n$ is equal to $\binom{n}{k}$.

Now we illustrate some of the really neat properties of Pascal's Triangle.

Theorem 1.

(i) The sum of the coefficients in the n^{th} row of Pascal's Triangle is 2^n.

(ii) If we alternately add and subtract the digits in the n^{th} row of Pascal's Triangle, we always arrive at zero. For example, $1 - 4 + 6 - 4 + 1 = 0$ for $n = 4$.

Proof. (i) Since

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{k}x^k + \cdots + \binom{n}{n}x^n,$$

substituting $x = 1$ into this expression yields

$$2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}.$$

(ii) The second part of the theorem is left as an exercise. It is the same technique as above; just substitute a different value for x. Can you guess which value of x we ought to substitute?

Theorem 2. If the binary representation of n contains p ones, then there are 2^p odd numbers in the n^{th} row of Pascal's Triangle. For example, since $9 = 1001_2$, there are $2^2 = 4$ odd numbers in the ninth row.

Proof. We shall analyze everything in modulo 2. For those of you not familiar with modular arithmetic, when we say that x is congruent to y modulo m, which is written $x \equiv y \pmod{m}$, we mean that x and y are numbers such that when both are divided by m, they give the same remainder. For example, 1998 is congruent to 2 modulo 4. Also, if $r \equiv 0 \pmod{2}$, then r is even.

We first show that

$$(1+x)^{2^n} \equiv 1 + x^{2^n} \pmod{2}$$

for all non-negative integers n. We proceed by induction. If $n = 0$, the claim is immediate. Suppose the claim is true for some $n = k$. Then

$$(1+x)^{2^{k+1}} \equiv ((1+x)^{2^k})^2 \equiv (1+x^{2^k})^2 \equiv 1+2x^{2^k}+x^{2^{k+1}} \equiv 1+x^{2^{k+1}} \pmod{2},$$

and so it is also true for $n = k + 1$. Hence, by induction, the claim has been verified.

Here is an example for a specific case. Let $n = 50$. Then $50 = 2^1 + 2^4 + 2^5 = 2 + 16 + 32$. Hence,

$$\begin{aligned}(1+x)^{50} &= (1+x)^2(1+x)^{16}(1+x)^{32} \\ &\equiv (1+x^2)(1+x^{16})(1+x^{32}) \\ &\equiv 1 + x^2 + x^{16} + x^{18} + x^{32} + x^{34} + x^{48} + x^{50} \pmod{2}.\end{aligned}$$

Hence, there are eight odd entries in the 50^{th} row of Pascal's Triangle, namely $\binom{50}{0}$, $\binom{50}{2}$, $\binom{50}{16}$, $\binom{50}{18}$, $\binom{50}{32}$, $\binom{50}{34}$, $\binom{50}{48}$, and $\binom{50}{50}$. Since $50 = 11010_2$, this verifies that there are exactly $2^3 = 8$ odd terms in this row.

For a general n, suppose there are p ones in the binary representation of n. Then

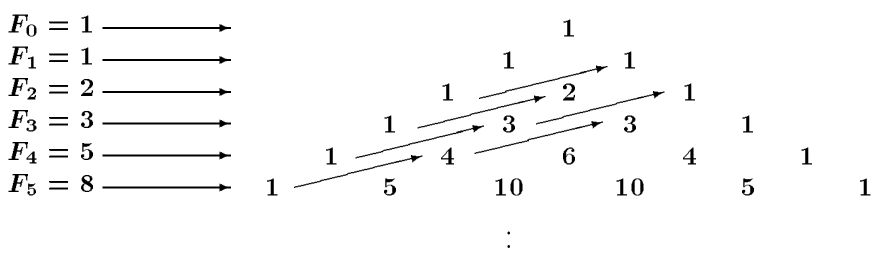
$$(1+x)^n = (1+x)^{2^{a_1}}(1+x)^{2^{a_2}}\cdots(1+x)^{2^{a_p}},$$

where $0 \le a_1 < a_2 < \cdots < a_p$ and the a_i^{th} digit of the binary representation of n is 1, starting from 0 at the right. The right side is congruent to

$$(1+x^{2^{a_1}})(1+x^{2^{a_2}})\cdots(1+x^{2^{a_p}})$$

modulo 2, and when we expand the right side, we will arrive at an expression with 2^p terms. Note that there must be exactly 2^p terms, since each exponent x^k can be formed in only one way by multiplying coefficients from the set $\{x^{2^{a_1}}, x^{2^{a_2}}, \ldots, x^{2^{a_p}}\}$. This is a direct result from the binary representation of k. Hence, if n has p ones in its binary representation, $(1+x)^n$ modulo 2 has 2^p terms, and hence there are 2^p odd entries in the n^{th} row of Pascal's Triangle.

Theorem 3. (The Fibonacci sequence is the sequence, 1, 1, 2, 3, 5, 8, 13, 21, 34, $\ldots$, where each element in the sequence is the sum of the two before it. More formally, we say that the Fibonacci sequence $\{F_n\}$ satisfies the conditions $F_0 = 1$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n > 1$.) We can derive the Fibonacci sequence from Pascal's Triangle in the following manner.



In other words, for all n, we have

$$F_{2n} = \binom{2n}{0} + \binom{2n \Leftrightarrow 1}{1} + \binom{2n \Leftrightarrow 2}{2} + \cdots + \binom{n}{n}$$

and

$$F_{2n+1} = \binom{2n+1}{0} + \binom{2n}{1} + \binom{2n \Leftrightarrow 1}{2} + \cdots + \binom{n+1}{n}.$$

For example,

$$F_7 = \binom{7}{0} + \binom{6}{1} + \binom{5}{2} + \binom{4}{3} = 1 + 6 + 10 + 4 = 21.$$

Proof. We prove this theorem using double induction: we know that the claim is true for F_0 and F_1. Suppose the claim is true for some F_{2k} and F_{2k+1}. Then we want to show that

$$F_{2k+2} = \binom{2k+2}{0} + \binom{2k+1}{1} + \binom{2k}{2} + \cdots + \binom{k+1}{k+1};$$

that is, the right side is equal to the $(2k+2)^{\text{th}}$ Fibonacci number. But

$$\begin{aligned} F_{2k+2} &= F_{2k} + F_{2k+1} \\ &= \binom{2k+1}{0} + \left[\binom{2k}{0} + \binom{2k}{1}\right] + \cdots + \left[\binom{k+1}{k \Leftrightarrow 1} + \binom{k+1}{k}\right] + \binom{k}{k} \\ &= \binom{2k+1}{0} + \binom{2k+1}{1} + \binom{2k}{2} + \cdots + \binom{k+2}{k} + \binom{k}{k}, \end{aligned}$$

by Pascal's Identity. Since $\binom{2k+1}{0} = 1 = \binom{2k+2}{0}$ and $\binom{k}{k} = 1 = \binom{k+1}{k+1}$, we have

$$F_{2k+2} = \binom{2k+2}{0} + \binom{2k+1}{1} + \binom{2k}{2} + \cdots + \binom{k+1}{k+1},$$

as required.

The second part of the induction follows similarly – assume the claim is true for F_{2k-1} and F_{2k}, and show that the claim is also true for F_{2k+1}. The proof is left as an exercise. Now we are done – since the proposition is true for $k = 0$ and $k = 1$, it is true for $k = 2$. Since it is true for $k = 1$ and $k = 2$, it is true for $k = 3$, etc. Then by double induction, the claim is true for all non-negative integers n.

Exercises

1. Show that $\binom{n}{k} = \binom{n}{n-k}$, and use this fact to show that Pascal's Triangle is symmetric about the vertical line that separates the triangle into two equal halves.

2. For which n is $\binom{2n}{n}$ odd?

3. Let a_n represent the number of elements in the n^{th} row of Pascal's Triangle that are congruent to 1 modulo 3. Let b_n represent the number of elements in the n^{th} row of Pascal's Triangle that are congruent to 2 modulo 3. Prove that for all n, $a_n \Leftrightarrow b_n$ is a power of two.

   (This problem was on the IMO short list one year – it is very tough!)

Swedish Mathematics Olympiad

1985 Qualifying Round

1. The real numbers a, b, and c satisfy the equations

$$\begin{aligned} ab + b &= -1 \\ bc + c &= -1 \\ ca + a &= -1 \end{aligned}$$

   Calculate the product abc.

2. 1985 runners reported for a marathon. They were assigned the numbers 1, 2, $\ldots$, 1985. However, a number of runners dropped out before the race. In fact, among the starting runners, there were no two for whom one runner's number was 10 times the other. What is the greatest number of runners that could have participated?

3. In the system of equations

$$\begin{aligned} a_1x + b_1y + c_1z + d_1u &= 0 \\ a_2x + b_2y + c_2z + d_2u &= 0 \\ a_3x + b_3y + c_3z + d_3u &= 0 \\ a_4x + b_4y + c_4z + d_4u &= 0 \end{aligned}$$

   the coefficients a_1, b_2, c_3, and d_4 are even integers and the other coefficients are odd integers. Prove that the only solution in integers is $x = y = z = u = 0$.

4. The non-negative integers p, q, r, and s satisfy the equality

$$(p + q)^2 + p = (r + s)^2 + r.$$

   Show that $p = r$ and $q = s$.

5. Let f be defined by

$$f(x) = \frac{4x^2 \sin^2 x + 9}{x \sin x}\,.$$

   Find the least value of f over the interval $0 < x < \pi$.

6. The point P lies on the perimeter or inside a given triangle T. The point P', in the plane of the triangle, lies at a distance d from P. Let r and r' be the radii of the smallest circles, with centres P and P' respectively, which contain T. Show that

$$r + d \leq 3r'.$$

   Give an example where equality holds.

1985 Final Round

1. Let $a > b > 0$. Prove that

$$\frac{(a \Leftrightarrow b)^2}{8a} < \frac{a+b}{2} \Leftrightarrow \sqrt{ab} < \frac{(a \Leftrightarrow b)^2}{8b}.$$

2. Find the least natural number such that if the first digit is placed last, the new number is $7/2$ times as large as the original number. (The numbers are written in the decimal system.)

3. A, B, and C are three points on a circle with radius r, and $AB = BC$. D is a point inside the circle such that the triangle BCD is equilateral. The line through A and D meets the circle at the point E. Show that $DE = r$.

4. The polynomial $p(x)$ of degree n has real coefficients, and $p(x) \geq 0$ for all x. Show that

$$p(x) + p'(x) + p''(x) + \cdots + p^{(n)}(x) \geq 0.$$

5. In a right-angled coordinate system, a triangle has vertices $A(a, 0)$, $B(0, b)$, and $C(c, d)$, where the numbers a, b, c, and d are positive. Show that if we denote the origin by O,

$$AB + BC + CA \geq 2CO.$$

6. X-wich has a vibrant club-life. For every pair of inhabitants there is exactly one club to which they both belong. For every pair of clubs there is exactly one person who is a member of both. No club has fewer than 3 members. At least one club has 17 members. How many people live in X-wich?

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}$"$\times$11" or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 November 1998**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in LaTeX). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

2306. *Proposed by Vedula N. Murty, Visakhapatnam, India.*

CORRECTION to (a) Give an elementary proof of the inequality:

$$\left(\sin\left(\frac{\pi x}{2}\right)\right)^2 > \frac{2x^2}{1+x^2}; \qquad (0 < x < 1). \qquad (1)$$

2326*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Prove or disprove that if A, B and C are the angles of a triangle, then

$$\frac{2}{\pi} < \sum_{\text{cyclic}} \frac{\left(1 \Leftrightarrow \sin\frac{A}{2}\right)\left(1+2\sin\frac{A}{2}\right)}{\pi \Leftrightarrow A} \leq \frac{9}{\pi}.$$

2327. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

The sequence $\{a_n\}$ is defined by $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, and

$$a_{n+1} = a_n \Leftrightarrow a_{n-1} + \frac{a_n^2}{a_{n-2}}, \qquad n \geq 3.$$

Prove that each $a_n \in \mathbb{N}$, and that no a_n is divisible by 4.

2328*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

It is known from Wilson's Theorem that the sequence $\{y_n : n \geq 0\}$, with $y_n = \dfrac{n!+1}{n+1}$, contains infinitely many integers; namely, $y_n \in \mathbb{N}$ if and only if $n+1$ is prime.

(a) Determine all integer members of the sequences $\{y_n(a) : n \geq 0\}$, with $y_n = \dfrac{n!+a}{n+a}$, in the cases $a = 2, 3, 4$.

(b) Determine all integer members of the sequences $\{y_n(a) : n \geq 0\}$, with $y_n = \dfrac{n!+a}{n+a}$, in the cases $a \geq 5$.

2329*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Suppose that p and $t > 0$ are real numbers. Define

$$\lambda_p(t) := t^p + t^{-p} + p^p \qquad \text{and} \qquad \kappa_p(t) := \left(t + t^{-1}\right)^p + 2\,.$$

(a) Show that $\lambda_p(t) \leq \kappa_p(t)$ for $p \geq 2$.

(b) Determine the sets of p: A, B and C, such that

1. $\lambda_p(t) \leq \kappa_p(t)$,
2. $\lambda_p(t) = \kappa_p(t)$,
3. $\lambda_p(t) \geq \kappa_p(t)$.

2330. *Proposed by Florian Herzig, student, Perchtoldsdorf, Austria.*

Prove that

$$e = 3 \Leftrightarrow \frac{1!}{1\cdot 3} + \frac{2!}{3\cdot 11} \Leftrightarrow \frac{3!}{11\cdot 53} + \frac{4!}{53\cdot 309} \Leftrightarrow \frac{5!}{309\cdot 2119} + \cdots\,,$$

where

$$\begin{array}{rcrcl} 11 &=& 3\cdot 3 &+& 2\cdot 1, \\ 53 &=& 4\cdot 11 &+& 3\cdot 3, \\ 309 &=& 5\cdot 53 &+& 4\cdot 11, \\ 2119 &=& 6\cdot 309 &+& 5\cdot 53, \\ & & \vdots & & \end{array}$$

[Ed: There is enough information here to deduce the general term.]

2331. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, Florida, USA.*

Let p be an odd prime. Show that there is at most one non-degenerate integer triangle with perimeter $4p$ and integer area. Characterize those primes for which such triangles exist.

2332. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*
Suppose x and y are integers. Solve the equation

$$x^2y^2 \Leftrightarrow 7x^2y + 12x^2 \Leftrightarrow 21xy \Leftrightarrow 4y^2 + 63x + 70y \Leftrightarrow 174 = 0.$$

2333. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*
You are given that D and E are points on the sides AC and AB respectively of $\triangle ABC$. Also, DE is not parallel to CB. Suppose F and G are points of BC and ED respectively such that

$$\overline{BF} : \overline{FC} = \overline{EG} : \overline{GD} = \overline{BE} : \overline{CD}.$$

Show that GF is parallel to the angle bisector of $\angle BAC$.

2334. *Proposed by Toshio Seimiya, Kawasaki, Japan.*
Suppose that ABC is a triangle with incentre I, and that BI, CI meet AC, AB at D, E respectively. Suppose that P is the intersection of AI with DE. Suppose that $PD = PI$.

Find angle ACB.

2335. *Proposed by Toshio Seimiya, Kawasaki, Japan.*
Triangle ABC has circumcircle Γ. A circle Γ' is internally tangent to Γ at P, and touches sides AB, AC at D, E respectively. Let X, Y be the feet of the perpendiculars from P to BC, DE respectively.

Prove that $PX = PY \sin \frac{A}{2}$.

2336. *Proposed by Toshio Seimiya, Kawasaki, Japan.*
The bisector of angle A of a triangle ABC meets BC at D. Let Γ and Γ' be the circumcircles of triangles ABD and ACD respectively, and let P, Q be the intersections of AD with the common tangents to Γ, Γ' respectively.

Prove that $PQ^2 = AB \cdot AC$.

2337. *Proposed by Iliya Bluskov, Simon Fraser University, Burnaby, BC.*
Let $F(1) = \left\lceil \dfrac{n^2+2n+2}{n^2+n+1} \right\rceil$, and, for each $i > 1$, let
$F(i) = \left\lceil \dfrac{n^2+2n+i+1}{n^2+n+i} F(i \Leftrightarrow 1) \right\rceil$.

Find $F(n)$.

Some readers have pointed out that problem 2287 [1997: 501] is the same as problem 2234 [1997: 168], and that problem 2288 [1997: 501] is the same as problem 2251 [1997: 299]. Also part (a) of problem 2306 [1998: 46; 175] is the same as the first part of 2296 [1997; 503]. The editors missed these duplications. Proposers are asked not to submit the same problem more than once.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2219. [1997: 110] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Show that there are an infinite number of solutions of the simultaneous equations:

$$\begin{aligned} x^2 - 1 &= (u+1)(v-1) \\ y^2 - 1 &= (u-1)(v+1) \end{aligned}$$

with x, y, u, v positive integers and $x \neq y$.

I. *Solution by Charles Ashbacher, Cedar Rapids, Iowa, USA; Edward J. Barbeau, University of Toronto, Toronto, Ontario; Charles R. Diminnie, Angelo State University, San Angelo, TX, USA; Florian Herzig, student, Perchtoldsdorf, Austria; Cyrus Hsia, student, University of Toronto, Toronto, Ontario; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; and Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*

For positive integers n, the quadruples

$$(x, y, u, v) = (1, 2n+1, 2n^2+2n+1, 1)$$

give an infinite set of solutions in which $x \neq y$, since

$$x^2 - 1 = (u+1)(v-1) = 0$$

and

$$y^2 - 1 = 4n^2 + 4n = (u-1)(v+1).$$

II. *Solution by Edward J. Barbeau, University of Toronto, Toronto, Ontario; Digby Smith, Mount Royal College, Calgary, Alberta; and the proposer.*

For any positive integer n, the quadruple

$$(x, y, u, v) = (2n^2 - n, 2n^2 + n, 4n^3 + n, n)$$

is a solution in which $x \neq y$, since

$$x^2 - 1 = (u+1)(v-1) = 4n^4 - 4n^3 + n^2 - 1$$

and

$$y^2 - 1 = (u-1)(v+1) = 4n^4 + 4n^3 + n^2 - 1.$$

III. *Solution by Charles R. Diminnie, Angelo State University, San Angelo, TX, USA; and Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

It is well known that the Pell equation $s^2 - 2t^2 = 1$ has infinitely many solutions in positive integers s, t. Clearly, $s > t > 1$.

If we set $u = s + t$, $v = s - t$, $x = t - 1$ and $y = t + 1$, then

$(u+1)(v-1) \;=\; s^2 - (t+1)^2 \;=\; t^2 - 2t \;=\; (t-1)^2 - 1 \;=\; x^2 - 1$, and
$(u-1)(v+1) \;=\; s^2 - (t-1)^2 \;=\; t^2 + 2t \;=\; (t+1)^2 - 1 \;=\; y^2 - 1$.

Also solved by MANSUR BOASE, student, St. Paul's School, London, England; FLORIAN HERZIG, student, Perchtoldsdorf, Austria (a second solution); RICHARD I. HESS, Rancho Palos Verdes, California, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece (two solutions); GERRY LEVERSHA, St Paul's School, London, England; and PANOS E. TSAOUSSOGLOU, Athens, Greece.

The families given by I, II *and* III *above, do not exhaust all the possible solutions. It is interesting to note that the "smallest" solution produced by all these families is* $(1, 3, 5, 1)$. *The next smallest ones are* $(1, 5, 13, 1)$, $(6, 10, 34, 2)$ *and* $(11, 13, 29, 5)$, *respectively. Both Herzig and Leversha obtained another infinite set of solutions in which* $v = 2$, *by considering the Pell equation* $3x^2 - y^2 = 8$. *Their "smallest" solution is* $(6, 10, 34, 2)$ *listed above. However, the next solution,* $(22, 38, 482, 2)$ *is not obtainable from any of the families given in* I, II *and* III.

2220. *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

Let V be the set of an icosahedron's twelve vertices, which can be partitioned into four classes of three vertices, each one in such a way that the three selected vertices of each class belong to the same face.

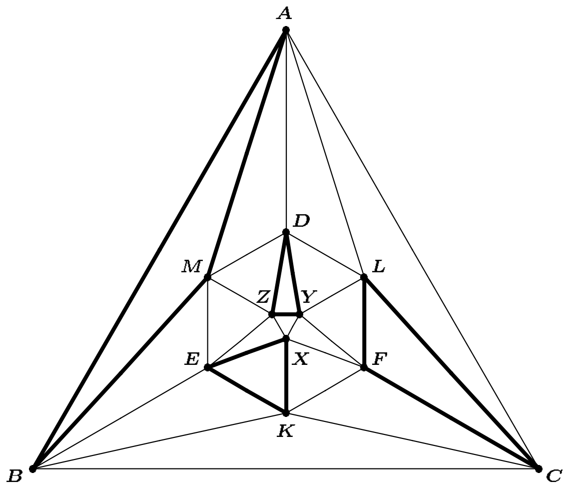
How many ways can this be done?

Solution by Florian Herzig, student, Perchtoldsdorf, Austria.

I will prove that there are 10 different ways of partitioning the vertices of an icosahedron in the described manner. Since only the topological properties of the icosahedron are important, consider the graph of the vertices: see figure on next page.

We want to find four triangles in this graph such that each vertex is used exactly once. Consider the vertex A. There are five triangles having A as vertex. Because of (spatial) symmetry we may assume that triangle ABM is chosen. Thus for the triangle containing point C only two choices remain: $\triangle CFL$ and $\triangle CFK$. Without loss of generality we may assume that triangle CFL is chosen. Now notice that the only "free" triangle containing D is $\triangle DZY$ and finally $\triangle EKX$ remains.

We have covered all possibilities already. For this case there are two different ways of finding "disjoint" triangles because we may choose from two equivalent triangles for vertex C. If all five possible triangles at vertex



A are considered, we get a total of 10 different configurations as claimed.

Also solved by MANSUR BOASE, student, St. Paul's School, London, England; JORDI DOU, Barcelona, Spain; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; and the proposer. There were two incorrect solutions.

Several solvers noted that the centres of the faces used in the partition are the vertices of a tetrahedron.

2221. [1997: 111] *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Find all members of the sequence $a_n = 3^{2n-1} + 2^{n-1}$ $(n \in \mathbb{N})$ which are the squares of any positive integer.

Solution by David Doster, Choate Rosemary Hall, Wallingford, Connecticut, USA.

We have $a_1 = 4$ and $a_2 = 29$. For $n \geq 3$, $3^{2n-1} \equiv 3 \pmod 4$ and $2^{n-1} \equiv 0 \pmod 4$. Thus, $a_n \equiv 3 \pmod 4$. But a positive integer is a square only if it is congruent to 0 or $1 \pmod 4$. Hence, $a_1 = 4$ is the only square in the sequence.

Also solved by SAM BAETHGE, Nordheim, Texas, USA; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain and MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; MANSUR BOASE, student, St. Paul's School, London, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; YEO KENG HEE, Hwa Chong Junior College, Singapore;

FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; ROBERT B. ISRAEL, University of British Columbia, Vancouver, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St Paul's School, London, England; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; BOB PRIELIPP, University of Wisconsin–Oshkosh, Wisconsin, USA; ISTVÁN REIMAN, Budapest, Hungary; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; ZUN SHAN and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; and the proposer.

2222. [1997: 111] *Proposed by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario.*

Find the value of the continued root:

$$\sqrt{4+27\sqrt{4+29\sqrt{4+31\sqrt{4+33\sqrt{\cdots}}}}}.$$

NOTE: This was inspired by the problems in chapter 26 "*Ramanujan, Infinity and the Majesty of the Quattuordecillion*", pp. 193–195, in "*Keys to Infinity*" by Clifford A. Pickover, John Wiley and Sons, 1995.

I. *Solution by Robert B. Israel, University of British Columbia, Vancouver, BC.*

The answer is 29. More generally, for any positive integer n, we claim that

$$\sqrt{4+n\sqrt{4+(n+2)\sqrt{4+(n+4)\sqrt{\cdots}}}} \;=\; n+2\,,$$

where the left side is defined as the limit of

$$F(n,m) \;=\; \sqrt{4+n\sqrt{4+(n+2)\sqrt{4+(n+4)\sqrt{\cdots\sqrt{4+m\sqrt{4}}}}}}$$

as $m \to \infty$ (where m is an integer and $(m \Leftrightarrow n)$ is even).

If $g(n,m) = F(n,m) \Leftrightarrow (n+2)$, we have

$$\begin{aligned} F(n,m)^2 \Leftrightarrow (n+2)^2 &= (4+nF(n+2,m)) \Leftrightarrow (4+n(n+4)) \\ &= n(F(n+2,m) \Leftrightarrow (n+4))\,, \end{aligned}$$

so

$$g(n,m) \;=\; \frac{n}{F(n,m)+n+2}\, g(n+2,m)\,.$$

Clearly $F(n,m) > 2$, so

$$|g(n,m)| \;<\; \frac{n}{n+4}\,|g(n+2,m)|\,.$$

By iterating this, we obtain

$$|g(n,m)| \;<\; \frac{n(n+2)}{m(m+2)}|g(m,m)| \;<\; \frac{n(n+2)}{m}\,.$$

Therefore $g(n,m) \to 0$ as $m \to \infty$.

II. *Solution by Efstratios Rappos, Girton College, University of Cambridge, England*

Let

$$S_n \;=\; \sqrt{4+(2n \Leftrightarrow 1)\sqrt{4+(2n+1)\sqrt{4+(2n+3)\sqrt{\cdots}}}}$$

S_n satisfies the recurrence relation

$$S_n \;=\; \sqrt{4+(2n \Leftrightarrow 1)S_{n+1}}$$

if and only if

$$(S_n \Leftrightarrow 2)(S_n+2) \;=\; (2n \Leftrightarrow 1)S_{n+1}\,.$$

By inspection, this admits $S_n = 2n+1$ as a solution. We only have to prove that $S_1 = 3$ to make this induction complete. Let

$$T_n \;=\; \sqrt{4+\sqrt{4+3\sqrt{\cdots(2n \Leftrightarrow 3)\sqrt{4+(2n \Leftrightarrow 1)\sqrt{2n+3)}}}}}$$

and

$$U_n \;=\; \sqrt{4+\sqrt{4+3\sqrt{\cdots(2n \Leftrightarrow 3)\sqrt{4+(2n \Leftrightarrow 1)(2n+3)}}}} = 3\,.$$

Clearly $T_n \le U_n$ and the latter is identically equal to 3. Therefore, using the fact that $B \ge A > 0$ implies that $\sqrt{(4+A)/(4+B)} \ge \sqrt{A/B}$,

$$\begin{aligned} 1 \;&\ge\; \frac{T_n}{3} \;=\; \frac{T_n}{U_n} \;=\; \frac{\sqrt{4+\sqrt{\cdots+(2n \Leftrightarrow 1)\sqrt{2n+3}}}}{\sqrt{4+\sqrt{\cdots(2n \Leftrightarrow 1))(2n+3)}}} \\ &\ge\; \frac{\sqrt{\sqrt{\cdots+(2n \Leftrightarrow 1)\sqrt{2n+3}}}}{\sqrt{\sqrt{\cdots+(2n \Leftrightarrow 1)(2n+3)}}} \;\ge\; \cdots \;\ge\; \sqrt[2^{n+1}]{\frac{1}{2n+3}} \\ &=\; \frac{1}{(2n+3)^{(\frac{1}{2})^{n+1}}} \;\Leftrightarrow\!\!\to\; 1 \end{aligned}$$

as $n \to \infty$ [for example, by rewriting as $\exp\{\Leftrightarrow \ln(2n+3)/2^{n+1}\}$ and using L'Hôpital's rule]. This proves that $S_1 = \lim_{n\to\infty} T_n = 3$. The required expression is precisely S_{14} and hence its value is 29.

Also solved or answered by MANSUR BOASE, student, St. Paul's School, London, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St Paul's School, London, England; J.A. MCCALLUM, Medicine Hat, Alberta; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer.

Several readers made reference to the July 1996 issue of the Mathematical Gazette, which contained a similar problem, Problem 80E posed by Tony Ward: evaluate:

$$\sqrt{1+1\sqrt{1+2\sqrt{1+3\sqrt{\cdots}}}}.$$

Bradley notes that the solution appeared in the March 1997 issue where the value of the continued root was proved to be 2, and to be "well-defined". He also adds that the editor of the Problems section of the Gazette concludes "Clearly, the problem raises some deep questions about the meaning of 'well-defined'." Along the same lines as this editorial comment, Murray S. Klamkin, University of Alberta, Edmonton, Alberta adds that "the value can be anything since there is no definition regarding the continuation of the root". Klamkin refers to A. Herschfeld, On Infinite Radicals, Amer. Math. Monthly, 42 (1935) 419-429 where Herschfeld notes that Ramanujan's solution for

$$\sqrt{1+2\sqrt{1+3\sqrt{1+\cdots}}} = 3$$

is incomplete since one may write similarly that

$$\begin{aligned} 4 &= \sqrt{1+2\cdot(15/2)} = \sqrt{1+2\sqrt{1+3\cdot(221/12)}} \\ &= \sqrt{1+2\sqrt{1+3\sqrt{1+\cdots}}}. \end{aligned}$$

Despite these comments most solvers expressed no difficulty in understanding the meaning of the continued root, and for that reason we have decided to print the above "solutions". Another reference to similar problems given by several readers was to J.M. Borwein, G. de Barra, Nested Radicals, Amer. Math. Monthly, 98 (1991) 735-739.

2224. [1997: 111] *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

Point P lies inside triangle ABC. Triangle BCD is erected outwardly on side BC such that $\angle BCD = \angle ACP$ and $\angle CBD = \angle ABC$. Prove that if the area of quadrilateral $PBDC$ is equal to the area of triangle ABC, then triangles ACP and BCD are similar.

Solution by Ian June L. Graces, Manila, The Philippines; and Giovanni Mazzarello, Firenze, Italy.

Let A' and P' be the respective images of A and P under reflection in the line BC. Note that B, D, A' are collinear (by the definition of D). Denoting by $[XYZ]$ the area of $\triangle XYZ$, we have

$$[P'CB] = \frac{1}{2}P'C \cdot BC \sin \angle P'CB,$$

and

$$[A'CD] = \frac{1}{2}A'C \cdot CD \sin \angle A'CD.$$

As a consequence of the given conditions ($[PBDC] = [ABC]$) and the effect of the reflection, $[P'CB] = [A'CD]$ (these are the complements of $\triangle BDC$ in $PBDC$ and in $\triangle A'CB$) and $\angle A'CD = \angle P'CB$. Thus

$$A'C \cdot CD = P'C \cdot BC.$$

In other words, (by SAS) we have $\triangle A'CP' \sim \triangle BCD$.

From the reflection we have $\triangle A'CP' \cong ACP$, and the desired result follows.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; MANSUR BOASE, student, St. Paul's School, London, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St Paul's School, London, England; ISTVÁN REIMAN, Budapest, Hungary; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

2225. [1997: 111] *Proposed by Kenneth Kam Chiu Ko, Mississauga, Ontario.*

(a) For any positive integer n, prove that there exists a unique n-digit number N such that:

(i) N is formed with only digits 1 and 2; and

(ii) N is divisible by 2^n.

(b) Can digits "1" and "2" in (a) be replaced by any other digits?

Solution by Robert B. Israel, University of British Columbia, Vancouver, BC.

We can use any two non-zero digits whose difference is odd. Let the digits be a and b, where $a, b \in \{1, 2, \ldots, 9\}$ and $a \Leftrightarrow b$ is odd.

There are 2^n different n-digit numbers formed with these two digits, and 2^n residue classes modulo 2^n. I claim that the 2^n numbers are all in distinct residue classes. By the Pigeonhole Principle, exactly one of these numbers must be in the residue class 0.

Consider two distinct n-digit numbers, N_1 and N_2, formed with the digits a and b. Suppose that the first digit, counting from right to left, where they differ, is in the 10^k position, $0 \le k \le n \Leftrightarrow 1$, where N_1 has a and N_2 has b. Then $N_1 \Leftrightarrow N_2 \equiv (a \Leftrightarrow b)10^k = (a \Leftrightarrow b)5^k 2^k \pmod{10^{k+1}}$, and thus modulo 2^{k+1}. Since $(a \Leftrightarrow b)5^k$ is odd, we have $N_1 \not\equiv N_2 \pmod{2^n}$.

Also solved by SAM BAETHGE, Nordheim, Texas, USA; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CURTIS COOPER, Central Missouri State University, Warrensburg, Missouri, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St Paul's School, London, England; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; JOEL SCHLOSBERG, student, Hunter College High School, New York, NY, USA; ZUN SHAN and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; KENNETH M. WILKE, Topeka, Kansas, USA; YEO KENG HEE, student, Hwa Chong Junior College, Singapore; and the proposer.

Besides Israel, only Lambrou, Leversha and Schlosberg proved the more general result presented above. Shan and Wang pointed out that if $\{a, b\} = \{2, 4\}, \{2, 8\}, \{4, 6\}, \{6, 8\}$ *or* $\{4, 8\}$, *then the "existence" claim is still true and the "uniqueness" claim would be true if one strengthens condition* (ii) *to* $2^{n+1} | N$ *for the first four pairs, and to* $2^{n+2} | N$ *for the pair* $\{4, 8\}$.

Diminnie asked whether any similar results are possible if 2^n is replaced by k^n for $3 \le k \le 9$.

2226. [1997: 166] *Proposed by K.R.S. Sastry, Dodballapur, India.*

An old man willed that, upon his death, his three sons would receive the u^{th}, v^{th}, w^{th} parts of his herd of camels respectively. He had $uvw \Leftrightarrow 1$ camels in the herd when he died. Obviously, their sophisticated calculator could not divide $uvw \Leftrightarrow 1$ exactly into u, v or w parts. They approached a distinguished ***CRUX*** problem solver for help, who rode over on his camel, which he added to the herd and then fulfilled the old man's wishes, and took the one camel that remained, which was, of course, his own.

Dear ***CRUX*** reader, how many camels were there in the herd?

I. *Solution by Robert B. Israel, University of British Columbia, Vancouver, BC.*

There were 41 camels. We may assume $u \le v \le w$. Of course $u \ge 2$, and $vw + uw + uv = uvw \Leftrightarrow 1$.

If $u = 2$, the equation becomes $(v \Leftrightarrow 2)(w \Leftrightarrow 2) = 5$. Since the only factorization of 5 is 1×5, this means $v = 3$, $w = 7$, and $uvw \Leftrightarrow 1 = 41$.

If $u = 3$, the equation becomes $(2v \Leftrightarrow 3)(2w \Leftrightarrow 3) = 11$. Again there is only one factorization, and $v = 2$ (which violates $u \le v$).

Finally, if $u \ge 4$ we have $1 \Leftrightarrow 1/(uvw) = 1/u + 1/v + 1/w \le 3/4$ so $uvw \le 4$, which is impossible.

II. *Solution by Michael Lambrou, University of Crete, Crete, Greece. This solution is so much in keeping with the spirit of the problem that the editor felt a need to share it with all **CRUX** readers.*

"In the holy name of the Almighty," said the distinguished problem solver who saw the ***CRUX*** of the matter, "if you add my humble camel to your herd then there will be uvw camels to share. Your kind selves, whose renowned hospitality offered me your well to quench my thirst, will receive uv, vw, and wu camels respectively. This amply fulfils the deceased's will, whose soul may rest in tranquility."

"So," interrupted the sophisticated calculator, a student of logistics (the art of the practical arithmetician), perpetuator of the Pythagorean doctrine that whole numbers are the essence of nature, "we must then have that $uv + vw + wu + 1$ equals uvw, since numbers are the balance of ideas, the epitome of fairness, and we must take into account the camel of our distinguished guest. We cannot let him leave our oasis for his long journey to redeem his pilgrimage pledge, without a fair chance to cross the desert."

The sophisticated calculator, well versed in the new art of Al-jabr, could easily re-write the condition as

$$u + v + w = (u \Leftrightarrow 1)(v \Leftrightarrow 1)(w \Leftrightarrow 1).$$

"What?" he exclaimed. "If any of u, v, w is 1, then the right hand side is a product of numbers, one of which is nothing!" He was perplexed although he had seen this 'nothing' number in Ptolemy's Almagest, the monumental astronomical work, in the Table of Chords in Chapter One. "I do not understand," he continued. "How can nothing exist? It is contrary to nature. Nature abhors void because it would make motion impossible, as falling bodies would have to have infinite speed."

"On the contrary," said the distinguished problem solver, "in my long travels I have heard that the wise men of the east have discovered a nothing number. They call it 'as-sifr', and it comes from the Sanskrit 'sunya'. It has the property that when multiplied by anything it gives as-sifr. So, mathematically speaking, we have to exclude equality of any of u, v, w to 1 because this would contradict the equation: $u + v + w = (u \Leftrightarrow 1)(v \Leftrightarrow 1)(w \Leftrightarrow 1)$."

"Perhaps mathematically we have to exclude this case but we have an inheritance problem," answered the calculator, trying to gain time, " and this nothing is not, philosophically speaking, on solid ground." He then turned to the respectful cadi, the assessor of values and conservator of culture, whose judgment had a Rhadamanthean wisdom. "What do you say, esteemed reverend?"

The cadi replied that "if any of u, v, w was 1, then one of the sons would take the entire herd, which is contrary to our sacred traditions. Surely it was not the intention of their late father to incite hatred in the thoughts of the two losers. Surely he did not want to upset the good values and bonds of his family." This answer satisfied the calculator.

"Fine! May your shadow never be less, but let us continue the analysis. We may assume that $w \geq v \geq u > 1$ since birth rites allow shares to be larger or lesser. But could u be 4 or more?"

"I hope not," continued the calculator, "because the u^{th} part would be too small, unworthy of the respect showed to his late father. Ah yes, if $u \geq 4$ then

$$3w \geq u + v + w = (u \Leftrightarrow 1)(v \Leftrightarrow 1)(w \Leftrightarrow 1) \geq 3 \cdot 3(w \Leftrightarrow 1)$$

giving $9 \geq 6w$, which cannot be, since then $w = 1$, but then he would take the entire herd, inciting hatred in the thoughts of the other two, as forewarned by the incontestable cadi."

"So we must have $u \leq 3$. Let us then see what happens if $u = 3$. Here $w \geq v \geq u = 3$ gives

$$3 + 2w \geq u + v + w = (u \Leftrightarrow 1)(v \Leftrightarrow 1)(w \Leftrightarrow 1) \geq 2 \cdot 2(w \Leftrightarrow 1);$$

that is, $7 \geq 2w$, giving $w = 1$, 2, or 3. The cases $w = 1, 2$ are excluded since $w \geq u = 3$, leaving $3 = w \geq u = 3$; that is, all shares equal, $u = v = w = 3$. But this does not satisfy the original equation and must be dropped."

The dropping of the equal shares possibility came as a relief to all. It must have been Almighty's wish since not all three sons deserved an equal share. One of the three was certainly more praiseworthy spiritually, as he attended prayers and consulted often the holy book.

"Last but not least we have to analyze the possibility $u = 2$." Everybody listened carefully, especially potential brides, because $u = 2$ meant that one son would take half the herd. That is, as much as the other two together. Wise is the Lord!

"So we have

$$2 + v + w = 1(v \Leftrightarrow 1)(w \Leftrightarrow 1)$$

which, after some Al-jabr, gives

$$(v \Leftrightarrow 2)w = 2v + 1."$$

At this point the problem solver interrupted again. "Observe," he said, "if $v = 2$ then the left hand side gives as-sifr, which is incompatible with the right, so this case must be dropped."

"I will do as you say," replied the calculator, "although I think there is a deeper reason for that. Harmony with nature does not allow $u = v = 2$ because the original equation then becomes

$$4 + w = w \Leftrightarrow 1$$

and, if anything is added to 4, be it something of substance or void, you get at least 4 more than the addend, and not one less than the addend."

"We, therefore, have," he continued:

$$\text{``}w = \frac{2v + 1}{v \Leftrightarrow 2} = 2 + \frac{5}{v \Leftrightarrow 2}$$

This is a difficult situation. How can an integer equal a fraction? Only when what seems a fraction is not really a fraction but an integer concealed. We are rescued from this difficult situation by appealing to the ideas of Diophantus. I am so glad I have a recent manuscript with a translation of his eternal book, because the original Greek is too difficult."

"This is how he approaches such problems: The denominator $v \Leftrightarrow 2$ must be a divisor of 5, a sacred number, the number of Platonic solids and the length of the hypotenuse of the eternal triangle. Divine wisdom arranged that 5 has the prime property that it possesses precisely two divisors, unity and itself. So v is either 3 or 7. It cannot be 7 because w would then be 3, a smaller number. This leaves $v = 3$ and $w = 7$."

"Thus the total number of camels in the original herd is 41. One camel is left over for our guest, which he can have back, as it is not counted in the 41. The shares are 21, 14, and 6 respectively," concluded the calculator boastfully.

Everybody applauded the sagacity of this artful manipulator of numbers whose eurhythmic mind interpreted the inheritance laws with the infallible ways of the mathematician.

The distinguished problem solver smiled to himself. He had succeeded again. In a true Socratic manner he led the dialogue by giving imperceptible hints, the ***CRUX*** of the matter, to his counterpart who then discovered the truth that had been known to the problem solver from the very beginning. He saddled his camel, thanked for the hospitality and the knowledge he acquired, savoured a sip of water and left for the next stage of his Promethean journey.

Also solved or answered by SAM BAETHGE, Nordheim, Texas, USA; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; MANSUR BOASE, student, St. Paul's School, London, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines and GIOVANNI MAZZARELLO, Ferrovie dello Stato, Florence, Italy; DAVID HANKIN, Hunter College Campus Schools, New York, NY, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; D. KIPP JOHNSON, Beaverton, Oregon, USA; GERRY LEVERSHA, St Paul's School, London, England; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; REZA SHAHIDI, student, University of Waterloo, Waterloo, Ontario; D.J. SMEENK, Zaltbommel, the Netherlands; DAVID STONE and VREJ ZARIKIAN, Georgia Southern University, Statesboro, Georgia; PANOS E. TSAOUSSOGLOU, Athens, Greece; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA; and the proposer. There were two incomplete solutions.

Lambrou also considers the general problem where the number of camels is $p \Leftrightarrow 1$, *and* u, v, w *all divide evenly into* p. *In addition to the solution given above he finds* 13 *other solutions* (u, v, w, p), *where* $w \leq v \leq w$: $(2,3,8,24)$, $(2,3,9,18)$, $(2,3,10,15)$, $(2,3,12,12)$, $(2,4,5,20)$, $(2,4,6,12)$, $(2,4,8,8)$, $(2,5,5,10)$, $(2,6,6,6)$, $(3,3,4,12)$, $(3,3,6,6)$, $(3,4,4,6)$, $(4,4,4,4)$.

2229. [1997: 167] *Proposed by Kenneth Kam Chiu Ko, Mississauga, Ontario.*

(a) Let m be any positive integer greater than 2, such that $x^2 \equiv 1 \pmod{m}$ whenever $(x, m) = 1$.

Let n be a positive integer. If $m|n+1$, prove that the sum of all divisors of n is divisible by m.

(b)* Find all possible values of m.

Solution by Kee-Wai Lau, Hong Kong (modified by the editor).

(a) We first show that n cannot be a perfect square.

Suppose that $n = k^2$. Then $k^2 \equiv -1 \pmod{m}$. But $k|n$, $m|n+1$ and $(n, n+1) = 1$ together imply that $(k, m) = 1$, and so, $k^2 \equiv 1 \pmod{m}$. Thus $1 \equiv -1 \pmod{m}$, which is false since $m > 2$. Therefore, all the divisors of n can be grouped into pairs (s, t), where $st = n$ and $s \neq t$. It then suffices to show that $m|(s+t)$. As above, $(s, m) = 1$ implies that $s^2 \equiv 1 \pmod{m}$. Adding $st \equiv -1 \pmod{m}$, we have that $s(s+t) \equiv 0 \pmod{m}$, or $s + t \equiv 0 \pmod{m}$.

(b) We show that the possible values of m are precisely 3, 4, 6, 8, 12, and 24.

For each m such that $3 \leq m \leq 24$, direct checking of those x with $1 < x < m$ and $(x, m) = 1$ reveals that these values are indeed the only ones that satisfy the described condition.

Assume then that $m > 24$ and let p_1, p_2, p_3, $\ldots$, denote the sequence of prime numbers. Then $2|m$, for otherwise $(2, m) = 1$ implies that $2^2 \equiv 1 \pmod{m}$, which is false. Similarly, $3|m$ and $5|m$. If $(7, m) = 1$, then $7^2 \equiv 1 \pmod{m}$, or $m|48$, which is impossible since $5|m$. Thus $7|m$.

Suppose that $p_i|m$ for all $i = 1, 2, \ldots, k$, for some $k \geq 4$. If $(m, p_{k+1}) = 1$, then $p_{k+1}^2 \equiv 1 \pmod{m}$, which implies that $p_{k+1}^2 > \prod_{i=1}^{k} p_i$. However, this contradicts the Bonse Inequality, which states that for all $k \geq 4$, $p_{k+1}^2 < \prod_{i=1}^{k} p_i$. (See, for example, chapter 27 of *The Enjoyment of Mathematics* by H. Rademacher and O. Toeplitz; Dover, 1990.)

It follows that $p_{k+1}|m$ and so m is divisible by any prime, which is clearly impossible. This shows that if $m > 24$, we cannot have $x^2 \equiv 1 \pmod{m}$ whenever $(x, m) = 1$, and the proof is complete.

Also solved by ADRIAN BIRKA, student, Lakeshore Catholic High School, Port Colbourne, Ontario; MANSUR BOASE, student, St. Paul's School, London, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; D. KIPP JOHNSON, Beaverton, Oregon, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; and HEINZ-JÜRGEN SEIFFERT, Berlin, Germany.

Part (a) *only was solved by JIMMY CHUI, student, Earl Haig Secondary School, North York, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SEAN MCILROY, student, University of British Columbia, Vancouver, BC; JOEL SCHLOSBERG, student, Hunter College High School, New York, NY, USA; and the proposer.*

Regarding the solution to (b), *the Bonse Inequality was also used, explicitly or implicitly, by Boase, Bradley, Herzig and Seiffert. This inequality is an easy consequence of Bertrand's Postulate as shown by Herzig and Seiffert. Johnson gave a solution using Bertrand's Postulate directly.*

2230. [1997: 167] *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

Triangles BCD and ACE are constructed outwardly on sides BC and CA of triangle ABC such that $AE = BD$ and $\angle BDC + \angle AEC = 180°$. The point F is chosen to lie on the segment AB so that

$$\frac{AF}{FB} = \frac{DC}{CE}.$$

Prove that

$$\frac{DE}{CD + CE} = \frac{EF}{BC} = \frac{FD}{AC}.$$

Solution by Toshio Seimiya, Kawasaki, Japan.

Let G be a point on AE produced beyond E such that $\angle ECG = \angle DCB$. Since $\angle CEG = 180° \Leftrightarrow \angle AEC = \angle CDB$, we have $\triangle CEG \sim \triangle CDB$, (directly similar) from which we have $\triangle CBG \sim \triangle CDE$. Thus

$$\angle BGC = \angle DEC. \qquad (1)$$

Since $\frac{AF}{FB} = \frac{CD}{CE} = \frac{BD}{EG} = \frac{AE}{EG}$, we get $FE \parallel BG$, so that

$$\angle AGB = \angle AEF. \qquad (2)$$

Hence we have from (1) and (2)

$$\begin{aligned} \angle FED &= \angle AEC \Leftrightarrow (\angle AEF + \angle DEC) \\ &= \angle AEC \Leftrightarrow (\angle AGB + \angle BGC) \\ &= \angle AEC \Leftrightarrow \angle AGC \\ &= \angle ECG \\ &= \angle BCD. \end{aligned} \qquad (3)$$

Similarly we have

$$\angle FDE = \angle ACE. \qquad (4)$$

Let H be a point on CD produced beyond D such that $DH = EC$. Since $BD = AE$ and $\angle BDH = 180° \Leftrightarrow \angle BDC = \angle AEC$, we have

$$\triangle BDH = \triangle AEC,$$

so that $BH = AC$, and $\angle BHD = \angle ACE$.

As $\angle FED = \angle BCD = \angle BCH$, and $\angle FDE = \angle ACE = \angle BHD = \angle BHC$, we have $\triangle FDE \sim \triangle BHC$.

Thus we get

$$\frac{DE}{HC} = \frac{EF}{BC} = \frac{FD}{BH}.$$

Since $HC = CD + DH = CD + CE$, and $BH = AC$, we have

$$\frac{DE}{CD + CE} = \frac{EF}{BC} = \frac{FD}{AC}.$$

Also solved by FLORIAN HERZIG, student, Perchtoldsdorf, Austria; ISTVÁN REIMAN, Budapest, Hungary; and the proposer.