

THE ACADEMY CORNER

No. 17

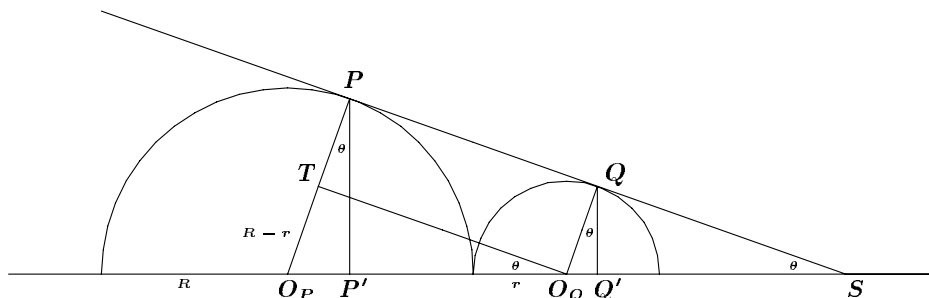
Bruce Shawyer

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This month, we present another solution to a university entrance scholarship examination paper from the 1940's, which appeared in the April 1997 issue of *CRUX with MAYHEM* [1997: 129].

10. Two unequal circles of radii R and r touch externally, and P and Q are the points of contact of a common tangent to the circles, respectively. Find the volume of the frustrum of a cone generated by rotating PQ about the line joining the centres of the circles.

Solution.



First, we note that triangle $\triangle TO_P O_Q$ has a right angle at T , with $O_P O_Q = R + r$ and $O_P T = R - r$. Hence $O_Q T = 2\sqrt{Rr}$.

Because of parallel and perpendicular lines, all of angles $\angle P S O_P$, $\angle T O_Q O_P$, $\angle O_P P P'$ and $\angle O_Q Q Q'$ are equal – we denote the common value by θ . From triangle $\triangle O_P O_Q T$, we note that

$$\sin \theta = \frac{R - r}{R + r} \quad \text{and} \quad \cos \theta = \frac{2\sqrt{Rr}}{R + r}.$$

Using various right triangles, we obtain:

$$\begin{aligned} OP'P' &= R \sin \theta = \frac{R(R-r)}{r+r}, \\ PP' &= R \cos \theta = \frac{2R\sqrt{Rr}}{R+r}, \\ P'S &= PP' \cot \theta \\ &= \frac{2R\sqrt{Rr}}{R+r} \frac{2\sqrt{Rr}}{R-r} \\ &= \frac{4rR^2}{R^2-r^2}, \end{aligned}$$

$$\begin{aligned} OQQ' &= r \sin \theta = \frac{r(R-r)}{r+r}, \\ QQ' &= r \cos \theta = \frac{2r\sqrt{Rr}}{R+r}, \\ Q'S &= QQ' \cot \theta \\ &= \frac{2r\sqrt{Rr}}{R+r} \frac{2\sqrt{Rr}}{R-r} \\ &= \frac{4r^2R}{R^2-r^2}. \end{aligned}$$

Hence

$$\begin{aligned} V &= \frac{\pi}{3} \left[(PP')^2 P'S - (QQ')^2 Q'S \right] \\ &= \frac{\pi}{3} \left[\frac{4R^3r}{(R+r)^2} \frac{4rR^2}{R^2-r^2} - \frac{4Rr^3}{(R+r)^2} \frac{4r^2R}{R^2-r^2} \right] \\ &= \frac{16\pi R^2 r^2 (R^3 - r^3)}{3(R+r)^2 (R^2 - r^2)} \\ &= \frac{16\pi R^2 r^2 (R^2 + Rr + r^2)}{3(R+r)^3}, \end{aligned}$$

which is the required volume.



11. Prove that

$$\begin{aligned} \sin^2(\theta + \alpha) + \sin^2(\theta + \beta) - 2 \cos(\alpha - \beta) \sin(\theta + \alpha) \sin(\theta + \beta) \\ = \sin^2(\alpha - \beta). \end{aligned} \quad (1)$$

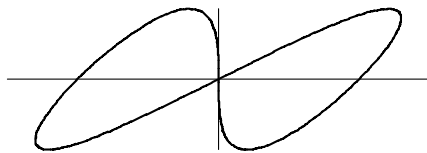
Solution.

$$\begin{aligned} & 2 \sin^2(\theta + \alpha) + \sin^2(\theta + \beta) \\ & \neq 2 - (\cos(2\theta + 2\alpha) + \cos(2\theta + 2\beta)) \\ & = 2 - 2 \cos(2\theta + \alpha + \beta) \cos(\alpha - \beta), \\ & \quad - 4 \cos(\alpha - \beta) \sin(\theta + \alpha) \sin(\theta + \beta) \\ & = -2 \cos(\alpha - \beta) (\cos(\alpha - \beta) - \cos(2\theta + \alpha + \beta)), \end{aligned}$$

so that

$$\begin{aligned} & 2 \times \text{LHS of (1)} \\ & = 2 \cancel{1} - \cos(2\theta + \alpha + \beta) \cos(\alpha - \beta) \\ & \quad \cancel{- \cos^2(\alpha - \beta)} + \cos(\alpha - \beta) \cos(2\theta + \alpha + \beta) \\ & = 2 \cancel{1} - \cos^2(\alpha - \beta) \\ & = 2 \sin^2(\alpha - \beta). \end{aligned}$$

What is the equation of this curve?



THE OLYMPIAD CORNER

No. 188

R.E. Woodrow

All communications about this column should be sent to Professor R. E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

This number we begin with the problems of The Final Round of the Japan Mathematical Olympiad. My thanks go to Richard Nowakowski for collecting these when he was Canadian Team Leader at the 35th IMO in Hong Kong.

JAPAN MATHEMATICAL OLYMPIAD

Final Round

February 1994 — 4 hours

1. For positive integer n , let a_n be the nearest positive integer to \sqrt{n} , and let $b_n = n + a_n$. Dropping all b_n ($n = 1, 2, \dots$) from the set of all positive integers N , we get a sequence of positive integers in ascending order $\{c_n\}$. Represent c_n by n .

2. Five points are located on the plane. Any three of those points are not collinear. Let l_1, \dots, l_{10} be the length of the ten segments obtained by joining every two points of the five points. Assume that l_1^2, \dots, l_9^2 are rational numbers. Prove that l_{10}^2 is also a rational number.

3. There is a triangle $A_0A_1A_2$ and seven points P_0, \dots, P_6 on the plane. Assume that any P_i and P_{i+1} are symmetric with center A_k , where k is the remainder of i divided by 3.

(a) Prove that $P_0 = P_6$.

(b) Describe the possible position of P_0 under the additional assumption that every segment connecting P_i and P_{i+1} does not intersect with the interior of the triangle $A_0A_1A_2$.

4. We consider a triangle ABC such that $\angle MAC = 15^\circ$, where M is the midpoint of BC . Determine the possible maximum value of $\angle B$.

5. There are N persons and N pieces of lot cards. Each number 1 through N is written on a card. When the N persons draw these cards, their order is determined by the numbers on their cards. After repeating this draw two times, we give gifts by the following rule.

Rule: A person X gets the gift, if there is no person Y such that Y is prior to X both times. Otherwise X cannot get the gift.

For example, if X is at the top in the first lot, X always gets the gift whatever he draws in the second lot.

Then determine the expected number of persons who get gifts.

As a second Olympiad for your puzzling pleasure we give problems of the 30th Spanish Mathematical Olympiad, First Round, November 26–27, 1993. My thanks go to Richard Nowakowski, Canadian Team Leader to the 35th IMO in Hong Kong and to Francisco Bellot Rosado, I. B. Emilio Ferrari, Valladolid, Spain, for copies of the contest.

30th SPANISH MATHEMATICAL OLYMPIAD
First Round — November 26–27, 1993
 Proposed by the Royal Spanish Mathematical Society

Time allowed: 4 hours each day. Each problem carries 10 points.

First Day

1. Show that, for all $n \in \mathbb{N}$, the fractions

$$\frac{n-1}{n}, \quad \frac{n}{2n+1}, \quad \frac{2n+1}{2n^2+2n}$$

are irreducible.

2. A sphere of radius R , and a right cone with base a meridian of the sphere and vertex external to the sphere, are given. Find the radius of the circle of intersection of the sphere and the cone, given that the volume of the cone is half of the volume of the sphere.

3. Solve the following system of equations:

$$x \cdot |x| + y \cdot |y| = 1, \quad [x] + [y] = 1,$$

in which $|t|$ and $[t]$ represent the absolute value and the integer part of the real number t .

4. Let AD the internal bisector of the triangle ABC ($D \in BC$), E the point symmetric to D with respect to the midpoint of BC , and F the point of BC such that $\angle BAF = \angle EAC$. Show that $\frac{BF}{FC} = \frac{c^3}{b^3}$.

Second Day

5. Find all the natural numbers n such that the number

$$n(n+1)(n+2)(n+3)$$

has exactly three prime divisors.

6. An ellipse is drawn taking as major axis the biggest of the sides of a given rectangle, such that the ellipse passes through the intersection point of the diagonals of the rectangle.

Show that, if a point of the ellipse, external to the rectangle, is joined to the extreme points of the opposite side, then three segments in geometric progression are determined on the major axis.

7. Let $a \in \mathbb{R}$ given. Find the real numbers x_1, \dots, x_n which satisfy the system of equations

$$\begin{aligned} x_1^2 + ax_1 + \left(\frac{a-1}{2}\right)^2 &= x_2 \\ x_2^2 + ax_2 + \left(\frac{a-1}{2}\right)^2 &= x_3 \\ \dots\dots\dots &\dots\dots\dots \\ x_{n-1}^2 + ax_{n-1} + \left(\frac{a-1}{2}\right)^2 &= x_n \\ x_n^2 + ax_n + \left(\frac{a-1}{2}\right)^2 &= x_1. \end{aligned}$$

8. (The Sisyphus's myth) There are 1001 steps going up, with rocks on some of them (no more than 1 rock on each step). Sisyphus may pick any rock and raise it one or more steps up to the nearest empty step. Then his opponent Hades rolls a rock (with an empty step directly below it) down one step. There are 500 rocks, originally located on the first 500 steps. Sisyphus and Hades move rocks in turn, Sisyphus making the first move. His goal is to place a rock on the top step. Can Hades stop him?

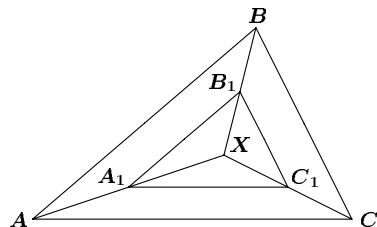


We now turn to readers' solutions received before January 1st, to problems for consideration by the International Jury at the 36th IMO in Canada [1996: 299–301].

4. Let A, B and C be non-collinear points. Prove that there is a unique point X in the plane of ABC such that $XA^2 + XB^2 + AB^2 = XB^2 + XC^2 + BC^2 = XC^2 + CA^2 + CA^2$.

Solutions by D.J. Smeenk, Zaltbommel, the Netherlands; and by Vasiliou Meletis, Elefsis, Greece. We give Meletis' solutions.

First Solution.



From the hypothesis we have

$$AX^2 + AB^2 = CX^2 + CB^2. \quad (1)$$

If B_1 is the midpoint of BX , applying the first theorem of the median in the triangles $\triangle ABX$, $\triangle CBX$ we get

$$\begin{aligned} 2AB_1^2 + 2BB_1^2 &= 2CB_1^2 + 2BB_1^2 \quad \text{or} \\ AB_1 &= CB_1. \end{aligned} \quad (2)$$

This indicates that the perpendicular bisector of the side AC passes through the point B_1 . Let A_1 , C_1 be the midpoints of AX and CX , respectively.

Similarly, we obtain that the perpendicular bisectors of BC and AB pass through the midpoints A_1 and C_1 , respectively. (3)

Furthermore we obtain $AB \parallel A_1B_1$, $AC \parallel A_1C_1$ and $BC \parallel B_1C_1$. (4)

From (3) and (4) we get that the circumcentre O of ABC is the orthocentre H_1 of $A_1B_1C_1$. (5)

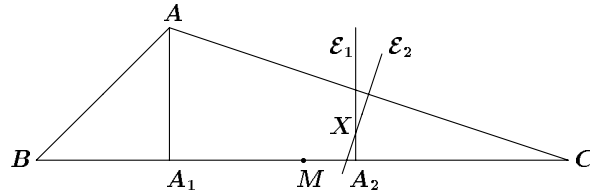
Also from (4) the triangles ABC and $A_1B_1C_1$ are similar with X the centre of similarity and ratio $\frac{1}{2}$. (6)

So, their orthocentres H and H_1 lie in the same straight line with the point X and $HH_1 = H_1X$. (7)

Combining (5) and (7) we get $HO = OX$; that is the point X is known (constant), because X is symmetric to H with respect to the orthocentre O of ABC .

Second Solution.

The conditions of the problem are equivalent to the system of equations



$$XB^2 - XC^2 = AC^2 - AB^2 \quad (1)$$

$$XC^2 - XA^2 = BA^2 - BC^2. \quad (2)$$

Now, taking equation (1) gives a locus of points X satisfying the condition. The relation reminds us of the second theorem of the median in a triangle.

Let AA_1 , XA_2 be the altitudes of the triangles ABC and XBC respectively on side BC (extended). Let M be the midpoint of the side BC .

If we suppose

$$AB \leq AC \leq BC, \quad (3)$$

for illustration, we get

$$XC \leq XB \leq XA,$$

and furthermore the point M lies between the points A_1 and A_2 (4)

But

$$\begin{aligned}XB^2 - XC^2 &= 2BC \cdot MA_2, \text{ and} \\AC^2 - AB^2 &= 2BC \cdot MA_1.\end{aligned}$$

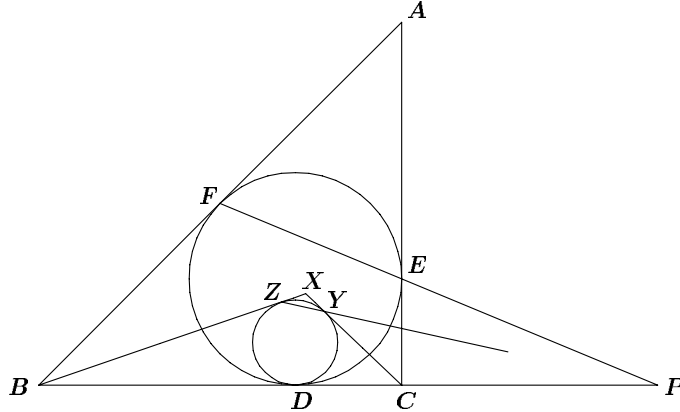
Hence $MA_1 = MA_2$ and A_2 is a constant point on BC because it is symmetric to A_1 with respect to the midpoint M .

Consequently, if (1) holds, the point X lies on the line \mathcal{E}_1 perpendicular to BC at A_2 . Similarly, if (2) holds, the point X lies on the line \mathcal{E}_2 perpendicular to AC at B_2 (where $BB_1 \perp AC$ and $AB_1 = CB_2$).

Hence, the required point X lies at the intersection of \mathcal{E}_1 and \mathcal{E}_2 .

5. The incircle of ABC touches BC , CA and AB at D , E and F respectively. X is a point inside ABC such that the incircle of XBC touches BC at D also, and touches CX and XB at Y and Z , respectively. Prove that $EFZY$ is a cyclic quadrilateral.

Solutions by Toshio Seimiya, Kawasaki, Japan; and by Vasiliou Meletis, Elefsis, Greece. We give Seimiya's argument.



Let P be the intersection of EF with BC . Then by Menelaus' Theorem we have

$$\frac{BP}{PC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1. \quad (1)$$

Since $CE = CD$, $EA = AF$, and $FB = BD$, we get

$$\frac{BP}{PC} \cdot \frac{CD}{BD} = 1$$

so that

$$\frac{BP}{PC} = \frac{BD}{CD}. \quad (2)$$

Since $XZ = XY$, $BZ = BD$ and $CY = CD$, we have from (2)

$$\frac{BP}{PC} \cdot \frac{CY}{YX} \cdot \frac{XZ}{ZB} = \frac{BD}{CD} \cdot \frac{CD}{YX} \cdot \frac{XY}{BD} = 1.$$

Hence by Menelaus' Theorem P , Z and Y are collinear. Since $PF \cdot PE = PD^2$ and $PZ \cdot PY = PD^2$ we have $PF \cdot PE = PZ \cdot PY$.

Hence $EFZY$ is a cyclic quadrilateral.

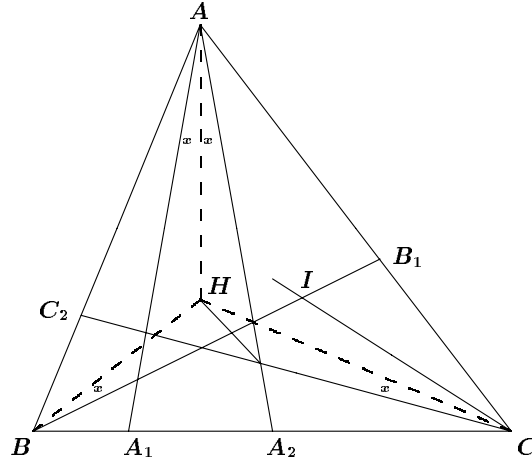
Comment. If $AB = AC$ then $BD = DC$ and then it can easily be proved that AD is the perpendicular bisector of EF and YZ so that $EFZY$ is an isosceles trapezoid, and is a cyclic trapezoid.

6. An acute triangle ABC is given. Points A_1 and A_2 are taken on the side BC (with A_2 between A_1 and C), B_1 and B_2 on the side AC (with B_2 between B_1 and A) and C_1 and C_2 on the side AB (with C_2 between C_1 and B) so that

$$\angle AA_1A_2 = \angle AA_2A_1 = \angle BB_1B_2 = \angle BB_2B_1 = \angle CC_1C_2 = \angle CC_2C_1.$$

The lines AA_1 , BB_1 , and CC_1 bound a triangle, and the lines AA_2 , BB_2 and CC_2 bound a second triangle. Prove that all six vertices of these two triangles lie on a single circle.

Solutions by Toshio Seimiya, Kawasaki, Japan; by D.J. Smeenk, Zaltbommel, the Netherlands; and by Vasiliou Meletis, Elefsis, Greece. We give Meletis' solution.



Let AA_1 , BB_1 meet at the point E ; AA_1 , CC_2 meet at the point F ; and BB_1 , CC_1 meet at the point I . Also

$$\angle A_1AA_2 = \angle B_1BB_2 = \angle C_1CC_2 = 2x. \quad (1)$$

The bisectors of the angles at A_1 , B_1 and C in triangles $\triangle A_1AA_2$, $\triangle B_1BB_2$ and $\triangle C_1CC_2$ respectively are perpendicular to their respective

bases. Hence they are the altitudes of $\triangle ABC$. Let H be the orthocentre of $\triangle ABC$.

Since $\angle A_1AH = \angle B_1BH = x$ and $\angle A_1AH = \angle C_1CH = x$ each one of the quadrilaterals $AHEB$, $AHDC$ is inscribable in a circle.

These two circles have a common chord, the segment AH and since $\angle ABH = \angle ACH = 90^\circ - \angle BAC$, then the circles have equal radii.

Thus, since the inscribed angles $\angle EAH$, $\angle DAH$ are equal, the corresponding chords HE and HD are equal.

Therefore $HE = HD$. Similarly, we prove that $HD = HI$, and so on for all six vertices of these two triangles of the problem.

Thus, all six vertices lie at the same distance from the point H , and the points are concyclic.

Next we give a rather novel solution by Meletis to the second problem of the 37th IMO itself [1996: 303].

2. Let P be a point inside triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let D, E be the incentres of triangles APB, APC respectively. Show that AP, BD and CE meet at a point.

Solution by Meletis Vasiliou, Elefsis, Greece.

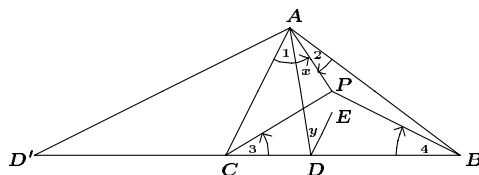


Figure 1.

We will first prove the converse of the proposition and then apply it to prove the stated problem.

Stage I. The equality $\angle APB - \angle APC = \angle ACB - \angle ABC$ is equivalent to

$$\angle A_1 - \angle A_2 = \angle C_3 - \angle B_4 \quad (1)$$

(see figure 1).

Stage II. Assume the conclusion of the problem; that is. assume that the bisectors of the angles $\angle PCA$ and $\angle PBA$ concur at a point I on AP . Then we have

$$\frac{AC}{PC} = \frac{AI}{PI} = \frac{AB}{PB}$$

or

$$\frac{AC}{AB} = \frac{PC}{PB}. \quad (2)$$

This ratio indicates that if AD is the bisector of the angle $\angle BAC$, then PD is the bisector of the angle $\angle BPC$, or equivalently

“the points A, P belong to the “circle of Apollonius” whose diameter DD' lies on the line CB , with D, D' harmonic conjugates to the points C, B ” (3)

Stage III. If (3) holds then (1) holds. Since AD bisects $\angle CAB$, then

$$\angle PAD = x = \frac{\angle A_1 - \angle A_2}{2}. \quad (4)$$

Since PD bisects $\angle CPB$ then

$$\angle PDB - \angle PDC = \angle C_3 - \angle B_4$$

and if we draw the bisector DE of the straight angle $\angle BDC$ we get

$$\angle PDE = y = \frac{\angle C_3 - \angle B_4}{2}. \quad (5)$$

Because of (3), and since $DE \perp CD$ we obtain that DE is tangent at D to the circumcircle of the triangle $\triangle APD$.

Hence $x = y$. (6)

Combining (4), (5) and (6) we get that

$$\angle A_1 - \angle A_2 = \angle C_3 - \angle B_4$$

so the converse of the proposition is true.

Stage IV. If (1) holds then (3) holds.

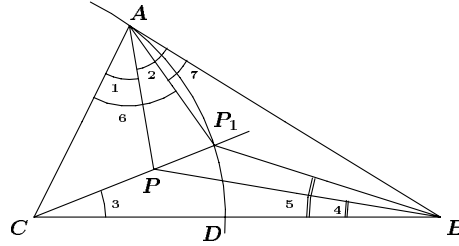


Figure 2.

Consider the circle of Apollonius with respect to the angle A of the triangle ABC with DD' as diameter. Let P_1 be the point at which the circle intersects line CP . We want to show that $P \equiv P_1$ to complete the proof.

Suppose $P \not\equiv P_1$. We distinguish two cases:

Case a. P_1 is external to the segment CP (see figure 2). Denote

$$\angle A_1 = \angle CAP, \quad \angle A_2 = \angle BAP,$$

$$\angle A_6 = \angle CAP_1, \quad \angle A_7 = \angle BAP_1,$$

$$\angle B_4 = \angle CBP \quad \text{and} \quad \angle B_5 = \angle CBP_1.$$

Then we have

$$\angle A_1 < \angle A_6 \tag{7}$$

$$\angle A_2 > \angle A_7 \tag{7}$$

$$\angle B_4 < \angle B_5. \tag{8}$$

So from the hypothesis and the conclusion of Stage III

$$\begin{aligned} \angle C_3 - \angle B_4 &= \angle A_1 - \angle A_2 \\ &< \angle A_6 - \angle A_7 \\ &= \angle C_3 - \angle B_5 \\ &< \angle C_3 - \angle B_4. \end{aligned}$$

or

$$\angle C_3 - \angle B_4 < \angle C_3 - \angle B_4,$$

a contradiction.

Case b. P_1 is internal to the segment CD . The proof is similar. This completes the proof.



To complete this number of the corner and our file of solutions submitted by the readers to problems from 1996 numbers of the Corner, we give one solution to a problem proposed to the jury but not used at the 36th IMO.

5. [1996: 348] 36th IMO Problems proposed to the jury and not used.

Let ABC be a triangle. A circle passing through B and C intersects the sides AB and AC again at C' and B' , respectively. Prove that BB' , CC' and HH' are concurrent, where H and H' are the orthocentres of triangles ABC and $AB'C'$ respectively.

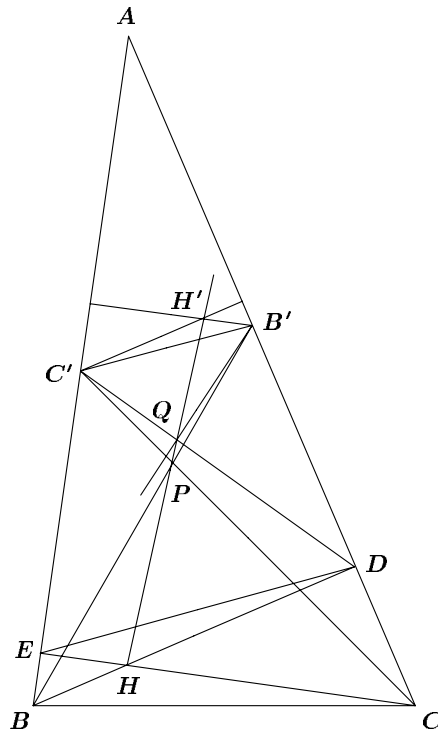
Solution by Toshio Seimiya, Kawasaki, Japan.

(See diagram on the next page.) Let BH , CH meet AC , AB at D , E respectively. Since $\angle BDC = \angle BEC = 90^\circ$, B , C , D , E are concyclic and thus $\angle ADE = \angle ABC$.

Now, since B , C , B' , C' are concyclic we have

$$\angle AB'C' = \angle ABC.$$

Thus $\angle ADE = \angle AB'C'$, so that $DE \parallel B'C'$. Let Q be the intersection of DC' with EB' . As $HE \perp AB$, $B'H' \perp AB$, we get $HE \parallel H'B'$. Similarly we have $HD \parallel H'C'$.



Since $DE \parallel C'B'$, $HD \parallel H'C'$ and $HE \parallel H'B'$, DC' , EB' and HH' are concurrent, so that H , Q , H' are collinear.

By Pappus' Theorem H , P , Q are collinear.

Hence H , P , Q , H' are collinear. So BB' , CC' , and HH' are concurrent.

That is the Corner for this issue! Send me your National and Regional Olympiad Contest materials for use in the Corner.

BOOK REVIEWS

Edited by ANDY LIU

From Erdős to Kiev: Problems of Olympiad Caliber, by Ross Honsberger, published by the Mathematical Association of America, 1996, ISBN# 0-88385-324-8, softcover, 257+ pages, \$31.00.
Reviewed by **Bill Sands**, University of Calgary, Calgary, Alberta.

This book is number 17 in the Dolciani Mathematical Expositions series of the MAA, and is the seventh in this series by Ross Honsberger. His previous book in this series, *More Mathematical Morsels* (reviewed in **Crux** on [1991: 235–236] by Andy Liu), contained mostly problems from **Crux**. The same is true of *From Erdős to Kiev* — and this time, all problems taken from **Crux** are accompanied by references (year and page number) to where in **Crux** the problem (and sometimes solution) appeared. Most of the **Crux** problems used here are from Rob Woodrow's Olympiad Corner columns from 1987 and 1988; thus they are mostly contest problems, some easy to find elsewhere, like the AIME and IMO contests, others quite obscure. The author has sometimes used solutions taken from **Crux**, duly crediting the original solvers, and other times has come up with his own, but in all cases he has written (or rewritten) the problems and solutions as he liked: a privilege not usually available to **Crux** editors! The result is a readable and enjoyable book, that every **Crux** fan will want to own.

Altogether the book has 89 problems collected into 46 unnumbered “chapters”, with no discernible reason for the order they are put in. Instead of a traditional index, at the end the problems are classified under three subjects (roughly: combinatorics, algebra/number theory, and geometry) with a one-line description of each problem. This reviewer can't single out many of the problems in this book for special mention — as the general editor of **Crux**, I was not as close to the material in the Olympiad Corners as I was to the regular Problems and Solutions columns. But Honsberger includes a few items from here too, and I recognized with pleasure one of Hidetosi Fukagawa's *Sangaku* problems on page 223 — one of several he submitted to **Crux** (and not the most striking one in my opinion) before publication of his book *Japanese Temple Geometry Problems* with Dan Pedoe in 1989.

Of course, none of you need to be sold on the virtues of **Crux**! One hopes, though, that other readers of this book would thus be drawn to subscribe to **Crux**, and maybe some of them will, but wouldn't it have been appropriate, given the book's debt to **Crux**, if the words “**Crux Mathematicorum**” had been more visible? Yes, **Crux** is acknowledged, and praised, in the Preface, and the book is even dedicated to **Crux**'s late founders Léo Sauvé and Fred Maskell, which is a thoughtful touch and is appreciated. But why couldn't **Crux** get some mention in the title or at least somewhere on the

cover? And the MAA's advertising for this book, as far as I have seen, does not mention **Crux** either. Come on, MAA — fair's fair!

As for other criticisms — well, we could start with the accent on the “o” of Erdős, which is wrong (I've used the incorrect umlaut till now in this review!), in the title and elsewhere in the book too. (As an aside, I might note that the title, though catchy, isn't especially appropriate, in that the book neither begins nor ends with a problem involving Erdős or Kiev; however, there are Erdős problems, and a problem from the Kiev Olympiad, in the book. Picky, picky)

I didn't notice many misprints or weaknesses of exposition liable to slow readers down. On the bottom of page 242 and the top of page 243, there are two displayed equations in which most of the terms have been left out, but the reader can probably reconstruct these with little trouble. On pages 116–117, to show that c_n is not a multiple root of the polynomial $P_n(x) = 0$, it is much simpler just to differentiate the polynomial $Q_n(x)$, which is a factor of $P_n(x)$; we get

$$(n + 1)x^n + nx^{n-1} + \cdots + 1$$

which is obviously positive for $x = c_n > 0$. Thus c_n is a single root of $Q_n(x)$ and so also of $P_n(x)$.

On page 107, Daniel Ropp's university should be Washington University, not Washington State University (the correct name is given in **Crux**). In oddly similar typos, the zeds in Bruce Reznick's last name and Zvi Margaliot's first name are replaced by s's, on pages 179 and 187 respectively.

In the second half of the book there is a rash of minor errors involving **Crux** references; for the benefit of readers rather than as criticism, I'll list them here. On page 153, the reference given [1987: 120] is to the original publication of the problem; the solution actually appeared on [1988: 182]. On page 159, the year for this reference should be 1988, not 1987. Also, the solution for this problem appeared on [1994: 191–193] with a further comment on [1995: 82] (both probably too recent to be picked up by Honsberger's searches). On page 167, the solution for this problem appeared on [1988: 199]. On page 177, again the year of publication of the original **Crux** problem should be 1988 not 1987, and the solution appeared on page 267 of the 1989 volume, not page 269. And by the way, these references all refer to problem 1 of this chapter, which contains two problems. On page 239, the page reference given as 491 should be just 49.

Having a whole chapter on Olympiad Corner solutions by George Evagelopoulos, the first chapter in fact, was, I think, a mistake. Some of these solutions were equally due to other readers, as reported in **Crux** at the time. In fact, problem 1 of this chapter is listed as coming from the 1983 Australian Olympiad, which indeed it does, but the **Crux** reference given [1985: 71] is to the same problem as proposed by Brazil (but not used) at some IMO. The correct reference for the Australian problem is [1983: 173], with

the *official* solution as supplied by Peter O'Halloran published on [1986: 22]. This is the same solution that Honsberger gives in his book and attributes to Evagelopoulos. Only on [1987: 43] is the Brazil version of the problem wrapped up, and here no solution is published, only the solvers are listed. (And, as Honsberger mentions, these include two others as well as Evagelopoulos.) Strangely, in two other cases Honsberger fails to mention solvers given equal credit in *Crux* for solutions he uses and ascribes to Evagelopoulos alone. The solution to problem 4 was credited (on [1989: 230]) also to Zun Shan and Ed Wang; what Honsberger calls a "brilliant observation" could just as well be attributed to them. And for problem 7, in the published solution in *Crux* [1990: 105] Duane Broline is also listed as a solver. There are also remarkable similarities between some of Evagelopoulos's solutions and earlier solutions published elsewhere. For example, for problem 3, a problem from the Russian journal *Kvant* and first published in *Crux* in 1988, Evagelopoulos's original solution in *Crux* [1990: 104] contained the same square diagram, complete with the same cells labelled "A" and "B", as is present in the *Kvant* solution (see page 24 of issue 12 of the 1987 *Kvant*); no doubt the *Kvant* solution is a "beautiful gem", as Honsberger calls Evagelopoulos's solution on page 5. And for problem 6, also from *Kvant*, Evagelopoulos's solution in *Crux* [1990: 102] contained the same terminology ("representative", translated from the Russian) and the same notation (" \sim ") as in the solution published on page 25 of issue 9 of the 1987 *Kvant*, to mention only two of the amazing resemblances between these two solutions.

The last chapter contains an exposition with proof of the power mean inequality, and so is of a different character from the rest of the book. While this is a useful inequality to know, it's a bit jarring to have it suddenly appear here, especially when there is only one small, brief, unattributed example given of a problem that can be solved with it. There must be lots of problems from *Crux* that could have been used. However, let none of the above reservations prevent anyone from buying this book.

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ONE PROBLEM – SIX SOLUTIONS

Georg Gunther

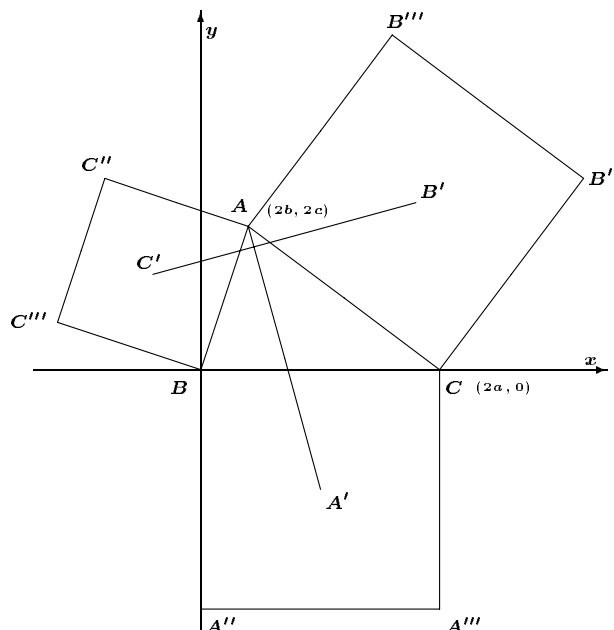
Sir Wilfred Grenfell College, Corner Brook, Newfoundland

The three ingredients necessary to solve a problem are insight, persistence and technique. In the following, all three of these play a role as six different solutions to the following problem are presented.

The Problem:

On the three sides of triangle ABC , construct squares facing outwards. Let A' , B' , C' be the centres of the squares constructed on sides BC , CA and AB , respectively. Prove that $\text{dist}(A, A') = \text{dist}(B', C')$ and $AA' \perp B'C'$.

Solution 1: (Analytic Geometry)

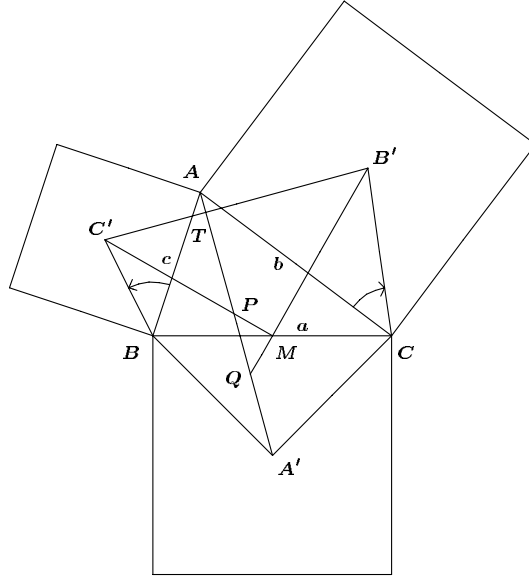


It is important to make the coordinates work for you. Place the origin at B , and the x -axis along BC . Let $C = (2a, 0)$ and $A = (2b, 2c)$. Then label the vertices of the three squares as shown.

It is easy to show that $B'' = (2a+2c, 2a-2b)$, $A'' = (0, -2a)$ and $C''' = (-2c, 2b)$. Now we quickly obtain $A' = (a, -a)$, $B' = (a+b+c, a-b+c)$ and $C' = (b-c, b+c)$.

Now, we have that $\text{dist}(A, A') = \sqrt{(2b-a)^2 + (2c+a)^2} = \text{dist}(B', C')$,
 $\text{slope } AA' = \frac{2c+a}{2b-a}$ and $\text{slope } B'C' = -\frac{2b-a}{2c+a}$, showing that the lines are
perpendicular.

Solution 2: (Geometry)



Let M be the midpoint of BC . Consider triangles ABA' and $C'BM$. We have $\widehat{ABA'} = \widehat{B} + 45^\circ = \widehat{C'BM}$, $\overline{AB} = c$, $\overline{C'B} = \frac{c}{\sqrt{2}}$, $\overline{BA'} = \frac{a}{\sqrt{2}}$ and $\overline{BM} = \frac{a}{2}$.

So these triangles are similar, and $\triangle C'BM$ is obtained from $\triangle ABA'$ by rotating through 45° about B , and scaling by a factor of $\frac{1}{\sqrt{2}}$.

This means that $\overline{C'M} = \frac{\overline{AA'}}{\sqrt{2}} = \frac{p}{\sqrt{2}}$, and that $\widehat{APC'} = 45^\circ$. Likewise, triangles ACA' and $B'CM$ are similar, and $\triangle B'CM$ is obtained from $\triangle ACA'$ by rotating through an angle of -45° about C and dilating by $\frac{1}{\sqrt{2}}$.

Hence $\overline{B'M} = \frac{p}{\sqrt{2}}$ and $\widehat{MQP} = 45^\circ$.

But this now tells us that $\widehat{B'MC'} = 90^\circ$, and so

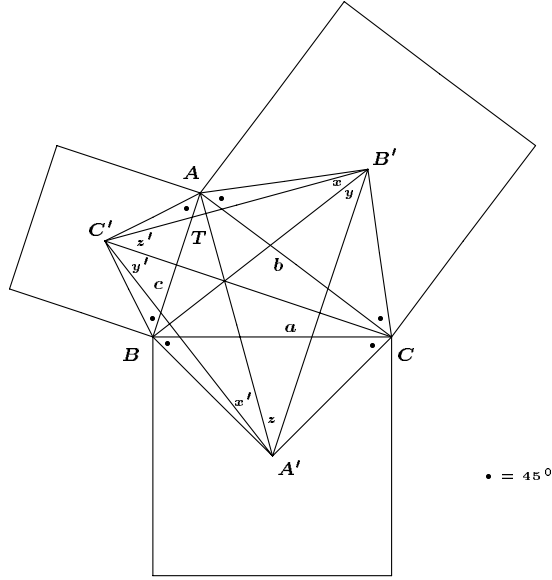
$$\overline{C'B'}^2 = \overline{C'M}^2 + \overline{B'M}^2 = \frac{p^2}{2} + \frac{p^2}{2} = p^2;$$

hence $\overline{C'B'} = p = \overline{AA'}$.

Also, since $\triangle C'MB'$ is a right-angled isosceles triangle, we note that $\widehat{MC'B'} = 45^\circ$.

In $\triangle TC'P$, we have $\hat{P} = \hat{C}' = 45^\circ$ and so $\hat{T} = 90^\circ$, telling us that $C'B' \perp AA'$.

Solution 3: (Trigonometry)



Consider the hexagon $AB'CA'BC'$ and all six of its diagonals.

Now we use the cosine law:

$$\begin{aligned}\overline{B'C'}^2 &= \overline{AC'}^2 + \overline{AB'}^2 - 2\overline{AC'} \cdot \overline{AB'} \cos(\hat{A} + 90^\circ) \\ &= \frac{c^2}{2} + \frac{b^2}{2} + bc \sin(\hat{A}).\end{aligned}$$

Also

$$\begin{aligned}\overline{AA'}^2 &= \overline{AC}^2 + \overline{CA'}^2 - 2\overline{AC} \cdot \overline{CA'} \cos(\hat{C} + 45^\circ) \\ &= b^2 + \frac{a^2}{2} - ab (\cos(\hat{C}) - \sin(\hat{C})) \\ &= \frac{b^2}{2} + \frac{c^2}{2} + ab \sin(\hat{C}).\end{aligned}$$

But recall that

$$bc \sin(\hat{A}) = ac \sin(\hat{C}) = 2 \times (\text{area} \triangle ABC).$$

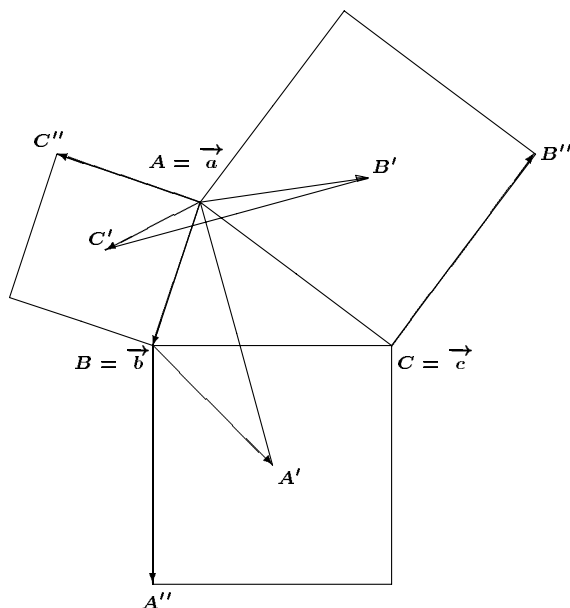
Hence $\overline{B'C'} = \overline{AA'}$.

Likewise, $\overline{C'A'} = \overline{BB'}$ and $\overline{A'B'} = \overline{CC'}$.

Now, consider triangles $B'C'B$ and $A'AC'$. These triangles are congruent since $\overline{B'C'} = \overline{AA'}$, $\overline{C'B} = \overline{AC'}$ and $\overline{BB'} = \overline{C'A'}$. Hence $x = x'$. Likewise,

$y = y'$ and $z = z'$. But then $2x + 2y + 2z = 180^\circ$ and hence $x + y + z = 90^\circ$. It follows, looking at triangle $A'B'T$, that $A'A \perp B'C'$.

Solution 4: (Vectors)



Here, we let the vertices of the triangle be represented by vectors \vec{a} , \vec{b} and \vec{c} .

Let \vec{n} be the unit vector which is perpendicular to the plane containing the diagram.

Recall that for any two vectors \vec{u} , \vec{v} , the vector $\vec{u} \times \vec{v}$ is perpendicular to both \vec{u} and \vec{v} , and points in the direction given by the right-hand rule. It follows that

$$\begin{aligned}\overrightarrow{AC''} &= \vec{n} \times \overrightarrow{BA} = \vec{n} \times (\vec{a} - \vec{b}), \\ \overrightarrow{BA''} &= \vec{n} \times \overrightarrow{CB} = \vec{n} \times (\vec{b} - \vec{c}), \\ \overrightarrow{CB''} &= \vec{n} \times \overrightarrow{AC} = \vec{n} \times (\vec{c} - \vec{a}).\end{aligned}$$

We next compute:

$$\begin{aligned}\overrightarrow{AC''} &= \frac{1}{2} \left[\vec{b} - \vec{a} + \vec{n} \times (\vec{a} - \vec{b}) \right], \\ \overrightarrow{AB''} &= \frac{1}{2} \left[\vec{c} - \vec{a} + \vec{n} \times (\vec{c} - \vec{a}) \right],\end{aligned}$$

and

$$\overrightarrow{BA''} = \frac{1}{2} \left[\vec{c} - \vec{b} + \vec{n} \times (\vec{b} - \vec{c}) \right].$$

Let \vec{u} be the vector from \vec{C}' to \vec{B}' ; let \vec{v} be the vector $\overrightarrow{AA'}$.

Now:

$$\vec{u} = \overrightarrow{AB'} - \overrightarrow{AC'} = \frac{1}{2} [\vec{c} - \vec{b} + \vec{n} \times (\vec{c} + \vec{b} - 2\vec{a})],$$

and

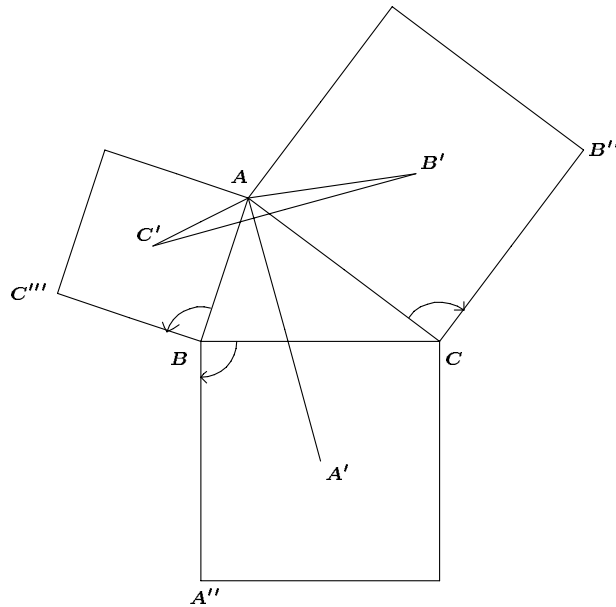
$$\begin{aligned} \vec{v} &= \frac{1}{2} [\vec{c} - \vec{b} + \vec{n} \times (\vec{b} - \vec{c})] - (\vec{a} - \vec{b}) \\ &= \frac{1}{2} [\vec{c} + \vec{b} - 2\vec{a} + \vec{n} \times (\vec{b} - \vec{c})]. \end{aligned}$$

But now:

$$\vec{n} \times \vec{v} = \frac{1}{2} [\vec{n} \times (\vec{c} + \vec{b} - 2\vec{a}) + (\vec{c} - \vec{b})] = \vec{u}.$$

And since $|\vec{n}| = 1$, we conclude that $|\vec{v}| = |\vec{u}|$ and that $\vec{v} \perp \vec{u}$.

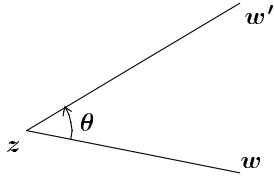
Solution 5: (Complex Numbers)



Place the diagram in the complex plane, with the origin at B . Let A and C represent the complex numbers at vertices A and C .

Recall that for any complex number z , the product $z' = ze^{i\theta}$ is the complex number which is obtained by rotating z about the origin through the angle θ .

If instead, we wish to rotate w about z through an angle θ , then $w' = z + (w - z)e^{i\theta}$.



Using this, we obtain the following:

$$B'' = C + (A - C)(-i) = C(1 + i) - Ai,$$

$$C''' = Ai,$$

$$A'' = -Ci.$$

Also, $A' = \frac{1}{2}C(1 - i)$, $B' = \frac{1}{2}A(1 - i) + C(1 + i)$ and $C' = \frac{1}{2}A(1 + i)$.

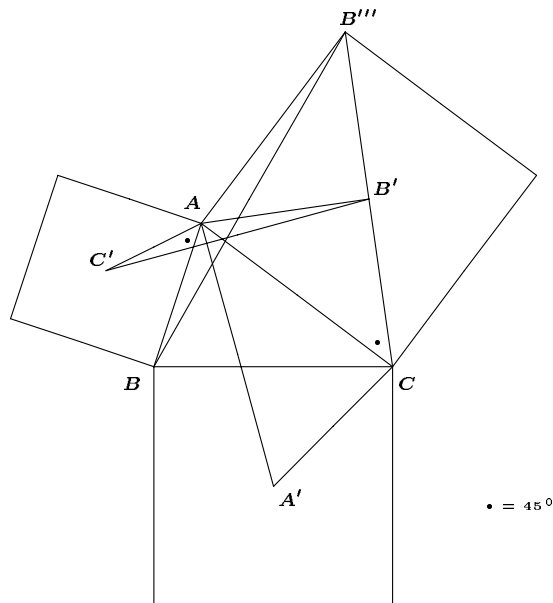
Let \vec{u} be the vector from C' to B' ; let \vec{v} be the vector from A' to A .

Now $\vec{u} = \frac{1}{2}C(1 + i) - Ai$ and $\vec{v} = A - \frac{1}{2}C(1 - i)$.

But now note that $\vec{u}i = A - \frac{1}{2}C(1 - i) = \vec{v}$.

So $A'A$ is obtained by rotating $C'B'$ through 90° , which proves both parts of the result that we seek.

Solution 6: (Transformations)



Rotate triangle ACA' about C through -45° and dilate by $\sqrt{2}$ to obtain triangle $B'''CB$.

This tells us that $\overline{B'''B}$ is inclined at 45° with respect to AA' .

Now rotate triangle $AB''B$ about A through -45° and dilate by $\frac{1}{\sqrt{2}}$ to obtain triangle $AB'C'$.

This tells us that $\overline{B'C'} = \frac{1}{\sqrt{2}}\overline{B''B}$, and that $B'C'$ is inclined at 45° with respect to $B''B$.

But this proves our result.

So, there it is — one problem, six different approaches. Which solution is the best? The late Professor Paul Erdős used to talk about “God’s Little Black Book”, in which could be found the perfect solution to every problem. Of the solutions presented here, perhaps the last one is the closest candidate for inclusion in this “Little Black Book”: it is economical, to the point and carries with it a wonderful element of surprise. Those are the properties of beautiful mathematics.

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This volume contains the problems and solutions from the 1995–1996 Mathematical Olympiads' Correspondence Program. This program has several purposes. It provides students with practice at solving and writing up solutions to Olympiad-level problems, it helps to prepare student for the Canadian Mathematical Olympiad and it is a partial criterion for the selection of the Canadian IMO team.

For more information, contact the Canadian Mathematical Society.

THE SKOLIAD CORNER

No. 28

R.E. Woodrow

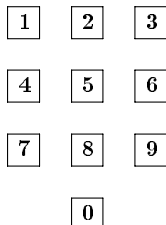
Last number we gave the Preliminary Round of the British Columbia Colleges Junior High School Mathematics Contest for 1997. To keep in the flow of the contest we give Part A and Part B of the Final Round. Students whose performance in class on problems of the Preliminary Round was exemplary were invited to nearby colleges to attempt the Final Round as part of a larger day-long mathematical event. Again my thanks go to John Grant McLoughlin, now of the Faculty of Education, Memorial University of Newfoundland, who participated in formulating the exams while he was at Okanagan College.

BRITISH COLUMBIA COLLEGES Junior High School Mathematics Contest Final Round 1997 — Part A

1. The buttons of a phone are arranged as shown below. If the buttons are one centimeter apart, centre-to-centre, when you dial the number 592-7018 the distance, in centimetres, traveled by your finger is:

- (a) $\sqrt{5}(3 + \sqrt{2}) + 2\sqrt{2}$
 (c) $2\sqrt{5}(1 + \sqrt{2}) + 2\sqrt{2}$
 (e) $\sqrt{5}(1 + \sqrt{2}) + 4\sqrt{10}$

- (b) $2\sqrt{5} + \sqrt{2}(3 + \sqrt{5})$
 (d) $\sqrt{5} + \sqrt{2}(2 + 3\sqrt{5})$



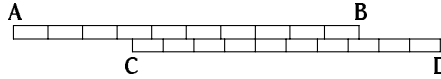
2. What is the total number of ones digits needed in order to write the integers from 1 to 100?

- (a) 11 (b) 18 (c) 20 (d) 21 (e) 100

3. The number of solutions (x, y, z) in positive integers for the equation $3x + y + z = 23$ is:

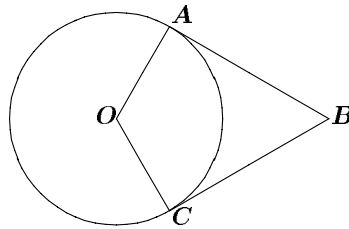
- (a) 86 (b) 50 (c) 60 (d) 70 (e) 92

4. In the diagram below the upper scale AB has ten 1 centimetre divisions. The lower scale CD also has ten divisions, but it is only 9 centimetres long. If the right hand end of the fourth division of scale CD coincides exactly with the right hand end of the seventh division of scale AB , what is the distance, in centimetres, from A to C ?



- (a) 3.2 (b) 3.25 (c) 3.3 (d) 3.35 (e) 3.4

5. Triangle ABC is equilateral with sides tangent to the circle with center at O and radius $\sqrt{3}$. The area of the quadrilateral $AOCB$, in square units is:



- (a) $3\sqrt{3}$ (b) 3 (c) $6\sqrt{3}$ (d) 3π (e) $2\sqrt{3}$

6. Times such as 1:01, 1:11, ... are called palindromic times because their digits read the same forwards and backwards. The number of palindromic times on a digital clock between 1:00 a.m. and 11:59 a.m. is:

- (a) 47 (b) 48 (c) 55 (d) 56 (e) 66

7. Ted's television has channels 2 through 42. If Ted starts on channel 15 and surfs, pushing the channel up button 518 times, when he stops he will be on channel:

- (a) 11 (b) 13 (c) 15 (d) 38 (e) 41

8. Consider a three-digit number with the following properties:

1. If its tens and ones digits are switched, its value would increase by 36.
2. Instead, if its hundreds and ones digits are switched its value would decrease by 198.

Suppose that only the hundreds and tens digits are switched. Its value would:

- (a) increase by 6 (b) decrease by 540 (c) increase by 540
(d) decrease by 6 (e) increase by 90

9. Speedy Sammy Seamstress sews seventy-seven stitches in sixty-six seconds. The time, in seconds, it takes Sammy to stitch fifty-five stitches is:

- (a) $43\frac{1}{3}$ (b) $44\frac{1}{4}$ (c) $45\frac{1}{5}$ (d) $46\frac{1}{6}$ (e) $47\frac{1}{7}$

10. How many positive integers less than or equal to 60 are divisible by 3, 4, or 5?

- (a) 25 (b) 35 (c) 36 (d) 37 (e) 44

Final Round 1997 — Part B

1. (a) The pages of a thick telephone directory are numbered from 1 to N . A total of 522 digits are required to print the pages. Find N .

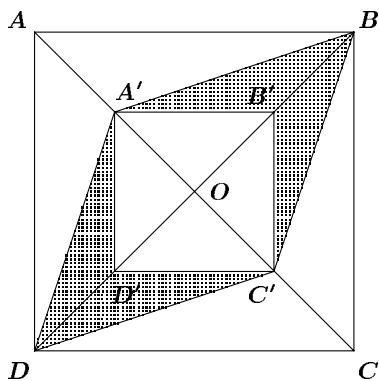
(b) There are 26 pages in the local newspaper. Suppose that you pull a sheet out and drop it on the floor. One of the pages facing you is numbered 19. What are the other page numbers on the sheet?

2. Using each of the digits 1, 9, 9, and 7 create expressions for the numbers 1, 2, 3, ..., 10. Note that the digits must appear separately; that is, numbers like 17 are not allowed. Only the basic operations $+$, $-$, \times , \div and brackets (if necessary) may be used. Other mathematical symbols such as $\sqrt{\quad}$ are *not* allowed. Every expression must include one 1, two 9's, and one 7, in any order.

3. (a) Decide which is greater: $\sqrt{6} + \sqrt{8}$ or $\sqrt{5} + \sqrt{9}$.

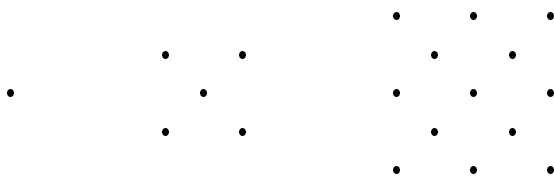
(b) Show that $\frac{x^2+1}{x} \geq 2$ for any real number $x > 0$.

4. In the plane figure shown on the right, $ABCD$ is a square with $AB = 12$. If A' , B' , C' , and D' are the mid-points of AO , BO , CO , and DO , respectively, then:



- (a) Find the area of the square $A'B'C'D'$.
 (b) Find the area of the shaded region.
 (c) Find the area of the trapezoid $AA'B'B$.

5. The figure below shows the first three in a sequence of square arrays of dots. The number of dots in the three arrays is 1, 5, and 13.



- Find the number of dots in the next array in the sequence.
- Find the number of dots in the sixth array in the sequence.
- Find an expression for a_n , the number of dots in the n^{th} array in the sequence, in terms of n alone.
- Find a relation between a_{n+1} , a_n , and n .

Last issue we gave the problems of the Preliminary Round of the British Columbia Colleges Junior High School Mathematics Contest 1997. Here are the answers:

1. E 2. A 3. D 4. A 5. C
 6. A 7. D 8. E 9. B 10. D
 11. C 12. B 13. D 14. C 15. E

That completes this number of the Corner. Send me your comments, suggestions, and suitable materials for use in the Skoliad Corner.

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All material intended for inclusion in this section should be sent to the **Mayhem Editor, Naoki Sato, Department of Mathematics, Yale University, PO Box 208283 Yale Station, New Haven, CT 06520-8283 USA**. The electronic address is still

mayhem@math.toronto.edu

The Assistant Mayhem Editor is Cyrus Hsia (University of Toronto). The rest of the staff consists of Adrian Chan (Upper Canada College), Jimmy Chui (Earl Haig Secondary School), Richard Hoshino (University of Waterloo), David Savitt (Harvard University) and Wai Ling Yee (University of Waterloo).

Shreds and Slices

The Fibonacci Triangle

We present an intriguing configuration of numbers, which might be called the Fibonacci triangle. Recall that the golden ratio is $\tau = \frac{1+\sqrt{5}}{2}$, and that $\lfloor x \rfloor$ is the greatest integer less than or equal to x . Starting with a 2, for any entry n , write $\lfloor \tau n \rfloor$ and $\lfloor \tau^2 n \rfloor$ under it. The first few rows we obtain are as follows:

				2											
			3				5								
		4		7		8		13							
	6	10	11	18	12	20	21	34							
9	15	16	26	17	28	29	47	19	31	32	52	33	54	55	89

A general portion of the triangle is the following:

			n			
		$\lfloor \tau n \rfloor$		$\lfloor \tau^2 n \rfloor$		
$\lfloor \tau \lfloor \tau n \rfloor \rfloor$		$\lfloor \tau^2 \lfloor \tau n \rfloor \rfloor$		$\lfloor \tau \lfloor \tau^2 n \rfloor \rfloor$		$\lfloor \tau^2 \lfloor \tau^2 n \rfloor \rfloor$

Some patterns are immediately apparent in the Fibonacci triangle; we present some now.

Proposition 1. $n + \lfloor \tau n \rfloor = \lfloor \tau^2 n \rfloor$.

Proof. $\lfloor \tau^2 n \rfloor = \lfloor (\tau + 1)n \rfloor = \lfloor \tau n + n \rfloor = \lfloor \tau n \rfloor + n$.

Proposition 2. $\lfloor \tau n \rfloor + \lfloor \tau^2 n \rfloor = \lfloor \tau \lfloor \tau^2 n \rfloor \rfloor$.

Proof. We first make a general observation: $\lfloor x \rfloor = a$ if and only if a is an integer and $a \leq x < a + 1$. Most problems of this form will use this fact. Let $a = \lfloor \tau n \rfloor$, so $a \leq \tau n < a + 1$. Note $\lfloor \tau n \rfloor + \lfloor \tau^2 n \rfloor = 2a + n$, and $\lfloor \tau \lfloor \tau^2 n \rfloor \rfloor = \lfloor \tau(a + n) \rfloor = \lfloor \tau a + \tau n \rfloor$. Therefore, we must prove $2a + n \leq \tau a + \tau n < 2a + n + 1$. For the left inequality, we have

$$2a + n \leq \tau a + \tau n$$

if and only if

$$(2 - \tau)a \leq (\tau - 1)n$$

if and only if

$$a \leq \frac{\tau - 1}{2 - \tau} n = \tau n,$$

which is true; and for right inequality,

$$\tau a + \tau n < 2a + n + 1$$

if and only if

$$(2 - \tau)a > (\tau - 1)n - 1$$

if and only if

$$a > \frac{\tau - 1}{2 - \tau} n - \frac{1}{2 - \tau} = \tau n - \tau - 1,$$

which is also true, and the statement is proved.

These results, for one, verify that the last two entries of each row do indeed form the Fibonacci sequence. We finally show the one conspicuous result in the triangle.

Proposition 3. Each positive integer greater than 1 appears exactly once in the table.

Proof. If a and b are distinct positive integers greater than 1, then so are $\lfloor \tau a \rfloor$ and $\lfloor \tau b \rfloor$. This is because $\tau > 1$, so τa and τb must differ by at least 1, and hence round down to different integers. The same argument works for τ^2 . Hence, the same integer cannot appear as a left leg or a right leg.

Now we invoke a classic result:

Beatty's Theorem. If α and β are positive, irrational numbers such that

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1,$$

then the sequences $\{\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor, \dots\}$, $\{\lfloor \beta \rfloor, \lfloor 2\beta \rfloor, \lfloor 3\beta \rfloor, \dots\}$ contain each positive integer exactly once.

We can take $\alpha = \tau$ and $\beta = \tau^2$. By Beatty's theorem, no integer can appear both in the form $\lfloor \tau a \rfloor$ and $\lfloor \tau^2 b \rfloor$. Hence, no integer appears more than once. But Beatty's also tells us that each integer greater than 1 must appear at least once. Hence, each integer greater than 1 appears exactly once.

Note that in this proof, we used only the fact that the sum of the reciprocals of τ and τ^2 is 1; they can be replaced with any numbers satisfying the conditions of Beatty's Theorem (the fact that each is greater than 1 also follows from this relationship).

Problems

1. (a) Prove that $\lfloor \tau^2 n \rfloor - \lfloor \tau \lfloor \tau n \rfloor \rfloor = 1$.
- (b) Prove that $\lfloor \tau \lfloor \tau^2 n \rfloor \rfloor - \lfloor \tau^2 \lfloor \tau n \rfloor \rfloor = 1$.

What do these say about the entries in the triangle?

2. An F-sequence is a sequence in which, except for the first and second terms, every term is the sum of the two previous terms.
 - (a) Show that the positive integers cannot be partitioned into a finite number of F-sequences.
 - (b) Show that the positive integers can be partitioned into an infinite number of F-sequences.

Mayhem Problems

The Mayhem Problems editors are:

Richard Hoshino	<i>Mayhem High School Problems Editor,</i>
Cyrus Hsia	<i>Mayhem Advanced Problems Editor,</i>
David Savitt	<i>Mayhem Challenge Board Problems Editor.</i>

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only solutions — the next issue will feature only problems.

We warmly welcome proposals for problems and solutions. With the new schedule of eight issues per year, we request that solutions from the previous issue be submitted by 1 April 1998, for publication in the issue 5 months ahead; that is, issue 6 of 1998. We also request that **only students** submit solutions (see editorial [1997: 30]), but we will consider particularly elegant or insightful solutions from others.

High School Solutions

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H217. Let a_1, a_2, a_3, a_4, a_5 be a five-term geometric sequence satisfying the inequality $0 < a_1 < a_2 < a_3 < a_4 < a_5 < 100$, where each term is an integer. How many of these five-term geometric sequences are there? (For example, the sequence 3, 6, 12, 24, 48 is a sequence of this type).

Solution by Joel Schlosberg, student, Hunter College High School, New York NY, USA.

Let $\frac{n}{m}$ be the common ratio of the geometric sequence, where n and m are relatively prime integers, with $n > m$. Now $a_5 = a_1 \times \frac{n^4}{m^4}$, so let $a_1 = km^4$, where k is a positive integer. Thus, our geometric series becomes $km^4, km^3n, km^2n^2, kmn^3, kn^4$, and $kn^4 < 100$. If $n > 4$, then $kn^4 \geq n^4 > 256 > 100$. So $n \leq 3$. Hence, there are three cases to consider:

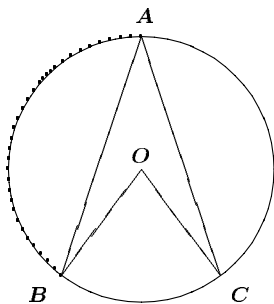
If $n = 3$ and $m = 2$, then $81k < 100$, so $k = 1$. The only solution is (16, 24, 36, 54, 81).

If $n = 3$ and $m = 1$, then $81k < 100$, so $k = 1$. The only solution is (1, 3, 9, 27, 81).

If $n = 2$ and $m = 1$, then $16k < 100$, so $k = 1, 2, \dots, 6$. There are six solutions, namely (1, 2, 4, 8, 16), (2, 4, 8, 16, 32), (3, 6, 12, 24, 48), (4, 8, 16, 32, 64), (5, 10, 20, 40, 80) and (6, 12, 24, 48, 96). In total, there are eight sequences.

There was one incorrect submission received, where the solver incorrectly assumed that the common ratio had to be an integer. Hence, the solution (16, 24, 36, 54, 81) was missed.

H218. A Star Trek logo is inscribed inside a circle with centre O and radius 1, as shown. Points A, B , and C are selected on the circle so that $AB = AC$ and arc BC is minor (that is, $ABOC$ is not a convex quadrilateral). The area of figure $ABOC$ is equal to $\sin m^\circ$, where $0 < m < 90$ and m is an integer. Furthermore, the length of arc AB (shaded as shown) is equal to $a\pi/b$, where a and b are relatively prime integers. Let $p = a + b + m$.



- (i) If $p = 360$, and m is composite, determine all possible values for m .
- (ii) If m and p are both prime, determine the value of p .

Solution.

(i) Let $\angle BOC = 2x^\circ$, where $0 < x < 90$. Otherwise, $\angle BOC$ would exceed 180° and $ABOC$ would be a convex quadrilateral (and not the figure of a Star Trek logo). Draw OA . Since $AB = AC$, OAB and OAC are congruent triangles, and so $\angle AOB = \angle AOC$. Since $\angle AOB + \angle AOC = (360 - 2x)^\circ$, we have $\angle AOB = (180 - x)^\circ$. Thus, the area of $ABOC$ is twice that of triangle OAB , so $\sin m^\circ = OA \cdot OB \cdot \sin(180 - x)^\circ = \sin x^\circ$. Since $0 < m, x < 90$, $\sin m^\circ = \sin x^\circ$ implies that $m = x$. Hence $\angle AOB = (180 - m)^\circ$ and the length of arc AB is

$$\frac{2\pi(180 - m)}{360} = \frac{\pi(180 - m)}{180}.$$

Thus, $\frac{180-m}{180} = \frac{a}{b}$, where a and b are relatively prime integers. Hence, $a \leq 180 - m$ and $b \leq 180$ with equality occurring iff $\frac{180-m}{180}$ is irreducible. In this case, $a + b + m = 180 - m + 180 + m = 360$, and so if $p = 360$, we must have $\gcd(m, 180) = 1$. But m is composite and cannot have any common divisors with $180 = 2^2 \times 3^2 \times 5$. So m must be a product of primes greater than 5. But $m < 90$, so we only have two possible choices for m : $7 \cdot 7$ and $7 \cdot 11$. Hence, if $p = 360$ and m is composite, then $m = 49$ or 77 .

(ii) If m is a prime larger than 5, we will have $p = 360$, since $\gcd(m, 180) = 1$. Since 360 is not prime, we must restrict our choices of m to 2, 3 and 5. If $m = 3$ or $m = 5$, we will find that p is composite. However, if $m = 2$, we have $\frac{a}{b} = \frac{178}{180} = \frac{89}{90}$, so $p = a + b + m = 89 + 90 + 2 = 181$, which is prime. Hence, $m = 2$ and $p = 181$.

H219. Consider the infinite sum

$$S = \frac{a_0}{10^0} + \frac{a_1}{10^2} + \frac{a_2}{10^4} + \frac{a_3}{10^6} + \cdots,$$

where the sequence $\{a_n\}$ is defined by $a_0 = a_1 = 1$, and the recurrence relation $a_n = 20a_{n-1} + 12a_{n-2}$ for all positive integers $n \geq 2$. If \sqrt{S} can be expressed in the form $\frac{a}{\sqrt{b}}$, where a and b are relatively prime positive integers, determine the ordered pair (a, b) .

Solution.

We have

$$\begin{aligned}
S - \frac{20S}{10^2} - \frac{12S}{10^4} &= \left(\frac{a_0}{10^0} + \frac{a_1}{10^2} + \frac{a_2}{10^4} + \frac{a_3}{10^6} + \cdots \right) \\
&\quad - \left(\frac{20a_0}{10^2} + \frac{20a_1}{10^4} + \frac{20a_2}{10^6} + \frac{20a_3}{10^8} + \cdots \right) \\
&\quad - \left(\frac{12a_0}{10^4} + \frac{12a_1}{10^6} + \frac{12a_2}{10^8} + \frac{12a_3}{10^{10}} + \cdots \right) \\
&= \frac{a_0}{10^0} + \frac{a_1}{10^2} - \frac{20a_0}{10^2} + \frac{a_2 - 20a_1 - 12a_0}{10^4} \\
&\quad + \frac{a_3 - 20a_2 - 12a_1}{10^6} + \frac{a_4 - 20a_3 - 12a_2}{10^8} + \cdots
\end{aligned}$$

Since $a_n - 20a_{n-1} - 12a_{n-2} = 0$ for all positive integers $n \geq 2$, we have

$$S - \frac{20S}{10^2} - \frac{12S}{10^4} = \frac{a_0}{10^0} + \frac{a_1}{10^2} - \frac{20a_0}{10^2},$$

and substituting in $a_0 = a_1 = 1$, we have $\frac{7988S}{10000} = \frac{81}{100}$, so $S = \frac{2025}{1997}$. Hence, $\sqrt{S} = \frac{45}{\sqrt{1997}}$, and so the desired ordered pair is $(a, b) = (45, 1997)$.

H220. Let S be the sum of the elements of the set $\{1, 2, 3, \dots, (2p)^n - 1\}$. Let T be the sum of the elements of this set whose representation in base $2p$ consists only of digits from 0 to $p - 1$. Prove that $2^n \times \frac{T}{S} = (p - 1)/(2p - 1)$.

Solution.

We have $S = \frac{[(2p)^n - 1](2p)^n}{2}$. Let R denote the set of numbers that have at most n digits in base $2p$ and contain only the digits from 0 to $p - 1$. Thus T is the sum of the elements of R . For example, when $p = 2$ and $n = 3$, we have $R = \{0, 1, 10, 11, 100, 101, 110, 111\}$ and $T = 1110_4 = 84$. Note R will have p^n elements, because each of the n digits (including leading zeros) can be any one of p different numbers. Since each number in $\{0, 1, \dots, p - 1\}$ appears the same number of times as a digit in R , there will be exactly p^{n-1} elements in R that have k as their t^{th} digit, for $k = 0, 1, \dots, p - 1$ and $t = 1, 2, \dots, n$. Thus,

$$\begin{aligned}
T &= \sum_{i=0}^{n-1} (2p)^i \cdot [p^{n-1} \times 0 + p^{n-1} \times 1 + \cdots + p^{n-1} \times (p - 1)] \\
&= \sum_{i=0}^{n-1} (2p)^i \cdot p^{n-1} \cdot \frac{p(p-1)}{2} = \frac{p^n(p-1)}{2} \cdot \sum_{i=0}^{n-1} (2p)^i \\
&= \frac{p^n(p-1)}{2} \cdot \frac{(2p)^n - 1}{2p - 1}.
\end{aligned}$$

Hence,

$$2^n \times \frac{T}{S} = \frac{2^n \cdot \frac{p^n(p-1)}{2} \cdot \frac{(2p)^n - 1}{2^{p-1}}}{\frac{[(2p)^n - 1](2p)^n}{2}} = \frac{p-1}{2p-1},$$

as required.

This can also be solved by induction – that is an exercise left for the reader.

Advanced Solutions

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A193. If $f(x, y)$ is a convex function in x for each fixed y , and a convex function in y for each fixed x , is $f(x, y)$ necessarily a convex function in x and y ?

Solution.

No; for example, take

$$f(x, y) = \begin{cases} e^{xy} - 1 & \text{if } x, y \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

Then f satisfies the given conditions. However, $f(0, 2) = f(2, 0) = 0$, and $f(1, 1) = e$, so f is not convex.

A194. Let H be the orthocentre (point where the altitudes meet) of a triangle ABC . Show that if $AH : BH : CH = BC : CA : AB$ then the triangle is equilateral.

Solution by Deepee Khosla, Ottawa, ON.

Let $a = BC$, $b = AC$, $c = AB$, as usual, and set $A' = AH \cap BC$, $B' = BH \cap AC$, $C' = CH \cap AB$. Since $\angle AA'B = \angle BB'A$, it follows that quadrilateral $ABA'B'$ is cyclic (with diameter AB). Thus $AH \times HA' = BH \times HB'$. Together with the givens, we have

$$\frac{HB'}{HA'} = \frac{AH}{BH} = \frac{a}{b},$$

so that

$$a \times BH = b \times AH, \tag{1}$$

$$a \times HA' = b \times HB'. \tag{2}$$

Also, $2K = a \times AA' = a(AH + HA')$, where K is the area of triangle ABC . This implies $a \times HA' = 2K - a \times AH$, and similarly for $b \times HB'$. Putting these into equation (2) gives

$$a \times AH = b \times BH. \quad (3)$$

Multiplying (1) and (3) and cancelling yields $a^2 = b^2$ or $a = b$ since $a, b > 0$. A similar argument shows that $a = c$, so triangle ABC is equilateral.

A195. Compute $\tan 20^\circ \tan 40^\circ \tan 60^\circ \tan 80^\circ$.

Solution I by D.J. Smeenk, Zaltbommel, the Netherlands.

Let $A = \tan 20^\circ \tan 40^\circ \tan 60^\circ \tan 80^\circ$. Then $A = \frac{B}{C}$, where $B = 8 \sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ$ and $C = 8 \cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ$. Then

$$\begin{aligned} B &= 8 \sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ \\ &= 2(\cos 60^\circ - \cos 100^\circ)(\cos 20^\circ - \cos 100^\circ) \\ &= (1 + 2 \sin 10^\circ)(\cos 20^\circ + \sin 10^\circ) \\ &= \cos 20^\circ + \sin 10^\circ + 2 \sin 10^\circ \cos 20^\circ + 2 \sin^2 10^\circ \\ &= \cos 20^\circ + \sin 10^\circ + \sin 30^\circ - \sin 10^\circ + (1 - \cos 20^\circ) = \frac{3}{2}, \end{aligned}$$

$$\begin{aligned} C &= 8 \cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ \\ &= 2(\cos 60^\circ + \cos 100^\circ)(\cos 20^\circ + \cos 100^\circ) \\ &= (1 - 2 \sin 10^\circ)(\cos 20^\circ - \sin 10^\circ) \\ &= \cos 20^\circ - \sin 10^\circ - 2 \sin 10^\circ \cos 20^\circ + 2 \sin^2 10^\circ \\ &= \cos 20^\circ - \sin 10^\circ - \sin 30^\circ + \sin 10^\circ + (1 - \cos 20^\circ) = \frac{1}{2}. \end{aligned}$$

Therefore, $A = \frac{B}{C} = 3$.

Solution II.

By a well-known identity,

$$\tan 9\theta = \frac{\binom{9}{1} \tan \theta - \binom{9}{3} \tan^3 \theta + \binom{9}{5} \tan^5 \theta - \binom{9}{7} \tan^7 \theta + \binom{9}{9} \tan^9 \theta}{\binom{9}{2} \tan^2 \theta - \binom{9}{4} \tan^4 \theta + \binom{9}{6} \tan^6 \theta - \binom{9}{8} \tan^8 \theta}.$$

The LHS is zero for the nine values $\theta = 0^\circ, 20^\circ, 40^\circ, \dots, 160^\circ$, so the roots of

$$\binom{9}{1}x - \binom{9}{3}x^3 + \binom{9}{5}x^5 - \binom{9}{7}x^7 + \binom{9}{9}x^9 = 0$$

are the nine distinct values $\tan 0^\circ = 0, \tan 20^\circ, \tan 40^\circ, \dots, \tan 160^\circ$. Note that $\tan 100^\circ = -\tan 80^\circ, \dots, \tan 160^\circ = -\tan 20^\circ$. By taking out a factor of x , we see the product of the roots is

$$\tan^2 20^\circ \tan^2 40^\circ \tan^2 60^\circ \tan^2 80^\circ = \binom{9}{1} = 9,$$

so that $\tan 20^\circ \tan 40^\circ \tan 60^\circ \tan 80^\circ = 3$.

Also solved by Deepee Khosla, Ottawa, ON, and Bob Prielipp, University of Wisconsin–Oshkosh, Wisconsin, USA.

A196. Show that $r^2 + r_a^2 + r_b^2 + r_c^2 \geq 4K$, where r , r_a , r_b , r_c , and K are respectively the inradius, exradii and area of a triangle ABC .

Solution by Deepee Khosla, Ottawa, ON, and Bob Prielipp, University of Wisconsin–Oshkosh, Wisconsin, USA.

We have

$$r = \frac{K}{s}, \quad r_a = \frac{K}{s-a}, \quad r_b = \frac{K}{s-b}, \quad \text{and } r_c = \frac{K}{s-c}.$$

Thus, by the AM-GM inequality,

$$\begin{aligned} r^2 + r_a^2 + r_b^2 + r_c^2 &= \frac{K^2}{s^2} + \frac{K^2}{(s-a)^2} + \frac{K^2}{(s-b)^2} + \frac{K^2}{(s-c)^2} \\ &\geq 4\sqrt[4]{\frac{K^8}{s^2(s-a)^2(s-b)^2(s-c)^2}} = 4\frac{K^2}{K} = 4K. \end{aligned}$$

Challenge Board Solutions

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C73. Proposed by Matt Szczesny, University of Toronto.

The sequence $\{a_n\}$ consists of positive reals such that the sum of the a_n diverges. Show that the sum of a_n/s_n diverges, where s_n is the n^{th} partial sum.

Solution by the proposer.

Since the a_n are positive, the sequence $\{s_n\}$ is increasing. Putting $s_0 = 0$, for any non-negative integer N and positive integer k , we therefore have

$$\sum_{n=N+1}^{N+k} \frac{a_n}{s_n} \geq \sum_{n=N+1}^{N+k} \frac{a_n}{s_{N+k}} = \frac{1}{s_{N+k}}(s_{N+k} - s_N) = 1 - \frac{s_N}{s_{N+k}}.$$

For every non-negative integer N there exists, by the divergence of $\{s_n\}$, some k such that $\frac{s_N}{s_{N+k}} < \frac{1}{2}$. It follows that for each such N we can obtain

a k so that $\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq \frac{1}{2}$, and hence the sum $\frac{a_1}{s_1} + \frac{a_2}{s_2} + \cdots$ must diverge.

The Order of a Zero

Naoki Sato

graduate student, Yale University

Recall that a root r of a polynomial $p(x)$ has multiplicity m if $(x - r)^m$ divides $p(x)$ but $(x - r)^{m+1}$ does not. In other words, $p(x) = (x - r)^m q(x)$ for some polynomial $q(x)$, and $q(r) \neq 0$. This means that $p(x)$ behaves much like $q(r)(x - r)^m$ around $x = r$, for example in graphing $p(x)$. In fact, in many applications, the values $q(r)$ and m are all the data we have to know. We extend these ideas analytically to calculus, where they can play a useful and powerful role. Keep in mind that the notion of multiplicity of a root will be our main motivation.

First, we give the analogue of multiplicity in calculus.

Definition. We say $f(x)$ has order r at $x = a$ if

$$\lim_{x \rightarrow a} \frac{f(x)}{(x - a)^r} = c$$

for some non-zero constant c . Let us call c the r -value of $f(x)$ at $x = a$.

First, it should be clear that the values of r and c are unique. To see this, assume that $f(x)$ has order r_1 and r_2 at a , so

$$\lim_{x \rightarrow a} \frac{f(x)}{(x - a)^{r_1}} = c_1, \quad \text{and} \quad \lim_{x \rightarrow a} \frac{f(x)}{(x - a)^{r_2}} = c_2.$$

Then

$$\lim_{x \rightarrow a} (x - a)^{r_1 - r_2} = \frac{c_2}{c_1}.$$

Since c_1 and c_2 are non-zero, we must have $r_1 = r_2$ (and of course, $c_1 = c_2$).

Thus, we can now speak of *the* order of $f(x)$ at a . However, the order at a need not exist (see Example 4 below).

From a numerical viewpoint, $f(x)$ behaves like $c(x - a)^r$ for x close to a , so the order gives a measure of how quickly $f(x)$ approaches (or recedes from) 0 as x approaches a . The greater r is, the faster $f(x)$ decreases. Note that if r is negative, then r must be an integer (why?), and $f(x)$ goes to $\pm\infty$ as x approaches a .

Examples.

1. If a root r of polynomial $p(x)$ has multiplicity m , and $p(x) = (x - r)^m q(x)$, then $p(x)$ has order m and r -value $q(r)$ at r . In particular, the order is a non-negative integer which is at most the degree of $p(x)$. For example, the polynomial $(x - 1)^2 x(x + 3)$ has order 2 and r -value 4 at $x = 1$.

2. If f is continuous at a and $f(a) \neq 0$, then $f(x)$ has order 0 and r -value $f(a)$ at a . It neither approaches nor recedes from 0 at $x = a$.

3. It is well known that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$. Hence, $\sin x$ has order 1 and r -value 1 at $x = 0$.

4. As mentioned, the order need not exist. For example, the somewhat trivial $f(x) = 0$ has no order at any point, and $f(x) = |x|$ has no order at $x = 0$. Also, $f(x) = x|x|$ is differentiable at $x = 0$, but has no order at $x = 0$. This shows that even regular behaviour such as differentiability cannot guarantee the existence of order!

We do have the following sufficient conditions for continuity and differentiability.

Proposition. If $f(x) - f(a)$ has order greater than 0 at a , then f is continuous at a . If $f(x) - f(a)$ has order at least 1 at a , then f is differentiable at a .

Proof. Let r be the order of $f(x) - f(a)$ at a . Assume that $r > 0$. Recall that f is continuous at a if and only if $f(a) = \lim_{x \rightarrow a} f(x)$, or equivalently $\lim_{x \rightarrow a} (f(x) - f(a)) = 0$. But we see

$$\lim_{x \rightarrow a} (f(x) - f(a)) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{(x - a)^r} \cdot \lim_{x \rightarrow a} (x - a)^r = 0.$$

In this product, the first limit exists by definition of order, and the second limit exists and is 0 since $r > 0$. Hence, f is continuous at a .

Similarly, recall that f is differentiable at a if and only if

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

exists, and the derivative $f'(a)$ is the value of this limit.

Assume that $r \geq 1$. Then

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{(x - a)^r} \cdot \lim_{x \rightarrow a} (x - a)^{r-1},$$

and in this product, both limits exist, so f is differentiable at a .

Remarks. Example 4 shows that the converse does not hold. Also, the function $f(x)$, which is 0 at $x = 0$ and 1 everywhere else, has order 1 at $x = 0$ but is not continuous at $x = 0$; hence, we do require that $r > 0$ for continuity.

So order can give some useful information, but this seems academic, since continuity and differentiability are easy to determine anyway. We now present a result which uses orders and r -values to compute limits.

Proposition. Let $f_1(x)$, $f_2(x)$ have orders r_1 , r_2 and r -value c_1 , c_2 at a respectively. Then $f_1(x)f_2(x)$ has order $r_1 + r_2$ and r -value c_1c_2 at a , and

$f_1(x)/f_2(x)$ has order $r_1 - r_2$ and r -value c_1/c_2 at a . Also,

$$\lim_{x \rightarrow a} \frac{f_1(x)}{f_2(x)} = \begin{cases} 0 & \text{if } r_1 > r_2 \\ c_1/c_2 & \text{if } r_1 = r_2 \end{cases}.$$

If $r_1 < r_2$, then

$$\lim_{x \rightarrow a^+} \frac{f_1(x)}{f_2(x)} = \operatorname{sgn}(c_1/c_2) \cdot \infty,$$

and if $r_1 - r_2$ is an integer, then

$$\lim_{x \rightarrow a^-} \frac{f_1(x)}{f_2(x)} = \operatorname{sgn}(c_1/c_2)(-1)^{r_2 - r_1} \cdot \infty.$$

Else, if $r_2 - r_1$ is not an integer, then this last limit does not exist.

A proof follows almost directly from the definition, and we will not include one here. Note that the evaluation of the limit only requires knowledge of r_1 , r_2 , c_1 , and c_2 , justifying the claim that these are the only values one usually needs to know. Actually finding these values may vary from trivial to difficult, which we will address in a moment.

The philosophy behind this concept is, as pointed out before, that if $f(x)$ has order r and r -value c at a , then $f(x)$ behaves like $c(x - a)^r$ at a . So, in evaluating the limit in the former proposition, the method that should work, and the one we should always pursue is to substitute $c(x - a)^r$ for $f(x)$ (since they behave similarly at $x = a$), cancel all factors of $x - a$, and see what is left, because what is left is what precisely determines the value of the limit.

For example, in taking the limit of a rational function as x approaches a , what we do is precisely as described above; that is, factor and cancel powers of $x - a$. Other limits are not so straightforward. For example, a recent final exam in a first year calculus course at the University of Toronto asked the following: Evaluate

$$\lim_{x \rightarrow 0} \frac{\sin x - \arctan x}{x^2 \ln(1 + x)}.$$

Analogously, what we would like to do is “factor” out the powers of x , or more precisely, find the orders and r -values of the numerator and denominator at 0. For other similar limits, we must also calculate the orders and r -values for fairly general functions. As of now, there does not seem to be a clear way of doing this. We give a systematic method for a reasonable class of functions, and we return to the polynomial case for motivation of a different description of order.

Lemma. Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$. Then there exist unique constants $c_n, c_{n-1}, \dots, c_1, c_0$ such that $p(x) = c_n(x - a)^n + c_{n-1}(x - a)^{n-1} + \cdots + c_1(x - a) + c_0$.

Proof. The c_i are determined by the following translation:

$$\begin{aligned} p(x + a) &= c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 \\ \iff p(x) &= c_n (x - a)^n + c_{n-1} (x - a)^{n-1} + \cdots + c_1 (x - a) + c_0. \end{aligned}$$

In particular, we see that by differentiating both sides k times, $c_k = p^{(k)}(a)/k!$.

Corollary. $(x - a)^r$ divides $p(x)$ if and only if $p^{(k)}(a) = 0$ for $k = 0, 1, \dots, r - 1$.

Proof. Both conditions are equivalent to $c_0 = c_1 = \dots = c_{r-1} = 0$.

Proposition. Let $p(x)$ be a non-zero polynomial. Then the order of $p(x)$ at a is the value of r such that $p^{(k)}(a) = 0$ for $k = 0, 1, \dots, r - 1$, and $p^{(r)}(a) \neq 0$, and the r -value at a is $p^{(r)}(a)/r!$.

Proof. Left as an exercise for the reader.

We now state the generalization we seek.

Theorem. Let $f(x)$ be a function which has a Taylor expansion around $x = a$, say

$$f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + \dots,$$

such that not all the coefficients c_i are zero, or in other words, $f(x)$ is not identical to zero around $x = a$. Then the order of $f(x)$ at a exists, and is the unique non-negative integer r such that $f^{(k)}(a) = 0$ for $k = 0, 1, \dots, r - 1$, and $f^{(r)}(a) \neq 0$, and the r -value at a is $f^{(r)}(a)/r!$.

Proof. For all $k \geq 0$, $f^{(k)}(a) = c_k$. Let c_r be the first coefficient which is non-zero. Then r is the order of $f(x)$ at a , as described, and

$$\lim_{x \rightarrow a} \frac{f(x)}{(x - a)^r} = c_r = f^{(r)}(a)/r! \neq 0.$$

Let us now apply this to the example above. Let $f_1(x) = \sin x - \arctan x$, $f_2(x) = \ln(1 + x)$. Then

$$\begin{aligned} f_1'(x) &= \cos x - \frac{1}{x^2 + 1}, & f_1'(0) &= 0, \\ f_1''(x) &= -\sin x + \frac{2x}{(x^2 + 1)^2}, & f_1''(0) &= 0, \\ f_1'''(x) &= -\cos x + \frac{2(x^2 + 1)^2 - 8x^2(x^2 + 1)}{(x^2 + 1)^4}, & f_1'''(0) &= 1. \end{aligned}$$

Hence, $f_1(x)$ has order 3 and r -value $1/6$ at 0, and

$$f_2'(x) = \frac{1}{1 + x}, \quad f_2'(0) = 1,$$

so $f_2(x)$ has order 1 and r -value 1 at 0. By a previous proposition,

$$\lim_{x \rightarrow 0} \frac{\sin x - \arctan x}{x^2 \ln(1 + x)} = \frac{1}{6}.$$

The orders cancel, and we obtain a non-zero number.

Remark. An approach using L'Hôpital's Rule does work, but one has to break the limit up as follows:

$$\lim_{x \rightarrow 0} \frac{\sin x - \arctan x}{x^2 \ln(1+x)} = \lim_{x \rightarrow 0} \frac{\sin x - \arctan x}{x^3} \cdot \lim_{x \rightarrow 0} \frac{x}{\ln(1+x)},$$

and then, the calculations actually amount to the same thing we have done (and the reader, if not sure, should check this).

Lastly, we present a method for determining partial fraction expansions, using the ideas so far.

Looking at one root, we wish to find constants A_1, A_2, \dots, A_n such that

$$\frac{p(x)}{(x-a)^n q(x)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \dots + \frac{A_n}{(x-a)^n} + \dots \quad (1)$$

We examine two cases.

Case I. $n = 1$

If the exponent is 1, then there is a very simple way of determining the coefficient. In fact, it is so easy one can usually do it in one's head. We illustrate with an example, where the exponent is 1 for all the roots.

We know

$$\frac{x-1}{(x-2)x(x+1)} = \frac{A}{x-2} + \frac{B}{x} + \frac{C}{x+1}$$

for some constants A, B , and C .

Multiplying by $x-2$, we obtain

$$\frac{x-1}{x(x+1)} = A + \frac{B(x-2)}{x} + \frac{C(x-2)}{x+1}.$$

Substituting $x = 2$, we automatically get $A = 1/6$. Similarly, we also obtain $B = 1/2$, and $C = -2/3$, and so

$$\frac{x-1}{(x-2)x(x+1)} = \frac{1}{6(x-2)} + \frac{1}{2x} - \frac{2}{3(x+1)}.$$

One must agree this is much easier and faster than the usual method by solving a system of linear equations.

Case II. $n > 1$

When there are multiple roots, the situation becomes much more complicated, and the method in Case I no longer works. For example, we know

$$\frac{x+1}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$$

for some constants A, B, C, D , and E . The values of B and E can be found by the method in Case I, but there is no way to directly isolate the other

constants in the same way. The key here, as in a previous proposition, is to make a smart substitution, to make these terms where the function goes to $\pm\infty$ into nicer terms.

Going back to (1), if we substitute $x = 1/t + a$, we obtain

$$\frac{p(1/t + a)t^n}{q(1/t + a)} = A_1t + A_2t^2 + \cdots + A_nt^n + \cdots .$$

Amazing! We get a garden variety polynomial (terms that blow up at a have become powers of t , with this substitution; why?). So, we make the substitution, and take the polynomial part. Furthermore, the A_i are simply the coefficients of the polynomial. But how do we know these are the right coefficients? Could there be more terms to the right, in that last equation, that contain polynomial terms?

No, not if we make sure that the expression remaining on the right stays bounded as t approaches $\pm\infty$. Then it can't contain any powers of t . We illustrate with an example.

Let

$$r(x) = \frac{1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

for some constants A , B , and C . Then

$$\begin{aligned} r\left(\frac{1}{t}\right) &= \frac{1}{\frac{1}{t}\left(\frac{1}{t}-1\right)^2} = \frac{t^3}{(1-t)^2} = \frac{(-y+1)^3}{y^2} \quad (\text{sub. } y = 1-t) \\ &= -y + 3 - \frac{3}{y} + \frac{1}{y^2} = 2 + t - \frac{3}{1-t} + \frac{1}{(1-t)^2}. \end{aligned}$$

The polynomial part of this expression is $2 + t$; we can be sure of this since the rest of the expression goes to 0 as t approaches $\pm\infty$. Similarly,

$$r\left(1 + \frac{1}{t}\right) = \frac{1}{\left(1 + \frac{1}{t}\right)\left(\frac{1}{t}\right)^2} = \frac{t^3}{t+1} = t^2 - t + 1 - \frac{1}{t+1}.$$

Hence, the polynomial part is $1 - t + t^2$.

Therefore, $A = 1$ (the coefficient of t in $2 + t$), and $B = -1$ and $C = 1$ (the coefficients of t , t^2 in $1 - t + t^2$). Hence,

$$\frac{1}{x(x-1)^2} = \frac{1}{x} - \frac{1}{x-1} + \frac{1}{(x-1)^2}.$$

We will leave it the reader to consider what happens in the case that the denominator of the fractions has non-real roots.



PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk () after a number indicates that a problem was submitted without a solution.*

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

*To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}'' \times 11''$ or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 October 1998**. They may also be sent by email to crux-editors@cms.math.ca. (It would be appreciated if email proposals and solutions were written in \LaTeX). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

2314. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given triangle ABC with $AB < AC$. The bisectors of angles B and C meet AC and AB at D and E respectively, and DE intersects BC at F .

Suppose that $\angle DFC = \frac{1}{2}(\angle DBC - \angle ECB)$. Determine angle A .

2315. *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*

Prove or disprove that $F(n) = \sqrt{n \left(1 - \frac{1}{n}\right)^{n-1}}$, where $F(n)$ is the maximum value of

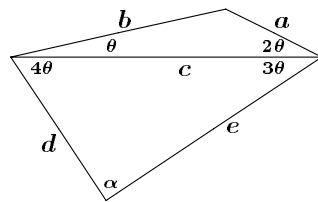
$$f(x_1, x_2, \dots, x_n) = \sin x_1 \cos x_2 \dots \cos x_n + \cos x_1 \sin x_2 \dots \cos x_n \\ + \dots + \cos x_1 \cos x_2 \dots \sin x_n,$$

$x_k \in [0, \pi/2]$, $k = 1, 2, \dots, n$, and $n > 1$ is a natural number.

2316. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given triangle ABC with angles B and C satisfying $C = 90^\circ + \frac{1}{2}B$. Suppose that M is the mid-point of BC , and that the circle with centre A and radius AM meets BC again at D . Prove that $MD = AB$.

2317. *Proposed by Richard I. Hess, Rancho Palos Verdes, California, USA.*



The quadrilateral shown at the left has integer elements a through e . The angles as shown are integer multiples of the smallest.

- What is the smallest possible value of c ?
- What is the smallest possible value of c if α must be obtuse?

2318. *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*

Suppose that ABC is a triangle with circumcentre O and circumradius R .

Consider the bisector (ℓ) of any side (say AC), and let P (the “pedal point”) be any point on ℓ inside the circumcircle.

Let K, L, M denote the feet of the perpendiculars from P to the lines AB, BC, CA respectively.

Show that $[KLM]$ (the area of the pedal triangle KLM) is a decreasing function of $\rho = \overline{OP}$, $\rho \in (0, R)$.

2319. *Proposed by Florian Herzig, student, Perchtoldsdorf, Austria.*

Suppose that UV is a diameter of a semicircle, and that P, Q are two points on the semicircle with $UP < UQ$. The tangents to the semicircle at P and Q meet at R . Suppose that S is the point of intersection of UP and VQ .

Prove that RS is perpendicular to UV .

2320. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Two circles on the same side of the line ℓ are tangent to it at D . The tangents to the smaller circle from a variable point A on the larger circle intersect ℓ at B and C . If b and c are the radii of the incircles of triangles ABD and ACD , prove that $b + c$ is independent of the choice of A .

2321. Proposed by David Doster, Choate Rosemary Hall, Wallingford, Connecticut, USA.

Suppose that $n \geq 2$. Prove that

$$\sum_{k=2}^n \left\lfloor \frac{n^2}{k} \right\rfloor = \sum_{k=n+1}^{n^2} \left\lfloor \frac{n^2}{k} \right\rfloor.$$

Here, as usual, $\lfloor x \rfloor$ means the greatest integer less than or equal to x .

2322. Proposed by K.R.S. Sastry, Dodballapur, India.

Suppose that the ellipse \mathcal{E} has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Suppose that Γ is any circle concentric with \mathcal{E} . Suppose that A is a point on \mathcal{E} and B is a point of Γ such that AB is tangent to both \mathcal{E} and Γ .

Find the maximum length of AB .

2323. Proposed by K.R.S. Sastry, Dodballapur, India.

Determine a positive constant c so that the Diophantine equation

$$uv^2 - v^2 - uv - u = c$$

has exactly four solutions in positive integers u and v .

2324. Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.

Find the exact value of $\sum_{n=1}^{\infty} \frac{1}{u_n}$, where u_n is given by the recurrence

$$u_n = n! \left(\frac{n-1}{n} \right) u_{n-1},$$

with the initial condition $u_1 = 2$.

2325* Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Suppose that q is a prime and n is a positive integer. Suppose that $\{a_k\}$ ($0 \leq k \leq n$) is given by

$$\sum_{k=0}^n a_k x^k = \frac{1}{q^n} \sum_{k=0}^n \binom{qn}{qk} (qx-1)^k.$$

Prove that each a_k is an integer.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2208. [1997: 47] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

1. Find a set of positive integers $\{x, y, z, a, b, c, k\}$ such that

$$\begin{aligned}y^2 z^2 &= a^2 + k^2 \\z^2 x^2 &= b^2 + k^2 \\x^2 y^2 &= c^2 + k^2\end{aligned}$$

2. Show how to obtain an infinite number of distinct sets of positive integers satisfying these equations.

I. Solution by C. Festraets-Hamoir, Brussels, Belgium.

1. Partons de triples pythagoriciens élémentaires

$$\begin{aligned}5^2 \cdot 1^2 &= 3^2 + 4^2 \\1^2 \cdot 5^2 &= 4^2 + 3^2 \\5^2 \cdot 5^2 &= 24^2 + 7^2\end{aligned}$$

Multiplications ces égalités respectivement par $3^2 \cdot 7^2$, $7^2 \cdot 4^2$ et $4^2 \cdot 3^2$; on obtient

$$\begin{aligned}15^2 \cdot 7^2 &= 63^2 + 84^2 \\7^2 \cdot 20^2 &= 112^2 + 84^2 \\20^2 \cdot 15^2 &= 288^2 + 84^2\end{aligned}$$

Donc $\{x, y, z, a, b, c, k\} = \{20, 15, 7, 63, 112, 288, 84\}$.

2. Etant donnée une solution:

$$\begin{aligned}y^2 z^2 &= a^2 + k^2 \\z^2 x^2 &= b^2 + k^2 \\x^2 y^2 &= c^2 + k^2\end{aligned}$$

il suffit de multiplier ces égalités respectivement par $b^2 c^2$, $c^2 a^2$, $a^2 b^2$ pour en obtenir une nouvelle:

$$\begin{aligned}b^2 c^2 y^2 z^2 &= a^2 b^2 c^2 + k^2 b^2 c^2 \\c^2 a^2 z^2 x^2 &= a^2 b^2 c^2 + k^2 c^2 a^2 \\a^2 b^2 x^2 y^2 &= a^2 b^2 c^2 + k^2 a^2 b^2\end{aligned}$$

On pose

$$\begin{array}{lll} b^2 y^2 = y_1^2 & a^2 b^2 c^2 = k_1^2 & k^2 b^2 c^2 = a_1^2 \\ c^2 z^2 = z_1^2 & & k^2 c^2 a^2 = b_1^2 \\ a^2 x^2 = x_1^2 & & k^2 a^2 b^2 = c_1^2 \end{array}$$

ce qui donne

$$\begin{array}{l} y_1^2 z_1^2 = k_1^2 + a_1^2 \\ z_1^2 x_1^2 = k_1^2 + b_1^2 \\ x_1^2 y_1^2 = k_1^2 + c_1^2 \end{array}$$

II. *Solution by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

We solve part 2 first. From our construction we can easily obtain particular solutions. Let q be any *even* natural number with the property that $q/2$ can be written as the product of two distinct factors in at least three different ways so that $q = 2u_1v_1 = 2u_2v_2 = 2u_3v_3$ where $u_i \neq u_j$, $v_i \neq v_j$, and $u_i \neq v_j$ for $i, j = 1, 2, 3$, $i \neq j$. (For example, we could let $q = 2p_1p_2p_3$ where p_1, p_2 , and p_3 are three distinct odd primes and let $u_i = p_i$, $v_i = q/2u_i$.) Let $a_1 = |u_1^2 - v_1^2|$, $b_1 = |u_2^2 - v_2^2|$, $c_1 = |u_3^2 - v_3^2|$. Then

$$\begin{array}{ll} a_1^2 + q^2 = (u_1^2 + v_1^2)^2 = l^2 & \text{where } l = u_1^2 + v_1^2 \\ a_2^2 + q^2 = (u_2^2 + v_2^2)^2 = m^2 & \text{where } m = u_2^2 + v_2^2 \\ a_3^2 + q^2 = (u_3^2 + v_3^2)^2 = n^2 & \text{where } n = u_3^2 + v_3^2 \end{array}$$

Now, let $d = lmn$, $a = a_1d$, $b = b_1d$, $c = c_1d$, $k = qd$, $x = mn$, $y = nl$, and $z = lm$. Then

$$\begin{array}{l} a^2 + k^2 = d^2(a_1^2 + q^2) = d^2l^2 = l^4m^2n^2 = y^2z^2 \\ b^2 + k^2 = d^2(b_1^2 + q^2) = d^2m^2 = l^2m^4n^2 = z^2x^2 \\ c^2 + k^2 = d^2(c_1^2 + q^2) = d^2n^2 = l^2m^2n^4 = x^2y^2 \end{array}$$

To get a particular solution, we could take $q = 24 = 2 \times 1 \times 12 = 2 \times 2 \times 6 = 2 \times 3 \times 4$ so $u_1 = 1$, $v_1 = 12$, $u_2 = 2$, $v_2 = 6$, $u_3 = 3$, $v_3 = 4$. Then $a_1 = 143$, $b_1 = 32$, $c_1 = 7$, $l = 145$, $m = 40$, $n = 25$, $d = 2^3 \times 5^4 \times 29$, which leads to $a = 2^3 \times 5^4 \times 11 \times 13 \times 29$, $b = 2^8 \times 5^4 \times 29$, $c = 2^3 \times 5^4 \times 7 \times 29$, $x = 2^3 \times 5^3$, $y = 5^3 \times 29$, $z = 2^3 \times 5^2 \times 29$ and $k = 2^6 \times 3 \times 5^4 \times 29$.

Also solved by GERALD ALLEN and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; ROBERT B. ISRAEL, University of British Columbia, Vancouver, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St Paul's School, London, England; DIGBY SMITH, Mount Royal College, Calgary, Alberta; PANOS E. TSAOUSSOGLU, Athens, Greece; and the proposer. There was one incomplete solution.

2209. [1997: 47] *Proposed by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.*

Let $ABCD$ be a cyclic quadrilateral having perpendicular diagonals crossing at P . Project P onto the sides of the quadrilateral.

1. Prove that the quadrilateral obtained by joining these four projections is inscribable and circumscribable.
2. Prove that the circle which passes through these four projections also passes through the mid-points of the sides of the given quadrilateral.

Comments. The recent solution to *Crux with Mayhem* 2194 [1997: 530-532] discussed the ambiguity of the word *circumscribable*. It is perhaps preferable to use the terminology *cyclic* and *circumscribing* for our *bicentric* quadrilateral; cf. *Crux with Mayhem* 2203.

Solution by Florian Herzig, student, Perchtoldsdorf, Austria.

Let the projections of P be K, L, M, N with $K \in AB$, etc. Furthermore, let the midpoints of AB, BC, CD, DA be Q, R, S and T . From one of Brahmagupta's theorems ([1] theorem 3.23), P is collinear with each projection (such as K) and the midpoint opposite that projection (namely S). Furthermore $QRST$ is a rectangle since its sides are parallel to the perpendicular diagonals (AC and BD) of the given quadrilateral. The diagonals QS and RT of the rectangle are diameters of its circumcircle, and this circle contains the projection points (such as K) since they form right triangles (namely QKS) that have a diameter as hypotenuse. In other words, K, L, M, N, Q, R, S, T are concyclic.

Next we have four cyclic quadrilaterals (each having an opposite pair of right angles): $KBLP, LCMP, MDNP$, and $NAKP$. Thus

$$\angle PKN = \angle PAN = \angle CAD = \angle CBD = \angle LBP = \angle LKP,$$

and hence, PK is an interior angle bisector of $KLMN$. Likewise PL, PM, PN are interior angle bisectors, whence $KLMN$ has an inscribed circle with incentre at P . [Note that the existence of an incircle does not require $AC \perp BD$.]

Reference.

- [1] H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited*. MAA, 1967.

Comments. The many references supplied by our solvers indicate that each part of problem 2209 can easily be found elsewhere; for example, see *Crux* 1836 [1993: 113; 1994: 84-85], *Crux* 1866 [1993: 203; 1994: 176], and problem 3 on the 1990 Canadian Mathematical Olympiad [1990: 198-199]. Seimiya found the problem itself as théorème 159, p. 319, in F.G.-M., *Exercices de géométrie*. (For the information of newcomers to *Crux with Mayhem*, F.G.-M. was a favourite reference of founding editor Léo Sauvé.)

The fifth edition of 1912 (Cours de mathematiques No. 267, Tours: Maison A. Mame et Fils, etc.) contains 2000 theorems with their proofs — nearly every result of elementary geometry known at that time. Because of his religious affiliation, the author had to remain anonymous.)

Lambrou mentions a converse: given a bicentric quadrilateral $KLMN$ there exists an appropriate cyclic quadrilateral $ABCD$ with orthogonal diagonals. (The sides of $ABCD$ are the perpendiculars at K, L, M, N to the lines from the incentre P .)

Also solved by CLAUDIO ARCONCHER, Jundiaí, Brazil; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; MANSUR BOASE, student, St. Paul's School, London, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; JORDI DOU, Barcelona, Spain; C. FESTAETS-HAMOIR, Brussels, Belgium; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St Paul's School, London, England; GOTTFRIED PERZ, Pestalozz gymnasium, Graz, Austria; ISTVÁN REIMAN, Budapest, Hungary; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; YEO KENG HEE, Hwa Chong Junior College, Singapore; and the proposer.

2211. [1997: 47] Proposed by Bill Sands, University of Calgary, Calgary, Alberta.

Several people go to a pizza restaurant. Each person who is “hungry” wants to eat either 6 or 7 slices of pizza. Everyone else wants to eat only 2 or 3 slices of pizza each. Each pizza in the restaurant has 12 slices.

It turns out that four pizzas are not sufficient to satisfy everyone, but that with five pizzas, there would be some pizza left over.

How many people went to the restaurant, and how many of these were “hungry”?

Solution by Kathleen E. Lewis, SUNY Oswego, Oswego, NY, USA.

I am interpreting the problem to mean, for example, that the “hungry” people would each be satisfied with 6 pieces, but would eat 7 if they were available.

If x represents the number of “hungry” people, and y the number of non-“hungry” people, then we know that $6x + 2y > 48$, but $7x + 3y < 60$. Because x and y must be integers, the quantities $6x + 2y$ and $7x + 3y$ are also integers and the first of these must be even. Therefore we can rewrite the inequalities as

$$6x + 2y \geq 50 \quad \text{and} \quad 7x + 3y \leq 59.$$

Taking the difference, we see that $x + y$, the total number of people, is at most 9. If x is seven or less, then $6x + 2y$ cannot be greater than 48, but if $x = 9$, then $7x > 60$, so x must be 8. If y were zero, then four pizzas would suffice, so y must be 1. Thus there are **eight** “hungry” people and **one** non-“hungry” person.

Also solved by MANSUR BOASE, student, St. Paul’s School, London, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; KEITH EKBLAW, Walla Walla, Washington, USA; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; ROBERT B. ISRAEL, University of British Columbia, Vancouver, BC; EDWARD J. KOSLOWSKA and ROSE MARIE SAENZ, students, Angelo State University, San Angelo, Texas, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St Paul’s School, London, England; SEAN MCILROY, student, University of British Columbia, Vancouver, BC; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; JOEL SCHLOSBERG, student, Hunter College High School, New York NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; ZUN SHAN and EDWARD T. H. WANG, Wilfrid Laurier University, Waterloo, Ontario; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; and the proposer. Also, one correct but anonymous solution was sent in. Two other readers misinterpreted the problem.

2212. [1997: 48] *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Let $S = \{1, 2, \dots, n\}$ where $n \geq 3$.

- (a) In how many ways can three integers x, y, z (not necessarily distinct) be chosen from S such that $x + y = z$? (Note that $x + y = z$ and $y + x = z$ are considered to be the same solution.)
- (b) What is the answer to (a) if x, y, z must be distinct?

Solution by William Moser, McGill University, Montreal, Quebec.

The answers to (a) and (b) are the cardinalities, s and t , of the sets:

$$\begin{aligned} S &= \{(x, y) \mid 1 \leq x \leq y, x + y \leq n\} \quad \text{and} \\ T &= \{(x, y) \mid 1 \leq x < y, x + y \leq n\}, \end{aligned}$$

where x and y denote integers.

Note first that the set

$$U = \{(x, y) \mid 1 \leq x, 1 \leq y, x + y \leq n\}$$

has cardinality $u = \binom{n}{2}$. Since the set

$$V = \{(x, y) \mid 1 \leq x = y, x + y \leq n\}$$

has cardinality $v = \lfloor \frac{n}{2} \rfloor$, it follows that the set

$$W = \{(x, y) \mid 1 \leq x, 1 \leq y, x \neq y, x + y \leq n\} = U - V$$

has cardinality $w = u - v = \binom{n}{2} - \lfloor \frac{n}{2} \rfloor$.

From $t = \frac{w}{2}$ and $s = t + v$, we then obtain that $s = \frac{1}{2} \binom{n}{2} + \lfloor \frac{n}{2} \rfloor$ and $t = \frac{1}{2} \binom{n}{2} - \lfloor \frac{n}{2} \rfloor$.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; KEITH EKBLAW, Walla Walla, Washington, USA; IAN JUNE L. GARCES, Manila, The Philippines, and GIOVANNI MAZZARELLO, Firenze, Italy; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St Paul's School, London, England; WILLIAM MOSER, McGill University, Montreal, Quebec (a second solution); GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; ZUN SHAN, Nanjing Normal University, Nanjing, China; DIGBY SMITH, Mount Royal College, Calgary, Alberta; and the proposer. One incorrect solution was sent in.

As expected, the answers can be given in many different forms. If we let $f(n)$ and $g(n)$ denote the numbers of ways for (a) and (b), respectively, then the most common answers are:

$$f(2n) = n^2, \quad f(2n+1) = n(n+1), \quad g(2n) = n(n-1) \text{ and } g(2n+1) = n^2,$$

which, as pointed out by many solvers, are equivalent to the following formulae:

$$f(n) = \begin{cases} \frac{n^2}{4} & \text{if } n \text{ is even} \\ \frac{n^2-1}{4} & \text{if } n \text{ is odd} \end{cases}; \quad g(n) = \begin{cases} \frac{n(n-2)}{4} & \text{if } n \text{ is even} \\ \frac{(n-1)^2}{4} & \text{if } n \text{ is odd} \end{cases}.$$

Janous and Seiffert observed that $g(n) = f(n-1)$ for $n \geq 4$ and Lambrou remarked that $f(n) = \frac{1}{4} n^2 - \frac{1}{2} (1 + (-1)^n)$.

Besides the solution given above (which was also obtained by Shan), there is a variety of other interesting "single expression" formulae involving the floor and/or the ceiling functions. These include:

$$1. \quad f(n) = \left\lfloor \frac{n^2}{4} \right\rfloor, \quad g(n) = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor = \left\lceil \frac{n(n-2)}{4} \right\rceil$$

(Garces and Mazzarello; Geretschläger; and the proposer).

$$2. f(n) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n+1}{2} \right\rfloor, g(n) = \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor$$

(Perz; and Seiffert),

$$3. f(n) = \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil, g(n) = \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{2} \right\rfloor$$

(Hsia),

$$4. f(n) = \left\lfloor \frac{n}{2} \right\rfloor \left(n - \left\lfloor \frac{n}{2} \right\rfloor \right), g(n) = (n-1) \left\lfloor \frac{n}{2} \right\rfloor - \left(\left\lfloor \frac{n}{2} \right\rfloor \right)^2$$

(Cabezón).

2213. [1997: 48] Proposed by Victor Oxman, University of Haifa, Haifa, Israel.

A generalization of problem 2095 [1995: 344, 1996: 373].

Suppose that the function $f(u)$ has a second derivative in the interval (a, b) , and that $f(u) \geq 0$ for all $u \in (a, b)$. Prove that

$$1. (y-z)f(x) + (z-x)f(y) + (x-y)f(z) > 0 \text{ for all } x, y, z \in (a, b), \\ z < y < x$$

if and only if $f''(u) > 0$ for all $u \in (a, b)$;

$$2. (y-z)f(x) + (z-x)f(y) + (x-y)f(z) = 0 \text{ for all } x, y, z \in (a, b), \\ z < y < x$$

if and only if $f(u)$ is a linear function on (a, b) .

Solution by Robert B. Israel, University of British Columbia, Vancouver, BC.

Part 1 is incorrect. The first statement is actually equivalent to " f is strictly convex", not " $f''(u) > 0$ for all $u \in (a, b)$ ". Strictly convex means that $f(tx + (1-t)y) < tf(x) + (1-t)f(y)$ for $x, y \in (a, b)$ and $0 < t < 1$. If f is twice differentiable this is equivalent to $f''(u) \geq 0$ everywhere with $f''(u) > 0$ on a dense set. For example, $f(x) = x^4$ is strictly convex everywhere but $f''(0) = 0$, and in this case

$$\begin{aligned} & (y-z)x^4 + (z-x)y^4 + (x-y)z^4 \\ &= (x-z)(y-z)(x-y) \left(\frac{(x+y)^2 + (y+z)^2 + (x+z)^2}{2} \right) \\ &> 0 \end{aligned}$$

if $z < y < x$.

More generally, let f be any continuous function on (a, b) , and $n \geq 3$ an integer. Given x_1, x_2, \dots, x_n let

$$L(f) = \sum_{i=1}^n (x_{i-1} - x_{i+1})f(x_i),$$

where we take $x_0 = x_n$ and $x_{n+1} = x_1$. Then the following are equivalent:

- (i) $L(f) > 0$ whenever $a < x_1 < x_2 < \dots < x_n < b$.
- (ii) f is strictly convex on (a, b) .

Similarly, for part 2, we will show that f is linear on (a, b) if and only if $L(f) = 0$ whenever $a < x_1 < x_2 < \dots < x_n < b$.

Proof of (i) \implies (ii): If f is not strictly convex, then either it is not convex at all or its graph includes a straight line segment. In the first case, there exist x, z , and t with $a < z < x < b$ and $0 < t < 1$ such that

$$f(tz + (1-t)x) > tf(z) + (1-t)f(x).$$

Taking $y = tz + (1-t)x$ and noting that $t = (x-y)/(x-z)$ we can rewrite this as

$$f(y) > \frac{x-y}{x-z}f(z) + \frac{y-z}{x-z}f(x)$$

or $(y-z)f(x) + (z-x)f(y) + (x-y)f(z) < 0$. Let $x_1 = z$, $x_2 = y$, and $x_3 = \dots = x_n = x$, and we have $L(f) < 0$ (if $n > 3$, note that the terms in $f(x_j)$ for $3 < j < n$ are 0 and the terms in $f(x_3)$ and $f(x_n)$ add to $(y-x)f(x)$). If $n = 3$ we must move x_3, \dots, x_n slightly so that $x_3 < \dots < x_n$, but by continuity the inequality will still be true if the changes are small enough. In the second case, we can take any $x_1 < \dots < x_n$ in the interval over which the graph is a straight line $f(x) = cx + d$, and then

$$L(f) = c \left(\sum_{i=1}^n x_{i-1}x_i - \sum_{i=1}^n x_i x_{i+1} \right) + d \left(\sum_{i=1}^n x_{i-1} - \sum_{i=1}^n x_{i+1} \right) = 0$$

Proof of (ii) \implies (i): As noted above, if g is linear on (a, b) then $L(g) = 0$. Thus we can add a linear function g to f without changing either (i) or (ii). With an appropriate choice of g we can make $f(x_1) = f(x_n) = 0$, and by strict convexity we then have $f(x_j) < 0$ for $1 < j < n$. Now write

$$\begin{aligned} L(f) &= \sum_{i=1}^n (x_{i-1} - x_i)f(x_i) + \sum_{i=1}^n (x_i - x_{i+1})f(x_i) \\ &= \sum_{i=1}^n (x_{i-1} - x_i)(f(x_i) + f(x_{i-1})) \end{aligned}$$

Note that the $i = 1$ term is 0, and each of the other terms is strictly positive, so the sum is strictly positive.

As noted above, if f is linear on (a, b) then $L(f) = 0$. Conversely, suppose $L(f) = 0$ whenever $a < x_1 < x_2 < \cdots < x_n < b$. By part 1, $L(h) > 0$ where $h(x) = x^2$, and so since L is a linear operator $L(f + ch) = cL(h) > 0$ for any $c > 0$. Again using part 1, $f + ch$ is strictly convex, and so $f = \lim_{c \rightarrow 0^+} (f + ch)$ is convex on (a, b) . By the same argument, $-f$ is convex, so f is concave on (a, b) . Any function that is both convex and concave on (a, b) is linear on (a, b) .

Also solved by FLORIAN HERZIG, student, Perchtoldsdorf, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece, (part 1 only). There were seven incorrect and two incomplete solutions. All of the incorrect solutions failed to notice the flaw in the statement of the problem, and then proceeded to treat as equivalent the two statements: “ f is strictly convex on (a, b) ” and “ $f''(u) > 0$ for all $u \in (a, b)$ ”.

Several solvers pointed out that the condition $f(u) \geq 0$ for all $u \in (a, b)$ is unnecessary.

Herzig remarks: If the interval considered is the set of real numbers instead, then it is sufficient to assume $y = \frac{x+z}{2}$ and f is a continuous function for part 2. The given equation reduces to Jensen's equation

$$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}$$

that can be solved using elementary methods (tricky substitution and Cauchy's method for solving functional equations).

Lambrou lists a series of six equivalent statements for part 1, where he assumes the existence of derivatives only when needed:

1. $(y - z)f(x) + (z - x)f(y) + (x - y)f(z) > 0$ for all $x, y, z \in (a, b)$ with $z < y < x$.
2. The first difference quotient $\frac{f(u) - f(v)}{u - v}$ ($u \neq v$) is strictly increasing in each variable (separately).
3. The second difference quotient

$$\left(\frac{f(u) - f(w)}{u - w} - \frac{f(v) - f(w)}{v - w} \right) / (u - v)$$

($u \neq w, v \neq w, u \neq v$) is strictly positive for all $u, v, w \in (a, b)$.

4. f' is strictly increasing on (a, b) .
5. f is strictly convex on (a, b) .

6. $f''(x) \geq 0$ for all $x \in (a, b)$ and there is no interval $I \subseteq (a, b)$ such that $f''(x) = 0$ for all $x \in I$. (In the language of topology, this says that $f''(x) > 0$ on a dense subset of (a, b) .)

2214. [1997: 109] Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Let $n \geq 2$ be a natural number. Show that there exists a constant $C = C(n)$ such that for all real $x_1, \dots, x_n \geq 0$ we have

$$\sum_{k=1}^n \sqrt{x_k} \leq \sqrt{\prod_{k=1}^n (x_k + C)}.$$

Determine the minimum $C(n)$ for some values of n .

[For example, $C(2) = 1$.]

Solution by Michael Lambrou, University of Crete, Crete, Greece.

We show that the inequality is valid for an aggregate of values of C of which the least is

$$C(n) = \frac{n-1}{\sqrt[n-1]{n^{n-2}}}, \quad n \geq 2.$$

Let us first do the easier task of proving the existence of C 's which make the inequality valid. Of course this part will be redundant as soon as we improve the technique to find the least C .

Setting $x_i = y_i^2$ where $y_i \geq 0$ ($i = 1, \dots, n$), we are to show, equivalently, that for some C we have

$$\left(\sum_{i=1}^n y_i \right)^2 \leq \prod_{i=1}^n (y_i^2 + C). \quad (1)$$

Treating the right hand side of (1) as a polynomial in C , we observe that all coefficients are non-negative and that the coefficient of C^{n-1} is $\sum y_i^2$. Thus

$$\prod_{i=1}^n (y_i^2 + C) \geq \left(\sum_{i=1}^n y_i^2 \right) C^{n-1}.$$

But by the Cauchy-Schwarz inequality we have

$$\left(\sum_{i=1}^n y_i \right)^2 \leq n \left(\sum_{i=1}^n y_i^2 \right),$$

so inequality (1) will be valid if we choose $C = n^{1/(n-1)}$ or larger. This completes the easier task.

It turns out that $n^{1/(n-1)}$ is only a slight overestimate of the minimum C , which we now seek. For any C for which (1) is valid, set $w_i = y_i\sqrt{n-1}/\sqrt{C}$, so that (1) becomes

$$\left(\sum_{i=1}^n w_i\right)^2 \leq \frac{C^{n-1}}{(n-1)^{n-1}} \prod_{i=1}^n (w_i^2 + n - 1)$$

or equivalently

$$\left(\sum_{i=1}^n w_i\right)^2 \leq \frac{C^{n-1}n^n}{(n-1)^{n-1}} \prod_{i=1}^n \left(\frac{w_i^2 - 1}{n} + 1\right). \quad (2)$$

To find the minimum C we shall first show that the following inequality is valid:

$$\left(\sum_{i=1}^n w_i\right)^2 \leq n^2 \prod_{i=1}^n \left(\frac{w_i^2 - 1}{n} + 1\right). \quad (3)$$

We shall use the Weierstrass inequality

$$\prod_{i=1}^m (1 + a_i) \geq 1 + \sum_{i=1}^m a_i,$$

which holds if all $a_i \geq 0$ or if $-1 < a_i < 0$ for all i [for example, see item 3.2.37, page 210 of D. S. Mitrinović, *Analytic Inequalities*]. Without loss of generality let $w_1, \dots, w_t \geq 1$ and $0 \leq w_{t+1}, \dots, w_n < 1$, where $t \in \{0, 1, \dots, n\}$. Then

$$\begin{aligned} \prod_{i=1}^n \left(\frac{w_i^2 - 1}{n} + 1\right) &= \prod_{i=1}^t \left(\frac{w_i^2 - 1}{n} + 1\right) \cdot \prod_{i=t+1}^n \left(\frac{w_i^2 - 1}{n} + 1\right) \\ &\geq \left(1 + \sum_{i=1}^t \frac{w_i^2 - 1}{n}\right) \left(1 + \sum_{i=t+1}^n \frac{w_i^2 - 1}{n}\right) \\ &= \frac{1}{n^2} \left(n - t + \sum_{i=1}^t w_i^2\right) \left(t + \sum_{i=t+1}^n w_i^2\right) \\ &= \frac{1}{n^2} \left(\sum_{i=1}^t w_i^2 + \sum_{i=t+1}^n 1^2\right) \left(\sum_{i=1}^t 1^2 + \sum_{i=t+1}^n w_i^2\right) \\ &\geq \frac{1}{n^2} \left(\sum_{i=1}^n w_i\right)^2 \end{aligned}$$

(the last inequality by the Cauchy–Schwarz inequality), which proves (3). Note that equality occurs for $w_1 = \dots = w_n = 1$. We conclude that (2)

is valid for any C with $C^{n-1}n^n/(n-1)^{n-1} \geq n^2$, i.e., with

$$C \geq \frac{n-1}{\sqrt[n-1]{n^{n-2}}}, \quad n \geq 2.$$

The minimum value $C(n)$ we seek is then as stated at the beginning, since for

$$x_i = y_i^2 = C \left(\frac{w_i}{\sqrt{n-1}} \right)^2 = \frac{C \cdot 1}{n-1}$$

the original inequality reduces to equality.

Remark. The above shows $C(2) = 1$,

$$C(3) = \frac{2}{\sqrt{3}} \approx 1.1547, \quad C(4) = \frac{3}{\sqrt[3]{4^2}} \approx 1.1905, \quad C(5) = \frac{4}{\sqrt[4]{5^3}} \approx 1.1963,$$

and generally $C(n) \approx n^{1/(n-1)}$ which approaches 1 in the limit.

The minimum $C(n)$ for all $n \geq 2$ was also found by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; KEE-WAI LAU, Hong Kong; and HEINZ-JÜRGEN SEIFFERT, Berlin, Germany. The existence of $C(n)$ was proved by FLORIAN HERZIG, student, Perchtoldsdorf, Austria; JOE HOWARD, New Mexico Highlands University, Las Vegas, NM, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; YEO KENG HEE, student, Hwa Chong Junior College, Singapore; and the proposer.

Of those readers who found the minimum $C(n)$, only Lambrou managed to do it without multivariable calculus. He also sent in a second solution, using calculus.

Herzig found the minimum $C(n)$ for $n = 2$ and 3, and Howard found it for $n = 2$. Konečný conjectured the correct minimum value of $C(n)$ for all $n \geq 2$, but proved it only for $n = 2$ and 3.

The proposer showed more generally that, for all real $\alpha > 0$ and integers $n \geq 2$, there is a constant $C = C_\alpha(n)$ such that

$$\sum_{i=1}^n x_i^\alpha \leq \left(\prod_{i=1}^n (x_i + C) \right)^\alpha$$

for all real $x_1, \dots, x_n \geq 0$.

2215*. [1997: 109] Proposed by Theodore Chronis, student, Aristotle University of Thessaloniki, Greece.

Let P be a point inside a triangle ABC . It is known how to determine P such that $PA + PB + PC$ is a minimum (known as Fermat's Problem for Torricelli).

Determine P such that $PA + PB + PC$ is a maximum.

We received two types of solutions to this problem:

I. P is considered to be an **interior** point of triangle ABC (this is the problem as it was posed);

II. P is considered to be inside or on the boundary of triangle ABC .

Solution by Michael Lambrou, University of Crete, Greece, [slightly modified by the editor].

(I) Let P be an internal point. We show the existence of another internal point Q such that $QA + QB + QC > PA + PB + PC$.

Consider the ellipse E through P with foci B and C and l the tangent to this ellipse at P . Then the circle centred at A with radius AP meets l at P . Therefore, the points on l to at least one side of P are outside the circle. Take Q to be a point on l that is outside the circle (hence $QA > PA$) but close to P so that it is inside triangle ABC . Since Q is on l , Q is outside E and $QB + QC > PB + PC$. Adding we get the required result.

Therefore, $PA + PB + PC$ has no maximum.

(II) Assume that $AC \geq AB \geq BC$ and P is an interior or boundary point of triangle ABC , $P \neq A$. Let the line parallel to BC through P meet AB at D and AC at E . By similarity $AE \geq AD \geq DE$ and $AE \geq AP$. Hence

$$\begin{aligned} PA + PB + PC &< PA + (BD + DP) + (PE + EC) \\ &= PA + DE + BD + EC \\ &\leq PA + AD + BD + EC \\ &= PA + AB + EC \\ &\leq AE + AB + EC \\ &= AC + AB \end{aligned}$$

[Seimiya and Tsaoussoglou both noted that this inequality, for an interior point P , is due to Visschers (1902) [see F.G.M. Exercices de Géométrie, p. 228, Th. 25 – IV]].

However, if $P = A$ then $PA + PB + PC = AB + AC$.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK (I and II); JORDI DOU, Barcelona, Spain (I and II); RICHARD I. HESS, Rancho Palos Verdes, California, USA (I and II); PETER HURTHIG, Columbia College, Burnaby, BC (II); ROBERT B. ISRAEL, University of British Columbia, Vancouver, BC (II); VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA (II); GERRY LEVERSHA, St Paul's School, London, England (I); ISTVÁN REIMAN, Budapest, Hungary (I); TOSHIO SEIMIYA, Kawasaki, Japan (II); PANOS E. TSAOUSSOGLOU, Athens, Greece (II).

2216. [1997: 110] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Suppose that $\lambda \geq 1$ is a natural number.

1. Determine the set of all λ 's such that the diophantine equation $x^\lambda + y^2 = z^2$ has infinitely many solutions.
2. * For any such λ , determine all solutions of this equation.

Solution by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

We show that $x^\lambda + y^2 = z^2$ has infinitely many solutions for *all* natural numbers λ by finding *all* solutions for any λ .

Suppose first that $x^\lambda + y^2 = z^2$. Let

$$x = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \quad (1)$$

where $p_1 < p_2 < \cdots < p_k$ are primes and $\alpha_i \in \mathbb{N}$ for $i = 1, 2, \dots, k$. Then $x^\lambda + y^2 = z^2$ is equivalent to $(z + y)(z - y) = \prod_{i=1}^k p_i^{\lambda \alpha_i}$ and so $z + y = \prod_{i=1}^k p_i^{\beta_i}$

and $z - y = \prod_{i=1}^k p_i^{\gamma_i}$ where β_i, γ_i are nonnegative integers satisfying

$$\beta_i + \gamma_i = \lambda \alpha_i \quad (2)$$

for $i = 1, 2, \dots, k$ and

$$\prod_{i=1}^k p_i^{\beta_i} > \prod_{i=1}^k p_i^{\gamma_i}. \quad (3)$$

It follows that

$$y = \frac{1}{2} \left(\prod_{i=1}^k p_i^{\beta_i} - \prod_{i=1}^k p_i^{\gamma_i} \right), \quad (4)$$

$$z = \frac{1}{2} \left(\prod_{i=1}^k p_i^{\beta_i} + \prod_{i=1}^k p_i^{\gamma_i} \right). \quad (5)$$

Conversely, if $p_1 \neq 2$ and x, y, z are given by (1), (4), (5) where $\alpha_i \in \mathbb{N}$ and β_i, γ_i are nonnegative integers satisfying (2) and (3) for $i = 1, 2, \dots, k$, then it is easy to see that $x^\lambda + y^2 = z^2$ holds. If $p_1 = 2$, then β_1 and γ_1 must both be *positive* integers, since x^λ even implies that $z + y$ and $z - y$ are both even. (In this case, $\beta_1 \geq 1$ and $\gamma_1 \geq 1$ and so $\lambda \alpha_1 \geq 2$ which is true for all $\lambda \geq 2$, while if $\lambda = 1$ we must assume $\alpha_1 \geq 2$.)

Note that condition (3) could be replaced by the condition that

$$\beta_i \neq \gamma_i \quad (6)$$

for at least one i if we replace (4) by

$$y = \frac{1}{2} \left| \prod_{i=1}^k p_i^{\beta_i} - \prod_{i=1}^k p_i^{\gamma_i} \right|. \quad (7)$$

In summary, if $p_1 \neq 2$ then all solutions to $x^\lambda + y^2 = z^2$ are given by (1), (5), and (7) where $\alpha_i \in \mathbb{N}$ and β_i, γ_i are nonnegative integers satisfying (2) and (6). If $p_1 = 2$ we must impose the condition that $\beta_1 \geq 1, \gamma_1 \geq 1$.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; YEO KENG HEE, Hwa Chong Junior College, Singapore; and the proposer. Four solvers solved part 1 correctly: FLORIAN HERZIG, student, Perchtoldsdorf, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; and GERRY LEVERSHA, St Paul's School, London, England. There was one incorrect solution, and one incomplete solution.

2217. [1997: 110] Proposed by Bill Sands, University of Calgary, Calgary, Alberta.

- (a) Prove that for every sufficiently large positive integer n , there are arithmetic progressions a_1, a_2, a_3 and b_1, b_2, b_3 of positive integers such that $n = a_1b_1 + a_2b_2 + a_3b_3$.
- (b) What happens if we require $a_1 = b_1 = 1$?

(This is a variation of problem 3 of the 1995/96 Alberta High School Mathematics Competition, Part II, which may appear in a future Skoliad Corner.)

I. Solution to part (a) by Joel Schlosberg, student, Bayside, New York, USA.

Let the progressions be $2, 3, 4$ and $a, a + b, a + 2b$. We have to show that

$$n = 2a + 3(a + b) + 4(a + 2b) = 9a + 11b$$

for sufficiently large n . There is a theorem of Euclid that any sufficiently large number can be represented in the form $ka + lb$ [for some positive integers a and b] if k and l are relatively prime. Since $\gcd(9, 11) = 1$, the answer follows immediately.

[*Editorial note.* Coincidentally, in a recent issue of *Mathematics and Informatics Quarterly* (Vol. 7, No. 4, 1997, p. 185), *Crux with Mayhem* regular K. R. S. Sastry gives a proof of the above "theorem of Euclid", in fact showing more generally the known result that, if k and l are relatively prime, every integer greater than kl can be expressed as $ka + lb$ for some positive integers a and b . Thus Schlosberg's proof shows that the statement of the problem holds for every integer $n \geq 100$. Actually, we can allow $b = 0$ in this problem, which (it can easily be checked) lowers the bound to $n \geq 89$. For more, see the editorial remarks below.]

II. *Solution to part (b) by Christopher J. Bradley, Clifton College, Bristol, UK.*

[Bradley first solved part (a). — Ed.]

If one requires $a_1 = 1$ and $b_1 = 1$ then only those n are representable for which non-negative x, y exist such that

$$1 + (1 + x)(1 + y) + (1 + 2x)(1 + 2y) = n; \quad (1)$$

that is, $n = 3 + 3x + 3y + 5xy$ or

$$5n - 6 = (5x + 3)(5y + 3).$$

Since neither $5x + 3$ nor $5y + 3$ can equal 1 this implies $5n - 6$ has to be composite as a necessary condition. When $5n - 6$ is prime, $n = 5$ or $n = 17$ for example, no x and y can possibly be found. Now, by Dirichlet's Theorem, since 5 and 6 are coprime there are indefinitely large primes of the form $5n - 6$. It follows that there are infinitely many n which cannot be expressed in the form (1).

III. *Editorial remarks.*

Here are extensions of this problem observed by various solvers. For the complete list of solvers, see below.

Shan and Wang calculated that the largest integer n **not** expressible as in part (a) is $n = 17$. Some other solvers gave larger "impossible" values for n because they did not allow constant arithmetic progressions.

Israel proved part (a) under the stronger condition that a_1 is any fixed positive integer. He also strengthened part (b) by showing that (a) fails whenever a_1 and b_1 are fixed positive integers.

Lambrou generalized (a) in another direction, namely he found all positive integers m so that every sufficiently large integer n can be written in the form

$$n = a_1b_1 + a_2b_2 + \cdots + a_mb_m$$

where a_1, \dots, a_m and b_1, \dots, b_m are m -term arithmetic progressions of positive integers. His answer: $m = 1, 2, 3$ or 6 .

Most solvers noted that one of the arithmetic progressions in part (a), say a_1, a_2, a_3 , can be fixed while the other varies, and the result is still true; for example see solution I, which uses $(a_1, a_2, a_3) = (2, 3, 4)$. Lambrou also investigated exactly which fixed (a_1, a_2, a_3) allow this. Letting $(a_1, a_2, a_3) = (a - h, a, a + h)$ where $a > h > 0$ are integers, he proved that every sufficiently large positive integer n can be expressed as $n = a_1b_1 + a_2b_2 + a_3b_3$ for some positive integer arithmetic progression b_1, b_2, b_3 if and only if $3a$ and $2h$ are relatively prime.

As a variation on part (b), Leversha noted that if we impose $a_1 = 1 = b_3$, then no integer of the form $p - 6$, where p is prime, can be written as in (a).

Readers may like to verify the above results for themselves! And there are still many related problems that await investigation. For example, some readers' solutions to (a) use arithmetic progressions in which the common differences are bounded (in absolute value). Bradley's solution seems to be the best in this respect, with his common differences always at most 7. Can this number be lowered? That is, can every sufficiently large positive integer be written as $a_1b_1 + a_2b_2 + a_3b_3$, where a_1, a_2, a_3 and b_1, b_2, b_3 are arithmetic progressions of positive integers with $|a_1 - a_2|$ and $|b_1 - b_2|$ both at most 6, say? Or what if the common differences of the arithmetic progressions in (a) are required to be bounded below by some constant? Finally, in (b), what if we only require $a_1 = b_1$?

Both parts also solved by FLORIAN HERZIG, student, Perchtoldsdorf, Austria; ROBERT B. ISRAEL, University of British Columbia, Vancouver, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St Paul's School, London, England; ZUN SHAN and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; and the proposer. Part (a) only solved by DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; RICHARD I. HESS, Rancho Palos Verdes, California, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; and YEO KENG HEE, student, Hwa Chong Junior College, Singapore.

All proofs of part (b) were similar, though Lambrou observed that the full statement of Dirichlet's Theorem is not needed, only that there are infinitely many primes of the form $10k - 1$ (as can be seen from Solution II above), and that this special case is a little easier to prove: for example, see T. Nagell's Introduction to Number Theory, Chapter 5, §50. Is there a solution to (b) that does not need Dirichlet's Theorem at all?

The original Alberta High School Mathematics Competition problem was to show that every integer n can be written in the form $a_1b_1 + a_2b_2 + a_3b_3$, where a_1, a_2, a_3 and b_1, b_2, b_3 are arbitrary arithmetic progressions of integers.

2218. [1997: 110] Proposed by Victor Oxman, University of Haifa, Haifa, Israel.

Suppose that a, b, c are positive real numbers and that

$$abc = (a + b - c)(b + c - a)(c + a - b).$$

Clearly $a = b = c$ is a solution. Determine all others.

I Solution by Goran Conar, student, Gymnasium Varaždin, Varaždin, Croatia.

If any of the three quantities, $a + b - c$, $b + c - a$ and $c + a - b$ is negative, say $a + b - c < 0$, then we have $c > a + b$, so that $b + c - a > 0$,

and $c + a - b > 0$. This implies that $abc < 0$, a contradiction. Hence $a + b - c > 0$, $b + c - a > 0$ and $c + a - b > 0$.

Note that

$$(a + b - c)(a - b + c) = a^2 - (b - c)^2 \leq a^2, \quad (1)$$

$$(b + c - a)(b - c + a) = b^2 - (c - a)^2 \leq b^2, \quad (2)$$

$$(c + a - b)(c - a + b) = c^2 - (a - b)^2 \leq c^2. \quad (3)$$

Multiply the three inequalities and take non-negative square roots. We get

$$(a + b - c)(b + c - a)(c + a - b) \leq abc. \quad (4)$$

Since equality holds in (4), it must also hold in (1), (2) and (3). Therefore, $a = b = c$ is the only solution.

II *Solution by Robert B. Israel, University of British Columbia, Vancouver, BC.*

Since the given equation is invariant under any permutation of a, b, c , we may assume that $0 < a \leq b \leq c$.

The equation can be re-written as

$$(b + c - a)(b - c)^2 + a(b - c)(c - a) = 0.$$

Since both terms on the left are non-negative, the only way to obtain equality is with $a = b = c$. Hence there are no other solutions in positive reals.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MANSUR BOASE, student, St. Paul's School, London, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; JOE HOWARD, New Mexico Highlands University, Las Vegas, NM, USA; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St Paul's School, London, England; CAN ANH MINH, University of California, Berkeley, CA, USA; BOB PRIELIPP, University of Wisconsin-Oshkosh, Wisconsin, USA; ISTVÁN REIMAN, Budapest, Hungary; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; ZUNSHAN and EDWARD T. H. WANG, Wilfrid Laurier University, Waterloo, Ontario; DIGBY SMITH, Mount Royal College, Calgary, Alberta; YEO KENG HEE, Hwa Chong Junior College, Singapore; and the proposer. There was also one incorrect solution.

A few solvers remarked that if a, b and c are non-negative, then we have the additional solutions: $(a, b, c) = (0, t, t), (t, 0, t), (t, t, 0)$ where $t \geq 0$ is arbitrary.

Howard pointed out that $(a + b - c)(b + c - a)(c + a - b) \leq abc$ is problem #3 of the 1981 British Mathematical Olympiad, and referred to Rabinowitz's Index to Mathematical Problems, 1980–1984, p. 49.

[Ed. — Actually, it is much “older” than that and dates back to at least 1925. See, for example, § 1.3 on p. 12 of Geometric Inequalities by O. Bottema et al.]

Many solvers, after establishing the fact that a, b, c are the sides of a triangle, say Δ , showed that the given equality is equivalent to the fact that $2r = R$, where r and R denote the inradius and circumradius of Δ respectively. Then the conclusion follows from a celebrated theorem of Euler, which states that $2r \leq R$, with equality if and only if Δ is equilateral.

[Ed. — This can be found, for example, in § 5.1 on p. 48 of the book mentioned in the previous paragraph.]

2210*. [1997: 47] Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.

Given $a_0 = 1$, the sequence $\{a_n\}$ ($n = 1, 2, \dots$) is given recursively by

$$\binom{n}{n} a_n - \binom{n}{n-1} a_{n-1} + \binom{n}{n-2} a_{n-2} - \dots \pm \binom{n}{\lfloor \frac{n}{2} \rfloor} a_{\lfloor \frac{n}{2} \rfloor} = 0.$$

Which terms have value 0?

No solutions have been received to date. The problem remains open.

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