THE ACADEMY CORNER

No. 14

Bruce Shawyer

All communications about this column should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7

Christopher Small writes:

> *I notice that in the Hints – 2 section of the Bernoulli Trials, the statement that the hint does not work for question 9 has been interpolated.*
>
> *It seems to me that the statement is correct as I originally gave it to you. A counterexample is easily found with $a = c$. For example, $a = c = 1$ and $b = 1/2$ is a counterexample.*

This month, we present solutions to some of the problems in a university entrance scholarship examination paper from the 1940's, which appeared in the April 1997 issue of ***CRUX with MAYHEM***.

1. Find all the square roots of

$$1 - x + \sqrt{22x - 15 - 8x^2}.$$

Solution.

First we note that the only way to generate the given expression as a square, is to square

$$\sqrt{ax + b} + \sqrt{cx + d}.$$

There are, of course, two square roots, being $\pm$ this quantity. So we have

$$\begin{aligned} &\left(\sqrt{ax+b} + \sqrt{cx+d}\right)^2 \\ = &\quad (a+c)x + (b+d) + 2\sqrt{acx^2 + (ad+bc)x + bd} \\ = &\quad -x + 1 + 2\sqrt{-2x^2 + \frac{11}{2}x - \frac{15}{4}}. \end{aligned}$$

Since this is an identity, we equate coefficients, yielding

$$\begin{aligned} a+c &= -1, & (1)\\ b+d &= 1, & (2)\\ ac &= -2, & (3)\\ ad+bc &= \tfrac{11}{2}, & (4)\\ bd &= -\tfrac{15}{4}. & (5)\end{aligned}$$

Solving (1) and (2) for c and d respectively, and substituting in (3) and (5) gives

$$\begin{aligned} a^2+a-2 &= (a+2)(a-1) = 0,\\ b^2-b-\tfrac{15}{4} &= (b-\tfrac{5}{2})(b+\tfrac{3}{2}) = 0.\end{aligned}$$

so that $a = -2, 1$ and $b = \frac{5}{2}, -\frac{3}{2}$, giving the corresponding values of $c = 1, -2$ and $d = -\frac{3}{2}, \frac{5}{2}$.

We must now substitute these into (4), and this leads to the solution that the square roots of the given expression are

$$\pm\left(\sqrt{-2x+\tfrac{5}{2}}+\sqrt{x-\tfrac{3}{2}}\right).$$

2. Find all the solutions of the equations:

$$\begin{aligned} x+y+z &= 2,\\ x^2+y^2+z^2 &= 14,\\ xyz &= -6.\end{aligned}$$

Solution.

First we recall the expressions involving the roots of a cubic:

$$(P-x)(P-y)(P-z) = P^3-(x+y+z)P^2+(xy+yz+zx)P-xyz.$$

We also note that

$$\frac{(x+y+z)^2-(x^2+y^2+z^2)}{2} = xy+yz+zx.$$

So, from two of the given three expressions, we get

$$xy+yz+zx = -5.$$

Thus, the solution of the given equations is the set of roots of the cubic equation

$$P^3-2P^2-5P+6 = 0.$$

It is easy to check that $P = 1$ is a root: so it is easy to factor into

$$(P-1)(P+2)(P-3) \;=\; 0,$$

giving that the other two roots are $P = -2$ and $P = 3$.

Thus, the solution set of the given equations is any permutation of $1, -2, 3$.

3. Suppose that n is a positive integer and that C_k is the coefficient of x^k in the expansion of $(1+x)^n$. Show that

$$\sum_{k=0}^{n}(k+1)C_k^2 \;=\; \frac{(n+2)\,(2n-1)!}{n!\,(n-1)!}.$$

Solution.

Consider $\left(\sum_{k=0}^{n}(k+1)C_k^2\right) x^n$. Since $C_{n-k} = C_k$, we can write this as

$$\sum_{k=0}^{n}\left((k+1)C_k x^k\right)\left(C_{n-k}x^{n-k}\right).$$

This is the term in x^k in the product

$$\left(\sum_{k=0}^{n}(k+1)C_k x^k\right)\left(\sum_{k=0}^{n}C_k x^k\right).$$

The right member of this product is $(1+x)^n$. There are several ways to determine the left member: for example,

$$x(1+x)^n \;=\; \sum_{k=0}^{n}C_k x^{k+1},$$

so that, on differentiating, we have that

$$(1+x)^n + nx(1+x)^{n-1} \;=\; \sum_{k=0}^{n}(k+1)C_k x^k.$$

Thus, the product above is

$$(1+x)^{n-1}(1+(n+1)x)\times(1+x)^n \;=\; (1+(n+1)x)(1+x)^{2n-1}.$$

The coefficient of x^n in this product is

$$\frac{(2n-1)!}{(n-1)!\,n!}(n+1) + \frac{(2n-1)!}{n!\,(n-1)!} \;=\; \frac{(n+2)\,(2n-1)!}{n!\,(n-1)!}.$$

THE OLYMPIAD CORNER

No. 185

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

To begin this number, we give the ten problems of the Seventh Irish Mathematical Olympiad written May 7, 1994. My thanks go to Richard Nowakowski for collecting the problems when he was Canadian Team leader at the IMO in Hong Kong.

IRISH MATHEMATICAL OLYMPIAD 1994

Second Paper — 7 May 1994

Time: 3 hours

1. Let x, y be positive integers with $y > 3$ and

$$x^2 + y^4 = 2\left[(x-6)^2 + (y+1)^2\right].$$

Prove that $x^2 + y^4 = 1994$.

2. Let A, B, C be three collinear points with B between A and C. Equilateral triangles ABD, BCE, CAF are constructed with D, E on one side of the line AC and F on the opposite side. Prove that the centroids of the triangles are the vertices of an equilateral triangle. Prove that the centroid of this triangle lies on the line AC.

3. Determine with proof all real polynomials $f(x)$ satisfying the equation

$$f(x^2) = f(x)f(x-1).$$

4. Consider the set of $m \times n$ matrices with every entry either 0 or 1. Determine the number of such matrices with the property that the number of "1"s in each row and in each column is even.

5. Let $f(n)$ be defined on the set of positive integers by the rules

$$f(1) = 2 \quad \text{and} \quad f(n+1) = (f(n))^2 - f(n) + 1; \quad n = 1, 2, 3, \ldots .$$

Prove that for all integers $n > 1$

$$1 - \frac{1}{2^{2^{n-1}}} < \frac{1}{f(1)} + \frac{1}{f(2)} + \cdots + \frac{1}{f(n)} < 1 - \frac{1}{2^{2^n}}.$$

6. A sequence x_n is defined by the rules

$$x_1 = 2$$

and

$$nx_n = 2(2n-1)x_{n-1}; \quad n = 2, 3, \ldots .$$

Prove that x_n is an integer for every positive integer n.

7. Let p, q, r be distinct real numbers which satisfy the equations

$$q = p(4-p)$$

$$r = q(4-q)$$

$$p = r(4-r).$$

Find all possible values of $p+q+r$.

8. Prove that for every integer $n > 1$

$$n((n+1)^{2/n} - 1) \ < \ \sum_{i=1}^{n} \frac{2k+1}{i^2} \ < \ n\left(1 - n^{-2/(n-1)}\right) + 4.$$

9. Let w, a, b, c be distinct real numbers with the property that there exist real numbers x, y, z for which the following equations hold:

$$\begin{aligned} x + y + z &= 1 \\ xa^2 + yb^2 + zc^2 &= w^2 \\ xa^3 + yb^3 + zc^3 &= w^3 \\ xa^4 + yb^4 + zc^4 &= w^4. \end{aligned}$$

Express w in terms of a, b, c.

10. If a square is partitioned into n convex polygons, determine the maximum number of edges present in the resulting figure.
[You may find it helpful to use a theorem of Euler which states that if a polygon is partitioned into n polygons, then $v - e + n = 1$ where v is the number of vertices and e is the number of edges in the resulting figure].

Next we turn to the "official" results of the 38th IMO which was written in Mar del Plato, Argentina, July 24 and 25, 1997. My source this year was the contest WEB site. I hope that I have made no serious errors in compiling the results and transcribing names.

This year a total of 460 students from 82 countries took part. This is somewhat up from last year. Sixty-five countries sent teams of 6 (the number invited to participate in recent years). But there were 13 teams of smaller size, 3 of five members, 2 of size four, 7 of size three, and 1 with two members.

The contest is officially an individual competition and the six problems were assigned equal weights of seven marks each (the same as the last 16 IMO for a maximum possible individual score of 42 and a total possible of 252 for a national team of six students). For comparison see the last 16 IMO reports in [1981: 220], [1982: 223], [1983: 205], [1984: 249], [1985: 202], [1986: 169], [1987: 207], [1988: 193], [1989: 193], [1990: 193], [1991: 257], [1992: 263], [1993: 256], [1994: 243], [1995: 267] and [1996: 301].

There were 4 perfect scores. The jury awarded first prize (Gold) to the thirty-nine students who scored 35 or more. Second (Silver) prizes went to the seventy students with scores from 25 to 34, and third (Bronze) prizes went to the one hundred and twenty-two students with scores from 15 to 24. Any student who did not receive a medal, but who scored full marks on at least one problem, was awarded honourable mention. This year there were seventy-eight honourable mentions awarded. The median score on the examination was 14.

Congratulations to the Gold Medalists, and especially to Ciprian Manolescu who scored a perfect paper last year as well.

Name	Country	Score
Eftekhari, Eaman	Iran	42
Manolescu, Ciprian	Romania	42
Bosley, Carleton	United States of America	42
Do Quoc Anh	Vietnam	42
Frenkel, Péter	Hungary	41
Pap, Gyula	Hungary	41
Ivanov, Ivan	Bulgaria	40
Terpai, Tamás	Hungary	40
Maruoka, Tetsuyuki	Japan	40
Curtis, Nathan	United States of America	40
Najnudel, Joseph	France	39
Dourov, Nikolai	Russia	39
Rullgøard, Hans	Sweden	39
Gyrya, Pavlo	Ukraine	39
Lam, Thomas	Australia	38
Han, Jia Rui	China	38
Ni, Yi	China	38
Zou, Jin	China	38
Woo, Jee Chul	Republic of Korea	38
Hornet, Stefan Laurentiu	Romania	38
Summers, Bennet	United Kingdom	38
Tchalkov, Rayko	Bulgaria	37
An, Jin Peng	China	37
Sun, Xiao Ming	China	37
Salmasian, Hadi	Iran	37
Leptchinski, Mikhail	Russia	37
Tcherepanov, Evgueni	Russia	37

Name	Country	Score
Herzig, Florian	Austria	36
Lippner, Gábor	Hungary	36
Bahramgiri, Mohsen	Iran	36
Battulga, Ulziibat	Mongolia	36
Farrar, Stephen	Australia	35
Zheng, Chang Jin	China	35
Podbrdský, Pavel	Czech Republic	35
Holschbach, Armin	Germany	35
Bayati, Mohsen	Iran	35
Merry, Bruce	Republic of South Africa	35
Grechuk, Bogdan	Ukraine	35
Kabluchko, Zajhar	Ukraine	35

Next we give the problems from this year's IMO Competition. Solutions to these problems, along with those of the 1997 USA Mathematical Olympiad will appear in a booklet entitled *Mathematical Olympiads* 1997 which may be obtained for a small charge from: Dr. W.E. Mientka, Executive Director, MAA Committee on HS Contests, 917 Oldfather Hall, University of Nebraska, Lincoln, Nebraska, 68588, USA.

38th INTERNATIONAL MATHEMATICAL OLYMPIAD

July 24–25, 1997 (Mar del Plata, Argentina)

First Day — Time: 4.5 hours

1. In the plane the points with integer coordinates are the vertices of unit squares. The squares are coloured alternately black and white (as on a chessboard).

For any pair of positive integers m and n, consider a right-angled triangle whose vertices have integer coordinates and whose legs, of lengths m and n, lie along edges of the squares.

Let S_1 be the total area of the black part of the triangle and S_2 be the total area of the white part. Let

$$f(m,n) \;=\; |S_1 - S_2|\,.$$

(a) Calculate $f(m,n)$ for all positive integers m and n which are either both even or both odd.

(b) Prove that $f(m,n) \le \frac{1}{2}\max\{m,n\}$ for all m and n.

(c) Show that there is no constant C such that $f(m,n) < C$ for all m and n.

2. Angle A is the smallest in the triangle ABC.

The points B and C divide the circumcircle of the triangle into two arcs. Let U be an interior point of the arc between B and C which does not contain A.

The perpendicular bisectors of AB and AC meet the line AU at V and W, respectively. The lines BV and CW meet at T.

Show that

$$AU = TB + TC.$$

3. Let $x_1, x_2, \dots, x_n$ be real numbers satisfying the conditions:

$$|x_1 + x_2 + \cdots + x_n| = 1$$

and

$$|x_i| \le \frac{n+1}{2} \quad \text{for} \quad i = 1, 2, \dots, n.$$

Show that there exists a permutation $y_1, y_2, \dots, y_n$ of $x_1, x_2, \dots, x_n$ such that

$$|y_1 + 2y_2 + \cdots + ny_n| \le \frac{n+1}{2}.$$

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Second Day — Time: 4.5 hours

4. An $n \times n$ matrix (square array) whose entries come from the set $S = \{1, 2, \dots, 2n-1\}$, is called a silver matrix if, for each $i = 1, \dots, n$, the ith row and the ith column together contain all elements of S. Show that

(a) there is no silver matrix for $n = 1997$;
(b) silver matrices exist for infinitely many values of n.

5. Find all pairs (a, b) of integers $a \ge 1$, $b \ge 1$ that satisfy the equation

$$a^{(b^2)} = b^a.$$

6. For each positive integer n, let $f(n)$ denote the number of ways of representing n as a sum of powers of 2 with nonnegative integer exponents.

Representations which differ only in the ordering of their summands are considered to be the same. For instance, $f(4) = 4$, because the number 4 can be represented in the following four ways:

$$4;\ 2+2;\ 2+1+1;\ 1+1+1+1.$$

Prove that, for any integer $n \ge 3$:

$$2^{n^2/4} < f(2n) < 2^{n^2/2}.$$

As the IMO is officially an individual event, the compilation of team scores is unofficial, if inevitable. These totals and the prize awards are given in the following table.

Rank	Country	Score	Gold	Silver	Bronze	Total
1.	China	223	6	–	–	6
2.	Hungary	219	4	2	–	6
3.	Iran	217	4	2	–	6
4.–5.	Russia	202	3	2	1	6
4.–5.	United States of America	202	2	4	–	6
6.	Ukraine	195	3	3	–	6
7.–8.	Bulgaria	191	2	3	1	6
7.–8.	Romania	191	2	3	1	6
9.	Australia	187	2	3	1	6
10.	Vietnam	183	1	5	–	6
11.	Republic of Korea	164	1	4	1	6
12.	Japan	163	1	3	1	5
13.	Germany	161	1	3	2	6
14.	Republic of China (Taiwan)	148	–	4	2	6
15.	India	146	–	3	3	6
16.	United Kingdom	144	1	2	2	5
17.	Belarus	140	–	2	4	6
18.	Czech Republic	139	1	2	2	5
19.	Sweden	128	1	3	–	4
20.–21.	Poland	125	–	2	2	4
20.–21.	Yugoslavia	125	–	2	3	5
22.–23.	Israel	124	–	1	5	6
22.–23.	Latvia	124	–	1	4	5
24.	Croatia	121	–	1	4	5
25.	Turkey	119	–	1	4	5
26.	Brazil	117	–	1	4	5
27.	Colombia	112	–	–	6	6
28.	Georgia	109	–	1	3	4
29.	Canada	107	–	2	2	4
30.–31.	Hong Kong	106	–	–	5	5
30.–31.	Mongolia	106	1	–	3	4
32.–33.	France	105	1	–	1	2
32.–33.	Mexico	105	–	1	3	4
34.–35.	Armenia	97	–	–	3	3
34.–35.	Finland	97	–	-	4	4
36.	Slovakia	96	–	1	2	3
37.–38.	Argentina	94	–	–	3	3
37.–38.	The Netherlands	94	–	2	–	2
39.	Republic of South Africa	93	1	–	2	3
40.	Cuba	91	–	1	2	3
41.–42.	Belgium	88	–	–	3	3
41.–42.	Singapore	88	–	–	4	4
43.	Austria	86	1	–	1	2
44.	Norway	79	–	–	3	3
45.	Greece	75	–	1	–	1

Rank	Country	Score	Gold	Silver	Bronze	Total
46.–47.	Kazakhstan	73	–	–	1	1
46.–47.	Former Yugoslav Republic of Macedonia	73	–	–	3	3
48.–49.	Italy	71	–	–	1	1
48.–49.	New Zealand	71	–	–	2	2
50.	Slovenia	70	–	–	2	2
51.	Lithuania	67	–	1	1	2
52.	Thailand	66	–	–	1	1
53.–54.	Estonia	64	–	–	2	2
53.–54.	Peru	64	–	–	2	2
55.	Azerbaijan	56	–	–	1	1
56.	Macao	55	–	–	–	–
57.–59.	Denmark	53	–	–	1	1
57.–59.	Moldova (team of 3)	53	–	–	2	2
57.–59.	Switzerland (team of 5)	53	–	–	2	2
60.–61.	Iceland	48	–	1	–	1
60.–61.	Morocco	48	–	–	–	–
62.	Bosnia and Herzegovina (team of 5)	45	–	–	1	1
63.	Indonesia	44	–	–	–	–
64.	Spain	39	–	–	–	–
65.	Trinidad and Tobago	30	–	–	–	–
66.	Chile	28	–	–	–	–
67.	Uzbekistan (team of 3)	23	–	–	–	–
68.	Ireland	21	–	–	–	–
69.–70.	Malaysia	19	–	–	–	–
69.–70.	Uruguay	19	–	–	–	–
71.–72.	Albania (team of 3)	15	–	–	–	–
71.–72.	Portugal (team of 5)	15	–	–	–	–
73.	Philippines (team of 2)	14	–	–	–	–
74.	Bolivia (team of 3)	13	–	–	–	–
75.	Kyrgyztan (team of 3)	11	–	–	–	–
76.–78.	Kuwait (team of 4)	8	–	–	–	–
76.–78.	Paraguay	8	–	–	–	–
76.–78.	Puerto Rico	8	–	–	–	–
79.	Guatemala	7	–	–	–	–
80.	Cyprus (team of 3)	5	–	–	–	–
81.	Venezuela (team of 3)	4	–	–	–	–
82.	Algeria (team of 4)	3	–	–	–	–

This year the Canadian Team slid to 29th place from 16th last year and 19th the previous year. The Team members were:

Byung Kyu Chun	29	Silver
Adrian Chan	25	Silver
Mihaela Enachescu	21	Bronze
Sabin Cautis	16	Bronze
Jimmy Chui	10	Honourable Mention
Adrian Birka	6	

The Canadian Team Leader was Richard Nowakowski, of Dalhousie University, and the Deputy Team Leader was Naoki Sato, currently a student at the University of Toronto and former Canadian Silver medalist.

The Chinese Team placed first this year. Its members were:

Jia Rui Han	38	Gold
Yi Ni	38	Gold
Jin Zou	38	Gold
Jin Peng An	37	Gold
Xiao Ming Sun	37	Gold
Chang Jin Zheng	35	Gold

Congratulations to the Chinese Team!!

Now we turn to solutions to problems of the Czechoslovakia Mathematical Olympiad 1993 [1996: 109].

CZECHOSLOVAK MATHEMATICAL OLYMPIAD 1993
Final Round

1. Find all natural numbers n for which $7^n - 1$ is a multiple of $6^n - 1$.

Solutions by Mansur Boase, student, St. Paul's School, London, England; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

No such n exists. Suppose n is a natural number for which $6^n - 1 \mid 7^n - 1$. Since $6^n \equiv 1 \bmod 5$ we have $5 \mid 7^n - 1$. From $7 \equiv 2$; $7^2 \equiv -1$, $7^3 \equiv 3$ and $7^4 \equiv 1 \bmod 5$, we see that $7^n - 1 \equiv 0 \bmod 5$ if and only if 4 divides n. In particular, n must be even, which implies, since $6^2 \equiv 1 \bmod 7$ that $6^n \equiv 1 \bmod 7$; that is, $7 \mid 6^n - 1$. However, obviously $7 \nmid 7^n - 1$, which is a contradiction.

2. A 19×19 table contains integers so that any two of them lying on neighbouring fields differ at most by 2. Find the greatest possible number of mutually different integers in such a table. (Two fields of the table are considered neighbouring if they have a common side.)

Solution by Mansur Boase, student, St. Paul's School, London, England.

We shall first prove that the value 71 for the number of different integers is attainable. We can assume, without loss of generality, that the number in the top left hand square is 0. We can then form the following array:

$$\begin{array}{ccccccc} 0 & 1 & 3 & 5 & \cdots & 31 & 33 & 34 \\ 2 & 3 & 5 & 7 & \cdots & 33 & 35 & 36 \\ 4 & 5 & 7 & 9 & \cdots & 35 & 37 & 38 \\ 6 & 7 & 9 & 11 & \cdots & 37 & 39 & 40 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 32 & 33 & 35 & 37 & & 63 & 65 & 66 \\ 34 & 35 & 37 & 39 & & 65 & 67 & 68 \\ 35 & 37 & 39 & 41 & & 67 & 69 & 70 \end{array}$$

The difference in the values of opposite corners of the array cannot exceed twice the number of moves to get from one to the other passing through adjacent squares. This equals

$$2 \times 36 = 72.$$

Thus there can be at most 73 different values since every square in the array can be reached from the top left square in at most 36 moves.

We need to prove that there is no array with 72 or 73 different numbers.

The value 73 can only occur if the difference between every pair of adjacent squares is exactly 2, for otherwise one of the shortest paths from opposite corners can pass through the pair of adjacent squares whose value is not 2, giving a contradiction.

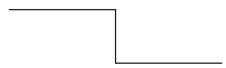
If the number of distinct values is 72, then the bottom right hand square must have value 71, so in any shortest path, there is a pair of adjacent squares with difference only 1.

To have 72 values, all the even and odd numbers less than or equal to 71 must be in the table.

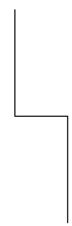
There is a (possibly broken) dividing line separating the odd and even numbers, all numbers on the side of 71 being odd.

If there is a gap on the line, then there is a shortest path from 71 to 0 without crossing the line, which is impossible.

Considering 1 and 71 we see that the shortest distance must be 35, so that one of the squares adjacent to 0 must contain 1. The broken line separating even and odd entries cannot have the form



or



for otherwise these would be a shortest path from 0 to 71 with more than one difference of 1. It follows that the even entries are confined to the first row or the first column, contradicting that there are 36 of them.

4. A sequence $\{a_n\}_{n=1}^{\infty}$ of natural numbers is defined recursively by $a_1 = 2$ and $a_{n+1} =$ the sum of 10th powers of the digits of a_n, for all $n \geq 1$. Decide whether some numbers can appear twice in the sequence $\{a_n\}_{n=1}^{\infty}$.

Solution by Mansur Boase, student, St. Paul's School, London, England.

Let a_n have d_n digits, so $d_n \leq 1 + \log_{10} a_n$.
Then $a_{n+1} \leq d_n 9^{10} \leq (1 + \log_{10} a_n)9^{10} < a_n$ for a_n sufficiently large, as $\frac{x}{1+\log x} \to \infty$ as $x \to \infty$. Thus $a_{n+1} < a_n$ if $\frac{a_n}{1+\log_{10} a_n} > 9^{10}$.

Let K be the first natural number for which $\frac{K}{1+\log_{10} K} > 9^{10}$.

Then if $a_i \geq K$, the sequence decreases until there is a term $< K$.

Hence the sequence has infinitely many terms $< K$, and there must be a term repeated (infinitely often).

5. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(-1) = f(1)$ and

$$f(x) + f(y) = f(x + 2xy) + f(y - 2xy)$$

for all integers x, y.

Solution by Mansur Boase, student, St. Paul's School, London, England.

$$f(1) + f(y) = f(1 + 2y) + f(-y) \qquad (1)$$
$$f(y) + f(-1) = f(-y) + f(-1 + 2y) \qquad (2)$$

$f(1) = f(-1)$ and $f(y) = f(y)$, so equating (1) and (2)

$$f(1 + 2y) + f(-y) = f(-y) + f(-1 + 2y).$$

Hence $f(2y + 1) = f(2y - 1)$ for all integers y. $(\star)$

Also $f(y) + f(1) = f(3y) + f(1 - 2y)$ and since $f(1) = f(1 - 2y)$ (making use of $(\star)$) $f(y) = f(3y)$.

Also, equating this with equation (1)

$$f(3y) + f(1 - 2y) = f(1 + 2y) + f(-y),$$

and since by $(\star)$ $f(1-2y) = f(1+2y)$

$$f(3y) = f(-y).$$

Thus $f(y) = f(3y) = f(-y)$ and $f(y) = f(-y)$ for all y.

If y is odd, then $f(y) = f(y - 2xy)$ so $f(x) = f(x + 2xy)$.

Thus $f(2a) = f(2a(1+2y))$.

Thus an odd multiple of a number and that number give the same value.

Therefore if $n = 2^k a$, a odd, then $f(n) = f(2^k)$. Therefore all functions must be such that $f(2^k a) = f(2^k)$, $k \geq 0$ where a is odd and $f(2^k)$ for each k can take any value as can $f(0)$.

These functions always satisfy the conditions since $f(-1) = f(1)$ and

$$\begin{aligned} f(x+2xy) + f(y-2xy) &= f(x(1+2y)) + f(y(1-2x)) \\ &= f(x) + f(y). \end{aligned}$$

For the remainder of this number of the Corner we turn to readers' solutions to 10 of the problems of the Baltic Way — 92 Contest given in the May 1996 number of the Corner [1996: 157–159].

MATHEMATICAL TEAM CONTEST
"BALTIC WAY — 92"
Vilnius, 1992 — November 5–8

1. Let p, q be two consecutive odd prime numbers. Prove that $p+q$ is a product of at least 3 positive integers > 1 (not necessarily different).

Solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; by Mansur Boase, student, St. Paul's School, London, England; by Christopher J. Bradley, Clifton College, Bristol, UK; by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario; by Bob Prielipp, University of Wisconsin–Oshkosh, Wisconsin, USA; by Michael Selby, University of Windsor, Windsor, Ontario; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Since p and q are odd, $p+q$ is even and has a factor of 2. The corresponding factor $\frac{1}{2}(p+q)$ lies strictly between p and q and is not prime since p and q are *consecutive* prime numbers. It must therefore have two factors at least. Hence $p+q$ is a product of at least 3 positive integers greater than 1.

2. Denote by $d(n)$ the number of all positive divisors of a positive integer n (including 1 and n). Prove that there are infinitely many n such that $\frac{n}{d(n)}$ is an integer.

Solutions by Mansur Boase, student, St. Paul's School, London, England; by Christopher J. Bradley, Clifton College, Bristol, UK; by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario; by Bob Prielipp, University of Wisconsin–Oshkosh, Wisconsin, USA; by Michael Selby, University of Windsor, Windsor, Ontario; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bradley's solution.

Let p be a prime. Then p^{p-1} has factors $1, p, p^2, \ldots, p^{p-1}$, which are p in number. Hence, for such n, $p^{p-1}/d(p^{p-1}) = p^{p-2}$ which is integral. Hence there are infinitely many n such that $n/d(n)$ is an integer.

3. Find an infinite non-constant arithmetic progression of positive integers such that each term is neither a sum of two squares, nor a sum of two cubes (of positive integers).

Solutions by Mansur Boase, student, St. Paul's School, London, England; by Christopher J. Bradley, Clifton College, Bristol, UK; by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario; by Michael Selby, University of Windsor, Windsor, Ontario; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Godin's solution.

Looking at squares $\bmod 4$ we see

x	$x^2 \bmod 4$
0	0
1	1
2	0
3	1

thus if $x \equiv 3 \bmod 4$, it is impossible to express x as a sum of two squares. Similarly, looking at cubes $\bmod 7$ we see

x	$x^3 \bmod 7$
0	0
1	1
2	1
3	6
4	1
5	6
6	6

and again if $x \equiv 3 \bmod 7$ it is impossible to express x as a sum of 2 cubes. Thus the sequence $s_0 = 3$, $s_{n+1} = s_n + 28$ has all $s_n \equiv 3 \bmod 4$ and $s_n \equiv 3 \bmod 7$ so all the terms are not the sum of 2 squares or cubes.

4. Is it possible to draw a hexagon with vertices in the knots of an integer lattice so that the squares of the lengths of the sides are six consecutive positive integers?

Solutions by Mansur Boase, student, St. Paul's School, London, England; by Christopher J. Bradley, Clifton College, Bristol, UK; and by Shawn

Godin, St. Joseph Scollard Hall, North Bay, Ontario. We give Bradley's solution.

No. It is not possible.

Six consecutive positive integers must contain three odd numbers and three even numbers. To form them as the sums of squares must involve an odd number of odd squares. The sum of all the numbers whose squares are made is therefore odd. But to move from $(0,0,0)$ to $(0,0,0)$ involves a total number which is even (going out and coming back to put it loosely). [A pentagon, on the other hand, can be made, for example,

$$(0,0,0) \to (0,1,1) \to (1,2,2) \to (1,2,4) \to (1,1,2) \to (0,0,0)$$

involving sides whose squares are $2, 3, 4, 5, 6$ — with two odd numbers.]

5. Given that $a^2 + b^2 + (a+b)^2 = c^2 + d^2 + (c+d)^2$, prove that $a^4 + b^4 + (a+b)^4 = c^4 + d^4 + (c+d)^4$.

Solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; by Mansur Boase, student, St. Paul's School, London, England; by Christopher J. Bradley, Clifton College, Bristol, UK; by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario; by Michael Selby, University of Windsor, Windsor, Ontario; by Panos E. Tsaoussoglou, Athens, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.

This follows immediately from the fact that

$$\begin{aligned} x^4 + y^4 + (x+y)^4 &= 2\left(x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4\right) \\ &= 2\left(x^2 + xy + y^2\right)^2 = \frac{1}{2}\left[x^2 + y^2 + (x+y)^2\right]^2 \end{aligned}$$

for all x, y.

6. Prove that the product of the 99 numbers $\frac{k^3-1}{k^3+1}$, $k = 2, 3, \ldots, 100$, is greater than $\frac{2}{3}$.

Solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; by Mansur Boase, student, St. Paul's School, London, England; by Christopher J. Bradley, Clifton College, Bristol, UK; by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario; by Michael Selby, University of Windsor, Windsor, Ontario; by Panos E. Tsaoussoglou, Athens, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution and comment.

Let $P(n) = \prod_{k=2}^{n} \frac{k^3-1}{k^3+1}$ where $n \geq 2$. We show that in general

$$P(n) = \frac{2(n^2+n+1)}{3n(n+1)}$$

from which it follows that $P(n) > \frac{2}{3}$. Since

$$(k+1)^2 - (k+1) + 1 = k^2 + k + 1,$$

we have

$$\begin{aligned} P(n) &= \prod_{k=2}^{n} \frac{(k-1)(k^2+k+1)}{(k+1)(k^2-k+1)} \\ &= \frac{\prod_{k=0}^{n-2}(k+1)}{\prod_{k=2}^{n}(k+1)} \cdot \frac{\prod_{k=2}^{n}(k^2+k+1)}{\prod_{k=1}^{n-1} k^2+k+1} \\ &= \frac{1\cdot 2}{n(n+1)} \cdot \frac{n^2+n+1}{3} = \frac{2(n^2+n+1)}{3n(n+1)}. \end{aligned}$$

Remark: It is easy to see that the sequence $\{P(n)\}$, $n = 2, 3, \ldots$ is strictly decreasing and from the result established above we see that $\lim_{n\to\infty} P(n) = \frac{2}{3}$. This result is well known and can be found in, for example, *Theory and Application of Infinite Series*, by K. Knopp (Ex. 85(2) on p. 28).

7. Let $a = \sqrt[1992]{1992}$. Which number is greater:

$$\left.a^{a^{a^{\cdot^{\cdot^{\cdot^{a}}}}}}\right\} 1992 \quad \text{or} \quad 1992?$$

Solutions by Mansur Boase, student, St. Paul's School, London, England; by Christopher J. Bradley, Clifton College, Bristol, UK; by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario; by Michael Selby, University of Windsor, Windsor, Ontario; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Godin's solution.

Let $f(x) = a^x$. Clearly $p > q$ if and only if $f(p) > f(q)$; that is, f is strictly increasing. Similarly if $f^n(x)$ is defined by $f^1(x) = f(x)$ and $f^{n+1}(x) = f(f^n(x))$, so $f^n(x) = \underbrace{f\, f\, \ldots\, f}_{n}(x)$ then $p > q$ if and only if $f^n(p) > f^n(q)$ for all n.

Now $1992 > a$.

Thus $f^{1992}(1992) > f^{1992}(a)$. But $f(1992) = 1992$, so $f^{1992}(1992) = 1992$. Now $f^{1992}(a)$ is the expression in question so 1992 is the larger.

Editor's Note. Selby points out that if $x_k = f^k(a)$ then $\lim_{k\to\infty} x_k = L$ where $a = L^{1/L}$ or $L = 1992$.

8. Find all integers satisfying the equation

$$2^x \cdot (4-x) = 2x+4.$$

Solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; by Mansur Boase, student, St. Paul's School, London, England; by Christopher J. Bradley, Clifton College, Bristol, UK; by Shawn Godin, *St. Joseph Scollard Hall, North Bay, Ontario; by Bob Prielipp, University of Wisconsin–Oshkosh, Wisconsin, USA; by Michael Selby, University of Windsor, Windsor, Ontario; by Panos E. Tsaoussoglou, Athens, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Tsaoussoglou's solution.*

$$2^x = \frac{2(x+2)}{4-x} > 0, \quad \text{so } -2 < x < 4.$$

Checking

$$\begin{array}{lll} x = -1 & \frac{1}{2} \neq \frac{2(-1)+4}{4-(-1)} = \frac{2}{5} & \\ x = 0 & 1 = \frac{2(0+2)}{4-0} & \text{is a solution} \\ x = 1 & 2 = \frac{2(1+2)}{4-1} & \text{is a solution} \\ x = 2 & 4 = \frac{2(2+2)}{4-2} & \text{is a solution} \\ x = 3 & 8 = \frac{2(3+2)}{4-3} & \text{is not a solution} \end{array}$$

9. A polynomial $f(x) = x^3 + ax^2 + bx + c$ is such that $b < 0$ and $ab = 9c$. Prove that the polynomial has three different real roots.

Solutions by Christopher J. Bradley, Clifton College, Bristol, UK; by Bob Prielipp, University of Wisconsin–Oshkosh, Wisconsin, USA; and by Michael Selby, University of Windsor, Windsor, Ontario. We give Selby's solution.

Suppose $a = 0$. Then $c = 0$ and $f(x) = x^3 + bx$. Therefore there are three roots: $x = 0$, $x = \sqrt{-b}$, $x = -\sqrt{-b}$. If $a > 0$, then $c = \frac{ab}{9} < 0$, since $b < 0$. Now $f(0) = c < 0$ and since $\lim_{x\to\infty} f(x) = \infty$, there is some r_1 in $(0, \infty)$ such that $f(r_1) = 0$.

Also $f(-a) = -a^3 + a^3 - ab + c = \frac{-8ab}{9} > 0$.

Since $f(0) < 0$ and $f(-a) > 0$, there is a root r_2 in the interval $(-a, 0)$. Finally, since $f(x) \to -\infty$ as $x \to -\infty$ and $f(-a) > 0$ there must be a root r_3 in $(-\infty, -a)$.

If $a < 0$, $c = \frac{ab}{9} > 0$. Hence $f(0) = c > 0$ while $f(x) \to -\infty$ as $x \to -\infty$. Therefore we have a root t_1 in $(-\infty, 0)$.

$f(-a) = -a^3 + a^3 - ab + c = -\frac{8}{9}ab < 0$. Hence there must be a root t_2 in $(0, -a)$. Finally $f(x) \to \infty$ as $x \to \infty$. Since $f(-a) < 0$ there need be a root t_3 in $(-a, \infty)$.

In all cases we have three distinct roots.

10. Find all fourth degree polynomials $p(x)$ such that the following four conditions are satisfied:

(i) $p(x) = p(-x)$, for all x,

(ii) $p(x) \geq 0$, for all x,

(iii) $p(0) = 1$,

(iv) $p(x)$ has exactly two local minimum points x_1 and x_2 such that $|x_1 - x_2| = 2$.

Solutions by Christopher J. Bradley, Clifton College, Bristol, UK; by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario; and by Michael Selby, University of Windsor, Windsor, Ontario. We give Selby's solution.

Condition (i) implies $p(x)$ is even.

$$p(x) = ax^4 + bx^3 + cx^2 + dx + e,$$

$$p(-x) = ax^4 - bx^3 + cx^2 - dx + e.$$

Hence we have $2bx^3 + 2dx = 0$ for all x. Thus $b = d = 0$.

Also $p(0) = 1$. Therefore $p(x)$ has the form

$$p(x) = ax^4 + cx^2 + 1.$$

Now $p'(x) = 4ax^3 + 2cx$. The critical points are $x = 0$, $x^2 = \frac{-c}{2a}$. Hence we must have $\frac{-c}{2a} > 0$.

$p''(x) = 12ax^2 + 2c$. Since we want exactly two local minima, we must have at $x^2 = \frac{-c}{2a}$, $p'(x) = 12a\left(\frac{-c}{2a}\right) + 2c = -4c > 0$. Therefore $c < 0$ and $a > 0$.

Further we want $p(x) \geq 0$. Since $p(x) \to \infty$ as $|x| \to \infty$ then since $p(0) = 1$, we need at $x^2 = \frac{-c}{2a}$, $p(x) = a\frac{c^2}{4a^2} + c\left(\frac{-c}{2a}\right) + 1 \geq 0$. So $\frac{-c^2}{4a} + 1 \geq 0$ or $4a \geq c^2$. Also $x_1 = \sqrt{\frac{-c}{2a}}$, $x_2 = -\sqrt{\frac{-c}{2a}}$, $|x_1 - x_2| = 2 \Rightarrow \left|\sqrt{\frac{-c}{2a}}\right| = 1$ or $-c = 2a$. Since $4a \geq c^2$, we have $4a \geq 4a^2$ or $a(1-a) \geq 0$.

Therefore since $a > 0$, $1 - a \geq 0$ and we have $0 < a \leq 1$, $c = -2a$.

The polynomials are of the form $ax^4 - 2ax^2 + 1$ where $0 < a \leq 1$.

That completes the Corner for this issue. Send me your nice solutions and contest materials.

BOOK REVIEWS

Edited by ANDY LIU

144 Problems of the Austrian-Polish Mathematics Competition 1978–1993,
compiled by **Marcin Emil Kuczma**,
University of Warsaw,
published by the Academic Distribution Center, 1993,
1216 Walker Road, Freeland, Maryland 21053, USA,
ISBN# 0-9640959-0-4, softcover, 157+ pages, US$20 plus handling.

40th Polish Mathematics Olympiad 1989/90,
edited by **Marcin Emil Kuczma**,
University of Warsaw,
published by Science, Culture, Technology Publishing,
AMK Central Post Office, Box 0581, Singapore 915603,
no ISBN#, softcover, 49+ pages, US$10 plus handling.

Reviewed by ***Andy Liu****, University of Alberta.*

Both books under review are by Marcin Kuczma, one of the world's leading problem proposers. In IMO95 in Canada, two of his problems made it and a third nearly did. He is responsible for a substantial number of problems in these two books, which come with the highest recommendation. While originality can never be guaranteed since great minds tend to think alike across space and time, the author strived to come up with something new and exciting, and has largely succeeded.

The first book contains all the problems of the Austrian-Polish Mathematics Competition since its inception in 1978, up to 1993. It is one of the world's oldest regional mathematics competitions, attesting to the close friendship of the mathematical communities in these two nations. Apart from six problems given over two days in the IMO format, there is a team competition in which the six members collaborate to solve three problems in three hours. These students are usually those ranked immediately below the IMO team members. While the APMC may be considered a consolation for a graduating high school student, it is a wonderful experience and excellent preparation for those who have further aspirations in making the IMO team in future years.

The second book is somewhat of an anomaly in publishing, in that it contains only the problems of one competition in one year. There are 24 problems in all, consisting of 12 from the First Round of the 1989/90 Polish Mathematics Olympiad, and 6 from each of the Second and the Third Rounds. Discounts are offered for orders of multiple copies. Contact the publishers for details.

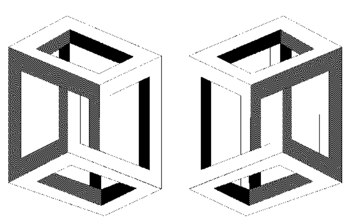
The Introduction in each of these two books provides valuable information about the respective competitions, but of course the main value lies in the problems. We give below a sample from each book. Both should be on the shelf of anyone serious about mathematics competitions.

Problem 5, Austrian-Polish Mathematics Competition, 1985.
We are given a certain number of identical sets of weights; each set consists of four different weights expressed by natural numbers (of weight units). Using these weights we are able to weigh out every integer mass up to 1985 (inclusive). How many ways are there to compose such a set of weights given that the joint mass of all weights is the least possible?

Problem 21, Third Round, Polish Mathematical Olympiad, 1989/90.
The edges of a cube are numbered 1 through 12.

(a) Show that for every such numbering, there exist at least eight triples of integers (i, j, k), with $1 \leq i < j < k \leq 12$, such that the edges labelled i, j and k are consecutive sides of a polygonal line.

(b) Give an example of a numbering for which a ninth triple with these properties does not exist.



MORE UNITARY DIVISOR PROBLEMS

K.R.S. Sastry

A (positive) integral divisor d of a (natural) number n is called a unitary divisor of n if and only if d and $\frac{n}{d}$ are relatively prime; that is $\left(d, \frac{n}{d}\right) = 1$. For example, 9 is a unitary divisor of 18 because $\left(9, \frac{18}{9}\right) = (9, 2) = 1$. But, 3 is **not** a unitary divisor of 18 because $(3, \frac{18}{3}) = (3, 6) = 3 \neq 1$. This novel definition of divisibility produces interesting analogues and contrasts with the results of ordinary divisibility. See [2, 3, 4]. In this paper we consider the solutions of two more new unitary divisor problems.

BACKGROUND MATERIAL: Let $n = p_1^{\alpha_1}\, p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ denote the prime decomposition of n. Let $d^*(n)$ denote the number of unitary divisors of n. Then, see [2, 3, 4], we have

$$\alpha^*(1) = 1,\ d^*(n) = 2^k \text{ for } n > 1. \tag{1}$$

The Euler ϕ function counts the number $\phi(n)$ of natural numbers that are less than and relatively prime to n. Then, see [1], we have

$$\phi(1) = 1,\ \phi(n) = \prod_{i=1}^{k}(p_i - 1)p_i^{\alpha_i - 1} \text{ for } n > 1. \tag{2}$$

The first problem asks us to determine the positive integers n for which $d^*(n) = \phi(n)$. The second asks us to establish a curious result concerning the **number** of unitary divisors of each unitary divisor of n.

THE FIRST PROBLEM: Determine the set $S = \{n : d^*(n) = \phi(n)\}$.
Solution: In view of equations (1) and (2), we have to solve the equation

$$(p_1 - 1)\, p_1^{\alpha_1 - 1}\, (p_2 - 1)\, p_2^{\alpha_2 - 1} \cdots (p_k - 1)\, p_k^{\alpha_k - 1} = 2^k. \tag{3}$$

The right hand side of equation (3) is a power of 2, so each factor on its left hand side is to be necessarily a power of 2. Two cases arise because of the even prime 2.

(I) $p_1 = 2$ and p_i are distinct odd primes for $2 \leq i \leq k$. Then $\alpha_2 - 1 = \alpha_3 - 1 = \cdots = \alpha_k - 1 = 0$ and $p_2 - 1 = 2^{\beta_2}, p_3 - 1 = 2^{\beta_3}, \cdots, p_k - 1 = 2^{\beta_k}$. Thus $p_i = 2^{\beta_i} + 1$ are distinct odd primes for

$$2 \leq i \leq k. \tag{4}$$

(II) p_i are all odd primes for $1 \leq i \leq k$. Then as in the preceding, $\alpha_i = 1$ and $p_i = 2^{\beta_i} + 1$ are distinct odd primes for

$$1 \leq i \leq k. \tag{5}$$

(**A**) First we note the obvious solution $n = 1$ that follows from $d^*(1) = 1$ and $\phi(1) = 1$.

(**B**) $k = 1$. This implies that n is formed from a single prime.

(I) $k = 1$, $p_1 = 2$. So $n = 2^{\alpha_1}$. This yields $2^{\alpha_1 - 1} = 2^1, \alpha_1 = 2$. Hence $n = 2^2 = 4$. It is easily verified that $d^*(4) = \phi(4) = 2$.

(II) $k = 1$, p_1 an odd prime. This yields $(p_1 - 1) = 2^1, p_1 = 3$. Hence $n = 3$. Again $d^*(3) = \phi(3) = 2$.

(**C**) $k = 2$. This implies that n is formed from powers of two distinct primes p_1, p_2.

(I) $k = 2$, $p_1 = 2$ and p_2 an odd prime. Hence $n = 2^{\alpha_1} p_2$. This yields $2^{\alpha_1 - 1} 2^{\beta_2} = 2^2$, $(\alpha_1 - 1) + \beta_2 = 2$. This equation has two solutions:

$$(\mathrm{C}_1) \quad \alpha_1 = 1,\ \beta_2 = 2 \qquad \Longrightarrow \qquad n = 2^{\alpha_1}\left(2^{\beta_2} + 1\right) = 10,$$

$$(\mathrm{C}_2) \quad \alpha_1 = 2,\ \beta_2 = 1 \qquad \Longrightarrow \qquad n = 2^{\alpha_1}\left(2^{\beta_2} + 1\right) = 12.$$

(II) $k = 2$, p_1, p_2 are both odd primes. Hence $n = p_1 p_2$, $2^{\beta_1} 2^{\beta_2} = 2^2$, $\beta_1 + \beta_2 = 2$. Now p_1 and p_2 are distinct odd primes so $\beta_1 + \beta_2 = 2$ are distinct natural numbers. Hence $\beta_1 + \beta_2 > 2$ and there is no solution in this case.

(**D**) $k = 3$. In this case, n is the product of powers of three distinct primes.

(I) $k = 3$, $p_1 = 2$, and p_2, p_3 are distinct odd primes. Hence $n = 2^{\alpha_1} p_2 p_3$, $2^{\alpha_1 - 1} 2^{\beta_2} 2^{\beta_3} = 2^3$, $(\alpha_1 - 1) + \beta_2 + \beta_3 = 3$. Since α_1 is at least 1 and only one of β_2 and β_3 can equal 1, we have the single solution $\alpha_1 = 1$, $\beta_2 = 1$, $\beta_3 = 2$. This gives $n = 2^{\alpha_1}\left(2^{\beta_1} + 1\right)\left(2^{\beta_2} + 1\right) = 30$.

(II) $k = 3$, p_1, p_2, p_3 are distinct odd primes. This leads to the equation $\beta_1 + \beta_2 + \beta_3 = 3$ that has no solution in natural numbers, as explained above.

(**E**) $k > 3$, $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$.

(I) $k > 3$, $p_1 = 2$ yields $(\alpha_1 - 1) + \beta_2 + \beta_3 + \cdots + \beta_k = k$, which has no solution for the reason explained above in (**D**).

(II) $k > 3$ yields $\beta_1 + \beta_2 + \cdots + \beta_k = k$, and this equation too does not have a solution.

Combining (**A**), $\cdots$, (**E**) we have determined $S = \{1, 3, 4, 10, 12, 30\}$. The verification that if $n \in S$ then $d^*(n) = \phi(n)$ is left to the reader.

THE SECOND PROBLEM: The fascinating property of the natural numbers that $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$ motivates us to look for a collection of integers with the similar property. For instance the collection of the integers 1, 2, 2, 2, 4, 4, 4, 8 exhibit that property:

$$1^3 + 2^3 + 2^3 + 2^3 + 4^3 + 4^3 + 4^3 + 8^3 = (1 + 2 + 2 + 2 + 4 + 4 + 4 + 8)^2.$$

How to generate collections of such integers exhibiting the above property? Here is an algorithm:

(A_1) Start with any integer n. Say $n = 30$.

(A_2) Write down the unitary divisors of n. The unitary divisors of 30 are $1, 2, 3, 5, 6, 10, 15, 30$.

(A_3) Now write down the **number** of unitary divisors of each unitary divisor of n. In view of (1) each of these integers will be a power of 2. In the present case we have respectively

$$1,\ 2,\ 2,\ 2,\ 4,\ 4,\ 4,\ 8.$$

Lo and behold! We have generated a collection of integers having the mentioned property. Intrigued? Try this on $n = 24$ or $n = 420$ or any other value. In general we have the following result:

Let $a_1,\ a_2,\ \cdots,\ a_\lambda$ denote the unitary divisors of an integer n. Let $d^*(a_i)$ denote the number of unitary divisors of a_i. Then

$$\sum_{i=1}^{\lambda} [d^*(a_i)]^3 = \left[\sum_{i=1}^{\lambda} d^*(a_i)\right]^2 .$$

Proof: Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ denote the prime decomposition of n. Then its $2^k = \lambda$ unitary divisors are:

$$\begin{array}{l} a_1, a_2, \cdots, a_\lambda = 1; \\ p_1^{\alpha_1},\ p_2^{\alpha_2},\ \cdots,\ p_k^{\alpha_k}; \\ p_1^{\alpha_1} p_2^{\alpha_2},\ p_1^{\alpha_1} p_3^{\alpha_3},\ \cdots,\ p_{k-1}^{\alpha_{k-1}} p_k^{\alpha_k}; \\ \vdots \\ p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}. \end{array}$$

Observe that the unitary divisors have been put together, for convenience, in collections containing

$$\binom{k}{0},\ \binom{k}{1},\ \binom{k}{2},\ \cdots,\ \binom{k}{k}$$

terms respectively. Here $\binom{k}{r} = \frac{k(k-1)\cdots(k-r+1)}{r!}$ denotes the binomial coefficient for $0 \leq r \leq k$. We now use (1) to write down $d^*(a_i)$ for $1 \leq i \leq 2^k$:

$$1;\ 2, 2, \cdots, 2;\ 2^2, 2^2, \cdots, 2^2;\ 2^3, 2^3, \cdots, 2^3;\ \cdots;\ 2^k.$$

Observe that these collections contain, respectively, $\binom{k}{0}$ term equalling 1: $\binom{k}{1}$ terms equalling 2; $\binom{k}{2}$ terms equalling $2^2; \cdots ; \binom{k}{k}$ term equalling 2^k. Hence

$$\begin{aligned} \sum [\alpha^*(a_i)]^3 &= 1^3 + \left[2^3 + 2^3 + \cdots \binom{k}{1} \text{ terms}\right] \\ &\quad + \left[2^6 + 2^6 + \cdots \binom{k}{2} \text{ terms}\right] + \cdots + \left[2^{3k}\right]. \end{aligned}$$

This is nothing but the expansion of

$$\left(1+2^3\right)^k = 9^k = 3^{2k}.$$

Also

$$\begin{aligned}\sum \alpha^*(a_i) &= 1 + \left[2 + 2 + \cdots \binom{k}{1}\text{terms}\right] \\ &\quad + \left[2^2 + 2^2 + \cdots \binom{k}{2} \text{ terms }\right] + \cdots + \left[2^k\right] \\ &= (1+2)^k = 3^k.\end{aligned}$$

It is easy to see that the assertion follows.

Finally we invite the reader to state and prove an analogous result for the ordinary (positive integral) divisors of (positive) integers n. For other problems on cubes of natural numbers see [5, 6].

References

[1] L.E. Dickson, *History of the Theory of Numbers*, Vol. I, Chelsea Publications, (1971), pp. 113–114.

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[5] K.R.S. Sastry, Fermat-Pell: An Observation and an Application, *Mathematics and Computer Education*, **28** (Winter 1994), pp. 24–35.

[6] K.R.S. Sastry, Cubes of Natural Numbers in Arithmetic Progression, *Crux Mathematicorum*, **18**, (1992), pp. 161–164.

K.R.S. Sastry
Jeevan Sandhya
Doddakalsandra Post
Raghuvanahalli
Bangalore 560 062, India

THE SKOLIAD CORNER

No. 25

R.E. Woodrow

This number we give the problems of the 1997 Alberta High School Mathematics Prize Exam, second round, written last February by students invited on the basis of the first round results from November. My thanks go to Ted Lewis, Chair of the Alberta High School Mathematics Prize Exam Board for forwarding the contest, which is partially supported by the Canadian Mathematical Society.

ALBERTA HIGH SCHOOL MATHEMATICS COMPETITION

February 11, 1997

Second Round

1. Find all real numbers x satisfying $|x - 7| > |x + 2| + |x - 2|$.

Remark. Note that $|a|$ is called the absolute value of the real number a. It has the same numerical value as a but is never negative. For example, $|3.5| = 3.5$, while $|-2| = 2$. Of course, $|0| = 0$.

2. Two lines b and c form a 60° angle at the point A, and B_1 is a point on b. From B_1, draw a line perpendicular to the line b meeting the line c at the point C_1. From C_1 draw a line perpendicular to c meeting the line b at B_2. Continue in this way obtaining points C_2, B_3, C_3, and so on. These points are the vertices of right triangles $AB_1C_1, AB_2C_2, AB_3C_3, \ldots$. If area $(AB_1C_1) = 1$, find

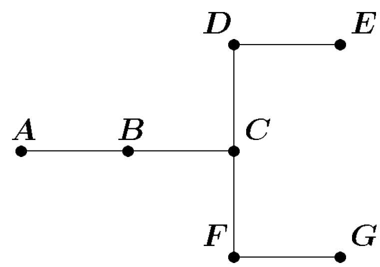
$$\text{area } (AB_1C_1) + \text{ area } (AB_2C_2) + \text{ area } (AB_3C_3) + \cdots + \text{ area } (AB_{1997}C_{1997}).$$

3. A and B are two points on the diameter MN of a semicircle. C, D, E and F are points on the semicircle such that $\angle CAM = \angle EAN = \angle DBM = \angle FBN$. Prove that $CE = DF$.

4. (a) Suppose that p is an odd prime number and a and b are positive integers such that p^4 divides $a^2 + b^2$ and p^4 also divides $a(a + b)^2$. Prove that p^4 also divides $a(a + b)$.

(b) Suppose that p is an odd prime number and a and b are positive integers such that p^5 divides $a^2 + b^2$ and p^5 also divides $a(a + b)^2$. Show by an example that p^5 does not necessarily divide $a(a + b)$.

5. The picture shows seven houses represented by the dots, connected by six roads represented by the lines. Each road is exactly 1 kilometre long. You live in the house marked B. For each positive integer n, how many ways are there for you to run n kilometres if you start at B and you never run along only part of a road and turn around between houses? You have to use the roads, but you may use any road more than once, and you do not have to finish at B. For example, if $n = 4$, then three of the possibilities are: B to C to F to G to F; B to A to B to C to B; and B to C to B to A to B.



Next we give the solutions to the 1995 Concours Mathématique du Québec which comes to us from Thérèse Ouellet, secretary to the contest. We give the solutions in French, the language of the competition.

CONCOURS MATHÉMATIQUE DU QUÉBEC 1995
February 2, 1995
Time: 3 hours

1. LA FRACTION À SIMPLIFIER

Simplifiez la fraction

$$\frac{1\ 358\ 024\ 701}{1\ 851\ 851\ 865}.$$

Solution.

$$\frac{1\ 358\ 024\ 701}{1\ 851\ 851\ 865} = \frac{11 \times 123\ 456\ 791}{15 \times 123\ 456\ 791} = \frac{11}{15}.$$

2. LA FORMULE MYSTÈRE

Considérons les équations suivantes

$$xy = p, \quad x + y = s, \quad x^{1993} + y^{1993} = t, \quad x^{1994} + y^{1994} = u.$$

En faisant appel aux lettres p, s, t, u mais pas aux lettres x, y, donnez une formule pour la valeur

$$x^{1995} + y^{1995}.$$

Solution. On a successivement

$$\begin{aligned} su &= (x+y)(x^{1994}+y^{1994}) \\ &= x^{1995}+xy^{1994}+x^{1994}y+y^{1995} \\ &= x^{1995}+y^{1995}+xy\cdot(x^{1993}+y^{1993}) \\ &= x^{1995}+y^{1995}+pt. \end{aligned}$$

D'où l'on tire

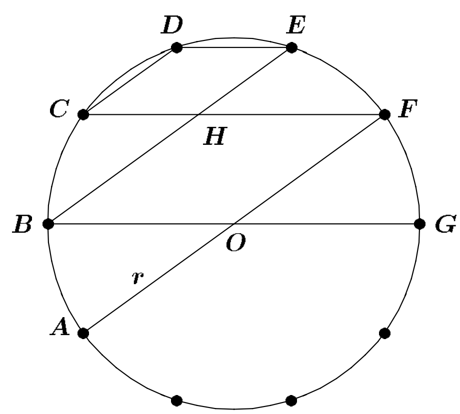
$$x^{1995}+y^{1995}=su-pt.$$

3. LA DIFFÉRENCE ÉTONNANTE

Lorsque la circonférence d'un cercle est divisée en dix parties égales, les cordes qui joignent les points de division successifs forment un décagone régulier convexe. En joignant chaque point de division au troisième suivant, on obtient un décagone régulier étoilé. Montrer que la différence entre les côtés de ces décagones est égale au rayon du cercle.

Solution. On considère le cercle divisé en dix parties égales. On joint les points CD et DE, par des segments égaux aux côtés du décagone convexe, les points BE et CF, par des segments égaux aux côtés du décagone étoilé, et les diamètres AF et BG. Alors $CD \parallel BE \parallel AF$ et $DE \parallel CF \parallel BG$. En vertu des propriétés des parallélogrammes, on a

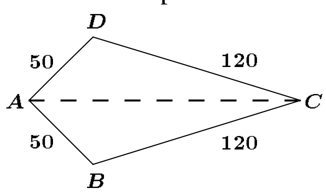
$$\begin{aligned} BE-CD &= BE-HE \\ &= BH=OF=r. \end{aligned}$$

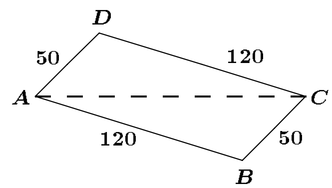


4. LE TERRAIN EN FORME DE CERF-VOLANT

Abel Belgrillet est membre du Club des aérocervidophiles (amateurs de cerfs-volants) du Québec. Il dispose de quatre tronçons de clôture rectilignes AB, BC, CD, DA pour délimiter un terrain (en forme de cerf-volant, voir figure) sur lequel il s'adonnera à son activité favorite cet été. Sachant que les tronçons AB et DA mesurent 50 m chacun et que les tronçons BC et CD mesurent 120 m chacun, déterminez la distance entre les points A et C qui maximisera l'aire du terrain.

Solution. Découpons le terrain selon AC:

l'aire de 

= l'aire de 

Mais, l'aire d'un parallélogramme de côtés donnés est maximale lorsque ce parallélogramme est un rectangle (car dans ce cas, la hauteur sera maximale). On doit donc avoir que l'angle $\widehat{B} = 90°$. Le triangle ABC est donc rectangle en B. Ainsi

$$\overline{AC} = \sqrt{\overline{AB}^2 + \overline{BC}^2} = \sqrt{2500 + 14400} = 130.$$

5. L'INÉGALITÉ MODIFIÉE D'AMOTH DIEUFUTUR

(a) (2 points) L'inégalité $x^2 + 2y^2 \geq 3xy$ est-elle vraie pour tous les entiers?

(b) (8 points) Montrez que l'inégalité $x^2 + 2y^2 \geq \frac{14}{5}xy$ est valide pour tous réels x et y.

Solution. (a) Non. Voici quelques couples qui fournissent un contre-exemple:

x	y
3	2
4	3
5	3
5	4
6	4

(b) 1$^{\text{ère}}$ **solution**. Considérons l'inégalité $(x - \alpha y)^2 \geq 0$, qui est vraie pour tout couple de réels x, y. Elle est équivalente à $x^2 + \alpha^2 y^2 \geq 2\alpha xy$. En posant $\alpha = \sqrt{2}$, on trouve $x^2 + 2y^2 \geq 2\sqrt{2}\,xy = (2,828\ldots)xy \geq \frac{14}{5}xy$. On peut aussi montrer que $2\sqrt{2} \geq \frac{14}{5}$ en élevant au carré de chaque côté.

2$^{\text{ième}}$ **solution**. Montrer que $x^2 + 2y^2 \geq \frac{14}{5}xy$ revient à démontrer l'inégalité $5x^2 + 10y^2 - 14xy \geq 0$. Or,

$$\begin{aligned} 5x^2 + 10y^2 - 14xy &= x^2 - 2xy + y^2 + 4x^2 - 12xy + 9y^2 \\ &= (x - y)^2 + (2x - 3y)^2 \\ &\geq 0 \end{aligned}$$

la dernière inégalité résultant du fait qu'une somme de carrés est toujours supérieure ou égale à zéro.

6. L'ÉCHIQUIER COQUET

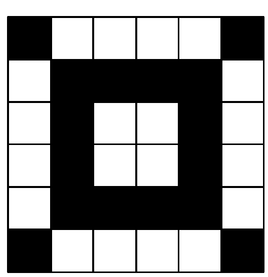
Trouvez l'unique façon de colorier les 36 cases d'un échiquier 6×6 en noir et blanc de sorte que chacune des cases soit voisine d'un nombre impair de cases noires.

(*Note:* deux cases sont voisine si elles se touchent par un côte-ou par un coin). On ne demande pas de démontrer que la solution est unique.

Solution. Ayant une solution, chacune des quatre symmétries possibles (les deux diagonales, la verticale et l'horizontale) engendre une nouvelle solution. Or, puisque la solution est unique, elle doit nécessairement être symétrique selon ces quatre axes de symétries. Il s'ensuit que les cases numérotées 1 ci-dessous sont toutes de la même couleur. De même pour les cases numérotées 2, numérotées 3, ... , numérotées 6.

1	2	3	3	2	1
2	4	5	5	4	2
3	5	6	6	5	3
2	4	5	5	4	2
1	2	3	3	2	1

Pour que les cases 1 touchent à un nombre impair de cases noires, les cases 4 doivent nécessairement être noires, puisque chacune d'elles est voisine de deux cases 2 et d'une case 4. De plus, les cases 4 étant voisines d'une case 1, d'une case 6, et de deux cases 2, 3 et 5, il est clair que les cases 1 et 6 doivent être de couleurs différentes. De plus, les cases 4 étant noires, les cases 6 sont nécessairement blanches, puisqu'elles sont voisines chacune d'une case 4, de deux cases 5 et de trois cases 6. Les cases 1 sont donc noires, puisque nous avons déjà établi qu'elles sont de couleurs différentes que les cases 6. En considérant maintenant les cases 3, et ce qu'on sait déjà, on conclut que les cases 2 et 3 doivent être de même couleur. Maintenant, en considérant les cases 2, sachant que 1 et 4 sont noires et que 2 et 3 sont de même couleur, on déduit que 5 doit être noir. En considérant la position 5, maintenant, on conclut finalement que les cases en 2 et en 3 sont blanches, ce qui nous donne finalement:



7. LA FRACTION D'ANNE GRUJOTE

En base 10, $\frac{1}{3} = 0.333\ldots$. Écrivons $0.\bar{3}$ pour ces décimales répétées. Comment écrit-on $\frac{1}{3}$ dans une base b, où b est de la forme

(i) $b = 3t$ (trois points),

(ii) $b = 3t + 1$ (trois points),

(iii) $b = 3t + 2$ (quatre points),

où t est un entier positif quelconque?

Solution. (i) Notons $\frac{1}{3} = \frac{t}{3t}$, d'ou $\frac{1}{3} = 0.t$ en base $3t$.

(ii) Supposons que $\frac{1}{3} = 0.d_1d_2d_3\ldots$. Alors $\frac{3t+1}{3} = d_1.d_2d_3\ldots$ et d_1 est égal à la partie entière de $\frac{3t+1}{3}$. Donc $d_1 = t$.

Soustrait $d_1 = t$ de $\frac{3t+1}{3}$. On obtient

$$\frac{3t+1}{3} - t = \frac{3t+1}{3} - \frac{3t}{3} = \frac{1}{3} = 0.d_2d_3\ldots .$$

Le procédé donne $t = d_2 = d_3 = \ldots$. Donc $\frac{1}{3} = 0.\bar{t}$.

(iii) Supposons encore $\frac{1}{3} = 0.d_1d_2d_3\ldots$. Alors $\frac{3t+2}{3} = d_1.d_2d_3\ldots$. La partie entière de $\frac{3t+2}{2}$ est t. Donc $d_1 = t$.

$$\frac{3t+2}{3} - \frac{3t}{3} = \frac{2}{3} = 0.d_2d_3\ldots .$$

On multiplie encore par $3t + 2$. On obtient $d_2.d_3d_4\ldots = \frac{2(3t+2)}{3} = \frac{6t+4}{3}$, dont la partie entière est $2t+1$. Donc $d_2 = 2t+1$. On soustrait $d_2 = 2t+1$ et on obtient

$$\begin{aligned} \frac{6t+4}{3} - \frac{3(2t+1)}{3} &= \frac{(6t+4)-(6t+3)}{3} \\ &= \frac{1}{3} = 0.d_3d_4\ldots . \end{aligned}$$

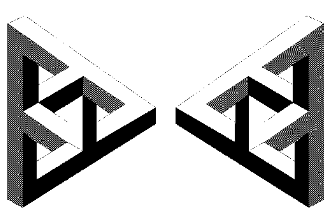
On voit que $t = d_1 = d_3 = d_5 = \ldots$ et $2t+1 = d_2 = d_4 = d_6 = \ldots$. Donc

$$\frac{1}{3} = 0.\overline{t\,(2t+1)}.$$

Vérification

$$
\begin{aligned}
0.\overline{t\,(2t+1)} &= \frac{(3t+2)t+(2t+1)}{(3t+2)^2}\left\{1+\frac{1}{(3t+2)^2}+\frac{1}{(3t+2)^4}+\cdots\right\}\\
&= \frac{3t^2+4t+1}{(3t+2)^2}\left\{\frac{1}{1-\frac{1}{(3t+2)^2}}\right\}\\
&= \frac{3t^2+4t+1}{(3t+2)^2}\left\{\frac{(3t+2)^2}{(3t+2)^2-1}\right\}\\
&= \frac{3t^2+4t+1}{9t^2+12t+3}\\
&= \frac{1}{3}.
\end{aligned}
$$

That completes the Skoliad Corner for this issue. Please send me suitable contest materials at this level, as well as your comments and suggestions for future columns.



MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to the Mayhem Editor, Naoki Sato, Department of Mathematics, Yale University, PO Box 208283 Yale Station, New Haven, CT 06520–8283 USA. The electronic address is still

mayhem@math.toronto.edu

The Assistant Mayhem Editor is Cyrus Hsia (University of Toronto). The rest of the staff consists of Richard Hoshino (University of Waterloo), Wai Ling Yee (University of Waterloo), and Adrian Chan (Upper Canada College).

Shreds and Slices

Factorial Fanaticism

If you've seen one too many math contests, then the following factorial problem will be old hat: "How many factors of 2 are there in 100!?" For those who don't recognize "!" as a mathematical symbol, here is its meaning: $n!$ is the product of all the integers from n down to 1, or

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1.$$

For example, $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$.

So how many factors of 2 are there in $100!$? Well, there are $\frac{100}{2} = 50$ even numbers giving 50 factors of 2. But every multiple of 4 gives an additional factor of 2. There are $\frac{100}{4} = 25$ of them. Continuing, each multiple of 8 gives another factor of 2, and there are $\lfloor\frac{100}{8}\rfloor = 12$ of them, and so on for multiples of 16, 32, and 64. This is enough, as the next power of 2, that is 128, is greater than 100, so there are no multiples of it which are less than 100. The total number of factors of 2 is then

$$50 + 25 + 12 + 6 + 3 + 1 = 97.$$

[Ed.: Coincidence?]

Now try and find how many factors of 5 there are $100!$. Find a general formula for finding the number of primes p in $n!$ for given integers p and n.

A similar problem to the one above is to calculate the number of consecutive zeros at the end of $100!$. It is equal to the number of factors of 5

in 100! (Why?). Once you have braved this question, you can easily figure out a method for finding the number of factors of m in $n!$ for given integers m and n.

In fact, it has been shown (and is an excellent exercise for the reader to show) that the number of factors of a prime number p in $n!$ is

$$\frac{n - s_n}{p - 1},$$

where s_n is the sum of the digits of n when expressed in base p.

A natural question to ask, after knowing all the zeros at the end of $n!$, is "What is the last non-zero digit in $n!$?" One way, perhaps not the most elegant, is to write $n!$ in the form $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where the p_i are all the distinct primes dividing n. The $\alpha_1, \ldots, \alpha_r$, are calculated by the method above. Since we don't care about the trailing zeros, say k of them, remove a factor of 10^k and then take the quotient modulo 10. For those who are not familiar with modular arithmetic, all this does is find the units digits.

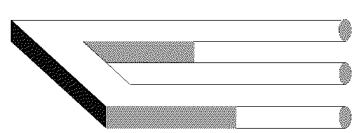
For example, $6! = 720 = 2^4 \times 3^2 \times 5$. Removing a factor of 10 to get rid of the trailing zeros gives $72 = 2^3 \times 3^2$. Then $2^3 \times 3^2$ is congruent to 2 modulo 10, that is, the units digit is 2. Thus, the last non-zero digit of 6! is 2.

Now for some more interesting questions.

1. How many digits does $n!$ have?

2. What is the initial digit of $n!$, or more generally its r^{th} digit?

3. How many digits are zeros in $n!$? How many are 1's? Likewise for the other digits.



Solving the Quartic

Cyrus Hsia

Most senior high school students are familiar with the general solution to the quadratic equation $ax^2 + bx + c = 0$. Either it is derived or given as a mere fact in class, and the notorious solution is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

It is rarely the case, however, that the solution to the general cubic equation $ax^3 + bx^2 + cx + d = 0$ is ever derived much less even mentioned. Here we do not tread this path, but refer the reader to [1] and [2] for its treatment. Instead, we go one step further and give the process of solving the general quartic (fourth degree) equation $ax^4 + bx^3 + cx^2 + dx + e = 0$.

Readers are encouraged to read [1] and [2] to learn or refresh their memories on solving the cubic equation. This is because we wish to reduce the quartic equation into that of a cubic equation. The process we give here is a variant of the solution credited to Lodovico Ferrari [3].

Motivations

Readers who are familiar with solving cubic equations know that the first step is to reduce the general cubic into the depressed cubic equation $x^3 + px + q = 0$. This may be of some value in the quartic case as well.

Also, if the quartic equation could be written as a difference of squares, that is $(x^2 + p)^2 - (qx + r)^2 = 0$, then solving the original quartic becomes an exercise in solving two quadratic equations — namely, solving

$$x^2 + p + qx + r = 0 \quad \text{and} \quad x^2 + p - qx - r = 0.$$

To this end, we try to convert the general quartic equation into this form somehow. Supposing we have reduced $ax^4 + bx^3 + cx^2 + dx + e = 0$ into the form $x^4 + ux^2 + vx + w = 0$. (How do we do this? See the method below.) Then we compare coefficients in

$$\begin{aligned} x^4 + ux^2 + vx + w &= (x^2 + p)^2 - (qx + r)^2 \\ &= x^4 + (2p - q^2)x^2 + (-2qr)x + (p^2 - r^2) \end{aligned}$$

to solve for p, q, and r in terms of the known u, v, and w. (How is this done? See the method below.) Voilà! The solution falls apart.

Method for Solving the General Quartic Equation

- Step 1: Reduce $ax^4 + bx^3 + cx^2 + dx + e = 0$ to the form $x^4 + ux^2 + vx + w = 0$. To do this, simply divide through by a. (We must have $a \neq 0$; else the equation would no longer be a quartic.) Next, make the substitution $x = t - \frac{b}{4a}$ to get $t^4 + ut^2 + vt + w = 0$.

- Step 2: Convert the reduced quartic from Step 1 to a difference of squares as described above. To do this, we need to solve for p, q, and r in terms of the known u, v, and w. By comparing coefficients, we have three equations in three unknowns:

$$\begin{aligned} 2p - q^2 &= u \\ -2qr &= v \\ p^2 - r^2 &= w \end{aligned}$$

  Solving for p, we have $v^2 = 4q^2r^2 = 4(2p-u)(p^2-w)$. This is a cubic equation in p, which can be solved.

- Step 3: Take the difference of squares found in Step 2 and solve for the two quadratic equations to get the four solutions of the quartic equation $t^4 + ut^2 + vt + w = 0$. Remember, these are **not** the desired solutions. Make the substitution $x = t - \frac{b}{4a}$ for each of the four solutions to get the four solutions of the original quartic equation. (Why didn't we multiply by a?)

Afterword

This seems quite tedious, and is thus one reason why it is not taught in classrooms. However, note how simple solving the quartic is if a black-box was given to automatically solve a cubic. The quartic is solved by using this black-box once and solving an additional two quadratic equations.

The best way to learn a technique is to use it. Here is a problem where the reader is encouraged to apply the above algorithm. Although there are more elegant ways to solve this, by using Ferrari's method the reader will gain a greater understanding of solving polynomial equations.

Solve the equation

$$x^2 + \frac{x^2}{(x+1)^2} = 3.$$

(1992 CMO Problem 4)

We end with a note that polynomial equations of degree five or higher cannot be solved, so readers can be relieved that there are not longer processes which reduce polynomials one degree at a time to reach a solution.

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[1] Hlousek, Daniela and Lew Dion, "Cardano's Little Problem", *Mathematical Mayhem*, Volume 6, Issue 3, March/April 1994, pp. 15-18.

[2] Dunham, William, *Journey Through Genius*, Penguin Books, New York, 1990, pp. 142-149.

[3] Dunham, William, *Journey Through Genius*, Penguin Books, New York, 1990, p. 151.

Pizzas and Large Numbers

Shawn Godin

The following is an excerpt from a television commercial for a certain unnamed pizza chain:

Customer: *So what's this new deal?*

Pizza Chef: *Two pizzas.*

Customer: *[Towards four-year-old Math Whiz] Two pizzas. Write that down.*

Pizza Chef: *And on the two pizzas choose any toppings - up to five [from the list of* 11 *toppings].*

Older Boy: *Do you...*

Pizza Chef: *...have to pick the same toppings on each pizza? No!*

Math Whiz: *Then the possibilities are endless.*

Customer: *What do you mean? Five plus five are ten.*

Math Whiz: *Actually, there are* 1 048 576 *possibilities.*

Customer: *Ten was a ballpark figure.*

Old Man: *You got that right.*

[At this point, the camera fades to a picture of hot pizzas; the announcer's voice and a very small Roman who says "Pizza! Pizza!"]

This simple advertisement caused an uproar in the mathematics community ([1], [2], [3], [4], [5]). The question is, should a four year old be trusted with higher mathematics? To answer the question, we should first do a little analysis.

Before jumping into the problem, let us state some assumptions:

(i) Each pizza may have up to five toppings on it (given).

(ii) Both pizzas do not have to have the same toppings (given).

(iii) A pizza may have no toppings (although a boring choice, it seems reasonable since most pizza places include a price for "sauce and cheese" as the basic "no item" pizza).

(iv) The order that the items are put on the pizza does not matter. So a pizza with ham and bacon is the same as a pizza with bacon and ham (it seems obvious, but this decision will affect our solution).

(v) The order that you order the pizzas, or the order that they are put in the box, or handed to you, does not matter. That is, ordering a pizza with ham and a second pizza with mushrooms is the same as ordering a pizza with mushrooms and another with ham (again, it seems obvious, but this decision will affect our solution).

(vi) Double items are not allowed (although it is allowed in most pizza places, we will consider this simpler problem first then extend it).

Now off to the solution. First, we will look at the number of ways that one pizza can be made. Our pizza may have zero, one, two, three, four, or five toppings, so the total number of pizzas available to us is:

$$\binom{11}{5} + \binom{11}{4} + \cdots + \binom{11}{0} = 1024.$$

Thus, if we are to choose two pizzas we have two possibilities:

Case 1: Both pizzas are the same. In this case, there are 1024 ways to pick the first pizza and, having picked the first one, we are locked in for our second one (it must be the same as the first) so there are 1024 ways to pick two pizzas if they must be the same.

Case 2: Both pizzas are different. In this case, there are 1024 pizzas to choose from and I want to pick two, so the number of ways to do this is:

$$\binom{1024}{2} = 523\,776.$$

So the total number of ways to pick the two pizzas is:

$$1024 + 523\,776 = 524\,800,$$

which does not agree with the four year old. Now, as luck would have it, if we disregard our assumption (v), the number of ways to create pizzas in Case 2 is $P(1024, 2) = 1\,047\,552$, giving a total of $1024 + 1\,047\,552 = 1\,048\,576$, which is the four year old's claim.

It would seem that the child regards the order that the pizzas are ordered in as important. Indeed, Jean Sherrod of Little Caesars Enterprises explained in a letter to *Mathematics Teacher* ([3]) that that is indeed what they assumed. If they are assuming this, that is disregarding our assumption (v), would not it also make sense to regard the order that the toppings were asked for as important? How many pizzas would be possible if we rewrote assumption (iv) so that order was important?

Let us pursue the truth even further. As is well known to pizza connoisseurs, and as is admitted by Sherrod ([3]), a pizza may have a multiple selection of an item. Double pepperoni would count as two choices, both of the same topping. Weber and Weber ([4]) consider the multiple topping question with an interesting technique.

Suppose our 11 toppings are as follows: pepperoni (P), bacon (B), sausage (S), ham (H), anchovies (A), mushrooms (M), green peppers (GP), tomatoes (T), green olives (GO), black olives (BO), and hot peppers (HP). When you order a pizza, the order is processed on an order form that lists all possible toppings separated by vertical lines. The order form may look like this:

P | B | S | H | A | M | GP | T | GO | BO | HP

and the person taking the order just marks an **X** over the symbol for each topping you order. If you wanted double pepperoni, two **X**'s would be placed in the P slot, counting as two toppings.

If we allow multiple toppings, we can calculate the total number of pizzas possible by considering the total number of ways that the order form can be filled out. For example, if you ordered your own favourite pizza, the "Cardiac Conspiracy", which has double bacon, double pepperoni, and ham, the order form (if we just look at the lines and **X**'s) would look like this:

XX | **XX** | | **X** | | | | | | |

Similarly, if you ordered a "Trip To The Sea" pizza (quintuple anchovies), the order form would look like this:

| | | | **XXXXX** | | | | | |

Thus, each 5 item pizza will be represented by an arrangement of 5 **X**'s and 10 lines. The number of ways to arrange these 15 symbols is:

$$\frac{15!}{5!10!} = 3003.$$

This is in fact the binomial coefficient $C(15,5)$, or $C(15,10)$. This result makes more sense if we think of the problem as picking 5 of the 15 positions to mark with an **X** (or alternately, pick 10 of the 15 spaces to mark with a line).

To account for all possible pizza choices, we need to consider pizzas with zero, one, two, three, four, or five toppings. If we do this we get:

$$\frac{15!}{10!5!} + \frac{14!}{14!4!} + \cdots + \frac{10!}{10!0!} = 4368.$$

So, the number of choices for two pizzas is then $C(4368,2) + 4368 =$ 9 541 896. So now we have a lot of numbers to deal with. In our first calculation, we saw that there are 1024 possible pizzas allowing no multiple toppings. In our latest calculation, we saw that there are a total of 4368 pizzas if multiple toppings are allowed. Thus, since the original 1024 has been accounted for in the 4368, it would seem that $4368 - 1024 = 3344$ pizzas have multiple toppings. Our next problem to consider is to classify these 3344 pizzas.

Consider the case where a double topping is allowed and is used. Then a five item pizza could be made up of a double item and three singles, or two double items and a single. The method of Weber and Weber for enumerating pizza choices here becomes too cumbersome to work with. For example, if I marked a double item with a D and a single with an X, then arranging 10 lines, a D and 3 $\mathbf{X}$'s to get our first case does not take into account $\mathbf{X}$'s beside the D (more than a double) or even two or three $\mathbf{X}$'s together (which means we recount later cases). So to solve the problem, we will have to go back to the fundamental counting theorem.

First, we will consider the pizza made up of a double item and three singles. Our first task is to select the item that will be double. This can be done in $C(11, 1) = 11$ ways. Next, pick the three singles from the other remaining toppings. This can be done in $C(10, 3) = 120$ ways. So the total number of pizzas with a double item and three singles is $C(11, 1)C(10, 3) = 1320$. Continuing in this manner, we can account for all pizzas with at least one double topping and no topping is allowed more than twice (can you identify each term with the case it represents?):

$$\binom{11}{1}\binom{10}{3}+\binom{11}{2}\binom{9}{1}+\binom{11}{1}\binom{10}{2}+\binom{11}{2}+\binom{11}{1}\binom{10}{1}+\binom{11}{1} = 2486.$$

Now we have accounted for $1024 + 2486 = 3510$ of the total 4368 possible single pizzas. At this point, the reader should have enough ammunition to attack the remaining problem. The following questions are still unanswered.

1. How many pizzas are possible if triple (quadruple, quintuple) toppings are allowed? (pick your cases carefully!)
2. How many choices of two pizzas do you have if you are allowed to pick an item at most twice (three, four, five times)?
3. Using local pizza company choices, how many pizzas (or pairs of pizzas) are possible from each pizza place?
4. Who in town makes the best pizza?

Have fun trying to solve these and other problems of your creation. Mathematics is a demanding endeavour, so if you feel hungry you may want to pick up the phone...

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1. N. Kildonan, problem 1946, *Crux Mathematicorum*, 20:5 (1994) 137.
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3. J. Sherrod, "Little Caesars Responds", *Mathematics Teacher*, 87:6 (1994) 474.
4. G. Weber and G. Weber, "Pizza Combinatorics", *The College Mathematics Journal*, 26:2 (1995) 141-143.
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Minima and Maxima of Trigonometric Expressions

Nicholae Gusita

Problem 1. Find the minimum and maximum values of

$$y_1 = \sin x + \cos x,$$

for $0^\circ \le x < 360^\circ$.

Solution. We write

$$y_1 = \sin x + \sin(90^\circ - x) = 2\sin 45^\circ \cos(x - 45^\circ) = \sqrt{2}\cos(x - 45^\circ)\,.$$

Then the minimum of $-\sqrt{2}$ occurs when $x - 45^\circ = 180^\circ$, so that $x = 225^\circ$, and the maximum of $\sqrt{2}$ occurs when $x - 45^\circ = 0$, so that $x = 45^\circ$.

Problem 2. Find the minimum and maximum values of

$$y_2 = 3\sin x + 4\cos x,$$

for $0^\circ \le x < 360^\circ$.

Solution. If we put $\phi = \tan^{-1}\left(\frac{4}{3}\right)$, then

$$y_2 = 3(\sin x + \tan\phi\cos x) = 3\left(\sin x + \frac{\sin\phi}{\cos\phi}\cos x\right) = \frac{3\sin(x+\phi)}{\cos\phi}.$$

But

$$\cos^2\phi = \frac{1}{1+\tan^2\phi} = \frac{9}{25},$$

so that

$$\cos\phi = \frac{3}{5},$$

since $0^\circ < \phi < 90^\circ$. Then $y_2 = 5\sin(x+\phi)$. It is clear now that the minimum of -5 occurs when $x + \phi = 270^\circ$, so that $x = 270^\circ - \phi$, and the maximum of 5 occurs when $x + \phi = 90^\circ$, so that $x = 90^\circ - \phi$.

Problem 3. Find the minimum and maximum values of

$$y_3 = a\sin x + b\cos x,$$

for $0^\circ \le x < 360^\circ$, $a, b \in \mathbb{R}\backslash\{0\}$.

Solution. Let $\phi = \tan^{-1}\left(\frac{b}{a}\right)$, $-90^\circ < \phi < 90^\circ$. Assume $a > 0$. Then

$$y_3 = a(\sin x + \tan\phi\cos x) = a\left(\sin x + \frac{\sin\phi}{\cos\phi}\cos x\right) = \frac{a\sin(x+\phi)}{\cos\phi}.$$

But as in Problem 2,

$$\cos^2\phi = \frac{1}{1+\tan^2\phi} = \frac{a^2}{a^2+b^2},$$

so that

$$\cos\phi = \frac{a}{\sqrt{a^2+b^2}}.$$

Therefore, $y_3 = \sqrt{a^2+b^2}\sin(x+\phi)$.

The minimum of $-\sqrt{a^2+b^2}$ occurs when $x+\phi = 270°$, so that $x = 270° - \phi$, and the maximum of $\sqrt{a^2+b^2}$ occurs when $x+\phi = 90°$, so that $x = 90° - \phi$.

If $a < 0$, then using similar reasoning, we find the minimum occurs when $x = 90° - \phi$, and maximum when $x = 270° - \phi$.

Problem 4. Given two positive angles a and b whose sum is $150°$, prove that the maximum value of $\sin a + \cos b$ is $\sqrt{3}$.

Solution. We have

$$\begin{aligned} y_4 &= \sin a + \cos b = \sin a + \sin(90° - b) \\ &= 2\sin\left(\frac{a-b}{2} + 45°\right)\cos\left(\frac{a+b}{2} - 45°\right) \\ &= 2\sin(a - 30°)\cos 30° = \sqrt{3}\sin(a - 30°). \end{aligned}$$

The maximum of $\sqrt{3}$ occurs when $a - 30° = 90°$, so that $a = 120°$, $b = 30°$.

Problem 5. Find the minimum and maximum values of

$$y_5 = a\sin^2 x + b\sin x\cos x + c\cos^2 x,$$

$0 \le x < 360°$, $a, b, c \in \mathbb{R}$.

Solution. Since $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$, and $\sin x \cos x = \frac{1}{2}\sin 2x$,

$$\begin{aligned} y_5 &= \frac{a}{2}(1 - \cos 2x) + \frac{b}{2}\sin 2x + \frac{c}{2}(1 + \cos 2x) \\ &= \frac{a+c}{2} + \frac{1}{2}\left[b\sin 2x + (c-a)\cos 2x\right]. \end{aligned}$$

By problem (3), the minimum and maximum values are

$$\frac{a+c-\sqrt{b^2+(a-c)^2}}{2} \quad \text{and} \quad \frac{a+c+\sqrt{b^2+(a-c)^2}}{2}$$

respectively.

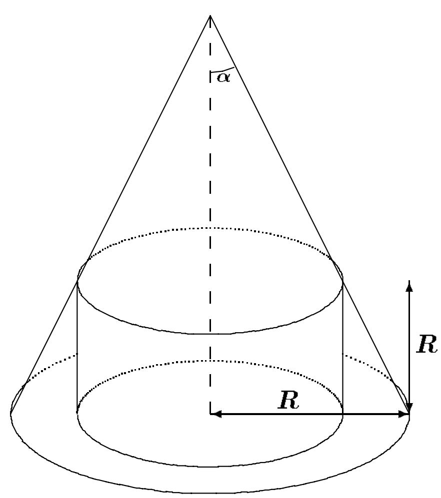
J.I.R. McKnight Problems Contest 1979

1. Sand is being poured on the ground forming a pile in the shape of a cone whose altitude is $\frac{2}{3}$ of the radius of its base. The sand is falling at the rate of 20 L/s. How fast is the altitude of the pile increasing when it is 42 cm high?

2. Solve:

$$\tan^{-1}\left(\frac{x-1}{x+1}\right) + \tan^{-1}\left(\frac{2x-1}{2x+1}\right) = \tan^{-1}\left(\frac{23}{36}\right).$$

3. A cylinder is inscribed in a cone, the altitude of the cylinder being equal to the radius of the base of the cone. Find the measure of angle α, if the ratio of the total surface area of the cylinder to the area of the base of the cone is $3:2$.



4. Solve the equation $|x| + |x-1| + |x-2| = 4$.

5. Solve for x and y.

$$\begin{aligned} 2^{x+y} &= 6^y \\ 3^x &= 3 \cdot 2^{y+1} \end{aligned}$$

6. A fixed circle of radius 3 has its centre at $(3, 0)$. A second circle has its centre at the origin, its radius is approaching zero. A line joins the point of intersection of the second circle and the y-axis to the intersection of the two circles. Find the limit of the point of intersection of this line with the x-axis.

Mayhem Problems

The Mayhem Problems editors are:

Richard Hoshino	*Mayhem High School Problems Editor,*
Cyrus Hsia	*Mayhem Advanced Problems Editor,*
Ravi Vakil	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only problems — the next issue will feature only solutions.

We warmly welcome proposals for problems and solutions. With the new schedule of eight issues per year, we request that solutions from this issue be submitted by 1 December 1997, for publication in the issue 5 months ahead; that is, issue 4 of 1998. We also request that **only students** submit solutions (see editorial [1997: 30]), but we will consider particularly elegant or insightful solutions from others. Since this rule is only being implemented now, you will see solutions from many people in the next few months, as we clear out the old problems from Mayhem.

High School Problems

Editor: Richard Hoshino, 17 Norman Ross Drive, Markham, Ontario, Canada. L3S 3E8 <rhoshino@undergrad.math.uwaterloo.ca>

H229. Here's a simple way to remember how many books there are in the Bible. Remember that there are x books in the Old Testament, where x is a two-digit integer. Then multiply the digits of x to get a new integer y, which is the number of books in the New Testament. Adding x and y, you end up with 66, the number of books in the Bible. What are x and y?

H230. Dick and Cy stand on opposite corners (on the squares) of a 4 by 4 chessboard. Dick is telling too many bad jokes, so Cy decides to chase after him. They take turns moving one square at a time, either vertically or horizontally on the board. To catch Dick, Cy must land on the square Dick is on. Prove that:

(i) If Dick moves first, Cy can eventually catch Dick.

(ii) If Cy moves first, Cy can never catch Dick.

(Can you generalize this to a $2m \times 2n$ chessboard?)

H231. Let O be the centre of the unit square $ABCD$. Pick any point P inside the square other than O. The circumcircle of PAB meets the circumcircle of PCD at points P and Q. The circumcircle of PAD meets the circumcircle of PBC at points P and R. Show that $QR = 2 \cdot OP$.

H232. Lucy and Anna play a game where they try to form a ten-digit number. Lucy begins by writing any digit other than zero in the first place, then Anna selects a different digit and writes it down in the second place, and they take turns, adding one digit at a time to the number. In each turn, the digit selected must be different from all previous digits chosen, and the number formed by the first n digits must be divisible by n. For example, 3, 2, 1 can be the first three moves of a game, since 3 is divisible by 1, 32 is divisible by 2 and 321 is divisible by 3. If a player cannot make a legitimate move, she loses. If the game lasts ten moves, a draw is declared.

(i) Show that the game can end up in a draw.

(ii) Show that Lucy has a winning strategy and describe it.

Advanced Problems

Editor: Cyrus Hsia, 21 Van Allan Road, Scarborough, Ontario, Canada. M1G 1C3 <hsia@math.toronto.edu>

A205. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \le x$ and $f(x+y) \le f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

A206. Let n be a power of 2. Prove that from any set of $2n-1$ positive integers, one can choose a subset of n integers such that their sum is divisible by n.

A207. Given triangle ABC, let A', B', and C' be points on sides BC, CA, and AB respectively such that $\triangle A'B'C' \sim \triangle ABC$. Find the locus of the orthocentre of all such triangles $A'B'C'$.

A208. Let p be an odd prime, and let S_k be the sum of the products of the elements $\{1, 2, \ldots, p-1\}$ taken k at a time. For example, if $p = 5$, then $S_3 = 1 \cdot 2 \cdot 3 + 1 \cdot 2 \cdot 4 + 1 \cdot 3 \cdot 4 + 2 \cdot 3 \cdot 4 = 50$. Show that $p \mid S_k$ for all $2 \le k \le p-2$.

Challenge Board Problems

Editor: Ravi Vakil, Department of Mathematics, Princeton University, Fine Hall, Washington Road, Princeton, NJ 08544–1000 USA <vakil@math.princeton.edu>

C74. Prove that the k-dimensional volume of a parallelopiped in $\mathbb{R}^n$ spanned by the vectors $\vec{v}_1, \ldots, \vec{v}_k$ is the determinant of the $k \times k$ matrix $\{v_i \cdot v_j\}_{i,j}$.

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk (⋆) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 May 1998**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in* LaTeX*). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication.*

Solutions submitted by FAX

There has been an increase in the number of solutions sent in by FAX, either to the Editor-in-Chief's departmental FAX machine in St. John's, Newfoundland, or to the Canadian Mathematical Society's FAX machine in Ottawa, Ontario. While we understand the reasons for solvers wishing to use this method, we have found many problems with it. The major one is that hand-written material is frequently transmitted very badly, and at times is almost impossible to read clearly. We have therefore adopted the policy that we will no longer accept submissions sent by FAX. We will, however, continue to accept submissions sent by email or regular mail. We do encourage email. Thank you for your cooperation.

2276. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Quadrilateral $ABCD$ is cyclic with circumcircle $\Gamma(0, R)$.

Show that the nine-point (Feuerbach) circles of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$ and $\triangle ABC$ have a point in common, and characterize that point.

2277. *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

For $n \geq 1$, define

$$u_n = \left[\frac{1}{(1,n)}, \frac{2}{(2,n)}, \ldots, \frac{n-1}{(n-1,n)}, \frac{n}{(n,n)}\right],$$

where the square brackets [ ] and the parentheses ( ) denote the **least common multiple** and **greatest common divisor** respectively.

For what values of n does the identity $u_n = (n-1)u_{n-1}$ hold?

2278. *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

Determine the value of a_n, which is the number of ordered n–tuples $(k_2, k_3, \ldots, k_n, k_{n+1})$ of non-negative integers such that

$$2k_2 + 3k_3 + \ldots + nk_n + (n+1)k_{n+1} = n+1.$$

2279. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

With the usual notation for a triangle, prove that

$$\sum_{\text{cyclic}} \sin^3 A \cos B \cos C = \frac{sr}{4R^4}\left(2R^2 - s^2 + (2R+r)^2\right).$$

2280. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle with incentre I. Let D be the second intersection of AI with the circumcircle of $\triangle ABC$. Let X, Y be the feet of the perpendiculars from I to BD, CD respectively.

Suppose that $IX + IY = \frac{1}{2}AD$. Find $\angle BAC$.

2281. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle, and D is a point on the side BC produced beyond C, such that $AC = CD$. Let P be the second intersection of the circumcircle of $\triangle ACD$ with the circle on diameter BC. Let E be the intersection of BP with AC, and let F be the intersection of CP with AB.

Prove that D, E, F, are collinear.

2282. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

A line, ℓ, intersects the sides BC, CA, AB, of $\triangle ABC$ at D, E, F respectively such that D is the mid-point of EF.

Determine the minimum value of $|EF|$ and express its length as elements of $\triangle ABC$.

2283. *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

You are given triangle ABC with $\angle C = 60^\circ$. Suppose that E is an interior point of line segment AC such that $CE < BC$. Suppose that D is an interior point of line segment BC such that

$$\frac{AE}{BD} = \frac{BC}{CE} - 1\,.$$

Suppose that AD and BE intersect in P, and the circumcircles of AEP and BDP intersect in P and Q. Prove that $QE \parallel BC$.

2284. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABCD$ is a rhombus with $\angle A = 60^\circ$. Suppose that E, F, are points on the sides AB, AD, respectively, and that CE, CF, meet BC at P, Q respectively. Suppose that $BE^2 + DF^2 = EF^2$.

Prove that $BP^2 + DQ^2 = PQ^2$.

2285. *Proposed by Richard I. Hess, Rancho Palos Verdes, California, USA.*

An isosceles right triangle can be 100% covered by two congruent tiles.

Design a connected tile so that two of them maximally cover a non-isosceles right triangle. (The two tiles must be identical in size and shape and may be turned over so that one is the mirror image of the other. They must not overlap each other or the border of the triangle.)

What coverage is achieved for a 30–60–90 right triangle?

2286.
Proposed by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

It is well known and easy to show that the product of any four consecutive positive integers plus one, is always a perfect square. It is also easy to show that the product of any two consecutive positive integers plus one is never a perfect square. It is possible that the product of three consecutive integers plus one is a perfect square. For example:

$$2 \times 3 \times 4 + 1 = 5^2 \qquad \text{and} \qquad 4 \times 5 \times 6 + 1 = 11^2.$$

(a) Find the next largest natural number n such that $n(n+1)(n+2)+1$ is a perfect square.

(b)* Are there any other examples?

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2090. [1995: 307] *Proposed by Peter Ivády, Budapest, Hungary.*

For $0 < x < \pi/2$ prove that

$$\left(\frac{\sin x}{x}\right)^2 < \frac{\pi^2 - x^2}{\pi^2 + x^2}\,.$$

Essentially the same solution was sent in by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; Václav Konečný, Ferris State University, Big Rapids, Michigan, USA; and Heinz-Jürgen Seiffert, Berlin, Germany.

For $0 < x < \frac{\pi}{2}$, we have

$$\left(\frac{\sin x}{x}\right)^2 = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{(n\pi)^2}\right)^2 < \left(1 - \frac{x^2}{\pi^2}\right)^2 < \frac{\pi^2 - x^2}{\pi^2 + x^2}.$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; LUIS V. DIEULEFAIT, IMPA, Rio de Janeiro, Brazil; PETER HURTHIG, Columbia College, Burnaby, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; CHRIS WILDHAGEN, Rotterdam, the Netherlands; and the proposer. Two incomplete solutions were received.

Most solvers proved a better inequality, and some commented that the interval could be extended to $(-\pi, \pi)$.

Klamkin and Manes both point out the complimentary inequality (which is not as easy to prove):

$$\frac{1 - t^2}{1 + t^2} \leq \frac{\sin \pi t}{\pi t} \qquad \text{for all real } t.$$

See American Mathematical Monthly, **76** (*1969*)*, pp. 1153-1154, R. Redheffer, problem 5642.*

Janous gave the extension to: what is the value of ρ that gives the best inequality of the type

$$\left(\frac{\sin x}{x}\right)^2 < \frac{\rho^2 - x^2}{\rho^2 + x^2}$$

which is valid for all $x \in (0, \pi/2)$? *We leave this as a challenge to our other readers*!

2169. [1996: 274] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

AB is a fixed diameter of circle $\Gamma_1(0, R)$. P is an arbitrary point of its circumference. Q is the projection onto AB of P. Circle $\Gamma_2(P_1PQ)$ intersects Γ_1 at C and D. CD intersects PQ at E. F is the midpoint of AQ. $FG \perp CD$, where $G \in CD$. Show that:

1. $EP = EQ = EG$,
2. A, G and P are collinear.

Solution by Toshio Seimiya, Kawasaki, Japan.

1. Let H be the second intersection of PQ with Γ_1. Since $PH \perp AB$, AB is the perpendicular bisector of PH, so that$PQ = QH$ and $\angle PAQ = \angle HAQ$. Since $PC = PQ = PD$, we get $\angle PHC = \angle PDC = \angle PCD$, so that $\triangle PHC \sim \triangle PCE$, from which we have

   $$PH : PC = PC : PE.$$

   Thus we have $PH \cdot PE = PC^2 = PQ^2$. As $PH = 2PQ$, we have $2PQ \cdot PE = PQ^2$, so that $2PE = PQ$; thus, $PE = EQ$.

   Since $FG \perp CD$ and $PQ \perp AB$, we have $\angle GFQ = \angle PEC$. As $\triangle PHC \sim \triangle PCE$ we get $\angle PEC = \angle PCH = \angle PAH = 2\angle PAQ$. Thus $\angle GFQ = 2\angle PAQ$. Since F, E are midpoints of AQ, PQ, we get $FE||AP$, so that $\angle PAQ = \angle EFQ$. Thus $\angle GFQ = 2\angle EFQ$, so that $\angle GFE = \angle EFQ$. Hence we have $\triangle GFE \equiv \triangle QFE$, so that $EG = EQ$. Therefore $EP = EQ = EG$.

2. Since $\triangle GFE \equiv \triangle QFE$ we have $FG = FQ = AF$, so that $\angle GAF = \frac{1}{2}\angle GFQ = \angle EFQ$. Thus we have $AG||FE$. Since $AP||FE$, A, G and P are collinear.

Also solved by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; CRISTÓBAL SÁNCHEZ-RUBIO, I.B. Penyagolosa, Castellón, Spain; and the proposer.

2170. [1996: 274] *Proposed by Tim Cross, King Edward's School, Birmingham, England.*

Find, with justification, the positive integer which comes next in the sequence $1411, 4463, 4464, 1412, 4466, 4467, 1413, 4469, \ldots$.

[Ed.: the answer is NOT 4470.]

Editor's summary based on the solutions and comments submitted by the solvers whose names appear below.

Most solvers felt (and the editors agree) that the answer could be anything, as stated in the following comment by Murray Klamkin:

> *Any number can be the next term! A set of numbers is a sequence mathematically if and only if a rule of formation is given. The given set of numbers is not a sequence and so the problem is meaningless. Given any finite set of n numbers, one can always find an infinite number of formulae which agree with the given n terms, and such that the $(n+1)$th term is completely arbitrary.*

However, some solvers did come up with interesting formulae or reasonings to "justify" the answer that they gave. Some examples:

I. (Diminnie) Let $g_n = n - 3\left\lfloor \frac{n}{3} \right\rfloor$, and let

$$x_n \;=\; 3\frac{1487 + \lfloor n/3 \rfloor}{2 - (-1)^{g(n)}} + \tfrac{1}{2}\left\{\left(1 + (-1)^{g(n)}\right) g(n) + 76\left((-1)^{g(n)} - 1\right)\right\}.$$

Then x_1, x_2, ..., x_8 agree with the given terms, and $x_9 = 4470$.

In general, if k is any number and if we define

$$y_n = x_n + \lfloor n/9 \rfloor (k - 4470),$$

then y_1, y_2, ..., y_8 agree with the given terms, and $y_9 = k$.

II. (Hurthig and the proposer) Squaring each of the given terms reveals

$$\underline{1990}21, \quad \underline{1991}8369, \quad \underline{1992}7296, \quad \underline{1993}744,$$
$$\underline{1994}5156, \quad \underline{1995}4089, \quad \underline{1996}569, \quad \underline{1997}1961,$$

and the given numbers a_n ($n = 1, 2, \ldots, 8$), are the least positive integers whose squares begin with the digits 1990, 1991, ..., 1997. That is, a_n is the smallest positive integer k such that k^2 begins with the same digits as $1989 + n$.

This leads to $a_9 = 447$.

III. (Bradley) The given numbers are the integer parts of the square roots of

$$1991 \times 10^3, \quad 1992 \times 10^4, \quad 1993 \times 10^4, \quad 1994 \times 10^3,$$
$$1995 \times 10^4, \quad 1996 \times 10^4, \quad 1997 \times 10^3, \quad 1998 \times 10^4.$$

Hence the next term would be $\left\lfloor\sqrt{1999 \times 10^4}\right\rfloor$, or 4471.

IV. (Hess) Let $f(n) = \left\lfloor n\sqrt{10} - \frac{1}{3}\right\rfloor$. The the given numbers are

$$n, f(n)+2, f(n+1), n+1, f(n+1)+2, f(n+2), n+2, f(n+2)+2,$$

with $n = 1411$. Thus the next term is $f(n+3) = f(1414) = 4471$.

Other submitted answers include 4479 (Konečný) and 44610 (Ortega and Gutiérrez).

Solved by HAYO AHLBURG, Benidorm, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, California, USA; PETER HURTHIG, Columbia College, Burnaby, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; SOLEDAD ORTEGA and JAVIER GUTIÉRREZ, students, University of La Roija, Logroño, Spain; and the proposer.

2171. [1996: 274] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let P be an arbitrary point taken on an ellipse with foci F_1 and F_2, and directrices d_1, d_2, respectively. Draw the straight line through P which is parallel to the major axis of the ellipse. This line intersects d_1 and d_2 at points M and N, respectively. Let P' be the point where MF_2 intersects NF_1.

Prove that the quadrilateral $PF_1P'F_2$ is cyclic.

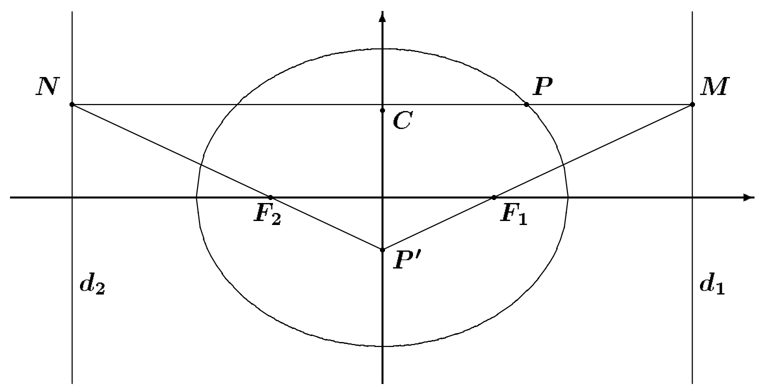
Does the result also hold in the case of a hyperbola?

Solution by Richard I. Hess, Rancho Palos Verdes, California, USA, modified by the editor.

I: The Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Let F_1, F_2 be $(\frac{a}{\kappa}, 0)$, $(-\frac{a}{\kappa}, 0)$, respectively, where $\kappa > 1$, and $b = \frac{\sqrt{\kappa^2-1}}{\kappa}a$. Let P be $(a\sin\theta, b\cos\theta)$ and M be $(\kappa a, b\cos\theta)$.

By symmetry, P' is on the y-axis. Let P' be $(0, -d)$.



Thus

$$\frac{d}{a/\kappa} \;=\; \frac{b\cos\theta}{\kappa a - a/\kappa},$$

so that $d = \dfrac{b\cos\theta}{\kappa^2-1}$.

Choose C to be the point $(0,\beta)$ such that $CF_1 = CF_2 = CP'$. Thus

$$(\beta+d)^2 \;=\; \beta^2 + \frac{a^2}{\kappa^2},$$

so that

$$\frac{2\beta b\cos\theta}{\kappa^2-1} + \frac{b^2\cos^2\theta}{(\kappa^2-1)^2} \;=\; \frac{a^2}{\kappa^2} \;=\; \frac{b^2}{\kappa^2-1},$$

giving

$$\beta \;=\; \frac{b}{2\cos\theta} - \frac{b\cos\theta}{2(\kappa^2-1)}.$$

We now show that $CP = CP'$. If this were true, we would have

$$(\beta - b\cos\theta)^2 + a^2 = \sin^2\theta \;=\; \left(\beta + \frac{b\cos\theta}{\kappa^2-1}\right)^2,$$

or

$$-2b\beta\cos\theta + b^2\cos^2\theta + \frac{\kappa^2 b^2}{\kappa^2-1}\sin^2\theta \;=\; \frac{2b\beta\cos\theta}{\kappa^2-1} + \frac{b^2\cos^2\theta}{(\kappa^2-1)^2},$$

or

$$-b^2 + \frac{b^2\cos^2\theta}{\kappa^2-1} + b^2\cos^2\theta + \frac{b^2\kappa^2}{\kappa^2-1} \;=\; \frac{b^2}{\kappa^2-1} + \frac{\kappa^2 b^2\cos^2\theta}{\kappa^2-1},$$

or

$$\frac{\kappa^2}{\kappa^2-1} - 1 \;=\; \frac{1}{\kappa^2-1},$$

which is clearly true.

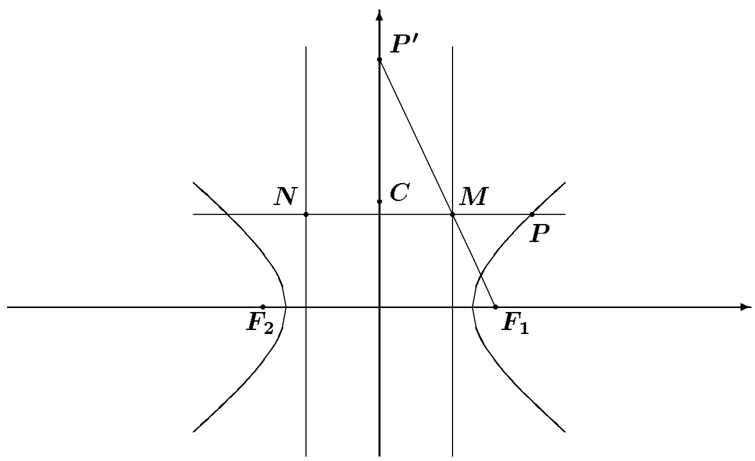
Thus C is the centre of the cyclic quadrilateral $PF_1P'F_2$.

II: The Hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Let F_1, F_2 be $\left(\frac{a}{\kappa}, 0\right)$, $\left(-\frac{a}{\kappa}, 0\right)$, respectively, where $0 < \kappa < 1$, and $b = \frac{\sqrt{1-\kappa^2}}{\kappa}a$. Let P be $(a\sinh\theta, b\cosh\theta)$ and M be $(\kappa a, b\cosh\theta)$.

By symmetry, P' is on the y–axis. Let P' be $(0, d)$.

Choose C to be the point $(0, \beta)$ such that $CF_1 = CF_2 = CP'$.



Follow the procedure in case I to show that C is the centre of the cyclic quadrilateral $PF_1P'F_2$.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer (for the ellipse only).

Smeenk notes that it is easy to verify that PP' is a normal to the ellipse. Bellot Rosado refers to E.A. Maxwell's Elementary Coordinate Geometry, *Oxford University Press, 1952, which contains the two following related problems:*

1. P, Q are two points on an ellipse with foci S, S', such that PQ is perpendicular to SS'.

   Prove that PS, QS, PS', QS' touch a circle, and identify its centre.

2. Tangents TP, TQ are drawn to an ellipse with foci S, S'. A line through S parallel to TQ meets $S'T$ in U, and a line through S' parallel to TP meets ST in V.

   Prove that S, S', U, V lie on a circle.

2172. [1996: 274] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $x, y, z \geq 0$ with $x + y + z = 1$. For fixed real numbers a and b, determine the maximum $c = c(a, b)$ such that $a + bxyz \geq c(yz + zx + xy)$.

Solution by Mihai Cipu, Institute of Mathematics, Romanian Academy, Bucharest, Romania.

The answer is

$$c(a, b) = \min\left(4a,\ 3a + \frac{b}{9}\right),$$

provided that $a \geq 0$.

For $x = y = 1/2$, $z = 0$ one obtains $c \leq 4a$, while by substituting $x = y = z = 1/3$ it follows that $c \leq 3a + b/9$. Thus $c(a, b) \leq \min(4a, 3a + b/9)$. Now to finish the proof we shall show that for all real numbers $a \geq 0$ and b,

$$a + bxyz \geq \min\left(4a,\ 3a + \frac{b}{9}\right) \cdot (xy + yz + zx) \tag{1}$$

for all $x, y, z \geq 0$, $x + y + z = 1$.

To this end we shall use the fact that for any such triple x, y, z there exists a Euclidean triangle whose sides have lengths $1 - x, 1 - y, 1 - z$. The triangle is degenerate if $xyz = 0$. Let us denote by r, resp. R, the radius of the incircle, resp. circumcircle, of the associated triangle. Using well-known formulae and the hypothesis, one easily finds

$$xyz = r^2 \qquad \text{and} \qquad xy + yz + zx = r^2 + 4Rr. \tag{2}$$

Here r and R have non-negative values subject to the restrictions $16Rr - 5r^2 \leq 1$ and $R \geq 2r$. [*Editor's note*. The hypothesis $x + y + z = 1$ means that the associated triangle has semiperimeter $s = 1$. Thus (2) follows from the known identities

$$(s - a_1)(s - a_2)(s - a_3) = r^2 s \qquad \text{and} \qquad \sum (s - a_1)(s - a_2) = r^2 + 4Rr$$

which hold for any triangle with sides a_1, a_2, a_3 — see for example equations (15) and (16), page 54 of Mitrinović, Pečarić and Volenec, *Recent Advances in Geometric Inequalities*. And the restriction $16Rr - 5r^2 \leq 1$ is just the known inequality $16Rr - 5r^2 \leq s^2$; see (3.6) on page 166 of *Recent Advances*, or item 5.8 of Bottema et al., *Geometric Inequalities*.]

We note that for $x = y = 0$, $z = 1$ one gets $a \geq 0$ [else there is no solution for c]. Using this fact, in the case $b \geq 9a$ we have

$$\min(4a, 3a + b/9) = 4a$$

and

$$a + br^2 \ \geq\ a(1 + 9r^2) \ \geq\ a(4r^2 + 16Rr),$$

so that (1) holds. In the opposite case $b \le 9a$ we get

$$a(1 - 3r^2 - 12Rr) \;\ge\; a(4Rr - 8r^2) \;\ge\; (4Rr - 8r^2)b/9$$

[since $4Rr - 8r^2 = 4r(R - 2r) \ge 0$], or equivalently

$$a + br^2 \;\ge\; (3a + b/9)(r^2 + 4Rr),$$

which is (1) again.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; and the proposer. One incorrect solution was sent in.

Most solvers noted that a solution exists only if $a \ge 0$.

2173. [1996: 275, 1997: 169] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $n \ge 2$ and $x_1, \ldots, x_n > 0$ with $x_1 + \ldots + x_n = 1$.

Consider the terms

$$l_n = \sum_{k=1}^{n} (1 + x_k)\sqrt{\frac{1 - x_k}{x_k}}$$

and

$$r_n = C_n \prod_{k=1}^{n} \frac{1 + x_k}{\sqrt{1 - x_k}}$$

where

$$C_n = (\sqrt{n-1})^{n+1}(\sqrt{n})^n/(n+1)^{n-1}.$$

1. Show $l_2 \le r_2$.

2. Prove or disprove: $l_n \ge r_n$ for $n \ge 3$.

I. *Solution to Part 1 by Richard I. Hess, Rancho Palos Verdes, California, USA.*

For $n = 2$, we get $C_2 = 2/3$ and [since $x_1 + x_2 = 1$]

$$l_2 \;=\; (1 + x_1)\sqrt{\frac{x_2}{x_1}} + (1 + x_2)\sqrt{\frac{x_1}{x_2}} \;=\; \frac{x_1 + x_2 + 2x_1x_2}{\sqrt{x_1x_2}}\,,$$

$$r_2 \;=\; \frac{2}{3}\left(\frac{1 + x_1}{\sqrt{x_2}}\right)\left(\frac{1 + x_2}{\sqrt{x_1}}\right).$$

Thus

$$\begin{aligned}\sqrt{x_1x_2}(r_2-l_2) &= \frac{2}{3}(1+x_1+x_2+x_1x_2)-x_1-x_2-2x_1x_2\\ &= \frac{1}{3}(1-4x_1x_2)\\ &= \frac{1}{3}\left[(x_1+x_2)^2-4x_1x_2\right]\\ &= \frac{1}{3}(x_1-x_2)^2 \geq 0.\end{aligned}$$

Therefore $l_2 \leq r_2$ with equality if and only if $x_1 = x_2 = 1/2$.

II. *Partial solution to Part 2 by the proposer.*

We show that $l_n \geq r_n$ in the case $n = 3$ (which indeed was the starting point for the whole problem).

Putting $x_1 = x$, $x_2 = y$, $x_3 = z$, the desired inequality $l_3 \geq r_3$ reads

$$(1+x)\sqrt{\frac{1-x}{x}}+(1+y)\sqrt{\frac{1-y}{y}}+(1+z)\sqrt{\frac{1-z}{z}}$$

$$\geq \frac{3\sqrt{3}}{4}\cdot\frac{(1+x)(1+y)(1+z)}{\sqrt{1-x}\sqrt{1-y}\sqrt{1-z}} \qquad (1)$$

where $x, y, z \in (0,1)$ such that $x+y+z=1$.

We now recall the difficult *Crux* problem 2029 of Jun-hua Huang, solved by Kee-Wai Lau on [1996: 129]:

$$w_bw_c + w_cw_a + w_aw_b \geq 3F\sqrt{3}, \qquad (2)$$

where w_a, w_b, w_c, F are the angle bisectors and the area of a triangle. We claim that this inequality is equivalent to inequality (1). Indeed, let us apply the transformation $a = y+z$, $b = z+x$, $c = x+y$ where $x, y, z > 0$, converting any triangle inequality into an algebraic inequality valid for positive numbers. Then it's not difficult to see that $x+y+z=s$ (the semiperimeter of the triangle), whence $x = s-a$, $y = s-b$ and $z = s-c$. Furthermore, due to homogeneity we may and do put $s = 1$, whence $F = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{xyz}$. Also, by the known formula

$$w_c = \frac{2ab\cos(C/2)}{a+b} = \frac{2ab}{a+b}\sqrt{\frac{s(s-c)}{ab}} = \frac{2}{a+b}\sqrt{s(s-c)ab}$$

(for example, see [1995: 321]) we get (using $x+y+z=1$)

$$w_c = \frac{2}{x+y+z+z}\sqrt{z(y+z)(z+x)} = \frac{2}{1+z}\sqrt{z(1-x)(1-y)},$$

and similarly for w_a and w_b. Hence (2) is equivalent to

$$\sum_{\text{cyclic}} 4 \cdot \frac{\sqrt{xy}\sqrt{(1-y)(1-x)}(1-z)}{(1+x)(1+y)} \;\geq\; 3\sqrt{3}\sqrt{xyz},$$

which is equivalent to (1). Since (2) is true, so is (1).

For $n \geq 4$ I do not have any idea of how to settle whether the inequality $l_n \geq r_n$ is true. It may be interesting and useful to see a purely algebraic proof of inequality (1).

Part 1 *also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; and the proposer. The proposer did Part* 2 *only in the case $n = 3$ (given above). One other reader sent in a solution to Part* 2 *which the editor considers to be faulty. Readers are invited to try finishing off this problem completely, or even just the special case $n = 4$.*

2174. [1996: 275] *Proposed by Theodore Chronis, student, Aristotle University of Thessaloniki, Greece.*

Let A be an $n \times n$ matrix. Prove that if $A^{n+1} = 0$ then $A^n = 0$.

Solution by John C. Tripp, Southeast Missouri State University, Cape Girardeau, Missouri.

We consider A as a linear transformation on an n-dimensional vector space. We assume that $A^{n+1} = 0$. Let x be any element of the vector space. The set of vectors

$$V = \{x, Ax, A^2x, \dots, A^nx\}$$

has $n+1$ elements, so it is linearly dependent. Let k be the smallest non-negative integer such that A^kx is a linear combination of the other vectors in V. We have

$$A^kx \;=\; c_1A^{k+1}x + c_2A^{k+2}x + c_3A^{k+3}x + \cdots + c_{n-k}A^nx,$$

for some scalars $c_1, c_2, c_3, \dots, c_{n-k}$, and

$$\begin{aligned} A^nx &= A^{n-k}A^kx \\ &= A^{n-k}(c_1A^{k+1}x + c_2A^{k+2}x + c_3A^{k+3}x + \cdots + c_{n-k}A^nx) \\ &= c_1A^{n+1}x + c_2A^{n+2}x + c_3A^{n+3}x + \cdots + c_{n-k}A^{2n-k}) \\ &= A^{n+1}(c_1x + c_2Ax + c_3A^2x + \cdots + c_{n-k}A^{n-k-1}x) \;=\; 0. \end{aligned}$$

Since x was arbitrary, we have $A^n = 0$.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIHAI CIPU, Romanian Academy, Bucharest, Romania; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; LUZ M. DeALBA, Drake University, Des Moines, Iowa; F.J.

FLANIGAN, San Jose State University, San Jose, California, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; WALDEMAR POMPE, student, University of Warsaw, Poland; E. RAPPOS, University of Cambridge, England; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; SKIDMORE COLLEGE PROBLEM GROUP, Saratoga Springs, New York, USA; DIGBY SMITH, Mount Royal College, Calgary, Alberta; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; and the proposer.

Most solvers used the Cayley-Hamilton Theorem. Wang comments that this problem is a special case of a more general and well-known result which states that "if A is an $n \times n$ complex matrix such that $A^k = 0$ for some $k \geq 1$ (that is, A is nilpotent), then $A^n = 0$". Indeed, some solvers proved this more general result.

2175. [1996: 275] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

The fraction $\frac{1}{6}$ can be represented as a difference in the following ways:

$$\frac{1}{6} = \frac{1}{2} - \frac{1}{3}; \quad \frac{1}{6} = \frac{1}{3} - \frac{1}{6}; \quad \frac{1}{6} = \frac{1}{4} - \frac{1}{12}; \quad \frac{1}{6} = \frac{1}{5} - \frac{1}{30}.$$

In how many ways can the fraction $\frac{1}{2175}$ be expressed in the form

$$\frac{1}{2175} = \frac{1}{x} - \frac{1}{y},$$

where x and y are positive integers?

Solution by D. Kipp Johnson, Valley Catholic High School, Beaverton, Oregon, USA.

Notice that

$$\frac{1}{2175} = \frac{1}{x} - \frac{1}{y}$$

so that

$$x = \frac{2175y}{y + 2175} = 2175 - \frac{2175^2}{y + 2175}.$$

Thus x will be an integer if and only if $y + 2175$ is a factor of 2175^2, and x will be positive whenever y is, since then $2175^2/(y + 2175) < 2175$, and y will be positive whenever $y + 2175 > 2175$, so we seek factors of 2175^2 which exceed 2175. But $2175^2 = 3^2 \cdot 5^4 \cdot 29^2$ has $(2+1)(4+1)(2+1) = 45$ positive factors, one of which is its square root, 2175. Since the factors of 2175^2 come in pairs whose product is 2175^2, exactly half of the other 44 factors exceed 2175, giving 22 solutions in positive integers. The smallest is $x = 300, y = 348$. This immediately generalizes to the solution in positive integers of $1/n = 1/x - 1/y$. Since $\tau(n^2)$ (the number of divisors of n^2) is odd for a perfect square, there will be $(\tau(n^2)-1)/2$ solutions to the equation $1/n = 1/x - 1/y$.

Also solved by HAYO AHLBURG, Benidorm, Spain; SAM BAETHGE, Nordheim, Texas, USA; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; TIM CROSS, King Edward's School, Birmingham, England; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; KEITH EKBLAW, Walla Walla, Washington, USA; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; F.J. FLANIGAN, San Jose State University, San Jose, California, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; KEE-WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; LAMARR WIDMER, Messiah College, Grantham, PA, USA; KENNETH M. WILKE, Topeka, Kansas, USA; and the proposer. There were 4 incorrect solutions.

Flanigan refers the interested reader to problem #10501 of the American Mathematical Monthly, *volume 103, number 2, February 1996, page 171.*

2176. [1996: 275] *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Prove that

$$\sqrt[n]{\prod_{k=1}^{n}(a_k+b_k)} \geq \sqrt[n]{\prod_{k=1}^{n}a_k} + \sqrt[n]{\prod_{k=1}^{n}b_k}$$

where $a_1, a_2, \ldots, a_n > 0$ and $n \in \mathbb{N}$.

Solution by Sai C. Kwok, San Diego, CA, USA.

Using the arithmetic-geometric mean inequality, we have

$$\left(\prod_{k=1}^{n}\frac{a_k}{a_k+b_k}\right)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{k=1}^{n}\frac{a_k}{a_k+b_k}$$

and

$$\left(\prod_{k=1}^{n}\frac{b_k}{a_k+b_k}\right)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{k=1}^{n}\frac{b_k}{a_k+b_k}$$

The result follows by adding the above two inequalities.

Also solved by THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; †MIHAI CIPU, Romanian Academy, Bucharest, Romania, and Concordia University, Montreal, Quebec; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; †WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; †CAN ANH MINH, student, University of California Berkeley, Berkeley, CA, USA; †SOLEDAD ORTEGA and JAVIER GUTIÉRREZ, students, University of La Rioja, Logroño, Spain; WALDEMAR POMPE, student, University of Warsaw, Poland; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; †HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; GEORGE TSAPAKIDIS, Agrinio, Greece; and the †proposer. A proof for the case $n = 4$ was submitted by V.N. Murty. (The symbol † before a solver's name indicates that the solver's solution was virtually the same as the one highlighted above.)

Clearly, the condition that $b_1, b_2, \ldots, b_n > 0$ was inadvertantly left out from the original statement. All solvers assumed, explicitly or implicitly, that this was the case. However, only Hess gave a simple example to show that the inequality need not be true without the aforementioned condition: take $n = 2$, $a_1 = a_2 = 1$ and $b_1 = b_2 = -1$.

Janous pointed out that equality holds if and only if the vectors $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are proportional.

Klamkin commented that the given inequality is an immediate special case of Jensen's generalization of Hölder's Inequality, and referred readers to D.S. Mitrinović et. al., Recent Advances in Geometric Inequalities, *Kluwer Academic Publishers, 1989, pp. 50–54.*

Konečný remarked that it is known that if A and B are $n \times n$ positive semi-definite Hermitian matrices, then

$$\sqrt[n]{\det(A+B)} \;\geq\; \sqrt[n]{\det A} + \sqrt[n]{\det B}$$

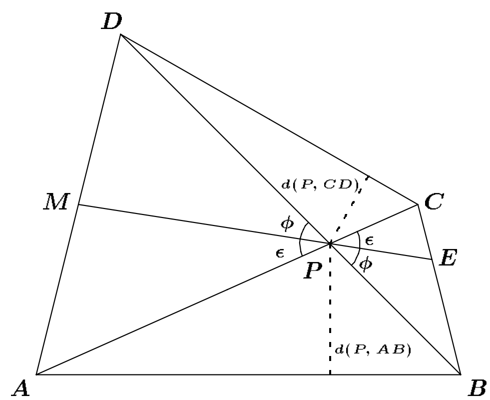
(see, for example, Inequalities: Theory of Majorization and Its Application *by Albert W. Marshall and Ingram Olkin, Academic Press Inc., 1979, p. 475). If we let A and B be the $n \times n$ diagonal matrices with diagonal entries a_k's and b_k's respectively ($i = 1, 2, \ldots, n$), then the proposed inequality follows immediately.*

Seiffert pointed out that the proof given above can be found on p. 178 of the book Classical and New Inequalities in Analysis, *Kluwer Academic Publishers, 1992.*

2177. [1996: 317] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABCD$ is a convex quadrilateral, with P the intersection of its diagonals and M the mid-point of AD. MP meets BC at E. Suppose that $BE : EC = (AB)^2 : (CD)^2$. Characterize quadrilateral $ABCD$.

Solution by Gottfried Perz, Pestalozzigymnasium, Graz, Austria.



Let

$$\epsilon = \angle MPA = \angle EPC,$$

$$\phi = \angle BPE = \angle DPM.$$

Then, applying the Sine Rule to the triangles $\triangle APM$ and $\triangle DPM$, we get

$$\sin\epsilon = \frac{AM \cdot \sin\angle AMP}{AP},$$

$$\sin\phi = \frac{DM \cdot \sin(180^\circ - \angle AMP)}{DP} = \frac{AM \cdot \sin\angle AMP}{DP},$$

whence

$$\frac{\sin\epsilon}{\sin\phi} = \frac{DP}{AP}. \tag{3}$$

Applying the law of sines to the triangles $\triangle CPE$, and $\triangle BPE$, we get

$$\sin\epsilon = \frac{CE \cdot \sin\angle CEP}{CP},$$

$$\sin\phi = \frac{BE \cdot \sin(180^\circ - \angle CEP)}{BP} = \frac{BE \cdot \sin\angle CEP}{BP},$$

whence

$$\frac{\sin\epsilon}{\sin\phi} = \frac{CE \cdot BP}{BE \cdot CP} = \frac{CD^2 \cdot BP}{AB^2 \cdot CP}. \tag{4}$$

From (1), (2) follows that quadrilateral $ABCD$ has the desired property if and only if

$$\frac{AB^2}{CD^2} = \frac{AP \cdot BP}{CP \cdot DP} = \frac{[ABP]}{[CDP]} = \frac{AB \cdot d(P, AB)}{CD \cdot d(P, CD)}$$

(where $[XYZ]$ denotes the area of $\triangle XYZ$ and $d(U, VW)$ the distance of U from VW); that is, if

$$AB : CD = d(P, AB) : d(P, CD).$$

This implies that $ABCD$ is characterized by the fact that in the triangles $\triangle ABP$ and $\triangle CDP$, having $\angle APB = \angle CPD$ in common ($ABCD$ is convex!), the ratio of the side opposite to P and the altitude passing through P is the same, which means that $\triangle ABP$ and $\triangle CDP$ are directly or inversely similar.

In the first case, we have $\angle BAP = \angle DCP$; that is, $ABCD$ is a trapezoid with parallel sides AB and CD. In the second case, we have $\angle BAP$ ($= \angle BAC$) $= \angle PDC$ ($= \angle BDC$), whence $ABCD$ is an inscribed quadrilateral (A and D are at the same side of BC).

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; SABIN CAUTIS, student, Earl Haig Secondary School, North York, Ontario; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; RICHARD I. HESS, Rancho Palos Verdes, California, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; WALDEMAR POMPE, student, University of Warsaw, Poland; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer. Four incomplete or incorrect solutions were received.

2178. [1996: 318] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

If A, B, C are the angles of a triangle, prove that

$$\begin{aligned} \sin A \sin B \sin C &\leq 8\left(\sin^3 A \cos B \cos C + \sin^3 B \cos C \cos A \right.\\ &\qquad \left. + \sin^3 C \cos A \cos B\right) \\ &\leq 3\sqrt{3}\left(\cos^2 A + \cos^2 B + \cos^2 C\right). \end{aligned}$$

Solution by Florian Herzig, student, Perchtoldsdorf, Austria.

I shall prove a stronger version with $6 \sin A \sin B \sin C$ of the left side.

We use the following known identities and inequalities:

$$\begin{aligned} \sin 2A + \sin 2B + \sin 2C &= 4 \sin A \sin B \sin C & (1)\\ \cos^2 A + \cos^2 B + \cos^2 C &= 1 - 2\cos A \cos B \cos C & (2)\\ \sin A \sin B \sin C &\leq \frac{3\sqrt{3}}{8} & (3)\\ \cos A \cos B \cos C &\leq \frac{1}{8} & (4) \end{aligned}$$

where A, B, C are the angles of a triangle. Therefore

$$\begin{aligned} &\sin^3 A \cos B \cos C + \sin^3 B \cos C \cos A + \sin^3 C \cos A \cos B \\ &\qquad = \sin A(1 - \cos^2 A) \cos B \cos C + \sin B(1 - \cos^2 B) \cos C \cos A \\ &\qquad\quad + \sin C(1 - \cos^2 C) \cos A \cos B \end{aligned}$$

$$\begin{aligned}
&= -\cos A\cos B\cos C(\sin A\cos A+\sin B\cos B+\sin C\cos C)\\
&\quad +\sin A\cos B\cos C+\cos A(\sin B\cos C+\sin C\cos B)\\
&= -\frac{1}{2}\cos A\cos B\cos C(\sin 2A+\sin 2B+\sin 2C)\\
&\quad +\sin A(\cos B\cos C+\cos A)\\
&= -2\cos A\cos B\cos C\sin A\sin B\sin C+\sin A\sin B\sin C\\
&= \sin A\sin B\sin C(1-2\cos A\cos B\cos C)\\
&= \sin A\sin B\sin C(\cos^2 A+\cos^2 B+\cos^2 C).
\end{aligned}$$

Combining (2) and (4) yields

$$\cos^2 A+\cos^2 B+\cos^2 \geq \frac{3}{4}.$$

By using this inequality and (3) we get

$$\begin{aligned}
6\sin A\sin B\sin C &\leq 8(\sin^3 A\cos B\cos C+\sin^3 B\cos C\cos A\\
&\qquad +\sin^3 C\cos A\cos B)\\
&= 8\sin A\sin B\sin C(\cos^2 A+\cos^2 B+\cos^2 C)\\
&\leq 3\sqrt{3}(\cos^2 A+\cos^2 B+\cos^2 C)
\end{aligned}$$

as we wanted to show. Equality holds as in (3) and (4) if and only if $A=B=C=60^\circ$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; MICHAEL LAMBROU, University of Crete, Crete, Greece; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

Janous, in his solution, used a new identity, which is presented as a problem 2279 in this issue.