THE ACADEMY CORNER

No. 10

Bruce Shawyer

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This month, we present a university entrance scholarship examination paper from the 1940's. Thanks to Georg Gunther, Sir Wilfred Grenfell College, Corner Brook, Newfoundland, for providing this. We challenge today's university students to solve these problems — send me your nice solutions!

1. Find all the square roots of

$$1 - x + \sqrt{22x - 15 - 8x^2}.$$

2. Find all the solutions of the system of equations:

$$\begin{aligned} x + y + z &= 2, \\ x^2 + y^2 + z^2 &= 14, \\ xyz &= -6. \end{aligned}$$

3. Suppose that n is a positive integer and that C_k is the coefficient of x^k in the expansion of $(1 + x)^n$. Show that

$$\sum_{k=0}^{n} (k+1)C_k^2 = \frac{(n+2)\,(2n-1)!}{n!\,(n-1)!}.$$

4. (a) Suppose that $a \neq 0$ and $c \neq 0$, and that $ax^3 + bx + c$ has a factor of the form $x^2 + px + 1$. Show that $a^2 - c^2 = ab$.
   (b) In this case, prove that $ax^3 + bx + c$ and $cx^3 + bx^2 + a$ have a common quadratic factor.

5. Prove that all the circles in the family defined by the equation

$$x^2 + y^2 - a(t^2 + 2)x - 2aty - 3a^2 = 0$$

(a fixed, t variable) touch a fixed straight line.

6. Find the equation of the locus of a point P which moves so that the tangents from P to the circle $x^2 + y^2 = r^2$ cut off a line segment of length $2r$ on the line $x = r$.

7. If the tangents at A, B and C to the circumcircle of triangle $\triangle ABC$ meet the opposite sides at D, E and F, respectively, prove that D, E and F are collinear.

8. Find the locus of P which moves so that the polars of P, with respect to three non-intersecting circles, are concurrent.

9. Suppose that P is a point within the tetrahedron $OABC$. Prove that $\angle AOB + \angle BOC + \angle COA$ is less than $\angle APB + \angle BPC + \angle CPA$.

10. Two unequal circles of radii R and r touch externally, and P and Q are the points of contact of a common tangent to the circles, respectively. Find the volume of the frustum of a cone generated by rotating PQ about the line joining the centres of the circle.

11. Prove that

$$\sin^2(\theta+\alpha)+\sin^2(\theta+\beta)-2\cos(\alpha-\beta)\sin(\theta+\alpha)\sin(\theta+\beta) = \sin^2(\alpha-\beta).$$

12. Three points A, B and C are on level ground. B is east of A, C is N. $49°$ E. of A, and C is N. $11°30'$ W. of B.

    Find the direction of C as seen from the mid-point of AB.

13. With each corner of a square of side r as a centre, four circles of radius r are drawn.

    Show that the area of the central curvilinear quadrilateral formed inside the square by the intersection of the four circles is

$$r^2\left(1-\sqrt{3}+\frac{\pi}{3}\right).$$

14. An observer on a boat is vertically beneath the centre of a bridge, which crosses a straight canal at right angles. Looking upwards, the observer sees that the angle subtended by the length of the bridge is 2α. The observer then rows a distance δ along the middle of the canal, and then finds that the length of the bridge now subtends an angle of 2β.

    Show that the length of the bridge is

$$\frac{2\delta}{\sqrt{\cot^2\beta-\cot^2\alpha}}.$$

THE OLYMPIAD CORNER

No. 181

R.E. Woodrow

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We begin this number with two contests. Thanks go to Richard Nowakowski, Canadian Team Leader to the 35th IMO in Hong Kong, for collecting them and forwarding them to us.

SELECTED PROBLEMS FROM THE ISRAEL MATHEMATICAL OLYMPIADS, 1994

1. p and q are positive integers. f is a function defined for positive numbers and attains only positive values, such that $f(xf(y)) = x^p y^q$. Prove that $q = p^2$.

2. The sides of a polygon with 1994 sides are $a_i = \sqrt{4+i^2}$, $i = 1, 2, \ldots, 1994$. Prove that its vertices are not all on integer mesh points.

3. A "standard triangle" in the plane is a (filled) isosceles right triangle whose sides are parallel to the x and y axes. A finite family of standard triangles, containing at least three, is given. Every three of this family have a common point. Prove that there is a point common to all triangles in that family.

4. A shape c' is called "a copy of the planar shape c" if the following conditions hold:

(i) There are two planes σ and σ' and a point P that does not belong to either of them.

(ii) $c \in \sigma$ and $c' \in \sigma'$.

(iii) A point X' satisfies $X' \in c'$ iff X' is the intersection of σ' with the line passing through X and P.

Given a planar trapezoid, prove that there is a square which is a copy of this trapezoid.

5. Find all polynomials $p(x)$, with real coefficients, satisfying

$$(x-1)^2 p(x) = (x-3)^2 p(x+2)$$

for all x.

PROBLEMS FROM THE BI-NATIONAL ISRAEL-HUNGARY COMPETITION, 1994

1. $a_1, \ldots, a_k, a_{k+1}, \ldots, a_n$ are positive numbers ($k < n$). Suppose that the values of $a_{k+1}, \ldots, a_n$ are fixed. How should one choose the values of $a_1, \ldots, a_k$ in order to minimize $\sum_{i,j,i\neq j} \frac{a_i}{a_j}$?

2. Three given circles pass through a common point P and have the same radius. Their other points of pairwise intersections are A, B, C. The 3 circles are contained in the triangle $A'B'C'$ in such a way that each side of $\triangle A'B'C'$ is tangent to two of the circles. Prove that the area of $\triangle A'B'C'$ is at least 9 times the area of $\triangle ABC$.

3. m, n are two different natural numbers. Show that there exists a real number x, such that $\frac{1}{3} \leq \{xn\} \leq \frac{2}{3}$ and $\frac{1}{3} \leq \{xm\} \leq \frac{2}{3}$, where $\{a\}$ is the fractional part of a.

4. An "n-m society" is a group of n girls and m boys. Show that there exist numbers n_0 and m_0 such that every n_0-m_0 society contains a subgroup of five boys and five girls in which all of the boys know all of the girls or none of the boys knows none of the girls.

Last issue we gave five more Klamkin Quickies. Next we give his "Quicky" solutions to these problems. Many thanks to Murray S. Klamkin, the University of Alberta, for creating the problems and solutions.

ANOTHER FIVE KLAMKIN QUICKIES

October 21, 1996

6. Determine the four roots of the equation $x^4 + 16x - 12 = 0$.

Solution. Since

$$x^4 + 16x - 12 = (x^2+2)^2 - 4(x-2)^2 = (x^2+2x-2)(x^2-2x+6) = 0,$$

the four roots are $-1 \pm \sqrt{3}$ and $1 \pm i\sqrt{5}$.

7. Prove that the smallest regular n-gon which can be inscribed in a given regular n-gon is one whose vertices are the midpoints of the sides of the given regular n-gon.

Solution. The circumcircle of the inscribed regular n-gon must intersect each side of the given regular n-gon. The smallest that such a circle can be is the inscribed circle of the given n-gon, and it touches each of its sides at its midpoints.

8. If 31^{1995} divides $a^2 + b^2$, prove that 31^{1996} divides ab.

Solution. If one calculates $1^2, 2^2, \ldots, 30^2 \bmod 31$ one finds that the sum of no two of these equals $0 \bmod 31$. Hence, $a = 31a_1$ and $b = 31b_1$ so that 31^{1993} divides $a_1^2 + b_1^2$. Then, $a_1 = 31a_2$ and $b_1 = 31b_2$. Continuing in this fashion (with $p = 31$), we must have $a = p^{998}m$ and $b = p^{998}n$ so that ab is divisible by p^{1996}.

More generally, if a prime $p = 4k + 3$ divides $a^2 + b^2$, then both a and b must be divisible by p. This follows from the result that "*a natural n is the sum of squares of two relatively prime natural numbers if and only if n is divisible neither by 4 nor by a natural number of the form $4k + 3$*" (see J.W. Sierpiński, Elementary Theory of Numbers, Hafner, NY, 1964, p. 170).

9. Determine the minimum value of

$$S = \sqrt{(a+1)^2 + 2(b-2)^2 + (c+3)^2} + \sqrt{(b+1)^2 + 2(c-2)^2 + (d+3)^2)} + \sqrt{(c+1)^2 + 2(d-2)^2 + (a+3)^2} + \sqrt{(d+1)^2 + 2(a-2)^2 + (b+3)^2}$$

where a, b, c, d are any real numbers.

Solution. Applying Minkowski's inequality,

$$S \geq \sqrt{(4+s)^2 + 2(s-8)^2 + (s+12)^2} = \sqrt{4s^2 + 288}$$

where $s = a + b + c + d$. Consequently, $\min S = 12\sqrt{2}$ and is taken on for $a = b = c = d = 0$.

10. A set of 500 real numbers is such that any number in the set is greater than one-fifth the sum of all the other numbers in the set. Determine the least number of negative numbers in the set.

Solution. Letting $a_1, a_2, a_3, \ldots$ denote the numbers of the set and S the sum of all the numbers in the set, we have

$$a_1 > \frac{S - a_1}{5}, \quad a_2 > \frac{S - a_2}{5}, \quad \ldots, \quad a_6 > \frac{S - a_6}{5}.$$

Adding, we get $0 > S - a_1 - a_2 - \cdots - a_6$ so that if there were six or less negative numbers in the set, the right hand side of the inequality could be positive. Hence, there must be at least seven negative numbers.

Comment. This problem where the "5" is replaced by "1" is due to Mark Kantrowitz, Carnegie–Mellon University.

First a solution to one of the 36th IMO problems:

2. [1995: 269] 36*th IMO*
Let a, b, and c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Solution by Panos E. Tsaoussoglou, Athens, Greece.
By the Cauchy–Schwartz inequality

$$[a(b+c) + b(c+a) + c(a+b)]\left[\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)}\right]$$
$$\geq \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2,$$

or

$$2(ab+ac+bc)\left[\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)}\right]$$
$$\geq \frac{(ab+ac+bc)^2}{(abc)^2},$$

or

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{ab+ac+bc}{2},$$

because $abc = 1$.
Also

$$\frac{ab+ac+bc}{3} \geq \sqrt[3]{a^2b^2c^2} = 1.$$

Therefore

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}$$

holds.

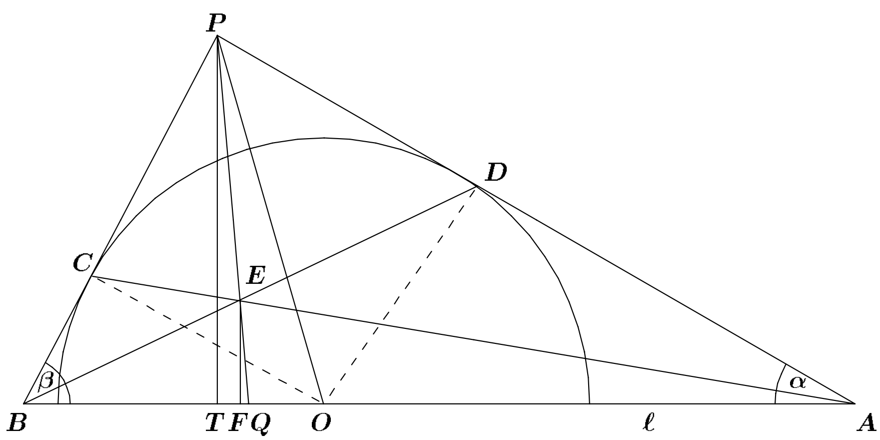
Now we turn to some of the readers' solutions to problems proposed to the jury but not used at the 35th IMO in Hong Kong [1995: 299–300].

PROBLEMS PROPOSED BUT NOT USED AT THE 35th IMO IN HONG KONG

Selected Problems

3. A semicircle Γ is drawn on one side of a straight line ℓ. C and D are points on Γ. The tangents to Γ at C and D meet ℓ at B and A respectively, with the center of the semicircle between them. Let E be the point of intersection of AC and BD, and F be the point on ℓ such that EF is perpendicular to ℓ. Prove that EF bisects $\angle CFD$.

Solutions by Toshio Seimiya, Kawasaki, Japan; and by D.J. Smeenk, Zaltbommel, the Netherlands. We give Seimiya's write-up.



Let P be the intersection of AD and BC. Then $\angle PCO = \angle PDO = 90°$, $\angle CPO = \angle DPO$ and $PC = PD$. Let Q be the intersection of PE with AB. Then by Ceva's Theorem, we get

$$\frac{BQ}{QA} \cdot \frac{AD}{DP} \cdot \frac{PC}{CB} = \frac{BQ}{QA} \cdot \frac{AD}{CB} = 1.$$

Thus we get

$$\frac{BQ}{QA} = \frac{BC}{AD}. \tag{1}$$

Since $\angle BPO = \angle APO$ we get

$$\frac{PB}{PA} = \frac{BO}{AO}. \tag{2}$$

We put $\angle PAB = \alpha$, $\angle PBA = \beta$.

Let T be the foot of the perpendicular from P to AB. Then from (1) and (2) we have

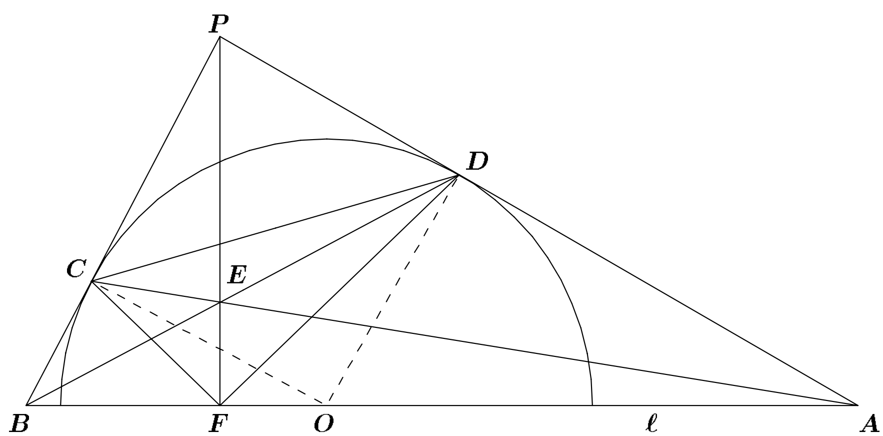
$$\frac{BC}{AD} = \frac{BO\cos\beta}{AO\cos\alpha} = \frac{PB\cos\beta}{PA\cos\alpha} = \frac{PT}{TA}. \tag{3}$$

From (1) and (3) we have

$$\frac{BQ}{QA} = \frac{PT}{TA}.$$

Hence Q coincides with T so that P, E, F are collinear. [See page 136.]

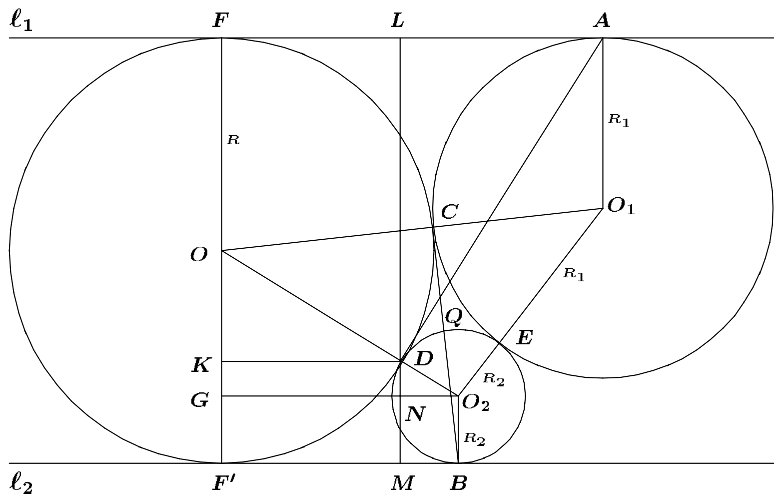
Because $\angle PCO = \angle PDO = \angle PFO = 90°$, P, C, F, O, D are concyclic. Hence $\angle CFE = \angle CFP = \angle CDP = \angle DCP = \angle DFP = \angle DFE$. Thus EF bisects $\angle CFD$.



4. A circle ω is tangent to two parallel lines ℓ_1 and ℓ_2. A second circle ω_1 is tangent to ℓ_1 at A and to ω externally at C. A third circle ω_2 is tangent to ℓ_2 at B, to ω externally at D and to ω_1 externally at E. AD intersects BC at Q. Prove that Q is the circumcentre of triangle CDE.

Solutions by Toshio Seimiya, Kawasaki, Japan; and by D.J. Smeenk, Zaltbommel, the Netherlands. We give Smeenk's solution.

We denote the three circles as $\omega(O, R)$, $\omega_1(O_1, R_1)$, $\omega_2(O_2, R_2)$. Now let ω touch ℓ_1 at F and ℓ_2 at F'. Let the line through O_2 parallel to ℓ_1 intersect FF' at G and the production of AO_1 at H.



Let the line through D parallel to ℓ_1 intersect FF' at K.

Let the line through D parallel to FF' intersect ℓ_1 at L, ℓ_2 at M and GO_2 at N. Now AF is a common tangent of ω and ω_1, so

$$AF = 2\sqrt{RR_1} \tag{1}$$

and

$$BF' = 2\sqrt{RR_2} = GO_2. \tag{2}$$

It follows that

$$HO_2 = |2\sqrt{RR_2} - 2\sqrt{RR_1}|;$$

$$HO_1 = 2R - R_1 - R_2.$$

In right triangle O_1HO_2,

$$(2\sqrt{RR_2} - 2\sqrt{RR_1})^2 + (2R - R_1 - R_2)^2 = (R_1 + R_2)^2.$$

After some reduction $R = 2\sqrt{R_1R_2}$.

Next consider triangle GOO_2.

$$GO = R - R_2, \quad GO_2 = 2\sqrt{RR_2}, \quad DO_2 = R_2, \quad DO = R, \quad KD \| GO_2.$$

We find that $GN = FL = \dfrac{R}{R + R_2} \cdot GO_2 = \dfrac{2R\sqrt{RR_2}}{R + R_2}$.

With (1) we have

$$AL = 2\sqrt{RR_1} - \frac{2R\sqrt{RR_2}}{R + R_2}. \tag{3}$$

Furthermore $DN = \dfrac{R_2}{R+R_2} \cdot GO = \dfrac{R_2(R-R_2)}{R+R_2}$ and

$$DL = 2R - R_2 - \frac{R_2(R-R_2)}{R+R_2} = \frac{2R^2}{R+R_2}. \tag{4}$$

Now $AD^2 = AL^2 + DL^2$. With (3) and (4),

$$\left(2\sqrt{RR_1} - \frac{2R\sqrt{RR_2}}{R_1+R_2}\right)^2 + \left(\frac{2R^2}{R_1+R_2}\right)^2 = 4RR_1 = AE^2.$$

So $AD = AF$.

That means that AD touches ω at D and AD is a common tangent and the radical axis of ω and ω_2.

In the same way BC is the radical axis of ω and ω_1 and Q is the radical point of ω, ω_1 and ω_2.

So $QC = QD = QE$, as required.

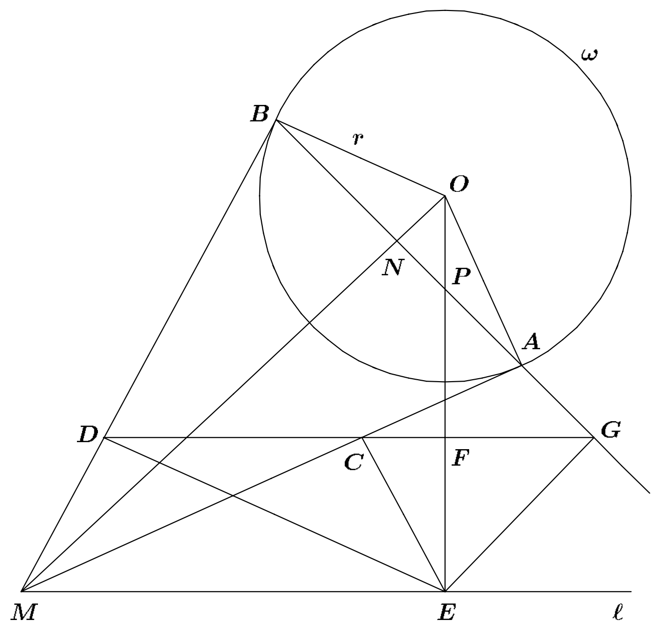
5. A line ℓ does not meet a circle ω with center O. E is the point on ℓ such that OE is perpendicular to ℓ. M is any point on ℓ other than E. The tangents from M to ω touch it at A and B. C is the point on MA such that EC is perpendicular to MA. D is the point on MB such that ED is perpendicular to MB. The line CD cuts OE at F. Prove that the location of F is independent of that of M.

Solution by Toshio Seimiya, Kawasaki, Japan.

As MA, MB are tangent to ω at A, B respectively, we get $\angle OAM = \angle OBM = 90°$ and $OM \perp AB$. Let N, P be the intersections of AB with OM and OE respectively.

Since M, E, P, N lie on the circle with diameter MP we get $ON \cdot OM = OB^2 = r^2$ where r is the radius of ω. Hence P is a fixed point. (P is the pole of ℓ.)

Let G be the foot of the perpendicular from E to AB. As $\angle OBM = \angle OAM = \angle OEM = 90°$, O, B, M, E, A are concyclic, so that by Simson's Theorem C, D, G are collinear.



Since A, C, E, G lie on the circle with diameter AE we get

$$\angle EGF = \angle EGC = \angle EAC = \angle EAM. \tag{1}$$

As O, M, E, A are concyclic and OM is parallel to EG we have

$$\angle EAM = \angle EDM = \angle DEG = \angle FEG. \tag{2}$$

From (1) and (2) we get

$$\angle EGF = \angle FEG. \tag{3}$$

Since $\angle EGP = 90°$ we get

$$\angle FGP = \angle FPG. \tag{4}$$

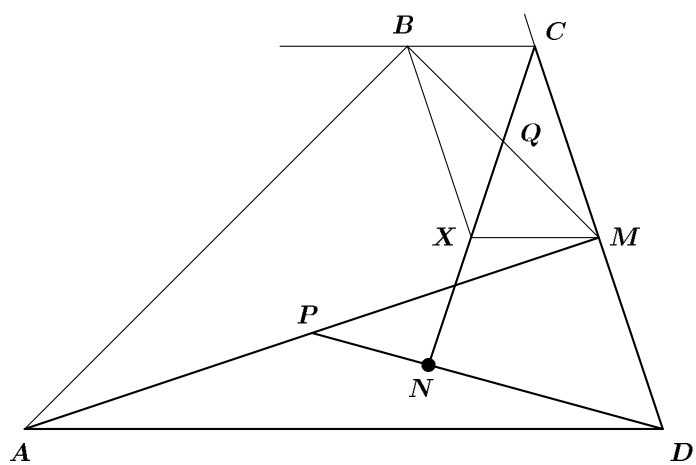
From (3) and (4) we have $EF = FG = FP$. Thus F is the midpoint of EP. Hence F is a fixed point.

Next, we give a counterexample to the first problem of the set of problems proposed to the jury, but not used at the 35th IMO in Hong Kong given in the December 1995 number of the corner.

1. [1995: 334] *Problems proposed but not used at the 35th IMO in Hong Kong.*

$ABCD$ is a quadrilateral with BC parallel to AD. M is the midpoint of CD, P that of MA and Q that of MB. The lines DP and CQ meet at N. Prove that N is not outside triangle ABM.

Counterexamples by Joanna Jaszuńska, student, Warsaw, Poland; and by Toshio Seimiya, Kawasaki, Japan. We give Jaszuńska's example.



We draw a triangle ADM and denote the midpoint of MA by P. Let C be a point on the half-line DM such that M is the midpoint of CD.

Let N be any point of the segment PD, *inside* triangle ADM.

We construct a parallelogram $MCBX$ such that MX and BC are parallel to AD and X lies on the segment CN.

Let us denote the point where the diagonal MB of this quadrilateral meets CN by Q. Q is then the midpoint of MB.

Connect points A and B. We have thus constructed a quadrilateral $ABCD$ with BC parallel to AD, M is the midpoint of CD, P that of MA and Q that of MB. Lines DP and CQ meet at N.

N is inside triangle ADM; hence it is outside triangle ABM.

Next we look back to some further solutions to problems of the Sixth Irish Mathematical Olympiad given in [1995: 151–152] and for which some solutions were given in [1997: 9–13]. An envelope from Michael Selby arrived which I misfiled. It contains solutions to problems 1, 2, and 4 of Day 1, and problems 1, 2, 3 and 4 of Day 2.

1. [1995: 152] *Second Paper, Sixth Irish Mathematical Olympiad.*

Given five points P_1, P_2, P_3, P_4, P_5 in the plane having integer coordinates, prove that there is at least one pair (P_i, P_j) with $i \neq j$ such that the line P_iP_j contains a point Q having integer coordinates and lying strictly between P_i and P_j.

Solution by Michael Selby, University of Windsor, Windsor, Ontario.

The points can be characterized according to the parity of their x and y coordinates. There are only four such classes: (even, even), (even, odd), (odd, even), (odd, odd).

Since we are given five such points, at least two must have the same parity of coordinates by the Pigeonhole Principle. Suppose they are P_i and P_j, $P_i = (x_i, y_i)$, $P_j = (x_j, y_j)$. Then $x_i + x_j$ is even and $y_i + y_j$ is even, since the x_i, x_j have the same parity and y_i, y_j have the same parity. Hence the midpoint

$$Q = \left(\frac{x_i + x_j}{2}, \frac{y_i + y_j}{2}\right)$$

has integral coordinates.

2. [1995: 152] *Second Paper, Sixth Irish Mathematical Olympiad.*

Let $a_1, a_2, \ldots a_n$, $b_1, b_2, \ldots b_n$ be $2n$ real numbers, where $a_1, a_2, \ldots, a_n$ are distinct, and suppose that there exists a real number α such that the product

$$(a_i + b_1)(a_i + b_2)\ldots(a_i + b_n)$$

has the value α for all i ($i = 1, 2, \ldots, n$). Prove that there exists a real number β such that the product

$$(a_1 + b_j)(a_2 + b_j)\ldots(a_n + b_j)$$

has the value β for all j ($j = 1, 2, \ldots, n$).

Solution by Michael Selby, University of Windsor, Windsor, Ontario.

Define

$$P_n(x) = (x + b_1)(x + b_2)\cdots(x + b_n) - \alpha. \qquad (1)$$

Then $P_n(a_i) = 0$ for $i = 1, 2, \ldots, n$.

Therefore $P_n(x) = (x - a_1)(x - a_2)\cdots(x - a_n)$ by the Factor Theorem.

Now $(-1)^n P_n(-x) = (x+a_1)(x+a_2)\cdots(x+a_n)$. So

$$\begin{aligned}(-1)^n P_n(-b_i) &= (b_i+a_1)(b_i+a_2)\cdots(b_i+a_n)\\ &= (-1)^{n+1}\alpha \quad by\ (1).\end{aligned}$$

Hence $(b_i+a_1)(b_i+a_2)\cdots(b_i+a_n) = (-1)^{n+1}\alpha$ for $i = 1, 2, \ldots, n$.

Thus, the result is true with $\beta = (-1)^{n+1}\alpha$.

That completes the Corner for this number. We are in high Olympiad season. Send me your nice solutions and contests.

Do you believe what occurs in print?

The last sentence of the quoted passage, taken from *The Daughters of Cain* by Colin Dexter (Macmillan, 1994), contains two factual erors. What are they?

> *'Have you heard of "Pythagorean Triplets"?'*
>
> *'We did Pythagoras Theorem at school.'*
>
> *'Exactly. The most famous of all the triplets, that is —*
>
> *"3, 4, 5" $3^2 + 4^2 = 5^2$. Agreed?'*
>
> *'Agreed.'*
>
> *'But there are more spectacular examples than that.*
>
> *The Egyptians, for example, knew all about "5961, 6480, 8161".'*

Contributed by J.A. McCallum, Medicine Hat, Alberta.

BOOK REVIEWS

Edited by ANDY LIU

The Lighter Side of Mathematics,
edited by Richard K. Guy and Robert E. Woodrow,
Mathematical Association of America, Washington DC,
1994, ISBN 0-88385-516-X, 376+ pages, softcover, US $38.50,
reviewed by ***Murray S. Klamkin***, University of Alberta.

This book is the proceedings of the **Eugene Strens Memorial Conference** on **Recreational Mathematics and its History** held at the University of Calgary in August 1986 to celebrate the founding of the Strens Collection which is now the most complete library of recreational mathematics in the world.

I had been invited to attend this conferences but unfortunately had a previous committment. To make up for missing this conference, the next best thing was reviewing this proceedings book which on doing so made me realize what I had missed, for example, some very interesting talks plus getting together with the leading practitioners of recreational mathematics, some of whom were long time colleagues.

One does not normally include a list of contents in a book review, but by doing so, it will give the reader a good indication of the wealth of recreational material here . So if you have any interest in recreational mathematics, this is a book for you. And even if you do not have such an interest, reading this book may give you one.

Contents

Postscript:

There is a typographical error in Andy Liu's article on page 206. The number of tetrominoes is **five** and not four. This is corrected in the reprint of this article as Appendix C in the new edition of Golomb's "Polyominoes".

In Memoriam — Leon Bankoff

We were saddened to learn of the recent death of Dr. Leon Bankoff, who has been a contributor to *CRUX* over many years. Sadly, he was not able to contribute recently, and some of our more recent readers may not know so much about him. We refer you to an excellent article in the March 1992 issue of the College Mathematics Journal, entitled ***A Conversation with Leon Bankoff***, written by G.L. Alexanderson.

One of Leon's long time friends, Dr. Clayton Dodge, has written the appreciation printed below.

Leon Bankoff practiced dentistry for sixty years in Beverly Hills, California, until his retirement just a few years ago. His patients included many Hollywood personalities whose names are household words. Among his several other interests, such as piano, guitar, calculators, and computers, he lectured and wrote papers both on dentistry and mathematics. His specialty was geometry, and the figure he loved best was the arbelos, or shoemaker's knife, which consists of three semicircles having a common diameter line. The two smaller semicircles are externally tangent to each other and internally tangent to the largest semicircle.

It is said that the test of a mathematician is not what he himself has discovered, but what he inspired others to do. Leon discovered a third circle congruent to the twin circles of Archimedes and published that result in the September 1974 issue of Mathematics Magazine ("Are the Twin Circles of Archimedes Really Twins?", pp. 214-218.) This revelation motivated the discovery by Leon and by others of several other members of that family of circles. An article on those circles is in progress.

Dr. Bankoff edited the Problem Department of the Pi Mu Epsilon Journal from 1968 to 1981, setting and maintaining a high standard of excellence in the more than 300 problems he included in its pages. Although the Journal has a relatively small circulation, its Problem Department grew to have a large number of regular contributors. He became acquainted with *Crux Mathematicorum* early in its history, when it was called *Eureka*, and made many contributions to its pages over the years, maintaining a close friendship with its founder and first editor Leo Sauvé. Like Leo, who started *Crux* to add some spice to his mathematical life of teaching basic post high school courses, Leon worked in mathematics for mental exercise and recreation, making friends with and earning the respect of many well known mathematicians.

Leon and I became good friends, first through correspondence regarding the Pi Mu Epsilon Journal Problem Department, and later through many personal meetings, including the August 1979 meeting of problemists in Ottawa, sponsored by Leo Sauvé and Fred Maskell of *Crux*. Following the formal sessions in Ottawa, seven of us drove to Quebec City for an enjoyable weekend of sightseeing and fellowship: Leo and Carmen Sauvé, Leon and Francine Bankoff, Charles and Avetta Trigg, and I. Since his retirement from his dentistry, Leon has worked on the manuscript for a proposed book on the properties of the arbelos, carrying on a monumental task started by him and the late Victor Thébault. Much material has been collected for this project and much remains to be done on it. Indeed, he asked me to finish the job.

At one time some years ago a schoolgirl wrote to Albert Einstein about a mathematical question she had. Apparently Einstein misinterpreted her question and gave an incorrect answer. Bankoff pointed out this error and in his mathematical museum he now has a copy of the Los Angeles Times with the front page headline "Local Dentist Proves Einstein Wrong."

Leon developed many physical problems in his later years. He was a fighter and he won several physical battles. When I last visited him at his home in Los Angeles in October 1996, he was fighting liver cancer, but still working on the Thébault material, in spite of failing eyesight. On Sunday afternoon, February 16, 1997, the cancer overtook him and he died at his home at the age of 88.

He was a gentleman, a scholar, and a true friend.

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Heronian Triangles with Associated Inradii in Arithmetic Progression

Paul Yiu
Department of Mathematics, Florida Atlantic University

In memory of Dr. Leon Bankoff

1. The area of a triangle is given in terms of its sides a, b, c by the Heron formula

$$\triangle = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s := \frac{1}{2}(a+b+c)$ is the semiperimeter. A triangle $(a, b, c; \triangle)$ is called Heronian if its sides and area are all integers. L. Bankoff [1] has made an interesting observation about the Heronian triangle $(13, 14, 15; 84)$. The height on the side 14 being 12, this triangle can be decomposed into two Pythagorean components, namely, $(5, 12, 13)$ and $(9, 12, 15)$. The inradii of these Pythagorean triangles, and that of the Heronian triangle, are respectively 2, 3, 4, three consecutive integers! (See Figure 1). Noting that the sides of the Heronian triangle are themselves three consecutive integers, Bankoff remarked that "no other Heronian triangle can claim that distinction".

Actually, apart from this, there are exactly two other Heronian triangles with three consecutive integers for the associated inradii. Each of these two Heronian triangles is decomposable into two Pythagorean components, namely,

$$(15, 20, 25; 150) = (9, 12, 15) \cup (16, 12, 20), \tag{2}$$
$$(25, 39, 56; 420) = (20, 15, 25) \cup (36, 15, 39). \tag{3}$$

The three inradii in these two cases are 3, 4, 5, and 5, 6, 7 respectively. The Heronian triangle $(15, 20, 25; 150)$ in (2) has an interesting property that no other Heronian triangle shares. Here, if the smaller Pythagorean component $(9, 12, 15)$ is excised from the larger one $(16, 12, 20)$, another Heronian triangle results, namely,

$$(15, 20, 7; 42) = (16, 12, 20) \setminus (9, 12, 15). \tag{4}$$

This has inradius 2. (See Figure 2). Note that the four inradii are consecutive integers! The same construction applied to Bankoff's example $(13, 14, 15; 84)$ gives $(13, 15, 4; 24)$, with inradius $\frac{3}{2}$, not an integer. For the Heronian triangle $(25, 39, 56; 420)$ in (3), this fourth inradius is 3, albeit not consecutive with the other three inradii 5, 6, and 7. In § 4 below, we shall show that, up to similarity, the configuration in Figure 2 is the only one with the four associated inradii in arithmetic progression.

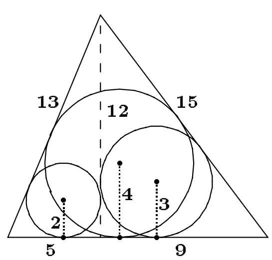


Figure 1.

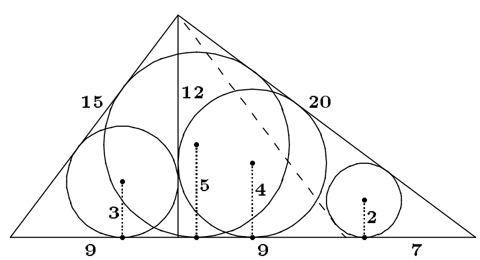
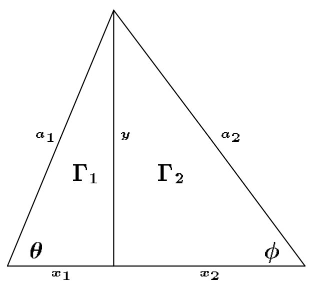


Figure 2.

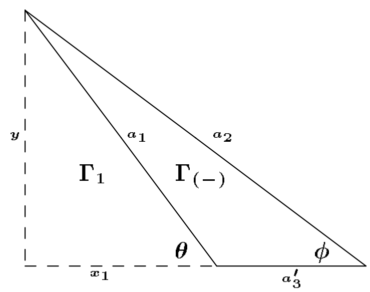
2. Consider two right triangles with a common side y and opposite acute angles θ and ϕ juxtaposed to form a triangle $\Gamma_{(+)}$. We shall assume $\theta > \phi$, so that Γ_1 can be excised from Γ_2 to form another triangle $\Gamma_{(-)}$. (See Figures 3 and 4). The inradius of a triangle with area $\triangle$ and semiperimeter s is given by $r = \frac{\triangle}{s}$. (See, for example, Coxeter [2, p. 12]). For a right triangle with legs a, b, and hypotenuse c, this is also given by the simpler formula

$r = s-c = \frac{1}{2}(a+b-c)$. We shall determine the similarity classes of triangles for which the inradii of the triangle and its two Pythagorean components are in arithmetic progression. The calculations can be made simple by making use of the fact that in a *right* triangle with an acute angle θ, the sides are in the ratio $2t : 1-t^2 : 1+t^2$, where $t = \tan\frac{\theta}{2}$. Let $t_1 := \tan\frac{\theta}{2}$ and $t_2 := \tan\frac{\phi}{2}$.



$\Gamma_{(+)} = \Gamma_1 \cup \Gamma_2$

Figure 3.



$\Gamma_{(-)} = \Gamma_2 \backslash \Gamma_1$

Figure 4.

By choosing $y = 2t_1t_2$, we have, in Figures 3 and 4,

$$\begin{aligned} a_1 &= t_2(1+t_1^2), & a_2 &= t_1(1+t_2^2);\\ x_1 &= t_2(1-t_1^2), & x_2 &= t_1(1-t_2^2);\\ a_3 &= x_1+x_2 = (t_1+t_2)(1-t_1t_2), & a_3' &= x_2-x_1 = (t_1-t_2)(1+t_1t_2). \end{aligned}$$

From these, we determine the inradii of the four triangles Γ_1, Γ_2, $\Gamma_{(+)}$ and $\Gamma_{(-)}$:

$$\begin{aligned} r_1 &= t_1t_2(1-t_1), & r_2 &= t_1t_2(1-t_2),\\ r_+ &= t_1t_2(1-t_1t_2), & r_- &= t_1t_2(1-\tfrac{t_2}{t_1}). \end{aligned} \tag{5}$$

Since $r_1 < r_2$, $r_- < r_2$, and $r_- < r_+$, there are only three cases in which three of these inradii can be in arithmetic progression:

(i) r_1, r_2, r_+ are in A.P. if and only if t_1, t_2, t_1t_2 are in A.P., that is, $t_1 + t_1t_2 = 2t_2$. From this, $t_2 = \frac{t_1}{2-t_1}$, and

$$r_1 : r_2 : r_+ = 2-t_1 : 2 : 2+t_1. \tag{6}$$

(ii) r_-, r_1, r_2 are in A.P. if and only if $\frac{t_2}{t_1}$, t_1, t_2 are in A.P., that is, $\frac{t_2}{t_1} + t_2 = 2t_1$. From this, $t_2 = \frac{2t_1^2}{1+t_1}$, and

$$r_- : r_1 : r_2 = 1 : 1+t_1 : 1+2t_1. \tag{7}$$

(iii) r_1, r_-, r_2 are in A.P. if and only if t_1, $\frac{t_2}{t_1}$, t_2 are in A.P., that is, $t_1 + t_2 = \frac{2t_2}{t_1}$. From this, $t_2 = \frac{t_1^2}{2-t_1}$, and

$$r_1 : r_- : r_2 = 2-t_1 : 2 : 2+t_1. \tag{8}$$

In each of these cases, with $t := t_1$, the proportions of the sides of $\Gamma_{(\pm)}$ are as follows. These triangles are all genuine for $0 < t < 1$.

	A.P.	$a_1 : a_2 : a_3$ (or a_3')
(i)	$r_1,\ r_2,\ r_+$	$(2-t)(1+t^2) : 2(2-2t+t^2) : (3-t)(1-t)(2+t)$
(ii)	$r_-,\ r_1,\ r_2$	$2t(1+t)(1+t^2) : 1+2t+t^2+4t^4 : (1-t)(1+t+2t^3)$
(iii)	$r_1,\ r_-,\ r_2$	$t(2-t)(1+t^2) : 4-4t+t^2+t^4 : 2(1-t)(2-t+t^3)$

3. Among the triangles constructed above with three associated inradii in A.P., the only cases in which the three sides also are in A.P. are tabulated below. In each case, we give the smallest Heronian triangle with three associated inradii in *integers*.

	(t_1, t_2)	Triangle with decomposition	inradii
(1)	$(\frac{2}{3},\ \frac{1}{2})$	$(13,15,14;84) = (5,12,13) \cup (9,12,15)$	$(r_1,r_2,r_+) = (2,3,4)$
(2)	$(\frac{1}{2},\ \frac{1}{3})$	$(15,20,25;150) = (9,12,15) \cup (16,12,20)$	$(r_1,r_2,r_+) = (3,4,5)$
(3)	$(\frac{1}{2},\ \frac{1}{6})$	$(15,37,26;156) = (35,12,37) \setminus (9,12,15)$	$(r_1,r_-,r_2) = (3,4,5)$
(4)	$(\frac{1}{5},\ \frac{1}{15})$	$(39,113,76;570) = (112,15,113) \setminus (36,15,39)$	$(r_-,r_1,r_2) = (5,6,7)$

Bankoff's observation [1] on the Heronian triangle $(13, 14, 15; 84)$ is case (1) in this table.

4. Finally, we consider the possibility for the four inradii r_-, r_1, r_2, r_+, to be in A.P. First, assume r_1, r_-, r_2 in A.P. By (8), $r_1 : r_- : r_2 = 2 - t_1 : 2 : 2 + t_1$; indeed, $t_2 = \frac{t_1^2}{2-t_1}$. From (5), we have, after simplification,

$$r_1 : r_- : r_2 : r_+ = 2 - t_1 : 2 : 2 + t_1 : 2 + t_1 + t_1^2.$$

Now, these four inradii are in A.P. if and only if $2 + t_1 + t_1^2 = 2 + 2t_1$. This is clearly impossible for $0 < t_1 < 1$.

It remains, therefore, to consider the possibility that r_-, r_1, r_2, r_+ be in A.P. This requires, by (7), $r_1 : r_2 = 1 + t_1 : 1 + 2t_1$, and also by (6), $r_1 : r_2 = 2 - t_1 : 2$. It follows that $1 + t_1 : 1 + 2t_1 = 2 - t_1 : 2$, from which $t_1 = \frac{1}{2}$. Consequently, $r_1 : r_2 = 3 : 4$. Also, $r_1 : r_+ = 3 : 5$ from (6), and $r_- : r_1 = 2 : 3$ from (7). Thus, we have the configuration in Figure 2, in which the four inradii are in the ratio

$$r_- : r_1 : r_2 : r_+ = 2 : 3 : 4 : 5.$$

The author thanks the referee for valuable comments and suggestions.

References

1. L. Bankoff, *An Heronian oddity*, Crux Math. 8 (1982) p.206.
2. H.S.M. Coxeter, *Introduction to Geometry*, Wiley, New York, 1961.

THE SKOLIAD CORNER

No. 21

R.E. Woodrow

This number we give the problems of the Manitoba Mathematical Contest for 1995. This is a two hour contest aimed primarily at grade 12 students, and sponsored by the Actuaries Club of Winnipeg, The Manitoba Association of Mathematics Teachers, The Canadian Mathematical Society and The University of Manitoba. My thanks go to Diane and Roy Dowling, organizers of the contest for supplying us with it.

THE MANITOBA MATHEMATICAL CONTEST 1995

For Students in Grade 12

Wednesday, February 22, 1995 — Time: 2 hours

1. (a) If a and b are real numbers such that $a + b = 3$ and $a^2 + ab = 7$ find the value of a.

(b) Noriko's average score on three tests was 84. Her score on the first test was 90. Her score on the third test was 4 marks higher than her score on the second test. What was her score on the second test?

2. (a) Find two numbers which differ by 3 and whose squares differ by 63.

(b) Find the real number which is a root of the equation

$$27(x-1)^3 + 8 = 0.$$

3. (a) Two circles lying in the same plane have the same centre. The radius of the larger circle is twice the radius of the smaller circle. The area of the region between the two circles is 7. What is the area of the smaller circle?

(b) The area of a right triangle is 5. Also, the length of the hypotenuse of this triangle is 5. What are the lengths of the other two sides?

4. (a) The parabola whose equation is $8y = x^2$ meets the parabola whose equation is $x = y^2$ at two points. What is the distance between these two points?

(b) Solve the equation $3x^3 + x^2 - 12x - 4 = 0$.

5. (a) Find the real number a such that $a^4 - 15a^2 - 16 = 0$ and $a^3 + 4a^2 - 25a - 100 = 0$.

(b) Find all positive numbers x such that $x^{x\sqrt{x}} = (x\sqrt{x})^x$.

6. If x, y and z are real numbers prove that

$$(x|y| - y|x|)\,(y|z| - z|y|)\,(x|z| - z|x|) = 0.$$

7. x and y are integers between 10 and 100. y is the number obtained by reversing the digits of x. If $x^2 - y^2 = 495$ find x and y.

8. Three points P, Q and R lie on a circle. If $PQ = 4$ and $\angle PRQ = 60^\circ$ what is the radius of the circle?

9. Three points are located in the finite region between the x-axis and the graph of the equation $2x^2 + 5y = 10$. Prove that at least two of these points are within a distance 3 of each other.

10. Three circles pass through the origin. The centre of the first circle lies in the first quadrant, the centre of the second circle lies in the second quadrant, and the centre of the third circle lies in the third quadrant. If P is any point that is inside all three circles, show that P lies in the second quadrant.

Last number we gave the problems of the Mathematical Association National Mathematics Contest 1994 from the United Kingdom. Here are the answers.

1.	C	**2.**	E	**3.**	C	**4.**	D	**5.**	E
6.	B	**7.**	A	**8.**	D	**9.**	B	**10.**	B
11.	B	**12.**	E	**13.**	E	**14.**	A	**15.**	A
16.	C	**17.**	D	**18.**	C	**19.**	B	**20.**	B
21.	C	**22.**	B	**23.**	E	**24.**	B	**25.**	E

That completes the Skoliad Corner for this issue. I need suitable contest materials and welcome your suggestions for the evolution of this feature.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to the Mayhem Editor, Naoki Sato, Department of Mathematics, University of Toronto, Toronto, ON Canada M5S 1A1. The electronic address is

mayhem@math.toronto.edu

The Assistant Mayhem Editor is Cyrus Hsia (University of Toronto). The rest of the staff consists of Richard Hoshino (University of Waterloo), Wai Ling Yee (University of Waterloo), and Adrian Chan (Upper Canada College).

A Journey to the Pole — Part II

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In this second (and last, for your relief) article, we look at some advanced topics like inversion or the applications of calculus to the theory of curves.

Inversion

Inversion is a transformation determined by a point called the *centre of inversion* O and an *inversion ratio* $\pm k^2$. The image of a point P is a point P' such that P' is on line OP and $|OP'| = \pm k^2|OP|$. It is evident that a curve $r = f(\theta)$ can be inverted by letting $r = \pm\dfrac{k^2}{f(\theta)}$. Inversion is a *conformal* transformation, meaning that the angles between intersecting curves are preserved. We will prove this in a later section.

Exercise 1. Prove that the inverse curve of a straight line is itself if it passes through the origin or a circle through the origin if it does not.

Example 1. By inspection of the equations of the conic $r = \dfrac{de}{1 - e\cos(\theta - \phi)}$ and the limaçon $r = b + a\cos\theta$, it is evident that the inverse of a conic about its focus is a limaçon. I imagine that trying to prove this theory with synthetic geometry would result in a severe headache.

This last result provides a different definition of conics as loci if we invert the definition of the limaçon given above. Consider a circle or straight line and a point O not on it. Draw a circle through O tangent to the circle or line. The diameter through O intersects the circle at P. The locus of all P's is a conic with O at one focus.

Tangent Lines

We leave the realm of elementary geometry to enter calculus, where we will teach the same old dog new tricks. The first is how to find the tangent line to a polar curve.

Our starting point will be the equation of the straight line $d = r\sin(\phi - \theta)$. The tangent line at θ_0 is a first-order approximation to the curve involving $r(\theta_0)$ and $\left.\frac{dr}{d\theta}\right|_{\theta_0}$. Differentiating the equation of the straight line with respect to θ at θ_0, we get $0 = \left.\frac{dr}{d\theta}\right|_{\theta_0} \sin(\phi - \theta_0) - r(\theta_0)\cos(\phi - \theta_0)$, which implies that $\tan(\phi - \theta_0) = \left.\frac{r}{(dr/d\theta)}\right|_{\theta_0}$. This quantity can sometimes be useful in itself, as $\phi - \theta_0$ represents the angle between the radius vector and the tangent line. We will make use of it in the next example.

The orientation ϕ is determined from its tangent, and

$$\tan(\phi - \theta_0 + \theta_0) = \frac{\tan(\phi - \theta_0) + \tan\theta_0}{1 - \tan(\phi - \theta_0)\tan\theta_0}$$

implies that

$$\tan\phi = \frac{r(\theta_0) + \frac{dr}{d\theta}|_{\theta_0}\tan\theta_0}{\frac{dr}{d\theta}|_{\theta_0} - r(\theta_0)\tan\theta_0}.$$

The parameter d in the equation of the tangent line is given (after some trigonometric manipulations) by

$$d = \frac{r(\theta_0)\tan(\phi - \theta_0)}{\sqrt{1 + \tan^2(\phi - \theta_0)}},$$

which gives

$$d = \frac{r^2(\theta_0)}{\sqrt{r^2(\theta_0) + (dr/d\theta)^2|_{\theta_0}}}.$$

Example 2. Proof that inversion preserves angles. Let $r = f(\theta)$ and $r = g(\theta)$ be two curves that intersect at θ_0. Their directions at θ_0 are ϕ_1 and ϕ_2, and the angle between them satisfies

$$\tan(\phi_2 - \phi_1) = \tan(\phi_2 - \theta_0 + \theta_0 - \phi_1) = \frac{\tan(\phi_2 - \theta_0) - \tan(\phi_1 - \theta_0)}{1 + \tan(\phi_2 - \theta_0)\tan(\phi_1 - \theta_0)}.$$

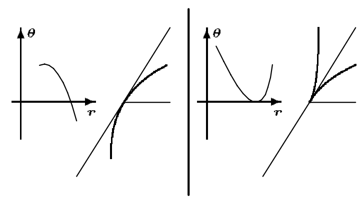
Now, the inverted curves, $r' = 1/f(\theta)$ and $r' = 1/g(\theta)$, also intersect at θ_0 and their directions ϕ_1' and ϕ_2' satisfy

$$\tan(\phi_1' - \theta_0) = \frac{1/f}{\frac{-1}{f^2}\left(\frac{df}{d\theta}\right)} = \frac{-f}{(df/d\theta)} = \tan(\theta_0 - \phi_1)$$

(and similarly for ϕ_2'). Hence, $\tan(\phi_2 - \phi_1) = \tan(\phi_1' - \phi_2')$ and we are done.

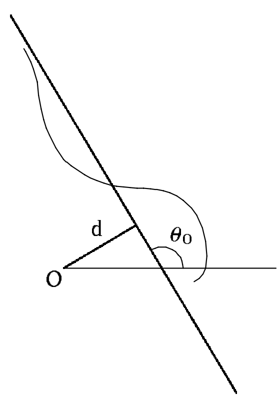
Tangent lines through the origin

An interesting special case is when $r(\theta_0) = 0$. In that case, $\tan\phi = \tan -\theta_0$ or $\phi = \theta_0$. In words, if the curve crosses the origin for a given θ_0, the equation of the tangent line at the origin is $\theta = \theta_0$.



When sketching curves, a useful result is that if $r(\theta)$ has an odd-order root at θ_0, then the curve is smooth at the origin, but if the root is even-order, then there is a cusp (see the figure).

Asymptotes



In certain cases r tends to infinity for some finite value of θ, signalling the possibility of an asymptote. In handling asymptotes, it is convenient to consider $s(\theta) = 1/r(\theta)$, in which case $\tan(\phi - \theta_0) = \dfrac{-s}{(ds/d\theta)}|_{\theta_0}$ and $\tan\phi = \dfrac{\frac{ds}{d\theta}|_{\theta_0}\tan\theta_0 - s(\theta_0)}{\frac{ds}{d\theta}|_{\theta_0} + s(\theta_0)\tan\theta_0}$. There is a possible asymptote if $s(\theta_0) = 0$ and its equation is: $s(\theta) = \dfrac{\sin(\theta - \theta_0)}{d}$, or $\theta = \theta_0$ if $d = 0$.

This is because, as in the case of tangent lines through the origin, the slope of the tangent line is $\tan\phi = \tan\theta_0$. The parameter d is given by

$$d = \lim_{\theta\to\theta_0} \frac{1}{\sqrt{s^2(\theta) + (ds/d\theta)^2}}.$$

If this does not diverge, there is an asymptote.

When sketching curves, it is useful to know from which side of the asymptote the curve approaches infinity. This is achieved by studying the sign of $s(\theta) - \frac{1}{d}\sin(\theta - \theta_0)$, which tends to 0 at $\theta = \theta_0$. If it tends to $0+$, the curve is closer to the origin than the asymptote (see the figure on the last page), and if it tends to $0-$, the curve is farther from the origin than the asymptote.

Exercise 2. Sketch the curve $r = \ln\theta$, its asymptote and the tangent line at the origin.

Example 3. The parabola $r = \dfrac{1}{1-\cos\theta}$ satisfies $\lim_{\theta\to 0} r(\theta) = \infty$, but it has no asymptotes since $\dfrac{1}{\sqrt{(1-\cos\theta_0)^2 + (\sin\theta_0)^2}} = \dfrac{1}{\sqrt{2(1-\cos\theta_0)}}$, which diverges at θ_0.

Arc Length

Another application of calculus is the computation of curve lengths. Usually one would take the expression for the line element in cartesian coordinates, $dl^2 = dx^2 + dy^2$ and transform it to polar coordinates. To use only polar coordinates, one could apply Pythagoras' Theorem to (dr) and $(r d\theta)$. Although this gives the right answer, it is not rigorous. A rigorous argument that does not rely on rectangular coordinates follows.

Applying the cosine rule to side PP' of POP' (see figure) we have

$$dl^2 = r^2(\theta + d\theta) + r^2(\theta) - 2r(\theta)r(\theta + d\theta)\cos(d\theta).$$

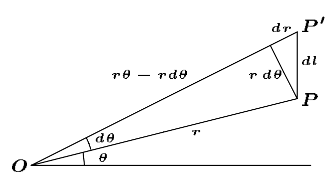
Expanding each term in a Taylor series up to the second order in $d\theta$, we get

$$\begin{aligned} dl^2 &= r^2 + 2r\frac{dr}{d\theta} + \left[\left(\frac{dr}{d\theta}\right)^2 + r\frac{d^2r}{d\theta^2}\right]d\theta^2 \\ &\quad + r^2 - 2r\left(r + \frac{dr}{d\theta} + \frac{1}{2}\frac{d^2r}{d\theta^2}\right)\left(1 - \frac{1}{2}d\theta^2\right). \end{aligned}$$

Keeping terms up to second order in $d\theta$ we have

$$dl^2 = \left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]d\theta^2.$$

We can thus write the expression for the length of a curve in polar coordinates as follows: $l = \displaystyle\int_{\theta_0}^{\theta}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta = \int_{\theta_0}^{\theta}\frac{1}{s^2}\sqrt{s^2 + \left(\frac{ds}{d\theta}\right)^2}\,d\theta$, where $s = \frac{1}{r}$.



Exercise 3. Derive the polar expression for arc length from the cartesian expression $dl^2 = dx^2 + dy^2$.

Curvature

It seems tautological to say that curvature is an important feature of curves, but the fact is that a planar curve is uniquely determined (up to translations and rotations) if its curvature is known as a function of arc length. This is generally of little practical importance, since the resulting differential equations can only be solved if you know the solution! We will give the formula for curvature in terms of $s(\theta) = l/r(\theta)$ and some applications.

Curvature, κ, can be defined as the rate of change of the direction of the tangent line per unit arc length. We have

$$\kappa = \frac{d\phi}{ds} = \frac{d\theta}{ds} \times \frac{d\phi}{d\theta} = \frac{s^2}{\sqrt{s^2 + \left(\frac{ds}{d\theta}\right)^2}} \times \left[1 + \frac{d(\phi - \theta)}{d\theta}\right].$$

Now, $\tan(\phi - \theta) = \dfrac{-s}{(ds/d\theta)}$, so that

$$\begin{aligned}\frac{d(\phi-\theta)}{d\theta} &= \frac{d\left(\arctan \frac{-s}{(ds/d\theta)}\right)}{d\theta} \\ &= \frac{1}{1 + \frac{s^2}{(ds/d\theta)}} \times \frac{-\left(\frac{ds}{d\theta}\right)^2 + s\left(\frac{d^2 s}{d\theta^2}\right)}{(ds/d\theta)^2}\end{aligned}$$

giving

$$\begin{aligned}\kappa &= \frac{s^2}{\sqrt{s^2 + \left(\frac{ds}{d\theta}\right)^2}} \times \left[1 + \frac{-\left(\frac{ds}{d\theta}\right)^2 + s\left(\frac{d^2 s}{d\theta^2}\right)}{s^2 + \left(\frac{ds}{d\theta}\right)^2}\right] \\ &= \frac{s^2}{\sqrt{s^2 + \left(\frac{ds}{d\theta}\right)^2}} \times \left[\frac{s + \left(\frac{d^2 s}{d\theta^2}\right)^2}{s^2 + \left(\frac{ds}{d\theta}\right)^2}\right] \\ &= \frac{s + \left(\frac{d^2 s}{d\theta^2}\right)}{\left[1 + \left(\frac{1}{2}\frac{ds}{d\theta}\right)^2\right]^{3/2}}.\end{aligned}$$

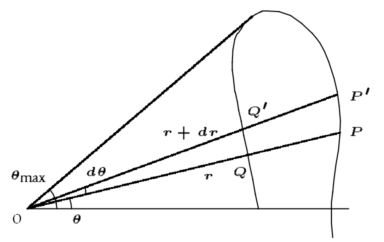
In terms or r, we have

$$\kappa = \frac{r^2 + 2(dr/d\theta)^2 - r(d^2r/d\theta^2)}{[r^2 + (dr/d\theta)^2]^{3/2}}.$$

Example 4. If curvature is zero we obtain the differential equation $s + s'' = 0$, with general solution $s = d\cos(\theta - \theta_0)$, that is, the equation of a straight line.

Exercise 4. Check that the curvature at each point of a lemniscate is proportional to the distance to the origin. (Hint: to simplify the algebra, divide numerator and denominator by r^3 in the curvature formula above.)

Area Enclosed by Polar Curves



The last application of calculus is the calculation of areas. The natural surface element is the 'triangle' defined by a segment of curve and the radius vectors at the endpoints (OPP' in the figure). The area of this triangle is, to first order in $d\theta$, $dA = \frac{1}{2}r^2 d\theta$.

If the origin does not lie inside the curve the equation $r = r(\theta)$ will have more than one branch, as shown, and the sign of the 'enclosed area' dA depends on the orientation given to the curve. In the figure the curve is traversed counterclockwise, and so the outer branch (PP') has positive sign (the positive sense of $d\theta$ coincides with the direction of the curve) and the inner branch (QQ') has negative sign (the positive sense of $d\theta$ as opposed to the direction of the curve). The same applies when calculating the area enclosed by two intersecting curves.

Example 5. As our last example, we will evaluate the area enclosed by the circle $(r - R\cos\theta)^2 = \rho^2 - R^2\sin^2\theta$. The area is given by

$$A = \int_{\sin\theta=-\rho/R}^{\sin\theta=\rho/R} \frac{1}{2}(r_+^2 - r_-^2)d\theta,$$

where $r_\pm = R\cos\theta \pm \sqrt{\rho^2 - R^2\sin^2\theta}$. We have

$$A = \int \frac{1}{2}(r_+ + r_-)(r_+ - r_-)d\theta = \int 2(R\cos\theta)\sqrt{\rho^2 - R^2\sin^2\theta d\theta}.$$

Letting $R\sin\theta = \rho\sin\phi$ and $R\cos\theta d\theta = \rho\cos\phi d\phi$, we have

$$A = \int_{\phi=-\pi/2}^{\pi/2} 2\rho^2\cos^2\phi d\phi = \pi\rho^2$$

as expected.

A Pattern in Permutations

John Linnell
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Let $p_n(k)$ be the number of permutations on n elements (say, n large anteaters) with exactly k fixed points (i.e. a permutation which takes k elements to themselves), so for example, $p_3(0) = 2$, $p_3(1) = 3$, $p_3(2) = 0$, and $p_3(3) = 1$. It should be clear that $\sum_{k=0}^{n} p_n(k) = n!$, but may be not so obvious that $\sum_{k=0}^{n} kp_n(k) = n!$ (this was problem 1 on the 1987 IMO). It may be even more surprising to learn that for $n \geq 2$, $\sum_{k=0}^{n} k^2 p_n(k) = 2n!$. What kind of pattern ensues? As my old analysis prof would no doubt say, "this is good exercise," and it is kind of fun to follow a trail like this and see where it leads.

Based on the above results, for $n \geq 1$ and $t \geq 0$, let

$$Q(n,t) = \frac{1}{n!} \sum_{k=0}^{n} k^t p_n(k).$$

Then $Q(n,0) = Q(n,1) = 1$ for all $n \geq 1$ and $Q(n,2) = 2$ for all $n \geq 2$ (not $n \geq 1$, and we will see why soon). Our first conjecture would probably be then that indeed each $Q(n,t)$ is an integer. But we will have to see a little more before we can prove anything.

Take $n = 5$. Then we can make the following table:

k	$p_5(k)$	$kp_5(k)$	$k^2p_5(k)$	$k^3p_5(k)$	$k^4p_5(k)$	$k^5p_5(k)$	$k^6p_5(k)$
0	44	0	0	0	0	0	0
1	45	45	45	45	45	45	45
2	20	40	80	160	320	640	1280
3	10	30	90	270	810	2430	7290
4	0	0	0	0	0	0	0
5	1	5	25	125	625	3125	15625
$\sum$	120	120	240	600	1800	6240	24240
$\sum/5!$	1	1	2	5	15	52	202

Table 1.

Thus, $Q(5,0) = 1$, $Q(5,1) = 1$, $Q(5,2) = 2$, $Q(5,3) = 5$, and so on. So far so good. We can make a second table with the actual $Q(n,t)$ values:

$n\backslash t$	0	1	2	3	4	5	6
1	1	1	1	1	1	1	1
2	1	1	2	4	8	16	32
3	1	1	2	5	14	41	122
4	1	1	2	5	15	51	187
5	1	1	2	5	15	52	202
6	1	1	2	5	15	52	203

Table 2.

Now we are getting somewhere. Notice how the rows seem to converge to a single sequence of integers, with one new term kicking in with each row. This sequence begins $1, 1, 2, 5, 15, 52, 203, \ldots$. I could not find this sequence in any of my references, so I did what any enterprising student would do. I sent an e-mail to `sequences@research.att.com`, with "`lookup 1 1 2 5 15 52 203`" in the body. For those not familiar, it is an on-line sequence server that tries to solve or match any sequence you might send it; I should also add that they ask that you send at most one request per hour. Soon enough, I had a response, which indicated that this was a sequence known as the Bell numbers, which satisfy

$$B(0) = 1, \quad B(n+1) = \sum_{k=0}^{n} \binom{n}{k} B(k).$$

This recursion is striking, because if one looks at the second row of table 2, it may remind one of the identity $2^n = \sum_{k=0}^{n} \binom{n}{k}$. Could it be? Yes, in fact the same recursion that generates the Bell numbers is what generates successive rows of table 2. Now we really have something.

Claim. For all $n \geq 1$ and $t \geq 0$, $Q(n+1, t+1) = \sum_{i=0}^{t} \binom{t}{i} Q(n,i)$.

Proof. First, note that $p_n(k) = \binom{n}{k} p_{n-k}(0)$ [why?]. Then

$$\begin{aligned} Q(n+1, t+1) &= \frac{1}{(n+1)!} \sum_{k=0}^{n+1} k^{t+1} p_{n+1}(k) \\ &= \frac{1}{(n+1)!} \sum_{k=0}^{n+1} k^{t+1} \binom{n+1}{k} p_{n+1-k}(0) \end{aligned}$$

$$\begin{aligned}
&= \frac{n+1}{(n+1)!}\sum_{k=1}^{n+1}\frac{n!}{(k-1)!(n+1-k)!}\frac{k^{t+1}}{k}p_{n+1-k}(0)\\
&= \frac{1}{n!}\sum_{k=1}^{n+1}k^t\binom{n}{k-1}p_{n+1-k}(0)\\
&= \frac{1}{n!}\sum_{k=1}^{n+1}k^t p_n(k-1)\\
&= \frac{1}{n!}\sum_{k=0}^{n}(k+1)^t p_n(k)\\
&= \frac{1}{n!}\sum_{k=1}^{n+1}\sum_{i=0}^{t}\binom{t}{i}k^i p_n(k)\\
&= \sum_{i=0}^{t}\binom{t}{i}\left(\frac{1}{n!}\sum_{k=0}^{n}k^i p_n(k)\right)\\
&= \sum_{i=0}^{t}\binom{t}{i}Q(n,i).
\end{aligned}$$

So we have proven quite a bit actually, including:

1. Each $Q(n,t)$ is an integer (i.e., $\sum_{k=0}^{n} k^t p_n(k)$ is divisible by $n!$), and
2. For fixed t, $Q(n,t)$ eventually becomes $B(t)$ for sufficiently high n.

The claim looks complicated, but we know what we want to prove, and it turns out to be just a little algebraic manipulation. So in the end, we have a nice result from a simple observation.

IMO CORRESPONDENCE PROGRAM

Canadian students wishing to participate in this program should first contact Professor Edward J. Barbeau, Department of Mathematics, University of Toronto, Toronto, Ontario. Please note that there is a fee for participation in the program: $12. Please make the cheque payable to Edward J. Barbeau.

PROBLEM SET 1

Algebra

1. Solve the system of equations

$$x^2 + 2yz = x,$$
$$y^2 + 2xz = z,$$
$$z^2 + 2xy = y.$$

2. Let m be a real number. Solve, for x, the equation

$$|x^2 - 1| + |x^2 - 4| = mx.$$

3. Let $\{x_1, x_2, \ldots, x_n, \ldots\}$ be a sequence of nonzero real numbers. Show that the sequence is an arithmetic progression if and only if, for each integer $n \geq 2$,

$$\frac{1}{x_1x_2} + \frac{1}{x_2x_3} + \cdots + \frac{1}{x_{n-1}x_n} = \frac{n-1}{x_1x_n}.$$

4. Suppose that x and y are two unequal positive real numbers. Let

$$r = \left(\frac{x^2 + y^2}{2}\right)^{1/2} \qquad g = (xy)^{1/2}$$
$$a = \frac{x+y}{2} \qquad h = \frac{2xy}{x+y}\ .$$

Which of the numbers $r - a$, $a - g$, $g - h$ is largest and which is smallest?

5. Simplify

$$\frac{x^3 - 3x + (x^2 - 1)\sqrt{x^2 - 4} - 2}{x^3 - 3x + (x^2 - 1)\sqrt{x^2 - 4} + 2}$$

to a fraction whose numerator and denominator are of the form $u\sqrt{v}$ with u and v each linear polynomials. For which values of x is the equation valid?

6. Prove or disprove: if x and y are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$.

7. X is a collection of objects upon which the operation of addition, subtraction and multiplication are defined so as to satisfy the following axioms:

   (1) if x, y belong to X, then $x + y$ and xy both belong to X;

   (2) for all x, y in X, $x + y = y + x$;

   (3) for all x, y, z in X, $x + (y + z) = (x + y) + z$ and $x(yz) = (xy)z$;

   (4) for all x, y, z in x, $x(y + z) = xy + xz$;

   (5) there is an element 0 such that $0+x = x+0 = x$ and for each x in X, there exists a unique element denoted by $-x$ for which $x + (-x) = 0$;

   (6) $x - y = x + (-y)$ for each pair x, y of elements of X;

   (7) $x^3 - x = x + x + x = 0$ for x in X.

   Note that these axioms do not rule out the possibility that the product of two non-zero elements of X may be zero, and so it may not be valid to cancel terms.

   On X, we define a relation $\leq$ by the following condition:

   $$x \leq y \text{ if and only if } x^2y - xy^2 - xy + x^2 = 0.$$

   Prove that the following properties obtain:

   (i) $x \leq x$ for each element x of X;

   (ii) if $x \leq y$ and $y \leq x$, then $x = y$;

   (iii) if $x \leq y$ and $y \leq z$, then $x \leq z$.

8. Let n be a positive integer and suppose that u and v are positive real numbers. Determine necessary and sufficient conditions on u and v such that there exist real numbers $a_1, a_2, \ldots, a_n$ satisfying

   $$a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$$

   $$u = a_1 + a_2 + \cdots + a_n$$
   $$v = a_1^2 + a_2^2 + \cdots + a_n^2.$$

   When such a representation is possible, determine the maximum and minimum values of a_1.

9. Suppose that $x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x} = t$, where x, y, z are not all equal. Determine xyz.

10. Let $a \geq 0$. The polynomial $x^3 - ax + 1$ has three distinct real roots. For which values of a does the root u of least absolute value satisfy $\frac{1}{a} < u < \frac{2}{a}$?

11. Determine the range of values of cd subject to the constraints $ab = 1$, $ac + bd = 2$, where a, b, c, d are real.

12. Find polynomials $p(x)$ and $q(x)$ with integer coefficients such that

$$\frac{p(\sqrt{2}+\sqrt{3}+\sqrt{5})}{q(\sqrt{2}+\sqrt{3}+\sqrt{5})} = \sqrt{2}+\sqrt{3}.$$

Mayhem Problems

The Mayhem Problems editors are:

Cyrus Hsia	*Mayhem Advanced Problems Editor,*
Richard Hoshino	*Mayhem High School Problems Editor,*
Ravi Vakil	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only problems — the next issue will feature only solutions.

We warmly welcome proposals for problems and solutions. With the new schedule of eight issues per year, we request that solutions to the new problems in this issue be submitted by 1 August 1997, for publication in the issue 5 months ahead; that is, issue 8. We also request that **only students** submit solutions (see editorial [1997: 30]), but we will consider particularly elegant or insightful solutions from others. Since this rule is only being implemented now, you will see solutions from many people in the next few months, as we clear out the old problems from Mayhem.

High School Problems

Editor: Richard Hoshino, 17 Norman Ross Drive, Markham, Ontario, Canada. L3S 3E8 <rhoshino@undergrad.math.uwaterloo.ca>

There is a correction for **H220**; the expression $2n \times \frac{T}{S}$ should be $2^n \times \frac{T}{S}$.

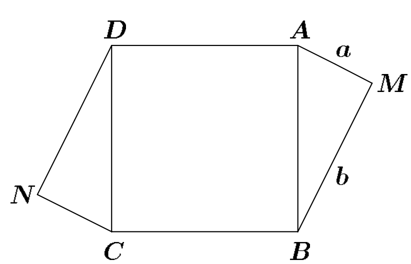
H221. Let $P = 19^5 + 660^5 + 1316^5$. It is known that 25 is one of the forty-eight positive divisors of P. Determine the largest divisor of P that is less than $10,000$.

H222. McGregor becomes very bored one day and decides to write down a three digit number ABC, and the six permutations of its digits. To his surprise, he finds that ABC is divisible by 2, ACB is divisible by 3, BAC is divisible by 4, BCA is divisible by 5, CAB is divisible by 6, and CBA is a divisor of 1995. Determine ABC.

H223. There are n black marbles and two red marbles in a jar. One by one, marbles are drawn at random out of the jar. Jeanette wins as soon as two black marbles are drawn, and Fraserette wins as soon as two red marbles are drawn. The game continues until one of the two wins. Let $J(n)$ and $F(n)$ be the two probabilities that Jeanette and Fraserette win, respectively.

1. Determine the value of $F(1) + F(2) + \cdots + F(3992)$.
2. As n approaches infinity, what does $J(2) \times J(3) \times J(4) \times \cdots \times J(n)$ approach?

H224. Consider square $ABCD$ with side length 1. Select a point M exterior to the square so that $\angle AMB$ is $90°$. Let $a = AM$ and $b = BM$. Now, determine the point N exterior to the square so that $CN = a$ and $DN = b$. Find, as a function of a and b, the length of line segment MN.



Advanced Problems

Editor: Cyrus Hsia, 21 Van Allan Road, Scarborough, Ontario, Canada. M1G 1C3 <hsia@math.toronto.edu>

A197. Calculate

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin(2N+1)\theta}{\sin\theta}\,d\theta,$$

where N is a non-negative integer.

A198. Given positive real numbers a, b, and c such that $a+b+c=1$, show that $a^a b^b c^c + a^b b^c c^a + a^c b^a c^b \le 1$.

A199. Let P be a point inside triangle ABC. Let A', B', and C' be the reflections of P through the sides BC, AC, and AB respectively. For what points P are the six points A, B, C, A', B', and C' concyclic?

A200. Given positive integers n and k, for $0 \le i \le k-1$, let

$$S_{n,k,i} = \sum_{j \equiv i \pmod k} \binom{n}{j}.$$

Do there exist positive integers n, $k > 2$, such that $S_{n,k,0}$, $S_{n,k,1}$, ... , $S_{n,k,k-1}$ are all equal?

Challenge Board Problems

Editor: Ravi Vakil, Department of Mathematics, One Oxford Street, Cambridge, MA, USA. 02138-2901 <ravi@math.harvard.edu>

There are no new Challenge Board Problems this month — we reprint those from issue 1 this year [1997: 44]'

C70. Prove that the group of automorphisms of the dodecahedron is S_5, the symmetric group on five letters, and that the rotation group of the dodecahedron (the subgroup of automorphisms preserving orientation) is A_5.

C71. Let L_1, L_2, L_3, L_4 be four general lines in the plane. Let p_{ij} be the intersection of lines L_i and L_j. Prove that the circumcircles of the four triangles $p_{12}p_{23}p_{31}$, $p_{23}p_{34}p_{42}$, $p_{34}p_{41}p_{13}$, $p_{41}p_{12}p_{24}$ are concurrent.

C72. A finite group G acts on a finite set X transitively. (In other words, for any $x, y \in X$, there is a $g \in G$ with $g \cdot x = y$.) Prove that there is an element of G whose action on X has no fixed points.

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 November 1997**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in LaTeX). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication.*

2226. *Proposed by K.R.S. Sastry, Dodballapur, India.*

An old man willed that, upon his death, his three sons would receive the u'th, v'th, w'th parts of his herd of camels respectively. He had $uvw-1$ camels in the herd when he died. Obviously, their sophisticated calculator could not divide $uvw-1$ exactly into u, v or w parts. They approached a distinguished ***CRUX*** problem solver for help, who rode over on his camel, which he added to the herd and then fulfilled the old man's wishes, and took the one camel that remained, which was, of course, his own.

Dear ***CRUX*** reader, how many camels were there in the herd?

2227. *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

Evaluate

$$\prod_p \left[\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2p)^{2k}}\right].$$

where the product is extended over all prime numbers.

2228. *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

Let A be the set of all real numbers from the interval $(0,1)$ whose decimal representation consists only of 1's and 7's; that is, let

$$A = \left\{ \sum_{k=1}^{\infty} \frac{a_k}{10^k} : a_k \in \{1,7\} \right\}.$$

Let B be the set of all reals that cannot be expressed as finite sums of members of A. Find $\sup B$.

2229. *Proposed by Kenneth Kam Chiu Ko, Mississauga, Ontario.*

(a) Let m be any positive integer greater than 2, such that $x^2 \equiv 1 \pmod{m}$ whenever $(x,m)=1$.

Let n be a positive integer. If $m|n+1$, prove that the sum of all divisors of n is divisible by m.

(b)* Find all possible values of m

2230. *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

Triangles BCD and ACE are constructed outwardly on sides BC and CA of triangle ABC such that $AE = BD$ and $\angle BDC + \angle AEC = 180^\circ$. The point F is chosen to lie on the segment AB so that

$$\frac{AF}{FB} = \frac{DC}{CE}.$$

Prove that

$$\frac{DE}{CD+CE} = \frac{EF}{BC} = \frac{FD}{AC}.$$

2231. *Proposed by Herbert Gülicher, Westfalische Wilhelms-Universität, Münster, Germany.*

In quadrilateral $P_1P_2P_3P_4$, suppose that the diagonals intersect at the point $M \neq P_i$ $(i = 1,2,3,4)$. Let $\angle MP_1P_4 = \alpha_1$, $\angle MP_3P_4 = \alpha_2$, $\angle MP_1P_2 = \beta_1$ and $\angle MP_3P_2 = \beta_2$.

Prove that

$$\lambda_{13} := \frac{|P_1M|}{|MP_3|} = \frac{\cot\alpha_1 \pm \cot\beta_1}{\cot\alpha_2 \pm \cot\beta_2},$$

where the $+(-)$ sign holds if the line segment P_1P_3 is located inside (outside) the quadrilateral.

2232. *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Find all solutions of the inequality:

$$n^2+n-5 < \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n+1}{3} \right\rfloor + \left\lfloor \frac{n+2}{3} \right\rfloor < n^2+2n-2, \ (n \in \mathbb{N}).$$

(Note: If x is a real number, then $\lfloor x \rfloor$ is the largest integer not exceeding x.)

2233. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let x, y, z be non-negative real numbers such that $x+y+z=1$, and let p be a positive real number.

(a) If $0 < p \le 1$, prove that

$$x^p+y^p+z^p \ \ge \ C_p\left((xy)^p+(yz)^p+(zx)^p\right),$$

where

$$C_p \ = \ \begin{cases} 3^p & \text{if } \ p \le \frac{\log 2}{\log 3 - \log 2}, \\ 2^{p+1} & \text{if } \ p \ge \frac{\log 2}{\log 3 - \log 2}. \end{cases}$$

(b)★ Prove the same inequality for $p > 1$.

Show that the constant C_p is best possible in all cases.

2234. *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

Given triangle ABC, its centroid G and its incentre I, construct, using only an unmarked straightedge, its orthocentre H.

2235. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Triangle ABC has angle $\angle CAB = 90°$. Let $\Gamma_1(O,R)$ be the circumcircle and $\Gamma_2(T,r)$ be the incircle. The tangent to Γ_1 at A and the polar line of A with respect to Γ_2 intersect at S. The distances from S to AC and AB are denoted by d_1 and d_2 respectively.

Show that

(a) $ST \| BC$,

(b) $|d_1 - d_2| = r$.

[For the benefit of readers who are not familiar with the term "polar line", we give the following definition as in, for example, *Modern Geometries*, 4th Edition, by James R. Smart, Brooks/Cole, 1994:

The line through an inverse point and perpendicular to the line joining the original point to the centre of the circle of inversion is called the polar of the original point, whereas the point itself is called the pole of the line.]

2236. *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

Let ABC be an arbitrary triangle and let P be an arbitrary point in the interior of the circumcircle of $\triangle ABC$. Let K, L, M, denote the feet of the perpendiculars from P to the lines AB, BC, CA, respectively.

Prove that $[KLM] \leq \frac{[ABC]}{4}$.

Note: $[XYZ]$ denotes the area of $\triangle XYZ$.

2237. *Proposed by Meletis D. Vasiliou, Elefsis, Greece.*

$ABCD$ is a square with incircle Γ. Let ℓ be a tangent to Γ. Let A', B', C', D' be points on ℓ such that AA', BB', CC', DD' are all perpendicular to ℓ.

Prove that $AA' \cdot CC' = BB' \cdot DD'$.

Correction

2173. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $n \geq 2$ and $x_1, \ldots, x_n > 0$ with $x_1 + \ldots + x_n = 1$.

Consider the terms

$$l_n = \sum_{k=1}^{n} (1 + x_k)\sqrt{\frac{1 - x_k}{x_k}}$$

and

$$r_n = C_n \prod_{k=1}^{n} \frac{1 + x_k}{\sqrt{1 - x_k}}$$

where

$$C_n = (\sqrt{n-1})^{n+1}(\sqrt{n})^n/(n+1)^{n-1}.$$

[Ed: there is no x in the line above.]

1. Show $l_2 \leq r_2$.
2. Prove or disprove: $l_n \geq r_n$ for $n \geq 3$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

1940. [1994: 108; 1995: 107; 1995: 205; 1996: 321] *Proposed by Ji Chen, Ningbo University, China.*

Show that if $x, y, z > 0$,

$$(xy + yz + zx)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \geq \frac{9}{4}\,.$$

Solution by Marcin E. Kuczma, Warszawa, Poland.

Let F be the expression on the left side of the proposed inequality. Assume without loss of generality $x \geq y \geq z \geq 0$, with $y > 0$ (not excluding $z = 0$), and define:

$$\begin{aligned}
A &= (2x + 2y - z)(x - z)(y - z) + z(x + y)^2,\\
B &= (1/4)z(x + y - 2z)(11x + 11y + 2z),\\
C &= (x + y)(x + z)(y + z),\\
D &= (x + y + z)(x + y - 2z) + x(y - z) + y(x - z) + (x - y)^2,\\
E &= (1/4)(x + y)z(x + y + 2z)^2(x + y - 2z)^2.
\end{aligned}$$

It can be verified that

$$C^2(4F - 9) = (x - y)^2\big((x + y)(A + B + C) + (x + z)(y + z)D/2\big) + E.$$

This proves the inequality and shows that it becomes an equality only for $x = y = z$ and for $x = y > 0$, $z = 0$.

Comment.

The problem is memorable for me! It was my "solution" [1995: 107] that appeared first. According to someone's polite opinion it was elegant, but according to the impolite truth, it was wrong. I noticed the fatal error when it was too late to do anything; the issue was in print already.

In [1995: 205] a (correct) solution by Kee-Wai Lau appeared. Meanwhile I found two other proofs, hopefully correct, and sent them to the editor. Like Kee-Wai Lau's, they required the use of calculus and were lacking "lightness", so to say, so the editor asked [1995:206] for a "nice" solution. I became rather sceptical about the possibility of proving the result by those techniques usually considered as "nice", such as convexity/majorization arguments — just because the inequality turns into equality not only for $x = y = z$, but also for certain boundary configurations.

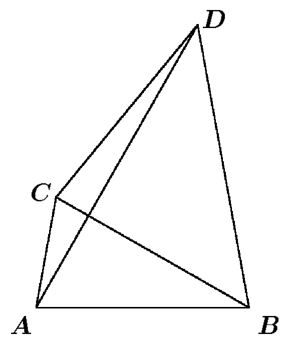
In response to the editor's prompt, Vedula Murty [1996: 321] proposed a short proof avoiding hard calculations. But I must frankly confess that I do not understand its final argument: I do not see why the sum of the first two terms in [1996: 321(3)] must be non-negative. While trying to clarify that, I arrived at the proof which I present here.

This proof can be called anything but "nice"! Decomposition into sums and products of several expressions, obviously nonnegative, and equally ugly, has the advantage that it provides a proof immediately understood and verified if one uses some symbolic calculation software (with some effort, the formula can be checked even by hand). But the striking disadvantage of such formulas is that they carefully hide from the reader all the ideas that must have led to them; they take the "background mathematics" of the reasoning away. In the case at hand I only wish to say that the equality I propose here has been inspired by Murty's brilliant idea to isolate the polynomial that appears as the third term in [1996: 321(3)] and to deal with the expression that remains.

I once overheard a mathematician problemist claiming lack of sympathy to inequality problems. In the ultimate end, he said, they all reduce to the only one fundamental inequality, which is $x^2 \geq 0$!

2124. [1996: 77] *Proposed by Catherine Shevlin, Wallsend, England.*

Suppose that $ABCD$ is a quadrilateral with $\angle CDB = \angle CBD = 50°$ and $\angle CAB = \angle ABD = \angle BCD$. Prove that $AD \perp BC$.



I. *Solution by Florian Herzig, student, Perchtoldsdorf, Austria.* (*Essentially identical solutions were submitted by Jordi Dou, Barcelona, Spain and Hans Engelhaupt, Franz-Ludwig-Gymnasium, Bamberg, Germany. The solution by Carl Bosley, student, Washburn Rural High School, Topeka, Kansas, USA was very similar.*)

Let F_1 and F_2 be the feet of the perpendiculars from D and A to BC respectively. Let $p = BC = CD$ and $q = AC$. Then, applying the Sine Rule to $\triangle ABC$, we have

$$CF_1 = p\cos 80°, \qquad CF_2 = q\cos 70° = \frac{p\sin 30°}{\sin 80°} = \frac{p\cos 70°}{2\sin 80°}.$$

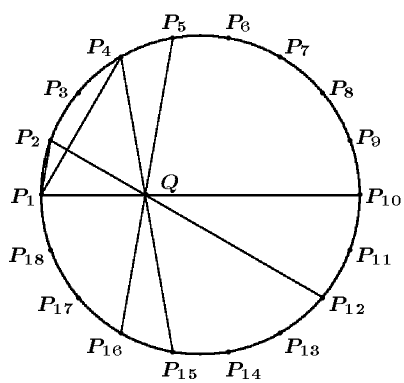
Thus we have

$$\frac{CF_1}{CF_2} = \frac{\cos 80^\circ}{\frac{\cos 70^\circ}{2\sin 80^\circ}} = \frac{2\sin 80^\circ \cos 80^\circ}{\cos 70^\circ} = \frac{\sin 160^\circ}{\sin 20^\circ} = 1.$$

Thus, $F_1 = F_2$, and this point is the intersection of AD and BC, whence $AD \perp BC$.

II. *Solution by Federico Ardila, student, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA.*

Consider a regular 18-gon $P_1P_2\ldots P_{18}$.



We will first show that P_1P_{10}, P_2P_{12} and P_4P_{15} concur.

By symmetry, P_1P_{10}, P_4P_{15} and P_5P_{16} are concurrent. Thus it is sufficient to prove that P_1P_{10}, P_2P_{12} and P_5P_{16} are concurrent.

Using the angles version of Ceva's theorem in triangle $\triangle P_1P_5P_{12}$, it if sufficient to prove that

$$\frac{\sin(\angle P_1P_{12}P_2)}{\sin(\angle P_2P_{12}P_5)} \cdot \frac{\sin(\angle P_{12}P_5P_{16})}{\sin(\angle P_{16}P_5P_1)} \cdot \frac{\sin(\angle P_5P_1P_{10})}{\sin(\angle P_{10}P_1P_{12})} = 1,$$

or

$$\frac{\sin(10^\circ)}{\sin(30^\circ)} \cdot \frac{\sin(40^\circ)}{\sin(30^\circ)} \cdot \frac{\sin(50^\circ)}{\sin(20^\circ)} = 1.$$

But this is true since

$$\begin{aligned} \sin 10^\circ \sin 40^\circ \sin 50^\circ &= \sin 10^\circ \sin 40^\circ \cos 40^\circ \\ &= \sin 10^\circ \left(\frac{\sin 80^\circ}{2}\right) = \frac{\sin 10^\circ \cos 10^\circ}{2} \\ &= \frac{\sin 20^\circ}{4} = (\sin 30^\circ)^2 \sin 20^\circ. \end{aligned}$$

So, P_1P_{10}, P_2P_{12} and P_4P_{15} concur at, say, Q.

Using this, it is easy to check that

$$\angle P_2P_4Q = \angle P_4QP_2 = 50^\circ,$$

and

$$\angle P_2P_1Q = \angle P_4QP_1 = \angle QP_2P_4\ (= 80^\circ).$$

This information clearly determines the quadrilateral $P_1P_2P_4Q$ up to similarity, so $P_1P_2P_4Q \sim ACDB$.

Since $P_1P_4 \perp P_2Q$, it follows that $AD \perp BC$.

Also solved by CLAUDIO ARCONCHER, Jundiaí, Brazil; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; SAM BAETHGE, Science Academy, Austin, Texas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; TIM CROSS, King Edward's School, Birmingham, England; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; RICHARD I. HESS, Rancho Palos Verdes, California, USA; PETER HURTHIG, Columbia College, Burnaby, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, the Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; MELETIS VASILIOU, Elefsis, Greece (two solutions); and the proposer.

The proposer writes: The genesis of this problem lies in a question asked by Junji Inaba, student, William Hulme's Grammar School, Manchester, England, in ***Mathematical Spectrum***, vol. 28 (1995/6), p. 18. He gives the diagram in my question, with the given information:

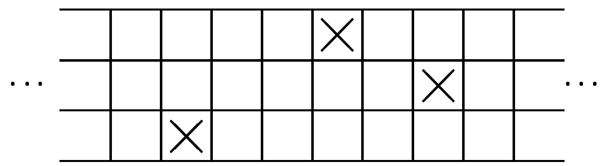
$$\begin{array}{rclcrcl} \angle CDA & = & 20^\circ, & \qquad & \angle DAB & = & 60^\circ, \\ \angle DBC & = & 50^\circ, & \qquad & \angle CBA & = & 30^\circ, \end{array}$$

and asks the question: "can any reader find $\angle CDB$ without trigonometry?" In fact, such a solution was given in the next issue of ***Mathematics Spectrum*** by Brian Stonebridge, Department of Computer Science, University of Bristol, Bristol, England.

The genesis of the diagram is much older, if one produces BD and AC to meet at E. See ***Mathematical Spectrum***, vol. 27 (1994/5), pp. 7 and 65–66. In one reference, the question of finding $\angle CDA$ is called "Mahatma's Puzzle", but no reference was available. Can any reader enlighten me on the origin of this puzzle?

2125. [1996: 122] *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

At Lake West Collegiate, the lockers are in a long rectangular array, with three rows of N lockers each. The lockers in the top row are numbered 1 to N, the middle row $N+1$ to $2N$, and the bottom row $2N+1$ to $3N$, all from left to right. Ann, Beth, and Carol are three friends whose lockers are located as follows:



By the way, the three girls are not only friends, but also next-door neighbours, with Ann's, Beth's, and Carol's houses next to each other (in that order) on the same street. So the girls are intrigued when they notice that Beth's house number divides into all three of their locker numbers. What is Beth's house number?

Solution by Han Ping Davin Chor, student, Cambridge, MA, USA.

From the diagram, it can be observed that the lockers have numbers

$$x + 3, \qquad N + x + 5 \qquad \text{and} \qquad 2N + x,$$

where $1 \leq x \leq N$, x a positive integer. Here locker $x + 3$ is in the first row, locker $N + x + 5$ is in the second row, and locker $2N + x$ is in the third row. Let y be Beth's house number, where y is a positive integer. Since y divides into $x + 3$, $N + x + 5$ and $2N + x$, y must divide into

$$2(N + x + 5) - (2N + x) - (x + 3) = 7.$$

Therefore $y = 1$ or 7. However, Beth's house is in between Ann's and Carol's. Assuming that 0 is not assigned as a house number, it means that Beth's house number cannot be 1 (else either Ann or Carol would have a house number of 0). Therefore Beth's house number is 7.

Also solved by SAM BAETHGE, Science Academy, Austin, Texas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; TIM CROSS, King Edward's School, Birmingham, England; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; J. K. FLOYD, Newnan, Georgia, USA; IAN JUNE L. GARCES, Ateneo de Manila University, Manila, the Philippines, and GIOVANNI MAZZARELLO, Ferrovie dello Stato, Firenze, Italy; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, New York, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; JOHN GRANT MCLOUGHLIN, Okanagan University College, Kelowna, B. C.; P. PENNING, Delft, the Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; CORY PYE, student, Memorial University of Newfoundland, St. John's, Newfoundland; JOEL SCHLOSBERG, student, Hunter College High School, New York NY, USA; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DAVID STONE, Georgia Southern University, Statesboro, Georgia, USA; EDWARD

T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; KENNETH M. WILKE, Topeka, Kansas, USA; and the proposer.

Two solvers eliminated 1 as a possible answer, because the problem said that the girls were "intrigued" that Beth's house number divided all their locker numbers, which would hardly be likely if Beth's house number were just 1! Thus they didn't need the information about the location of Beth's house at all. Another solver, to whom the editor has therefore given the benefit of the doubt, merely stated that "the location of Ann's and Carol's houses doesn't enter into the problem".

2126. [1996: 123] *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

At Lake West Collegiate, the lockers are in a long rectangular array, with three rows of N lockers each, where N is some positive integer between 400 and 450. The lockers in the top row were originally numbered 1 to N, the middle row $N+1$ to $2N$, and the bottom row $2N+1$ to $3N$, all from left to right. However, one evening the school administration changed around the locker numbers so that the first column on the left is now numbered 1 to 3, the next column 4 to 6, and so forth, all from top to bottom. Three friends, whose lockers are located one in each row, come in the next morning to discover that each of them now has the locker number that used to belong to one of the others! What are (were) their locker numbers, assuming that all are three-digit numbers?

Solution by Ian June L. Garces, Ateneo de Manila University, Manila, the Philippines, and Giovanni Mazzarello, Ferrovie dello Stato, Firenze, Italy.

The friends' locker numbers are **246**, **736** and **932**.

To show this, first consider any particular locker. Then the original (before the change) number of this locker can be written as $iN+j$, where $0 \le i \le 2$ (the row) and $1 \le j \le N$ (the column). With respect to this original locker number, this particular locker has a new (after the change) number $3(j-1)+(i+1) = 3j+i-2$.

Consider now the three friends' lockers. Since the three lockers are located one in each row, we can let them be j_1, $N+j_2$ and $2N+j_3$ where $1 \le j_1, j_2, j_3 \le N$. For each of these lockers, the corresponding new locker numbers will be $3j_1-2$, $3j_2-1$ and $3j_3$. Then there will be two possibilities for how their original locker numbers and their new locker numbers were "properly" interchanged:

Possibility 1. The first possibility is when

$$j_1 = 3j_3, \tag{1}$$

$$N + j_2 = 3j_1 - 2, \tag{2}$$

$$2N + j_3 = 3j_2 - 1. \qquad (3)$$

Substituting (1) into (2) and solving for j_2, we have $j_2 = 9j_3 - 2 - N$. Substituting this last equality into (3) and solving for j_3, we have

$$j_3 = \frac{5N + 7}{26}$$

which implies that $N \equiv 9 \bmod 26$. Choosing N between 400 and 450, we have the unique $N = 425$ and thus $j_3 = 82$, $j_2 = 311$ and $j_1 = 246$. Hence the original locker numbers are 246, 736 and 932 which, after the change, will respectively be 736, 932 and 246 which satisfy what we want.

Possibility 2. The other possibility is when

$$j_1 = 3j_2 - 1, \qquad N + j_2 = 3j_3, \qquad 2N + j_3 = 3j_1 - 2.$$

Similar computation as in Possibility 1 yields $N = 425$, $j_2 = 115$, $j_3 = 180$ and $j_1 = 344$. But this means that one of the lockers will have number 1030 which is contrary to the assumption.

Therefore, the only possible locker numbers of the three friends are 246, 736 and 932.

Also solved by SAM BAETHGE, Science Academy, Austin, Texas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JOSEPH CALLAGHAN, student, University of Waterloo, Waterloo, Ontario; HAN PING DAVIN CHOR, student, Cambridge, MA, USA; TIM CROSS, King Edward's School, Birmingham, England; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California, USA; PETER HURTHIG, Columbia College, Burnaby, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, New York, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; P. PENNING, Delft, the Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; DAVID STONE, Georgia Southern University, Statesboro, Georgia, USA; and the proposer.

Many solvers mentioned that the other set of locker numbers arising from the problem is 344, 540 *and* 1030. *Some remarked that the value of* N *was* 425 *in both cases. However, apparently nobody noticed that these two triples of numbers enjoy a curious relationship:*

$$246 + 1030 \quad = \quad 736 + 540 \quad = \quad 932 + 344\ !$$

So now readers are challenged to figure out why this relationship is true.

When $N = 425$, *the problem says that the numbers* $246, 736, 932$ *are interchanged when the lockers are renumbered. So let's call this set of numbers a "swapset" for* $N = 425$; *that is, for a particular* N, *a swapset is* *any set of numbers which get swapped among each other by the renumbering. We want true swapping; so we don't allow the sets* $\{1\}$ *or* $\{3N\}$ *(or the "middle" locker* $\{(3N+1)/2\}$ *when* N *is odd), which are obviously unchanged by the renumbering, to be in swapsets. Lots of problems concerning swapsets could be looked at. For example, one of the solvers (Stone) points out that there are no swapsets of two numbers when* $N = 425$, *but there are when* $N = 427$: *lockers* 161 *and* 481 *get swapped. Which values of* N *have swapsets of size two? Here's another problem. It's clear that the set of all numbers from* 1 *to* $3N$, *minus the two or three numbers that stay the same, will be a swapset for every* N. *But are there any numbers* N *which have* **no other** *swapsets? If so, can you describe all such* N*?*

2127. [1996: 123] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is an acute triangle with circumcentre O, and D is a point on the minor arc AC of the circumcircle ($D \neq A, C$). Let P be a point on the side AB such that $\angle ADP = \angle OBC$, and let Q be a point on the side BC such that $\angle CDQ = \angle OBA$. Prove that $\angle DPQ = \angle DOC$ and $\angle DQP = \angle DOA$.

Solution by Florian Herzig, student, Perchtoldsdorf, Austria.

First I prove that B is an excentre of $\triangle PDQ$.

$$\begin{aligned} \angle ABC &= 180^\circ - \angle ADC \\ &= 180^\circ - (\angle ADP + \angle CDQ + \angle PDQ) \\ &= 180^\circ - (\angle CBO + \angle ABO + \angle PDQ) \\ &= 180^\circ - \angle ABC - \angle PDQ, \qquad (1) \\ \Rightarrow \angle ABC &= 90^\circ - \frac{\angle PDQ}{2}, \end{aligned}$$

$$\angle PDB = \angle ADB - \angle ADP = \angle ACB - \angle OCB = \angle ACO,$$

and

$$\angle QDB = \angle CDB - \angle CDQ = \angle CAB - \angle OAB = \angle CAO.$$

Since $\triangle OAC$ is isosceles, we have that $\angle PDB = \angle QDB$ and thus BD is the internal angle bisector of $\angle PDQ$. (2)

What is more, we know that, in any $\triangle XYZ$, the excentre, M, (whose excircle touches YZ), is exactly the point on the internal angle bisector of $\angle YXZ$ outside the triangle for which

$$\begin{aligned} \angle YMZ &= 180^\circ - \angle MZY - \angle MYZ \\ &= \frac{\angle Y}{2} + \frac{\angle Z}{2} = \frac{180^\circ - \angle X}{2} = 90^\circ - \frac{\angle X}{2}. \end{aligned}$$

Therefore B is an excentre of $\triangle PDQ$ because of (1) and (2). Then BP and BQ are the external angle bisectors of $\angle DPQ$ and $\angle DQP$, respectively, whence

$$\angle APD = \angle BPQ \qquad \text{and} \qquad \angle CQD = \angle BQP. \qquad (3)$$

Starting with

$$\angle BOC = 2\angle BDC$$

we obtain

$$\begin{aligned}
180^\circ - 2\angle OBC &= 2\angle BDC, \\
90^\circ - \angle OBC &= \angle BDC, \\
180^\circ - \angle BDC &= 90^\circ + \angle OBC, \\
\angle BCD + \angle DBC &= 90^\circ + \angle ADP, \\
(180^\circ - \angle DAP) + \angle DBC &= 90^\circ + (180^\circ - \angle DAP - \angle APD), \\
\angle DBC &= 90^\circ - \angle APD, \\
\angle DOC &= 180^\circ - (\angle APD + \angle BPQ) \\
&\qquad\qquad\qquad\qquad [\text{ because of (3) }] \\
&= \angle DPQ,
\end{aligned}$$

and analogously $\angle DOA = \angle DQP$.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; HAN PING DAVIN CHOR, student, Cambridge, MA, USA; P. PENNING, Delft, the Netherlands; WALDEMAR POMPE, student, University of Warsaw, Poland; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

2128. [1996: 123] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABCD$ is a square. Let P and Q be interior points on the sides BC and CD respectively, and let E and F be the intersections of PQ with AB and AD respectively. Prove that

$$\pi \le \angle PAQ + \angle ECF < \frac{5\pi}{4}.$$

Solution by Heinz-Jürgen Seiffert, Berlin, Germany.

In cartesian coordinates, let $A = (0,0)$, $B = (1,0)$, $C = (1,1)$, $D = (0,1)$, $P = (1,p)$ and $Q = (q,1)$, where $0 < p, q < 1$.

Then $E = \left(\frac{1-pq}{1-p}, 0\right)$ and $F = \left(0, \frac{1-pq}{1-q}\right)$, $\tan\angle PAB = p$, $\tan\angle DAQ = q$, $\tan\angle DCF = FD = \frac{q(1-p)}{1-q}$, $\tan\angle BCE = BE = \frac{p(1-q)}{1-p}$.

Since

$$\angle PAQ = \frac{\pi}{2} - \angle PAB - \angle DAQ \text{ and } \angle ECF = \frac{\pi}{2} + \angle DCF + \angle BCE,$$

it follows that

$$\begin{aligned}
&\angle PAQ + \angle ECF \\
&\quad = \pi + \arctan\frac{q(1-p)}{1-q} - \arctan q + \arctan\frac{p(1-q)}{1-p} - \arctan p \\
&\quad = \pi + \arctan\left(\frac{(1-pq)(p-q)^2}{(1-p)(1-q)(1-pq)^2 + (p(1-q)^2 + q(1-p)^2)(p+q)}\right)
\end{aligned}$$

by the addition formula for arctangents. Since $0 < p, q < 1$, it suffices to show that

$$0 \leq (1-pq)(p-q)^2 < (p(1-q)^2 + q(1-p)^2)(p+q).$$

The left inequality is obviously true, while the right follows from the identity

$$(p(1-q)^2+q(1-p)^2)(p+q) = (1-pq)(p-q)^2+2pq((1-p)^2+(1-q)^2).$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; NIELS BEJLEGAARD, Stavanger, Norway; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JOSEPH CALLAGHAN, student, University of Waterloo; RICHARD I. HESS, Rancho Palos Verdes, California, USA; VICTOR OXMAN, University of Haifa, Haifa, Israel; and the proposer.

2130. [1996: 123] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

A and B are fixed points, and ℓ is a fixed line passing through A. C is a variable point on ℓ, staying on one side of A. The incircle of ΔABC touches BC at D and AC at E. Show that line DE passes through a fixed point.

Solution by Mitko Kunchev, Baba Tonka School of Mathematics, Rousse, Bulgaria.

We choose the point P on ℓ with $AP = AB$. Let C be an arbitrary point of ℓ, different from P but on the same side of A. The incircle of $\triangle ABC$ touches the sides BC, AC, AB in the points D, E, F respectively. Let ED meet PB in the point Q. According to Menelaus' Theorem applied to $\triangle CBP$ and the collinear points E, D, Q, we get

$$\frac{PE}{EC} \cdot \frac{CD}{DB} \cdot \frac{BQ}{QP} = 1. \qquad (1)$$

We have $EC = CD$ (because they are tangents from C). Similarly, $AF = AE$, so that $FB = EP$ (since $AB = AP$). But also, $FB = DB$, so that $DB = PE$. Setting $EC = CD$ and $DB = PE$ in (1), we conclude that $BQ = QP$; therefore Q is the mid-point of BP. Hence the line DE passes through the fixed point Q.

Also solved by NIELS BEJLEGAARD, Stavanger, Norway; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; P. PENNING, Delft, the Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA (two solutions); and the proposer.

Seimiya and Yiu used the same argument as Kunchev. Seimiya mentions that the result is easily shown to hold also when C coincides with P (even though the featured argument breaks down). Yiu extends the result to include excircles: The line joining the points where an excircle touches the segment BC and the line ℓ also passes through Q.

2131. [1996: 124] *Proposed by Hoe Teck Wee, Singapore.*

Find all positive integers $n > 1$ such that there exists a cyclic permutation of $(1, 1, 2, 2, \ldots, n, n)$ satisfying:

(i) no two adjacent terms of the permutation (including the last and first term) are equal; and

(ii) no block of n consecutive terms consists of n distinct integers.

Solution by the proposer.

It is clear that 2 does not have the desired property.

Suppose 3 has the specified property. So there exists a permutation of $(1, 1, 2, 2, 3, 3)$ satisfying the two conditions. WLOG assume that the first term is 1. From (ii) we know that the second term is not 1, say it is 2. From (i) the third term must be 1. From (i) and (ii) the fourth term must be 2. This leaves the two 3s as the last two terms, contradicting (i).

Suppose 4 has the specified property. So there exists a permutation of $(1, 1, 2, 2, 3, 3, 4, 4)$ satisfying the two conditions. Arrange these eight (permuted) numbers in a circle in that order so that they are equally spaced. Then the two conditions still hold. Now consider any four consecutive numbers on the circle. If they consist of only two distinct integers, we may assume by (i) that WLOG these four numbers are $1, 2, 1, 2$ in that order, and that the other four numbers are $3, 4, 3, 4$. Then (ii) does not hold. If they consist of three distinct integers, by (i) and (ii) we may assume WLOG that these four numbers are (a) $1, 2, 3, 1$ or (b) $1, 2, 1, 3$ or (c) $1, 2, 3, 2$, in these orders. By reversing the order, (c) reduces to (b). Next consider (a). If the next number is 2, then by (ii) we have $1, 2, 3, 1, 2, 3$, and the two 4s are adjacent, contradicting (i). If the next number is 3, reversing the order to obtain $3, 1, 3, 2, 1$ reduces it to (b). Finally consider (b). By (i) and (ii) the next number must be 2, followed by 3, so the two 4s are adjacent, contradicting (ii).

Next consider the following permutation for $n > 4$:

$$(4, 5, \ldots, n, 1, 2, 3, 2, 3, 4, 5, \ldots, n).$$

Clearly, (i) is satisfied. (ii) follows from the fact that there does not exist a set of four consecutive terms which is a permutation of $(1, 2, 3, 4)$.

In conclusion, the answer is: $n > 4$.

Also solved by HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; and DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA. There was one incomplete solution.

2132. [1996: 124] *Proposed by Šefket Arslanagić, Berlin, Germany.*

Let n be an even number and z a complex number.

Prove that the polynomial $P(z) = (z+1)^n - z^n - n$ is not divisible by $z^2 + z + n$.

I. *Solution by Richard I. Hess, Rancho Palos Verdes, California, USA.*

Let $Q(z) = z^2 + z + n$. For $n = 0$ or 1, we have that $P(z) = 0$, which is clearly divisible by $Q(z)$. For *any* $n > 1$, suppose that $P(z)$ is divisible by $Q(z)$. Then $Q(n)$ divides $P(n)$.

But $Q(n) = n(n+2) \equiv 0 \pmod{n}$, while $P(n) = (n+1)^n - n^2 - n \equiv 1 \pmod{n}$. Thus $P(z)$ is not divisible by $Q(z)$.

II. *Composite solution by F.J. Flanigan, San Jose State University, San Jose, California, USA and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Let $D(z) = z^2 + z + n$. If $n = 0,\ 1$, then $P(z) = 0$, which is divisible by $D(z)$. If $n = 2$, then $P(z) = 2z - 1$, which is clearly not divisible by $z^2 + z + 2$.

For $n > 2$, suppose that $D(z)$ divides $P(z)$. Then, since $D(z)$ is monic, $P(z) = Q(z)D(z)$, where $Q(z)$ is a polynomial of degree $n-3$ with integer coefficients. Thus $P(0) = Q(0)D(0)$, or $1 - n = nQ(0)$, which is clearly impossible.

III. *Solution and generalization by Heinz-Jürgen Seiffert, Berlin, Germany.*

Let $n \geq 2$ be an even integer. We shall prove that if a, b, c, are complex numbers such that $a \neq 0$, then the polynomial

$$P(z) = (z+b)^n - z^n - a$$

is not divisible by $z^2 + bz + c$.

The proposer's result, which does not hold for $n = 0$, is obtained when $a = c = n$ and $b = 1$.

Let z_1 and z_2 denote the (not necessarily distinct) roots of $z^2 + bz + c$. The $z_1 + z_2 = -b$, so that $P(z_1) = z_2^n - z_1^n - a$, and $P(z_2) = z_1^n - z_2^n - a$. Since $P(z_1) + P(z_2) = -2a \neq 0$, our result follows.

The example $(z+1)^6 - z^6 = (z^2 + z + 1)(6z^3 + 9z^2 + 5z + 1)$ shows that the condition $a \neq 0$ cannot be dropped.

Also solved by: CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; NORVALD MIDTTUN (two solutions), Royal Norwegian Naval Academy, Norway; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; and the proposer.

Besides the solvers listed in Solutions I *and* II *above, only Janous observed and showed that the assertion holds for all $n \geq 2$.*

2134*. [1996: 124] *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

Let $\{x_n\}$ be an increasing sequence of positive integers such that the sequence $\{x_{n+1} - x_n\}$ is bounded. Prove or disprove that, for each integer $m \geq 3$, there exist positive integers $k_1 < k_2 < \ldots < k_m$, such that $x_{k_1}, x_{k_2}, \ldots, x_{k_m}$ are in arithmetic progression.

Solution by David R. Stone, Georgia Southern University, Statesboro, Georgia, USA, and Carl Pomerance, University of Georgia, Athens, Georgia, USA.

An old and well-known result of van der Waerden [4] is that if the natural numbers are partitioned into two subsets, then one of the subsets has arbitrarily long arithmetic progressions. It is not very difficult to show [1] that van der Waerden's theorem has the following equivalent formulation:

> for every number B and positive integer m, there is a number $W(m, B)$ such that if $n \geq W(m, B)$ and $0 < a_1 < a_2 < \ldots < a_n$ are integers with each $a_{i+1} - a_i \leq B$, then m of the a_i's form an arithmetic progression.

Thus, for the problem as stated, if we let B be the bound on the differences $x_{n+1} - x_n$, then for any given $m \geq 3$, there exists a $W(m, B)$ with the property stated above. Then, for any $n \geq W(m, B)$, any finite subsequence of length n will have an arithmetic progression of length m as a sub-subsequence. That is, the original sequence contains infinitely many arithmetic progressions of length m.

In 1975, Szémeredi [3] proved a conjecture of Erdős and Turán which improves on van der Waerden's Theorem, relaxing the condition that the sequence's differences have a uniform upper bound, requiring only that the sequence have a positive upper density. Hence the problem posed here also follows from the theorem of Szémeredi, who, we believe, received (for this result) the highest cash prize ever awarded by Pál Erdős — $1,000.

Comment by the solvers.

Do we know how Pompe became interested in this problem?

References

[1] T.C. Brown, Variations on van der Waerden's and Ramsey's theorems, *Amer. Math. Monthly* **82** (1975), 993–995.

[2] Carl Pomerance, Collinear subsets of lattice point sequences — an analog of Szemeredi's Theorem, *J. Combin. Theory* **28** (1980), 140–149.

[3] E. Szemeredi, On sets of integers containing no k elements in arithmetic progression, *Acta Arith.* **27** (1975), 199–245.

[4] B.L. van der Waerden, Beweis einer Baudetschen Vermutung, *Nieuw. Arch. Wisk.* **15** (1928), 212–216.

Also solved by THOMAS LEONG, Staten Island, NY, USA; and JOEL SCHLOSBERG, student, Hunter College High School, New York NY, USA; both using van der Waerden's theorem or its variation. Leong gave the reference: Ramsey Theory *by R.L. Graham, B.L. Rothschild and J.H. Spencer. Schlosberg remarked that van der Waerden's theorem was discussed in the July 1990 issue of* Scientific American.

The proposer showed that van der Waerden's theorem follows easily from the statement of his problem. His intention (and hope) was to find a proof independent of van der Waerden's theorem. This would establish a new "proof" of the latter. In view of his comment and the solution above, it should be obvious that the two statements are equivalent, and hence such a proof is unlikely.

2135. [1996: 124] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

Let n be a positive integer. Find the value of the sum

$$\sum_{k=1}^{\lfloor n/2 \rfloor} \frac{(-1)^k(2n-2k)!}{(k+1)!(n-k)!(n-2k)!}.$$

Solution by Florian Herzig, student, Perchtoldsdorf, Austria. [Modified slightly by the editor.]

Let S_n denote the given summation. Note that S_1 is an "empty" sum, which we shall define to be zero. We prove that $S_n = -\binom{2n}{n+2}$.

Since $\binom{2}{3} = 0$, this is true for $n = 1$. Assume that $n \geq 2$. Following standard convention, for $k = 0, 1, 2, \ldots$, let $[x^k]\,(f(x))$ denote the coefficient of x^k in the series expansion of the function $f(x)$. Let

$$P(x) = (1 - x^2)^{n+1}(1-x)^{-(n+1)}.$$

Then, by the binomial expansion, its generalization (by Newton), and the well-known fact that $\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$, we have:

$$\begin{aligned} [x^{2k+2}]\left((1-x^2)^{n+1}\right) &= \left[(x^2)^{k+1}\right]\left((1-x^2)^{n+1}\right) \\ &= (-1)^{k+1}\binom{n+1}{k+1}, \end{aligned}$$

for $k = -1, 0, 1, 2, \ldots$, and

$$\begin{aligned} [x^{n-2k}]\left((1-x)^{-(n+1)}\right) &= (-1)^{n-2k}\binom{-n-1}{n-2k} \\ &= \binom{2n-2k}{n-2k} \end{aligned}$$

for $k \le n/2$. Hence

$$\begin{aligned} [x^{n+2}]\,(P(x)) &= [x^{2k+2} \cdot x^{n-2k}]\,(P(x)) \\ &= \sum_{k=-1}^{\lfloor n/2 \rfloor} (-1)^{k+1}\binom{n+1}{k+1}\binom{2n-2k}{n-2k}. \end{aligned}$$

On the other hand, since $P(x) = (1+x)^{n+1}$, we have $[x^{n+2}]\,(P(x)) = 0$. Therefore

$$\sum_{k=-1}^{\lfloor n/2 \rfloor} (-1)^{k+1}\binom{n+1}{k+1}\binom{2n-2k}{n-2k} = 0.$$

Since

$$S_n = -\frac{1}{n+1}\sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^{k+1}\binom{n+1}{k+1}\binom{2n-2k}{n-2k},$$

we get

$$\begin{aligned} S_n &= -\frac{1}{n+1}\left\{\binom{n+1}{1}\binom{2n}{n} - \binom{n+1}{0}\binom{2n+2}{n+2}\right\} \\ &= \frac{(2n+2)!}{(n+1)(n+2)!n!} - \frac{(2n)!}{n!n!} \\ &= \frac{\{2(2n+1) - (n+2)(n+1)\}}{(n+2)!n!} \times (2n)! \\ &= \frac{-n(n-1)\,(2n)!}{(n+2)!n!} = \frac{-(2n)!}{(n+2)!(n-2)!} \\ &= -\binom{2n}{n+2}. \end{aligned}$$

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer. The correct answer, without a proof, was sent in by RICHARD I. HESS, Rancho Palos Verdes, California, USA.

If, in the given summation, one lets k start from zero (this was, in fact, the proposer's original idea), then it is easy to see that the answer becomes

$$\frac{1}{n+2}\binom{2n+2}{n+1},$$

the $(n+1)$-th Catalan number.

2136. [1996: 124] *Proposed by G.P. Henderson, Campbellcroft, Ontario.*

Let a, b, c be the lengths of the sides of a triangle. Given the values of $p = \sum a$ and $q = \sum ab$, prove that $r = abc$ can be estimated with an error of at most $r/26$.

Solution by P. Penning, Delft, the Netherlands.

Scale the triangle down by a factor $(a+b+c)$. The value of p then becomes 1, the value of q becomes $Q = \dfrac{q}{(a+b+c)^2}$, and $R = \dfrac{r}{(a+b+c)^3}$.

Introduce $s = \dfrac{a+b}{2}$ and $v = \dfrac{a-b}{2}$:

$$a = s+v; \qquad b = s-v; \qquad c = 1-2s;$$

$$Q = 2s - 3s^2 - v^2; \qquad R = (1-2s)(s^2-v^2).$$

Since a, b, c, represent the sides of a triangle, we must require

$$0 < c < a+b \qquad \text{and} \qquad -c < a-b < c.$$

[Ed: in other words, the triangle is not degenerate — a case which must be discarded as inappropriate.]

This translates to

$$\frac{1}{4} < s < \frac{1}{2} \qquad \text{and} \qquad |v| < \frac{1}{2} - s.$$

Lines of constant Q are ellipses in the s–v plane, with centre $s = \frac{1}{3}$, $v = 0$. So we write:

$$s = \frac{1}{3} + A\cos(x); \quad v = \sqrt{3}A\sin(x),$$

with $A = \dfrac{\sqrt{(1-3Q)}}{3}$ replacing Q.

Very symmetrical expressions are now obtained for a, b, c:

$$a = \frac{1}{3} - 2A\cos(120° + x); \quad b = \frac{1}{3} - 2A\cos(120° - x); \quad c = \frac{1}{3} - 2A\cos(x).$$

$$R = abc - \frac{1}{27} - A^2 - 2A^3\cos(3x).$$

Now, R is minimal for $x = 0$:

$$R_{\text{min}} = \frac{1}{27} - A^2 - 2A^3.$$

R is maximal for $x = 60°$ provided that $a \le \frac{1}{2}$, $A \le \frac{1}{12}$:

$$R_{\text{max}} = \frac{1}{27} - A^2 + 2A^3.$$

For $\frac{1}{12} \le A \le \frac{1}{6}$, we have $\cos(120° + x_{\text{max}}) = -\frac{1}{12A}$, since the maximum of a is $\frac{1}{2}$. So

$$\cos(3x_{\text{max}}) = -\frac{4}{(12A)^3} + \frac{3}{(12A)};$$

$$R_{\text{max}} = \frac{1}{27} - A^2 - 2A^3\left(-\frac{4}{(12A)^3} + \frac{3}{(12A)}\right) = \frac{1}{24} - \frac{3A^2}{2}.$$

We must determine the reciprocal of the relative spread in R:

$$F = \frac{R_{\text{max}} + R_{\text{min}}}{R_{\text{max}} - R_{\text{min}}}.$$

For $A \le \frac{1}{12}$, we have

$$F = \frac{\frac{1}{27} - A^2}{2A^3}.$$

The minimum in F is reached at $A = \frac{1}{12}$, so that $F_{\text{min}} = 26$.

For $\frac{1}{12} \le A \le \frac{1}{6}$, both R_{max} and R_{min} are zero for $A = \frac{1}{6}$. So

$$\begin{aligned} R_{\text{min}} &= \left(\frac{1}{6} - A\right)\left(\frac{2}{9} + \frac{4A}{3} + 2A^2\right), \\ R_{\text{max}} &= \left(\frac{1}{6} - A\right)\left(\frac{1+6A}{4}\right). \end{aligned}$$

The minimum in F is also at $A = \frac{1}{6}$ and yields the same value for $F_{\text{min}} = 26$.

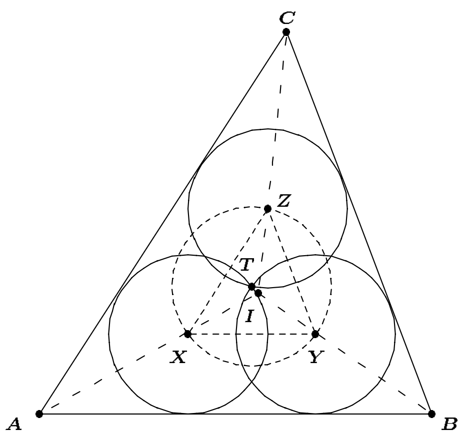
Also solved by NIELS BEJLEGAARD, Stavanger, Norway; and the proposer. One incorrect submission was received in that the sender assumed that a degenerate triangle disproved the proposition.

2137. [1996: 124, 317; 1997: 48] *Proposed by Aram A. Yagubyants, Rostov na Donu, Russia.*

Three circles of (equal) radius t pass through a point T, and are each inside triangle ABC and tangent to two of its sides. Prove that:

(i) $t = \dfrac{rR}{R+r}$, (ii) T lies on the line segment joining the centres of the circumcircle and the incircle of $\triangle ABC$.

Solution by Gottfried Perz, Pestalozzigymnasium, Graz, Austria.



We denote the centres of the three circles by X, Y and Z. Since the three circles pass through a common point T and have equal radius t, it follows that X, Y and Z lie on the circle with centre T and radius t. Since each of the circles is tangent to two sides of $\triangle ABC$, it follows that X, Y and Z lie on the internal bisectors of $\angle A$, $\angle B$ and $\angle C$. Since AB is a common tangent of two intersecting circles with radius t, it follows that $AB \| XY$, and analogously, we have $YZ \| BC$ and $ZX \| AC$. This implies that the lines AX, BY and CZ are bisectors of the angles of $\triangle XYZ$ as well, and so $\triangle ABC$ and $\triangle XYZ$ have the same incentre I.

Thus we conclude that triangles $\triangle ABC$ and $\triangle XYZ$ are homothetic with I as centre of similitude. This implies that:

(i) the ratio of the radii of the circumcircles of $\triangle ABC$ and $\triangle XYZ$ equals the ratio of the radii of the incircles of the triangles; that is

$$R : t \;=\; r : (r - t) \qquad Rr - Rt \;=\; rt$$
$$t(R + r) \;=\; Rr \qquad t = \frac{Rr}{R+r};$$

(ii) as corresponding points in the homothety, T (the circumcentre of $\triangle XYZ$) and the circumcentre of $\triangle ABC$ lie collinear with I, as desired.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; NIELS BEJLEGAARD, Stavanger, Norway; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; HAN PING DAVIN CHOR, student, Cambridge, MA, USA; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA;

P. PENNING, Delft, the Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA; and the proposer.

Janous has seen both parts of the problem before; although unable to provide a reference to part (i), *he reconstructed the argument that he had seen, which was much like our featured solution. He, among several others, noted that* (ii) *is essentially problem 5 of the 1981 IMO* [1981: 223], *solution on pp. 35–36 of M.S. Klamkin,* International Mathematical Olympiads 1979–1985, *MAA, 1986. See also the "generalization" 694* [1982: 314] *and the related problem 1808* [1993: 299].

2138. [1996: 169] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

ABC is an acute angle triangle with circumcentre O. AO meets the circle BOC again at A', BO meets the circle COA again at B', and CO meets the circle AOB again at C'.

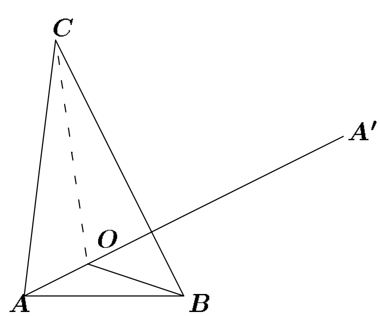
Prove that $[A'B'C'] \geq 4[ABC]$, where $[XYZ]$ denotes the area of triangle XYZ.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

There is an even sharper inequality:

$$[A'B'C'] \geq 3\sqrt[3]{\prod_{cyclic} \frac{\cos^2(B-C)}{\sin A \sin 2A}}\,[ABC].$$

For this, we first represent $[A'B'C']$ as a function of A, B, C and R (the circumradius).



We have: $\angle AOB = 2C$, so that $\angle BOA' = 180^\circ - 2C$ and $\angle BA'O = \angle BCO = 90^\circ - A$.
[Both angles subtend the line BO on circle BOC!] Thus,

$$\begin{aligned}\angle A'BO &= 180^\circ - (180^\circ - 2C) - (90^\circ - A)\\ &= A + 2C - 90^\circ\\ &= 180^\circ - B + C - 90^\circ\\ &= 90^\circ - (B - C).\end{aligned}$$

Hence, using the law of sines in $\triangle OBA'$, we get

$$\frac{|OA'|}{\sin(90^\circ - (B-C))} = \frac{R}{\sin(90^\circ - A)};$$

that is, $|OA'| = \dfrac{R\cos(B-C)}{\cos A}$.

Similarly, $|OB'| = \dfrac{R\cos(C-A)}{\cos B}$ and $|OC'| = \dfrac{R\cos(A-B)}{\cos C}$.
Now, since $\angle A'OB' = \angle AOB = 2C$, we get, via the trigonometric area formula of triangles, that

$$\begin{aligned}[A'B'O] &= \tfrac{1}{2}|OA'|\cdot|OB'|\cdot\sin(\angle A'OB') \\ &= \frac{R^2}{2}\frac{\cos(B-C)\cos(C-A)}{\cos A\cos B}\sin 2C,\end{aligned}$$

and similarly for $[B'C'O]$ and $[C'A'O]$. Thus

$$\begin{aligned}[A'B'C'] &= [A'B'O]+[B'C'O]+[C'A'O] \\ &= \left(\sum_{cyclic}\frac{\cos(C-A)\cos(C-B)}{\cos A\cos B}\sin 2C\right)\times\frac{R^2}{2}. \qquad (1)\end{aligned}$$

Next, we recall the formula

$$[ABC] = 2R^2\prod_{cyclic}\sin A \qquad (2)$$

From (1), we get, via the arithmetic-geometric-mean inequality:

$$\begin{aligned}&\sum_{cyclic}\frac{\cos(C-A)\cos(C-B)}{\cos A\cos B}\sin 2C \\ &\geq 3\left[\prod_{cyclic}\left(\frac{\cos^2(B-C)}{\cos^2 A}\cdot\sin 2A\right)\right]^{\frac{1}{3}} \\ &= 6\left[\prod_{cyclic}\frac{\cos^2(B-C)}{\cos A}\cdot\sin A\right]^{\frac{1}{3}} \\ &= 6\left[\prod_{cyclic}\frac{\cos^2(B-C)}{\sin^2 A\cos A}\right]^{\frac{1}{3}}\times\prod_{cyclic}\sin A \\ &= 12\left[\prod_{cyclic}\frac{\cos^2(B-C)}{\sin A\sin 2A}\right]^{\frac{1}{3}}\times\prod_{cyclic}\sin A;\end{aligned}$$

so that, using (1) and (2),

$$[A'B'C'] \geq 3\left[\prod_{cyclic}\frac{\cos^2(B-C)}{\sin A\sin 2A}\right]^{\frac{1}{3}}\times[ABC]$$

as claimed.

Finally, we recall the angle inequality:

$$\prod_{cyclic} \cos^2(B-C) \geq \frac{512}{27} \prod_{cyclic} (\sin^2 A \cos A) \left[= \frac{64}{27} \prod_{cyclic} (\sin A \sin 2A) \right]$$

which is valid for all triangles (but interesting only for acute triangles) with equality if and only if $A = B = C = 60^\circ$, or the degenerate cases with two of A, B, C being right angles. This immediately yields

$$\left[\prod_{cyclic} \frac{\cos^2(B-C)}{\sin A \sin 2A} \right]^{\frac{1}{3}} \geq \sqrt[3]{\frac{64}{27}} = \frac{4}{3},$$

and the original inequality follows.

Also solved by D.J. SMEENK, Zaltbommel, the Netherlands.

2139. [1996: 169, 219] *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

Point P lies inside triangle ABC. Let D, E, F be the orthogonal projections from P onto the lines BC, CA, AB, respectively. Let O' and R' denote the circumcentre and circumradius of the triangle DEF, respectively. Prove that

$$[ABC] \geq 3\sqrt{3}R'\sqrt{R'^2 - (O'P)^2},$$

where $[XYZ]$ denotes the area of triangle XYZ.

Solution by the proposer.

Let $\mathcal{C}$ denote the circumcircle of DEF. Let P' be the symmetric point to P with respect to O'. Let $\mathcal{E}$ be the ellipse with foci P and P' tangent (internally) to $\mathcal{C}$. The diameter of the ellipse $\mathcal{E}$ is $2R'$, and its area is equal to $\pi R'\sqrt{R'^2 - (O'P)^2}$. Since the locus of the orthogonal projections from P onto tangents to the ellipse $\mathcal{E}$ is the circle $\mathcal{C}$, the sides of ABC must be tangent to $\mathcal{E}$. Thus $\mathcal{E}$ is inscribed in the triangle ABC. Let L be an affine mapping which takes $\mathcal{E}$ to some circle of radius R, and let it take the triangle ABC to the triangle $A'B'C'$. Since L preserves the ratio of areas, we obtain

$$\frac{[ABC]}{\pi R'\sqrt{R'^2 - (O'P)^2}} = \frac{[ABC]}{\text{area of } \mathcal{E}} = \frac{[A'B'C']}{\pi R^2} \geq \frac{3\sqrt{3}R^2}{\pi R^2}, \qquad (1)$$

since among all triangles circumscribed about the given circle, the one of smallest area is the equilateral triangle. Thus (1) is equivalent to the desired inequality, so we are done.

Remarks: The same proof works also for an n-gon which has an interior point whose projections onto the sides of the n-gon are concyclic. The analogous inequality will be

$$[A_1 \ldots A_n] \geq n \tan\left(\frac{\pi}{n}\right) R' \sqrt{R'^2 - (O'P)^2}\,.$$

Note that as a special case, when the n-gon has the incircle and $P = O'$ we obtain the well-known result that *among all n-gons circumscribed about a given circle, the one of smallest area is the regular one*, though it is used in the proof.

2140. [1996: 169] *Proposed by K.R.S. Sastry, Dodballapur, India.*

Determine the quartic $f(x) = x^4 + ax^3 + bx^2 + cx - c$ if it shares two distinct integral zeros with its derivative $f'(x)$ and $abc \neq 0$.

Solution by Florian Herzig, student, Perchtoldsdorf, Austria.

Let the zeros of $f(x)$ be the integers p and q; without loss of generality $p > q$. It is a well-known theorem that if a polynomial $Q(x)$ divides the polynomial $P(x)$ as well as the derivative $P'(x)$, then $(Q(x))^2$ divides $P(x)$. Applying the theorem for this problem, we obtain

$$f(x) = (x-p)^2(x-q)^2 = x^4 + ax^x + bx^2 + cx - c.$$

Comparing coefficients of x and the constant term yields

$$0 = c + (-c) = -2(p+q)pq + p^2q^2.$$

As $pq = 0$ implies $abc = 0$, we may divide by pq

$$\begin{aligned} pq - 2p - 2q &= 0 \\ (p-2)(q-2) &= 4 = 4 \cdot 1 = (-1)(-4) \end{aligned}$$

Hence $(p,q) = (6,3) \vee (1,-2)$ (since $p \neq q$) and the two possible polynomials are

$$\begin{aligned} f_1(x) &= (x-6)^2(x-3)^2 = x^4 - 18x^3 + 117x^2 - 324x + 324, \\ f_2(x) &= (x+2)^2(x-1)^2 = x^4 + 2x^3 - 3x_4^2x + 4. \end{aligned}$$

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; F.J. FLANIGAN, San Jose State University, San Jose, California, USA; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State

University, Big Rapids, Michigan, USA; BEATRIZ MARGOLIS, Paris, France; L. RICE, Woburn Collegiate, Scarborough, Ontario; HARRY SEDINGER, St. Bonaventure University, St. Bonaventure, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; SKIDMORE COLLEGE PROBLEM GROUP, Saratoga Springs, New York, USA; DIGBY SMITH, Mount Royal College, Calgary, Alberta; and the proposer. There were eight incorrect or incomplete solutions.

Do you know the equation of this graph?

Contributed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.