

2026 Canada Lynx Mathematical Competition

Official Solutions



A competition of the Canadian Mathematical Society.

Part A: 5 marks each

1. If x is the positive integer for which $x^2 = 13^2 - 4^2 - 3^2$, what is x ?
(A) 4 (B) 6 (C) 8 (D) 10 (E) 12

Solution: We have $x^2 = 13^2 - 4^2 - 3^2 = 13 \times 13 - 4 \times 4 - 3 \times 3 = 169 - 16 - 9 = 144$.

Since $x^2 = 144 = 12^2$ and x is a positive integer, we must have $x = 12$.

We can also solve this problem by noting that $3^2 + 4^2 = 5^2$ and $5^2 + 12^2 = 13^2$, which are the smallest two “Pythagorean triplets”. This implies that $13^2 = 5^2 + 12^2 = (3^2 + 4^2) + 12^2$, from which we notice that $12^2 = 13^2 - 4^2 - 3^2$.

Yet another alternate solution is

$$x^2 = 13^2 - 3^2 - 4^2 = (13 - 3)(13 + 3) - 4^2 = 10 \cdot 16 - 16 = 9 \cdot 16 = (3 \cdot 4)^2 = 12^2$$

Answer: (E)

2. If $\frac{1}{2} + \frac{3}{4} + \frac{5}{6} = \frac{n}{12}$, what is n ?

- (A) 9 (B) 13 (C) 19 (D) 25 (E) 29

Solution: Let us represent each fraction with 12 as its denominator.

$$\text{The left side is } \frac{1}{2} + \frac{3}{4} + \frac{5}{6} = \frac{6}{12} + \frac{9}{12} + \frac{10}{12} = \frac{6+9+10}{12} = \frac{25}{12}.$$

We have now shown that this equation simplifies to $\frac{25}{12} = \frac{n}{12}$, from which we see that $n = 25$.

We can also solve this problem by multiplying both sides of the equation by 12, which gives us $12 \cdot \frac{1}{2} + 12 \cdot \frac{3}{4} + 12 \cdot \frac{5}{6} = n$, which implies that $6 + 9 + 10 = n$, or $n = 25$.

Answer: (D)

3. Let $S = 2026 + 202.6 + 20.26 + 2.026$. Which of these integers is closest to S ?

- (A) 2050 (B) 2250 (C) 2450 (D) 2650 (E) 2850

Solution: Let us represent each number using three decimal places. Adding the four numbers, we get:

$$\begin{array}{r} 2026.000 \\ +202.600 \\ +20.260 \\ +2.026 \\ \hline = 2250.866 \end{array}$$

Of the five options, 2250 is the integer closest to $S = 2250.866$.

We can also solve this problem by estimating the sum S . We see that the first two numbers add up to approximately $2000 + 200 = 2200$, and that S is slightly larger than 2200 since the remaining two numbers contribute little to the sum S . This implies that the answer has to be the option that is slightly more than 2200.

Answer: (B)

4. The line with equation $y = 3x - 7$ intersects the line with equation $y = -\frac{1}{2}x + 7$ at the point (a, b) . What is the value of $\frac{b}{a}$?

- (A) $\frac{2}{7}$ (B) $\frac{4}{5}$ (C) 1 (D) $\frac{5}{4}$ (E) $\frac{7}{2}$

Solution: Let (a, b) be the intersection point of the two lines. We have $b = 3a - 7$ and $b = -\frac{a}{2} + 7$.

Therefore, $3a - 7 = -\frac{a}{2} + 7$, which implies $3a + \frac{a}{2} = 7 + 7$, or $\frac{7a}{2} = 14$. Multiplying both sides by 2, we get $7a = 28$, or $a = 4$. Substituting $a = 4$ into the equation $b = 3a - 7$, we find that $b = 3 \cdot 4 - 7 = 12 - 7 = 5$, and so $(a, b) = (4, 5)$ must be the intersection point of the two lines.

It follows that $\frac{b}{a} = \frac{5}{4}$.

We can also get the correct answer in a simpler way without having to solve for a and b . From above, we have $b = 3a - 7$ and $b = -\frac{a}{2} + 7$. Adding the two equations, we get $2b = 3a - \frac{a}{2} = \frac{5a}{2}$, which implies that $2b = \frac{5a}{2}$, which is equivalent to $\frac{b}{a} = \frac{5}{4}$.

Answer: (D)

5. Let $\{10, 12, 14, \dots, 94, 96, 98\}$ be the set of even two-digit numbers. How many even two-digit numbers contain the digit 6?

(A) 13 (B) 14 (C) 15 (D) 16 (E) 17

Solution: We are looking for all even two-digit numbers that contain the digit 6.

Let's break our analysis into two cases: when 6 is the first digit, and when 6 is the last digit.

There are 5 options for when 6 is the first digit: 60, 62, 64, 66, 68.

There are 9 options for when 6 is the last digit: 16, 26, 36, 46, 56, 66, 76, 86, 96.

However, notice that the number 66 appears in both lists, and so we need to ensure this option is not counted twice. To find the correct answer, we add 5 and 9 and then subtract 1 to account for the number that appears twice. (This is an illustration of the Inclusion-Exclusion Principle used in combinatorics problems.)

Therefore, the correct answer is $5 + 9 - 1 = 13$.

Answer: (A)

Part B: 5 marks each

1. The distance from Toronto to Ottawa is exactly 430 kilometres. Sally left Toronto at exactly 12:00 and drove towards Ottawa at a constant speed of 100 kilometres per hour. When she passed a certain Tim Hortons coffee shop, she accelerated to 120 kilometres per hour and drove the rest of the trip at this speed.

If Sally's trip took 4 hours, at what time did Sally pass the Tim Hortons coffee shop?

- (A) 2:00 (B) 2:15 (C) 2:30 (D) 2:45 (E) 3:00

Solution: Suppose Sally reaches the coffee shop after exactly x hours. Since her trip from Toronto to Ottawa took exactly 4 hours, she drove for x hours at 100 kilometres per hour, and for $4 - x$ hours at 120 kilometres per hour.

Since the distance from Toronto to Ottawa is 430 kilometres, we have $430 = 100x + 120(4 - x) = 100x + 480 - 120x = 480 - 20x$, which simplifies to $20x = 480 - 430 = 50$, or $x = \frac{50}{20} = 2.5$.

Thus, Sally reached the coffee shop exactly 2.5 hours after she began to drive. Since she left Toronto at 12:00, she arrived at Tim Hortons at 2:30.

Answer: (C)

2. There is one real value of x for which $4^x - 2^{x+2} = 12$. What is the value of 8^x ?

- (A) 36 (B) 64 (C) 108 (D) 128 (E) 216

Solution: To simplify the problem, let $2^x = a$. Notice that $4^x = (2^2)^x = 2^{2x} = 2^x \cdot 2^x = a \cdot a = a^2$ and $2^{x+2} = 2^x \cdot 2^2 = a \cdot 4 = 4a$.

Therefore, our equation $4^x - 2^{x+2} = 12$ is equivalent to $a^2 - 4a = 12$, which is the same thing as $0 = a^2 - 4a - 12 = (a - 6)(a + 2)$. Thus, $a = 6$ or $a = -2$.

Since $a = 2^x$ where x is a real number, a cannot be negative. It follows

that $a = 6$, which implies $2^x = 6$.

At this point, we can solve for x and show that $x = \log_2 6$. But we don't need to do this as the question asks for the value of 8^x .

Since $8^x = (2^3)^x = 2^{3x} = (2^x)^3 = a^3$, we conclude that $8^x = 6^3 = 216$.

Answer: (E)

3. Let b be a 2-digit positive integer. The line $y = x + b$ intersects the parabola $y = x^2$ at points P and Q . If the distance of line segment PQ is $9\sqrt{2}$, what is the sum of the digits of b ?
- (A) 2 (B) 3 (C) 6 (D) 9 (E) 11

Solution: Given that the points P and Q lie on the parabola $y = x^2$, we must have $P = (p, p^2)$ and $Q = (q, q^2)$ for some real numbers p and q . Let's assume that $q > p$.

Furthermore, we are given that these two points lie on the straight line $y = x + b$. Therefore, $p^2 = p + b$ and $q^2 = q + b$.

Subtracting the second equation from the first, we have $p^2 - q^2 = (p + b) - (q + b) = p - q$, which implies $(p + q)(p - q) = p - q$. Therefore, either $p + q = 1$ or $p - q = 0$.

If $p - q = 0$, then P and Q represent the same point, and therefore its distance cannot be $9\sqrt{2}$. It follows that $p + q = 1$.

We now solve for p and q .

One approach is to let $q = 1 - p$, so that $Q = (1 - p, (1 - p)^2)$, and use the Pythagorean distance formula to solve for p . We know that the distance between points $P = (p, p^2)$ and $Q = (1 - p, (1 - p)^2)$ is $9\sqrt{2}$, and so $((1 - p) - p)^2 + ((1 - p)^2 - p^2)^2 = (9\sqrt{2})^2$. This simplifies to $(1 - 2p)^2 + (1 - 2p)^2 = 81 \cdot 2 = 162$, which implies $(1 - 4p + 4p^2) + (1 - 4p + 4p^2) = 162$, or $8p^2 - 8p - 160 = 0$. Dividing both sides by 8 and factoring, we get $0 = p^2 - p - 20 = (p - 5)(p + 4)$, and so $p = 5$ or $p = -4$.

Since we are assuming that $q > p$, we must have $p = -4$, and so $q = 1 - p = 5$.

Therefore, the two points have coordinates $(-4, 16)$ and $(5, 25)$.

The much faster solution is to notice that P and Q lie on the straight line $y = x + b$, which has slope 1. Therefore, if the line segment PQ has length $9\sqrt{2}$, this must be the hypotenuse of an isosceles right-angled triangle with side lengths $9, 9, 9\sqrt{2}$. So the horizontal distance (i.e., the “run”) between P and Q is equal to 9, which implies $q - p = 9$.

We already know $p + q = 1$ from above, and so adding this equation to $q - p = 9$ yields $(p + q) + (q - p) = 1 + 9$, which implies $2q = 10$, or $q = 5$. Since $p + q = 1$, we get $p = -4$, and so we conclude, as before, that $P = (-4, 16)$ and $Q = (5, 25)$.

Finally, if $y = x + b$ is the equation of the line through P and Q , we have $16 = -4 + b$ and $25 = 5 + b$, from which we conclude that $b = 20$. The sum of the digits of b is $2 + 0 = 2$.

Answer: (A)

4. Consider a 2 by 3 array of cells.

If we randomly place the numbers 1, 2, 3, 4, 5, 6 into this array, what is the probability that the product of each row is an even number?

- (A) $\frac{3}{4}$ (B) $\frac{4}{5}$ (C) $\frac{5}{6}$ (D) $\frac{9}{10}$ (E) $\frac{19}{20}$

Solution: Let p be the probability that the product of each row is an even number. Therefore $q = 1 - p$ is the probability that the product of (at least) one row is an odd number.

There are $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ ways we can place the six numbers into the array. There are three odd numbers and three even numbers in the set $\{1, 2, 3, 4, 5, 6\}$. To ensure one row multiplies to an odd number, we must have the three odd numbers in one row, and the three even numbers in the other row.

Here is one example for how one row can multiply to an odd number.

5	1	3
4	2	6

There are $3! = 3 \times 2 \times 1 = 6$ ways we can permute the odd numbers in the top row, and similarly $3! = 6$ ways we can permute the even numbers in the bottom row, leading to $6 \times 6 = 36$ cases. Similarly, there are $6 \times 6 = 36$ cases when the odd numbers are in the bottom row and the even numbers are in the top row.

We conclude that $q = \frac{36+36}{6!} = \frac{72}{720} = \frac{1}{10}$, and so $p = 1 - \frac{1}{10} = \frac{9}{10}$.

An alternate way to count the bad cases is as follows: there are 6 different ways of putting a number in the top right corner. After this, the remaining two numbers in the top row must have the same parity with it (and can be placed in 2×1 ways) and the three remaining numbers must be placed (in $3 \times 2 \times 1$ ways) in the second row. This gives us $6 \times 2 \times 1 \times 3 \times 2 \times 1 = 72$ cases when one of the rows multiplies to an odd number.

Here is a much faster solution: we represent the six numbers $\{1, 2, 3, 4, 5, 6\}$ as $\{O, O, O, E, E, E\}$, i.e., three odd numbers and three even numbers. There are $\binom{6}{3} = \frac{6!}{3!3!} = \frac{720}{6 \times 6} = 20$ ways we can select three of the six letters to form the top row of our array.

By the argument above, the only way one of the rows multiplies to an odd number is if the top row is OOO or if the top row is EEE . Therefore, $q = \frac{1+1}{20} = \frac{2}{20} = \frac{1}{10}$, and we conclude that $p = 1 - \frac{1}{10} = \frac{9}{10}$.

Answer: (D)

5. Let $f(x) = (x^2 + 4x + 3)(x^2 - 4x + 3)$. Over all possible values of x , what is the minimum possible value of $f(x)$?

- (A) -16 (B) -15 (C) -9 (D) 0 (E) 9

Solution: One possible way to solve this problem involves expanding $f(x) = (x^2 + 4x + 3)(x^2 - 4x + 3)$ and using calculus to determine the value of x that minimizes the function $f(x)$. However, a much more insightful way to solve this problem is to exploit the symmetry in the two terms that multiply to $f(x)$.

Since $x^2 + 4x + 3 = (x + 1)(x + 3)$ and $x^2 - 4x + 3 = (x - 1)(x - 3)$, we have

$$f(x) = (x+1)(x+3)(x-1)(x-3) = (x+1)(x-1)(x+3)(x-3) = (x^2-1)(x^2-9).$$

Let $a = x^2 - 5$. Then,

$$\begin{aligned} f(x) &= (x^2 - 1)(x^2 - 9) \\ &= x^4 - 10x^2 + 9 \\ &= x^4 - 10x^2 + 25 - 16 \\ &= (x^2 - 5)^2 - 16 \\ &= a^2 - 16 \end{aligned}$$

Since $a^2 \geq 0$, by definition, we see that the function $f(x)$ attains its minimum value of $0^2 - 16 = -16$ precisely when $a = 0$, i.e., when $x = \pm\sqrt{5}$.

We have shown that $f(x) = -16$ when $x = \pm\sqrt{5}$, and for all other values of x , $f(x) > -16$.

Answer: (A)

Part C: 6 marks each

1. Let $f(x) = ax^2 + bx + c$ be the quadratic function for which $f(1) = 19$, $f(2) = 37$, and $f(3) = 61$.

For each positive integer n , define $S(n) = f(1) + f(2) + f(3) + \dots + f(n)$. For example, $S(3) = f(1) + f(2) + f(3) = 19 + 37 + 61 = 117$.

What is the smallest integer n for which $S(n) > 1,000,000$?

- (A) 98 (B) 99 (C) 100 (D) 101 (E) 102

Solution: We first determine the quadratic function $f(x) = ax^2 + bx + c$. Since we are given $f(1) = 19$, $f(2) = 37$, and $f(3) = 61$, we set up a system of three equations in three unknown variables:

$$\begin{aligned}a + b + c &= 19 \\4a + 2b + c &= 37 \\9a + 3b + c &= 61\end{aligned}$$

Subtracting the first equation from the second, we get $3a + b = 18$. Subtracting the second equation from the third, we get $5a + b = 24$.

From the previous step, we see that $b = 18 - 3a$ and $b = 24 - 5a$. It follows that $18 - 3a = 24 - 5a$, which implies $2a = 6$, or $a = 3$.

Since $a = 3$, we have $b = 18 - 3a = 18 - 9 = 9$ and $c = 19 - a - b = 19 - 3 - 9 = 7$. Therefore, our quadratic function is $f(x) = 3x^2 + 9x + 7$.

Now, we can determine $S(n) = f(1) + f(2) + f(3) + \dots + f(n)$, which is equal to:

$$S(n) = 3(1^2 + 2^2 + 3^2 + \dots + n^2) + 9(1 + 2 + 3 + \dots + n) + 7(1 + 1 + 1 + \dots + 1)$$

But to solve the problem this way, we need to know the formulas $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$ and $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$. If we do know

these formulas, we get

$$S(n) = 3 \cdot \frac{n(n+1)(2n+1)}{6} + 9 \cdot \frac{n(n+1)}{2} + 7 \cdot n$$

From here, a cumbersome algebraic calculation gives us $S(n) = n^3 + 6n^2 + 12n$.

We now present a more elegant solution. The crucial observation is that each $f(x)$ term can be expressed as the difference of two consecutive cubes. We have $f(1) = 19 = 27 - 8 = 3^3 - 2^3$, $f(2) = 37 = 64 - 27 = 4^3 - 3^3$, and $f(3) = 61 = 125 - 64 = 5^3 - 4^3$.

In general, $f(x) = 3x^2 + 9x + 7 = (x^3 + 6x^2 + 12x + 8) - (x^3 + 3x^2 + 3x + 1) = (x+2)^3 - (x+1)^3$.

Since $S(n) = f(1) + f(2) + \dots + f(n-1) + f(n) = f(n) + f(n-1) + \dots + f(2) + f(1)$, we have

$$\begin{aligned} S(n) &= f(n) + f(n-1) + f(n-2) + \dots + f(2) + f(1) \\ &= [(n+2)^3 - (n+1)^3] + [(n+1)^3 - n^3] + [n^3 - (n-1)^3] + \\ &\quad \dots + [4^3 - 3^3] + [3^3 - 2^3] \\ &= (n+2)^3 - (n+1)^3 + (n+1)^3 - n^3 + n^3 - (n-1)^3 + \\ &\quad \dots + 4^3 - 3^3 + 3^3 - 2^3 \\ &= (n+2)^3 - 2^3 \end{aligned}$$

This technique is known as a *telescoping series*, where the sum collapses to the difference between the largest term and smallest term, with all of the middle terms cancelling out.

Since $S(n) = (n+2)^3 - 2^3$, we observe that $S(98) = 100^3 - 2^3$ is just less than $100^3 = 1,000,000$, while $S(99) = 101^3 - 2^3$ is definitely larger than $1,000,000$.

Therefore, $n = 99$ is the smallest integer n for which $S(n) > 1,000,000$.

Answer: (B)

2. Let $S = 2^{2026} + 0^{2026} + 2^{2026} + 6^{2026}$. What are the last two digits of S ?

(A) 04

(B) 24

(C) 44

(D) 64

(E) 84

Solution: To determine the last two digits of S , we calculate the last two digits of each term in our sum.

Our first claim is that each element in $\{2^6, 2^{26}, 2^{46}, 2^{66}, \dots\}$ has the same last two digits. In other words, $2^{20n+6} \equiv 2^{20(n+1)+6} \pmod{100}$ for all integers $n \geq 0$.

To prove that $2^{20n+6} \equiv 2^{20(n+1)+6} \pmod{100}$ for all integers $n \geq 0$, it suffices to show that $2^{20(n+1)+6} - 2^{20n+6}$ is always a multiple of 100. This difference simplifies to:

$$\begin{aligned} 2^{20(n+1)+6} - 2^{20n+6} &= 2^{20n+6} \cdot (2^{20} - 1) \\ &= 2^{20n+4} \cdot 2^2 \cdot (2^{10} + 1) \cdot (2^{10} - 1) \\ &= 2^{20n+4} \cdot 4 \cdot (1024 + 1) \cdot (2^{10} - 1) \\ &= 2^{20n+4} \cdot 4 \cdot 1025 \cdot (2^{10} - 1) \\ &= 2^{20n+4} \cdot 4100 \cdot (2^{10} - 1) \end{aligned}$$

Since this difference is a multiple of 4100, it must also be a multiple of 100. This proves the claim. Thus, the last two digits of 2^{2026} are 64 since $2^6 = 64$.

Our second claim is that the sequence $6^2, 6^3, 6^4, 6^5, 6^6, 6^7, \dots$ has this repeating pattern as its last two digits: 36, 16, 96, 76, 56, 36, $\dots$

To justify this, we now prove that $6^{n+5} \equiv 6^n \pmod{100}$ for all integers $n \geq 2$.

To prove that $6^{n+5} \equiv 6^n \pmod{100}$ for all integers $n \geq 2$, it suffices to show that $6^{n+5} - 6^n$ is a multiple of 100. This difference simplifies to:

$$\begin{aligned}
6^{n+5} - 6^n &= 6^n \cdot (6^5 - 1) \\
&= 6^{n-2} \cdot 6^2 \cdot (6^5 - 1) \\
&= 6^{n-2} \cdot 6^2 \cdot (6^5 - 1) \\
&= 6^{n-2} \cdot 9 \cdot 4 \cdot (7776 - 1) \\
&= 6^{n-2} \cdot 9 \cdot 4 \cdot 7775 \\
&\equiv 6^{n-2} \cdot 9 \cdot 4 \cdot 75 \pmod{100} \\
&\equiv 6^{n-2} \cdot 9 \cdot 300 \pmod{100} \\
&\equiv 6^{n-2} \cdot 9 \cdot 0 \pmod{100} \\
&\equiv 0 \pmod{100}
\end{aligned}$$

We can also show that $6^{n+5} - 6^n = 6^n(6^5 - 1)$ is a multiple of 100 as follows: for all $n \geq 2$, 6^n is a multiple of 4. It suffices to show that $6^5 - 1$ is a multiple of 25, which we can do via the Binomial Theorem:

$$(5 + 1)^5 - 1 = \sum_{k=0}^5 \binom{5}{k} 5^k - 1 = \sum_{k=0}^5 \binom{5}{k} 5^k - \binom{5}{0} = \sum_{k=1}^5 \binom{5}{k} 5^k$$

For each $k \geq 1$, the term $\binom{5}{k} 5^k$ is divisible by 25, which proves the claim.

Therefore, we have shown that 6^{2026} has the same last two digits as 6^6 , which equals $6^3 \cdot 6^3 = 216 \cdot 216 \equiv 16 \cdot 16 = 256 \equiv 56 \pmod{100}$. We conclude that

$$\begin{aligned}
S &= 2^{2026} + 0^{2026} + 2^{2026} + 6^{2026} \\
&\equiv 64 + 0 + 64 + 56 \pmod{100} \\
&\equiv 184 \pmod{100} \\
&\equiv 84 \pmod{100}
\end{aligned}$$

We have shown that the last two digits of S are 84.

Answer: (E)

3. Let p and q be positive real numbers. The points $(0, 0)$, $(p, 1)$, and $(2, q)$ form an equilateral triangle.

The area of this triangle is A square units. What is A , rounded to the nearest integer?

- (A) 9 (B) 11 (C) 13 (D) 15 (E) 17

Solution: Label the three points $O = (0, 0)$, $P = (p, 1)$, and $Q = (2, q)$.

Since $\triangle OPQ$ is an equilateral triangle, we have $OP^2 = OQ^2$, which implies $(p - 0)^2 + (1 - 0)^2 = (2 - 0)^2 + (q - 0)^2$. This simplifies to $p^2 + 1 = 4 + q^2$, or $p^2 = q^2 + 3$.

Also, we have $OP^2 = PQ^2$, which implies $(p - 0)^2 + (1 - 0)^2 = (2 - p)^2 + (q - 1)^2$. This simplifies to $p^2 + 1 = 4 - 4p + p^2 + q^2 - 2q + 1$, or $4p = q^2 - 2q + 4$.

From the above two equations, we have $(q^2 - 2q + 4)^2 = (4p)^2 = 16p^2 = 16(q^2 + 3)$. Therefore,

$$\begin{aligned} (q^2 - 2q + 4)^2 &= 16(q^2 + 3) \\ ((q^2 + 4) - 2q)^2 &= 4q^2 + 12q^2 + 48 \\ (q^2 + 4)^2 - 4q(q^2 + 4) + 4q^2 &= 4q^2 + 12(q^2 + 4) \\ (q^2 + 4)^2 - 4q(q^2 + 4) - 12(q^2 + 4) &= 0 \\ (q^2 + 4)(q^2 + 4 - 4q - 12) &= 0 \\ (q^2 + 4)(q^2 - 4q - 8) &= 0 \end{aligned}$$

Since q is a positive real number, it is impossible for $q^2 + 4 = 0$. Therefore, $q^2 - 4q - 8 = 0$, from which the Quadratic Formula gives us $q = 2 + \sqrt{12} = 2 + 2\sqrt{3}$ since we know that $q > 0$.

At this point, we can substitute $q = 2 + 2\sqrt{3}$ into the equation $4p = q^2 - 2q + 4$ to show that $p = 4 + \sqrt{3}$, but we can solve this problem without calculating the value of p .

Let x be the side length of $\triangle OPQ$. By drawing the height of this triangle and applying the Pythagorean Theorem, it is straightforward to show that the

height is $\frac{\sqrt{3}x}{2}$. And so the area of this equilateral triangle is $\frac{1}{2} \cdot x \cdot \frac{\sqrt{3}x}{2} = \frac{\sqrt{3}}{4}x^2$.

Since $x = OQ$, by definition, we have $x^2 = OQ^2 = 4 + q^2 = 4 + (2 + 2\sqrt{3})^2 = 4 + (4 + 8\sqrt{3} + 12) = 20 + 8\sqrt{3}$. Therefore, the area of this triangle is $\frac{\sqrt{3}}{4}x^2 = \frac{\sqrt{3}}{4}(20 + 8\sqrt{3}) = 5\sqrt{3} + 6$.

Since $\sqrt{3}$ is approximately 1.73, the area of the triangle is approximately $5\sqrt{3} + 6 = 5 \times 1.73 + 6 = 8.65 + 6 = 14.65$, which rounds up to 15.

While the above solution is correct, here is a clever solution that involves complex numbers. We represent each two-dimensional coordinate (a, b) as the vector $a + bi$, where i is the complex number satisfying $i^2 = -1$. We define $\text{cis}(\theta) = \cos(\theta) + i \cdot \sin(\theta)$.

Let $OP = p + 1 \cdot i = x \cdot \text{cis}(\theta)$. Then $OQ = 2 + q \cdot i = x \cdot \text{cis}(\theta + 60^\circ) = x \cdot \text{cis}(\theta) \cdot \text{cis}(60^\circ)$.

It follows that $2 + q \cdot i = x \cdot \text{cis}(\theta) \cdot \text{cis}(60^\circ) = (p + 1 \cdot i)(\cos 60^\circ + i \cdot \sin 60^\circ) = (p + 1 \cdot i)(\frac{1}{2} + \frac{\sqrt{3}}{2}i)$.

Expanding the right side, we get $2 + q \cdot i = \frac{p}{2} + \frac{1}{2} \cdot i + \frac{\sqrt{3}p}{2} \cdot i - \frac{\sqrt{3}}{2}$.

Equating the real and imaginary parts, we get $2 = \frac{p}{2} - \frac{\sqrt{3}}{2}$ and $q = \frac{1}{2} + \frac{\sqrt{3}p}{2}$. This gives us $p = 4 + \sqrt{3}$, from which we can show that $q = \frac{1}{2} + \frac{\sqrt{3}p}{2} = \frac{1}{2} + \frac{\sqrt{3}}{2}(4 + \sqrt{3}) = 2 + 2\sqrt{3}$.

From here, we proceed as above to show that the area of the triangle is approximately $5\sqrt{3} + 6 = 5 \times 1.73 + 6 = 8.65 + 6 = 14.65$, which rounds up to 15.

Answer: (D)

4. Consider a 3 by 3 grid where each entry is a positive integer.

a	b	c
d	e	f
g	h	i

In this grid, the product of each row, each column, and each diagonal is equal to the constant P .

Let S be the sum of these nine numbers.

If the grid consists of nine *different* positive integers, each less than 2026, what is the smallest possible value of $\frac{S}{\sqrt[3]{P}}$ rounded to two decimal places?

- (A) 9.01 (B) 9.15 (C) 9.17 (D) 9.19 (E) 9.21

Solution: Since the product of each row, each column, and each diagonal is equal to the constant P , we have $P = def = beh = aei = ceg$. It follows that $P^4 = (def)(beh)(aei)(ceg) = (abcdefghi) \cdot e^3$.

Since the product of each row is equal to P , we have $P = abc = def = ghi$, and so $P^3 = (abc)(def)(ghi) = (abcdefghi)$.

From the above two equations, we have $P^4 = (abcdefghi) \cdot e^3 = P^3 \cdot e^3$, which implies $P = e^3$.

Let $a = ex$ and $c = ey$, where x and y are real numbers. Since the product of each row and column and diagonal is equal to $P = e^3$, we can quickly determine each of the entries in our grid in terms of x and y .

ex	$\frac{e}{xy}$	ey
$\frac{ey}{x}$	e	$\frac{ex}{y}$
$\frac{e}{y}$	exy	$\frac{e}{x}$

Letting S be the sum of these nine numbers, we have

$$\begin{aligned} S &= ex + \frac{e}{xy} + ey + \frac{ey}{x} + e + \frac{ex}{y} + \frac{e}{y} + exy + \frac{e}{x} \\ &= e\left(1 + x + \frac{1}{x}\right) + ey\left(1 + x + \frac{1}{x}\right) + \frac{e}{y}\left(1 + x + \frac{1}{x}\right) \\ &= e\left(1 + x + \frac{1}{x}\right)\left(1 + y + \frac{1}{y}\right). \end{aligned}$$

And so the quantity we wish to minimize, $\frac{S}{\sqrt[3]{P}}$, is equal to $\frac{S}{e} = \left(1 + x + \frac{1}{x}\right)\left(1 + y + \frac{1}{y}\right)$.

It is straightforward to verify that $z + \frac{1}{z} \geq 2$ for all real numbers z , with equality if and only if $z = 1$. We can do this by applying the Arithmetic Mean - Geometric Mean (AM-GM) Inequality, or showing that the above inequality is equivalent to $(z - 1)^2 \geq 0$.

Therefore, $\frac{S}{e} = \left(1 + x + \frac{1}{x}\right)\left(1 + y + \frac{1}{y}\right) \geq (1 + 2)(1 + 2) = 9$. The closer x and y are to 1, the closer $\frac{S}{e}$ will be to 9. If all nine entries in the grid could be the same number, then we have $\frac{S}{e} = 9$. However, since the entries in the grid are nine *different* integers, the quantity $\frac{S}{e}$ will be greater than 9.

Before we find the correct minimum value of $\frac{S}{e}$, let us use an example to illustrate the proof that will follow.

Suppose $x = \frac{3}{2}$ and $y = \frac{4}{3}$. Then $\frac{S}{e} = \left(1 + \frac{3}{2} + \frac{2}{3}\right)\left(1 + \frac{4}{3} + \frac{3}{4}\right) = \frac{19}{6} \cdot \frac{37}{12}$, which is approximately equal to 9.76. The nine entries of our grid are

$\frac{3e}{2}$	$\frac{e}{2}$	$\frac{4e}{3}$
$\frac{8e}{9}$	e	$\frac{9e}{8}$
$\frac{3e}{4}$	$2e$	$\frac{2e}{3}$

Finally, we set e to be any number that ensures all nine entries are *integers*.

For example, if we set $e = 72$, our final grid is:

108	36	96
64	72	81
54	144	48

Our intuition is as follows: we set x and y to be as close to 1 as possible; this will ensure that our desired quantity, $\frac{S}{e}$, is as close to 9 as possible. But we must ensure that all nine entries in our grid are positive integers *less than* 2026. So when we find a value of e that makes all of the entries an integer, we must ensure that the largest number is smaller than 2026.

Let $x = \frac{p}{q}$ and $y = \frac{r}{s}$. Without loss of generality, assume that $p \geq r$, $p > q$, and $r > s$. Furthermore, assume that x and y are reduced to lowest terms, i.e., $\gcd(p, q) = \gcd(r, s) = 1$.

We claim that e must be a multiple of $pqrs$. (This means, $e = pqrs$ is the smallest number e that guarantees that all nine entries are integers).

To justify this claim, let v be a prime number. Since $\gcd(p, q) = 1$, the prime v can be a divisor of p or q but not both. Suppose w_1 is the multiplicity of v in p or q . (In other words, v^{w_1} divides p or q but v^{w_1+1} does not). Similarly, let w_2 be the multiplicity of v in r or s .

Our grid consists of nine different entries, including $b = \frac{e}{xy} = \frac{eqs}{pr}$, $d = \frac{ey}{x} = \frac{eqr}{ps}$, $f = \frac{ex}{y} = \frac{eps}{qr}$, and $h = exy = \frac{epr}{qs}$. Looking at the four denominators qs , ps , qr , pr , we know that one of them is a multiple of $v^{w_1+w_2}$.

For example, assume $h = exy = \frac{epr}{qs}$ is the fraction for which the denominator is a multiple of $v^{w_1+w_2}$. Since $\gcd(p, q) = 1$ and $\gcd(r, s) = 1$, the only way h can be an integer is if e is a multiple of $v^{w_1+w_2}$. The same argument holds for each of the other entries b , d , and f .

The multiplicity of v in pq is w_1 and the multiplicity of v in rs is w_2 . Therefore, the multiplicity of v in $pqrs$ is $w_1 + w_2$. From the above paragraph, we have shown that the multiplicity of v in e is *at least* $w_1 + w_2$.

Since this statement holds for *all* prime numbers in the factorization of $pqrs$, we conclude that e must be a multiple of $pqrs$.

As we want all of our entries to be as small as possible, it makes sense to simply let $e = pqrs$, and ensure that our values of p, q, r, s are small enough so that all entries in our grid are less than 2026. Our grid becomes

p^2rs	q^2s^2	pqr^2
q^2r^2	$pqrs$	p^2s^2
pqs^2	p^2r^2	q^2rs

Recall that we assumed $p \geq r$, $p > q$, and $r > s$. This means the largest entry in our grid is $p^2r^2 = (pr)^2$ which we know must be smaller than $2026 > 2025 = 45^2$. So $pr \leq 45$.

To make our quantity $\frac{S}{e} = (1 + \frac{p}{q} + \frac{q}{p})(1 + \frac{r}{s} + \frac{s}{r})$ as close to 9 as possible, we want the fractions $\frac{p}{q}$ and $\frac{r}{s}$ to be as close to 1 as possible. Thus, it follows that we want $q = p - 1$ and $s = r - 1$.

To ensure the fractions $\frac{p}{q} = \frac{p}{p-1} = 1 + \frac{1}{p-1}$ and $\frac{r}{s} = \frac{r}{r-1} = 1 + \frac{1}{r-1}$ is as close to 1 as possible, we want p and r to be as large as possible, while still ensuring that $pr \leq 45$.

Since we are assuming $p \geq r$, this implies that the desired pair (p, r) must be $(9, 5)$ or $(7, 6)$. Any other option, such as $(9, 4)$, $(9, 3)$, $(8, 5)$, $(8, 4)$, or $(7, 5)$ is guaranteed to ensure a larger value of $\frac{S}{e}$ than either $(9, 5)$ or $(7, 6)$.

The case $(p, r) = (9, 5)$ makes $x = \frac{9}{8}$ and $y = \frac{5}{4}$, and this gives us $\frac{S}{e} = (1 + \frac{9}{8} + \frac{8}{9})(1 + \frac{5}{4} + \frac{4}{5}) = \frac{13237}{1440}$, which equals 9.19 rounded to two decimal places.

The case $(p, r) = (7, 6)$ makes $x = \frac{7}{6}$ and $y = \frac{6}{5}$, and this gives us $\frac{S}{e} = (1 + \frac{7}{6} + \frac{6}{7})(1 + \frac{6}{5} + \frac{5}{6}) = \frac{11557}{1260}$, which equals 9.17 rounded to two decimal places.

Therefore, the correct answer must be 9.17. We can verify that each entry in our grid is less than 2026 for the case $(p, q, r, s) = (7, 6, 6, 5)$.

Answer: (C)

1470	900	1512
1296	1260	1225
1050	1764	1080

5. Hexagon $ABCDEF$ is inscribed in a circle. This hexagon has the property that $AB = BC = CD$ and $DE = EF = FA$.

Given that AB, AC, AD, AE, AF all have integer side lengths, what is the minimum possible value of $|BE - AD|$?

- (A) 1 (B) 15 (C) 16 (D) 32 (E) 64

Solution: Let $AB = BC = CD = p$, and $DE = EF = FA = q$. From the wording of the question, we know that p and q are positive integers.

Let O be the centre of the circle, and let $\angle AOB = x$. Then if we draw the six radii from O to each of the six points in the hexagon, we see that $\angle AOB = \angle BOC = \angle COD = x$ and $\angle DOE = \angle EOF = \angle FOA = 120^\circ - x$.

Since $\triangle OAF$ and $\triangle OAB$ are isosceles, we have $\angle OAF = \angle OFA = 30^\circ + \frac{x}{2}$ and $\angle OAB = \angle OBA = 90^\circ - \frac{x}{2}$. Therefore, $\angle FAB = \angle OAF + \angle OAB = (30^\circ + \frac{x}{2}) + (90^\circ - \frac{x}{2}) = 120^\circ$.

By the Cosine Law on $\triangle BAF$, we know that $\cos 120^\circ = \frac{p^2 + q^2 - BF^2}{2pq}$. Since $\cos 120^\circ = -\frac{1}{2}$, this implies that $BF^2 = p^2 + pq + q^2$. By symmetry, $BF^2 = CE^2$.

Any four points on the hexagon form a *cyclic quadrilateral*. We now apply Ptolemy's Theorem to express each of AC, AD, AE , in terms of p and q .

By applying Ptolemy's Theorem on quadrilateral $BCEF$, we have

$$\begin{aligned} BF \cdot CE + BC \cdot FE &= BE \cdot CF \\ BF \cdot BF + p \cdot q &= BE^2 \quad \text{since } BE = CF \text{ by symmetry} \\ (p^2 + pq + q^2) + pq &= BE^2 \\ (p + q)^2 &= BE^2 \end{aligned}$$

Therefore, we have $BE = CF = p + q$.

By applying Ptolemy's Theorem on quadrilateral $ABEF$, we have

$$\begin{aligned} AF \cdot BE + AB \cdot FE &= AE \cdot BF \\ q(p + q) + pq &= AE \sqrt{p^2 + pq + q^2} \\ q(2p + q) &= AE \sqrt{p^2 + pq + q^2} \end{aligned}$$

Therefore, we have $AE = \frac{q(2p+q)}{\sqrt{p^2+pq+q^2}}$.

By a similar application of Ptolemy's Theorem on quadrilateral $ABCF$, we get $AC = \frac{p(2q+p)}{\sqrt{p^2+pq+q^2}}$.

Notice that $AC = BD$, by symmetry. Finally, by applying Ptolemy's Theorem on quadrilateral $ABCD$, we have

$$\begin{aligned} AB \cdot CD + BC \cdot AD &= AC \cdot BD \\ p \cdot p + p \cdot AD &= AC^2 \\ p^2 + p \cdot AD &= \frac{p^2(2q+p)^2}{p^2 + pq + q^2} \\ p + AD &= \frac{p(2q+p)^2}{p^2 + pq + q^2} \\ AD &= \frac{p(2q+p)^2}{p^2 + pq + q^2} - p = p \left(\frac{(2q+p)^2}{p^2 + pq + q^2} - 1 \right) \\ AD &= p \left(\frac{4q^2 + 4qp + p^2}{p^2 + pq + q^2} - \frac{p^2 + pq + q^2}{p^2 + pq + q^2} \right) \\ AD &= p \left(\frac{3q^2 + 3qp}{p^2 + pq + q^2} \right) = \frac{3pq(p+q)}{p^2 + pq + q^2} \end{aligned}$$

We are given that $AB = p$, and $AF = q$ have integer side lengths, as do each of AC , AD , and AE . From our formulas for AC and AE , we know that $p^2 + pq + q^2$ must be a perfect square.

As an example, suppose $p = 3$ and $q = 5$, which makes $p^2 + pq + q^2 = 9 + 15 + 25 = 49 = 7^2$. By substituting these values of p and q into our above formulas, we can show that $(AB, AC, AD, AE, AF) = (3, \frac{39}{7}, \frac{360}{49}, \frac{55}{7}, 5)$.

We want each number in this 5-tuple to be an integer. The simplest way to do this is to multiply p and q by 49. If we substitute $p = 3 \cdot 49 = 147$ and $q = 5 \cdot 49 = 245$ into our above formulas, we can show that $(AB, AC, AD, AE, AF) = (147, 273, 360, 385, 245)$.

For these two values of p and q , we have $AD = 360$ and $BE = p + q = 147 + 245 = 392$. And so, $|BE - AD|$ is equal to 32.

But can we do better than this? We now prove that the minimum possible value of $|BE - AD|$ is 15.

Let $p = ka$ and $q = kb$, where $\gcd(a, b) = 1$. Then $BE = p + q = k(a + b)$ and $AD = \frac{3kab(a+b)}{a^2+ab+b^2} = 3k \cdot \frac{ab(a+b)}{a^2+ab+b^2}$.

Since $\gcd(a, b) = 1$, it is straightforward to verify that each of a and b is relatively prime to $a^2 + ab + b^2$. We now show that $a + b$ is also relatively prime to $a^2 + ab + b^2$.

By the Euclidean Algorithm, $\gcd(a^2+ab+b^2, a+b) = \gcd((a+b)^2-ab, a+b) = \gcd(ab, a+b)$. If some prime p divides ab , then p divides either a or b but not both. So p is a divisor of ab but cannot be a divisor of $a+b$. This implies that $\gcd(ab, a+b)$ must be 1, implying that $a+b$ is relatively prime to a^2+ab+b^2 .

We have now shown that $AD = 3k \cdot \frac{ab(a+b)}{a^2+ab+b^2}$ can only be an integer if $3k$ is a multiple of $a^2 + ab + b^2$.

This implies that $k \geq \frac{1}{3}(a^2 + ab + b^2)$ if $a^2 + ab + b^2$ is a multiple of 3, and $k \geq (a^2 + ab + b^2)$ if $a^2 + ab + b^2$ is not a multiple of 3.

From our formulas for BE and AD , we have

$$\begin{aligned} BE - AD &= k(a + b) - 3k \cdot \frac{ab(a + b)}{a^2 + ab + b^2} \\ &= k(a + b) \left(1 - \frac{3ab}{a^2 + ab + b^2} \right) \\ &= \frac{k(a + b)(a - b)^2}{a^2 + ab + b^2}. \end{aligned}$$

From above, if $a^2 + ab + b^2$ is a multiple of 3,

$$\begin{aligned} |BE - AD| &= BE - AD \\ &= \frac{k(a+b)(a-b)^2}{a^2 + ab + b^2} \geq \frac{1}{3}(a^2 + ab + b^2) \frac{(a+b)(a-b)^2}{a^2 + ab + b^2} \\ &= \frac{(a+b)(a-b)^2}{3}. \end{aligned}$$

And if $a^2 + ab + b^2$ is not a multiple of 3,

$$\begin{aligned} |BE - AD| &= BE - AD \\ &= \frac{k(a+b)(a-b)^2}{a^2 + ab + b^2} \geq (a^2 + ab + b^2) \frac{(a+b)(a-b)^2}{a^2 + ab + b^2} \\ &= (a+b)(a-b)^2. \end{aligned}$$

To complete the proof, we just need to find positive integers a and b , with $\gcd(a, b) = 1$, for which $p^2 + pq + q^2 = k^2(a^2 + ab + b^2)$ is a perfect square. Our goal is to make $|BE - AD|$ as small as possible. Without loss of generality, assume $a < b$.

In our earlier example, if $a = 3$ and $b = 5$, then $a^2 + ab + b^2 = 9 + 15 + 25 = 49 = 7^2$. Since 7^2 is not a multiple of 3, we have $|BE - AD| \geq (a+b)(a-b)^2 = (3+5)(3-5)^2 = 32$.

If $b - a = 1$, then a simple inspection shows that $(a, b) = (7, 8)$ is the smallest pair for which $(a^2 + ab + b^2)$ is a perfect square. Indeed, $a^2 + ab + b^2 = 49 + 56 + 64 = 169 = 13^2$. Since 13^2 is not a multiple of 3, we have $|BE - AD| \geq (a+b)(a-b)^2 = (7+8)(7-8)^2 = 15$. Equality occurs when $k = a^2 + ab + b^2 = 169$.

We now prove that the smallest possible value of $|BE - AD|$ is 15.

If $b - a = 1$, then we see that $a \geq 7$ in order for $(a^2 + ab + b^2)$ to be a perfect square. If $a > 7$, then $|BE - AD| \geq (a+b)(a-b)^2 = (2a+1) \cdot 1^2 = 2a+1 > 15$.

If $b - a = 2$, then

$$\begin{aligned} a^2 + ab + b^2 &= a^2 + a(a+2) + (a+2)^2 \\ &= a^2 + a^2 + 2a + a^2 + 4a + 4 \\ &= 3a^2 + 6a + 4 \\ &= 3(a+1)^2 + 1. \end{aligned}$$

Since this quantity cannot be a multiple of 3, we have $|BE - AD| \geq (a+b)(a-b)^2 = (a+a+2)(2)^2 = (2a+2) \cdot 4 = 8a+8$. Since $a \geq 1$, we have $|BE - AD| \geq 16$.

If $b - a = 3$, then the case $(a, b) = (1, 4)$ does not make $(a^2 + ab + b^2)$ a perfect square. So $a \geq 2$ and $b \geq 5$. And $|BE - AD| \geq \frac{(a+b)(a-b)^2}{3} \geq \frac{(2+5)(3)^2}{3} = 21 > 15$.

Finally, if $b - a \geq 4$, then $|BE - AD| \geq \frac{(a+b)(a-b)^2}{3} \geq \frac{(1+5)(4)^2}{3} > 15$.

We conclude that the minimum value of $|BE - AD| = 15$ occurs when $a = 7$, $b = 8$, and $k = a^2 + ab + b^2 = 49 + 56 + 64 = 169 = 13^2$.

From $p = ka = 169 \cdot 7 = 1183$ and $q = kb = 169 \cdot 8 = 1352$, we get $(AB, AC, AD, AE, AF) = (1183, 2093, 2520, 2288, 1352)$. Since $BE = p + q = 1183 + 1352 = 2535$, we conclude that $|BE - AD| = 2535 - 2520$ indeed is equal to 15.

Answer: (B)