

Solution:

Problem 4 of the Moldova Team Selection Test 2011, Day 2, which appeared in Crux Mathematicorum [2011:496]. We present the solution by Oliver Geupe which appeared at [2013:77-78], modified.

At each moment we have 20 numbers between 1 and 20 on the board. Since each move picks two numbers a, b and replaces them with two other numbers $a + 1, b - 1$ of the same sum

$$(a + 1) + (b - 1) = a + b,$$

the sum of the numbers on the board never changes. In particular, at any moment, the twenty numbers on the board will have an average of

$$10.5.$$

Consider now the list of numbers on the board after some moves. To any such list, associate a state $S = (s_1, s_2, \dots, s_{20})$, where s_k represents the number of copies of the number k on the board.

We clearly have

$$\begin{aligned} s_j &\geq 0 & \forall 1 \leq j \leq 20 \\ s_1 + \dots + s_{20} &= 20. \end{aligned}$$

The initial state is

$$S_0 = (1, 1, \dots, 1).$$

Now, we define the value $f(S)$ of the state S by

$$f(S) = s_1 + 4s_2 + 9s_3 + 16s_4 + \dots + 400s_{20} = \sum_{k=1}^{20} k^2 \cdot s_k,$$

which is the sum of the squares of all numbers on the board.

If state S can be moved to state T by picking $a \leq b - 2$ and replacing a, b with $a + 1$ and $b - 1$, then

$$f(S) - f(T) = a^2 + b^2 - ((a + 1)^2 - (b - 1)^2) = 2b - 2a - 2.$$

Since $b - a \geq 2$ we get that

$$f(S) - f(T) \geq 2$$

with equality if and only if $a + 2 = b$.

This shows that after each move, the value of the state decreases by at least 2. Since $f(T) \geq 0$ for each state T , any series of moves must eventually terminates.

Now, consider a sequence $S_0, S_1, \dots, S_n$ of moves which terminates. We can easily determine the final state S_n .

Since the process terminates at S_n , we cannot find on the board two numbers a, b with $a + 2 \leq b$. This means that either all number of the board are equal, or they take only two consecutive values.

Since the average of the numbers on the board is always 10.5, and the final state contains numbers taking only one or two consecutive values, all the numbers in the final state must be 10 or 11. Moreover, since their average is exactly 10.5, there must be ten numbers equal to 10, and ten numbers equal to 11 on the board in the final state.

Therefore, the final state is

$$S_n = (\underbrace{0, 0, \dots, 0}_9, 10, 10, \underbrace{0, 0, \dots, 0}_9).$$

Let us find an upperbound for n . Consider any series of moves $S_0, S_1, \dots, S_n$. By eventually making more moves, we can assume without loss of generality that S_n terminates (as we are looking for an upperbound).

Then,

$$\begin{aligned} f(S_0) - f(S_1) &\geq 2 \\ f(S_1) - f(S_2) &\geq 2 \\ &\dots\dots\dots \\ f(S_{n-1}) - f(S_n) &\geq 2. \end{aligned}$$

Adding those together, we get

$$f(S_0) - f(S_n) \geq 2n.$$

Since the initial state is $S_0 = (1, 1, \dots, 1)$ we have

$$f(S_0) = 1 + 2^2 + \dots + 20^2 = 2870.$$

Similarly, since the final state is $S_n = (\underbrace{0, 0, \dots, 0}_9, 10, 10, \underbrace{0, 0, \dots, 0}_9)$, we have

$$f(S_n) = 10 \cdot 10^2 + 10 \cdot 11^2 = 2210.$$

In particular,

$$2n \leq f(S_0) - f(S_n) = 2870 - 2210 = 660.$$

This gives us an upperbound

$$n \leq 330.$$

To complete the proof, we show that 330 is achievable.

To get $n = 330$ we must have $f(S_0) - f(S_n) = 2n$, which is only possible when

$$f(S_0) - f(S_1) = f(S_1) - f(S_2) = \dots = f(S_{n-1}) - f(S_n) = 2$$

or equivalently, if in all moves we select integers a, b satisfying $a + 2 = b$.

Therefore, to show that $n = 330$ is achievable, we must find a series of moves which at every step change a pair (a, b) satisfying $b = a + 2$ with $(a + 1, b - 1)$.

Let us introduce the shortcut $s \rightarrow_a t$ to denote a move with $b = a + 2$.

Then, the following series of moves takes us from

$$S_0 = (\underbrace{1, 1, \dots, 1}_{20})$$

to

$$S_n = (\underbrace{0, 0, \dots, 0}_9, 10, 10, \underbrace{0, 0, \dots, 0}_9)$$

by only using such moves:

$$\begin{aligned} & (\underbrace{1, 1, \dots, 1}_{20}) \rightarrow_1 \rightarrow_2 \cdots \rightarrow_{18} (0, 2, \underbrace{1, 1, \dots, 1}_{16}, 2, 0) \\ & (\rightarrow_2 \rightarrow_3 \cdots \rightarrow_{17})^2 (0, 0, 3, \underbrace{1, 1, \dots, 1}_{14}, 3, 0, 0) \\ & (\rightarrow_3 \rightarrow_4 \cdots \rightarrow_{16})^3 (0, 0, 0, 4, \underbrace{1, 1, \dots, 1}_{12}, 4, 0, 0, 0) \\ & \dots \\ & (\rightarrow_8 \rightarrow_9 \rightarrow_{10} \rightarrow_{11})^8 (\underbrace{0, 0, \dots, 0}_8, 9, 1, 1, 9, \underbrace{0, 0, \dots, 0}_8) \\ & (\rightarrow_9 \rightarrow_{10})^9 (\underbrace{0, 0, \dots, 0}_9, 10, 10, \underbrace{0, 0, \dots, 0}_9). \end{aligned}$$

Therefore, this is a sequence of 330 moves, and hence, the maximum number of possible moves is 330.

Editor note: More generally, Geupel has shown that if we start with the numbers $1, 2, \dots, n$ on the board, the maximum possible number of moves is

$$\frac{1}{6} \left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n+1}{2} \right\rceil n.$$

See if you can prove this.