

Solution:

Problem 1 of the Team Selection Contest. Italy, which appeared in Crux Mathematicorum [2002:356]. We present the solution by Edward T. H. Wang which appeared at [2004:491], slightly modified.

Let us assume to the contrary that there exists a prime p , an integer $n \geq 2$, and an integer x such that

$$x^n = 2^p + 3^p.$$

Note first that $p \neq 2$ since otherwise we would have

$$x^n = 13,$$

which has no integer solution x when $n \geq 2$.

Therefore, $p = 2m + 1$ is an odd integer.

Then

$$x^n = 2^{2m+1} + 3^{2m+1} = (2+3)(2^{2m} - 2^{2m-1}3 + 2^{2m-2}3^2 + \dots + 3^{2m}).$$

This shows that 5 divides x^n . Since 5 is prime, 5 divides x . Thus, $x = 5y$ for an integer y .

It follows that

$$5^{n-1}y^n = 2^{2m} - 2^{2m-1} \cdot 3 + 2^{2m-2} \cdot 3^2 + \dots + 3^{2m}.$$

Since $n \geq 2$, it follows that $2^{2m} - 2^{2m-1} \cdot 3 + 2^{2m-2} \cdot 3^2 + \dots + 3^{2m}$ is divisible by 5. Thus 5 divides

$$2^{2m} - 2^{2m-1} \cdot (5-2) + 2^{2m-2} \cdot (5-2)^2 + \dots + (5-2)^{2m} = \sum_{j=0}^{2m} (-1)^j 2^{2m-j} \cdot (5-2)^j.$$

A short computation shows that for all $0 \leq j \leq 2m$ we have

$$(-1)^j 2^{2m-j} \cdot (5-2)^j = \text{multiple of } 5 + 2^{2m}.$$

Therefore,

$$2^{2m} - 2^{2m-1} \cdot (5-2) + 2^{2m-2} \cdot (5-2)^2 + \dots + (5-2)^{2m} = \text{multiple of } 5 + (2m+1) \cdot 2^{2m}.$$

Since this is divisible by 5, it follows that $2^{2m} \cdot (2m+1) = 2^{2m} \cdot p$ is divisible by 5 and hence $p = 5$.

This implies that

$$x^n = 2^5 + 3^5 = 275 = 5^2 \cdot 11$$

which contradicts the fact that x, n are integers and $n \geq 2$.

Since we got a contradiction, our assumption that the equation has an integer solution x is wrong.