

Solution:

Problem 1 of the 15th Balkan Math Olympiad 1998, which appeared in Crux Mathematicorum [2001:357]. We present the solution by Mohammed Aassila which appeared at [2003:499-500].

We first observe that

$$\frac{(n+1)^2}{1998} - \frac{n^2}{1998} = \frac{2n+1}{1998} \begin{cases} < 1, & \text{if } n \leq 998 \\ > 1, & \text{if } n \geq 999 \end{cases}.$$

This implies that for all $1 \leq n \leq 998$ we have

$$x_{n+1} = \lfloor \frac{(n+1)^2}{1998} \rfloor \leq \frac{(n+1)^2}{1998} < \frac{n^2}{1998} + 1 \leq \left(\lfloor \frac{n^2}{1998} \rfloor + 1 \right) + 1 = x_n + 2.$$

Therefore, when $1 \leq n \leq 998$ we have $x_{n+1} - x_n < 2$. Since $x_{n+1} - x_n$ is an integer, we get $x_{n+1} - x_n \leq 1$ and hence

$$x_n \leq x_{n+1} \leq x_n + 1 \quad \forall 1 \leq n \leq 998.$$

This means that for this range, x_n does not "jump" over any integer value.

Since

$$x_1 = \lfloor \frac{1}{1998} \rfloor = 0$$

$$x_{999} = \lfloor \frac{999^2}{1998} \rfloor = \lfloor \frac{999}{2} \rfloor = 499,$$

the sequence x_n must take all values between 0 and 499 inclusive, when $1 \leq n \leq 999$.

Next, for all $n \geq 999$ we have $\frac{(n+1)^2}{1998} - \frac{n^2}{1998} > 1$. Therefore

$$x_n = \lfloor \frac{n^2}{1998} \rfloor \leq \frac{n^2}{1998} < \frac{(n+1)^2}{1998} - 1 < \left(\lfloor \frac{(n+1)^2}{1998} \rfloor + 1 \right) - 1 = x_{n+1}.$$

It follows that for $999 < n \leq 1997$, x_n takes distinct values.

Thus, in total, x_n takes

$$500 + 998 = 1498$$

distinct values.