

Solution:

Problem 11 of the 19th Lithuanian Team Contest, which appeared in Crux Mathematicorum [2008:284]. We present the solution by George Apostolopoulos which appeared at [2009:300], expanded by the editor.

Let $a_1, a_2, \dots, a_n$ be positive integers such that

$$a_1 + \dots + a_n = 2004.$$

We want to maximize

$$P = a_1 \cdot a_2 \cdot \dots \cdot a_n.$$

Let us first note that since $a_1, \dots, a_n \geq 1$ we have

$$2004 = a_1 + \dots + a_n \geq n.$$

Therefore,

$$1 \leq n \leq 2004.$$

For each $1 \leq n \leq 2004$ set

$$T_n := \{(a_1, \dots, a_n) : a_1, \dots, a_n \text{ are positive integers and} \\ a_1 + \dots + a_n = 2004\}.$$

Let

$$T := \bigcup_{n=1}^{2004} T_n.$$

It is clear that

$$T_n \subseteq \{(a_1, \dots, a_n) : 1 \leq a_j \leq 2004 \quad \forall j\}$$

and hence T_n is finite. This implies that T is a finite set.

Since T is finite so is the set

$$S = \{a_1 \cdot a_2 \cdot \dots \cdot a_n : (a_1, \dots, a_n) \in T\}$$

of products of elements in T . In particular, S has a largest element.

This shows that there exists some element $(a_1, \dots, a_n) \in T$ with maximal product.

Consider one such element $(a_1, \dots, a_n)$.

We next show that there exists some (possibly different) element in T with maximal product, such that all its elements are either 2 or 3.

First, it is easy to see that none of $a_1, a_2, \dots, a_n$ can be 1.

Indeed, if some $a_i = 1$, then picking another element a_j with $j \neq i$ and replacing the pair (a_i, a_j) by the single number $a_i + a_j = a_j + 1$ would produce another element in T with larger product, which is not possible. Therefore, none of $a_1, a_2, \dots, a_n$ can be 1.

Next, if some a_k is so that $a_k \geq 4$, then we can replace a_k by the two numbers 2 and $a_k - 2$. Since

$$2 \cdot (a_k - 2) \geq a_k$$

this cannot decrease the product. Since the original product was the maximum, so must be the new product.

Therefore, if some $a_k \geq 4$, this process produces a new element in T of maximal product, where one number $a_k \geq 4$ is replaced by two smaller numbers, and all other numbers are unchanged. Since this process cannot continue forever [for example, each time we use this process, the following sum decreases: $\sum_{a_k \geq 4} a_k$], by applying this process until it ends we produce an element $(b_1, b_2, \dots, b_m) \in T$ of maximal product, such that for all $1 \leq j \leq m$ we have

$$2 \leq b_j \leq 3.$$

Finally, let us note that at most two of $b_1, b_2, \dots, b_m$ can be equal to 2. Indeed, if there exists three or more two's among the b_j 's, then, since

$$2 + 2 + 2 = 3 + 3$$

$$2 \cdot 2 \cdot 2 < 3 \cdot 3,$$

we can replace the three 2's by two 3's and get an element of T with larger product, which is not possible.

This shows that there exists some $r \in \{0, 1, 2\}$ such that exactly r of $b_1, b_2, \dots, b_m$ are equal to 2 and the remaining $m - r$ are equal to 3.

It follows that

$$3 \cdot 668 = 2004 = b_1 + \dots + b_m = 2 \cdot r + 3 \cdot (m - r).$$

This gives that $2r$ is a multiple of 3 and hence $r = 0$. Therefore,

$$m = 668.$$

The above shows that the element

$$\underbrace{(3, 3, \dots, 3)}_{668}$$

is one of the elements of maximal product. Therefore, greatest value for the product is

$$P = 3^{668}.$$

Editor note: Reading very carefully the solution, one can argue that $\underbrace{(3, 3, \dots, 3)}_{668}$

is the only element in T of maximal product. In fact, if 2024 is replaced by a positive integer $n > 1$, one can show that:

- If n has a remainder of 0 or 2 when divided by 3, then there exists a unique element in T of maximal product.
- If n has a remainder of 1 when divided by 3, then there exists exactly two elements in T of maximal product.

See if you can prove those two claims.