

Solution: Problem 3 of the Junior Level of the 2015 Azerbaijan National Olympiad, which appeared in *Crux Mathematicorum* [2016:202]. We present the solution by Pedro Acosta which appeared at [2017:287], modified by the editor.

First let us note that the zero polynomial satisfies the given relation.

We next find all non-zero solutions. Let $P(X)$ be any non-zero polynomial and let $d = \deg(P)$. Then

$$d^2 = \deg\left(P(P(X))\right) = \deg((X^2 + X + 1) \cdot P(X)) = d + 2.$$

Solving, we get $d = 2$ or $d = -1$. Since $d = \deg(P) \geq 0$ we get $d = 2$.

Therefore, there must exist $a, b, c \in \mathbb{R}$ such that $a \neq 0$ and

$$P(X) = aX^2 + bX + c.$$

The given equation then gives

$$aP(X)^2 + bP(X) + c = (X^2 + X + 1) \cdot P(X)$$

and hence $P(X)$ must divide c . This implies that $c = 0$.

Therefore,

$$(aP(X) + b) \cdot P(X) = (X^2 + X + 1) \cdot P(X).$$

Since $P(X)$ is a non-zero polynomial, we can divide by $P(X)$ and get

$$aP(X) + b = X^2 + X + 1.$$

Recalling that

$$P(X) = aX^2 + bX,$$

we get

$$X^2 + X + 1 = a(aX^2 + bX) + b = a^2X + abX + b.$$

This gives

$$\begin{cases} a^2 &= 1 \\ ab &= 1 \\ b &= 1 \end{cases}$$

and hence $a = b = 1$. This shows that

$$P(X) = X^2 + X.$$

Checking, this polynomial satisfies the given equation. Indeed

$$\begin{aligned} P(P(X)) &= P(X)^2 + P(X) = (X^2 + X)^2 + (X^2 + X) = (X^2 + X)(X^2 + X + 1) \\ &= (X^2 + X + 1) \cdot P(X). \end{aligned}$$

Therefore, the only polynomials satisfying the given equation are

$$P_1(X) = 0 \quad \text{and} \quad P_2(X) = X^2 + X .$$