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MathemAttic

No. 76

The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **August 15, 2026**.

MA376. Let A, B, C be positive integers that satisfy

$$A^2 + B^2 = C^2$$

and suppose that A is prime. Show that B must be larger than A.

MA377. Observe that (1, 25, 49) is a triple of perfect squares for which the middle number is the average of the other two numbers. Show that there are infinitely many such triples.

MA378. Suppose $ABCD$ is a square with centre O. A' is the midpoint of AO and B', C' and D' are the midpoints of BO, CO and DO respectively. The two parallelograms $AB'CD'$ and $A'BC'D$ overlap. How big is this overlap as a fraction of the square $ABCD$?

MA379. I write out the numbers 1, 2, 3, 4 in a circle. Starting at 1, I cross out every second integer till just one number remains: 2 goes first, then 4, leaving 1 and 3; 3 goes next, leaving 1 - so '1' is the last number left. Suppose I start with $1, 2, 3, \ldots, n$ in a circle. For which values of n will the number '1' be the last number left?

MA380. A sequence of terms, chosen from {1, 2, 3, 4, 5, 6, 7, 8} is such that the second term is bigger than the first, and each term after the first two is greater than the sum of the previous two terms. How many possible sequences of this kind are there with at least three terms?

. . . .

Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 août 2026**.

MA376. Soient A, B et C des entiers positifs tels que

$$A^2 + B^2 = C^2$$

Supposons de plus que A soit un nombre premier. Montrez que B est strictement plus grand que A.

MA377. On remarque que (1,25,49) est un triplet de carrés parfaits dont le nombre du milieu est la moyenne des deux autres nombres. Montrez qu'il existe une infinité de tels triplets.

MA378. Soit $ABCD$ un carré de centre O. Soit A' le milieu de AO et soient B', C' et D' les milieux respectifs de BO, CO et DO. Les deux parallélogrammes $AB'CD'$ et $A'BC'D$ se chevauchent. Quelle fraction de l'aire du carré représente cette région de chevauchement ?

MA379. J'écris les nombres 1, 2, 3 et 4 sur un cercle. En partant de 1, je barre un entier sur deux jusqu'à ce qu'il ne reste plus qu'un seul nombre : 2 est éliminé en premier, puis 4, ce qui laisse 1 et 3; ensuite 3 est éliminé, de sorte que 1 est le dernier nombre restant. Supposons maintenant que l'on écrive $1, 2, 3, \dots, n$ sur un cercle. Pour quelles valeurs de n le nombre 1 sera-t-il le dernier nombre restant ?

MA380. Une suite de termes, choisis parmi $\{1, 2, 3, 4, 5, 6, 7, 8\}$, est telle que le deuxième terme est strictement plus grand que le premier, et que chaque terme à partir du troisième est strictement plus grand que la somme des deux termes précédents. Combien existe-t-il de suites de ce type comportant au moins trois termes ?

MathemAttic SOLUTIONS

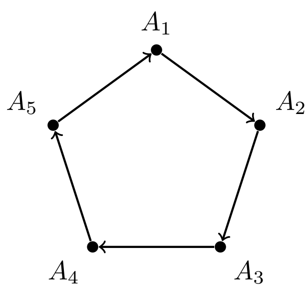
Statements of the problems in this section originally appear in 2026: 52(1), p. 8–10.

MA351. The pairwise distances between nine cities are distinct. From each city, a traveller departs to visit the nearest city. Prove that some city is visited by at least two travellers.

Originally from K. Garaschuk, A. Liu, Grade Five Competition from the Leningrad Mathematical Olympiad: 1979–1992*, example from set V.*

We received 5 submissions, 2 of which were correct and complete. We present the solution by Vibhu Konduru.

Each city sends its traveler to the nearest other city. We define an n-cycle as a bijection map f on n distinct cities such that, for some city A, $f^n(A) = A$ and $f^k(A) \neq A$ for all $k \in \{1, \ldots, n-1\}$. In other words, starting from A, one returns to A after exactly n iterations, and not before. An example of a 5-cycle is shown below.



Assume, for contradiction, that there exists a cycle

$$A_1 \mapsto A_2 \mapsto \ldots \mapsto A_n \mapsto A_1$$

with $n \geq 3$. Among the pairwise distances A_1A_2, A_2A_3, ..., A_nA_1 in the cycle, choose the smallest one; without loss of generality, let it be A_1A_2 (else, just relabel the cities). Since the traveler from A_2 goes to A_3, the city A_3 must be closer to A_2 than A_1 is, so $A_2A_3 < A_2A_1$. But this contradicts the choice of A_1A_2 as the smallest distance on the cycle. Therefore, no cycle of length greater than 2 can exist, and the only possible cycle is a 2-cycle.

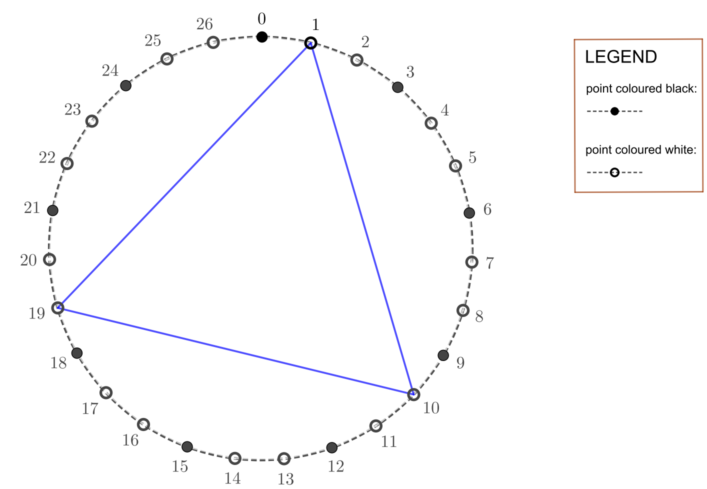
Since the only possible cycle length is 2, every cycle is a 2-cycle. Thus, if every city belonged to a cycle, the cities would be partitioned into disjoint pairs, which is only possible when n is even. But here $n = 9$, which is odd. Therefore, not every city can lie on a cycle. Starting from a city not on a cycle and repeatedly following the travelers, we must eventually enter a 2-cycle, since there is a traveler leaving from every city. When this happens, one of the two cities in that cycle receives a traveler from outside the cycle, in addition to the traveler it already receives from the other city in the cycle. Hence, there is at least one city that receives two travelers.

Editor's comments. It is also possible to solve the problem by induction, an approach which is similar to the one taken by the other solver, Jianlin Zhang. To do so, the basic idea is the following. Suppose that the statement is true for an odd number n of cities. With $n+2$ cities, pick the two cities that are separated by the smallest pairwise distance between all pairs of cities. Then these two cities will certainly exchange their travelers, so either one of them receives another traveler, or they are completely separated from the other n cities. In the latter case, the induction hypothesis ensures that one of these n cities receives two travelers. To complete this proof, it suffices to verify that when there are three cities, one of them is visited by two travelers.

MA352. A circle is divided by 27 points into 27 equal arcs. Each point is black or white. No two black points are adjacent or separated by one white point. Prove that there are 3 white points that are the vertices of an equilateral triangle.

Originally from the 1995 Leningrad Math Olympiad, problem 5.

We received 6 submissions of which 4 were correct and complete. We present 2 solutions.



Solution 1, by Vibhu Konduru.

Number the points 0 through 26; then the points n, $n+9$ and $n+18$ split the circle into three equal arcs and hence are the vertices of an equilateral triangle (we perform the addition mod 27 if necessary). It follows that there are 9 distinct equilateral triangles we can form in this manner.

Next we consider the possible colourings of the points. Note that the maximum number of black points is 9: since each black point must be separated from the next black point by more than 1 white point, there must be at least 2 white points for every black point. When there are exactly two white points for every black point we have 9 black points, as shown below (we number the points 0 through 26 starting at a black point):

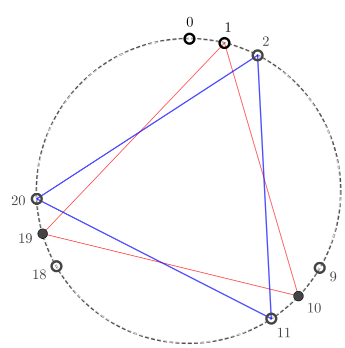
Note that in the diagram below, the points 1, 10 and 19 are white and also the vertices of an equilateral triangle. In all other cases (when there are more white points), we must have 8 or fewer black points. Since there are 9 equilateral triangles, there must be one such triangle which has no black points – that is, whose vertices are all white.

Solution 2, by Chilltime Chillizar.

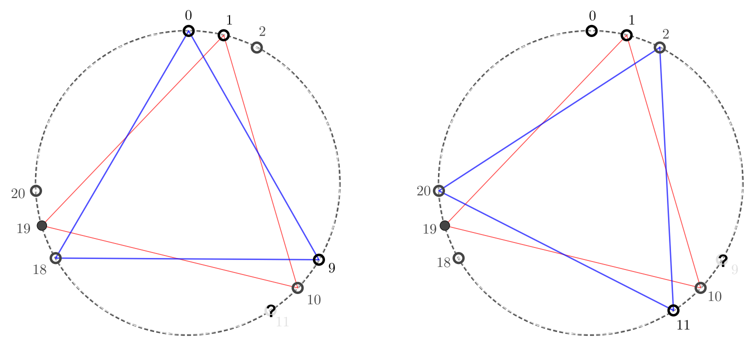
Note that the colouring restrictions mean that there are at least two white points between any two black points. Consider first the case when there are exactly two white points between any pair of consecutive black points, and number the points by 0 through 26, starting at a black point. This is the situation shown in the figure included with Solution 1: points 1, 10 and 19 are points coloured white and also the vertices of an equilateral triangle.

In all other cases, there must be a sequence of length 3 or more consisting of white points. Number the points by starting at the beginning of such a sequence, so that points 0, 1 and 2 are all coloured white. Consider cases based on the colours of points 10 and 19:

1. *Points 10 and 19 are both white.* Then points 1, 10 and 19 are white vertices of an equilateral triangle.
2. *Points 10 and 19 are both black.* Since two black points cannot be next to each we know that 9, 11, 18 and 20 are all white (see figure below); so, in particular, points 2, 11 and 20 are white vertices of an equilateral triangle.



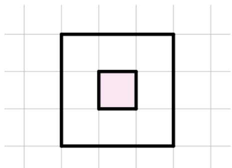
3. *Point 10 is white and point 19 is black.* Both points 18 and 20 must be white (no two black points can be next to each other). Also, the neighbours of point 10 cannot both be black, since there cannot be exactly one white point between two black ones; in other words, either 9 or 11 is white (the two cases are illustrated below). It follows that either 0, 9 and 18 or 2, 11 and 20 are white vertices of an equilateral triangle.



4. *Point 10 is black and point 19 is white.* By reflection symmetry, this case is mutatis mutandis the same as the previous one.

Therefore, we can always find three white points which are the vertices of an equilateral triangle.

MA353. Min and Max each have a 4×4 grid of 16 unit squares. Each of them removes three of the unit squares in their grid, and then computes the perimeter of their resulting shape. What is the maximum possible difference in their answers? Note: The perimeter of a shape is the sum of lengths of all the line segments that border the shape. For example, the following 3×3 square with the middle 1×1 square missing has the perimeter 16.



Originally from the 2025 Canadian Open Mathematics Challenge, problem B1.

There were 9 submissions, 7 of them were complete and correct. We present the solution by Imana Vellathottam.

We can approach this problem by analyzing Min and Max's solutions individually, then performing the subtraction.

Min's solution. Since every square shares at least two sides with adjacent squares, removing a square adds back at least two new edges on the perimeter. To reduce the perimeter as much as possible, we must remove either three corners from the 4×4 grid or three consecutive squares from the boundary. This results in losing two sides per square, and adding back two sides per square. Similar arguments apply to other cases, such as removing two corner squares and a square adjacent to exactly one of these corners. Thus, we obtain a perimeter of 16 units.

Max's solution. To maximize the perimeter, we must create as many internal sides as possible. By removing squares that don't lie on the boundary of the 4×4 grid, we get four additional sides to add to our perimeter. We can remove two interior squares that are not adjacent (opposite to each other) to increase the perimeter by $4 + 4 = 8$ units. However, the third square cannot be removed without sharing at least two sides with one of the removed squares. The next best option is to remove a square that only shares one side with the boundary of the original grid and no sides with our removed squares. This adds an additional two sides to the perimeter. Thus, the maximum perimeter we can obtain is $16 + 4 + 4 + 2 = 26$ units.

Therefore, the difference between Min and Max's solution is $26 - 16 = 10$ units.

Editor's Comments. Special mention to Konstantine Zelator for looking into a generalisation of the problem.

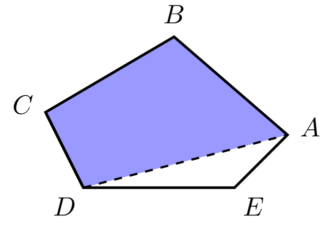
MA354. Show that, given five points in the plane in general position (that is, no three points are collinear), the number of convex quadrilaterals formed by these points is odd.

Originally exercise 6.1 from V. Jungic, "Basics of Ramsey Theory", Routledge, 2023.

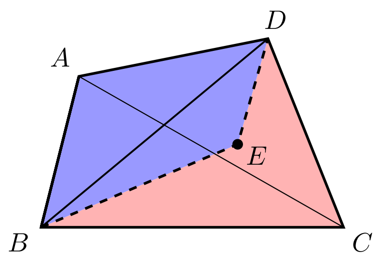
We received 6 solutions, none of which were correct and complete. We present the solution by the editor with elements taken from reader submissions.

First note that by the noncollinearity condition, the convex hull of five points in the plane is necessarily a polygon. Consider three cases according to the number of vertices of the convex hull.

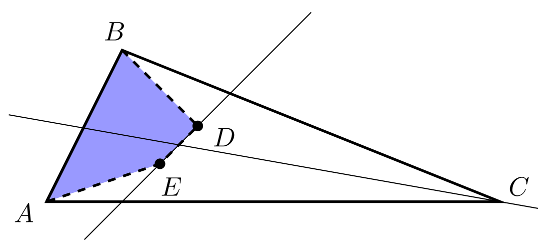
If the convex hull is a pentagon, then any four points form a convex quadrilateral. Thus in this case, there are $\binom{5}{4} = 5$ convex quadrilaterals formed.



If the convex hull is itself a quadrilateral, one of the points will be in its interior. Let the vertices of the convex hull be A, B, C, D, the interior point be E, and consider the diagonal line segments AC and BD. The point E cannot lie on either of these diagonals (else there would be three points collinear). Thus for each of the two diagonals, there is exactly one vertex on the opposite side of the line segment from E, yielding a convex quadrilateral. Thus there are two convex quadrilaterals with E as a vertex, and one more ($ABCD$) by assumption, making 3 total. (Example below: A and E are on opposite sides of BD. This makes $ABED$ and $ABCE$ convex quadrilaterals, while $BCDE$ is a nonconvex quadrilateral.)



Finally if the convex hull is a triangle, two points will be in its interior. Let the vertices of the convex hull be A, B, C and the interior points be D, E. The line passing through D and E cannot contain any of A, B, C by noncollinearity, and furthermore not all three of these points can lie on the same side of the line since D and E are assumed to be interior points. Thus there are some two points on one side of the line (say A and B), which together with D and E form a convex quadrilateral. Consider the bisector of the line segment DE which passes through point C, so that A and B are on opposite sides of this line. Without loss of generality, suppose A and E are on the same side of this line.



Any other choice of four points includes C and exactly three of the points A, B, D, E. Note that both D and E lie in the convex hull of ABC, that D lies in the convex hull of BCE, and that E lies in the convex hull of ACD. Thus the only convex quadrilateral is $ABDE$.

In each case the number of convex quadrilaterals is odd, as desired.

Editor's Comments. Some submissions were partially correct, but missing at least one of the above cases or incorrectly counting/justifying counts of convex quadrilaterals.

MA355.

(a) How many ways are there to pair up the elements of $1, 2, \ldots, 14$ into seven pairs so that each pair has sum at least 15?

(b) How many ways are there to pair up the elements of $1, 2, \ldots, 14$ into seven pairs so that each pair has sum at least 13?

(c) How many ways are there to pair up the elements of $1, 2, \ldots, 2024$ into 1012 pairs so that each pair has sum at least 2001?

Originally from the 2024 Canadian Open Mathematics Challenge, problem C1.

We received 15 submissions of which 12 were correct and complete. We present the solution by Imana Vellathottam, lightly modified.

(a) There is only 1 such way to pair the elements. Since the number 1 must pair with a number ≥ 14, the only element that works is 14. Similarly, the number 2 must be paired with 13, 3 with 12,..., 7 with 8, to result in a sum of at least 15. Hence, the only way to pair the elements 1, 2,..., 14 into seven pairs with a sum at least 15 is as follows:

$$\{1, 14\}, \{2, 13\}, \{3, 12\}, \{4, 11\}, \{5, 10\}, \{6, 9\}, \{7, 8\}.$$

(b) Consider pairing the numbers in increasing order. We can pair the number 1 with any number ≥ 12 to have a sum at least 13. The three options are 12, 13, and 14. Now consider the possible partners for the number 2. The options are 11, 12, 13 and 14. However, one of those numbers has already been paired with 1. Therefore, there are only three candidates remaining. Similarly, we can pair 3 with 10, 11, 12, 13, or 14. Given the previous pairings, two of these numbers have been used, leaving only three valid choices. This process continues for pairing numbers 1, 2, ..., 6. After 6 has been paired, the two remaining numbers are forced to select each other and satisfy the required conditions. Therefore, by the multiplication principle, there are 3^6 ways to pair up the elements of 1, 2,..., 14 into seven pairs so that each pair has sum at least 13.

(c) Consider how many choices of numbers we can pair with the number 1 to have a pair with a sum at least 2001. The valid candidates are

$$\{2000, 2001, ..., 2024\} = 25 \text{ options.}$$

Now consider the valid candidates for the number 2. The valid candidates are

$$\{1999, 2000, ..., 2024\} = 26 \text{ options.}$$

However, one of those numbers has already been paired with 1. Thus, there are only 25 choices. This continues for numbers 1, 2, ..., 1000, as expressed by 25^{1000}. At that point, we begin running out of numbers as 1001 only has

$$\{1000, 1002, 1003, ..., 2024\} = 1024$$

numbers as potential candidates. However, among the numbers larger than 1000, exactly 1000 numbers already exist in pairs. This leaves 1024 - 1000 = 24 remaining numbers. Since they are all at least 1001, they can be paired up arbitrarily, with each move resulting in a decrease of 2 options. Hence, the 24 remaining numbers can be paired in

$$23 \times 21 \times 19 \times ... \times 3 \times 1 = 23!!$$

ways. By the multiplication principle, there are

$$25^{1000} \times 23!!$$

ways to pair up the elements of 1, 2,..., 2024 into 1012 pairs each with a sum at least 2001.

Competition Highlights

The European Girls' Mathematical Olympiad

by Kateryna Tretiakova

Behind every Olympiad problem lies not only a solution but a story about how that solution earns its points. Here, we explore a problem from the 2026 Canadian Team Selection Test for the European Girls' Mathematical Olympiad (EGMO) and offer the full solution. For the first time, we also include the marking schemes used during grading. Our aim is to demystify the evaluation process and give students and teachers a clearer sense of what strong mathematical writing looks like under contest conditions.

Problem

While waiting for his train after the Winter Camp, Anji plays the following game. He writes 2^n non-negative integers $a_1, a_2, \ldots a_{2^n}$ in a circle. Each minute, each number a_i is replaced by $|a_{i+1} - a_i|$, where $a_{2^n+1} = a_1$. Show that, regardless of the initial numbers, Anji will be staring at a circle of zeros by the time his train leaves.

Note 1. The problem statement was reworded from the 5^{th} qualifying problem for Grade 10 in 2026 Innopolis Open Mathematics Olympiad.

Note 2. I would like to personally thank Anji Wang, who was indeed on the train with me after the camp, for his agreement to become the main character in this problem.

Solution

This solution is modified from the official solution by *Innopolis Open*.

Let us start by noting that the problem is identical to proving that all the numbers become zeros after finitely many steps. We first observe that the maximum of the numbers never increases. If it becomes zero, then all numbers are zero. Thus, it suffices to show that the maximum eventually becomes zero.

Define a new sequence $(b_i)_{i=1}^{2^n}$ of integers around a circle by $b_i := a_i \pmod 2$, and let us denote the value at b_i spot after $k = 0, 1, 2, \ldots$ steps of the game by $b_i^{(k)}$ with $b_i^{(0)}$ corresponding to the initial parity of a_i.

Note that the following is true for indices being taken modulo 2^n:

$$\begin{aligned} b_i^{(k+1)} = a_i^{(k+1)} \pmod 2 = |a_{i+1}^{(k)} - a_i^{(k)}| \pmod 2 \\ \equiv a_{i+1}^{(k)} + a_i^{(k)} \pmod 2 \equiv b_{i+1}^{(k)} + b_i^{(k)} \pmod 2 \,. \end{aligned}$$

Let us call this changing rule $(*)$.

Claim 1. *For all $i = 1, 2, 3, \ldots, 2^n$ and any arbitrary positive integer k, the value b_i after k steps can be written as*

$$b_i^{(k)} \equiv \sum_{j=0}^{k} \binom{k}{j} b_{i+j}^{(0)} \pmod 2 \,, \tag{1}$$

where the indices are taken modulo 2^n.

Proof. Let us prove this claim by induction on k.

Base step for $k = 1$: From the changing rule $(*)$, we have

$$b_i^{(1)} \equiv b_i^{(0)} + b_{i+1}^{(0)} \pmod 2 = \binom{1}{0} b_i^{(0)} + \binom{1}{1} b_{i+1}^{(0)} \pmod 2 \,,$$

as required.

Induction step: Assume the formula (1) holds for a natural number k. Using the changing rule $(*)$, we have that

$$\begin{aligned} b_i^{(k+1)} \equiv b_i^{(k)} + b_{i+1}^{(k)} \pmod 2 &\equiv \sum_{j=0}^{k} \binom{k}{j} b_{i+j}^{(0)} + \sum_{j=0}^{k} \binom{k}{j} b_{(i+1)+j}^{(0)} \pmod 2 \\ &\equiv \sum_{j=0}^{k} \binom{k}{j} b_{i+j}^{(0)} + \sum_{j=1}^{k+1} \binom{k}{j-1} b_{i+j}^{(0)} \pmod 2 \,. \end{aligned}$$

Then the coefficient of $b_{i+j}^{(0)}$ in the sum is

$$\begin{aligned} \sum_{j=0}^{k} \binom{k}{j} + \sum_{j=1}^{k+1} \binom{k}{j-1} &= \binom{k}{0} + \sum_{j=1}^{k} \binom{k}{j} + \sum_{j=1}^{k} \binom{k}{j-1} + \binom{k}{k} \\ &= \binom{k+1}{0} + \sum_{j=1}^{k} \binom{k+1}{j} + \binom{k+1}{k+1} = \sum_{j=0}^{k+1} \binom{k+1}{j}, \end{aligned}$$

which completes the induction. $\square$

When $k = 2^n$, by Lucas's Theorem, from Pascal's triangle, or from considering $(1+x)^{2^n} \pmod 2$, we know that all binomial coefficients $\binom{2^n}{j}$ with $0 < j < 2^n$ are even. Hence by Claim 1 at step 2^k, we have the following modulo 2:

$$b_i^{(2^n)} \equiv \binom{2^n}{0} b_{i+0}^{(0)} + \sum_{j=1}^{2^n-1} \binom{2^n}{j} b_{i+j}^{(0)} + \binom{2^n}{2^n} b_{i+2^n}^{(0)} \equiv b_i^{(0)} + b_i^{(0)} \equiv 0.$$

In other words, after 2^n steps every a_i is even.

Now consider the process in blocks of 2^n steps. After the first block, all numbers are multiples of 2, so let's write $a_i^{(2^n)} = 2c_i$ with c_i integers. Note that the evolution of the $(c_i)_{i=1}^{2^n}$ follows the same rule as $(a_i)_{i=1}^{2^n}$ since

$$|c_{i+1} - c_i| = \frac{|a_{i+1}^{(2^n)} - a_i^{(2^n)}|}{2}.$$

Thus the configuration (c_i) is exactly the state of the process after the first block, scaled down by a factor of 2. Applying the same reasoning to (c_i), after another 2^n steps all c_i become even. Consequently, the original a_i after $2 \cdot 2^n$ steps become multiples of 4. Thus, after t blocks of 2^n steps, every a_i is divisible by 2^t.

To finish the problem, let us consider $m = \max\{a_1, \ldots, a_{2^n}\}$ in the original circle. If $m = 0$, then we are done. Otherwise, choose t such that $2^t > m$, i.e., an arbitrary positive integer $t > \log_2 m$. Since every a_i is a nonnegative integer less or equal to m and after $t \cdot 2^n$ steps it must be divisible by 2^t, we must have $a_i = 0$ for all i.

Remark: *The circle with all zeros is reached in at most* $2^n \lceil \log_2(m+1) \rceil$ *steps.*

Marking Scheme

Any complete solution is awarded 7 points. Partial solutions should be evaluated according to the following scheme. Points for a part are awarded only if all claims in that item have been justified, unless otherwise specified in a subpoint.

Part A Noting that the maximum among the a_i never increases . . . . 1 point

Part B Showing that all a_i become even after finitely many steps . . . . 4 points

- **(B1)** Considering the sequence $b_i \equiv a_i \pmod 2$ . . . . 1 point
- **(B2)** Proving Claim 1 . . . . 2 points
- **(B3)** Proving that after 2^n steps all a_i are even . . . . 1 point

Part C Concluding that all entries eventually become zero . . . . 2 points

Part D Observations not worth points

- **(D1)** Claiming the statement of Claim 1 without proof . . . . 0 points

(D2) Considering the number of occurrences of the maximum value in the circle, possibly claiming that this number must decrease over time . . . 0 points

(D3) Considering the sum of all entries, possibly claiming that this quantity must decrease over time . . . . 0 points

(D4) Considering the $\gcd(a_1, a_2, \ldots, a_{2^n})$ or similar . . . . 0 points

Remarks.

- This scheme applies strictly to the presented solution. Any alternative *correct* solution must be mapped to the logical equivalents of Parts A, B, C.
- Merely stating Claim 1 or mentioning *induction* earns 0 points for **Part B2**. A proof with minor errors (e.g., off-by-one in indices, missing modulo notation) earns 1 point at the grader's discretion.
- **Part C** requires at least 3 points from Part B. The standard completion as well as any *correct* way to finish the solution earns 2 points. An argument that correctly shows iteration but omits the finiteness justification earns 1 point. Claiming immediate zero after 2^n steps earns 0 points.

Acknowledgements

I would like to thank the deputy leader of the team, Emily Ma, together with James Rickards, Luciano Salvetti, and Eric Shen for proof-reading the materials either during the TST preparation period or during the preparation of the article.

OLYMPIAD CORNER

No. 444

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **August 15, 2026**.

OC786. Let ABC be an obtuse triangle with orthocenter H and centroid S. Let D, E and F be the midpoints of segments BC, AC, AB, respectively. Show that the circumcircle of triangle ABC, the circumcircle of triangle DEF and the circle with diameter HS have two distinct points in common.

OC787. Let n be a positive integer and let $z_1, z_2, \ldots, z_n$ be positive integers such that for $j = 1, 2, \ldots, n$ the inequalities $z_j \leq j$ hold and $z_1 + \cdots + z_n$ is even. Prove that the number 0 occurs among the values of

$$z_1 \pm z_2 \pm \cdots \pm z_n,$$

where $+$ or $-$ can be chosen independently for each operation.

OC788. For each prime number p, determine the number of residue classes modulo p which can be represented as a^2+b^2 modulo p, where a and b are arbitrary integers.

OC789. We say that a real number V is *good* if there exist two closed convex subsets X, Y of the unit cube in $\mathbb{R}^3$ with volume V each, such that for each of the three coordinate planes (that is, the planes spanned by any two of the three coordinate axes), the projections of X and Y onto that plane are disjoint. Find $\sup\{V \mid V \text{ is good}\}$.

OC790. For every positive integer n, let $f(n)$, $g(n)$ be the minimal positive integers such that

$$1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} = \frac{f(n)}{g(n)}.$$

Determine whether there exists a positive integer n for which $g(n) > n^{0.999n}$.

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 août 2026**.

OC786. Soit ABC un triangle obtusangle, d'orthocentre H et de centre de gravité S. Soient D, E et F les milieux respectifs des segments BC, AC et AB. Montrez que le cercle circonscrit au triangle ABC, le cercle circonscrit au triangle DEF et le cercle de diamètre HS ont deux points distincts en commun.

OC787. Soit n un entier positif et soient $z_1, z_2, \ldots, z_n$ des entiers positifs tels que, pour tout $j = 1, 2, \ldots, n$ on ait $z_j \leq j$ et tels que $z_1 + \cdots + z_n$ soit pair. Montrez que le nombre 0 figure parmi les valeurs de

$$z_1 \pm z_2 \pm \cdots \pm z_n,$$

où les signes $+$ et $-$ peuvent être choisis indépendamment à chaque opération.

OC788. Pour chaque nombre premier p, déterminez le nombre de classes de résidus modulo p qui peuvent être représentées sous la forme $a^2 + b^2$ modulo p, où a et b sont des entiers arbitraires.

OC789. On dit qu'un nombre réel V est *bon* s'il existe deux sous-ensembles convexes fermés X et Y du cube unité de $\mathbb{R}^3$, chacun de volume V, tels que, pour chacun des trois plans de coordonnées (c'est-à-dire les plans engendrés par deux des trois axes de coordonnées), les projections de X et de Y sur ce plan soient disjointes. Déterminez $\sup\{V \mid V \text{ est bon}\}$.

OC790. Pour tout entier positif n, soient $f(n)$ et $g(n)$ les plus petits entiers positifs tels que

$$1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} = \frac{f(n)}{g(n)}.$$

Déterminez s'il existe un entier positif n tels que $g(n) > n^{0.999n}$.

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2026: 52(1), p. 33–35.

OC761. At a school, there are a number of clubs. A club is a set of students. Each club contains at least one student. A student may be in more than one club, but cannot be in every club. Surprisingly, for any two clubs A and B at the school, their union $A \cup B$ is also a club. Is it guaranteed that there is a club containing an even number of students?

Originally from the AMOC Senior Contest 2024, Problem 4.

We received 10 solutions, all correct. We present the solution by Oliver Geupel.

The answer is yes. For the sake of obtaining a contradiction, suppose that all clubs $C_1, C_2, \dots, C_n$ have odd cardinalities. Since the collection of all clubs is closed under union, the union of all clubs

$$\Omega = \bigcup_{k=1}^{n} C_k$$

is once again a club, so that $|\Omega|$ is odd. Every intersection of Ω-complements of clubs

$$\bigcap_{j\in J} \overline{C_j} = \overline{\bigcup_{j\in J} C_j}$$

is once again the complement of a club, from which we deduce that its cardinality is even. Since no student is in every club, we have $\Omega = \bigcup_{k=1}^{n} \overline{C_k}$. The inclusion-exclusion principle states that

$$|\Omega| = \left|\bigcup_{k=1}^{n} \overline{C_k}\right| = \sum_{\varnothing \neq J \subseteq \{1,\dots,n\}} (-1)^{|J|+1} \left|\bigcap_{j\in J} \overline{C_j}\right|$$

While $|\Omega|$ is odd, the terms of the sum are even. This is the desired contradiction.

OC762. Consider a 2024×2024 grid of unit squares. Two distinct unit squares are *adjacent* if they share a common side. Each unit square is to be colored either black or white. Such a colouring is called *evenish* if every unit square in the grid is adjacent to an even number of black unit squares. Determine the number of evenish colorings.

Originally from the Australian Mathematical Olympiad 2024, Problem 4.

We received no submissions for this problem.

OC763. In a school, there are 1000 students in each year level, from Year 1 to Year 12. The school has 12000 lockers, numbered from 1 to 12000. The school principal requests that each student is assigned their own locker, so that the following condition is satisfied:

> For every pair of students in the same year level, the difference between their locker numbers must be divisible by their year-level number.

Can the principal's request be satisfied?

Originally from the Australian Mathematical Olympiad 2024, Problem 6.

We received 1 correct solution by Aaron Thiessen, which we present below.

We first want to show that all locker numbers in a certain year must have the same residue mod that year number. Suppose there are two lockers that have a different residue mod n. The difference of these locker numbers must be divisible by n, or congruent to 0 mod n. If the two residues are different, their difference must not be congruent to 0 mod n, meaning that all locker numbers must have the same residue.

We start by assigning lockers backwards, assigning 1000 year 12 lockers, then 1000 year 11, and so on. Since there are 1000 lockers for each residue mod 12, we can pick an arbitrary residue mod 12.

We now will use induction to prove the rest of the lockers can be assigned. For the inductive hypothesis, suppose that all of the lockers for every year until x are assigned, satisfying the conditions. We now want to assign lockers for the next year, $x-1$. There would already be $12000-1000(x-1)$ lockers assigned, meaning there are $1000(x-1)$ lockers left. There are $x-1$ possible residues mod $x-1$, so by the Pigeonhole Principle, there must be at least 1 residue that has at least 1000 lockers. We can then assign these lockers to the next year, satisfying the condition.

This proves the inductive hypothesis, and since the base case is proved, it is possible to assign all 12000 lockers.

OC764. The set of natural numbers from 1 to 1000 inclusive is partitioned into two groups A and B of 500 numbers each. For an integer k, let N_k be the number of pairs (a, b) of a number $a \in A$ and a number $b \in B$ such that $a - b = k$. Prove that:

(a) in any such partition there exists k such that $N_k \geq 126$;

(b) there exists a partition such that $N_k \leq 250$ for every k.

Originally from the Bulgarian Mathematical Olympiad 2024, Second Round, Grade 8, Problem 4.

We received 5 solutions. We present the solution by Roy Barbara.

(a) More generally, if we replace 1000 (in the hypothesis) by $2n$, $n > 1$ ($n \in \mathbb{N}$), then, given any such partition, there exists k such that

$$N_k \geq \left\lfloor \frac{n^2}{2(2n-1)} \right\rfloor + 1.$$

Indeed, consider the map $f : A \times B \to \mathbb{Z}$, $f((a, b)) = a - b = k$. As $1 - 2n \leq a - b \leq 2n - 1$ and $a \neq b$, then,

$$\operatorname{Im} f \subseteq \{-(2n-1), \ldots, -1\} \cup \{1, \ldots, 2n-1\}.$$

Hence,

$$|\operatorname{Im} f| \leq 2(n-1) \tag{3}$$

Note that for $k \in \operatorname{Im} f$, $|f^{-1}(\{k\})| = N_k$. Hence by (1) we may write,

$$|A \times B| = \sum_{k \in \operatorname{Im} f} |f^{-1}(\{k\})| = \sum_{k \in \operatorname{Im} f} N_k.$$

In view of contradiction, suppose that

$$N_k \leq \left\lfloor \frac{n^2}{2(2n-1)} \right\rfloor$$

for all k, then, using (2) and (3) we get,

$$\begin{aligned} n^2 = |A \times B| = \sum_{k \in \operatorname{Im} f} N_k \leq \sum_{k \in \operatorname{Im} f} \left\lfloor \frac{n^2}{2(2n-1)} \right\rfloor &< \sum_{k \in \operatorname{Im} f} \frac{n^2}{2(2n-1)} \\ &\leq 2(n-1) \frac{n^2}{2(2n-1)} = n^2. \end{aligned}$$

We get $n^2 < n^2$, a contradiction.

(b) More generally, if we replace 1000 (in the hypothesis) by $4n$ ($n \in \mathbb{N}$), then, there exists a partition such that $N_k \leq n$ for every k. Indeed, set

$$A = \{n+1, n+2, \ldots, 3n\}$$

and

$$B = B_1 \cup B_2, \ B_1 = \{1, 2, \ldots, n\}, \ B_2 = \{3n+1, 3n+2, \ldots, 4n\}.$$

Let $(a, b) \in A \times B$. Clearly, if $b \in B_1$, then, $a - b = k \geq 1$ and if $b \in B_2$, then

$$a - b = k \leq -1. \tag{4}$$

- Let $k \in \mathbb{Z}$. If $k = 0$, as $a \neq b$, then $N_0 = 0$.
- Let $k \geq 1$ be given: Let $b \in B$ be given. If $b \in B_2$, then by (4), there is no $a \in A$ such that $a - b = k$. If $b = b_1 \in B_1$, there is at most one $a \in A$ such that $a - b = k$, namely $a = b_1 + k$. (for a to exist, the value $b_1 + k$ should lie in the range $A = \{n+1, n+2, \ldots, 3n\}$). Since $|B_1| = n$, we conclude that $N_k \leq n$.

- Let $k \leq -1$ be given: Let $b \in B$ be given. If $b \in B_1$, then by (4), there is no $a \in A$ such that $a - b = k$. If $b = b_2 \in B_2$, there is at most one $a \in A$ such that $a - b_2 = k$, namely $a = b_2 + k$. Since $|B_2| = n$, we conclude that $N_k \leq n$.

OC765. Given a family $\mathcal{F}$ of 4-element subsets (4-tuples) of a given set of 5^m elements, where m is a fixed natural number. It is known that the intersection of no two 4-tuples in $\mathcal{F}$ consists of exactly two elements. Find the maximum possible value for the number of 4-tuples in $\mathcal{F}$.

Originally from the Bulgarian Mathematical Olympiad 2024, Final Round, Grades 9-12, Problem 5.

We received 1 solution. We present the solution by Sicheng Du, modified by the Editorial Board.

The construction. Let the universal set be a set of 5^m points $U = \{0, 1, 2, 3, 4\}^m$. We will regard all points in $\mathbb{Z}^m$ as an element in U by replacing the coordinates with their remainders modulo 5.

Define a *line* to be a set of 5 points

$$\ell(\boldsymbol{a}, \boldsymbol{v}) = \{\boldsymbol{a} + t\boldsymbol{v} \mid t = 0, 1, 2, 3, 4\}$$

for any $\boldsymbol{a} \in U$ and $\boldsymbol{v} \in U \setminus \{\boldsymbol{0}\}$. We claim that there is a unique line through any pair of distinct points $\boldsymbol{a}$, $\boldsymbol{b} \in U$.

Existence. Let $\boldsymbol{v} = \boldsymbol{b} - \boldsymbol{a}$, then $\ell(\boldsymbol{a}, \boldsymbol{v})$ is a line passing $\boldsymbol{a}$ and $\boldsymbol{b}$.

Uniqueness. Assume that there is a second line $\ell(\boldsymbol{a}', \boldsymbol{v}')$ passing $\boldsymbol{a}$ and $\boldsymbol{b}$. If $\boldsymbol{a} = \boldsymbol{a}' + t_1\boldsymbol{v}'$ and $\boldsymbol{b} = \boldsymbol{a}' + t_2\boldsymbol{v}'$, then $\boldsymbol{v} = (t_2 - t_1)\boldsymbol{v}'$. It follows from 5 being prime that

$$\{(t + t_1)\boldsymbol{v}' \mid t = 0, 1 \ldots, 4\} = \{t\boldsymbol{v}' \mid t = 0, 1 \ldots, 4\} = \{t\boldsymbol{v} \mid t = 0, 1 \ldots, 4\}.$$

Hence, $\ell(\boldsymbol{a}', \boldsymbol{v}') = \ell(\boldsymbol{a}, \boldsymbol{v}') = \ell(\boldsymbol{a}, \boldsymbol{v})$ and the claim is proven.

Hence, every line contains $\binom{5}{2}$ unordered pairs of points, and every unordered pair of distinct points lies on a unique line. Therefore the number of distinct lines is

$$\frac{\binom{5^m}{2}}{\binom{5}{2}} = \frac{5^m(5^m - 1)}{20} = \frac{5^{m-1}(5^m - 1)}{4}.$$

Each line gives exactly 5 distinct 4-subsets by deleting one of its points. Also, no 4-subset obtained from one line is obtained from another line, since such a 4-subset contains two points, and any two points determine the line uniquely. Thus we obtain

$$5 \cdot \frac{5^{m-1}(5^m - 1)}{4} = \frac{5^m(5^m - 1)}{4}$$

distinct 4-subsets. Let $\mathcal{F}$ be the family of all these 4-subsets.

For any two elements of $\mathcal{F}$, if they come from the same line, then they share 3 points. If they come from different lines, then they share at most 1 point, because two common points would determine the same line. Therefore $\mathcal{F}$ is valid and has size
$$\frac{5^m(5^m-1)}{4}.$$

Proof. Let $\mathcal{F}$ be a valid family, and put $n = 5^m$. For every unordered pair $p = \{\boldsymbol{a}, \boldsymbol{b}\}$ of distinct points of U, let $f(p)$ be the number of members of $\mathcal{F}$ containing both points of p. By double counting pairs contained in members of $\mathcal{F}$, we have
$$\sum_{p\in\binom{U}{2}} f(p) = \binom{4}{2}|\mathcal{F}| = 6|\mathcal{F}|.$$

We will prove that
$$\sum_{p\in\binom{U}{2}} f(p) \leq 3\binom{n}{2}.$$

Then $6|\mathcal{F}| \leq 3\binom{n}{2}$, so
$$|\mathcal{F}| \leq \frac{1}{2}\binom{n}{2} = \frac{n(n-1)}{4} = \frac{5^m(5^m-1)}{4}.$$

Call a pair p heavy if $f(p) \geq 4$. Suppose $f(\{\boldsymbol{a}, \boldsymbol{b}\}) = x \geq 4$. Let $s_1, \ldots, s_x$ be the members of $\mathcal{F}$ containing both $\boldsymbol{a}$ and $\boldsymbol{b}$. Since two distinct members of $\mathcal{F}$ cannot intersect in exactly 2 points, after renaming points we may write
$$s_1 = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}_1\}, \qquad s_2 = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}_2\}.$$

We claim that every s_i contains $\boldsymbol{c}$. Indeed, suppose some s_i does not contain $\boldsymbol{c}$. Since s_i intersects both s_1 and s_2 in at least $\boldsymbol{a}, \boldsymbol{b}$, it must contain both $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$. Thus
$$s_i = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}_1, \boldsymbol{d}_2\}.$$

But then any further member containing $\boldsymbol{a}$ and $\boldsymbol{b}$ would have to be one of
$$\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}_1\}, \qquad \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}_2\}, \qquad \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}_1, \boldsymbol{d}_2\},$$

otherwise it would intersect one of these sets in exactly 2 points. This contradicts $x \geq 4$. Therefore all x members containing $\boldsymbol{a}$ and $\boldsymbol{b}$ have the form
$$\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}_i\}, \qquad i = 1, \ldots, x,$$

where the points $\boldsymbol{d}_1, \ldots, \boldsymbol{d}_x$ are distinct.

It follows that
$$f(\{\boldsymbol{a}, \boldsymbol{b}\}) = f(\{\boldsymbol{a}, \boldsymbol{c}\}) = f(\{\boldsymbol{b}, \boldsymbol{c}\}) = x.$$

Indeed, all x sets above contain these three pairs. Conversely, if another member contained $\boldsymbol{a}$ and $\boldsymbol{c}$ but not $\boldsymbol{b}$, then it would have to contain every $\boldsymbol{d}_i$ in order to avoid intersecting $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}_i\}$ in exactly 2 points. This is impossible since $x \geq 4$. Hence $f(\{\boldsymbol{a}, \boldsymbol{c}\}) = x$, and similarly $f(\{\boldsymbol{b}, \boldsymbol{c}\}) = x$.

Now we show that for every $i = 1, \ldots, x$,

$$f(\{\boldsymbol{a}, \boldsymbol{d}_i\}) = f(\{\boldsymbol{b}, \boldsymbol{d}_i\}) = f(\{\boldsymbol{c}, \boldsymbol{d}_i\}) = 1.$$

For example, suppose some member $T \neq \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}_i\}$ contains $\boldsymbol{a}$ and $\boldsymbol{d}_i$. Since T intersects $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}_i\}$ in at least 2 points, it must contain $\boldsymbol{b}$ or $\boldsymbol{c}$. If it contains $\boldsymbol{b}$, then it contains the pair $\{\boldsymbol{a}, \boldsymbol{b}\}$, so it must be one of the sets

$$\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}_j\}.$$

Since it also contains $\boldsymbol{d}_i$, it must be $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}_i\}$, a contradiction. The case where T contains $\boldsymbol{c}$ is identical. Therefore $f(\{\boldsymbol{a}, \boldsymbol{d}_i\}) = 1$. The same argument gives

$$f(\{\boldsymbol{b}, \boldsymbol{d}_i\}) = f(\{\boldsymbol{c}, \boldsymbol{d}_i\}) = 1.$$

Thus every heavy pair belongs to a unique heavy triangle

$$\{\boldsymbol{a}, \boldsymbol{b}\}, \qquad \{\boldsymbol{a}, \boldsymbol{c}\}, \qquad \{\boldsymbol{b}, \boldsymbol{c}\},$$

whose three pairs all have the same value $x \geq 4$. This triangle is accompanied by the $3x$ pairs

$$\{\boldsymbol{a}, \boldsymbol{d}_i\}, \qquad \{\boldsymbol{b}, \boldsymbol{d}_i\}, \qquad \{\boldsymbol{c}, \boldsymbol{d}_i\}, \qquad i = 1, \ldots, x,$$

all of which have f-value 1.

The collections obtained from different heavy triangles are disjoint. Indeed, a heavy pair determines its heavy triangle uniquely, and if a pair of f-value 1 belonged to two such collections, then the unique member of $\mathcal{F}$ containing it would contain two different heavy triangles, which is impossible by the preceding paragraph.

For one such collection, the total contribution to $\sum f(p)$ is $3x + 3x = 6x$, coming from three heavy pairs of value x and $3x$ pairs of value 1. The number of pairs in the collection is $3 + 3x$, so the average value of f on this collection is

$$\frac{6x}{3+3x} = \frac{2x}{x+1} < 3.$$

All remaining pairs are not heavy, so their f-values are at most 3. Therefore

$$\sum_{p \in \binom{U}{2}} f(p) \leq 3\binom{n}{2}.$$

Hence

$$|\mathcal{F}| \leq \frac{5^m(5^m - 1)}{4}.$$

Together with the construction, the maximum possible value is $\dfrac{5^m(5^m-1)}{4}$.

The α-method for solving and refining symmetrical and homogeneous inequalities

Mihály Bencze and Marius Drăgan

Abstract

The new α-method for solving and refining symmetrical and homogenous algebraic inequalities in three variables was proposed in [1]. In this work we present more details about this method, and apply it to solve and generalise a wide range of inequalities published in ***Crux***.

1 Introduction

Recently, M. Drăgan has proposed a new method for solving and refining symmetrical and homogeneous algebraic inequalities in three variables [1]. This method links the symmetric sums of three variables in a way which enables fine comparisons between expressions which involve them.

In the first part of this paper we present the "α-method", which we apply in the second part to solve some inequalities published in this journal.

Theorem 1. *[The α-method] Let x, y, z be positive numbers. We denote:*

$$s = x+y+z, \quad p = xyz, \quad \alpha = (x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 9,$$

$$t_{1,2} = \frac{\alpha^2+18\alpha-27 \pm \sqrt{(\alpha-1)(\alpha-9)^3}}{8}$$

$$t_3 = 4\alpha-9, \quad t_4 = \frac{\alpha^3}{4\alpha-9}, \quad t = \frac{(x+y+z)^3}{xyz} \geq 27.$$

The following inequalities hold:

$$t_3 \leq t_1 \leq t \leq t_2 \leq t_4. \tag{1}$$

Proof. We first prove that $t_3 \leq t \leq t_4$. By Schur's inequality we have

$$s^3 + 9p \geq 4s\sum xy \quad \text{or} \quad t = \frac{s^3}{p} \geq 4\alpha - 9 \quad \text{or} \quad t \geq t_3.$$

Also from Schur's inequality, we get

$$\left(\sum \frac{1}{x}\right)^3 + \frac{9}{xyz} \geq 4\sum \frac{1}{xy}\sum \frac{1}{x}$$

or $\dfrac{\alpha^3}{4\alpha-9} = t_4 \geq t$.

We now prove that $t_1 \leq t \leq t_2$. Indeed, one can check that x, y, z are the positive roots of the equation

$$x^3 - \sigma_1 x^2 + \sigma_2 x - \sigma_3 = 0,$$

where $\sigma_1, \sigma_2, \sigma_3$ are the symmetric sums

$$\sigma_1 = x + y + z, \quad \sigma_2 = xy + yz + zx, \quad \sigma_3 = xyz.$$

Then, it is known (see [4]) that:

$$18\sigma_1\sigma_2\sigma_3 - 4\sigma_1^3\sigma_3 + \sigma_1^2\sigma_2^2 - 4\sigma_2^3 - 27\sigma_3^2 \geq 0.$$

If we replace $\sigma_1 = s$, $\sigma_2 = \dfrac{p\alpha}{s}$, $\sigma_3 = p$, this becomes

$$18\alpha - 4t + \alpha^2 - \frac{4\alpha^3}{t} - 27 \geq 0,$$

which is equivalent to $t \in [t_1, t_2]$. $\square$

Using the previous notations, the symmetric and homogeneous inequalities in three variables of the form $F(a, b, c) \geq 0$, can be rewritten as $f(t) = g(t, \alpha) \geq 0$, where $t \in [t_1, t_2]$ or $t \in [t_3, t_4]$ with $\alpha \geq 9$ fixed.

The monotonicity of the function f (defined in various problems) on the intervals $[t_1, t_2]$ or $[t_3, t_4]$ will allow us to prove many symmetric inequalities. In particular, we recover some results published in ***Crux***.

2 Applications

Application 2.1. Let a, b, c be positive numbers and $\alpha = \left(\sum a\right)\left(\sum \dfrac{1}{a}\right)$.

Prove that

$$\frac{4(a^2+b^2+c^2)}{ab+bc+ca} + \frac{2(ab+bc+ca)}{a^2+b^2+c^2} \leq \frac{4\alpha^3}{4\alpha-9} + \frac{8\alpha-18}{\alpha^2-8\alpha+18} - 8.$$

Solution. Writing the left-hand side in the α notation we obtain

$$\frac{4\sum a^2}{\sum ab} + \frac{2\sum ab}{\sum a^2} = \frac{4(s^3-2p\alpha)}{p\alpha} + \frac{2p\alpha}{s^3-2p\alpha} = \frac{4t}{\alpha} - 8 + \frac{2\alpha}{t-2\alpha}.$$

Let us consider the function $f : [t_3, t_4] \to \mathbb{R}$ defined as

$$f(t) = \frac{4t}{\alpha} - 8 + \frac{2\alpha}{t-2\alpha}. \tag{2}$$

Clearly for $t \geq t_3$, we have

$$f'(t) = \frac{4}{\alpha} - \frac{2\alpha}{(t-2\alpha)^2} \geq \frac{4}{\alpha} - \frac{2\alpha}{(2\alpha-9)^2}.$$

But

$$\frac{4}{\alpha} - \frac{2\alpha}{(2\alpha-9)^2} \geq 0$$

if

$$\alpha \geq \frac{36+9\sqrt{2}}{7} \simeq 6,36,$$

so true as $\alpha \geq 9$.

Since $f'(t) \geq 0$, $\forall\, t \in [t_3, t_4]$, function f is increasing on $[t_3, t_4]$. Hence

$$f(t) \leq f(t_4) = \frac{4\alpha^2}{4\alpha-9} + \frac{8\alpha-18}{\alpha^2-8\alpha+18} - 8.$$

Remark 2.1. Since

$$\frac{4\alpha^2}{4\alpha-9} + \frac{8\alpha-18}{\alpha^2-8\alpha+18} - 8 - \alpha + 3 = \frac{-(\alpha-9)(11\alpha^2-66\alpha+108)}{(4\alpha-9)(\alpha^2-8\alpha+18)} \leq 0,\ \forall\,\alpha \geq 9,$$

we obtain

$$\frac{4\alpha^2}{4\alpha-9} + \frac{8\alpha-18}{\alpha^2-8\alpha+18} - 8 \leq \alpha - 3 = \sum \frac{b+c}{a}.$$

Application 2.1 represents a refinement of Problem 4497 from ***Crux*** 45(10), authored by Hoang Le Nhat Tung, with the statement:

Consider a, b, c positive numbers. Prove that

$$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq \frac{4(a^2+b^2+c^2)}{ab+bc+ca} + \frac{2(ab+bc+ca)}{a^2+b^2+c^2}.$$

Application 2.2. Let a, b, c be positive real numbers and define the expression $\alpha = \left(\sum a\right)\left(\sum \frac{1}{a}\right)$. Prove that

$$\sum \frac{1}{a^2} - 27\left(\sum \frac{ab}{c}\right)^{-2} \geq \frac{1}{3}\sum\left(\frac{1}{a} - \frac{1}{b}\right)^2 + \frac{4(a+b+c)(\alpha-9)(4\alpha-9)}{3abc(2\alpha-9)^2}.$$

Solution. Clearly, we have the identities

$$\sum \frac{1}{a^2} = \left(\sum \frac{1}{a}\right)^2 - 2\sum \frac{1}{bc} = \frac{\alpha^2}{s^2} - \frac{2s}{p}$$

$$\left(\frac{\sum ab}{c}\right)^{-2} = \left(\frac{\prod a}{\sum a^2b^2}\right)^{-2} = \frac{s^4}{(p\alpha^2-2s^3)^2}$$

$$\sum\left(\frac{1}{a} - \frac{1}{b}\right)^2 = \frac{2\alpha^2}{s^2} - \frac{6s}{p}.$$

In this notation, for $t \in [t_3, t_4]$ we obtain

$$\begin{aligned}\sum \frac{1}{a^2} - 27\left(\sum \frac{ab}{c}\right)^{-2} - \frac{1}{3}\sum\left(\frac{1}{a}-\frac{1}{b}\right)^2 &= \frac{1}{3}\,\frac{\alpha^2}{s^2} - \frac{27s^4}{(p\alpha^2-2s^3)^2}\\ &= \frac{s}{p}\left(\frac{\alpha^2}{3t} - \frac{27t}{(\alpha^2-2t)^2}\right)\\ &= \frac{s}{p} f(t),\end{aligned}$$

where the function $f : [t_3, t_4] \to \mathbb{R}$ is defined by

$$f(t) = \frac{\alpha^2}{3t} - \frac{27t}{(\alpha^2-2t)^2}. \tag{3}$$

If follows that

$$f'(t) = \frac{-\alpha^2}{3t^2} - \frac{27(\alpha^2+2t)}{(\alpha^2-2t)^3}, \quad t \in [t_3, t_4].$$

Since

$$t \le t_4 = \frac{\alpha^3}{4\alpha-9} \le \frac{\alpha^2}{2}, \quad \forall\, \alpha \ge 9,$$

we have $\alpha^2 - 2t > 0$.

Since $f'(t) < 0$, $\forall\, t \in [t_3, t_4]$, function f is decreasing on $[t_3, t_4]$ or

$$f(t) \ge f(t_4) = \frac{4(\alpha-9)(4\alpha-9)(a+b+c)}{3(2\alpha-9)^2 abc}\,,\ \forall\, \alpha \ge 9,$$

which proves the inequality.

Application 2.2 is a refinement of Problem 2930 of ***Crux*** 49(3), authored by José Díaz Barrero:

> Suppose that a, b, c are positive real numbers. Prove that
>
> $$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}-27\left(\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\right)^2 \ge \frac{1}{3}\left[\left(\frac{1}{a}-\frac{1}{b}\right)^2+\left(\frac{1}{b}-\frac{1}{c}\right)^2+\left(\frac{1}{c}-\frac{1}{a}\right)^2\right].$$

Application 2.3. Let a, b and c be positive real numbers such that $\sum a^2 = 1$ and consider $\alpha = \left(\sum a\right)\left(\sum \frac{1}{a}\right)$. Prove that

$$\sum a^2 \sum \frac{1}{a^2} \ge 3 + \frac{2\sum a^3}{abc} + \frac{(\alpha-9)\left(\alpha-5-\sqrt{(\alpha-1)(\alpha-9)}\right)}{4}.$$

Solution. One can write

$$\sum a^2 \sum \frac{1}{a^2} - 3 - \frac{2\sum a^3}{abc} = \frac{s^3-2\alpha p}{s}\cdot\frac{\alpha^2-2s^3}{ps^2} - 3 - \frac{2(s^3-3\alpha p+3p)}{p}.$$

Consider the function $f : [t_1, t_2] \to \mathbb{R}$ defined as

$$f(t) = \frac{(t-2\alpha)(\alpha^2 - 2t)}{t} - 3 - 2(t - 3\alpha + 3),$$

for which

$$f'(t) = \frac{2(\alpha^3 - 2t^2)}{t}$$

and $t_0 = \sqrt{\dfrac{\alpha^3}{2}}$ is a critical point.

We have $t_2 \le t_0$. We distinguish two cases, according to Wolfram Alpha.

Case 1. $9 \le \alpha \le \alpha_0 = 3\left(3 + \sqrt{3} + \sqrt{9 + 6\sqrt{3}}\right) \simeq 27.40$.

In this case $t_0 \le t_1 \le t_2$, hence f is decreasing, so

$$f(t) \ge f(t_2) = \frac{(\alpha-9)(\alpha - 5 - \sqrt{(\alpha-1)(\alpha-9)}}{4}.$$

Case 2. $\alpha > \alpha_0$.

In this case $t_1 \le t_0 \le t_2$, hence $f(t) \ge \min\{f(t_1), f(t_2)\} = f(t_2)$ since

$$f(t_1) = \frac{(\alpha-9)(\alpha - 5 + \sqrt{(\alpha-1)(\alpha-5)}}{4}.$$

So in both cases $f(t) \ge \dfrac{(\alpha-9)(\alpha - 5 - \sqrt{(\alpha-1)(\alpha-5)}}{4} \ge 0.$

Application 2.3 is a refinement of Problem 2532 in ***Crux*** 26(3) by Ho-joo Lee with the statement:

> Suppose that a, b, c are positive real numbers satisfying $a^2+b^2+c^2 = 1$. Prove that
> $$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \ge 3 + \frac{2(a^3+b^3+c^3)}{abc}.$$

Application 2.4. Let a, b, c be the side lengths of a triangle with inradius r and circumradius R, semiperimeter s and $\alpha = s\sum \dfrac{1}{s-a}$. Prove that

$$\frac{12\alpha^3 + 28\alpha^2 - 75\alpha + 27}{18\alpha^3 + 12\alpha^2 - 31\alpha + 9} \le \sum_{cyclic} \frac{a}{2a+b+c} \le \frac{55\alpha - 111}{75\alpha - 163}.$$

Solution. Consider the substitution $a = y+z$, $b = z+x$, $c = x+y$.

We have

$$\frac{R}{r} = \frac{\prod(y+z)}{4xyz}, \quad \alpha = s\sum\frac{1}{s-a} = \sum x \cdot \sum \frac{1}{x}.$$

Then

$$\begin{aligned}\sum_{cyclic} \frac{a}{2a+b+c} &= \sum \frac{y+z}{2x+3y+3z}\\ &= \frac{\sum\limits_{cyclic} (y+z)(2y+3z+3x)(2z+3x+3y)}{\prod(2x+3y+3z)}\\ &= \frac{12\sigma_1^3+7\sigma_2\sigma_1-3\sigma_3}{18\sigma_1^3+3\sigma_1\sigma_2-\sigma_3} = \frac{12s^3+7p\alpha-3p}{18s^3+3p\alpha-p} = \frac{12t+7\alpha-3}{18t+3\alpha-1}\,.\end{aligned}$$

Considering the function $f : [t_3, t_4] \to \mathbb{R}$ defined as

$$f(t) = \frac{12t+7\alpha-3}{18t+3\alpha-1},$$

we get

$$f'(t) = \frac{-6(15\alpha-7)}{(3\alpha+18t-1)^2} < 0,$$

so f is decreasing for $t \in [t_3, t_4]$ or $f(t_4) \le f(t) \le f(t_3)$, which can be written as

$$\frac{12\alpha^3+28\alpha^2-75\alpha+27}{18\alpha^3+12\alpha^2-31\alpha+9} \le f(t) \le \frac{55\alpha-111}{75\alpha-163}.$$

The desired inequality is obtained.

Remark 2.2. Since $\forall\,\alpha \ge 9$ and $\dfrac{R}{r} = \dfrac{\prod(y+z)}{4xyz} = \dfrac{p\alpha-p}{4p} = \dfrac{\alpha-1}{4}$, one has the inequalities

$$\frac{6}{\alpha-1} \le \frac{12\alpha^3+28\alpha^2-75\alpha+27}{18\alpha^3+12\alpha^2-31\alpha+9}, \quad \frac{55\alpha-111}{75\alpha-163} \le \frac{3\alpha-3}{32},$$

hence we obtain

$$\frac{3}{2}\,\frac{r}{R} \le \frac{12\alpha^3+28\alpha^2-75\alpha+27}{18\alpha^3+12\alpha^2-31\alpha+9} \le \sum_{cyclic} \frac{a}{2a+b+c} \le \frac{55\alpha-111}{75\alpha-163} \le \frac{3}{8}\,\frac{R}{r}\,.$$

Hence, Application 2.4 is a refinement of Problem 4508 of ***Crux*** 46(9) by George Apostolopoulos, with the statement:

> Let a, b, c be the side lengths of a triangle with inradius r and circumradius R. Prove that
>
> $$\frac{3}{2}\,\frac{r}{R} \le \sum_{cyclic} \frac{a}{2a+b+c} \le \frac{3}{8}\,\frac{R}{r}\,.$$

3 Problems

The following problems can be solved by the proposed method.

Problem 1. Let a, b, c be positive real numbers and $\alpha = (\sum a)\left(\sum \frac{1}{a}\right)$.
Prove that

$$\frac{(\sum a)^2}{\sum a^2} + \frac{1}{2}\left(\frac{\sum a^3}{abc} - \frac{\sum a^2}{\sum ab}\right) \geq 4 + \frac{(\alpha-9)(2\alpha^2-15\alpha+9)}{2\alpha(2\alpha-9)} \geq 4$$

(Refinement of Problem Klamkin-03, ***Crux*** 31(5), by Pham Van Thuan).

Problem 2. Let x, y, z be positive real numbers with $\sum x^2 = 1$ and $\alpha = (\sum x)\left(\sum \frac{1}{x}\right)$.
Prove the following two compound inequalities:

$$\sum \frac{1}{x} - \sum x \geq 2\sqrt{3} + \frac{xyz}{(x+y+z)^2}\left(2\alpha^2 - 13\alpha + 9 - 2\sqrt{3}\sqrt{8\alpha^2 - 54\alpha + 81}\right) \geq 2\sqrt{3}$$

and

$$\sum \frac{1}{x} + \sum x \geq 4\sqrt{3} + \frac{1}{xyz(x+y+z)^2}\left(2\alpha^2 - 5\alpha - 9 - 4\sqrt{24\alpha^2 - 162\alpha + 243}\right) \geq 4\sqrt{3}.$$

(Refinement of Problem 2946, ***Crux*** 30(4), by Panos E. Tsaoussoglou).

Problem 3. Let x, y and z be positive numbers and $\alpha = (\sum x)\left(\sum \frac{1}{x}\right)$.
Prove that

$$\sqrt[3]{\frac{x^3+y^3+z^3}{xyz}} + \sqrt{\frac{xy+yz+zx}{x^2+y^2+z^2}} \geq \sqrt[3]{\alpha-6} + \sqrt{\frac{4\alpha-9}{\alpha^2-8\alpha+18}} \geq \sqrt[3]{3} + 1$$

(Refinement of Problem 3467, ***Crux*** 35(9), by Tuon Le).

Problem 4. Let h_a, h_b, h_c the altitudes, r the inradius and s the semiperimeter of a triangle, and define $\alpha = s\sum \frac{1}{s-a}$. Prove that

$$\frac{h_a-2r}{h_a+2r} + \frac{h_b-2r}{h_b+2r} + \frac{h_c-2r}{h_c+2r} \geq \frac{3}{5} + \frac{6(\alpha-9)}{90\alpha-185} \geq \frac{3}{5}$$

(Refinement of Problem 3993, ***Crux*** 40(10), by Dragoljub Miloševic).

Problem 5. Suppose x, y and z are real numbers and $\alpha = (\sum x)\left(\sum \frac{1}{x}\right)$. Prove that

$$\begin{aligned}(x^3+y^3+z^3)^2 + 3(xyz)^2 &\geq 4(y^3z^3 + z^3x^3 + x^3z^3) + x^2y^2z^2 f(\alpha)\\ &\geq 4(x^3y^3 + y^3z^3 + z^3x^3),\end{aligned}$$

where

$$f(\alpha) = \frac{1}{32}(\alpha - 9)\left(\alpha^3 - 11\alpha^2 + 35\alpha - 57 - (\alpha^2 - 6\alpha + 13)\sqrt{(\alpha-1)(\alpha-9)}\right).$$

(Refinement of Problem 2839, ***Crux*** 29(4), by Murray Klamkin).

References

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[2] Bencze, M. and Drăgan, M., Some rational symmetrical and homogenous inequalities, *Arhimede Mathematical Journal*, **9**(2022), 159–171.

[3] Bencze, M. and Drăgan, M., A new method to solve and refine algebraic inequalities, *Math. Reflection*, **5**(2023), 1–15.

[4] Niculescu, C. P., A new look at Newton's inequalities, *J. Inequal. Pure Appl. Math.*, Vol 1, **2**(2000), Article 17, 33pp.

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **August 15, 2026**.

5151. *Proposed by Robert Szasz.*

Define the sequence $(x_n)_{n\geq 1}$ by

$$x_1 = 1, \quad x_{n+1} = x_n + \frac{1}{\sum_{k=1}^{n} e^{x_k}}.$$

Compute the limit $\lim_{n\to\infty}(x_n - \ln(\ln n))$.

5152. *Proposed by Michel Bataille.*

Let m, n, r, s be non-negative integers with $r \geq s$. Prove that

$$\sum_{j=0}^{n}\binom{r+j}{s}2^{m+n-j}+2^m\sum_{j=0}^{s}\binom{r+n+1}{j} = \sum_{j=0}^{m}\binom{r+j}{s}2^{m+n-j}+2^n\sum_{j=0}^{s}\binom{r+m+1}{j}.$$

5153. *Proposed by Shilong Zhuang.*

A review website collects a sequence of k ratings from customers over a period of time, where k is the total number of ratings submitted. Each rating is an integer from 1 to 5 stars and is submitted independently at a distinct timestamp. It is observed that at a specific point in time, the average of all submitted ratings is less than 4.8, and later, at another specific point in time, the average exceeds 4.8. Let $P(k)$ denote the number of possible sequences of k ratings that satisfies this pattern. Find

$$\sum_{k=1}^{15} P(k)$$

5154. *Proposed by Tatsunori Irie.*

Let $n \geq 4$ be an integer and let $a_1, \ldots, a_n$ be positive real numbers such that $a_1 + \cdots + a_n = 1$. Prove that

$$\prod_{i=1}^{n} a_i^{a_{i+1}} \leq \frac{e^{\frac{n}{4}-1}}{n},$$

where we adopt the convention that $a_{n+1} = a_1$.

5155. *Proposed by Nguyen Minh Ha and Tran Quang Hung.*

Given a tetrahedron $SABC$. A sphere (O) passes through A, B, C and meets the segments SA, SB, SC at X, Y, Z, respectively. Prove that the line joining the Lemoine points of triangles ABC and XYZ passes through S.

5156. *Proposed by Vasile Cîrtoaje.*

Let a, b, c be real numbers such that $a \geq 1 \geq b \geq c \geq 0$ and $ab + bc + ca = 3$. Prove that

$$\sqrt{8a+1} + \sqrt{8b+1} + \sqrt{8c+1} \geq 9.$$

5157. *Proposed by David Palaghianu.*

Let A and B be square matrices of order 2 with real entries such that A is not invertible and B is invertible, satisfying

$$A^2 + B^2 + 2AB^2A = BA^2B.$$

Determine the minimum value that $\det(B)$ can have, as well as the range of values that $\mathrm{Tr}(A)$ can take.

5158. *Proposed by Xicheng Peng.*

Let A, B, C be the interior angles of triangle ABC and let θ be any nonzero angle. Prove the identity

$$\sin^3\theta - \sin(A-\theta)\sin(B-\theta)\sin(C-\theta) = \sin A \sin B \sin C \sin\theta(\cot A + \cot B + \cot C - \cot\theta).$$

5159. *Proposed by Michael Friday.*

Let ABC be a triangle with $B - A = 90^\circ$, F the foot of the altitude from C, and O the circumcenter. Tangents to the circumcircle of ABC at A and B meet at T. Prove that the foot of the symmedian from C is the orthocenter of triangle OFT.

5160. *Proposed by Nguyen Viet Hung.*

Prove the following inequality holds for all positive real numbers a, b, c

$$\frac{ab+bc+ca}{a^2+b^2+c^2} + \frac{1}{6}\left(\frac{(a-b)^2}{a^2+b^2} + \frac{(b-c)^2}{b^2+c^2} + \frac{(c-a)^2}{c^2+a^2}\right) \leq 1.$$

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 août 2026**.

5151. *Soumis par Robert Szasz.*

Soit $(x_n)_{n\geq 1}$ la suite définie par

$$x_1 = 1, \quad x_{n+1} = x_n + \frac{1}{\sum_{k=1}^{n} e^{x_k}}.$$

Calculez la limite $\lim_{n\to\infty}(x_n - \ln(\ln n))$.

5152. *Soumis par Michel Bataille.*

Soient m, n, r et s des entiers non négatifs tels que $r \geq s$. Montrez que

$$\sum_{j=0}^{n} \binom{r+j}{s} 2^{m+n-j} + 2^m \sum_{j=0}^{s} \binom{r+n+1}{j} = \sum_{j=0}^{m} \binom{r+j}{s} 2^{m+n-j} + 2^n \sum_{j=0}^{s} \binom{r+m+1}{j}.$$

5153. *Soumis par Shilong Zhuang.*

Un site d'évaluation recueille une suite de k notes attribuées par des utilisateurs au cours d'une certaine période, où k désigne le nombre total de notes soumises. Chaque note est un entier compris entre 1 et 5 étoiles et est attribuée indépendamment des autres à un instant distinct. On observe qu'à un certain instant, la moyenne de toutes les notes soumises est strictement inférieure à 4,8, puis qu'à un instant ultérieur, cette moyenne devient strictement supérieure à 4,8. On note $P(k)$ le nombre de suites de k notes satisfaisant cette propriété. Déterminez

$$\sum_{k=1}^{15} P(k)$$

5154. *Soumis par Tatsunori Irie.*

Soit $n \geq 4$ un entier, et soient $a_1, \ldots, a_n$ des nombres réels positifs tels que $a_1 + \cdots + a_n = 1$. Montrez que

$$\prod_{i=1}^{n} a_i^{a_{i+1}} \leq \frac{e^{\frac{n}{4}-1}}{n},$$

où, par convention, $a_{n+1} = a_1$.

5155. *Soumis par Nguyen Minh Ha et Tran Quang Hung.*

Soit $SABC$ un tétraèdre. Une sphère (O) passe par les points A, B et C, et coupe respectivement les segments SA, SB et SC en X, Y et Z. Montrez que la droite joignant les points de Lemoine des triangles ABC et XYZ passe par S.

5156. *Soumis par Vasile Cîrtoaje.*

Soient a, b et c des nombres réels avec $a \geq 1 \geq b \geq c \geq 0$ et $ab + bc + ca = 3$. Montrez que

$$\sqrt{8a+1} + \sqrt{8b+1} + \sqrt{8c+1} \geq 9.$$

5157. *Soumis par David Palaghianu.*

Soient A et B des matrices carrées d'ordre 2 à coefficients réels, telles que A ne soit pas inversible et que B soit inversible, et vérifiant

$$A^2 + B^2 + 2AB^2A = BA^2B.$$

Déterminez la plus petite valeur que peut prendre $\det(B)$, ainsi que l'ensemble des valeurs possibles de $\mathrm{Tr}(A)$.

5158. *Soumis par Xicheng Peng.*

Soient A, B et C les angles intérieurs d'un triangle ABC, et soit θ un angle non nul. Montrez l'identité

$$\sin^3\theta - \sin(A-\theta)\sin(B-\theta)\sin(C-\theta) = \sin A \sin B \sin C \sin\theta(\cot A + \cot B + \cot C - \cot\theta).$$

5159. *Soumis par Michael Friday.*

Soit ABC un triangle tel que $B - A = 90^\circ$. Soit F le pied de la hauteur issue de C, et soit O le centre du cercle circonscrit. Les tangentes au cercle circonscrit à ABC en A et en B se coupent au point T. Montrez que le pied de la symédiane issue de C est l'orthocentre du triangle OFT.

5160. *Soumis par Nguyen Viet Hung.*

Montrez que l'inégalité suivante est vérifiée pour tous nombres réels positifs a, b et c :

$$\frac{ab+bc+ca}{a^2+b^2+c^2} + \frac{1}{6}\left(\frac{(a-b)^2}{a^2+b^2} + \frac{(b-c)^2}{b^2+c^2} + \frac{(c-a)^2}{c^2+a^2}\right) \leq 1.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2026: 52(1), p. 41–45.

5101. *Proposed by Mihaela Berindeanu.*

Let ABC be an acute triangle with the orthocenter H and the incenter I. Let H_a, H_b and H_c be the orthocenters of $\triangle IBC$, $\triangle IAC$ and $\triangle IAB$, respectively, and let

$$HA + HB + HC = IH_a + IH_b + IH_c.$$

Show that $\triangle ABC$ is an equilateral triangle.

Of the 19 submissions received, 18 were correct. Our featured solution, which is based on the work of C. R. Pranesachar, is one version similar solutions submitted by a majority of the correspondents.

In any triangle ΔABC (with orthocenter H, circumradius R), the distances from the orthocenter to a vertex are $HA = 2R\cos A$, etc. (where A is used to represent both a vertex and the measure of the angle at that vertex). Hence, the expression on the left of the given equation is

$$HA + HB + HC = 2R(\cos A + \cos B + \cos C). \tag{1}$$

Note that because we assume that the angles are acute, all terms in (1) are positive. Since H_a is the orthocentre of triangle BIC, it follows that I is the orthocentre of triangle BH_aC. In ΔBIC we have

$$\angle BIC = 90^\circ + \frac{A}{2}.$$

Since $\angle BH_aC$ and $\angle BIC$ are oppoisite angles in the circle on diameter H_aI, they are supplementary, whence

$$\angle BH_aC = 180^\circ - \angle BIC = 90^\circ - \frac{A}{2}.$$

Consequently, the circumradius R_a of triangle BH_aC is

$$R_a = \frac{BC}{2\sin\angle BH_aC} = \frac{2R\sin A}{2\sin\left(90^\circ - \frac{A}{2}\right)} = \frac{2R\cdot 2\sin\frac{A}{2}\cos\frac{A}{2}}{2\cos\frac{A}{2}} = 2R\sin\frac{A}{2}.$$

Hence,

$$IH_a = 2\cdot 2R\sin\frac{A}{2}\cdot\cos\left(90^\circ - \frac{A}{2}\right) = 4R\sin^2\frac{A}{2} = 2R(1-\cos A),$$

and similarly,

$$IH_b = 2R(1-\cos B),\ IH_c = 2R(1-\cos C).$$

The right-hand side of the given equation is, thus,

$$IH_a + IH_b + IH_c = 2R[3-(\cos A+\cos B+\cos C)] \tag{2}$$

Our given hypothesis says that sums in equations (1) and (2) are equal; specifically,

$$2R(\cos A+\cos B+\cos C) = 2R[3-(\cos A+\cos B+\cos C)],$$

or,

$$\cos A+\cos B+\cos C = \frac{3}{2}.$$

But, it is known that in any triangle ABC with inradius r and circumradius R,

$$\cos A+\cos B+\cos C = 1+\frac{r}{R};$$

therefore,

$$1+\frac{r}{R} = \frac{3}{2},$$

so that $R = 2r$. Euler's triangle formula says that $R-2r$ is the distance between the circumcenter and incenter. Because an equilateral triangle is the only triangle for which that distance is 0, we conclude that ΔABC is equilateral as claimed.

Editor's Comments. After completing his solution, Walther Janous observed that because

$$HA^2 = 4R^2\cos^2 A \quad \text{and} \quad IH_a^2 = 4R^2(1-2\cos A+\cos^2 A),$$

it follows that

$$\begin{aligned}\sum_{cyclic} HA^2 = \sum_{cyclic} IH_a^2 \quad &\Longleftrightarrow \quad \sum_{cyclic}\cos^2 A = 3-2\sum_{cyclic}\cos A+\sum_{cyclic}\cos^2 A\\ &\Longleftrightarrow \quad \sum_{cyclic}\cos A = \frac{3}{2}\\ &\Longleftrightarrow \quad \sum_{cyclic} HA = \sum_{cyclic} IH_a.\end{aligned}$$

In other words, for $t \in 1, 2$, ΔABC is equilateral if and only if

$$\sum_{cyclic} HA^t = \sum_{cyclic} IH_a^t.$$

The natural question: Which values of the positive integer t result in the corresponding equivalence?

5102. *Proposed by Michel Bataille.*

Let n be a positive integer. Evaluate

$$\sum_{k=1}^{n} \binom{n+k}{2k} (-1)^{k-1} k.$$

We received 14 submissions, all correct. We present the solution by Theo Koupelis.

Let S_n denote the sum in the problem statement. Define

$$f_n(x) := \sum_{k=0}^{n} \binom{n+k}{2k} (-1)^k x^k.$$

Then

$$f_n'(x) = \sum_{k=1}^{n} \binom{n+k}{2k} (-1)^k k x^{k-1},$$

hence $S_n = -f_n'(1)$.

We first show that

$$f_{n+1}(x) = (2-x) f_n(x) - f_{n-1}(x) \qquad (n \geq 1).$$

Indeed, comparing the coefficients of x^k, it is enough to verify that

$$\binom{n+k+1}{2k} = 2\binom{n+k}{2k} - \binom{n+k-1}{2k-2} - \binom{n+k-1}{2k}.$$

Now

$$\binom{n+k+1}{2k} = \binom{n+k}{2k} + \binom{n+k}{2k-1},$$

and, by Pascal's identity,

$$\binom{n+k}{2k-1} = \left(\binom{n+k}{2k} - \binom{n+k-1}{2k} - \binom{n+k-1}{2k-1}\right) + \left(\binom{n+k-1}{2k-1} + \binom{n+k-1}{2k-2}\right).$$

Therefore

$$\binom{n+k+1}{2k} = 2\binom{n+k}{2k} - \binom{n+k-1}{2k} - \binom{n+k-1}{2k-2},$$

which proves the recurrence.

For fixed x, the characteristic equation is $r^2 - (2-x)r + 1 = 0$, whose roots are

$$r_{\pm} = \frac{2 - x \pm \sqrt{x^2 - 4x}}{2}.$$

Hence

$$f_n(x) = A(x)\, r_+^n + B(x)\, r_-^n,$$

where $A(x)$ and $B(x)$ are determined by the initial conditions $f_0(x) = 1$ and $f_1(x) = 1 - x$. Thus

$$A(x) + B(x) = 1, \qquad A(x)r_+ + B(x)r_- = 1 - x.$$

Solving this system gives

$$A(x) = \frac{\sqrt{x^2 - 4x} - x}{2\sqrt{x^2 - 4x}}, \qquad B(x) = \frac{\sqrt{x^2 - 4x} + x}{2\sqrt{x^2 - 4x}}.$$

Therefore

$$f_n(x) = \frac{\sqrt{x^2 - 4x} - x}{2\sqrt{x^2 - 4x}} \left(\frac{2 - x + \sqrt{x^2 - 4x}}{2} \right)^n + \frac{\sqrt{x^2 - 4x} + x}{2\sqrt{x^2 - 4x}} \left(\frac{2 - x - \sqrt{x^2 - 4x}}{2} \right)^n.$$

Now differentiate and evaluate at $x = 1$. Set $\Delta(x) := \sqrt{x^2 - 4x}$. Then

$$\Delta(1) = i\sqrt{3}, \qquad \Delta'(1) = \frac{1 - 2}{\Delta(1)} = \frac{i}{\sqrt{3}}.$$

Also

$$r_\pm(1) = \frac{1 \pm i\sqrt{3}}{2} = e^{\pm i\pi/3} \quad \text{and} \quad r'_\pm(1) = \frac{-1 \pm \Delta'(1)}{2} = \frac{-1 \pm i/\sqrt{3}}{2}.$$

Moreover,

$$A(1) = \frac{i\sqrt{3} - 1}{2i\sqrt{3}} = \frac{1}{2} + \frac{i}{2\sqrt{3}}, \qquad B(1) = \frac{1}{2} - \frac{i}{2\sqrt{3}},$$

and

$$A'(1) = \frac{i}{3\sqrt{3}}, \qquad B'(1) = -\frac{i}{3\sqrt{3}}.$$

Using

$$f_n(x) = A(x)r_+^n + B(x)r_-^n,$$

we obtain

$$f'_n(1) = A'(1)r_+(1)^n + A(1)nr_+(1)^{n-1}r'_+(1) + B'(1)r_-(1)^n + B(1)nr_-(1)^{n-1}r'_-(1).$$

A straightforward simplification yields

$$f'_n(1) = -\frac{n}{3}\left(e^{i(n-1)\pi/3} + e^{-i(n-1)\pi/3}\right) + \frac{i}{3\sqrt{3}}\left(e^{in\pi/3} - e^{-in\pi/3}\right).$$

Hence

$$S_n = -f'_n(1) = \frac{n}{3}\left(e^{i(n-1)\pi/3} + e^{-i(n-1)\pi/3}\right) - \frac{i}{3\sqrt{3}}\left(e^{in\pi/3} - e^{-in\pi/3}\right).$$

Using $e^{i\theta} + e^{-i\theta} = 2\cos\theta$ and $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$, we get

$$S_n = \frac{2n}{3}\cos\left(\frac{(n-1)\pi}{3}\right) + \frac{2}{3\sqrt{3}}\sin\left(\frac{n\pi}{3}\right).$$

Finally,

$$\cos\left(\frac{(n-1)\pi}{3}\right) = \cos\left(\frac{n\pi}{3}\right)\cos\left(\frac{\pi}{3}\right) + \sin\left(\frac{n\pi}{3}\right)\sin\left(\frac{\pi}{3}\right),$$

so

$$S_n = \frac{n}{3}\cos\left(\frac{n\pi}{3}\right) + \frac{3n+2}{3\sqrt{3}}\sin\left(\frac{n\pi}{3}\right).$$

Editor's Comments. It is worth noting that this problem was solved using a variety of different methods. Among the methods used were recurrences, generating functions, Chebyshev polynomials, hypergeometric identities, and even contour integration. The solution presented above was selected because it is especially concise and elegant.

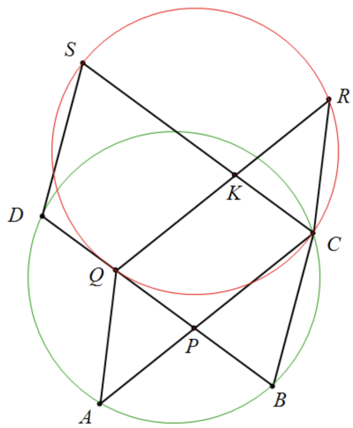
Following the publication of the problem, the Editorial Board has also received a related proposal by Paul Bracken, who asked for the evaluation of the sum

$$\sum_{k=1}^{n} \binom{n+k}{2k-1} k,$$

which we propose as a follow-up problem.

5103. *Proposed by Xicheng Peng.*

As shown in the figure, diagonals AC and BD of quadrilateral $ABCD$ intersect at point P. Let Q be a point on line BD (not coinciding with B) such that $PQ = PB$. Construct parallelograms $CAQR$ and $DBCS$. Prove that A, B, C, D are concyclic if and only if Q, C, R, S are concyclic.



We received 21 submissions, of which 13 were complete and correct. We present the solution by Janssen Samuel Halim.

Denote by K the intersection of SC and QR, as in the diagram provided. Note that, because by construction $KC \parallel QP$ and $QK \parallel CP$, we have $KQPC$ is a parallelogram. Thus

$$\begin{aligned}QK &= PC \text{ and}\\ KC &= QP,\end{aligned}$$

which we can use to conclude that

$$\begin{aligned}KC &= PB \qquad \text{(since } PQ = PB \text{ by construction)}\\ SK &= SC - KC = DB - PB = DP\\ KR &= QR - QK = AC - PC = AP,\end{aligned}$$

where in the last two lines we also used the fact that $DBCS$ and $CAQR$ respectively are parallelograms. Hence,

$$PA \cdot PC = PB \cdot PD \iff KR \cdot KQ = KC \cdot KS.$$

By the Power of a Point Theorem, this equivalence shows that $ABCD$ is cyclic if and only if $QCRS$ is cyclic, as desired.

5104. *Proposed by Vasile Cirtoaje.*

Prove that 8 is the smallest positive value of k such that

$$\frac{a^2+b^2+c^2+d^2}{4} \leq \left(\frac{a+b+c+d}{4}\right)^k$$

for all nonnegative real numbers a, b, c, d with $a \geq b \geq c \geq d$ and $(a+b)(c+d) = 4$.

There were 7 solutions, of which 6 were correct and one was incomplete. We follow the solution of Michal Adamaszek.

We first show that the inequality holds when $k \geq 8$. Since

$$1 = \frac{\sqrt{(a+b)(c+d)}}{2} \leq \frac{(a+b)+(c+d)}{4},$$

it suffices to deal with $k = 8$. Since $(a-c)(b-d) + (a-d)(b-c) \geq 0$,

$$(ab+cd) \geq \frac{1}{2}(bc+ad+bd+ac) = \tfrac{1}{2}(a+b)(c+d) = 2.$$

Let $s = \frac{1}{4}(a+b+c+d)$. Then

$$\begin{aligned}\frac{a^2+b^2+c^2+d^2}{4} &= \left(\frac{a+b+c+d}{2}\right)^2 - \frac{1}{2}[(a+b)(c+d) + (ab+cd)]\\ &= (2s)^2 - \tfrac{1}{4}[4 + (ab+cd)] \leq 4s^2 - 3.\end{aligned}$$

Since

$$s^8 + 3 = s^8 + 1 + 1 + 1 \geq 4s^2,$$

it follows that

$$s^8 \geq 4s^2 - 3 \geq \tfrac{1}{4}(a^2 + b^2 + c^2 + d^2),$$

as desired.

Now suppose that $t \leq 1$, that $b = c = d = t$ and $a = 2t^{-1} - t$. Then $(a+b)(c+d) = 4$ and $ab + cd = (2 - t^2) + t^2 = 2$. In this case, the left side of the inequality is actually equal to $4s^2 - 3$. Since $s \downarrow 1$ as $t \uparrow 1$, we can write $s = 1 + x$ for $x > 0$. Since $4s^2 - 3 = 1 + 8x + 4x^2$ and $s^k = 1 + kx + o(x)$ as $x \downarrow 0$, when x is sufficiently small, $s^k < 4s^2 - 3$ when $k < 8$.

Thus, for the inequality to hold under all conditions, $k \geq 8$.

Editor's Comments. In showing that k could not be less than 8, Theo Koupelis and Walther Janous looked at the special case where three of the variables were equal to t and the remaining one equal to $2t^{-1} - t$ and rendered the inequality in the form

$$k \geq \frac{\log(t^4 - t^2 + 1) - 2\log t}{\log(t^2 + 1) - \log t - \log 2} \equiv f(t).$$

Janous used l'Hôpital's Rule to show that the limit of this as t goes to 1 is equal to 8. Koupelis noted that the inequality $8 \geq f(t)$ was equivalent to

$$\left(\frac{t^2 + 1}{2t}\right)^8 - \left(\frac{t^4 - t^2 + 1}{t^2}\right) \geq 0.$$

The left side is equal to

$$(t^2 - 1)^4(t^8 + 12t^2 + 70t^4 + 12t^2 + 1) \geq 0$$

divided by $256t^8$.

5105. *Proposed by Tatsunori Irie.*

Find all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that for every $m, n \in \mathbb{N}_0$,

$$f(m^2 + mn + n^2) = f(m)^2 + f(m)f(n) + f(n)^2.$$

We received 5 submissions, of which 3 were correct and complete. We present the solution by the proposer.

Define

$$\mathsf{Q}(x, y) = x^2 + xy + y^2.$$

Write $P(m, n)$ for the identity

$$f(\mathsf{Q}(m, n)) = \mathsf{Q}(f(m), f(n)).$$

From $P(0, 0)$ we get $f(0) = 3f(0)^2$, hence $f(0) = 0$.

From $P(1,0)$ we get $f(1) = f(1)^2$, hence $f(1) \in \{0,1\}$. We also use that if $u, v \in \mathbb{N}_0$ and $\mathsf{Q}(u,v) = 0$, then $u = v = 0$.

Lemma 1 (Balancing identity)**.** *For all $x, a \in \mathbb{N}_0$,*

$$\mathsf{Q}(f(7x+2a), f(a)) = \mathsf{Q}(f(5x+a), f(3x+2a)).$$

Proof. A direct expansion gives

$$\mathsf{Q}(7x+2a, a) = 49x^2 + 35ax + 7a^2 = \mathsf{Q}(5x+a, 3x+2a).$$

Apply f and use $P(7x+2a, a)$ and $P(5x+a, 3x+2a)$. □

Lemma 2 (Pull-back identity)**.** *For every integer $x \geq 5$,*

$$\mathsf{Q}(f(2x-3), f(x-5)) = \mathsf{Q}(f(2x-7), f(x)).$$

Proof. One checks

$$\mathsf{Q}(2x-3, x-5) = \mathsf{Q}(2x-7, x) = 7x^2 - 21x + 19.$$

Apply f and use $P(2x-3, x-5)$ and $P(2x-7, x)$. □

Lemma 3 (Coarse growth)**.** *For every $\varepsilon > 0$ there exists $C = C(\varepsilon)$ such that*

$$f(n) \leq Cn^{1+\varepsilon} \qquad (\forall n \in \mathbb{N}_0).$$

Proof sketch (strong induction). Write $k = 7x + 2a$ with $x \geq 0$ and $0 \leq a \leq 6$. By Lemma 1,

$$\mathsf{Q}(f(k), f(a)) = \mathsf{Q}(f(5x+a), f(3x+2a)).$$

Since $f(\cdot) \geq 0$ and $\mathsf{Q}(u,v) \leq (u+v)^2$ for $u, v \geq 0$,

$$f(k)^2 \leq \mathsf{Q}(f(k), f(a)) \leq (f(5x+a) + f(3x+2a))^2.$$

Apply the induction hypothesis to $5x + a$ and $3x + 2a$ and absorb finitely many base cases into C. □

Lemma 4 (Decisive bound)**.** *For all $n \in \mathbb{N}_0$, $f(n) \leq n$.*

Proof. Assume that $f(n) > n$ for some n. Choose $\varepsilon > 0$ and $t > 1$ such that $f(n) \geq tn^{1+\varepsilon}$. From $P(n,0)$ we have $f(n^2) = f(n)^2$, hence by iteration

$$f(n^{2^k}) = f(n)^{2^k} \geq t^{2^k} n^{(1+\varepsilon)2^k}.$$

By Lemma 3 with $\varepsilon/2$, we also have

$$f(n^{2^k}) \leq C\, n^{(1+\varepsilon/2)2^k}.$$

For k large this is impossible because t^{2^k} dominates C. □

Lemma 5. *For every integer $N \geq 14$ there exist integers $a \in \{0, 1, 2, 3, 4, 5, 6\}$ and $x \geq 1$ such that*

$$N = 7x + 2a.$$

Proof. Since $\gcd(2, 7) = 1$, there is a unique $a \in \{0, \ldots, 6\}$ with $N \equiv 2a \pmod 7$. Then $x = (N - 2a)/7$ is an integer. Because $2a \leq 12$ and $N \geq 14$, we have $x \geq 1$. $\square$

Theorem 2. *If $f(1) = 0$, then $f \equiv 0$.*

Proof. From $P(1, 1)$ we get $f(3) = \mathsf{Q}(0, 0) = 0$. From $P(3, 0)$, $f(9) = f(3)^2 = 0$.

Apply Lemma 2 with $x = 6$. Then $\mathsf{Q}(f(9), f(1)) = \mathsf{Q}(f(5), f(6))$ and hence $\mathsf{Q}(f(5), f(6)) = 0$ and $f(5) = f(6) = 0$.

Apply Lemma 2 with $x = 5$. Then $\mathsf{Q}(f(7), f(0)) = \mathsf{Q}(f(3), f(5)) = 0$, hence $f(7)^2 = 0$ and $f(7) = 0$.

Now $P(2, 1)$ gives $f(7) = \mathsf{Q}(f(2), f(1)) = f(2)^2$. Thus $f(2) = 0$. Then $P(2, 0)$ gives $f(4) = 0$.

We now have $f(a) = 0$ for all $a \in \{0, 1, 2, 3, 4, 5, 6\}$.

We prove by strong induction on $N \geq 0$ that $f(N) = 0$. The claim holds for $0 \leq N \leq 13$ by the values above (and repeated use of $P(\cdot, \cdot)$ if desired).

Fix $N \geq 14$ and assume $f(t) = 0$ for all $t < N$. Choose $x \geq 1$ and $a \in \{0, \ldots, 6\}$ with $N = 7x + 2a$ from Lemma 5. By Lemma 1,

$$\mathsf{Q}(f(N), f(a)) = \mathsf{Q}(f(5x + a), f(3x + 2a)).$$

We have $a \leq 6$, so $f(a) = 0$. Also $5x + a < N$ and $3x + 2a < N$, so the induction hypothesis gives $f(5x + a) = f(3x + 2a) = 0$. Hence $f(N)^2 = \mathsf{Q}(f(N), 0) = 0$, so $f(N) = 0$. $\square$

Theorem 3. *If $f(1) = 1$, then $f(n) = n$ for all $n \in \mathbb{N}_0$.*

Proof. We first determine the small values needed for the induction.

From $P(1, 1)$ we get $f(3) = \mathsf{Q}(1, 1) = 3$. From $P(3, 0)$ we get $f(9) = f(3)^2 = 9$. From $P(3, 1)$ we get $f(13) = \mathsf{Q}(3, 1) = 13$.

Apply Lemma 2 with $x = 6$. Then $\mathsf{Q}(f(9), f(1)) = \mathsf{Q}(f(5), f(6))$, hence

$$f(5)^2 + f(5)f(6) + f(6)^2 = \mathsf{Q}(9, 1) = 91.$$

A short check in $\mathbb{N}_0$ shows the only solutions are $(f(5), f(6)) = (5, 6)$ or $(6, 5)$.

Apply Lemma 2 with $x = 5$. Then $f(7)^2 = \mathsf{Q}(f(3), f(5)) = \mathsf{Q}(3, f(5))$. If $f(5) = 6$ then $\mathsf{Q}(3, 6) = 63$ is not a square. Hence $f(5) = 5$ and $f(6) = 6$. Then $\mathsf{Q}(3, 5) = 49$, so $f(7) = 7$.

Now $P(2,1)$ gives

$$7 = f(7) = \mathsf{Q}(f(2), f(1)) = f(2)^2 + f(2) + 1,$$

so $f(2) = 2$. Then $P(2,0)$ gives $f(4) = 4$.

Apply Lemma 2 with $x = 7$. Then $\mathsf{Q}(f(11), f(2)) = \mathsf{Q}(f(7), f(7)) = \mathsf{Q}(7,7) = 147$. Since $f(2) = 2$, we get $f(11)^2 + 2f(11) + 4 = 147$, hence $f(11) = 11$.

Apply Lemma 2 with $x = 8$. Then $\mathsf{Q}(f(13), f(3)) = \mathsf{Q}(f(9), f(8))$. The left-hand side is $\mathsf{Q}(13,3) = 217$. Thus $f(8)^2 + 9f(8) + 81 = 217$, hence $f(8) = 8$.

Finally $P(2,2)$ gives $f(12) = \mathsf{Q}(2,2) = 12$.

We now prove by strong induction on $N \geq 0$ that $f(N) = N$. The claim holds for $0 \leq N \leq 13$ by the computed values.

Fix $N \geq 14$ and assume $f(t) = t$ for all $t < N$. Choose $x \geq 1$ and $a \in \{0, \dots, 6\}$ with $N = 7x + 2a$ from Lemma 5. By Lemma 1,

$$\mathsf{Q}(f(N), f(a)) = \mathsf{Q}(f(5x + a), f(3x + 2a)).$$

Since $a \leq 6$, the hypothesis gives $f(a) = a$. Also $5x + a < N$ and $3x + 2a < N$, so $f(5x + a) = 5x + a$ and $f(3x + 2a) = 3x + 2a$. Therefore

$$\mathsf{Q}(f(N), a) = \mathsf{Q}(5x + a, 3x + 2a) = \mathsf{Q}(7x + 2a, a) = \mathsf{Q}(N, a).$$

Expanding yields

$$f(N)^2 + af(N) = N^2 + aN,$$

so

$$(f(N) - N)(f(N) + N + a) = 0.$$

The second factor is positive in $\mathbb{N}_0$. Hence $f(N) = N$. $\square$

By Theorems 2 and 3, the only solutions are

$$f(n) \equiv 0 \qquad \text{or} \qquad f(n) = n \quad (\forall n \in \mathbb{N}_0).$$

5106. *Proposed by Minh Ha Nguyen, modified by the Editorial Board.*

Let ABC be a scalene triangle and let (I) be its incircle. A point M varies on (I). Denote by H, K, L the perpendicular projections of M onto BC, CA, AB, respectively. Find the position of M such that the expression $MH + MK + ML$ attains its maximum (or minimum) value.

We received 8 submissions, all of which were correct, and shall sample 2 of them.

Solution 1, by Theo Koupelis.

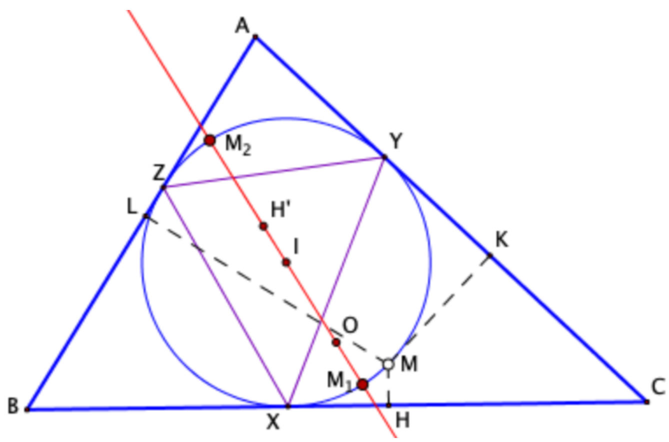
Let O be the circumcenter of triangle ABC, and let r, R be its inradius and circumradius, respectively. Let $d = MH + MK + ML$. We know that for any point M on the plane of the triangle we have

$$d = \frac{MI^2 - MO^2}{2R} + \frac{R}{2} + 2r.$$

(See, for example, https://community.wolfram.com/groups/-/m/t/2510942.) In the special case when M is on the incircle, we have $MI = r$, and therefore,

$$d = \frac{r^2 + R^2 + 4rR}{2R} - \frac{MO^2}{2R}.$$

It follows that d attains its maximum value when MO attains its minimum value, and vice versa. Let the line IO intersect the incircle at points M_1 and M_2, where $OM_1 < OM_2$. Then d attains its maximum value when $M = M_1$, and it attains its minimum value when $M = M_2$.



Solution 2, by the proposer.

Let r be the radius of the incircle, and X, Y, Z be the points where it touches the sides BC, CA, AB of ΔABC (as in the accompanying figure). Denote by H' the orthocenter of ΔXYZ, and by M_1 and M_2 the points where the line IH' (namely, the Euler line of ΔXYZ) meets the incircle, labeled so that the vectors $\overrightarrow{IH'}$ and $\overrightarrow{IM_1}$ point in opposite directions. Observe that the vectors $\overrightarrow{MH}, \overrightarrow{MK}, \overrightarrow{ML}$ are parallel and point in the same direction as $\overrightarrow{IX}, \overrightarrow{IY}, \overrightarrow{IZ}$, respectively. Recall that the circumcenter I and orthocenter H' of ΔXYZ satisfy

$$\overrightarrow{IH'} = \overrightarrow{IX} + \overrightarrow{IY} + \overrightarrow{IZ}.$$

With this notation we have

$$\begin{aligned}
&MH + MK + ML \\
&= \frac{1}{r}\left(|\overrightarrow{MH}|\cdot|\overrightarrow{IX}| \;+\; |\overrightarrow{MK}|\cdot|\overrightarrow{IY}| \;+\; |\overrightarrow{ML}|\cdot|\overrightarrow{IZ}|\right) \\
&= \frac{1}{r}\left(\overrightarrow{MH}\cdot\overrightarrow{IX} \;+\; \overrightarrow{MK}\cdot\overrightarrow{IY} \;+\; \overrightarrow{ML}\cdot\overrightarrow{IZ}\right) \\
&= \frac{1}{r}\left((\overrightarrow{MI}+\overrightarrow{IX})\cdot\overrightarrow{IX} \;+\; (\overrightarrow{MI}+\overrightarrow{IY})\cdot\overrightarrow{IY} \;+\; (\overrightarrow{MI}+\overrightarrow{IZ})\cdot\overrightarrow{IZ}\right) \\
&= \frac{1}{r}\left(\overrightarrow{MI}\cdot(\overrightarrow{IX}+\overrightarrow{IY}+\overrightarrow{IZ}) + 3r^2\right) \\
&= -\frac{3}{r}\left(\overrightarrow{IM}\cdot\overrightarrow{IH'}\right) \;+\; 3r.
\end{aligned}$$

Consequently, $MH+MK+ML$ achieves its maximum value if and only if $\overrightarrow{IM}\cdot\overrightarrow{IH'}$ reaches its minimum value, if and only if the vectors $\overrightarrow{IM}$ and $\overrightarrow{IH'}$ point in opposite directions, if and only if M coincides with M_1.

Editor's comments.

(1) The restriction of the result to scalene triangles is too severe. The above arguments fail only when ΔABC is equilateral, in which case the sum $MH + MK + ML$ is constant by Viviani's theorem (The sum of the shortest distances from any interior point to the sides of an equilateral triangle equals the length of the triangle's altitude). Wikipedia provides two very simple proofs of Viviani's theorem, but note that it is confirmed by the formula for d in solution 1 (where for an equilateral triangle the points I and O coincide, while $R = 2r$ and the altitude has length $3r$).

(2) Solutions 1 and 2 combine to prove that the Euler line of the intouch triangle XYZ (aka the contact triangle, aka the Gergonne triangle) contains the circumcenter of the original triangle ABC.

(3) An informative version of our problem 5106 appeared as *Crux* Problem 3819 [Vol. 40(2), February 2014, pages 87-89]. By coincidence, a simple pictorial proof of a companion result appeared at the same time in *Mathematics Magazine* [Burkard Polster, “Viviani á la Kawasaki: Take Two”, Vol. 87(4), October 2014, 280-283]: The sum of the distances from a point inside a triangle to the three sides takes every value from the smallest altitude of the triangle to the largest; moreover the sum is constant along parallel line segments that are perpendicular to OI. According to four of the correspondents who used coordinates, the maximum and minimum values of the sum when M is restricted to the incircle are

$$d = r\left(3 \pm \sqrt{3 - 2(\cos A + \cos B + \cos C)}\right) = r\left(3 \pm \sqrt{1 - \frac{2r}{R}}\right).$$

5107. *Proposed by Nikolai Osipov and Alex Chen.*

The base-b repunits are defined as

$$R_d(b) = 1 + b + \cdots + b^{d-1} = \frac{b^d - 1}{b - 1}.$$

For positive integers m, k, d, a such that $d \geqslant 2$, $a \geqslant 2$, prove that $R_d(a^m)$ is divisible by $R_d(a^k)$ if and only if m is divisible by k and $\gcd(m/k, d) = 1$.

We received 1 solution. We present the solution by the proposers, slightly modified by the editor.

Assume first that $m = kn$ for some positive integer n with $\gcd(n, d) = 1$. Put $b = a^k$. Then

$$\frac{R_d(a^m)}{R_d(a^k)} = \frac{(b^{nd} - 1)(b - 1)}{(b^n - 1)(b^d - 1)}.$$

Since $\gcd(b^n - 1, b^d - 1) = b^{\gcd(n,d)} - 1 = b - 1$, we have

$$\frac{R_d(a^m)}{R_d(a^k)} = \frac{b^{nd} - 1}{\operatorname{lcm}(b^n - 1, b^d - 1)},$$

which is an integer, since $b^{nd} - 1$ is divisible by both $b^n - 1$ and $b^d - 1$.

Conversely, suppose $R_d(a^k) \mid R_d(a^m)$. Then

$$Q' = \frac{(a^{md} - 1)(a^k - 1)}{a^{kd} - 1}$$

is an integer, because it equals $(a^m - 1)R_d(a^m)/R_d(a^k)$. Hence

$$(a^{kd} - 1) \mid (a^{md} - 1)(a^k - 1).$$

Let $g = \gcd(m, k)$. Since

$$\gcd(a^{md} - 1, a^{kd} - 1) = a^{\gcd(md,kd)} - 1 = a^{dg} - 1,$$

we can write $a^{md} - 1 = (a^{dg} - 1)A'$ and $a^{kd} - 1 = (a^{dg} - 1)A''$, with $\gcd(A', A'') = 1$. The divisibility above gives $A'' \mid A'(a^k - 1)$, hence $A'' \mid a^k - 1$. Therefore

$$a^{kd} - 1 = (a^{dg} - 1)A'' \mid (a^{dg} - 1)(a^k - 1).$$

Thus

$$a^{kd} - 1 \leq (a^{dg} - 1)(a^k - 1) < a^{dg+k} - 1,$$

so $kd < dg + k$. Hence $k/g < d/(d - 1) \leq 2$. Since $g \mid k$, it follows that $k = g$ and thus $k \mid m$.

Now write $m = kn$ and $b = a^k$. The assumed divisibility says that

$$\frac{(b^{nd} - 1)(b - 1)}{(b^n - 1)(b^d - 1)} \in \mathbb{Z}.$$

Let $h = \gcd(n, d)$ and $M = (b^h - 1)/(b - 1)$. Then we have

$$M \mid \frac{b^n - 1}{b - 1} \qquad \text{and} \quad M \mid \frac{b^d - 1}{b - 1}.$$

It follows that

$$M \mid \frac{b^{nd} - 1}{b^n - 1} \qquad \text{and} \quad M \mid \frac{b^{nd} - 1}{b^d - 1}.$$

But $b^n \equiv b^d \equiv 1 \pmod{M}$, so

$$\frac{b^{nd} - 1}{b^n - 1} = 1 + b^n + \cdots + b^{(d-1)n} \equiv d \pmod{M},$$

and similarly $(b^{nd} - 1)/(b^d - 1) \equiv n \pmod{M}$. Hence $M \mid d$ and $M \mid n$, so $M \mid h$.

If $h > 1$, then $M = 1 + b + \cdots + b^{h-1} \geq 2^h - 1 > h$, contradicting $M \mid h$. Thus $h = 1$, i.e. $\gcd(n, d) = 1$. Hence $m = kn$ with $\gcd(n, d) = 1$, as claimed.

5108. *Proposed by Huseyin Yigit Emekci.*

Let a, b, c, d be positive real numbers such that $a+b+c+d = 16$. Find the smallest value of

$$T = \frac{a}{b^3 + 32} + \frac{b}{c^3 + 32} + \frac{c}{d^3 + 32} + \frac{d}{a^3 + 32}$$

We received 13 submissions of which 9 were correct and complete. We present the solution by Sicheng Du.

By the AM-GM inequality,

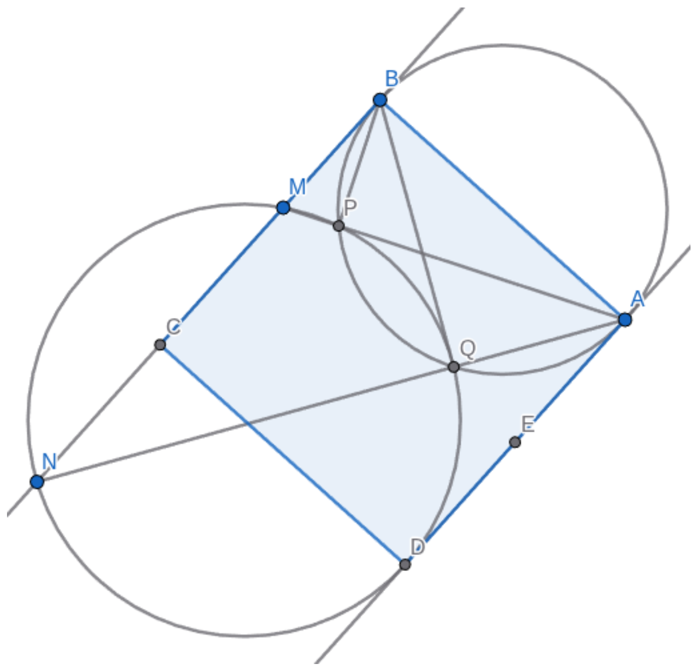
$$\begin{aligned}
\sum_{\text{cyc}} \frac{a}{b^3 + 32} &= \sum_{\text{cyc}} \left[\frac{a}{32} - \frac{ab^3}{32\,(b^3 + 32)} \right] \\
&= \frac{1}{2} - \frac{1}{32} \sum_{\text{cyc}} \frac{ab^3}{\frac{1}{2}b^3 + \frac{1}{2}b^3 + 32} \\
&\geq \frac{1}{2} - \frac{1}{32} \sum_{\text{cyc}} \frac{ab^3}{3\sqrt[3]{\frac{1}{2}b^3 \cdot \frac{1}{2}b^3 \cdot 32}} \\
&= \frac{1}{2} - \frac{1}{32} \sum_{\text{cyc}} \frac{ab}{6} \\
&= \frac{1}{2} - \frac{(a+c)(b+d)}{192} \\
&\geq \frac{1}{2} - \frac{1}{192} \left(\frac{a+b+c+d}{2} \right)^2 \\
&= \frac{1}{6}.
\end{aligned}$$

The equality holds when $a = b = c = d = 4$.

5109. *Proposed by Ion Patrascu.*

Let ABC be an isosceles right-angled triangle with $BA = BC$. On the segment BC we consider the point M to be mobile and denote by N its symmetry with respect to C. The points P and Q are the projections of B onto AM and AN respectively. Prove that the line PQ passes through a fixed point.

We received 20 submissions, all correct and complete. We present the solution by Michal Adamaszek.



Complete the triangle to a square $ABCD$. Let Γ_1 be the circle through D, M, N. Since DMN is isosceles with $DM = DN$, that circle is tangent to AD at D. Let Γ_2 be the circle with diameter AB. It is tangent to AD at A.

If XYZ is a triangle with $\angle X = 90°$ and $H \in YZ$ is the foot of the altitude from X then $YH \cdot YZ = XY^2$. Using that observation in triangles BAM and BAN we get

$$AP \cdot AM = AB^2 = AD^2, \ AQ \cdot AN = AB^2 = AD^2,$$

which implies that the points P, Q belong to Γ_1. Since $\angle BPA = \angle BQA = 90°$ the points P, Q belong also to Γ_2. It follows that PQ is the radical axis of Γ_1 and Γ_2. Since AD is a common tangent of these circles, the midpoint E of segment AD also lies on the radical axis of Γ_1 and Γ_2. Therefore the line PQ always passes through E, which is a fixed point independent of M.

5110. *Proposed by Paul Bracken.*

Define the sequence of integrals I_n for $n \in \mathbb{N}$ as

$$I_n = \int_0^{\pi/2} (\sin^n x + \cos^n x)^{1/n} dx$$

and let $\Lambda = \lim_{n\to\infty} I_n$. Determine the limit

$$\lim_{n\to\infty} n^2(I_n - \Lambda).$$

We received 6 submissions of which 3 were correct and complete. We present an edited solution proposed by Sean Stewart.

For all $x \in [0, \pi/4]$, $0 \le \sin x \le \cos x$, therefore, for all $n \in \mathbb{N}$

$$\cos x \le (\sin^n x + \cos^n x)^{1/n} \le 2^{1/n} \cos x. \tag{1}$$

Moreover,

$$I_n = 2\int_0^{\pi/4} (\sin^n x + \cos^n x)^{1/n} dx$$

Applying (1) yields,

$$2\int_0^{\pi/4} \cos x dx \le I_n \le 2^{1/n+1} \int_0^{\pi/4} \cos x dx,$$
$$\sqrt{2} \le I_n \le 2^{1/n+1} \frac{1}{\sqrt{2}}.$$

Taking the limit, the squeeze theorem allows us to conclude that $I_n \to \sqrt{2}$.

Additionally, factoring through $\cos^n$, we may rewrite the integrals as,

$$I_n = 2\int_0^{\pi/4} \cos x(1 + \tan^n x)^{1/n} dx.$$

This form is suitable for the substitution $u = \tan$, which results in,

$$I_n = 2\int_0^1 \frac{(1 + u^n)^{1/n}}{(1 + u^2)^{3/2}} du.$$

We used that $\frac{d}{dx}\tan x = \frac{1}{\cos^2 x}$ and that $\cos x = \frac{1}{\sqrt{1+\tan^2 x}}$ in the first quadrant. Now using the uniform expansion

$$(1 + u^n)^{1/n} = \exp(\log(1 + u^n)1/n) = 1 + 1/n \log(1 + u^n) + O(u^{2n}/n^2).$$

This allows one to write

$$I_n = 2\int_0^1 \frac{du}{(1 + u^2)^{3/2}} + \frac{2}{n}\int_0^1 \frac{\log(1 + u^n)du}{(1 + u^2)^{3/2}} + O(n^{-3}).$$

For the first integral we have,

$$2\int_0^1 \frac{du}{(1+u^2)^{3/2}} = 2\left[\frac{u}{\sqrt{1+u^2}}\right]_0^1 = \sqrt{2}.$$

Substituting $u^n = t$ in the second integral results in,

$$I_n = \sqrt{2} + \frac{2}{n^2}\int_0^1 \frac{\log(1+t)}{t}\frac{t^{1/n}}{(1+t^{2/n})^{3/2}}dt + O(n^{-3}).$$

Recalling that $\Lambda = \lim_{n\to\infty} I_n = \sqrt{2}$, the desired limit becomes,

$$\lim_{n\to\infty} n^2(I_n - \Lambda) = \lim_{n\to\infty}\left(2\int_0^1 \frac{\log(1+t)}{t}\frac{t^{1/n}}{(1+t^{2/n})^{3/2}} + O(n^{-1})dt\right). \tag{2}$$

Consider the sequence of functions of $t \in [0,1]$

$$f_n(t) = \frac{\log(1+t)}{t}\frac{t^{1/n}}{(1+t^{2/n})^{3/2}}.$$

It has pointwise limit

$$f(t) = \frac{\log(1+t)}{2t\sqrt{2}}.$$

Moreover, the sequence is dominated by the integrable function

$$|f_n(t)| \le \frac{\log(1+t)}{t} =: g(t).$$

Thus, up to adjusting g by a finite constant, we may use the Dominated Convergence Theorem to evaluate (2) resulting in,

$$\lim_{n\to\infty} n^2(I_n - \Lambda) = 2\int_0^1 \lim_{n\to\infty} f_n(t)dt = \frac{1}{\sqrt{2}}\int_0^1 \frac{\log(1+t)}{t}dt.$$

To compute the last integral, we use the power series expansion

$$\frac{\log(1+t)}{t} = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}t^{k-1}.$$

Which results in the famous series,

$$\int_0^1 \frac{\log(1+t)}{t}dt = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{12}.$$

We conclude,

$$\lim_{n\to\infty} n^2(I_n - \Lambda) = \frac{\pi^2}{12\sqrt{2}}.$$