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Editorial Board

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MathemAttic

No. 75

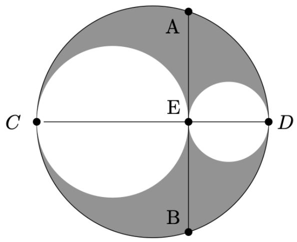
The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

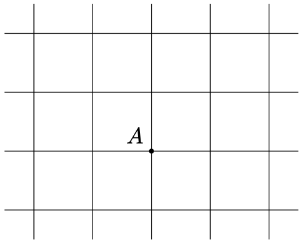
To facilitate their consideration, solutions should be received by **July 15, 2026**.

MA371. 20 lines are drawn in the plane so that no two are parallel and so no three meet in a single point. They divide the plane in to 211 non-overlapping regions. Some of these regions have finite area and others have infinite area. How many of the regions have finite area?

MA372. In the diagram, the line AB is tangent to the unshaded circles, and has length 10 cm. The centers of all 3 circles lie on the line CD, which has length 11 cm. Find the area of the shaded region.



MA373. In a town Squareville the streets form a square grid (all the streets intersect at a 90 degree angle, and the neighboring parallel streets are equidistant). A car starts driving from a point A, with constant speed. Every 15 seconds, the car turns at a 90 degree angle, to the left or to the right. Prove that the car can come back to point A only after a whole number of minutes.



MA374. A, B, C are positive integers that satisfy

$$A^2 + B^2 = C^2$$

and A is prime. Show that B must be larger than A.

MA375. Each of the numbers 1, 2, 3, 4, 5 and 6 are painted on the faces of a cube, one number per face. (The result is similar to a die, but may have a different ordering of the numbers.) The cube is placed on a table. From each of three chairs, a person can see the top and two other faces of the cube. The sums of the numbers on the visible faces, from each of these three chairs, are 9, 14 and 15. What is the number on the bottom face?

. . . .

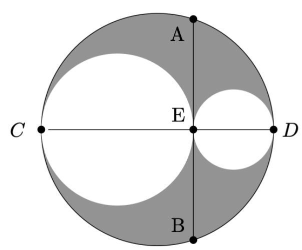
Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

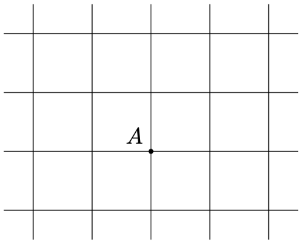
Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 juillet 2026**.

MA371. Vingt droites sont tracées dans le plan de telle sorte qu'aucune paire ne soit parallèle et qu'aucun triplet ne soit concourant. Elles divisent le plan en 211 régions disjointes. Certaines de ces régions sont d'aire finie, tandis que d'autres sont d'aire infinie. Combien de ces régions sont d'aire finie ?

MA372. Dans la figure, la droite AB est tangente aux cercles non ombrés et sa longueur est de 10 cm. Les centres des trois cercles sont tous situés sur la droite CD, dont la longueur est de 11 cm. Déterminez l'aire de la région ombrée.



MA373. Dans la ville de Carréville, les rues forment une grille carrée (toutes les rues se coupent à angle droit et les rues parallèles voisines sont équidistantes). Une voiture part d'un point A et roule à vitesse constante. Toutes les 15 secondes, la voiture effectue un virage de 90 degrés, soit vers la gauche, soit vers la droite. Montrez que la voiture ne peut revenir au point A qu'après un nombre entier de minutes.



MA374. Soient A, B et C des entiers positifs tels que

$$A^2 + B^2 = C^2$$

Supposons de plus que A soit un nombre premier. Montrez que B est strictement plus grand que A.

MA375. Les nombres 1, 2, 3, 4, 5 et 6 sont inscrits sur les faces d'un cube, à raison d'un nombre par face. (Le résultat est semblable à un dé, mais l'ordre des nombres peut être différent.) Le cube est placé sur une table. Depuis chacune de trois chaises, une personne peut voir la face supérieure ainsi que deux autres faces du cube. Les sommes des nombres apparaissant sur les faces visibles, depuis chacune de ces trois chaises, sont respectivement 9, 14 et 15. Quel est le nombre inscrit sur la face inférieure ?

MathemAttic
SOLUTIONS

Statements of the problems in this section originally appear in 2025: 51(10), p. 451-452.

MA333. The problem-solving group The Ring Lords pointed out an error in the published solution to MA333 b), which should have been $6\left[\binom{3n}{n}-\binom{2n}{n}\right]$. Their solution is included in the Problem Solving Vignettes in this issue.

MA346. Find the primes p, q, r, given that one of the numbers pqr and $p+q+r$ is 101 times the other.

Originally from the 29th Nordic Mathematical Contest, 2015, Problem 2.

We received 18 solutions, 15 of which were completely correct; we present the one by Catherine Jian.

Because $\min(p,q,r)\geq 2$, we must have $pqr>p+q+r$. Therefore

$$pqr=101(p+q+r)$$

from the given condition.

Without loss of generality, let $p=101$. Then, $qr=101+q+r$. Rearranging, we get

$$(q-1)(r-1)=1\times 102=2\times 51=4\times 34=6\times 17.$$

Since q and r are primes, the only solutions are $q=2, r=103$ or $q=103, r=2$. Therefore the solutions p, q, r are permutations of $(2,101,103)$.

Editor's Comments. The same result holds with 101 replaced by the smaller of any pair of twin primes s congruent to 1 (mod 4). Indeed, because $(q-1)(r-1)=s+1$ is singly-even, q and r cannot both be odd—hence one of them must be 2.

MA347. There are 20 sweets on a table. A game consists of two players taking turns to choose some sweets. On each move, a player must choose at least one sweet but never more than half of what remains. The loser is the one who has no valid move. How many sweets should the first player choose on the opening move to ensure that they may always win the game? Justify your answer.

Originally from Peter's Problem 2025, Problem 13.

We received 7 solutions, 5 of which were correct and complete. We present the solution by The Ring Lords Problem Solving Group, slightly modified by the edtior.

On each move, when n sweets remain, a player must take x sweets, $1\leq x\leq \lfloor \frac{n}{2}\rfloor$. We analyze the game by classifying positions as *winning* or *losing*:

- A *losing position* is one in which either no move is possible, or every legal move leads to a winning position for the next player.
- A *winning position* is one in which at least one legal move leads to a losing position for the next player.

We start by analyzing the first cases.

- $n = 0$ and $n = 1$ are losing positions, since there is no legal move in either case.
- $n = 2$ is a winning position, since the only possible move is to take one sweet, leaving $n = 1$, a losing position, for the next player.
- $n = 3$ is a losing position, since the only legal move is to take one sweet, leaving $n = 2$, a winning position, for the next player.

Thus, the first losing positions are 0, 1 and 3, all of which have the form $2^k - 1$ for a positive integer k. This suggests that *all numbers of the form* $2^k - 1$ *are losing positions*. We now justify this claim.

Suppose there are $2^k - 1$ sweets on the table. The maximum number that may be taken is

$$\left\lfloor \frac{2^k - 1}{2} \right\rfloor = 2^{k-1} - 1.$$

Therefore, after any legal move, the number of remaining sweets is at least

$$(2^k - 1) - (2^{k-1} - 1) = 2^{k-1}.$$

In particular, it is impossible to move from a position of the form $2^k - 1$ to another position of the same form $2^j - 1$ for $j < k$. Hence, whatever move is made, the next player will then always be able to move to the position $2^{k-1} - 1$. Because the positions $2^j - 1$ are losing for $j \in \{0, 1, 2\}$, by induction, this shows that the position $2^k - 1$ is losing.

Now, consider the initial position with 20 sweets. The largest number of the form $2^k - 1$ that is less than or equal to 20 is $2^4 - 1 = 15$. Therefore, the first player should leave 15 sweets on the table for the second player to be in a losing position. To do this, they must take $20 - 15 = 5$ sweets on the opening move. From that point onward, the first player can always respond to the opponent's move so as to return the game to a position of the form $2^k - 1$, ensuring a win.

MA348. Find all positive integers $n < 200$ such that $n^2 + (n+1)^2$ is a perfect square.

Originally from the 8th Nordic Mathematical Contest, 1994, Problem 4.

We received 10 submissions, of which 8 were correct and complete. We present the solution by Catherine Jian.

We can temporarily rename n as a because n will be used to represent another value. We see that a and $a+1$ have to be the legs of a right triangle. All primitive

Pythagorean triples (that is those with greatest common divisor of 1) can be written as m^2-n^2, $2mn$, and m^2+n^2, where m^2-n^2 and $2mn$ are the legs.

Since a could be m^2-n^2 or $2mn$ and $a+1$ could be $2mn$ or m^2-n^2 we have two cases

$$a=m^2-n^2, \quad a+1=2mn \quad \Rightarrow \quad m^2-n^2=2mn-1$$

or

$$a=2mn, \quad a+1=m^2-n^2 \quad \Rightarrow \quad m^2-n^2=2mn+1.$$

Solving for m in the first case, we get

$$m^2-n^2=2mn-1 \Rightarrow m^2-2mn-(n^2-1)=0 \Rightarrow m=\frac{2n\pm\sqrt{4n^2+4(n^2-1)}}{2}$$

Since m must be positive,

$$m=n+\sqrt{2n^2-1}.$$

Since m has to be an integer, $\sqrt{2n^2-1}$ is also an integer, so $2n^2-1$ can be written as k^2 and $m=n+k$. Then we can solve $2n^2-1=k^2$ as a Pell's Equation in a series of implications:

$$\begin{aligned}
&2n^2-1=k^2\\
\Longrightarrow&k^2-2n^2=-1\\
\Longrightarrow&(k-\sqrt{2}n)(k+\sqrt{2}n)=-1\\
\Longrightarrow&(k-\sqrt{2}n)^3(k+\sqrt{2}n)^3=-1\\
\Longrightarrow&(k^3-3\sqrt{2}nk^2+6n^2k-2\sqrt{2}n^3)(k^3+3\sqrt{2}nk^2+6n^2k+2\sqrt{2}n^3)=-1\\
\Longrightarrow&(6n^2k+k^3)^2-2(3nk^2+2n^3)^2=-1
\end{aligned}$$

Therefore, we see that if n_i and k_i satisfy $k^2-2n^2=-1$, then $k_{i+1}=6n_i^2k_i+k_i^3$ and $n_{i+1}=3n_ik_i^2+2n_i^3$ also satisfy the equation.

We can find a base solution of $n=1$ and $k=1$ that satisfies $2n^2-1=k^2$. In this case , $n=1$, and $m=2$, so $a=4-1=3$ and the corresponding Pythagorean triple is $(3,4,5)$.

Then, another solution is $k=6+1=7$ and $n=3+2=5$. In this case, $n=5$ and $m=7+5=12$ so $a=144-25=119$ and the corresponding Pythagorean triple is $(119,120,169)$.

We see that continuing this sequence results in values of a that are greater than 200, so these are the only two solutions for this case.

We can use the same technique for the second case. Solving for m in the second case, we get

$$m^2-n^2=2mn+1 \Rightarrow m^2-2mn-(n^2+1)=0 \Rightarrow m=\frac{2n\pm\sqrt{4n^2+4(n^2+1)}}{2}$$

Since m must be positive,

$$m = n + \sqrt{2n^2+1}.$$

Then, since m has to be an integer, $\sqrt{2n^2+1}$ is also an integer, so $2n^2+1$ can be written as k^2 and $m = n+k$.

Similarly, $2n^2+1 = k^2$ is a Pell's Equation and can be solved as below:

$$\begin{aligned} &k^2 - 2n^2 = 1 \\ \Longrightarrow &(k-\sqrt{2}n)(k+\sqrt{2}n) = 1 \\ \Longrightarrow &(k-\sqrt{2}n)^2(k+\sqrt{2}n)^2 = 1 \\ \Longrightarrow &(k^2+2n^2-2\sqrt{2}nk)(k^2+2n^2+2\sqrt{2}nk) \\ \Longrightarrow &(k^2+2n^2)^2 - 2(2nk)^2 = 1. \end{aligned}$$

Therefore, we see that if n_i and k_i satisfy $2n^2+1 = k^2$, then $k_{i+1} = k_i^2 + 2n_i^2$ and $n_{i+1} = 2n_ik_i$ also satisfy the equation.

We can find a base solution of $n = 2, k = 3$. In this case, $m = 2+3 = 5$, so $a = 2mn = 2 \cdot 4 \cdot 5 = 20$ and the corresponding Pythagorean triple is $(20, 21, 29)$.

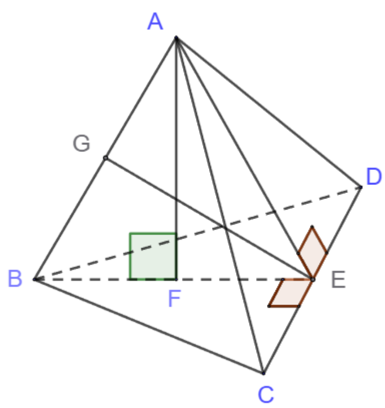
Then, another solution is $n = 12$, $k = 9+8 = 17$. In this case, $m = 29$, so $a = 2 \cdot 12 \cdot 29 = 696$, which is greater than 200.

Therefore, $3, 20, 119$ are the only positive integers $n < 200$ such that $n^2 + (n+1)^2$ is a perfect square.

MA349. Let $ABCD$ be a regular tetrahedron whose edges are of length 6. Let E be the mid-point of CD so that $|CE| = |ED|$ and let F be a point on BE such that $AF \perp BE$. Find $|AF|$.

Originally from the Irish Mathematics Teachers' Association, Team Maths National Final, 2011, Round 8, Question 3.

We received 4 submissions of which 3 were correct and complete. We present the solution by Corneliu Manescu-Avram.



The segments AE, BE are altitudes in the equilateral triangles ACD, BCD respectively, so we have

$$AE = BE = \frac{AB\sqrt{3}}{2} = \frac{6\sqrt{3}}{2} = 3\sqrt{3}.$$

The altitude

$$GE = \sqrt{AE^2 - \left(\frac{AB}{2}\right)^2} = \sqrt{(3\sqrt{3})^2 - 3^2} = 3\sqrt{2}.$$

Now, $2 \cdot \text{area}(ABE) = AF \cdot BE$ and $= GE \cdot AB$ so

$$AF = \frac{AB \cdot GE}{BE} = \frac{6 \cdot 3\sqrt{2}}{3\sqrt{3}} = 2\sqrt{6}.$$

MA350. For a given arithmetic sequence, the ratio of the sum of the first m terms to the sum of the first n terms is $m^2 : n^2$. Find, in simplest form, the ratio of the m^{th} term to the n^{th} term in terms of m and n, where $m \neq n$.

Originally from the Irish Mathematics Teachers' Association (IMTA) - Team Maths National Final, 2015, Round 6, Question 2.

We received 15 solutions, all of which were correct and complete. We present the solution by Corneliu Manescu-Avram with minor notational edits for clarity.

Let a and d be the first term and common difference of the arithmetic sequence, respectively. The sum S_k of the first k terms of the sequence is given by $S_k = \frac{1}{2}\big[2a + (k-1)d\big]k$, so

$$\begin{aligned}
\frac{S_m}{S_n} = \frac{m^2}{n^2} \qquad &\implies \qquad \frac{\big[2a+(m-1)d\big]m}{2} \cdot \frac{2}{\big[2a+(n-1)d\big]n} = \frac{m^2}{n^2}\\
&\implies \qquad \big[2a+(m-1)d\big]n = \big[2a+(n-1)d\big]m\\
&\implies \qquad (2a-d)(n-m) = 0\\
&\implies \qquad d = 2a,
\end{aligned}$$

since $m \neq n$. Thus the arithmetic sequence is $a, 3a, 5a, \ldots$ with $a \neq 0$ by the condition from the hypothesis. That is, the general form of the sequence is $a_k = (2k-1)a$, and it follows that

$$\frac{a_m}{a_n} = \frac{(2m-1)a}{(2n-1)a} = \frac{2m-1}{2n-1}.$$

Editor's Comments. Some submissions interpreted the assumption given as holding for all $m \neq n$ whereas others took m, n to be fixed. Both interpretations lead to the same answer.

PROBLEM SOLVING VIGNETTES

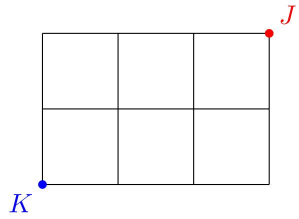
No. 42

Shawn Godin

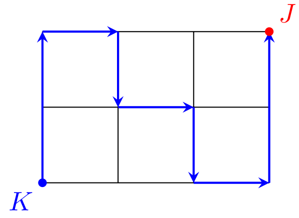
Counting Paths

How to get from point A to point B is a common problem we run into in life. Whether we are thinking philosophically about reaching a goal or literally trying to find our way to some destination, we've all run into situations where we are searching for a path. In mathematics, we seize upon this idea and often create problems involving determining the *number* of possible paths we can take to reach our target. As an example, consider the elementary example below.

> *Kseniya is meeting up with John to have a coffee (which is what mathematicians do when they are not solving problems ... and sometimes when they are solving problems). John is located three blocks east and two blocks north of Kseniya's current location. How many different paths can Kseniya take enroute to meet John?*



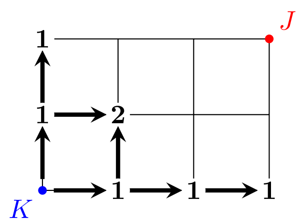
First of all we need some assumptions. In some cases, these are explicitly stated in the problem although they may just be implied. Since we have been given a "road map", we will assume that Kseniya will keep her route to the streets, and not wander through buildings and people's back yards. Secondly, we will assume that since Kseniya is a mathematician, she is anxious to get her coffee and, as such, follows an *optimal* path. That is, Kseniya will take the shortest possible route and do no backtracking like the case shown below.



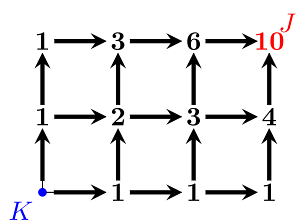
In the path above, Kseniya has travelled nine blocks. However, from the problem statement we know that she only needs to travel five blocks to meet John. So, we will keep her path to five blocks. Enforcing this restriction, this forces Kseniya to always be travelling either north or east on her way to John's house.

With all of our restrictions in mind, we can start solving the problem. A common method is to systematically label the number of paths to each intersection along the way. That is, to get to the intersection one block north of her place, Kseniya has only one path—walk north one block. Similarly there is one path to get to the intersection one block to the east. This idea can be extended to see that to reach any intersection either directly north or directly east of Kseniya's house, there is only one path.

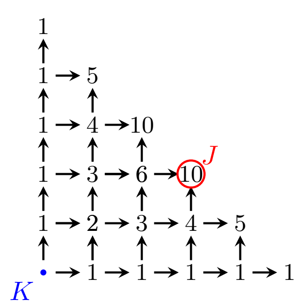
However, there are *two* paths to the intersection one block north and one block east of her place: she could go north then east, or east then north. If we label these numbers on the map, it seems that we get the two, by adding the two ones leading directly into it, as shown below.



Thus, we should be able to fill in the rest of grid and get our answer, ten paths.



Some readers may see something familiar in this pattern. For those that need a little more information, let's assume that Kseniya is going for a longer trip on a larger grid. If we follow the same process, we can see where she has made it after walking five blocks, and how many paths lead to these destinations.



Hopefully now, the familiar pattern of *Pascal's triangle*—although turned on its side—becomes apparent. Recalling that the entries of Pascal's triangle are equal to the binomial coefficients, $\binom{n}{r}$, we can see that our number of paths to John's place corresponds to $\binom{5}{2} = 10$. Let's try to determine why this should be so.

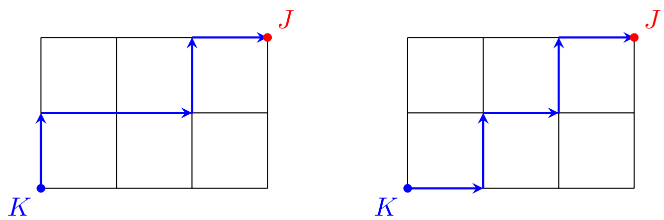
Recall that we can rewrite the binomial coefficients in terms of factorials as

$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5!}{2!3!}$$

We know from the original problem that Kseniya had to travel three blocks east and two blocks north. It would seem from our result, that if she wanted to travel e blocks east and n blocks north, there would be

$$\binom{e+n}{e} = \binom{e+n}{n} = \frac{(e+n)!}{e!n!}$$

paths that she could follow. Why should this be true? Consider labelling her paths using E and N to represent blocks travelled east and north respectively. Then, in the diagram below, the path on the left could be labelled $NEENE$, while the one on the right could be labelled $ENENE$.



As such, all possible paths could be described as all possible arrangements of three E's and two N's. Since we have five letters there are 5! ways to arrange them. However, since all the E's are indistinguishable, as are the N's, rearranging like letters amongst themselves does not change the appearance of our five letter arrangement. Thus, since there are 3! and 2! ways to arrange the E's and N's amongst themselves, respectively, then our total number of paths is

$$\frac{5!}{3!2!}$$

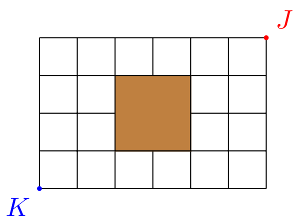
or in general

$$\frac{(e+n)!}{e!n!} = \binom{e+n}{e} = \binom{e+n}{n}.$$

MathemAttic readers may recognize the last result as being equivalent to the solution to part a) of problem MA333 [2025: 51(7), 306-308] and [2026: 52(2), 67]. However, the solution to part b), published in issue 2 of this year, was incorrect. The given solution is for the number of paths from $(0, 0, 0)$, to (n, n, n) by moving unit steps in the positive direction. That is, it includes paths that pass *through* the interior of the cube, whereas the problem stipulated that the paths should be restricted to the *surface* of the cube.

Whenever one encounters a counting problem, one should proceed with caution. These occasionally easy *looking* problems are often filled with subtle details that can lead the careless problem-solver to commit the sins of missing cases, over-counting, or both.

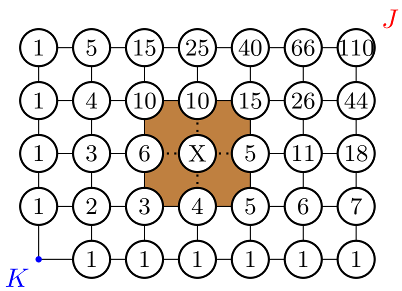
Let's return to Kseniya and John and examine a related problem. Again, Kseniya is going to meet up with John, but in this case he is six blocks east and four blocks north of her. On top of that, in the middle of the path, there is an area where paths do not pass through, as shown in the diagram below.



If we just assume that it is the same as the previous problem, we would get

$$\binom{6+4}{6} = \frac{10!}{6!4!} = 210.$$

However, if we go back and count the paths as we did at the start of the article, we get 110, as shown in the diagram below. Where did the 100 missing paths go?



A hint comes from the diagram above, where I have labelled, with an X, the "missing" intersection from our street grid. The $\binom{10}{6}$ that we jumped to originally included paths that pass through the unavailable intersection marked X. As such, if we could count the number of paths that pass through X, we could just take them from our original! Let's give it a try.

If our path is going to pass through X, we can treat it as *two* paths: one from Kseniya's starting point to X, the other from X to John's location. Both of these paths involve a trip three blocks east and two blocks north, of which there are $\binom{5}{3}$.

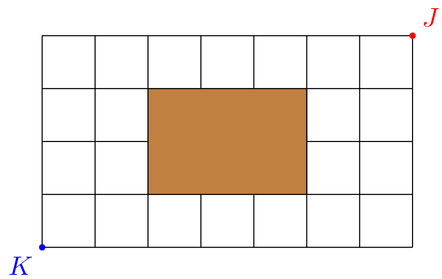
Thus, our total number of allowed paths should be

$$\binom{10}{6} - \binom{5}{3}^2 = 210 - 10^2 = 110.$$

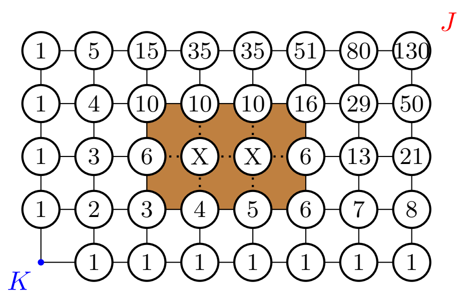
Bingo! So we should be able to fix the solution to MA33 b) by subtracting the unwanted paths.

Before you run off trying this new strategy, a moment of reflection is in order. For one thing, there are a lot of interior points in an $n \times n \times n$ cube that we will have to account for, each with different—although related—numbers of paths through them. Although this logistical nightmare is the type of thing some people love to wrestle with, let's return to Kseniya and John for one more cautionary tale.

Suppose Kseniya is still trying to meet up with John for coffee, however this time she has even further to go with an even larger "restricted" zone.



Using the basic strategy, we see that Kseniya has 130 different routes to follow to meet up with John, as shown in the diagram below.



If we use the strategy that we just developed and take off the paths that pass through the missing intersections we get

$$\binom{11}{7} - \binom{5}{3}\binom{6}{4} - \binom{6}{4}\binom{5}{3} = 330 - 10 \times 15 - 15 \times 10 = 30$$

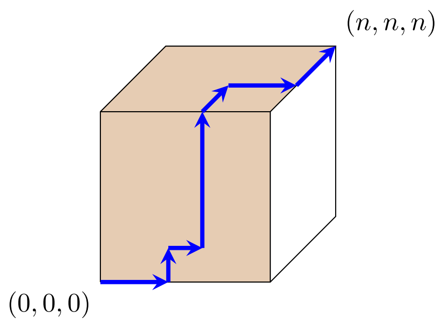
which undercounts by 100. What happened this time? By now, I am sure you are wise to my tricks. We have inadvertently removed those paths that pass through *both* missing intersections *twice*. As such, we need to add back in the paths we've over-removed. The number of paths that pass through both missing intersections is

$$\binom{5}{3} \times 1 \times \binom{5}{3} = 100$$

which brings us to our desired result. This means, if we want to use a strategy of removing the paths through the interior in MA333 b) it is going to become overly complex as many paths that pass through the interior pass through multiple interior points. Thus we would have to worry about over subtracting paths that pass through two points, but then maybe overcorrecting for paths that pass through three points, etc. Yuck!

As with many problems, when our current strategy runs into a roadblock or becomes unwieldy, its time to come up with a new strategy. Our new strategy comes from *The Ring Lords* problem-solving group, who pointed out the error in the published solution to MA333 b). *The Ring Lords* are a team of highly gifted students from various high schools across Baden-Württemberg, Germany. The group includes (in alphabetical order of their last names) Emilia Brix, Avaneesh Jawalkar, Lilli Kappler, Benedikt Kuhn, Tianqi Li, Johanna Prietz, and Letong Zhong. The solution that follows is an edited version of the solution received by the Editor-in-Chief of ***Crux***.

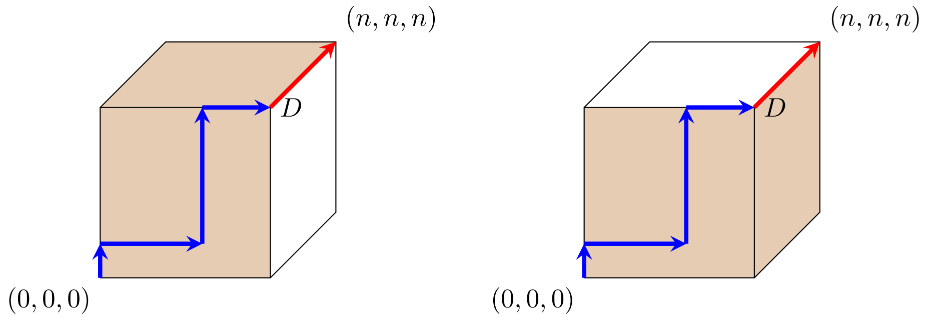
First we note that the path must lie on two adjacent faces. Picture the $n \times n \times n$ cube facing us with $(0, 0, 0)$ at the bottom left corner and (n, n, n) at the top right corner of the back side. In the diagram below, a possible path from $(0, 0, 0)$ to (n, n, n) is shown and is contained on the front and top faces.



As $(0, 0, 0)$ is at the intersection of the front, bottom and left faces, and (n, n, n) is at the intersection of the back, top and right faces, a path will be contained on two faces, the first from the front, bottom and left faces and the second from the back, top and right faces. However, each face in each group of three only attaches to two other faces in the other group. That is, our path must be contained to one of the pairs of faces: front-top, front-right, bottom-back, bottom-right, left-back, and left-top. In each of these cases, we can consider a path on a $n \times 2n$ grid, where we are going to diagonally opposite corners. Hence, the number of paths across all of these six pairs is

$$6\binom{2n+n}{n} = 6\binom{3n}{n}.$$

We can see that we haven't missed any cases, but have any been overcounted? Consider the path shown in the diagram below. It would be counted in *both* the front-top group and the front-right group.



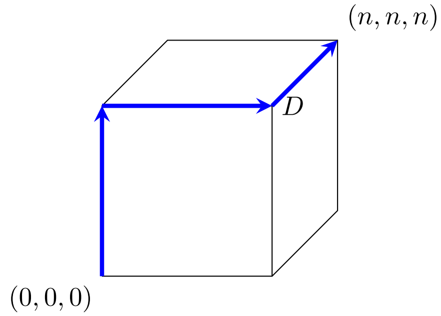
The blue part of the path is on the front face, and since the red part follows the edge that is common to both the top and right face, it is counted in both groups. As such, each face will contribute paths that will be double counted. This only happens when the path goes from one vertex to the diagonally opposite vertex and the rest of the path (either before or after) follows the edge of the cube that is common between the two other faces (from the other group) that the face in question is attached to. That is, in our example, there are

$$\binom{n+n}{n} = \binom{2n}{n}$$

paths on the front face from $(0,0,0)$ to D, hence there are $\binom{2n}{n} \times 1$ paths from $(0,0,0)$ to (n,n,n) that are counted both on the front-top group and the front-right group. Each face contributes this same number of double counted paths, which means we need to subtract $6\binom{2n}{n}$ from our original total, giving the number of unique paths on the surface of the cube as

$$6\binom{3n}{n} - 6\binom{2n}{n} = 6\left[\binom{3n}{n} - \binom{2n}{n}\right].$$

There is just one more type of overcounted path we need to make sure hasn't been under or over adjusted for, that is paths that completely follow edges. Consider the path shown in the diagram below.

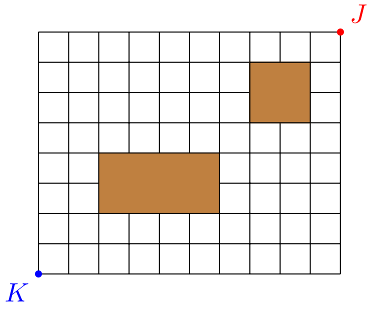


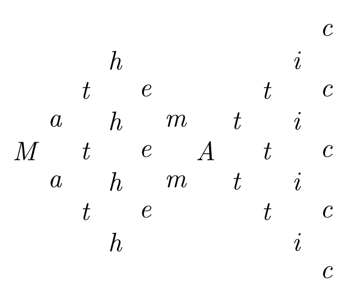
In our original counting, this path is counted *three* times, in the front-top, left-top, and front-right groups. When we do our adjustment, it would be subtracted from the front face (edge between top and right) and top face (edge between front and left) groups, so it would be subtracted twice, leaving each such path counted once, as desired. Thus our result is correct.

. . . .

My thanks to *The Ring Lords* for pointing out the error with the original solution and for providing us with the solution featured above. I hope we will see more solutions from you in the future! Below are a few more path counting problems for you to consider.

1. How many paths, on the map below (on the left), can Kseniya follow to meet John for a coffee?
2. Determine the number of ways to spell *MathemAttic* on the diagram below (on the right). You may proceed from letter to letter by going to one of the two (or the only) letter to the right and directly above or below the previous letter.
3. Solve the four-dimensional version of MA333 b).





OLYMPIAD CORNER

No. 443

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **July 15, 2026**.

OC781. Two countries agreed to resolve a border issue by talk and have sent (each of them) a delegation of $n > 2$ diplomats, all of different heights. For each diplomat, strict protocol regulates the number of delegation members from the other country's contingent that he or she needs to shake hands with. This number is 1 for the shortest diplomats of each contingent, 2 for the next ones in tallness and so on; the only exception being both delegations' tallest diplomats, who need to shake hands with $n-1$, rather than with n members of the other delegation. Give an explicit formula for the number c_n of ways the regulations of the protocol can be met!

OC782. Let $A_1, \ldots, A_m$ be a collection of real, symmetric, $n \times n$ matrices such that $\sum_{k=1}^{m} A_k^2 = I$. Decide whether the equality $\sum_{k=1}^{m} A_k X A_k = X$ implies that the (real, $n \times n$) matrix X commutes with each A_k.

OC783. Prove that $\frac{2}{2^{2n}} \sum_{k=0}^{n} \binom{2n}{2k} 13^k$ is an integer equal to 1 modulo 3 for every $n = 1, 2, \ldots$

OC784. In $\triangle ABC$, let D, E, F be the midpoints of BC, AC, AB respectively and let G be the centroid of the triangle. For each value of $\angle BAC$, how many non-similar triangles are there in which $AEGF$ is a cyclic quadrilateral?

OC785. Let $P(x)$ be a non-constant polynomial with integer coefficients such that $P(0) \neq 0$. Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of integers such that $P(i-j)$ divides $a_i - a_j$ for all distinct positive integers i, j. Prove that the sequence $a_1, a_2, a_3, \ldots$ must be constant, that is, a_n equals a constant c for all positive integers n .

. . . .

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 juillet 2026**.

OC781. Deux pays ont convenu de régler un différend frontalier par la négociation et ont chacun envoyé une délégation de $n > 2$ diplomates, tous de tailles différentes. Pour chaque diplomate, un protocole strict détermine le nombre de membres de la délégation adverse avec lesquels il ou elle doit serrer la main. Ce nombre est égal à 1 pour les plus petits diplomates de chaque délégation, à 2 pour les suivants par ordre de taille, et ainsi de suite ; la seule exception concerne les plus grands diplomates des deux délégations, qui doivent serrer la main avec $n-1$ membres de l'autre délégation plutôt qu'avec n membres de l'autre délégation. Donnez une formule explicite pour le nombre c_n de façons dont les règles du protocole peuvent être respectées.

OC782. Soient $A_1, \ldots, A_m$ une famille de matrices réelles symétriques de taille $n \times n$ telle que $\sum_{k=1}^m A_k^2 = I$. Déterminez si l'égalité $\sum_{k=1}^m A_k X A_k = X$ implique ou non que la matrice réelle X de taille $n \times n$ commute avec chacune des matrices A_k.

OC783. Montrez que $\frac{2}{2^{2n}} \sum_{k=0}^n \binom{2n}{2k} 13^k$ est un entier congru à 1 modulo 3 pour tout $n = 1, 2, \ldots$

OC784. Dans $\triangle ABC$, soient D, E et F les milieux respectifs de BC, AC et AB, et soit G le centre de gravité du triangle. Pour chaque valeur de $\angle BAC$, déterminez le nombre de triangles non semblables tels que $AEGF$ soit un quadrilatère cyclique.

OC785. Soit $P(x)$ un polynôme non constant à coefficients entiers tel que $P(0) \neq 0$. Soit $a_1, a_2, a_3, \ldots$ une suite infinie d'entiers telle que, pour tous entiers positifs distincts i et j on a que $P(i-j)$ divise $a_i - a_j$. Montrez que la suite $a_1, a_2, a_3, \ldots$ est nécessairement constante, c'est-à-dire qu'il existe une constante c telle que $a_n = c$ pour tout entier positif n.

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2025: 51(10), p. 464–465.

OC756. Let $A \in \mathcal{M}_2(\mathbb{R})$ be a matrix with real entries such that

$$\det(A^{2014} - I_2) = \det(A^{2014} + I_2)$$

and

$$\det(A^{2016} - I_2) = \det(A^{2016} + I_2).$$

Prove that $\det(A^n - I_2) = \det(A^n + I_2)$, for any $n \in \mathbb{N}$. Above $\mathbb{N}$ is the set of positive integers and I_2 is the 2×2 identity matrix.

Proposed by Vlad Mihaly, Romanian Mathematical Olympiad-Final Round, 2016.

We received 8 submissions, all of which were correct and complete. We present the solution by Theo Koupelis.

Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$\det(X \pm I_2) = (a \pm 1)(d \pm 1) - bc = \det X \pm \operatorname{Tr} X + 1. \qquad (1)$$

Using (1) we get

$$\det(A^{2014} - I_2) = \det(A^{2014} + I_2) \iff \operatorname{Tr} A^{2014} = 0, \qquad (2)$$
$$\det(A^{2016} - I_2) = \det(A^{2016} + I_2) \iff \operatorname{Tr} A^{2016} = 0. \qquad (3)$$

Let λ_1, λ_2 be the eigenvalues of A. Then from (2) and (3) we get

$$\lambda_1^{2014} + \lambda_2^{2014} = 0 \quad \text{and} \quad \lambda_1^{2016} + \lambda_2^{2016} = 0,$$

which implies

$$0 = (\lambda_1^{2014} + \lambda_2^{2014})(\lambda_1^2 + \lambda_2^2) = \lambda_1^2\lambda_2^2(\lambda_1^{2012} + \lambda_2^{2012}).$$

Thus, if $\lambda_1\lambda_2 \neq 0$, we get $\lambda_1^{2012} + \lambda_2^{2012} = 0$. Proceeding inductively as above we finally get $\lambda_1^2 + \lambda_2^2 = 0$ and

$$\lambda_1^4 + \lambda_2^4 = 0 = (\lambda_1^2 + \lambda_2^2)^2 - 2\lambda_1^2\lambda_2^2,$$

which is a contradiction. Therefore,

$$\lambda_1 = \lambda_2 = 0 \Longrightarrow \operatorname{Tr} A^n = 0, \text{ for any } n \in \mathbb{N}. \qquad (4)$$

From (1) and (4) we get the desired result, namely that for any $n \in \mathbb{N}$

$$\det(A^n - I_2) = \det(A^n + I_2) = \det(A^n + 1).$$

OC757. Find all continuous bijective functions $f : [0,1] \to [0,1]$ such that

$$\int_0^1 g(f(x))dx = \int_0^1 g(x)dx,$$

for any continuous function $g : [0,1] \to \mathbb{R}$.

Proposed by Dorin Andrica and Mihai Piticari, Romanian Mathematical Olympiad-Final Round, 2009.

We received 6 submissions, all of which were correct and complete. We present the solution by Corneliu Manescu-Avram.

We claim that there are two such functions: $f(x) = x \ \forall x \in [0,1]$ and $f(x) = 1-x$, $\forall x \in [0,1]$.

As the function $f : [0,1] \to [0,1]$ is continuous and bijective it must be strictly monotone. If it is strictly increasing, then select as g the function

$$g_c(x) = \begin{cases} x - c, & x \in [0,c) \\ 0, & x \in [c,1]. \end{cases}$$

It implies that

$$\int_0^1 g_c(f(x))dx = \int_0^1 g_c(x)dx \text{ and}$$
$$\int_0^{f^{-1}(c)} [f(x) - c]dx = \int_0^c (x-c)dx \text{ and}$$
$$\int_0^{f^{-1}(c)} f(x)dx = cf^{-1}(c) - \frac{c^2}{2}.$$

Let $f^{-1}(c) = t$, then

$$\int_0^t f(x)dx = tf(t) - \frac{[f(t)]^2}{2} \text{ and } \int_0^t [f(x) - x]dx = -\frac{[f(t)-t]^2}{2}.$$

Let $h(x) = f(x) - x$ for $x \in [0,1]$, we get:

$$\int_0^t h(x)dx = -\frac{[h(t)]^2}{2}, \forall t \in [0,1]. \quad (*)$$

The function h is continuous and $h(0) = h(1) = 0$. Let $t_0 \in [0,1]$ be a point of global minimum of h. If $h(t_0) < 0$ then $t_0 < 1$. Moreover, there exists $\delta > 0$ such that $h(x) < 0$ for any $x \in [t_0, t_0 + \delta]$, therefore

$$\int_0^{t_0+\delta} h(x)dx < \int_0^{t_0} h(x)dx \implies -\frac{[h(t_0+\delta)]^2}{2} < -\frac{[h(t_0)]^2}{2} \implies h(t_0+\delta) < h(t_0).$$

This contradicts the fact that t_0 is a point of global minimum. It follows that $h(t_0) \geq 0$, hence $h(x) \geq 0$ for any $x \in [0,1]$. From $(*)$ we deduce that $h(x) = 0$, so $f(x) = x$ for any $x \in [0,1]$.

If f is strictly decreasing, then $1-f$ is strictly increasing and satisfies the condition stated by the question. Then we deduce that $1 - f(x) = x$, so $f(x) = 1 - x$ for any $x \in [0,1]$.

OC758. Consider $A, B \in \mathcal{M}_n(\mathbb{C})$ such that $AB = BA$ and $\det B \neq 0$.

a) If $|\det(A + zB)| = 1$, for all $z \in \mathbb{C}$ with $|z| = 1$, prove that $A^n = 0_n$;

b) Is the conclusion true if the commutative condition is dropped?

Proposed by Vasile Pop, Romanian Mathematical Olympiad-Final Round, 2009.

We received 3 submissions, all of which were correct and complete, and we present the solution by Corneliu Mănescu-Avram.

a) The function $f(z) = \det(A + zB) = a_0 + a_1 z + \cdots + a_n z^n$ is a polynomial function of degree n, since $a_n = \det B \neq 0$. We have

$$\begin{aligned}
& |\det(A + zB)| = 1, \forall z, |z| = 1 \\
\iff\quad & f(z)\overline{f(z)} = 1, \forall z, \bar{z} = \frac{1}{z} \\
\iff\quad & (a_0 + a_1 z + \cdots + a_n z^n)\left(\bar{a}_0 + \bar{a}_1 \frac{1}{z} + \cdots + \bar{a}_n \frac{1}{z^n}\right) = 1 \\
\iff\quad & (a_0 + a_1 z + \cdots + a_n z^n)\left(\bar{a}_0 z^n + \bar{a}_1 z^{n-1} + \cdots + \bar{a}_n\right) = z^n.
\end{aligned}$$

This equality holds for infinitely many z, so it is an identity. Identifying coefficients and proceeding inductively, we get $a_0 = a_1 = \cdots = a_{n-1} = 0$ and $|a_n| = 1$. Hence $|\det B| = 1$. Now, we have

$$\begin{aligned}
& f(z) = a_n z^n \\
\implies\quad & \det(A + zB) = (\det B) z^n \\
\implies\quad & \det[B(B^{-1}A + zI_n)] = (\det B) z^n \\
\implies\quad & \det B \cdot \det(B^{-1}A + zI_n) = (\det B) z^n \\
\implies\quad & \det(B^{-1}A + zI_n) = z^n.
\end{aligned}$$

It follows that $g(z) = z^n$ is the characteristic polynomial of $C = -B^{-1}A$. By the Cayley-Hamilton theorem,

$$C^n = (-B^{-1}A)^n = O_n \Rightarrow A^n = O_n.$$

b) The condition $AB = BA$ is necessary, as the following counterexample shows. Define the following matrices using block notation:

$$A = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & I_{n-1} \end{array}\right), \text{ and } B = \left(\begin{array}{c|c} 0 & 1 \\ \hline I_{n-1} & 0 \end{array}\right).$$

It is easy to check that $AB \neq BA$. Also,

$$\det(A+zB) = -(-z)^n$$

and

$$|\det(A+zB)| = 1$$

for all z with $|z|=1$. But $A^n = A \neq O_n$.

OC759. Let ABC be a scalene triangle, let I be its incentre, and let A_1, B_1, and C_1 be the points of contact of the excircles with the sides BC, CA, and AB, respectively. Prove that the circumcircles of the triangles AIA_1, BIB_1, and CIC_1 have a common point different from I.

Proposed by Cezar Lupu and Vlad Matei, Balkan Mathematical Olympiad 2010.

We received 6 submissions, of which 5 were correct and complete. We present the solution by Theo Koupelis.

Let s be the semiperimeter of the triangle with $a=BC$, $b=AC$, and $c=AB$. Let the circles (AIA_1), (BIB_1), and (CIC_1) intersect the circle (ABC) again at points A_2, B_2, and C_2, respectively. Using barycentric coordinates, let $A=(1,0,0)$, $B=(0,1,0)$, $C=(0,0,1)$, and thus $A_1=(0:s-b:s-c)$, $B_1=(s-a:0:s-c)$, $C_1=(s-a:s-b:0)$, and $I=(a:b:c)$. The general equation of a circle is given by $a^2yz+b^2zx+c^2xy=(x+y+z)(ux+vy+wz)$, where u,v,w are real numbers. Substituting the coordinates (x,y,z) for the points A, B, C, A_1, B_1, C_1, and I, we get the equations for the corresponding circles:

$$\begin{aligned}
(ABC):&\ a^2yz+b^2zx+c^2xy=0,\\
(AIA_1):&\ a^2yz+b^2zx+c^2xy=\\
&\qquad (x+y+z)\left(\frac{ac(s-c)(2b-s)}{s(b-c)}\cdot y+\frac{ab(s-b)(s-2c)}{s(b-c)}\cdot z\right),\\
(BIB_1):&\ a^2yz+b^2zx+c^2xy=\\
&\qquad (x+y+z)\left(\frac{bc(s-c)(2a-s)}{s(a-c)}\cdot x+\frac{ab(s-a)(s-2c)}{s(a-c)}\cdot z\right),\\
(CIC_1):&\ a^2yz+b^2zx+c^2xy=\\
&\qquad (x+y+z)\left(\frac{bc(s-b)(2a-s)}{s(a-b)}\cdot x+\frac{ac(s-a)(s-2b)}{s(a-b)}\cdot y\right).
\end{aligned}$$

By subtracting each of the equations for the circles (AIA_1), (BIB_1), and (CIC_1) from that of the circle (ABC), and rearranging, we get the equations of the lines AA_2, BB_2, CC_2, which are the equations of the corresponding radical axes. Thus,

$$\begin{aligned}
AA_2:&\ c(s-c)(2b-s)\cdot y+b(s-b)(s-2c)\cdot z=0,\\
BB_2:&\ c(s-c)(2a-s)\cdot x+a(s-a)(s-2c)\cdot z=0,\\
CC_2:&\ b(s-b)(2a-s)\cdot x+a(s-a)(s-2b)\cdot y=0.
\end{aligned}$$

It is easy to show that the determinant of the above equations in x, y, z is zero, and thus the lines AA_2, BB_2, CC_2 are concurrent. Let the intersection point be Q. Let the line IQ intersect the circles (AIA_1), (BIB_1), and (CIC_1), at Q_1, Q_2, and Q_3, respectively. Considering the powers of Q with respect to the circles (ABC), (AIA_1), (BIB_1), and (CIC_1), we get

$$QA \cdot QA_2 = QI \cdot QQ_1, \quad QB \cdot QB_2 = QI \cdot QQ_2, \quad QC \cdot QC_2 = QI \cdot QQ_3.$$

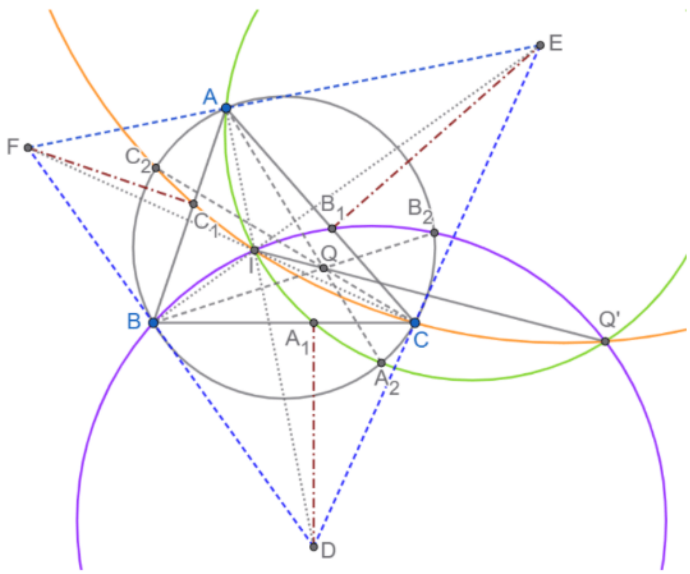
But

$$QA \cdot QA_2 = QB \cdot QB_2 = QC \cdot QC_2 \neq 0,$$

and thus

$$Q_1 \equiv Q_2 \equiv Q_3 \equiv Q'.$$

Therefore, the circumcircles of the triangles AIA_1, BIB_1, and CIC_1 have a common point different from I.



OC760. Let $\mathbb{N}$ denote the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$\gcd(f(x), y) f(xy) = f(x)f(y)$$

for all x and y in $\mathbb{N}$.

Originally from the Nordic Mathematical Contest, March 2023.

We received 2 submissions, both of which were correct and complete, and we present the solution by Theo Koupelis.

Let $P(x, y)$ be the assertion that $\gcd(f(x), y)\, f(xy) = f(x)f(y)$ for all x and y in $\mathbb{N}$. From $P(1, 1)$ we get $f(1) = 1$. From $P(y, x)$ and $P(x, y)$ we get

$$\gcd(f(y), x) = \gcd(f(x), y).$$

Setting $y = f(x)$ we get

$$\gcd(f(f(x)), x) = \gcd(f(x), f(x)) = f(x).$$

Thus, $f(x)$ divides x, and specifically, when x is a prime p, we get $f(p) = 1$ or $f(p) = p$.

(i) Let $f(p) = 1$. From $P(p, y)$ we get $\gcd(1, y)f(py) = f(y)$ and thus we obtain $f(py) = f(y)$. By induction, if $y = p^k \cdot z$, where $k \geq 0, z > 0$ are integers and $(p, z) = 1$, then $f(py) = f(z)$. Also, if $y = p^k \cdot z$, where $k \geq 1, z \geq 1$ are integers and $(p, z) = 1$, then $f(y) = f(z) = f(p)f(z)$.

(ii) Let $f(p) = p$. If $(p, y) = 1$, then from $P(p, y)$ we get $f(py) = pf(y)$. If $(p, y) = p$, then from $P(p, y)$ we get $f(py) = f(y)$; thus, if $y = p^k \cdot z$, where $(p, z) = 1$, then $f(p^{k+1}z) = f(p^k z)$, and thus by induction we get $f(py) = f(pz) = pf(z)$. If $y = p^k \cdot z$, where $(p, z) = 1$, $k \geq 1, z \geq 1$ then $f(y) = pf(z) = f(p)f(z)$.

From the above we get that if the prime factorization of y is

$$y = p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n},$$

where k_i are positive integers for $i = 1, \ldots, n$, then

$$f(y) = f(p_1)f(p_2) \cdots f(p_n),$$

where each of the terms $f(p_i)$ can take either the value 1 or p_i. Indeed, let

$$x = p_1^{m_1} p_2^{m_2} \cdots p_n^{m_n} p_{n+1}^{m_{n+1}} \cdots p_{n+\ell}^{m_{n+\ell}},$$

where m_i are non-negative integers. Then $\gcd(f(y), x) = f(p_1)f(p_2) \cdots f(p_n)$, where for any primes p_i that are not in the prime factorization of x but are in y (because some m_i could be zero) we have $f(p_i) = 1$. If all primes p_i, where $i = 1, 2, \ldots, n$, are included in the prime factorization of x, then $P(y, x)$ holds because $\gcd(f(y), x) = f(x)$, and $f(xy) = f(x)$. If, say, p_1 is not included in the prime factorization of x, then

$$\begin{aligned} \gcd(f(y), x) &= [f(p_2) \cdots f(p_n)] \cdot [f(p_1) \cdots f(p_{m+\ell})] \\ &= [f(p_2) \cdots f(p_{m+\ell})][f(p_1) \cdots f(p_n)] = f(x)f(y), \end{aligned}$$

and thus $P(y, x)$ holds. This is true for any prime that is included in the prime factorization of y but not in x.

The Termwise Bounding Technique

Sicheng Du

1 Introduction

Termwise bounding† is a technique used to prove inequalities. It works by bounding certain terms with newly constructed expressions that are either simpler or more closely related to other parts of the inequality. A typical example would be the official solution to Problem 2 of IMO 2001, which asks to prove that for all positive real numbers a, b, c,

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\geq 1.$$

The official solution proceeds by establishing the bound

$$\frac{a}{\sqrt{a^2+8bc}}\geq\frac{a^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}},$$

and taking its cyclic sum, which completes the proof.

To grasp the technique's main idea, we start by summarizing its application on three variable cyclic inequalities. To prove that $\sum\limits_{\text{cyc}} f(a,b,c)\geq p$, we take the following two steps:

(a) find and prove an inequality $f(a,b,c)\geq g(a,b,c)$ which makes the next step feasible;

(b) prove that $\sum\limits_{\text{cyc}} g(a,b,c)\geq p$ and conclude $\sum\limits_{\text{cyc}} f(a,b,c)\geq\sum\limits_{\text{cyc}} g(a,b,c)\geq p$.

From this abstraction, it is evident that step (a) has no explicit objective and that we have no way of assessing its feasibility before proceeding to step (b). As a result, the form of g has a high degree of uncertainty which makes it hard to search for. Nevertheless, a pursuit for simpler inequalities guides us to use expressions of relatively determined shapes as bounds, enabling termwise bounding to function as a technique, rather than a total mystery. Still, the processes of constructing and proving partial bounds involve seemingly intimidating asymmetric inequalities. This article will reconstruct the motivations and detailed steps behind several problems solved using the technique. Note that the search for g is an exploration

† The name "isolated fudging" used by Yufei Zhao in an inequality handout (accessible at `https://yufeizhao.com/olympiad/wc08/ineq.pdf`) may be more familiar to some readers. The same technique also appeared in [1], where it is given a more concrete name: *establishing new effective inequalities*.

instead of an indispensable part of the proof logic, so it need not be absolutely rigorous and sometimes even a guess can be very helpful.

In what follows, the reader will be assumed to have knowledge of simple calculus and inequalities commonly used in math Olympiads, such as AM-GM, Cauchy-Schwarz and so on.

2 Example Problems

The problems in this section are ordered roughly by their difficulty. The accompanied solutions showcase thinking processes in order to give the reader an impression on what inequalities might one apply termwise bounding to, and what common methods there are to find and prove the specific bounds.

Cyclic expressions or summations naturally indexed by a group of symmetries in general, are our primary focus. The bound of each term can derive from that of a particular term. The method of *undetermined coefficients* (also known as "bacteria") is a standard part of the technique: we introduce parameters into the bound and determine their values by assigning certain values to the variables (e.g. the equality case).

Problem 1. *Let a, b, and c be nonnegative real numbers. Prove that*

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}.$$

Proof. At first glance, this is a cyclic sum that seems hard enough to try termwise bounding on. We will first try to find a lower bound for $\frac{2a}{3(b+c)}$ whose cyclic sum is 1. The family $\frac{a^r}{a^r+b^r+c^r}$ is a natural candidate because it sums cyclically to 1 and the parameter r provides flexibility to make the bound work.

Learning from the IMO problem mentioned above, we may try to compare $\frac{2a}{3(b+c)}$ with $\frac{a^r}{a^r+b^r+c^r}$ for some r in order to prove that $\sum_{\text{cyc}}\frac{2a}{3(b+c)}\geq 1$. In particular, we need

$$2a\left(a^r+b^r+c^r\right)\geq 3a^r(b+c),$$

Approaching from the equality case, set $b=c=1$ to get $a\left(a^r+2\right)\geq 3a^r$. This inequality can be written as $f(a):=a\left(a^r+2\right)-3a^r\geq 0$. Since $f(1)=0$, then $f(x)$ has a local minimum at $x=1$, so $f'(1)=0$, from which we solve out $r=\frac{3}{2}$.

The fitting process above — using the equality case to determine parameters — is an important part of the technique. The parameter r leaves room for adjustment in the $\frac{a^r}{a^r+b^r+c^r}$ family.

In addition, according to AM-GM inequality,

$$2\sum_{\text{cyc}} a^{\frac{3}{2}} = \left(a^{\frac{3}{2}} + 2b^{\frac{3}{2}}\right) + \left(a^{\frac{3}{2}} + 2c^{\frac{3}{2}}\right) \geq 3a^{\frac{1}{2}}b + 3a^{\frac{1}{2}}c = 3a^{\frac{1}{2}}(b+c),$$

so indeed

$$\frac{a}{b+c} \geq \frac{3a^{\frac{3}{2}}}{2\left(a^{\frac{3}{2}} + b^{\frac{3}{2}} + c^{\frac{3}{2}}\right)},$$

the cyclic sum of which completes the proof. □

Problem 2 (2025 IMOC)**.** *Let $a+b+c=1$, where a, b, $c \geq 10^{-4}$. Prove that:*

$$\sum_{\text{cyc}} \frac{141a^2b + 3a^3}{\sqrt{ab + 3\sqrt{a^3b}}} \leq 24.$$

Proof. The gnarly denominator, and the odd coefficients in the numerator, suggests that they are set up rather specifically to render some trick usable. One approach would be to try to reverse-engineer the trick.

While the purpose of the condition a, b, $c \geq 10^{-4}$ is currently unclear, we still try to homogenize the inequality, into

$$\sum_{\text{cyc}} \frac{47a^2b + a^3}{\sqrt{ab + 3\sqrt{a^3b}}} \leq 8(a+b+c)^2.$$

(Also a, b, $c \geq 10^{-4}$ can be homogenized into a, b, $c \geq 10^{-4}(a+b+c)$, which is a constraint on the relative sizes of a, b, c.)

Conceivably, we wish to upper bound $\dfrac{47a^2b + a^3}{\sqrt{ab + 3\sqrt{a^3b}}}$ by a quadratic expression whose cyclic sum is $8(a+b+c)^2$. In general this can be expressed as

$$\frac{47a^2b + a^3}{\sqrt{ab + 3\sqrt{a^3b}}} \leq k_1a^2 + k_2b^2 + k_3c^2 + k_4ab + k_5bc + k_6ca,$$

where $k_1 + k_2 + k_3 = 8$ and $k_4 + k_5 + k_6 = 16$.

The equality case at $a = b = c$ is already satisfied, so we approach from the border. When $c = 10^{-4}$ and $a = b = \dfrac{1-c}{2}$, the inequality transforms into

$$\begin{aligned}&\frac{c^2}{4}(k_1 + k_2 + 4k_3 + k_4 - 2k_5 - 2k_6 - 24)\\ &- \frac{c}{2}(k_1 + k_2 + k_4 - k_5 - k_6 - 24) + \frac{k_1 + k_2 + k_4 - 24}{4} \geq 0.\end{aligned}$$

Because c is a small quantity, we guess that $k_1 + k_2 + k_4 = 24$ and $k_3 = k_5 = k_6 = 0$, thereby $k_1 + k_2 = 8$ and $k_4 = 16$. The inequality to fulfill becomes

$$\frac{47a^2b + a^3}{\sqrt{ab + 3\sqrt{a^3b}}} \leq (8 - k_2)a^2 + k_2b^2 + 16ab.$$

Substitute $b = at^2$ and square both sides, so that the variable a cancels out and the last inequality simplifies to

$$\left(47t^2+1\right)^2 \le \left(t^2+3t\right)\left(8-k_2+k_2t^4+16t^2\right)^2.$$

Similar to the previous problem, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(t^2+3t\right)\left(8-k_2+k_2t^4+16t^2\right)^2-\left(47t^2+1\right)^2\right]=768k_2=0,$$

so $k_2 = 0$, for which the last inequality factorizes into

$$(t-1)^2\left(256t^4+1280t^3+351t^2+190t-1\right)\ge 0.$$

Since $b \ge 10^{-4}$ and $a \le 1$, then $t \ge \sqrt{\frac{10^{-4}}{1}} = 10^{-2}$, so the inequality holds. □

Remark. *The problem proposer probably designed the coefficients and structure of the original expression so that anyone who thinks of termwise bounding can work out a proof smoothly. This is usually the case for contest problems, so one should remember not to overthink.*

Problem 3. *(Mathematical Reflections S627, [2].) Let a, b, and c be nonnegative real numbers. Prove that*

$$\sum_{\text{cyc}} 2a\sqrt{9b^2+16c^2}+15abc\sum_{\text{cyc}}\frac{1}{2b+3c}\ge 13\sum_{\text{cyc}} ab.$$

Proof. We hope to find a bound for the radical $\sqrt{9b^2+16c^2}$ in the form of a rational expression to simplify the inequality, because the simultaneous existence of square roots, fractions, and polynomials would otherwise be too intractable.

Considering homogeneity, it could be $\sqrt{9b^2+16c^2} \ge \frac{f(b,c)}{g(b,c)}$, where f and g is a quadratic and a linear polynomial respectively. Moreover, to facilitate further combination of terms, we want shared denominators, so $g(b,c)$ had better be $2b+3c$. Set $f(b,c) = ub^2+vc^2+wbc$, then the following inequality must hold.

$$X(b,c) := \left(9b^2+16c^2\right)(2b+3c)^2-\left(ub^2+vc^2+wbc\right)^2\ge 0.$$

To preserve the equality case at $a = b = c$, we need $X(1,1) = 0$, or $u+v+w = 25$. Moreover, since $X(b,1) \ge 0$, then a local minimum of the function $f(b) = X(b,1)$ is obtained at $b = 1$, so $f'(1) = \frac{\partial X}{\partial b}\mid_{b,c=1} = 0$, or

$$950-2(2u+w)(u+v+w)=950-50(2u+w)=0.$$

The two equations imply that $v = 6+u$ and $w = 19-2u$. Plugging back, we get

$$X(b,c)=(b-c)^2\begin{bmatrix} b^2(6-u)(6+u)+c^2(6-u)(18+u) \\ \qquad\qquad +\,bc\left(2u^2-38u+180\right)\end{bmatrix}.$$

It is easy to see that when $u = 6$, the coefficients before b^2 and c^2 in the second parentheses vanish, leaving only $24bc$, which is indeed nonnegative.

Now that the bound $\sqrt{9b^2+16c^2} \geq \frac{6b^2+12c^2+7bc}{2b+3c}$ is established, what remains is

$$\sum_{\text{cyc}} 2a \cdot \frac{6b^2+12c^2+7bc}{2b+3c} + 15abc \sum_{\text{cyc}} \frac{1}{2b+3c} \geq 13 \sum_{\text{cyc}} ab,$$

This can be transformed into

$$ab+bc+ca \geq abc\left(\frac{5}{2b+3c}+\frac{5}{2c+3a}+\frac{5}{2a+3b}\right),$$

or $\sum\limits_{\text{cyc}} \frac{1}{a} \geq \sum\limits_{\text{cyc}} \frac{5}{2b+3c}$, which follows from the Cauchy-Schwarz inequality. □

Remark. *From this example we can see that termwise bounding is a useful approach to radical inequalities, because it helps us escape the predicament of having to handle multiple root expressions at the same time.*

Problem 4 (17th XMO† Problem 14)**.** *Let $x_1, x_2, \ldots, x_{2024}$ be nonnegative real numbers such that $x_1 + x_2 + \cdots + x_{2024} = 2024$. Prove that*

$$\left(x_1^2+x_2\right)\left(x_2^2+x_3\right)\cdots\left(x_{2024}^2+x_1\right) \leq 2^{3036}.$$

Proof. The structure of the expression is easy to partition, so we use termwise bounding and try to bound each segment. To make the objective consistent with the condition (as summations), we take logarithms to get

$$\ln\left(x_1^2+x_2\right)+\ln\left(x_2^2+x_3\right)+\cdots+\ln\left(x_{2024}^2+x_1\right) \leq 3036\ln 2.$$

Equality holds when $x_i = 2i \bmod 2$ or $x_i = 2(1+i) \bmod 2$.

In the first attempt, we assume that $\ln\left(x_1^2+x_2\right) \leq Ax_1 + Bx_2 + C$ holds for variables x_1, x_2 and parameters A, B, C. Let

$$F(x_1,x_2) = Ax_1 + Bx_2 + C - \ln\left(x_1^2+x_2\right),$$

then‡

$$F(2,0) = F(0,2) = 0, \quad \left.\frac{\partial F(x_1,x_2)}{\partial x_1}\right|_{(2,0)} = \left.\frac{\partial F(x_1,x_2)}{\partial x_2}\right|_{(0,2)} = 0,$$

or

$$2A + C = 2\ln 2,\ 2B + C = \ln 2,\ A - 1 = 0,\ B - \frac{1}{2} = 0.$$

Unfortunately this equation system has no solution, so the attempt failed.

† XMO is a series of mock math Olympiads organized by Xueersi, an education enterprise in China.

‡ Be careful that $F_{x_1} = 0$ at $(x_1,x_2) = (0,2)$ is *not* necessarily true, because 0 is the boundary of the range of x_1, hence not necessarily a point of local extreme value.

In the second attempt, consider 2 logarithmic terms and introduce more parameters. Assume that

$$\ln\left(x_1^2+x_2\right)+\ln\left(x_2^2+x_3\right)\le Ax_1+Bx_2+Cx_3+D,$$

or $G(x_1,x_2,x_3)\ge 0$. Hence,

$$\begin{aligned}&G(2,0,2)=G(0,2,0)=0,\\ &\left.\tfrac{\partial G(x_1,x_2,x_3)}{\partial x_1}\right|_{(2,0,2)}=\left.\tfrac{\partial G(x_1,x_2,x_3)}{\partial x_3}\right|_{(2,0,2)}=\left.\tfrac{\partial G(x_1,x_2,x_3)}{\partial x_2}\right|_{(0,2,0)}=0,\end{aligned}$$

which yields $A=1$, $B=\frac{3}{2}$, $C=\frac{1}{2}$, $D=3\ln 2-3$.

The bounding inequality is not trivially true, so further analysis is required. The partial derivatives are

$$G_{x_1}=1-\frac{2x_1}{x_1^2+x_2},\quad G_{x_3}=\frac{1}{2}-\frac{1}{x_2^2+x_3}$$

and we will do casework as follows.

1. If $x_2\ge\sqrt{2}$, then G_{x_1}, $G_{x_3}\ge 0$, so

$$G(x_1,x_2,x_3)\ge G(0,x_2,0)=3\left(\frac{x_2}{2}-\ln\frac{x_2}{2}-1\right)\ge 0.$$

2. If $1\le x_2\le\sqrt{2}$, then $G_{x_1}\ge 0$, so

$$G(x_1,x_2,x_3)\ge G\left(0,x_2,2-x_2^2\right)\ge\frac{3}{2}\left(\sqrt{2}+\ln 2-2\right)>0,$$

because $\frac{\mathrm{d}}{\mathrm{d}x_2}G\left(0,x_2,2-x_2^2\right)=-\frac{2x_2^2-3x_2+2}{2x_2}<0$, and we take $x_2=\sqrt{2}$.

3. If $y<1$, then

$$G(x_1,x_2,x_3)\ge G\left(x_1,x_2,2-x_2^2\right)=:p(x_2).$$

- If $x_1\ge 1$, then $p'(x_2)=1-\frac{1}{x_1^2+x_2}\ge 0$, so

$$p(x_2)\ge p(0)=2\left(\frac{x}{2}-\ln\frac{x}{2}-1\right)\ge 0.$$

- If $x_1<1$, then

$$p(x_2)\ge p\left(1-x_1^2\right)=\ln 4-1+x_1-\frac{x_1^2+x_1^4}{2}>0.$$

Lastly, we apply the derived inequality cyclically to get

$$\begin{aligned}\sum_{i=1}^{2024} \ln\left(x_i^2 + x_{i+1}\right) &= \sum_{i=1}^{1012} \left[\ln\left(x_{2i-1}^2 + x_{2i}\right) + \ln\left(x_{2i}^2 + x_{2i+1}\right)\right] \\ &\le \sum_{i=1}^{1012} \left(x_{2i-1} + \frac{3}{2}x_{2i} + \frac{1}{2}x_{2i+1} + 3\ln 2 - 3\right) \\ &= \frac{3}{2}\sum_{i=1}^{2024} x_i + 3036\ln 2 - 3036 = 3036\ln 2,\end{aligned}$$

where subscripts are taken modulo 2024. The proof is complete. □

Remark. *We demonstrated an application of termwise bounding in a difficult math Olympiad problem. Surprisingly, almost no guessing is involved in the process, so the proof should turn out quite natural to anyone aware of the technique. In addition, the* 2024*-variable inequality is broken down to* 3*-variable ones, reducing the difficulty significantly.*

3 More Complicated Applications

The AM-GM inequality and the method of undetermined coefficients are two potent tools for devising termwise bounding solutions in more complicated cases. The related problems such as [3] may deviate from the typical genre of contest math or the popular concern of readers, but are available in the Extended Material for interested readers.

4 Exercise Problems

Problem 5. *Find an alternative proof of Problem 1 via rational fractional bounds.*

Problem 6 (Yunnan High School League Preliminaries 2025, Problem 13)**.** *Let* a*,* b*, and* c *be nonnegative real numbers such that* $a^2 + b^2 + c^2 = 1$*. Prove that*

$$\frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} \ge 1.$$

Problem 7. *Let* a*,* b*, and* c *be nonnegative real numbers such that* $a+b+c=3$*. Prove that*

$$\frac{1}{\sqrt{a^2+b^2+1}} + \frac{1}{\sqrt{b^2+c^2+1}} + \frac{1}{\sqrt{c^2+a^2+1}} \le \sqrt{3}.$$

The proof can be found in [4].

Problem 8. *Let a, b, and c be nonnegative real numbers such that $a+b+c>0$. Prove that*
$$\sum_{\text{cyc}} \frac{12a^2+4ab+4bc+4ca}{28a^2+b^2+c^2+14ab+14bc+14ca} \le 1.$$

5 Conclusion

Despite the "revelations" in this article, it is undeniable that instincts are a vital factor in proving inequalities. Persistent accumulation of these instincts can improve your overall perception of inequalities, so that what may seem a mystery to an average mind may gradually become natural to you.

For this purpose, I recommend further studying of inequalities related to the "termwise bounding" technique. Good problem sources include Math Stack Exchange, Nguyenhuyen_AG's blog, [1], and more contest problems to be discovered!

References

[1] Chen Ji and Ji Chaocheng, Establishing new valid inequalities, *Algebraic Inequalities*, 2009, Shanghai Scientific & Technological Education Publishing House, pp. 61–87.

[2] P. Perfetti, Mathematical Reflections S627, *AwesomeMath*.

[3] youthdoo, How to prove $\sqrt{a^2+bc}+\sqrt{b^2+ca}+\sqrt{c^2+ab}+\frac{2abc}{(a+b+c)^2} \le \frac{3}{2}(a+b+c)$, *Mathematics Stack Exchange*, retrieved on 2025-03-06.

[4] youthdoo, Prove $\frac{1}{\sqrt{a^2+b^2+1}}+\frac{1}{\sqrt{b^2+c^2+1}}+\frac{1}{\sqrt{c^2+a^2+1}} \le \sqrt{3}$ for $a+b+c=3$, *Mathematics Stack Exchange*, retrieved on 2025-02-25.

From the lecture notes of . . .

Mark Hamilton

*In this feature of **Crux**, we share some of our favourite problems from first and second year undergraduate courses. If you have a problem you would like to share (and it fits on one or so page), please send it along with its solution and a description of the course/audience it is intended for to `crux.eic@gmail.com`.*

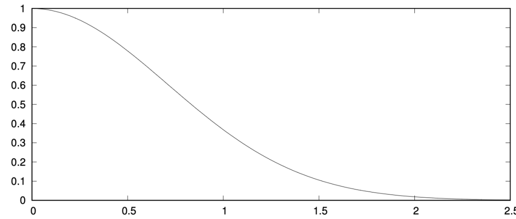
This month's column is brought to you by Mark Hamilton. Mark grew up in Vancouver and did his PhD in Toronto. His research is in symplectic geometry and geometric quantization; he is also passionate about teaching and coming up with interesting problems. In his spare time he enjoys baking and singing, occasionally at the same time. Since 2011 he has worked at Mount Allison University, where (among other activities) he teaches a lot of calculus.

. . . .

I like to give a very challenging problem as the Last Question on tests and midterms, to give the strongest students something to keep them interested (and to separate the A's from the A+'s). The following is a problem I have used a number of times in first-year Calculus II classes, and this is perhaps my favourite problem of all of them. I love the interplay between geometry and analysis, and I like how it is possible to get a fairly accurate guess for the correct answer visually — and then the correct answer turns out to be a non-obvious number that is very close to the "nice" estimate one can get by looking.

Most students usually get part (a) correct, but few of them are able to apply that knowledge to help with part (b).

Problem. A graph of the function e^{-t^2} for $0 \le t \le 2.5$ is shown below.

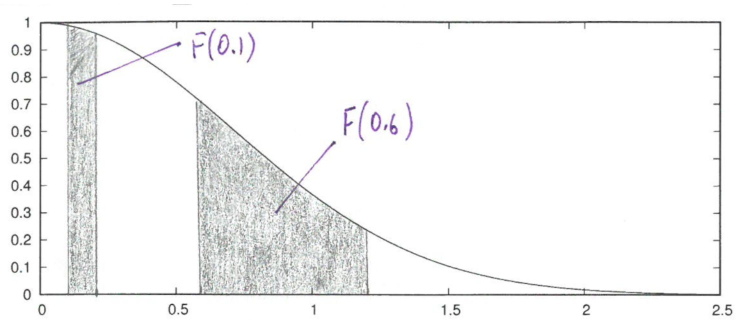


Define a function F for $x \ge 0$ by $F(x) = \int_x^{2x} e^{-t^2}\,dt$.

(a) Explain what $F(x)$ represents in terms of the graph above.

(b) Find the value of x where F is maximum. (Do not try to compute the integral explicitly; the function e^{-t^2} is not integrable in finite terms. For part marks, explain, perhaps using part (a), why F has a maximum value at some $x > 0$, and/or estimate x from the graph.)

Solution



(a) $F(x)$ represents the area under the curve between x and $2x$.

The areas corresponding to $F(0.1)$ and $F(0.6)$ are sketched on the graph.

(b) From the picture, we can see that if x is small, the strip between x and $2x$ will be narrow, so $F(x)$ will be small. (In fact, $F(0) = 0$.) On the other hand, if x is large, the graph is close to the x-axis, so the strip is not very high, so $F(x)$ will again be small. If we imagine the vertical bars at x and $2x$ sliding along as x increases, we can see that there will be some point where the area between them is a maximum. We can even estimate visually that it will probably be somewhere around $x = 0.5$.

To find the point precisely, we want to find x where $F'(x) = 0$. To find F', let

$$G(u) = \int_0^u e^{-t^2}\,dt,$$

so that $F(x) = G(2x) - G(x)$. By the Fundamental Theorem of Calculus we know that $G'(u) = e^{-u^2}$. Then

$$F'(x) = 2G'(2x) - G'(x) = 2\big(e^{-(2x)^2}\big) - e^{-x^2} = 2e^{-4x^2} - e^{-x^2} = e^{-x^2}\big(2e^{-3x^2} - 1\big).$$

This will be zero only when $2e^{-3x^2} = 1$, which solves to $x = \sqrt{\frac{1}{3}\ln 2}$.

Numerically, this is about 0.48, so our visual estimate was close!

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **July 15, 2026**.

5141. *Proposed by Kang Hyeonbin.*

Let

$$f(x) = (x+a)(x-a), \qquad g(x) = x(x+2b)(x-2b),$$

where $a, b > 0$ are real constants. Suppose there exists a constant k such that the equation

$$|f(g(x))| = k$$

has exactly seven distinct real solutions. Find $\dfrac{b^3}{a}$.

5142. *Proposed by Tatsunori Irie, modified by the Editorial Board.*

Let n be a positive integer. In 3D space, consider four planes: $z = n$, $z = n + 1$, $z = -n$ and $z = -n - 1$. Find a positive integer n and a regular tetrahedron whose four vertices lie on the four given planes such that the edge length of the tetrahedron is an integer.

5143. *Proposed by Xicheng Peng.*

Let D be an interior point of the acute triangle ABC. Construct parallelograms $DAZB$, $DBXC$, and $DCYA$. Suppose that

$$\angle XAC = \angle BAD \quad \text{and} \quad \angle YBA = \angle CBD.$$

Prove that D is the orthocenter of ΔABC.

5144. *Proposed by Michel Bataille.*

Prove or disprove:

i) For all $n \in \mathbb{N}$, $\lfloor \sqrt[4]{n} + \sqrt[4]{n+1} \rfloor = \lfloor \sqrt[4]{2^4 n + 4} \rfloor$

ii) For all $n \in \mathbb{N}$, $\lfloor \sqrt[5]{n} + \sqrt[5]{n+1} \rfloor = \lfloor \sqrt[5]{2^5 n + 5} \rfloor$.

(Note that it is known that for $k = 2$ or $k = 3$, $\lfloor \sqrt[k]{n} + \sqrt[k]{n+1} \rfloor = \lfloor \sqrt[k]{2^k n + k} \rfloor$ holds for all $n \in \mathbb{N}$.)

5145. *Proposed by Chikara Tsugawa.*

Recall that a pair of lines passing through the vertex of a given angle are called *isogonal* if they make equal angles with the bisector of the angle. For a given triangle ABC, define X_1 and X_2 to be a pair of points on BC for which the lines AX_1 and AX_2 are isogonal, and similarly for Y_1 and Y_2 on CA and for Z_1 and Z_2 on AB. Finally define

$$P = Z_2X_1 \cap X_2Y_1, \quad Q = X_2Y_1 \cap Y_2Z_1, \quad R = Y_2Z_1 \cap Z_2X_1.$$

Prove that the lines AP, BQ, CR are either concurrent or parallel.

5146. *Proposed by Vasile Cîrtoaje.*

Find the smallest positive value of k such that

$$\frac{1}{(a+b)^2+k} + \frac{1}{(b+c)^2+k} + \frac{1}{(c+a)^2+k} \leq \frac{3}{4+k}$$

for all nonnegative real numbers a, b, c satisfying $ab+bc+ca = 3$.

5147. *Proposed by Wanlong Han.*

Let $alpha \neq 0$ be a constant and let S be the set of all positive real roots of the equation

$$e^{-x^2} + \cos(\alpha x) = 0,$$

arranged in increasing order. Prove that $\sum_{n=1}^{\infty} \frac{1}{x_n^2}$ converges and find its value.

5148. *Proposed by José Luis Díaz-Barrero.*

Calculate

$$\lim_{n\to\infty} \left(-n + \sum_{k=1}^{n} e^{\frac{k}{k^2+n^2}} \right).$$

5149. *Proposed by Zon Zon and Baogian Liu.*

In $\triangle ABC$, show that

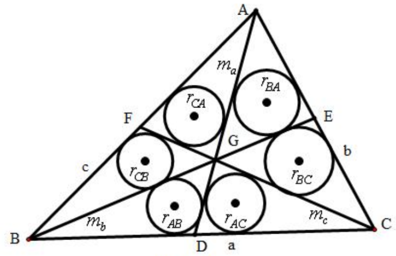
$$\sqrt{\cos B + \cos C}\cos A + \sqrt{\cos C + \cos A}\cos B + \sqrt{\cos A + \cos B}\cos C$$
$$\leqslant \frac{\cos(B-C) + \cos(C-A) + \cos(A-B)}{2}.$$

5150. *Proposed by Zhang Yun.*

Let the three medians AD, BE and CF of a triangle ABC intersect at one point G as they divide the original triangle ABC into six small triangles. Let the inradii of these six small triangles GBD, GCD, GCE, GAE, GAF, GBF be r_{AB}, r_{AC},

r_{BC}, r_{BA}, r_{CA}, r_{CB}. The circumradius of triangle ABC is R and the inradius is r. Prove that

$$\frac{12(1+\sqrt{3})}{R} \le \frac{1}{r_{AB}} + \frac{1}{r_{AC}} + \frac{1}{r_{BC}} + \frac{1}{r_{BA}} + \frac{1}{r_{CA}} + \frac{1}{r_{CB}} \le \frac{3(1+\sqrt{3})R}{r^2}.$$



. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 juillet 2026**.

5141. *Soumis par Kang Hyeonbin.*

Soient

$$f(x) = (x+a)(x-a), \qquad g(x) = x(x+2b)(x-2b),$$

où $a, b > 0$ sont des constantes réelles. Supposons qu'il existe une constante k telle que l'équation

$$|f(g(x))| = k$$

admette exactement sept solutions réelles distinctes. Déterminez $\dfrac{b^3}{a}$.

5142. *Soumis par Tatsunori Irie, modifié par le comité de rédaction.*

Soit n un entier positif. Dans l'espace tridimensionnel, considérons les quatre plans $z = n$, $z = n+1$, $z = -n$ et $z = -n-1$. Trouvez un entier positif n ainsi qu'un tétraèdre régulier dont les quatre sommets appartiennent aux quatre plans donnés, de sorte que la longueur des arêtes du tétraèdre soit un entier.

5143. *Soumis par Xicheng Peng.*

Soit D un point intérieur du triangle acutangle ABC. Construisons les parallélogrammes $DAZB$, $DBXC$ et $DCYA$. Supposons que

$$\angle XAC = \angle BAD \quad \text{and} \quad \angle YBA = \angle CBD.$$

Montrez que D est l'orthocentre du triangle ΔABC.

5144. *Soumis par Michel Bataille.*

Montrez ou réfutez les affirmations suivantes :

i) Pour tout $n \in \mathbb{N}$, $\lfloor \sqrt[4]{n} + \sqrt[4]{n+1} \rfloor = \lfloor \sqrt[4]{2^4 n + 4} \rfloor$,

ii) Pour tout $n \in \mathbb{N}$, $\lfloor \sqrt[5]{n} + \sqrt[5]{n+1} \rfloor = \lfloor \sqrt[5]{2^5 n + 5} \rfloor$.

(On sait que, pour $k = 2$ ou $k = 3$, $\lfloor \sqrt[k]{n} + \sqrt[k]{n+1} \rfloor = \lfloor \sqrt[k]{2^k n + k} \rfloor$ est vérifiée pour tout $n \in \mathbb{N}$.)

5145. *Soumis par Chikara Tsugawa.*

Rappelons qu'une paire de droites passant par le sommet d'un angle est dite *isogonale* si ces droites forment des angles égaux avec la bissectrice de cet angle. Soit un triangle ABC. Soient X_1 et X_2 deux points de BC tels que les droites AX_1 et AX_2 soient isogonales. De manière analogue, soient Y_1 et Y_2 deux points de CA, ainsi que Z_1 et Z_2 deux points de AB, satisfaisant la même propriété.

$$P = Z_2X_1 \cap X_2Y_1, \quad Q = X_2Y_1 \cap Y_2Z_1, \quad R = Y_2Z_1 \cap Z_2X_1.$$

Montrez que les droites AP, BQ et CR sont soit concourantes, soit parallèles.

5146. *Soumis par Vasile Cîrtoaje.*

Déterminez la plus petite valeur positive de k telle que

$$\frac{1}{(a+b)^2+k} + \frac{1}{(b+c)^2+k} + \frac{1}{(c+a)^2+k} \leq \frac{3}{4+k}$$

pour tous les nombres réels non négatifs a, b et c satisfaisant $ab + bc + ca = 3$.

5147. *Soumis par Wanlong Han.*

Soit $\alpha \neq 0$ et $\{x_n\}$ la suite formée de toutes les racines réelles positives de l'équation

$$e^{-x^2} + \cos(\alpha x) = 0,$$

rangées par ordre croissant. Montrez que $\sum_{n=1}^{\infty} \frac{1}{x_n^2}$ converge et déterminez-en la valeur.

5148. *Soumis par José Luis Díaz-Barrero.*

Calculez

$$\lim_{n\to\infty}\left(-n+\sum_{k=1}^{n} e^{\frac{k}{k^2+n^2}}\right).$$

5149. *Soumis par Zon Zon et Baogian Liu.*

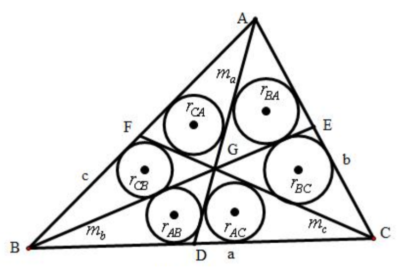
Dans $\triangle ABC$, montrez que

$$\sqrt{\cos B+\cos C}\cos A+\sqrt{\cos C+\cos A}\cos B+\sqrt{\cos A+\cos B}\cos C$$
$$\leqslant \frac{\cos(B-C)+\cos(C-A)+\cos(A-B)}{2}.$$

5150. *Soumis par Zhang Yun.*

Soient AD, BE et CF les trois médianes d'un triangle ABC, qui se coupent en un point G, divisant ainsi le triangle initial ABC en six petits triangles. Soient r_{AB}, r_{AC}, r_{BC}, r_{BA}, r_{CA} et r_{CB} les rayons des cercles inscrits respectifs des triangles GBD, GCD, GCE, GAE, GAF et GBF. Notons R le rayon du cercle circonscrit au triangle ABC, et r le rayon de son cercle inscrit. Montrez que

$$\frac{12(1+\sqrt{3})}{R}\leq \frac{1}{r_{AB}}+\frac{1}{r_{AC}}+\frac{1}{r_{BC}}+\frac{1}{r_{BA}}+\frac{1}{r_{CA}}+\frac{1}{r_{CB}}\leq \frac{3(1+\sqrt{3})R}{r^2}.$$



SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2025: 51(10), p. 472–476.

5091. *Proposed by Tatsunori Irie.*

On a circle there are 2025 places. The numbers $1, 2, \dots, 2025$ are written on them in some order (one number per place). A move consists of choosing three consecutive places with entries (x, y, z) in this cyclic order and replacing them by $(x+1, 2-y, z+1)$. In other words, we give 1 to each neighbour and then replace the middle entry by the negative of what remains. Show that, no matter how the numbers are arranged at the start, it is possible by finitely many moves to make all 2025 entries equal. Decide whether the common value is uniquely determined.

We received 2 solutions, both of which were correct and complete. We present the solution by Michal Adamaszek.

We call the middle position y the *pivot* of the move. By a slight abuse of language we will identify a position with the value in that position. Note that a move with pivot y changes the parity of x and z and preserves the parity of all other numbers. Also, repeating the move with pivot y twice:

$$(x, y, z) \to (x+1, 2-y, z+1) \to (x+2, y, z+2) \tag{1}$$

increases the numbers in x, z by 2 while preserving the values of all other numbers. We generalize this to the following observation:

Lemma. For any two different positions a and b there are move sequences which:

(A) change the parity of a and b while preserving the parity of all other numbers.

(B) increase the numbers in a and b by 2 while preserving the values of all other numbers.

Proof (A). There are an odd number of places, so in one of the directions around the circle there is an odd number of places between a and b. Let us consider that segment:

$$(a = a_0, a_1, \dots, a_{2k-1}, a_{2k} = b).$$

We perform the moves with pivots in $a_1, a_3, \dots, a_{2k-1}$. Then the parity in a and b will change once, the parity in the pivots will not change, the parity in each of $a_2, \dots, a_{2k-2}$ will change twice, hence staying unchanged, and all numbers outside the segment will also remain unchanged.

Proof (B). Consider a segment of the circle joining a and b and containing at least one place in between (in the extreme case when a and b are neighbors we take the long segment going all around the circle):

$$(a = a_0, a_1, \ldots, a_k, a_{k+1} = b),\ k \geq 1.$$

We now perform a sequence of moves with pivots $a_1, \ldots, a_k$ and then repeat the same sequence of moves again. If $k = 1$, this is the double move shown in Equation (1), so assume $k \geq 2$. The various positions in the sequence transform as follows (the label on the arrow indicates the pivot of the move affecting the position):

- $a \xrightarrow{a_1} a + 1 \xrightarrow{a_1} a + 2.$
- $a_1 \xrightarrow{a_1} 2 - a_1 \xrightarrow{a_2} 3 - a_1 \xrightarrow{a_1} a_1 - 1 \xrightarrow{a_2} a_1.$
- for $2 \leq n \leq k - 1$:

  $$a_n \xrightarrow{a_{n-1}} a_n + 1 \xrightarrow{a_n} 1 - a_n \xrightarrow{a_{n+1}} 2 - a_n \xrightarrow{a_{n-1}} 3 - a_n \xrightarrow{a_n} a_n - 1 \xrightarrow{a_{n+1}} a_n$$

- $a_k \xrightarrow{a_{k-1}} a_k + 1 \xrightarrow{a_k} 1 - a_k \xrightarrow{a_{k-1}} 2 - a_k \xrightarrow{a_k} a_k.$
- $b \xrightarrow{a_k} b + 1 \xrightarrow{a_k} b + 2.$

The numbers outside the segment are unchanged.

Now that we have shown how to operate on arbitrary pairs of positions, we can forget the cyclic ordering and just consider the 2025 numbers as a multiset.

We proceed in two stages. In the first stage we make all the numbers odd. There are 1012 even and 1013 odd numbers to begin with, so we can group the even numbers into 506 pairs and apply operation (A) sequentially to all of those pairs, turning each of them into an odd pair one by one, without affecting the parity of the other numbers.

In the second stage we will increase the numbers using operation (B) until they are all equal. Suppose the largest number is M. For any two numbers in the circle $a, b < M$, we apply operation (B) to replace them with $a + 2, b + 2$. Since a, b, M are all odd, after this operation the maximum of the multiset is still M and all numbers are still odd. We repeat this operation until no longer possible due to one of two situations: (1) all numbers are equal to M and we are done or (2) there are 2024 numbers equal to M and one number equal to $m < M$.

In case (2) we pair up m with each of the 2024 other M positions and apply operation (B) 2024 times. This way we obtain a configuration with 2024 values $M + 2$ and one value $m + 2 \cdot 2024 = m + 4048$. After sufficiently many repetitions of this process we get to the point when the "lonely" value exceeds the 2024 equal values, say we have 2024 values k and one value $K \geq k$, still all odd. We group the 2024 values k into 1012 pairs and in each of them apply operation (B) $\frac{K-k}{2}$ times, until they become equal K. Now we have all 2025 places equal to K as needed.

This common value is not unique: given a constant sequence with value c we apply operation (B) to any pair, increasing its values to $c+2$, and then repeat the process from the second stage to arrive at another constant sequence at some value $C > c$.

5092. *Proposed by Khuong Trang Tran Ngoc.*

Prove that the following inequality

$$\frac{1}{x^2+y^2+3z^2}+\frac{1}{y^2+z^2+3x^2}+\frac{1}{z^2+x^2+3y^2}\le\frac{3}{5}$$

holds for all positive real numbers $x\ge y\ge z$ such that $x^2+y^2+z^2+xyz=4$. When does equality occur?

We received 6 submissions, of which 2 were correct and complete. We present the solution by Sicheng Du.

The solution utilizes so-called *pqr-method*, see for example ***Crux*** 43(5) for the details. For convenience, the assumption $x\ge y\ge z$ is ignored in the following solution.

Let $a=x^2$, $b=y^2$, and $c=z^2$, then $a+b+c+\sqrt{abc}=4$ and we need to prove that

$$\frac{1}{3a+b+c}+\frac{1}{a+3b+c}+\frac{1}{a+b+3c}\le\frac{3}{5}. \tag{1}$$

In terms of $p=a+b+c$, $q=ab+bc+ca$, $r=abc$, (1) can be written as

$$9p^3-35p^2+(12p-20)q+24r\ge 0, \tag{2}$$

a linear inequality in q. Let p and r take fixed values such that the condition $p+\sqrt{r}=4$ is satisfied, then it suffices to prove (2) when q reaches its extreme values. By the q-lemma, this occurs when two of a, b, c are equal.

Without loss of generality, let $a=b$. Set $c=t^2$, then we get $a=b=2-t$. Now (1) reduces to

$$9t^4-42t^3+73t^2-56t+16\ge 0,$$

that is

$$(t-1)^2(3t-4)^2\ge 0,$$

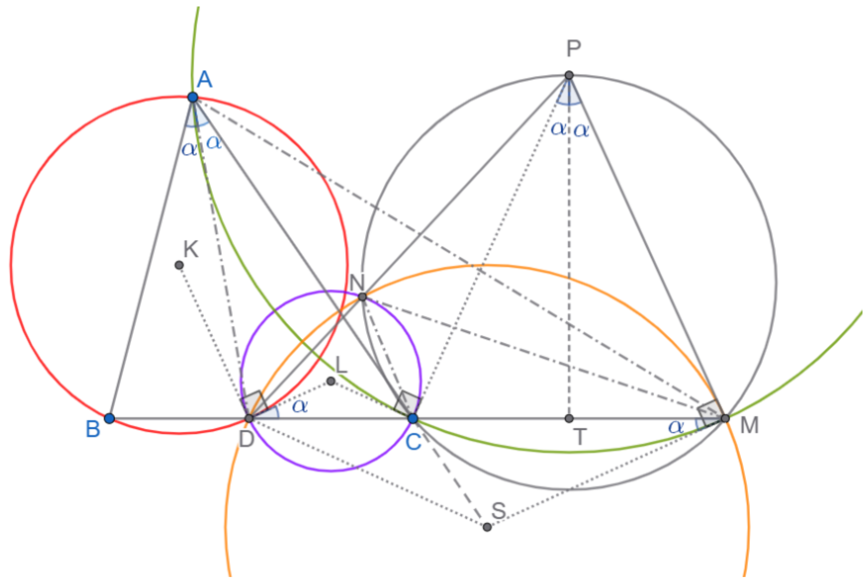
which is true.

The equality holds when $t=1$ or $\frac{4}{3}$, which corresponds to $(x,y,z)=(1,1,1)$, or $\left(\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}},\frac{4}{3}\right)$ and its permutations.

5093. *Proposed by Michel Bataille.*

Let ABC be a triangle with $\angle A\ne 2\angle C$ and let D on side BC be such that AD bisects $\angle A$. Let Γ_1 be the circumcircle of ΔADB, Γ_2 be the circle through C and D orthogonal to Γ_1 and Γ_3 be the circle through A and C orthogonal to Γ_2. Let Γ_3 intersect the line BC at $M\ne C$ and its diameter through D intersect Γ_2 at $N\ne D$. Prove that the circumcenter of ΔDNM is on the line AC.

All 6 of the submissions that we received were correct; we feature the solution by Theo Koupelis.



Let $\alpha = \angle A/2$, and let K, L, P, S be the centers of the circles $\Gamma_1, \Gamma_2, \Gamma_3$, and (DNM), respectively. We now need to show that S lies on AC. Note that when $\angle A = 2\angle C$, the points P, M, S are at infinity, and the line PND is the perpendicular bisector to AC. To avoid having to consider whether $\angle A$ is greater or less than $2\angle C$, we shall use directed angles, where $\angle XYZ$ represents the angle through which the line YX must be rotated in the positive direction about Y in order to coincide with YZ.

By construction LD is tangent to Γ_1 at D so that the angle it makes with the chord BD is

$$\alpha = \angle CDL = \angle LCD,$$

(the latter equality because $LD = LC$). Let T be the midpoint of the chord CM. By construction, $PC \perp LC$, and thus

$$\alpha = \angle LCD = \angle CPT = \angle TPM.$$

Considering the power of point P with respect to circles Γ_2 and (DNM), we get

$$PN \cdot PD = PC^2 = PM^2,$$

and thus PM is tangent to the circle (DNM). Therefore, $PM \perp SM$, and thus

$$\alpha = \angle DMS = \angle SDM.$$

Let S' be the intersection point of the line MS with AC. Then the points D, A, M, S' are concyclic because

$$\angle DAS' = \angle DAC = \angle DMS = \angle DMS' = \alpha.$$

But $\angle S'AM = \angle CAM = \angle CPM/2 = \alpha$, and thus

$$\angle S'DM = \angle S'AM = \alpha = \angle DMS'.$$

Therefore, $S \equiv S'$, and the circumcenter of $\triangle DNM$ is on the line AC.

Editor's comment. Four of the submitted solutions used coordinates, with barycentric coordinates involving somewhat easier calculations than cartesian coordinates.

5094. *Proposed by Benjamin Braiman.*

Find a dense sequence of real numbers $\{x_n\}_{n\geq 1}$ such that

$$\lim_{n\to\infty} \frac{x_1 + \cdots + x_n}{n} = 0,$$

or show no such sequence exists. (Recall that a sequence is said to be dense if every open interval (a, b) contains an element of the sequence.)

We received 7 submissions of which 6 were correct and complete. We present an edited solution proposed by Michal Adamaszek.

Let $q_1, q_2, \ldots$ be a sequence of all ordered positive rational numbers and let $k_1, k_2, \ldots$ be any sequence of integers such that $k_i \geq i \cdot q_i + 10$ for all i. Now define the sequence x by consecutively concatenating for $i = 1, 2, \ldots$ blocks of length k_i having the form

$$(\underbrace{0, \ldots, 0}_{k_i-2}, q_i, -q_i).$$

This sequence is dense as it contains all rational numbers. The sum $x_1 + \cdots + x_n$ equals 0 most of the time, except only for the cases when the last term x_n happens to be q_m at the $(k_m - 1)$-th position of the m-th block for some m, that is when $n = k_1 + \cdots + k_m - 1$. But then:

$$0 \leq \frac{x_1 + \cdots + x_n}{n} = \frac{q_m}{k_1 + \cdots + k_m - 1} \leq \frac{q_m}{k_m - 1} < \frac{1}{m}$$

and as $m \to \infty$ together with n that proves convergence of $\frac{1}{n}\sum_{i=1}^{n} x_i$ to 0.

5095. *Proposed by Nikolai Osipov.*

(a) Let x, y, z be positive integers such that $(x^2 - 1)(y^2 - 1) = z^2 - 1$. Show that x, y, z are pairwise coprime.

(b) Let $a \geqslant 2$ and $b \geqslant 1$ be integers such that b is a divisor of $a^2 - 2$. Prove that the equation

$$x^2 - (a^2 - 1)y^2 = \frac{2 - a^2}{b}$$

is solvable in integers x, y if and only if $b = 1$.

We did not receive any correct submission. We present the solution by the proposer, slightly modified by the editor.

Note that the main proof below is that of part (b) of the problems, whereas part (a) will be explicitly proved as part 1 of the upcoming Lemma.

If $b = 1$, then clearly $x = y = 1$ is a solution of the given equation.

Conversely, suppose that a solution exists. Without loss of generality, we may assume $x, y > 0$. Put $D = a^2 - 1$. Multiplying by b, we get

$$D(by^2 - 1) = bx^2 - 1. \tag{1}$$

Next, we prove the following lemma using the method of descent.

Lemma. Let $X, Y, Z > 1$ with $(X^2 - 1)(Y^2 - 1) = Z^2 - 1$. Then:

1. if X, Y, Z are positive integers, they are pairwise coprime;
2. if $n > 1$ is squarefree, there is no solution for which exactly two of X, Y, Z lie in $\sqrt{n}\,\mathbb{Z} = \{m\sqrt{n} \mid m \in \mathbb{Z}\}$ and the remaining one lies in $\mathbb{Z}$.

Proof. We prove both assertions by descent. Suppose that a counterexample exists, and choose one with $X + Y$ minimal. By symmetry, assume $1 < X \le Y$. Define

$$Y' = XY - Z, \qquad Z' = XZ - (X^2 - 1)Y.$$

Since

$$Z^2 = X^2Y^2 - X^2 - Y^2 + 2 < X^2Y^2,$$

we have $Y' > 0$. Also

$$Z^2 - (X - 1)^2Y^2 = 2(X - 1)Y^2 - X^2 + 2 \ge 2(X - 1)X^2 - X^2 + 2 > 0,$$

so $Z > (X - 1)Y$, and hence $Y' < Y$. Similarly,

$$X^2Z^2 - (X^2 - 1)^2Y^2 = (X^2 - 1)Y^2 - X^4 + 2X^2 \ge X^2 > 0,$$

so $Z' > 0$. Finally,

$$Z'^2 - (X^2 - 1)Y'^2 = Z^2 - (X^2 - 1)Y^2 = 2 - X^2,$$

and therefore

$$(X^2 - 1)(Y'^2 - 1) = Z'^2 - 1.$$

We claim that $Y' \ne 1$. Indeed, if $Y' = 1$, then $Z = XY - 1$, and substitution gives

$$(Y - X)^2 = 1.$$

In the integer case this forces $Y = X + 1$ and $Z = X^2 + X - 1$, whence X, Y, Z are pairwise coprime, contrary to being a counterexample. In the squarefree case,

$(Y-X)^2 = 1$ is incompatible with the condition that exactly two of X, Y, Z lie in $\sqrt{n}\,\mathbb{Z}$. Thus $Y' > 1$, and then also $Z' > 1$.

In the integer case, non-coprimality is preserved: if $d > 1$ divides one of the pairs among X, Y, Z, then

$$d \mid X, Y \Rightarrow d \mid X, Z', \qquad d \mid X, Z \Rightarrow d \mid X, Y', \qquad d \mid Y, Z \Rightarrow d \mid Y', Z'.$$

Hence (X, Y', Z') is a smaller integer counterexample.

In the squarefree case, the formulas for Y' and Z' plainly preserve the property that exactly two entries lie in $\sqrt{n}\,\mathbb{Z}$ and the remaining one lies in $\mathbb{Z}$. Thus (X, Y', Z') is again a smaller counterexample. In both cases this contradicts the minimality of $X + Y$. □

Now write $b = ns^2$, where n is squarefree. With $Y = sy$ and $Z = sx$, equation (1) becomes

$$(a^2-1)(nY^2-1) = nZ^2 - 1. \tag{2}$$

Equivalently,

$$(a^2-1)((Y\sqrt{n})^2-1) = (Z\sqrt{n})^2 - 1.$$

If $n > 1$, then $(X, U, V) = (a, Y\sqrt{n}, Z\sqrt{n})$ contradicts part 2 of the Lemma. Hence $n = 1$, so $b = s^2$.

Now (1) gives

$$(a^2-1)((sy)^2-1) = (sx)^2 - 1.$$

Thus $(X, Y, Z) = (a, sy, sx)$ is an integer solution of $(X^2-1)(Y^2-1) = Z^2-1$. By part 1 of the Lemma, X, Y, Z are pairwise coprime. But $s \mid sy$ and $s \mid sx$, so $s = 1$. Therefore $b = s^2 = 1$.

Hence the equation is solvable if and only if $b = 1$.

5096. *Proposed by Vasile Cîrtoaje.*

Let a, b, c, d be nonnegative real numbers such that at most one of them is larger than 1 and $ab + ac + ad + bc + bd + cd = 6$. Prove that

$$\frac{1}{(a+b+c)^2} + \frac{1}{(b+c+d)^2} + \frac{1}{(c+d+a)^2} + \frac{1}{(d+a+b)^2} \geq \frac{4}{9}.$$

We received 3 submissions. The only correct and complete solution is by the problem proposer, which we present here, slightly edited.

Proof. Without loss of generality, assume that

$$a = \max\{a, b, c, d\} \geq 1, \quad d = \min\{a, b, c, d\} \leq 1, \quad b, c \in [d, 1].$$

Write the inequality as $E \geq 0$, where

$$E = \frac{1}{(p-a)^2} + \frac{1}{(p-b)^2} + \frac{1}{(p-c)^2} + \frac{1}{(p-d)^2} - \frac{4}{9}, \quad p = a + b + c + d.$$

If $a = d$, then $a = b = c = d = 1$ and $E = 0$. Assume further that $a > d$. For c and p fixed, consider a, d, and E as functions of b. By differentiating the equality constraints, we get

$$a' + d' + 1 = 0, \quad (p-a)a' + (p-d)d' + p - b = 0.$$

Hence,

$$a'(b) = \frac{-(b-d)}{a-d} \le 0, \quad \text{and} \quad d'(b) = \frac{-(a-b)}{a-d} \le 0.$$

Since $a'(b) \le 0$ and $d'(b) \le 0$, the functions $a(b)$ and $d(b)$ are decreasing. We have

$$\frac{E'(b)}{2} = \frac{a'}{(p-a)^3} + \frac{d'}{(p-d)^3} + \frac{1}{(p-b)^3}.$$

Therefore,

$$\begin{aligned}\frac{-(a-d)E'(b)}{2} &= \frac{b-d}{(p-a)^3} + \frac{a-b}{(p-d)^3} - \frac{a-d}{(p-b)^3} \\ &= (b-d)f(a) + (a-b)f(d) - (a-d)f(b),\end{aligned}$$

where $f(x) = \dfrac{1}{(p-x)^3}$. Since $f(x)$ is convex on $[0, p)$, Jensen's inequality yields

$$(b-d)f(a) + (a-b)f(d) \ge [(b-d)+(a-b)]f\left(\frac{(b-d)a + (a-b)d}{(b-d)+(a-b)}\right) = (a-d)f(b).$$

Thus, $E'(b) \le 0$, i.e. the function $E(b)$ is decreasing. By increasing b, we can achieve either $b = 1$ or $d = 0$. Similarly, fixing b and p and increasing c, we can achieve either $c = 1$ or $d = 0$. Thus, it suffices to consider two cases: $b = c = 1$ and $d = 0$.

Case 1: $b = c = 1$. We need to show that $2(a+d) + ad = 5$ implies $F \ge 0$, where

$$F = \frac{1}{(a+2)^2} + \frac{1}{(d+2)^2} + \frac{2}{(a+d+1)^2} - \frac{4}{9}.$$

Let $x = \dfrac{a+d}{2}$. We have $ad = 5 - 4x$ and

$$\begin{aligned}\frac{1}{(a+2)^2} + \frac{1}{(d+2)^2} &= \frac{a^2 + d^2 + 4(a+d) + 8}{(ad + 2a + 2d + 4)^2} \\ &= \frac{4x^2 + 8x + 8 - 2ad}{(ad + 4x + 4)^2} \\ &= \frac{2(2x^2 + 8x - 1)}{81}.\end{aligned}$$

Therefore,

$$\begin{aligned}F = \frac{2(2x^2+8x-1)}{81} + \frac{2}{(2x+1)^2} - \frac{4}{9} &= \frac{4(4x^4 + 20x^3 - 21x^2 - 34x + 31)}{81(2x+1)^2} \\ &= \frac{4(x-1)^2(4x^2 + 28x + 31)}{81(2x+1)^2} \ge 0.\end{aligned}$$

Case 2: $d = 0$. We need to show that

$$\frac{1}{(a+b+c)^2} + \frac{1}{(a+b)^2} + \frac{1}{(a+c)^2} + \frac{1}{(b+c)^2} \geq \frac{4}{9},$$

where $a \geq 1$ and $b, c \leq 1$ satisfy $ab + bc + ca = 6$. Let $x = \dfrac{b+c}{2} \leq 1$. By Jensen's inequality, we have

$$\frac{1}{(a+b)^2} + \frac{1}{(a+c)^2} \geq \frac{2}{(a+x)^2}.$$

Therefore, it suffices to show that

$$\frac{1}{(a+2x)^2} + \frac{2}{(a+x)^2} + \frac{1}{4x^2} \geq \frac{4}{9},$$

where $a \geq 1 \geq x > 0$ satisfy $bc + 2ax = 6$. This inequality is true if

$$\frac{1}{(a+2)^2} + \frac{2}{(a+1)^2} + \frac{1}{4x^2} \geq \frac{4}{9}.$$

From

$$6 = bc + 2ax = (1-b)(1-c) + b + c - 1 + 2ax \geq 2x - 1 + 2ax$$

we get

$$a \leq \frac{7-2x}{2x}.$$

Therefore,

$$a + 2 \leq \frac{7-2x}{2x} + 2 = \frac{7+2x}{2x} \leq \frac{9}{2x}, \quad \frac{1}{(a+2)^2} \geq \frac{4x^2}{81}$$

and

$$a + 1 \leq \frac{7-2x}{2x} + 1 = \frac{7}{2x}, \quad \frac{1}{(a+1)^2} \geq \frac{4x^2}{49} > \frac{2x^2}{27}.$$

So, it suffices to show that

$$\frac{4x^2}{81} + \frac{4x^2}{27} + \frac{1}{4x^2} \geq \frac{4}{9},$$

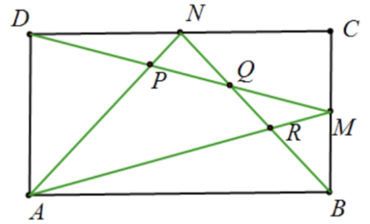
which is equivalent to

$$\frac{16x^2}{81} + \frac{1}{4x^2} \geq \frac{4}{9}, \quad (8x^2 - 9)^2 \geq 0.$$

The proof is finished. The equality occurs when $a = b = c = d = 1$.

5097. *Proposed by Xicheng Peng.*

Let $ABCD$ be a parallelogram. Let M and N be the midpoints of BC and CD, respectively. Let AM intersect BN at R, DM intersect BN at Q, and AN intersect DM at P. Prove that points A, P, Q, R are concyclic if and only if $BA \perp BC$.



We received 15 solutions, of which 14 were correct. We present 4 solutions.

Solution 1, by the proposer.

The result is an immediate consequence of this striking (unknown?) result:

$$\tfrac{10}{3}(\cos\angle BAD) = \cot\angle MAN - \cot\angle BQM.$$

For, $\angle BAD = 90^\circ \Leftrightarrow \angle MAN = \angle BQM \Leftrightarrow APQR$ is concyclic.

To establish this, place the configuration in the complex plane with A, B, C at $a = 0, b = 1, c$ respectively, so that D, M, N are at

$$a + c - b = c - 1, \quad \frac{1}{2}(b + c) = \frac{1}{2}(c + 1) \quad \text{and} \quad \frac{1}{2}(d + c) = \frac{1}{2}(2c - 1).$$

Recall that, if $z = \cos\theta + i\sin\theta$, then

$$\cot\theta = \frac{i(z + \bar{z})}{z - \bar{z}}.$$

Apply this to

$$z = \frac{a - d}{a - b} = d = c - 1, \quad z = \frac{a - n}{a - m} = \frac{2c - 1}{c + 1}, \quad z = \frac{d - m}{n - b} = \frac{c - 3}{2c - 3}$$

in turn, to find that both sides of the equation are equal to i times

$$\frac{10(c + \bar{c} - 2)}{3(c - \bar{c})}.$$

Solution 2, by Chikara Tsugawa.

Let $NA = x$ and $NB = y$. Then $ABCD$ is a rectangle if and only if the median of the triangle NAB from N is perpendicular to $AB \Leftrightarrow x = y$.

Let $AM \cap DC = T$ and $DM \cap AB = S$. Since triangles DPN and SPA are similar, then $NP : PA = ND : SA = 1 : 4$ whence $NP = \frac{1}{5}x$. Since triangles ARB and TRN are similar, we have

$$NR : RB = NT : BA = 3 : 2$$

whence $NR = \frac{3}{5}y$. Finally, since triangles NQD and BQS are similar,

$$NQ : QB = ND : BS = 1 : 2$$

whence $NQ = \frac{1}{3}y$.

$APQR$ is concyclic iff $NP \cdot NA = NQ \cdot NR$ iff $x = y$. The result follows.

Solution 3, by Michal Adamaszek.

Let $E = AC \cap BD$, $S = AM \cap BD$ and $O = AD \cap ME$, where E is the midpoint of AC. Since Q is on the medians from the vertices B and D of triangle BCD, it lies on the third median, namely $CE = CA$.

Applying Menelaus' Theorem to triangle DQC with transversal APN and to triangle BQC with transversal ARM, we find that

$$\frac{QP}{PD} \cdot \frac{DN}{NC} \cdot \frac{CA}{AQ} = 1 = \frac{QR}{RB} \cdot \frac{BM}{MC} \cdot \frac{CA}{AQ},$$

whence $PR \| BD$.Applying Ceva's Theorem to the cevians through E in triangle MDA, we get

$$\frac{MQ}{QD} \cdot \frac{DO}{OA} \cdot \frac{AS}{SM} = 1,$$

whence $MQ : QD = MS : SA$ and $QS \| AD$. We now have a chain of equivalences:

$$\begin{aligned} APQR \text{ concyclic} &\Leftrightarrow \angle QAR = \angle QPR \Leftrightarrow \angle QAS = \angle QDS \\ &\Leftrightarrow \text{trapezoid } ASQD \text{ concyclic} \\ &\Leftrightarrow QD = SA \Leftrightarrow MD = MA \\ &\Leftrightarrow MO \perp AD \Leftrightarrow AB \perp AD. \end{aligned}$$

Solution 4, by Giuseppe Fera.

We use barycentric coordinates $(x : y : z)$ relative to the triangle BCD with b, c, d the lengths of the sides CD, DB, BC respectively. Since Q is the intersection of two medians, it is the centroid of triangle BCD. The barycentric coordinates of the various points are given by

$$A(1 : -1 : 1);\ B(1 : 0 : 0);\ C(0 : 1 : 0);\ D(0 : 0 : 1);$$

$$Q(1 : 1 : 1);\ M(1 : 1 : 0);\ N(0 : 1 : 1).$$

The equations of the various lines are given by

$$AM : x - y - 2z = 0; \quad BN : y - z = 0; \quad DM : x - y = 0; \quad AN : 2x + y - z = 0.$$

The barycentric coordinates of $P = AN \cap DM$ are $(1 : 1 : 3)$ and of $R = AM \cap BN$ are $(3 : 1 : 1)$ respectively.

The general equation of a circle in the CBD plane is

$$b^2yz + c^2zx + d^2xy - (x + y + z)(fx + gy + hz) = 0.$$

This circle passes through the points A, P, R if and only if

$$(f, g, h) = \left(\frac{-3b^2 + 4c^2}{15}, \frac{12b^2 - 7c^2 + 12d^2}{15}, \frac{4c^2 - 3d^2}{15}\right).$$

The condition that Q lies on the circle is equivalent to $c^2 = b^2 + d^2$, *i.e.*, that triangle BCD is right.

Comments by the editor. Some solvers broke the solution into two halves. The symmetry of the situation allows for a straightforward angle-chasing argument that $APQR$ is concyclic when $ABCD$ is a rectangle. However, the proof in the opposite direction is much more complicated. One other solver used barycentric coordinates.

Five solvers used analytic geometry. After identifying the coordinates of all the points in question, there were essentially three approaches: (1) finding the circumcircle of triangle APQ and checking when it contained R; (2) using the slopes to examine the tangents of the angles PAR and PQR or PQN; (3) determining when $MQ \cdot MP = MR \cdot MA$, in which case $APQR$ is concyclic.

One solver used the Law of Cosines to calculate the lenths of AC, AM, AN, BD, BM and BN. Various similar triangles were exploited to determine the other lengths involved which were then plugged into the two sides of Ptolemy's equation $AP \cdot PR = AP \cdot RQ + AR \cdot PQ$. An interesting proportion that came out of this was

$$DP : PQ : QM = BR : RQ : QN = 6 : 4 : 5.$$

5098. *Proposed by Mihaela Berindeanu, modified by the Editorial Board.*

Given a cyclic quadrilateral $BCAD$ (with A and B separating C from D) such that $CB = CD$, define A_1 and B_1 to be the feet of the altitudes from A and B in triangle ABC. Prove that the orthocenter H of ΔABC is the midpoint of AA_1 if and only if DB_1 is perpendicular to DB.

All of the 12 submissions that we received were correct, and we shall sample two of them.

Solution 1 by Chikara Tsugawa.

We shall use complex coordinates with the circumcircle of the given quadrilateral as the unit circle, and C chosen to be the point where the unit circle meets the positive x-axis; we use lower case letters to denote the complex number that represents the point denoted by the corresponding capital letter. Thus we have $c = 1$ and

$|a| = |b| = |d| = 1$. Furthermore, since $CB = CD$, we have $d = \overline{b} = \frac{1}{b}$. It is known (and easily verified) that $h = a + b + c = a + b + 1$, while the feet of the altitudes from A and B are given by

$$a_1 = \frac{1}{2}\left(a + b + c - \overline{a}bc\right) = \frac{1}{2}\left(a + b + 1 - \overline{a}b\right)$$

and

$$b_1 = \frac{1}{2}\left(a + b + c - a\overline{b}c\right) = \frac{1}{2}\left(a + b + 1 - a\overline{b}\right).$$

Consequently, H is the midpoint of AA_1, if and only if

$$\begin{aligned} h = \frac{1}{2}(a + a_1) \quad &\iff \quad a + b + 1 = \frac{1}{2}\left(a + \frac{1}{2}\left(a + b + 1 - \overline{a}b\right)\right) \\ &\iff \quad a + 3b + 3 = -\overline{a}b \\ &\iff \quad a + 3b + 3 = -\frac{b}{a} \\ &\iff \quad a^2 + 3ab + 3a + b = 0 \end{aligned}$$

On the other hand, DB_1 is perpendicular to DB, if and only if

$$\begin{aligned} \mathfrak{Re}\left(\frac{b_1 - d}{b - d}\right) = 0 \quad &\iff \quad \frac{b_1 - d}{b - d} = -\overline{\left(\frac{b_1 - d}{b - d}\right)} \\ &\iff \quad \frac{b_1 - \overline{b}}{b - \overline{b}} = -\frac{\overline{b_1} - b}{\overline{b} - b} \\ &\iff \quad b_1 - \overline{b} = \overline{b_1} - b \\ &\iff \quad \frac{1}{2}\left(a + b + 1 - a\overline{b}\right) - \frac{1}{b} = \frac{1}{2}\left(\overline{a} + \overline{b} + 1 - \overline{a}b\right) - b \\ &\iff \quad (b - 1)(a^2 + 3ab + 3a + b) = 0 \\ &\iff \quad a^2 + 3ab + 3a + b = 0 \end{aligned}$$

This completes the proof.

Solution 2 by Michal Adamaszek.

Denote by B' and C' the other ends of the diameters through B and C. Note that because we assume that $CB = CD$, the arcs $CB', BC', C'D$ are equal; denote by α the measure of the angles subtended by these arcs. Because AH and $B'C$ are both perpendicular to BC, while CH and $B'A$ are both perpendicular to AB, it follows that $CHAB'$ is a parallelogram and

$$AH = B'C,$$

while $\angle HCA = \angle B'AC = \alpha$. Consequently, in the right triangle B_1HC we have

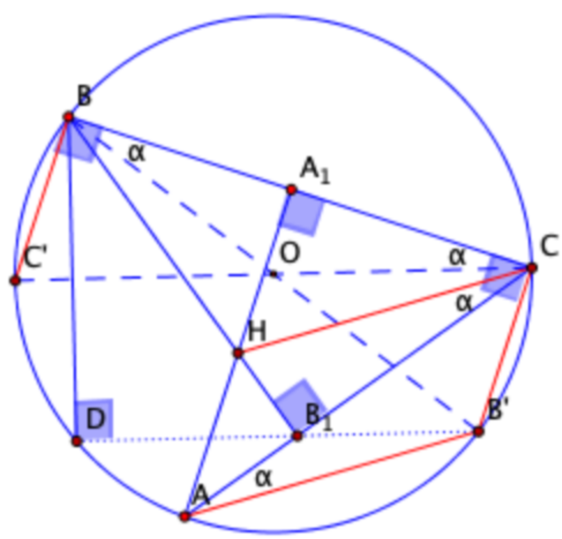
$$\angle B_1HC = 90^\circ - \alpha.$$

Finally,

$$\angle CB'D = \angle CB'C' + \angle C'B'D = 90^\circ + \alpha.$$

From the results thus far we see that

- The condition that DB_1 is perpendicular to DB is equivalent to the statement that B_1 lies on DB' (because DB' is the unique line perpendicular to DB at D).
- The condition that H is the midpoint of AA_1 is equivalent to the statement HA_1CB' is a rectangle (because of the right angles at A_1 and C, while $HA_1 = HA = B'C$.



We can now prove the desired equivalence. First suppose that B_1 lies on $B'D$. Then

$$\angle CB'B_1 + \angle B_1HC = \angle CB'D + \angle B_1HC = (90^\circ + \alpha) + (90^\circ - \alpha) = 180^\circ,$$

so the circumcircle of CHB_1 (which has diameter CH) passes through B'. It passes also through A_1, so HA_1CB' is cyclic; moreover, since it has right angles at A_1 and C it is a rectangle, as required.

Conversely, suppose that HA_1CB' is a rectangle. Its circumcircle (with diameter CH) passes through B_1 since $\angle HB_1C = 90^\circ$. Furthermore, $H \in B'C'$ because HB' (in the circle with diameter CH) and $C'B'$ (in the circle with diameter CC') are both perpendicular to CB' at B'. It follows that

$$\angle C'B'B_1 = \angle HB'B_1 = \angle HCB_1 = \angle HCA = \alpha = \angle C'B'D;$$

hence, B_1 lies on $B'D$, as required.

5099. *Proposed by Michel Bataille.*

Let s be a positive integer and let $a_0 = 1,\ a_k = \prod_{j=1}^{k} (s+j)$ for $k \geq 1$. Evaluate

$$\lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{a_k}{(2ns)^k}.$$

We received 7 solutions, all correct and complete. Each solution follows one of two approaches. Both approaches appear in the submission of M. Bello, M. Benito, Ó. Ciaurri, and E. Fernández, which we present here.

Denote by $S_n(s)$ the sum within the proposed limit and assume that $s > 1/2$. We will show that

$$\lim_{n\to\infty} S_n(s) = \left(\frac{2s}{2s-1}\right)^{s+1}.$$

Solution 1.

Using that

$$\lim_{n\to\infty} \binom{n}{k} \frac{1}{n^k} = \frac{1}{k!} \lim_{n\to\infty} 1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) = \frac{1}{k!},$$

Tannery's theorem (the dominated convergence theorem in $\ell^1(\mathbb{N})$) implies that

$$\begin{aligned}\lim_{n\to\infty} S_n(s) &= \lim_{n\to\infty} \sum_{k=0}^{\infty} \binom{n}{k} \frac{a_k}{(2ns)^k} \chi_{[0,n]}(k)\\ &= \sum_{k=0}^{\infty} \lim_{n\to\infty} \binom{n}{k} \frac{a_k}{(2ns)^k} \chi_{[0,n]}(k)\\ &= \sum_{k=0}^{\infty} \binom{k+s}{k} \frac{1}{(2s)^k}\\ &= \left(\frac{2s}{2s-1}\right)^{s+1},\end{aligned}$$

where we used that

$$\sum_{k=0}^{\infty} \binom{k+s}{k} \frac{1}{(2s)^k} = \left(1-\frac{1}{2s}\right)^{-s-1} \quad \text{for} \quad s > \frac{1}{2}.$$

Solution 2. It is clear that

$$a_k = (s+1)(s+2)\cdots(s+n) = \frac{\Gamma(s+k+1)}{\Gamma(s+1)}.$$

Then, from the well-known identity (Laplace transform)

$$\frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}} = \int_0^\infty e^{-\lambda t} t^\alpha dt, \qquad \lambda > 0, \quad \alpha > -1 \tag{1}$$

we obtain

$$\begin{aligned} S_n(s) &= \frac{(2ns)^{s+1}}{\Gamma(s+1)} \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(s+k+1)}{(2ns)^{s+k+1}} \\ &= \frac{(2ns)^{s+1}}{\Gamma(s+1)} \int_0^\infty e^{-2nst} \sum_{k=0}^{n} \binom{n}{k} t^{s+k} dt \\ &= \frac{(2ns)^{s+1}}{\Gamma(s+1)} \int_0^\infty e^{-2nst} t^s (1+t)^n dt \\ &= \frac{1}{\Gamma(s+1)} \int_0^\infty e^{-w} w^s \left(1 + \frac{w}{2ns}\right)^n dw. \end{aligned}$$

In the last step we have applied the change of variable $2nst = w$.

Since $(1+z/n)^n < e^z$ for $z > 0$, $n \geq 1$, and $\lim_{n\to\infty}(1+z/n)^n = e^z$, we can apply the dominated convergence theorem, and (1) will give

$$\lim_{n\to\infty} S_n(s) = \frac{1}{\Gamma(s+1)} \int_0^\infty e^{-w(1-1/(2s))} w^s dw = \left(\frac{2s}{2s-1}\right)^{s+1},$$

(The last integral converges if and only if $s > 1/2$.) The result holds for any real number $s > 1/2$.

Editor's Comments. It is not difficult to see that $S_n(s)$ diverges as $n \to \infty$ whenever $s \leq 1/2$.

5100. *Proposed by Huseyin Yigit Emekci.*

Let $n \geq 2$ be an integer and let $a_i, \dots, a_n$ be positive real numbers such that $a_1 + \cdots + a_n = 1$. Prove that

$$\sum_{k=2}^{n} \frac{a_k}{1-a_k}(a_1 + a_2 + \cdots + a_{k-1})^4 < \frac{1}{5}.$$

We received 5 solutions, 4 of which were correct; we present 2 solutions.

Solution 1, by Sicheng Du & Hyeonbin Kang, done independently.

Lemma. *Let $x > 0$ and $t \in (0,1)$. If $x \leq 1-t$, then*

$$\frac{t}{1-t} x^4 < \frac{(x+t)^5 - x^5}{5}.$$

Proof. Let $y := x + t$ and $u := x/y$. Note that

$$\frac{1}{\frac{t}{1-t}} = \frac{1-t}{t} = \frac{1}{t} - 1 = \frac{1}{y-x} - 1$$

so that

$$\begin{aligned}\frac{y^5 - x^5}{\frac{t}{1-t}} &= y^4 + y^3x + y^2x^2 + yx^3 + x^4 - y^5 + x^5 \\ &= y^4(1 + u + u^2 + u^3 + u^4 - y + yu^5).\end{aligned}$$

But

$$\begin{aligned}1 + u + u^2 + u^3 + u^4 - y + yu^5 &\geq u + u^2 + u^3 + u^4 + u^5 && [\text{since } u, y \leq 1] \\ &\geq 5\sqrt[5]{u^{1+2+3+4+5}} && [\text{by AM–GM}] \\ &= 5u^3 \\ &> 5u^4 && [\text{since } 0 < u < 1]\end{aligned}$$

and so

$$\frac{y^5 - x^5}{\frac{t}{1-t}} > 5u^4y^4 = 5x^4. \qquad \square$$

Now if $t = a_k$ and $x = s_{k-1} = a_1 + a_2 + \cdots + a_{k-1}$ then $x + t = s_k \leq 1$ so the Lemma yields

$$\sum_{k=2}^{n} \frac{a_k}{1 - a_k}(a_1 + a_2 + \cdots + a_{k-1})^4 \leq \sum_{k=2}^{n} \frac{s_k^5 - s_{k-1}^5}{5} = \frac{s_n^5 - s_1^5}{5} < \frac{1}{5}$$

by telescoping.

Solution 2, by Theo Koupelis.

Let

$$f(a_1, \ldots, a_n) = \sum_{k=2}^{n} \frac{a_k}{1 - a_k}(a_1 + \cdots + a_{k-1})^4$$

and $g(a_1, \ldots, a_n) = a_1 + \cdots + a_n - 1$. Since g has nonzero gradient everywhere, the method of Lagrange multipliers implies that the extreme points $\vec{a}$ of f satisfy satisfy

$$\frac{(a_1 + \cdots + a_{i-1})^4}{(1 - a_i)^2} + 4\sum_{k=i+1}^{n} \frac{a_k}{1 - a_k}(a_1 + \cdots + a_{k-1})^3 = \lambda \qquad (1 \leq i \leq n)$$

for some real number λ. Multiplying the ith equation by a_i and adding them using the condition $g(\vec{a}) = 1$ we get

$$\lambda = 4f(\vec{a}) + \sum_{i=1}^{n} \frac{a_i}{(1 - a_i)^2}(a_1 + \cdots + a_{i-1})^4$$

after interchanging the order of summation. But $1/(1-a_i)^2 > 1/(1-a_i)$ because $0 < a_i < 1$ for all i, and therefore $\lambda > 5f(\vec{a})$. On the other hand,

$$\lambda = \frac{(a_1 + \cdots + a_{n-1})^4}{(1-a_n)^2} = (1-a_n)^2 < 1$$

by the nth equation. Thus $f(\vec{a}) < \lambda/5 < 1/5$ as desired.

Editor's Comments. One solver remarked that this inequality is asymptotically sharp, in the sense that

$$\lim_{n\to\infty} \max_{\substack{g(\vec{a})=1 \\ a_i \geq 0}} f(\vec{a}) = \frac{1}{5};$$

can you prove this?

More generally, under the same hypotheses on $a_1, \ldots, a_n$ we have

$$\sum_{k=2}^{n} \frac{a_k}{1-a_k}(a_1 + \cdots + a_{k-1})^{\alpha} < \frac{1}{1+\alpha}$$

for all positive real exponents $\alpha > 0$. This can be shown using the same approach as Solution 1, replacing the AM–GM step by the inequality

$$\frac{1-u^{\alpha+1}}{1-u} - 1 + u^{\alpha+1} > (\alpha+1)u^{1+\alpha/2}$$

which itself follows e.g. from convexity of the hyperbolic sine on $(0, \infty)$.