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MathemAttic

No. 74

The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **June 15, 2026**.

MA366. *Proposed by Neculai Stanciu.*

Determine all five-digit numbers that, when divided by 4, result in their reversal.

MA367. Suppose the perimeter of a right triangle is p and the hypotenuse has length h.

a) Find the area of the triangle in terms of p and h.

b) Find the smallest and largest values of p (in terms of h) such that there actually is a right triangle with the given perimeter and hypotenuse.

MA368. A box contains an even number n of balls numbered $1, 2, 3, \ldots, n$. If three balls are randomly taken out of the box, without replacement, what is the probability that the number on one of the balls will be the average of the other two?

MA369. An arithmetic series and a geometric series have r as the common difference and the common ratio, respectively. The first term of the arithmetic series is 1 and the first term of the geometric series is 2. If the fourth term of the arithmetic series is equal to the sum of the third and fourth terms of the geometric series, find the three possible values of r.

MA370. Show that if a prime number is divided by 30, the remainder is either one or a prime number.

Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 juin 2026**.

MA366. *Soumis par Neculai Stanciu.*

Déterminez tous les nombres à cinq chiffres qui, lorsqu'on les divise par 4, donnent leur renversement.

MA367. Le périmètre d'un triangle rectangle est p et l'hypoténuse a pour longueur h.

a) Exprimez l'aire du triangle en fonction de p et de h;

b) Déterminez les plus petites et plus grandes valeurs possibles de p (en fonction de h) pour lesquelles il existe effectivement un triangle rectangle ayant ce périmètre et cette hypoténuse.

MA368. Une boîte contient un nombre pair n de boules numérotées $1, 2, 3, \ldots, n$. Si l'on tire au hasard trois boules de la boîte, sans remise, quelle est la probabilité que le numéro inscrit sur l'une des boules soit la moyenne des deux autres ?

MA369. Une suite arithmétique et une suite géométrique ont respectivement r pour différence commune et pour rapport commun. Le premier terme de la suite arithmétique est 1 et le premier terme de la suite géométrique est 2. Si le quatrième terme de la suite arithmétique est égal à la somme des troisième et quatrième termes de la suite géométrique, déterminez les trois valeurs possibles de r.

MA370. Montrez que si l'on divise un nombre premier par 30, le reste est soit égal à 1, soit un nombre premier.

MathemAttic SOLUTIONS

Statements of the problems in this section originally appear in 2025: 51(9), p. 403–404.

MA341. In a soccer game the home team won by 5 goals to 4. The home team scored first and were never behind in the game. In how many ways could the goals have been scored?

Originally Peter's Problem 2025, Problem 11.

We received 7 submissions, of which 6 were correct. We present the solution by Melih Unal.

We model the match as a sequence of 9 goals consisting of 5 home-goals (H) and 4 away-goals (A). The condition "the home team scored first and were never behind" is equivalent to requiring that in every prefix of the sequence the number of H's is at least the number of A's.

Let $p = 5$ and $q = 4$. The total number of sequences with 5 H's and 4 A's is $\binom{9}{4} = 126$. By the reflection principle (Ballot argument), the number of sequences in which some prefix has more A's than H's equals

$$\binom{p+q}{q-1} = \binom{9}{3} = 84.$$

Therefore the number of admissible sequences (never behind) is

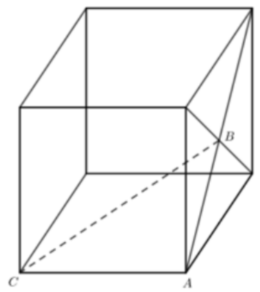
$$\binom{9}{4} - \binom{9}{3} = 126 - 84 = 42.$$

Alternatively, using the ballot formula directly:

$$\left(\frac{p-q+1}{p+1}\right)\binom{p+q}{q} == \left(\frac{5-4+1}{5+1}\right)\binom{9}{4} = \frac{2}{6} \times 126 = 42.$$

Editor's Comments. The solution correctly models the situation by representing the match as a sequence of home (H) and away (A) goals and applies the ballot (reflection) principle to count the admissible sequences. A minor shortcoming is that the reflection-principle step is presented somewhat abruptly: the argument moves directly to the count $\binom{9}{3}$ without briefly explaining the bijective reasoning behind this value. The solution then strengthens the approach by invoking the general ballot formula, which applies to arbitrary values of p and q with $p \geq q$. This generalization both confirms the numerical result and places the solution within a broader combinatorial framework.

MA342. Points A and C are vertices of a cube with side length 2 and B is the point of intersection of the diagonals of one face of the cube, as shown. Find the length $|CB|$.



We received 12 submissions, 11 of which were correct and complete. We present the most direct solution, which was made more complicated by most solvers.

First note that the diagonals of a square intersect at a right angle and since the cube's side length is 2, we get that $AB = \sqrt{2}$.

Next up, notice that the points A, B, C lie in a plane and, since AC and AB lie on adjacent faces of the cube, $\angle A = 90^\circ$. Therefore triangle ABC is a right-angle triangle and the Pythagoras' Theorem gives:

$$CB^2 = AC^2 + AB^2 = 2 + 4 = 6,$$

giving $CB = \sqrt{6}$.

MA343. If $2025 = a^b c^d$, what is the minimum value of $a + b + c + d$, where a, b, c, d are positive integers?

Originally Peter's Problem 2025, Problem 5.

We received 18 submissions, 3 of which are correct and complete. Most solvers found that the minimum value is at most 14, but did not proceed to justify that it is indeed 14. We present the solution by The Ring Lords.

We begin with the prime factorization

$$2025 = 3^4 \cdot 5^2$$

This gives one representation of the form $2025 = a^b c^d$, namely

$$(a, b, c, d) = (3, 4, 5, 2)$$

for which

$$a + b + c + d = 3 + 4 + 5 + 2 = 14$$

Hence the minimum possible value is at most 14.

Now consider any representation $2025 = a^b c^d$ with positive integers a, b, c, d. Since $a^b \mid 2025$ and $c^d \mid 2025$, both a and c must be divisors of 2025. Moreover, if

$$a + b + c + d \leq 14,$$

then necessarily $a \leq 14$ and $c \leq 14$.

The divisors of 2025 that are at most 14 are $1, 3, 5, 9$. Therefore, the only possible bases a and c that can appear in such a decomposition are among these numbers. Checking all ways to write 2025 as a product of two perfect powers with bases in $\{1, 3, 5, 9\}$, we obtain

$$2025 = 3^4 \cdot 5^2 = 5^2 \cdot 3^4 = 9^2 \cdot 5^2 = 5^2 \cdot 9^2$$

Computing $a+b+c+d$ in each case shows that the minimum value is 14, achieved for

$$(a, b, c, d) = (3, 4, 5, 2) \quad \textit{or} \quad (5, 2, 3, 4)$$

MA344. There are 32 competitors in a tournament. No two of them are equal in playing strength, and in a one against one match the better one always wins. Show that the gold, silver, and bronze medal winners can be found in 39 matches.

Originally from the 23rd Nordic Mathematical Contest, 2009, Problem 4.

We received 8 solutions, 5 of which were correct and complete. We present the solution by Shaurya Patil, modified by the editor.

To find the gold medalist, we can have all 32 competitors play a single-elimination bracket tournament (if a player loses, they're out), with the winner being the gold medalist. This takes 31 matches, since we need one match to put each player (except for the gold medalist) out of the competition.

Now, we also know that the silver medalist can only lose to the gold medalist. In the bracket tournament of the first part, the gold medalist plays five matches in total (round of 32, round of 16, quaterfinals, semifinals and final), and their opponents in these five matches are the candidates for the silver medal. We can find the right person if we make these five competitors play a single elimination tournament. The tournament is structured as follows. Assuming the gold medalist beat, in order, Player 1, Player 2, Player 3, Player 4 and Player 5, then the tournament for the silver medal starts with a match between Player 1 and Player 2, then the winner plays against Player 3, and so on until Player 5. The winner of the last match is the silver medalist, and this adds a total of 4 matches.

We also know that the bronze medalist can only lose to the gold or silver medalists. This means that the bronze medalist must be either

1. A competitor who lost to the silver medalist in the bracket tournament,
2. A competitor who lost to the gold medalist in the bracket tournament, and then to the silver medalist in the 5-player elimination tournament.

Now, suppose that the silver medalist got out in the nth round of the bracket tournament. This means that there are $n-1$ competitors in category 1. Reflecting back on the 5-player tournament, we notice that the silver medalist beat $6-n$ competitors if $n \geq 2$ or 4 competitors if $n = 1$. Thus, we know that for $n \geq 2$, the bronze medalist has to be one of $n-1+6-n = 5$ competitors, while for $n = 1$, they must be one of $0+4 = 4$ competitors. This means that we need a maximum of 4 additional matches to find the bronze medalist.

Thus, we need 31 matches to find the gold medalist, 4 more to find the silver medalist, and a maximum of 4 more to find the bronze medalist. This is a total of $31+4+4 = 39$ matches.

MA345. Find the altitude of the equilateral triangle whose area and perimeter have the same numerical value.

Originally from the Irish Mathematics Teachers' Association - Team Maths National Final, 2023, Tiebreak Round, #10.

We received 19 submissions, of which 17 were correct and complete. We present 2 solutions.

Solution 1, by Henry Ricardo.

If s denotes the side length of an equilateral triangle, then the area, perimeter and altitude of the triangle are given by the well-known formulas $A = \frac{\sqrt{3}s^2}{4}$, $P = 3s$ and $h = \frac{\sqrt{3}s}{2}$, respectively. Therefore

$$A = P \ \Rightarrow s = 4\sqrt{3} \ \Rightarrow \ h = 4\sqrt{3} \cdot \frac{\sqrt{3}}{2} = 6.$$

Solution 2, by Aaron Thiessen.

Let p be the perimeter and h be the height of the triangle. The base of the triangle is then $\frac{p}{3}$, because the triangle is equilateral. Therefore,

$$\frac{1}{2} \cdot \frac{p}{3} \cdot h = p$$

and hence $h = 6$.

TEACHING PROBLEMS

No. 31

John McLoughlin

How many triangles are there?

> In triangle ABC, the side AB has length 20 and angle ABC is equal to 90 degrees. If the lengths of the other sides must be positive integers, how many such possible triangles are there?

This problem has surprised me while working with teachers including those at the secondary school level. It seems that there is a novelty nestled in the problem statement. Perhaps it is an unusual counting problem as it neither lends itself to permutations and combinations nor a logical mental counting process. The math in the problem is straight forward enough in that (math) teachers can readily construct a suitable diagram, thus, reframing the given information into an equation that needs to be satisfied. It becomes evident though that it is not a simple case of solving for an unknown, but rather something requiring deeper insight. Here we take a closer look at the problem.

Readers are encouraged to draw a triangle while assigning the hypotenuse AC a length of y and the unknown third side BC a length of x. The assignment of variables in this manner is not required, though encouraged, for consistency with the discussion here. Applying the Pythagorean Theorem followed by some rearrangement gives us the equation $y^2 - x^2 = 400$. That is, $(y-x)(y+x) = 400$.

Pausing here, my observations suggest that many people have difficulty bringing this solution home, so to speak. This has surprised me in that few have trouble understanding "my solution" while acknowledging that the approach is new to them, or that a roadblock of sorts appeared along their own route with respect to how to proceed. The math involved in the problem is neither complicated nor difficult to access. Hence, it appears that exposure to such problems has been minimal. Keeping this in mind, let us proceed to complete the solution.

Some observations are helpful prior to listing values. Note that the hypotenuse, y, must be greater than x. So $(y-x)$ is positive, and clearly $(y+x)$ as the sum of two positive values is positive and greater than $(y-x)$. Since we require that x and y are positive integers, both $(y-x)$ and $(y+x)$ are positive integers. Considering the divisors of 400 and these observations leads to the list of possible values below:

$y-x$	$y+x$
1	400
2	200
4	100
5	80
8	50
10	40
16	25

A quick teaching point is to consider why x and y must have the same parity as in both being odd or both being even. This can be reasoned algebraically. Observe that $y^2 - x^2 = 400$, an even difference. Therefore, it follows that y^2 and x^2 must both be even (odd) as the difference between an even perfect square and an odd perfect square would always be odd, thus, making 400 impossible. Likewise, the values of y and x must therefore both be even (odd) and hence have the same parity. Alternatively, we could have considered the original fact that $x^2+20^2 = y^2$. One can see that an odd (even) value of x would imply a necessarily odd (even) value of y for any feasible pair of values (x, y). Further, it follows that $(y - x)$ and $(y + x)$ must have the same parity. This makes a pair of values like 1 and 400 implausible.

Note that $(y - x) + (y + x) = 2y$, and must be even. Hence, the pairings on the list with an odd number and an even number such as 1 and 400, 5 and 80, along with 16 and 25 will not give integer values for x and y. Hence, there are only four possible triangles meeting the conditions.

Examining the side lengths

The problem does not ask for the side lengths, but they could readily be found by solving the pairs of equations. For example, $(y - x) = 2$ and $(y + x) = 200$ gives $y = 101$ and $x = 99$. The other satisfactory pairs of values are given by $(x, y) = (48, 52), (29, 21)$, and $(25, 15)$. Recall that the given side length in the problem was 20. What do we observe if the side lengths are written in increasing order as triples?

The triples would be (20, 48, 52), (20, 21, 29), (15, 20, 25) and (20, 99, 101). The second and fourth triples on this list are examples of primitive Pythagorean triples as the three numbers, namely, 20, 21, and 29 or 20, 99, and 101 share no common factors greater than 1. In the other cases, dividing by the common factors takes us back to two more familiar primitive Pythagorean triples, namely, (5, 12, 13) and (3, 4, 5). Since the length of 20 was given as one of the legs of a right angled triangle, the satisfying triples for this problem could have 20 as a multiple of one of the legs of a primitive Pythagorean triple, as with $20 = 4 \times 5$ or $20 = 5 \times 4$, thus, giving $4 \times (5, 12, 13)$ or $5 \times (3, 4, 5)$ as workable lengths. Alternatively, 20 could be one of the lengths in a primitive Pythagorean triple as with the other two plausible cases.

Closing Comments

The problem discussed here appeared as Question 17 on the 2016 PRISM (Problem Solving for Irish Second level Mathematicians) contest paper. This contest has an unusual personal connection in that Jerome Sheehan of University of Galway reached out to me that September (prior to the contest offering in Maths Week mid-October) with a request for assistance in putting some problems together. The challenge was met as we took one week of our math education course with a focus on preparing problems for the purpose of compiling a complete 20 question contest.

This experience of solving and proposing problems while attending to language was enriching for the group of prospective secondary mathematics teachers. Anyone wishing to see the complete 2016 contest may reach out to me `johngm@unb.ca` to request a copy. The PRISM contest is no longer running and it has become increasingly difficult to locate old papers. Question 18, a more familiar style of counting problem, is shared here. Note that while some may know Letterkenny as a Canadian television show, it is a town in County Donegal, Ireland.

> How many ways can the letters of the word LETTERKENNY be arranged in a row if the R must stay in the middle position and the letters L, R, K, and Y must remain in their current order LRKY? (An example of an arrangement that meets the requirements is ELTTERENKYN.)
>
> (A) 50400 (B) 25200 (C) 10500 (D) 900 (E) 840

In closing, the discussion of problems and solutions raises surprising observations at times. For one, the core problem discussed in this feature is not one that seemed as rich when initially shared with others. Since initially sharing the problem with teachers it has consistently offered a challenge. The nuances are informative as is the solution process including the ways that others may attack the problem. Perhaps the most curious aspect of my observations relates to the fact that many can effectively set up the problem while not seeing it through to solution. That is in stark contrast to the more familiar issue with problems where it is the setup that is the stumbling block. If you have an opportunity to share this problem with a group of students at a suitable secondary level, know that feedback on your experience would be welcomed.

Competition Highlights

The Canada Lynx Mathematical Competition

by Richard Hoshino

The third annual Canada Lynx Mathematical Competition (CLMC) took place on October 2 and 3, 2025.

This new contest was created by the Canadian Mathematical Society (CMS), and is aimed at students in grades 7 to 12. CMS created CLMC to foster interest in mathematics among students regardless of their skill level, to increase student confidence in their math abilities, and to present math as a fun and playful subject.

The contest is 90 minutes long, consisting of 15 multiple-choice questions. The first five questions are worth 4 points each, the next five questions are worth 5 points each, and the final five questions are worth 7 points each. Thus, the maximum possible score is 80 points.

The 2025 CLMC attracted 3,092 participants from Canada, and a dozen countries around the world, including Bulgaria, France, Australia, New Zealand, and Thailand. The median score was 47 out of 80, a big improvement from the 39 out of 80 median from the previous year. Furthermore, there were 42 students achieving perfect scores, including 31 perfect scores from Canada.

The CLMC is a fun contest taking place at the start of the year, giving students a chance to assess their skills in preparation for other contests later in the Fall semester, most notably the Canadian Open Mathematics Challenge (COMC). The COMC, in turn, is the primary gateway to the elite invitation-only Canadian Mathematical Olympiad (CMO) contest.

Here, we present Question #15 of the 2025 CLMC.

Question #15: Let $f(x) = px^4 + 2025x^3 + qx^2 + 2025x + p$, where p and q are positive integers. If $f(x) = 0$ has exactly three solutions, all of which are rational numbers, determine the minimum possible value of p.

(a) 1 (b) 25 (c) 44 (d) 45 (e) 46

Solution.

First notice that $f(0) = 0 + 0 + 0 + 0 + p = p$. Since p is a positive integer, $f(0)$ cannot be 0 and so $x = 0$ cannot be a root of $f(x)$.

Let $g(x) = \frac{f(x)}{x^2}$. Since $x = 0$ is not a root of $f(x)$, we see that $x = r$ is a root of $f(x)$ if and only if $x = r$ is a root of $g(x)$.

Notice that

$$g(x) = \frac{f(x)}{x^2} = px^2 + \frac{p}{x^2} + 2025x + \frac{2025}{x} + q.$$

For all $r \neq 0$, we see that $g(r) = g(\frac{1}{r})$. In other words, r is a root if and only if $\frac{1}{r}$ is a root as well.

For example, if 3 and -7 are roots of $g(x)$, then that would imply that $g(x)$ has four roots: $\{3, -7, \frac{1}{3}, -\frac{1}{7}\}$. The only way $g(x)$ can have *three* roots is if one of them is a double root, i.e., $r = \frac{1}{r}$.

Since $f(x)$ has exactly three roots, this means that $g(x)$ has exactly three roots as well. This implies that one of these roots must satisfy $r^2 = 1$, i.e., $r = 1$ or $r = -1$. Let us consider both of these cases.

In our first case, we assume $r = 1$ is a root of $f(x)$. Then

$$0 = f(1) = p + 2025 + q + 2025 + p = 2p + q + 4050,$$

and so $q = -2p - 4050$. We can rewrite the polynomial $f(x)$ as

$$\begin{aligned} f(x) &= px^4 + 2025x^3 + qx^2 + 2025x + p \\ &= px^4 + 2025x^3 + (-2p - 4050)x^2 + 2025x + p \\ &= (x^2 - 2x + 1)(px^2 + (2025 + 2p)x + p). \\ &= (x - 1)^2(px^2 + (2025 + 2p)x + p). \end{aligned}$$

If $f(x)$ has exactly three *rational* roots, then the two solutions of

$$px^2 + (2025 + 2p)x + p = 0$$

must both be rational. This occurs precisely when the discriminant is a perfect square. The discriminant is equal to

$$(2025+2p)^2 - 4\cdot p\cdot p = 2025^2 + 2025\cdot 4p + 4p^2 - 4p^2 = 2025(2025+4p) = 45^2(2025+4p)$$

From above, we see that $f(x)$ has exactly three rational roots when $2025 + 4p$ is a perfect square. We know that $2025 + 4p$ is an odd integer larger than $2025 = 45^2$, and so we set $2025 + 4p = 47^2$ to find the *smallest* positive integer p in this case. We find that $2025 + 4p = 2209$ implies $p = 46$.

In our second case, we assume $r = -1$ is a root of $f(x)$. Then

$$0 = f(-1) = p - 2025 + q - 2025 + p = 2p + q - 4050,$$

and so $q = -2p + 4050$. We can rewrite the polynomial $f(x)$ as

$$\begin{aligned} f(x) &= px^4 + 2025x^3 + qx^2 + 2025x + p \\ &= px^4 + 2025x^3 + (-2p + 4050)x^2 + 2025x + p \\ &= (x^2 + 2x + 1)(px^2 + (2025 - 2p)x + p). \\ &= (x + 1)^2(px^2 + (2025 - 2p)x + p). \end{aligned}$$

If $f(x)$ has exactly three *rational* roots, then the two solutions of

$$px^2 + (2025 - 2p)x + p = 0$$

must both be rational. This occurs precisely when the discriminant is a perfect square. The discriminant is equal to

$$(2025-2p)^2-4\cdot p\cdot p = 2025^2-2025\cdot 4p+4p^2-4p^2 = 2025(2025-4p) = 45^2(2025-4p).$$

From above, we see that $f(x)$ has exactly three rational roots when $2025 - 4p$ is a perfect square. We know that $2025 - 4p$ is an odd integer less than $2025 = 45^2$, and so we set $2025 - 4p = 43^2$ to find the *smallest* positive integer p in this case. We find that $2025 - 4p = 1849$ implies $p = 44$.

Our question asks for the smallest positive integer p for which there exists a polynomial $f(x)$ with exactly three rational roots. From our analysis, we see that $p = 44$ is the correct answer, and in this case $q = -2 \cdot 44 + 4050 = 3962$.

Indeed, we can check that

$$f(x) = 44x^4 + 2025x^3 + 3962x^2 + 2025x + 44 = (x + 1)^2(x + 44)(44x + 1),$$

so this polynomial has exactly three rational roots: $x = -1, -44, -\frac{1}{44}$.

The answer is (c).

OLYMPIAD CORNER

No. 442

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **June 15, 2026**.

OC776. Given a $n \times n$ table with non-negative real entries such that the sums of entries in each column and row are equal, a player plays the following game: The step of the game consists of choosing n cells no two of which share a column or a row, and subtracting the same number from each of the entries of the n cells, provided that the resulting table has all non-negative entries. Prove that the player can change all entries to zeros.

OC777. In the complex plane, consider a square having the following property: the complex numbers to which its vertices correspond are exactly the roots of an equation with integer coefficients $x^4 + px^3 + qx^2 + rx + s = 0$. Find the minimum area of such a square.

OC778. Given a trapezoid $ABCD$ with $AD \parallel BC$, E is a moving point on the side AB. Let O_1, O_2 be the circumcenters of triangles AED and BEC, respectively. Prove that the length of O_1O_2 is a constant value.

OC779. Find all positive integers n such that $n^4 + 4n^3 + 22n^2 + 36n + 18$ is a perfect square.

OC780. Consider a convex 2025-gon. We draw a subset of its diagonals such that each drawn diagonal (excluding the two endpoints) intersects exactly one of the other drawn diagonals. What is the maximum possible number of such diagonals that can be drawn?

. . . .

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 juin 2026**.

OC776. On considère un tableau $n \times n$ à entrées réelles non négatives tel que les sommes des entrées de chaque ligne et de chaque colonne soient égales. Un joueur effectue le jeu suivant : un coup consiste à choisir n cases, dont aucune ne se trouve sur une même ligne ou une même colonne, puis à soustraire un même nombre à chacune des entrées correspondantes, à condition que le tableau obtenu conserve uniquement des entrées non négatives. Montrez que le joueur peut transformer toutes les entrées en zéros.

OC777. Dans le plan complexe, on considère des carrés possédant la propriété suivante : les nombres complexes correspondant à leurs sommets sont exactement les racines d'une équation à coefficients entiers $x^4 + px^3 + qx^2 + rx + s = 0$. Déterminez la valeur minimale de l'aire de tels carrés.

OC778. Soit $ABCD$ un trapèze tel que $AD \parallel BC$. Soit E un point mobile sur le côté AB. On note O_1 et O_2 les centres des cercles circonscrits aux triangles AED et BEC, respectivement. Montrez que la longueur de O_1O_2 est constante.

OC779. Déterminez tous les entiers positifs n tels que $n^4 + 4n^3 + 22n^2 + 36n + 18$ soit un carré parfait.

OC780. Considérons un polygone convexe à 2025 côtés. On trace un certain sous-ensemble de ses diagonales de sorte que chaque diagonale tracée (à l'exclusion de ses extrémités) coupe exactement l'une des autres diagonales tracées. Quel est le nombre maximal de telles diagonales que l'on peut tracer ?

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2026: 51(9), p. 420–421.

OC751. Let $A, B \in \mathcal{M}_n(\mathbb{R})$ be matrices. Consider the matrix function $f : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ defined by

$$f(Z) = AZ + B\overline{Z}, \qquad Z \in \mathcal{M}_n(\mathbb{C}),$$

where $\overline{Z}$ is the matrix having as elements the conjugates of the elements of Z. Prove that the following statements are equivalent:

(i) f is injective;

(ii) f is surjective;

(iii) the matrices $A + B$ and $A - B$ are invertible.

Originally from the Romanian Mathematical Olympiad 2024 - Final Round, Grade 11, Problem 3.

We received 5 solutions and we present 2 of them.

Solution 1, by Michel Bataille.

Clearly, any matrix Z in $\mathcal{M}_n(\mathbb{C})$ can be written as $Z = X + iY$ for a unique pair (X, Y) of matrices with real entries and then $\overline{Z} = X - iY$. It follows that

$$f(Z) = f(X + iY) = (A + B)X + i(A - B)Y.$$

Suppose that f is surjective. There exists $Z_0 = X_0 + iY_0 \in \mathcal{M}_n(\mathbb{C})$ such that $f(Z_0) = I_n$, the identity $n \times n$ matrix ($X_0, Y_0 \in \mathcal{M}_n(\mathbb{R})$). We deduce that

$$(A + B)X_0 + i(A - B)Y_0 = I_n + iO_n,$$

so that $(A+B)X_0 = I_n$. We conclude that the real matrix $A+B$ is invertible (with inverse X_0). Similarly, there exist $X_0', Y_0' \in \mathcal{M}_n(\mathbb{R})$ such that $f(X_0' + iY_0') = iI_n$ and we deduce that $(A - B)Y_0' = I_n$. Thus, $A - B$ is invertible (with inverse Y_0'). We have proved that (ii) implies (iii).

Suppose (iii) and let $Z = X + iY$, $Z' = X' + iY'$ with $X, X', Y, Y' \in \mathcal{M}_n(\mathbb{R})$ and $f(Z) = f(Z')$. Then

$$(A + B)(X - X') + i(A - B)(Y - Y') = O_n$$

so that

$$(A + B)(X - X') = (A - B)(Y - Y') = O_n.$$

Since $A+B$ and $A-B$ are invertible, it follows that $X-X' = O_n$ and $Y-Y' = O_n$ and therefore $Z = Z'$. Thus, (iii) implies (i).

Suppose that f is injective. Consider the functions $g, h : \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$ defined by $g(X) = (A+B)X,\ h(Y) = (A-B)Y$ so that

$$f(X+iY) = g(X) + ih(Y).$$

If $X, X' \in \mathcal{M}_n(\mathbb{R})$ and $g(X) = g(X')$, then

$$f(X + iO_n) = f(X' + iO_n)$$

and since f is injective, we must have $X = X'$. Therefore g is injective, hence g is surjective (since g is an endomorphism of the finite-dimension space $\mathcal{M}_n(\mathbb{R})$). Similarly, h is injective, hence surjective and we immediately deduce that f is surjective. Thus, (i) implies (ii) and the proof is complete.

Solution 2, by Henry Díaz Bordón.

Consider the vector space of $n \times n$ complex matrices with $\mathbb{R}$ as the field of scalars, henceforth denoted $\mathcal{M}_n(\mathbb{C})$ for simplicity. Any $M \in \mathcal{M}_n(\mathbb{C})$ can be split into its real and complex part: $M = \operatorname{Re} M + i \operatorname{Im} M$, where $\operatorname{Re} M$ and $i \operatorname{Im} M$ are linearly independent.So then it holds that $\dim \mathcal{M}_n(\mathbb{C}) = 2n^2$, as

$$\left\{ \begin{pmatrix} 1 & & \\ & & \\ & & \end{pmatrix}, \begin{pmatrix} & 1 & \\ & & \\ & & \end{pmatrix}, \dots, \begin{pmatrix} & & \\ & & \\ & & 1 \end{pmatrix}, \begin{pmatrix} i & & \\ & & \\ & & \end{pmatrix}, \begin{pmatrix} & i & \\ & & \\ & & \end{pmatrix}, \dots, \begin{pmatrix} & & \\ & & \\ & & i \end{pmatrix} \right\}$$

is a (canonical) basis of the space.

Notice that f inherits the property of being a linear map from the complex conjugate being additive and multiplicative, because from this operation follows that $\overline{X+Y} = \overline{X} + \overline{Y}$, specifically

$$\overline{X+Y}_{ij} := \overline{(X+Y)_{ij}} = \overline{X_{ij}} + \overline{Y_{ij}} = \overline{X}_{ij} + \overline{Y}_{ij}.$$

Thus

$$f(X+Y) = A(X+Y) + B(\overline{X+Y}) = (AX + B\overline{X}) + (AY + B\overline{Y}) = f(X) + f(Y);$$

an analogous reasoning applies to scalar product (the fact that $\alpha \in \mathbb{R}$ justifies the step $\overline{\alpha} = \alpha$):

$$f(\alpha Z) = A(\alpha Z) + B(\overline{\alpha Z}) = \alpha AZ + \overline{\alpha} B\overline{Z} = \alpha(AZ + B\overline{Z}) = \alpha f(Z).$$

Thus, the results of linear algebra can be applied to f, and in particular one can derive the equivalence between statements (i) and (ii) from the rank-nullity theorem:

Lemma 1. *f is injective if and only if f is surjective.*

Proof. Assume first that f is injective, then $\ker f = \{0\}$ and $\dim \ker f = 0$, which by the aforementioned result implies that $\dim \mathrm{range} f = \dim \mathcal{M}_n(\mathbb{C})$; however, since $\mathrm{range} f$ is a subspace of $\mathcal{M}_n(\mathbb{C})$, then clearly $\mathrm{range} f = \mathcal{M}_n(\mathbb{C})$ and f is surjective.

Proving the converse is equally straightforward, as the same argument can be invoked. f being surjective is equivalent to $\mathrm{range} f = \mathcal{M}_n(\mathbb{C})$, and by rank-nullity $\dim \ker f = 0$, hence f is injective and the proposition is proven. □

In order to now relate statements (ii) and (iii), it is convenient to think of the behavior of the transformation f with respect to the real and imaginary part of the input matrix; for that, recall that given $Z \in \mathcal{M}_n(\mathbb{C})$, $Z = \mathrm{Re}\, Z + i \,\mathrm{Im}\, Z$, and then

$$f(Z) = A(\mathrm{Re}\, Z + i\,\mathrm{Im}\, Z) + B(\overline{\mathrm{Re}\, Z + i\,\mathrm{Im}\, Z}) = (A+B)\,\mathrm{Re}\, Z + i(A-B)\,\mathrm{Im}\, Z.$$

Lemma 2. *f is surjective if and only if the matrices $A+B$ and $A-B$ are invertible.*

Proof. First, suppose $A+B$ and $A-B$ are invertible, then for any matrix $W \in \mathcal{M}_n(\mathbb{C})$, let $Z = (A+B)^{-1}\,\mathrm{Re}\, W + i(A-B)^{-1}\,\mathrm{Im}\, W$, thus

$$f(Z) = (A+B)(A+B)^{-1}\,\mathrm{Re}\, W + i(A-B)(A-B)^{-1}\,\mathrm{Im}\, W = \mathrm{Re}\, W + i\,\mathrm{Im}\, W = W,$$

and so f is surjective because any W has a preimage.

Now, for the converse assume f is surjective and fix a $W \in \mathcal{M}_n(\mathbb{C})$, then

$$(A+B)\,\mathrm{Re}\, Z + i(A-B)\,\mathrm{Im}\, Z = W = \mathrm{Re}\, W + i\,\mathrm{Im}\, W$$

for some Z, yet notice that the leftmost matrix is purely real while the rightmost is only imaginary in both left- and right-hand sides: $\mathcal{M}_n(\mathbb{R})$ is closed under addition and product, so $A+B, A-B \in \mathcal{M}_n(\mathbb{R})$ and $(A+B)\,\mathrm{Re}\, Z, (A-B)\,\mathrm{Im}\, Z \in \mathcal{M}_n(\mathbb{R})$. So it must be the case that

$$\begin{cases} (A+B)\,\mathrm{Re}\, Z &= \mathrm{Re}\, W, \\ (A-B)\,\mathrm{Im}\, Z &= \mathrm{Im}\, W. \end{cases}$$

Since the choice of W was arbitrary, the only way for the system above to be always consistent is for $A+B$ and $A-B$ to be invertible, given that $\mathrm{Re}\, Z, \mathrm{Re}\, W \in \mathcal{M}_n(\mathbb{R})$ and also $\mathrm{Im}\, Z, \mathrm{Im}\, W \in \mathcal{M}_n(\mathbb{R})$. Overall, this shows the equivalence of the two statements. □

Finally, joining Lemma 1 and 2,

$$f \text{ injective} \iff f \text{ surjective} \iff A+B, A-B \text{ invertible.}$$

OC752. Let $f : [0,1] \to \mathbb{R}$ be a continuous function with $f(1) = 0$. Prove the existence and determine the value of the limit

$$\lim_{\substack{t\to 1\\ t<1}} \left(\frac{1}{1-t} \int_0^1 x\left(f(tx) - f(x)\right)\, dx \right).$$

Originally from the Romanian Mathematical Olympiad 2024 - Final Round, Grade 12, Problem 3.

We received 7 solutions. We present 2 solutions.

Solution 1, by Kee-Wai Lau.

We show that the required limit is $2L$, where $L = \displaystyle\int_0^1 xf(x)\,dx$. For $0 < t < 1$, we substitute $y = tx$ to obtain

$$\int_0^1 xf(tx)\,dx = \frac{1}{t^2}\int_0^t yf(y)\,dy,$$

so that

$$\frac{1}{1-t}\int_0^1 x\big(f(tx) - f(x)\big)\,dx = \frac{\displaystyle\int_0^t yf(y)\,dy - t^2L}{(1-t)t^2}.$$

Hence, by l'Hôpital's rule, we have

$$\begin{aligned}
\lim_{\substack{t\to 1\\ t<1}} \left(\frac{1}{1-t}\int_0^1 x\big(f(tx) - f(x)\big)\,dx \right) &= \lim_{\substack{t\to 1\\ t<1}} \left(\frac{1}{1-t}\left(\int_0^t yf(y)\,dy - t^2L \right)\right) \\
&= \lim_{\substack{t\to 1\\ t<1}} \big(-(tf(t) - 2tL)\big) \\
&= 2L - f(1) = 2L,
\end{aligned}$$

as claimed.

Solution 2, by Michel Bataille.

Let $F(x) = \int_0^x f(u)\,du$. The function F is differentiable on $[0,1]$ with $F'(x) = f(x)$. First, we prove the following lemma:

$$\lim_{t\to 1^-} \frac{1}{1-t}\int_0^1 [F(x) - F(tx)]\,dx = \int_0^1 xf(x)\,dx. \tag{1}$$

Proof. Let $t \in (0,1)$ and $x \in [0,1]$. From the Mean Value Theorem, we have

$$F(x) - F(tx) = (1-t)xF'(\theta) = (1-t)xf(\theta)$$

for some $\theta \in (tx, x)$. Let $\varepsilon > 0$. The continuous function f is uniformly continuous on $[0,1]$, hence there exists $\alpha > 0$ such that $|f(u) - f(v)| \le 2\varepsilon$ whenever $|u-v| \le \alpha$ for $u, v \in [0,1]$. Suppose that $0 < 1-t < \alpha$. Then

$$|x - \theta| < x - tx = (1-t)x \le 1-t \le \alpha,$$

hence $|f(x) - f(\theta)| \leq 2\varepsilon$ and we deduce that

$$\begin{aligned}\left|\frac{1}{1-t}\int_0^1 [F(x) - F(tx)]\,dx - \int_0^1 xf(x)\,dx\right| &= \left|\int_0^1 x(f(\theta) - f(x))\,dx\right| \\ &\leq \int_0^1 x\,|f(\theta) - f(x)|\,dx \\ &\leq 2\varepsilon \int_0^1 x\,dx = \varepsilon.\end{aligned}$$

Thus,

$$\left|\frac{1}{1-t}\int_0^1 [F(x) - F(tx)]\,dx - \int_0^1 xf(x)\,dx\right| \leq \varepsilon$$

whenever $0 < 1 - t < \alpha$, which means (1). Now, let $\phi(t) = \int_0^1 x(f(tx) - f(x))\,dx$ ($t \in (0,1)$). Integrating by parts leads to

$$\begin{aligned}\phi(t) &= \left[x\left(\frac{F(tx)}{t} - F(x)\right)\right]_{x=0}^{x=1} - \frac{1}{t}\int_0^1 (F(tx) - tF(x))\,dx \\ &= \frac{F(t) - tF(1)}{t} + \frac{1}{t}\int_0^1 (F(x) - F(tx))\,dx - \frac{1-t}{t}\int_0^1 F(x)\,dx.\end{aligned}$$

We have

- $\frac{F(t)-tF(1)}{1-t} = \frac{F(t)-F(1)}{1-t} + F(1)$, hence

$$\lim_{t\to 1^-} \frac{F(t) - tF(1)}{t(1-t)} = -F'(1) + F(1) = -f(1) + F(1) = F(1).$$

- $\lim\limits_{t\to 1^-} \frac{1}{t(1-t)}\int_0^1 (F(x) - F(tx))\,dx = \int_0^1 xf(x)\,dx$ (from the lemma) and
- $\lim\limits_{t\to 1^-} \frac{1}{t}\int_0^1 F(x)\,dx = [xF(x)]_0^1 - \int_0^1 xf(x)\,dx = F(1) - \int_0^1 xf(x)\,dx.$

It readily follows that

$$\lim_{t\to 1^-}\left(\frac{1}{1-t}\int_0^1 x(f(tx) - f(x))\,dx\right) = \lim_{t\to 1^-}\frac{\phi(t)}{1-t} = 2\int_0^1 xf(x)\,dx.$$

OC753. Determine all pairs of prime numbers (p, q) with the following property: there exist positive integers a, b, c satisfying the equality

$$\frac{p}{a} + \frac{p}{b} + \frac{p}{c} = 1, \qquad \frac{a}{p} + \frac{b}{p} + \frac{c}{p} = q + 1.$$

Originally from the Polish Mathematical Olympiad 2024 - Final Round, Problem 3.

We received 7 solutions. We present the solution by Theo Koupelis.

The given equations are equivalent to $p(ab + bc + ca) = abc$ and $a + b + c = p(q + 1)$. Thus, p divides at least one of the positive integers a, b, c. Without loss of generality, let $c = px$, where x is a positive integer. Then the given equations become

$$ab(x - 1) = px(a + b), \qquad (1)$$

and

$$a + b = p(q + 1 - x). \qquad (2)$$

(i) If $p \mid a$ then from (2) we get that $p \mid b$. Let $a = py, b = pz$, where y, z are positive integers. Then from (1) we get

$$yz(x - 1) = x(y + z),$$

and from (2) we get

$$y + z = q + 1 - x.$$

But $(x, x - 1) = 1$, and thus $(x - 1) \mid (y + z)$. Setting

$$y + z = (x - 1)w,$$

where w is a positive integer, we get $yz = xw$ and $(x - 1)(w + 1) = q$. But q is a prime and thus $x = 2$ and therefore $y = 2 + 4/(z - 2)$. The only solutions on the positive integers are $(y, z) = (6, 3), (4, 4), (3, 6)$, for which we get $q = 10, 9, 10$, respectively, none of which is a prime.

(ii) Let $p \nmid a$ and $p \nmid b$. From (2) we get $p \mid (a + b)$. Setting $a + b = pt$, where t is a positive integer, from (1) and (2) we get

$$ab(x - 1) = p^2 xt$$

and

$$t = q + 1 - x,$$

and thus

$$ab(q - t) = p^2 xt = p^2 t(q + 1 - t).$$

Clearly $t \neq q$; if $t = 1$ then $ab(q - 1) = q(a + b)^2 \geq 4abq$, which is not possible over the positive integers. Thus, $t \nmid (q - t)$ and $t \mid (ab)$. Setting $ab = ts$, we get

$$s(q - t) = p^2(q + 1 - t) \quad \text{or} \quad (q - t)(s - p^2) = p^2.$$

But $p \nmid (ab)$ and thus $p \nmid s$. Therefore, $s - p^2 = 1$ and $q - t = p^2$. From the above we get

$$c = p(p^2 + 1), \qquad a + b = p(q - p^2), \qquad \text{and} \qquad ab = (q - p^2)(p^2 + 1).$$

Thus, a and b are roots of the quadratic equation $t^2 - p(q - p^2)t + (q - p^2)(p^2 + 1) = 0$. The discriminant of this quadratic must be a square. That is, we must have

$$\left[p(q - p^2) - 2p\right]^2 - 4q = \ell^2,$$

where ℓ is a positive integer, or

$$\left[p(q-p^2)-2p-\ell\right]\left[p(q-p^2)-2p+\ell\right]=4q.$$

But q is a prime, with $q>p^2$, and therefore, the possible solutions for the quantities in brackets are $(\pm 1, \pm 4q)$, or $(\pm 2, \pm 2q)$, or $(\pm 4, \pm q)$. But the difference between the two quantities is equal to 2ℓ, an even positive integer. Thus,

$$p(q-p^2)-2p-\ell=\pm 2 \quad \text{and} \quad p(q-p^2)-2p+\ell=\pm 2q.$$

Adding the quantities, we get

$$p(q-p^2)-2p=\pm(q+1).$$

Solving for q we get

$$q=p^2+p+3+4/(p-1) \quad \text{or} \quad q=p^2-p+3-4/(p+1).$$

For the former, p can take only the values $2, 3, 5$, with corresponding acceptable solutions $(p,q)=(2,13)$ and $(3,17)$; for the latter, p can only take the value 3 but then $q=8$, which is not a prime.

Therefore, the only solutions are $(p,q)=(2,13)$ and $(3,17)$, for which we get $(a,b,c)=(15,3,10)$ and $(a,b,c)=(20,4,30)$, respectively, and cyclic permutations.

OC754. Solve in real numbers the equation

$$3^{\log_5(5x-10)}-2=5^{-1+\log_3 x}.$$

Originally from the Romanian Mathematical Olympiad 2024 - Final Round, Grade 10, Problem 1.

We received 9 submissions of which 7 were correct and complete. We present the solution by Brian D. Beasley.

We show that $x=3$ or $x=27$.

Since $5x-10>0$, we must have $x>2$. This allows us to write $x=3^m$ for some real number $m>\log_3 2$. Similarly, we express $5x-10=5^n$ for some positive real number n. Then we obtain

$$3^n-2=5^{m-1} \quad \text{and} \quad 5^n=5\cdot 3^m-10,$$

so $5^m=5\cdot 3^n-10$ as well. Hence $5^m-5^n=5\cdot 3^n-5\cdot 3^m$, or equivalently

$$5^m+5\cdot 3^m=5^n+5\cdot 3^n.$$

Since the function $f(t)=5^t+5\cdot 3^t$ is increasing on $\mathbb{R}$, we conclude that $m=n$. To solve the equation $5^n=5\cdot 3^n-10$, we let $g(n)=5^n-5\cdot 3^n+10$. Then $g'(n)=5^n\ln 5-3^n(5\ln 3)$, so there is a unique real number r with $g'(r)=0$, namely

$$r = \frac{\ln(5\ln 3/\ln 5)}{\ln(5/3)} = \frac{\ln 5 + \ln(\ln 3) - \ln(\ln 5)}{\ln 5 - \ln 3} \approx 2.40317.$$

Thus g is decreasing on $(-\infty, r)$ and increasing on (r, ∞), so that $g(n) = 0$ has at most two real solutions. Since $g(1) = g(3) = 0$, we have $n = 1$ or $n = 3$, which completes the proof.

OC755. Consider the inscribable pentagon $ABCDE$ in which $AB = BC = CD$ and the centroid of the pentagon coincides with the center of the circumscribed circle. Show that the pentagon $ABCDE$ is regular.

Originally from the Romanian Mathematical Olympiad 2024 - Final Round, Grade 10, Problem 2.

We received 10 solutions. We present 2 of them.

Solution 1, by Theo Koupelis.

Let the center O of the circle be the origin of the Cartesian coordinate system. If R is the radius of the circle, let z_A, z_B, z_C, z_D, and z_E be the complex numbers denoting the coordinates of the vertices of the pentagon. Since $AB = BC = CD$, without loss of generality we set

$$z_A = Re^{i0} = R,\ z_B = Re^{i\theta},\ z_C = Re^{i2\theta},\ z_D = Re^{i3\theta},\ z_E = Re^{i\phi},$$

where $0 < 3\theta < \phi < 2\pi$. With the centroid of the pentagon being at O we get $z_O = (z_A + z_B + z_C + z_D + z_E)/5 = 0$. Thus,

$$e^{i\phi} = -1 - e^{i\theta} - e^{i2\theta} - e^{i3\theta} \Longrightarrow \begin{cases} \cos\phi & = -\left[1 + \cos\theta + \cos(2\theta) + \cos(3\theta)\right] \\ \sin\phi & = -\left[\sin\theta + \sin(2\theta) + \sin(3\theta)\right]. \end{cases}$$

Therefore,

$$\begin{aligned} \cos\phi &= -\left[2\cos^2\theta + 2\cos(2\theta)\cos\theta\right] = -4\cos\theta\cos\frac{3\theta}{2}\cos\frac{\theta}{2}, \\ \sin\phi &= -\left[2\sin\frac{3\theta}{2}\cos\frac{3\theta}{2} + 2\sin\frac{3\theta}{2}\cos\frac{\theta}{2}\right] = -4\sin\frac{3\theta}{2}\cos\theta\cos\frac{\theta}{2}, \quad (*) \end{aligned}$$

and thus $\tan\phi = \tan\frac{3\theta}{2}$. But $3\theta < \phi < 2\pi$ and thus $\phi = \pi + \frac{3\theta}{2}$. Substituting into $(*)$ and eliminating the term $\sin\frac{3\theta}{2}$ (because $0 < 3\theta < 2\pi$), we get

$$\cos\theta\cos\frac{\theta}{2} = \frac{1}{4} \Longleftrightarrow \left(2\cos\frac{\theta}{2} + 1\right)\left(4\cos^2\frac{\theta}{2} - 2\cos\frac{\theta}{2} - 1\right) = 0.$$

But $0^\circ < \theta < 120^\circ$ and thus $\cos\frac{\theta}{2} > 0$, and therefore

$$\cos\frac{\theta}{2} = \frac{1+\sqrt{5}}{4} \Longrightarrow \theta = 72^\circ \Longrightarrow \phi = 288^\circ = 4\theta = 360^\circ - \theta.$$

Thus, the pentagon $ABCDE$ is regular.

Solution 2, by Michel Bataille.

We embed the configuration in the complex plane and denote by a, b, c, d, e the respective affixes of the (distinct) points A, B, C, D, E. Without loss of generality, we suppose that these vertices are on the circle centered at the origin O and with radius 1, so that $|a| = |b| = |c| = |d| = |e| = 1$. From the hypotheses we have $a+b+c+d+e = 0$ and $|b-a| = |c-b| = |d-c|$. Since the pentagon is inscribable, the problem reduces to showing that $DE = EA = AB$.

If $|u| = |v| = 1$, then $|u-v|^2 = (u-v)\left(\frac{1}{u} - \frac{1}{v}\right) = 2 - \frac{u^2+v^2}{uv}$; it follows that

$$\frac{a^2+b^2}{ab} = \frac{b^2+c^2}{bc} = \frac{c^2+d^2}{cd} = \lambda, \text{ say.}$$

We readily deduce that $b^2 = ac,\ c^2 = bd$ and $\lambda = \frac{a+c}{b} = \frac{b+d}{c}$. Note that

$$\overline{\lambda} = \frac{a+c}{ac} \cdot b = \frac{a+c}{b} \cdot \frac{b^2}{ac} = \lambda,$$

hence λ is a real number. Also, we have

$$a+d = \frac{b^2}{c} + \frac{c^2}{b} = \frac{(b+c)(b^2+c^2-bc)}{bc} = \frac{(b+c)(\lambda bc - bc)}{bc} = (\lambda-1)(b+c).$$

If $\lambda = 1$, then $a + d = 0$, hence $e = -(b+c)$, in contradiction with $|e| = 1$ (since $|b+c|^2 = 2 + \lambda = 3$). Thus $\lambda \neq 1$ and we obtain $e = \frac{\lambda}{1-\lambda}(a+d)$. Since $\lambda \in \mathbb{R}$, this means that $\overrightarrow{OE}$ is collinear with $\overrightarrow{OM}$ where M is the midpoint of AD. As a result, E is on the perpendicular bisector of AD and $DE = EA$.

From

$$1 = |e|^2 = |\lambda(b+c)|^2 = \lambda^2(b+c)\left(\frac{1}{b} + \frac{1}{c}\right) = \lambda^2(2+\lambda),$$

we obtain

$$(\lambda+1)(\lambda^2+\lambda-1) = 0,$$

hence $\lambda^2 + \lambda - 1 = 0$. Note that $\lambda = -1$ implies

$$a^2 + ab + b^2 = b^2 + bc + c^2 = c^2 + cd + d^2 = 0,$$

hence $a^3 = b^3 = c^3 = d^3$, contradicting the fact that a complex number has at most three cubic roots.

We complete the proof by showing that OA is perpendicular to BE (then A is on the perpendicular bisector of BE and $EA = AB$). We want to prove that

$$\frac{e-b}{a} + \frac{\overline{e} - \overline{b}}{\overline{a}} = 0.$$

A short calculation shows that it suffices to prove that $a^2 = eb$. We are done since

$$a^2 - eb = a^2 + b \cdot \lambda(b+c) = a^2 + (a+c)(b+c) = ac\left(\frac{a^2+c^2}{ac} + \frac{b^2}{ac} \cdot \frac{a+c}{b} + 1\right)$$

$$= ac(\lambda^2 - 2 + \lambda + 1) = 0.$$

Geometric, Arithmetic and Root-Mean Squares

Ed Barbeau

Every once in a while, the editor of *Crux* receives a problem proposal subject to a hypothesis which may not be satisfiable. One recent such required that the geometric and arithmetic means, along with the root-mean-square, of two positive integers themselves be integers.

After a fruitless search for an example, it seemed possible that no such pair existed. Suppose that a and b are the positive integers and that g, u and v are respectively the geometric mean, arithmetic mean and root-mean square of the two numbers:

$$g = \sqrt{ab}; \qquad u = \tfrac{1}{2}(a+b); \qquad w = \sqrt{\tfrac{1}{2}(a^2+b^2)}.$$

Exercise 1. Prove that, if there exists a pair of positive integers a and b whose three means are integers, then there exists a coprime pair. (This will be assumed from now on.)

Exercise 2. Show that a and b are both odd and that there exists a positive integer v for which $(a, b) = (u + v, u - v)$. Verify that $g^2 = u^2 - v^2$ and $w^2 = u^2 + v^2$.

Exercise 3. Prove that u is the root-mean-square of g and w.

Exercise 2 tells us that g^2, u^2 and w^2 are three squares in arithmetic progression. The determination of such a trio of squares is an ancient problem that goes back to Fibonacci; the common difference $u^2 - g^2 = w^2 - u^2$ is called the *congruum*, and there are lots of triples such as $(1, 25, 49)$ and $(49, 169, 289)$ that have nonsquare congrua.

As we did with a and b, we set $g = r - s$ and $w = r + s$, where r and s are positive integers.

Exercise 4. Verify that g, u and w are pairwise coprime, as are r and s. Verify that $u^2 = r^2 + s^2$ and $v^2 = 2rs$.

Since (r, s, u) is a primitive Pythagorean triple, there exists a coprime pair of integers m, n of opposite parity for which

$$\{r, s, u\} = \{2mn, m^2 - n^2, m^2 + n^2\}.$$

Exercise 5. Prove that $v^2 = 4mn(m + n)(m - n)$ where each of the numbers m, n, $m + n$, $m - n$ are square.

It is at this point that we let Fermat take over and make use of *proof by infinite descent* that he pioneered. The idea is that, if there is an example of three squares in arithmetic progression with a square congruum, then there must be one for which the positive integers involved are minimum. Then, we can obtain a contradiction, if for whatever example we have, we can construct one with even smaller numbers.

We make a key observation. The expression for the area of a right triangle can always be put into the form $mn(m+n)(m-n)$, so that there are positive integers m, n for which $mn(m+n)(m-n)$ is square if and only if there is a right triangle with integer sides with a square area.

Suppose that m and n are squares, and that $m+n = p^2$ and $m - n = q^2$ for some integers p and q.

Exercise 6. Prove that p and q are odd and that one of $p+q$ and $p-q$ is divisible by 4. What is the parity of n?

Let $h = \frac{1}{2}(p+q)$ and $k = \frac{1}{2}(p-q)$.

Exercise 7. Show that h and k have opposite parity and that they are the smaller two numbers of a Pythagorean triple. Prove that the area of the right triangle for this triple is $n/4$.

Since $\frac{1}{4}n$ is less that $mn(m+n)(m-n)$ and can be put in the same form, the desired result follows by infinite descent. Thus, g, u and w cannot all be integers.

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **June 15, 2026**.

5131. *Proposed by Tigran Hakobyan.*

Describe all the triplets (a, b, c) of positive integers greater than 1 having the following property: for any positive integers k and m there exists a positive integer l such that

$$\gcd(a^k - 1, b^m - 1) = c^l - 1$$

5132. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Calculate

$$\lim_{n\to\infty} \sqrt[n]{\int_0^1 \left(x + \frac{x^2}{2^2} + \cdots + \frac{x^n}{n^2}\right)^n \mathrm{d}x}.$$

5133. *Proposed by Ziji Hu.*

Let S be a set of points on a plane with the following properties:

(i) S has at least 3 points;

(ii) No three points in S are collinear;

(iii) If $A, B,$ and C are three distinct points in S, then the circumcenter of $\triangle ABC$ is in S.

Find all such finite sets S or prove that such a set does not exist.

5134. *Proposed by Vasile Cîrtoaje.*

For $n \geq 3$, let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 \geq a_2 \geq \cdots \geq a_n$ and $a_1a_2 + a_2a_3 + \cdots + a_na_1 = n$. Prove that

$$\frac{1}{a_1+2} + \frac{1}{a_2+2} + \cdots + \frac{1}{a_n+2} \geq \frac{n}{3}.$$

5135. *Proposed by Tatsunori Irie.*

A transparent cubic box with a side length of 20 contains 1050 fireflies, which we consider as geometric points. Prove that if one covers a part of the interior of this box with a hemispherical dome of radius 2, there exists a position for the dome such that it contains at least 3 fireflies. (Note: Fireflies located on the boundary of the dome are considered to be inside.)

5136. *Proposed by Marius Stănean.*

Let $a \geq b \geq c \geq 0$ be real numbers such that $a+b+c = ab+bc+ca > 0$. Prove that
$$a^2 + b^2 + c^2 + 5abc \geq 8 + (a-b)^2.$$

5137. *Proposed by Pelegrí Viader.*

Given $k \in \mathbb{N}$, find the solution of the difference equation
$$x_{n+k} + nx_{n+1} - nx_n = 0 \quad \text{for } n = 1, 2, \ldots, \tag{1}$$
such that $x_n \to 0$ and $nx_n \to 1$.

5138. *Proposed by Nazar Kirgizbaev.*

Let x, y, z be positive real numbers such that $x^2 + y^2 + z^2 = 3$. Prove that
$$\sum_{cyc} \left(\frac{2xy}{xy+z}\right)^2 \geq 1 + \frac{2}{3}(xy + yz + zx.)$$

5139. *Proposed by Giuseppe Fera, modified by the Editorial Board.*

Find infinitely many cubes in three-dimensional space so that

(i) all vertices have integer coordinates,

(ii) no edge is parallel to a coordinate axis, and

(iii) no cube is an integer multiple of another.

5140. *Proposed by Phan Ngoc Chau.*

Prove that the following inequality
$$\frac{1}{\left(\sqrt{a}+\sqrt{bc}\right)^2} + \frac{1}{\left(\sqrt{b}+\sqrt{ca}\right)^2} + \frac{1}{\left(\sqrt{c}+\sqrt{ab}\right)^2} \geq \frac{2}{abc+1}$$
holds for all positive real numbers a, b, c with $ab+bc+ca = 1$. When does equality occur?

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 juin 2026**.

5131. *Soumis par Tigran Hakobyan.*

Décrivez tous les triplets (a, b, c) d'entiers positifs strictement supérieurs à 1 ayant la propriété suivante : pour tous entiers positifs k et m, il existe un entier positif l tel que

$$\mathrm{PGCD}(a^k - 1, b^m - 1) = c^l - 1$$

5132. *Soumis par Ovidiu Furdui et Alina Sîntămărian.*

Calculez

$$\lim_{n\to\infty} \sqrt[n]{\int_0^1 \left(x + \frac{x^2}{2^2} + \cdots + \frac{x^n}{n^2}\right)^n \mathrm{d}x}.$$

5133. *Soumis par Ziji Hu.*

Soit S un ensemble de points du plan ayant les propriétés suivantes :

(i) S contient au moins 3 points ;

(ii) aucun triplet de points de S n'est colinéaire ;

(iii) si A, B et C sont trois points distincts de S, alors le centre du cercle circonscrit au triangle $\triangle ABC$ appartient à S.

Déterminez tous tels ensembles finis S, ou prouver qu'un tel ensemble n'existe pas.

5134. *Soumis par Vasile Cîrtoaje.*

Soit $n \geq 3$. Soient $a_1, a_2, \ldots, a_n$ des nombres réels non négatifs tels que $a_1 \geq a_2 \geq \cdots \geq a_n$ et $a_1a_2 + a_2a_3 + \cdots + a_na_1 = n$. Montrez que

$$\frac{1}{a_1+2} + \frac{1}{a_2+2} + \cdots + \frac{1}{a_n+2} \geq \frac{n}{3}.$$

5135. *Soumis par Tatsunori Irie.*

Une boîte cubique transparente d'arête 20 contient 1050 lucioles, que l'on considère comme des points géométriques. Montrez que, si l'on recouvre une partie de l'intérieur de cette boîte avec un dôme hémisphérique de rayon 2, il existe une position du dôme telle qu'il contienne au moins 3 lucioles. (Remarque : les lucioles situées sur la frontière du dôme sont considérées comme étant à l'intérieur.)

5136. *Soumis par Marius Stănean.*

Soient a, b et c des nombres réels tels que $a \geq b \geq c \geq 0$ et $a + b + c = ab + bc + ca > 0$. Montrez que

$$a^2 + b^2 + c^2 + 5abc \geq 8 + (a - b)^2.$$

5137. *Soumis par Pelegrí Viader.*

Soit $k \in \mathbb{N}$. Déterminez la solution de l'équation aux différences

$$x_{n+k} + nx_{n+1} - nx_n = 0 \quad \text{pour } n = 1, 2, \ldots,$$

telle que $x_n \to 0$ et $nx_n \to 1$.

5138. *Soumis par Nazar Kirgizbaev.*

Soient x, y et z des nombres réels positifs tels que $x^2 + y^2 + z^2 = 3$. Montrez que

$$\sum_{cyc} \left(\frac{2xy}{xy + z} \right)^2 \geq 1 + \frac{2}{3}(xy + yz + zx.)$$

5139. *Soumis par Giuseppe Fera, modifié par le comité de rédaction.*

Trouvez une infinité de cubes dans l'espace tridimensionnel tels que

(i) tous les sommets aient des coordonnées entières ;

(ii) aucune arête ne soit parallèle à un axe de coordonnées ;

(iii) aucun cube ne soit un multiple entier d'un autre.

5140. *Soumis par Phan Ngoc Chau.*

Montrez que l'inégalité suivante

$$\frac{1}{\left(\sqrt{a} + \sqrt{bc}\right)^2} + \frac{1}{\left(\sqrt{b} + \sqrt{ca}\right)^2} + \frac{1}{\left(\sqrt{c} + \sqrt{ab}\right)^2} \geq \frac{2}{abc + 1}$$

est vérifiée pour tous nombres réels positifs a, b et c tels que $ab + bc + ca = 1$. Dans quels cas l'égalité a-t-elle lieu ?

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2026: 51(9), p. 428–431.

5081. *Proposed by Alex Roberts.*

Oscar bought a set of blank playing cards. He puts stamps on each card such that each card has $k \geq 4$ different stamps and

1. every two cards have exactly one stamp in common;
2. every stamp is used at least twice.

Show that the number of different stamps v he can use is in the range

$$\binom{k+1}{2} \leq v \leq k^2 - k + 1,$$

the number of different cards b he can use is in the range

$$k + 1 \leq b \leq k^2 - k + 1,$$

and that $V = \max v$ is at least $k^2 - 2k + 5$ for any k and equals $k^2 - k + 1$ for infinitely many k.

We received no correct submissions to the problem apart from the proposer's, whose solution is presented below.

We will show three of the four inequalities. The proof of $v \leq k^2 - k + 1$ was unclear to the editor.

Consider a card A_0 with stamps $a_1, \ldots, a_k$. Since every stamp is used at least twice there are cards $A_1, \ldots, A_k$ such that card A_i has stamp a_i. Note that each A_i only shares stamp a_i with A_0 since every two cards have exactly one stamp in common. Therefore all the A_i are distinct. Thus $b \geq k+1$. Furthermore, counting the number of stamps on $A_1, \ldots, A_k$ we note that there are k stamps on each of the k cards; however, any two cards A_i and A_j share exactly one stamp. Thus the number of distinct stamps on $A_1, \ldots, A_k$ is $k^2 - \binom{k}{2} = \binom{k+1}{2}$ and therefore $v \geq \binom{k+1}{2}$.

To show that $b \leq k^2 - k + 1$, consider a card A_0 containing a stamp a and let $A_1, \ldots, A_n$ be all the other cards that have a as a stamp. Since every two cards have exactly one stamp in common, $A_0, A_1, \ldots, A_n$ do not share any stamps apart from a. Now let B be a card that does not have a as a stamp. Then B shares a different stamp with each of $A_0, A_1, \ldots, A_n$. Since B has k stamps, we get that $n \leq k - 1$. As every card shares a stamp with A_0 and every stamp of A_0 appears on at most $k - 1$ other cards, we get that $b \leq 1 + k(k-1) = k^2 - k + 1$.

5082. *Proposed by Michel Bataille.*

Let $H_m = \sum\limits_{k=1}^{m} \frac{1}{k}$. Evaluate

$$\lim_{n\to\infty} \frac{1}{(\ln n)^2} \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k-1} H_k}{k^3}.$$

We received 3 submissions, all of which were correct and complete. We present the solution by Giuseppe Fera.

First, we prove that

$$H_k = -\int_0^1 kx^{k-1} \ln(1-x)\,\mathrm{d}x. \tag{1}$$

We know for $0 < x < 1$

$$\sum_{j=0}^{k-1} x^j = \frac{1-x^k}{1-x} \implies \int_0^1 \sum_{j=0}^{k-1} x^j \mathrm{d}x = \int_0^1 \frac{1-x^k}{1-x}\,\mathrm{d}x.$$

Exchanging the sum and integral and integrating by parts on the right hand side, we get

$$\sum_{j=0}^{k-1} \frac{1}{j+1} = -\int_0^1 kx^{k-1} \ln(1-x)\,\mathrm{d}x,$$

proving Equation (1). Thus,

$$\begin{aligned}\sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k-1} H_k}{k^3} &= \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k-1}}{k^3} \cdot \left[-\int_0^1 kx^{k-1} \ln(1-x)\,\mathrm{d}x\right] \\ &= \int_0^1 \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k}{k^2} x^{k-1} \ln(1-x)\,\mathrm{d}x \\ &= \int_0^1 \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k}{k^2} x^k \frac{\ln(1-x)}{x}\,\mathrm{d}x.\end{aligned}$$

Let $\mathrm{Li}_s(z)$ be the polylogarithm function of order s and argument z. Using the fact that $\int \frac{\ln(1-x)}{x}\mathrm{d}x = -\mathrm{Li}_2(x) + C$ and integrating by parts, we get

$$\begin{aligned}&\int_0^1 \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k}{k^2} x^k \frac{\ln(1-x)}{x}\,\mathrm{d}x \\ &= -\mathrm{Li}_2(1) \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k}{k^2} + \int_0^1 \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k}{k} x^k \frac{\mathrm{Li}_2(x)}{x}\,\mathrm{d}x.\end{aligned}$$

Similarly, since $\int \frac{\mathrm{Li}_2(x)}{x}\mathrm{d}x = \mathrm{Li}_3(x) + C$, we integrate by parts again and get

$$-\mathrm{Li}_2(1) \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k}{k^2} + \mathrm{Li}_3(1) \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k}{k} - \int_0^1 \sum_{k=1}^{n} \binom{n}{k} (-1)^k x^{k-1} \mathrm{Li}_3(x)\,\mathrm{d}x.$$

From the binomial theorem, $(1-x)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k$, so the last integral can be simplified to

$$\int_0^1 \sum_{k=1}^{n} \binom{n}{k} (-1)^k x^{k-1} \mathrm{Li}_3(x)\,\mathrm{d}x = \int_0^1 \frac{(1-x)^n - 1}{x} \mathrm{Li}_3(x)\,\mathrm{d}x.$$

To simplify the term $\sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k}{k}$, note that for $0 < x < 1$, we have that

$$\sum_{k=0}^{n-1} (1-x)^k = \frac{1-(1-x)^n}{x},$$

and so

$$\begin{aligned}
\sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k}{k} &= \int_0^1 \sum_{k=1}^{n} \binom{n}{k} (-1)^k x^{k-1}\,\mathrm{d}x \\
&= \int_0^1 \frac{(1-x)^n - 1}{x}\,\mathrm{d}x \\
&= \int_0^1 \sum_{k=0}^{n-1} (1-x)^k\,\mathrm{d}x \\
&= \sum_{k=0}^{n-1} \int_0^1 (1-x)^k\,\mathrm{d}x \\
&= \sum_{k=0}^{n-1} \frac{1}{k+1} = H_n.
\end{aligned}$$

To simplify the term $\sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k}{k^2}$, we prove by induction that

$$\sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k}{k^2} = -\frac{1}{2}\left(H_n^2 + K_n\right),$$

where $K_n = \sum_{k=1}^{n} \frac{1}{k^2}$. The base case is trivial. Consider

$$\begin{aligned}
\sum_{k=1}^{n+1} \binom{n+1}{k} \frac{(-1)^k}{k^2} &= -\frac{(-1)^n}{(n+1)^2} + \sum_{k=1}^{n} \left(1 + \frac{k}{n-k+1}\right) \binom{n}{k} \frac{(-1)^k}{k^2} \\
&= -\frac{(-1)^n}{(n+1)^2} + \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k}{k^2} + \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k}{k(n-k+1)}.
\end{aligned}$$

Note that the last sum is

$$\begin{aligned}
\sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k}{k(n-k+1)} &= \frac{1}{n+1} \sum_{k=1}^{n} \binom{n}{k} \left(\frac{1}{k} + \frac{1}{n-k+1}\right) (-1)^k \\
&= \frac{1}{n+1} \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k}{k} + \frac{1}{n+1} \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k}{n-k+1}
\end{aligned}$$

where

$$\begin{aligned}\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^k}{n-k+1} &= \sum_{k=1}^{n}\binom{n}{n-k}\frac{(-1)^k}{n-k+1}\\ &= \sum_{k=1}^{n-1}\binom{n}{k}\frac{(-1)^{n-k}}{k+1}\\ &= \sum_{k=0}^{n-1}\binom{n+1}{k+1}\frac{(-1)^{n-k}}{n+1}\\ &= -\sum_{k=1}^{n}\binom{n+1}{k}\frac{(-1)^{n-k}}{n+1}\\ &= -\sum_{k=0}^{n+1}\binom{n+1}{k}\frac{(-1)^{n-k}}{n+1}+\frac{(-1)^n}{n+1}+\frac{-1}{n+1}\\ &= \frac{(-1)^n-1}{n+1}.\end{aligned}$$

Consequently,

$$\sum_{k=1}^{n+1}\binom{n+1}{k}\frac{(-1)^k}{k^2} = -\frac{H_n^2}{2}-\frac{K_n}{2}-\frac{H_n}{n+1}-\frac{1}{(n+1)^2}.$$

Since

$$\begin{aligned}-\frac{1}{2}\left(H_{n+1}^2+K_{n+1}\right) &= -\frac{1}{2}\left(H_n+\frac{1}{n+1}\right)^2-\frac{K_n}{2}-\frac{1}{2(n+1)^2}\\ &= -\frac{H_n^2}{2}-\frac{K_n}{2}-\frac{H_n}{n+1}-\frac{1}{(n+1)^2},\end{aligned}$$

we have proved the inductive step, and so

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^k}{k^2} = -\frac{1}{2}\left(H_n^2+K_n\right).$$

Putting this all together, we have

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k-1}H_k}{k^3} = \frac{1}{2}\left(H_n^2+K_n\right)\mathrm{Li}_2(1)-H_n\mathrm{Li}_3(1)+\int_0^1\frac{1-(1-x)^n}{x}\mathrm{Li}_3(x)\,\mathrm{d}x.$$

Since $H_n\sim\ln n$ as $n\to\infty$, $0<K_n<\frac{\pi^2}{6}$, and

$$0<\int_0^1\frac{1-(1-x)^n}{x}\mathrm{Li}_3(x)\,\mathrm{d}x<\int_0^1\frac{\mathrm{Li}_3(x)}{x}\,\mathrm{d}x=\mathrm{Li}_4(1),$$

we have

$$\lim_{n\to\infty}\frac{K_n}{(\ln n)^2}=0,\quad \lim_{n\to\infty}\frac{H_n}{(\ln n)^2}=0,\quad \lim_{n\to\infty}\frac{\int_0^1\frac{1-(1-x)^n}{x}\mathrm{Li}_3(x)\,\mathrm{d}x}{(\ln n)^2}=0.$$

Thus,

$$\lim_{n\to\infty}\frac{1}{(\ln n)^2}\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k-1}H_k}{k^3}=\lim_{n\to\infty}\frac{\frac{1}{2}H_n^2\mathrm{Li}_2(1)}{(\ln n)^2}=\frac{1}{2}\mathrm{Li}_2(1)=\frac{\pi^2}{12}.$$

5083. *Proposed by Vasile Cîrtoaje.*

Let a, b, c, d be nonnegative real numbers such that $a \geq b \geq c \geq d$ and

$$ab + bc + cd + da = 4.$$

Prove that

$$abc + bcd + cda + dab \leq 4.$$

We received 5 submissions, of which 3 were correct and complete. We present two solutions.

Solution 1, by Michal Adamaszek.

Let $p = a + c$, $q = b + d$, and $x = b - c$. Then, we have the following:

$$pq = (a + c)(b + d) = ab + bc + cd + da = 4$$

and

$$x = b - c = b - (p - a) = a + b - p.$$

We rewrite the sum as follows:

$$\begin{aligned}
abc + bcd + cda + dab &= ac(b + d) + bd(a + c)\\
&= qac + pbd\\
&= qa(p - a) + pb(q - b)\\
&= pqa - qa^2 + pqb - pb^2\\
&= pq(a + b) - qa^2 - pb^2\\
&= 4(a + b) - qa^2 - pb^2\\
&= 4(x + p) - q(x + p - b)^2 - pb^2\\
&= 4x + 4p - qx^2 - qp^2 - qb^2 - 2pqx + 2pqb + 2qxb - pb^2\\
&= 4x + 4p - qx^2 - 4p - qb^2 - 8x + 8b + 2qxb - pb^2\\
&= 8b - (p + q)b^2 - x(qx + 4 - 2qb)\\
&= 8b - (p + q)b^2 - x(4 - q(2b - x)).
\end{aligned}$$

Since $a \geq b$, we have that

$$q(2b - x) = q(2b - (b - c)) = q(b + c) \leq q(a + c) = qp = 4,$$

and since $x = b - c \geq 0$ this implies

$$x(4 - q(2b - x)) \geq 0.$$

Moreover, we have $p+q \geq 2\sqrt{pq} = 4$, and therefore

$$8b - (p+q)b^2 \leq 8b - 4b^2 \leq 4.$$

By combining the last two inequalities, we get that

$$abc + bcd + cda + dab = 8b - (p+q)b^2 - x(4 - q(2b-x)) \leq 8b - (p+q)b^2 \leq 4,$$

and this completes the proof.

Solution 2, by Jianlin Zhang.

Let a, b, c, d be positive real numbers, and let t be a Lagrange multiplier to define the Lagrange function

$$F(a,b,c,d,t) = abc + bcd + acd + abd + t(ab + bc + cd + ad - 4).$$

We differentiate F with respect to the variables a, b, c, d to obtain the following homogeneous system of equations

$$F_a = bc + cd + bd + t(b+d) = 0 \tag{1}$$
$$F_b = ac + cd + ad + t(a+c) = 0 \tag{2}$$
$$F_c = ab + bd + ad + t(b+d) = 0 \tag{3}$$
$$F_d = bc + ac + ab + t(a+c) = 0 \tag{4}$$

Equating Equation (1) and Equation (3) we get that

$$bc + cd + bd = ab + bd + ad \Rightarrow c(b+d) = a(b+d)$$

Since $b + d > 0$, then $a = c$. In a similar way, we equate (2) and (4) to get $b = d$. We rewrite (1) as $2ab + b^2 + 2tb = 0$ and (2) as $2ab + a^2 + 2ta = 0$ and subtract them to get

$$b^2 + 2tb - a^2 - 2ta = (b-a)(b+a) + 2t(b-a) = (b-a)(b+a+2t) = 0 \Rightarrow a = b.$$

We find that the critical point is obtained when $a = b = c = d = 1, t = \frac{-3}{2}$. For these values we get $abc + bcd + acd + abd = 4$, and thus, the critical point is a maximum. We conclude that $abc + bcd + acd + abd \leq 4$ for all nonnegative real values of a, b, c, and d.

5084. *Proposed by Xicheng Peng.*

Let ABC be a triangle. Let its incircle O touch the side BA at point F. Prove that the area of ABC is $AF \cdot FB \cdot \cot \frac{C}{2}$.

There were 24 correct solutions from 22 solvers (one submitting three).

Let a, b, c, s, r have their usual meanings. Since $r\cot(C/2) = (s-c)$ and

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)} = rs,$$

$$[ABC] = \frac{[ABC]^2}{[ABC]} = (s-a)(s-b)\left(\frac{s-c}{r}\right) = (AF)(FB)\cot\tfrac{C}{2}.$$

Comment from the editor. *This solution was provided by a majority of the solvers. Others relied on a variety of less familiar trigonometric formulae and identities. The equality of the two expressions for $[ABC]$ is equivalent to*

$$\cot\tfrac{A}{2} + \cot\tfrac{B}{2} + \cot\tfrac{C}{2} = \cot\tfrac{A}{2}\cot\tfrac{B}{2}\cot\tfrac{C}{2}.$$

One solver used analytic geometry.

5085. Proposed by Tran Ngoc Khuong Trang.

Prove that the following inequality

$$\frac{\sqrt{2-bc}}{a+1} + \frac{\sqrt{2-ca}}{b+1} + \frac{\sqrt{2-ab}}{c+1} \geq 1 + \sqrt{2}$$

holds for all non-negative real numbers a, b, c satisfying $ab + bc + ca = 1$. When does equality occur?

We received 2 submissions. The only correct solution was by the problem proposer Khuong Trang Tran Ngoc. A slightly edited version of this solution is presented here.

The given condition implies that $0 \leq ab, bc, ca \leq 1$.

We'll prove the following estimate

$$\sqrt{2-bc} \geq \left(1-\sqrt{2}\right)bc + \sqrt{2}. \tag{$*$}$$

It is equivalent to

$$(1-bc)\left(\frac{bc+2}{\sqrt{2-bc}+bc} - \sqrt{2}\right) \geq 0.$$

Thus, it suffices to prove

$$\frac{bc+2}{\sqrt{2-bc}+bc} - \sqrt{2} \geq 0$$

or

$$bc\left(1-\sqrt{2}+\frac{2}{\sqrt{4-2bc}+2}\right) \geq 0,$$

which is true since $0 \leq bc \leq 1$ and

$$1-\sqrt{2}+\frac{2}{\sqrt{4-2bc}+2} \geq 1-\sqrt{2}+\frac{1}{2} > 0.$$

Hence, the estimate $(*)$ *is proved. Similarly, we obtain the estimates*

$$\frac{\sqrt{2-bc}}{a+1} \geq \frac{\left(1-\sqrt{2}\right)bc+\sqrt{2}}{a+1},$$
$$\frac{\sqrt{2-ca}}{b+1} \geq \frac{\left(1-\sqrt{2}\right)ca+\sqrt{2}}{b+1},$$
$$\frac{\sqrt{2-ab}}{c+1} \geq \frac{\left(1-\sqrt{2}\right)ab+\sqrt{2}}{c+1}.$$

Therefore, it's enough to prove that

$$\frac{\left(1-\sqrt{2}\right)ab+\sqrt{2}}{c+1}+\frac{\left(1-\sqrt{2}\right)bc+\sqrt{2}}{a+1}+\frac{\left(1-\sqrt{2}\right)ca+\sqrt{2}}{b+1} \geq 1+\sqrt{2}. \qquad (**)$$

We may simplify $(**)$ *by denoting*

$$x=1-\sqrt{2}; \quad y=\sqrt{2}; \quad t=1+\sqrt{2}; \quad a+b+c=p; \quad abc=r.$$

Now, we will prove

$$\frac{xbc+y}{a+1}+\frac{xca+y}{b+1}+\frac{xab+y}{c+1} \geq t.$$

We have

$$\begin{aligned}\sum_{\text{cyc}}(b+1)(c+1)(xbc+y) &= \sum_{\text{cyc}} xb^2c^2+xbc(b+c)+xbc+y(b+c+bc+1)\\ &= x\left(1-2pr\right)+x\left(p-3r\right)+x+2yp+4y,\\ t(a+1)(b+1)(c+1) &= t\left(2+p+r\right).\end{aligned}$$

Finally, $(**)$ *rewrites as*

$$x\left(1-2pr\right)+x\left(p-3r\right)+x+2yp+4y \geq t\left(2+p+r\right),$$

which reduces to

$$r(2xp+3x+t) \leq 0,$$

or

$$2\left(\sqrt{2}-1\right)\cdot abc\left(a+b+c-\sqrt{2}\right) \geq 0.$$

The last inequality is true since

$$a+b+c \geq \sqrt{3(ab+bc+ca)}=\sqrt{3}>\sqrt{2}.$$

The proof is completed. Equality holds if and only if one of the numbers a, b, c *is equal to zero.*

5086. Proposed by Mihaela Berindeanu.

Let ABC be an acute triangle. The bisector of angle B cuts the perpendicular bisector of segment AB at C_1. The projection of point C_1 onto AB, respectively AC is C_2, respectively C_3. The points B_1, B_2, B_3 and A_1, A_2, A_3 are defined similarly. Show that:

$$\frac{AC_3}{AC_2}+\frac{BA_3}{BA_2}+\frac{CB_3}{CB_2}\le\frac{BC}{BA}+\frac{CA}{CB}+\frac{AB}{AC}.$$

We received 6 solutions, all correct and complete. We present the solution by Theo Koupelis (slightly edited), which is very close to the solution by the problem proposer Mihaela Berindeanu. A similar solution was proposed by C. R. Pranesachar.

Let $(BC, AC, AB) = (a, b, c)$ and let E, R, s be the area, the circumradius, and the semiperimeter of the triangle, respectively. By construction, C_2 is the midpoint of AB. Thus, $\angle BAC_1 = \angle ABC_1 = B/2$. From the right triangles AC_2C_1 and AC_3C_1 we get $AC_2 = AC_1\cos\frac{B}{2}$ and $AC_3 = AC_1\cos\big(A-\frac{B}{2}\big)$. Thus,

$$\frac{AC_3}{AC_2}=\cos A+\sin A\cdot\tan\frac{B}{2}. \qquad (*)$$

Using the laws of sines and cosines, one can show that

$$\sin\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{ac}},\quad \cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ac}},\quad \text{and}\quad \tan\frac{B}{2}=\frac{E}{s(s-b)}.$$

Also, $\sin A=\dfrac{a}{2R}=\dfrac{2E}{bc}$. *Substituting into $(*)$, we get*

$$\frac{AC_3}{AC_2}=\frac{b^2+c^2-a^2}{2bc}+\frac{2(s-a)(s-c)}{bc}=\frac{b^2-a^2+ac}{bc},$$

with similar expressions for BA_3/BA_2 and CB_3/CB_2. Thus, the desired inequality is equivalent to

$$\frac{b^2-a^2+ac}{bc}+\frac{c^2-b^2+ab}{ac}+\frac{a^2-c^2+bc}{ab}\le\frac{a}{c}+\frac{b}{a}+\frac{c}{b},$$

or, clearing the denominators, to

$$a(a-c)^2+b(a-b)^2+c(b-c)^2\ge 0,$$

which is obvious. Equality occurs when the triangle is equilateral.

Editor's Comments.

Oliver Geupel pointed out that, in fact, the inequality

$$\frac{AC_3}{AC_2}+\frac{BA_3}{BA_2}+\frac{CB_3}{CB_2}\le 3 \qquad (**)$$

holds. This inequality is stronger than the initial one because

$$\frac{BC}{BA}+\frac{CA}{CB}+\frac{AB}{AC}\geq 3\sqrt[3]{\frac{BC}{BA}\cdot\frac{CA}{CB}\cdot\frac{AB}{AC}}=3$$

by AM-GM. To show $(**)$*, we note that, reasoning in the same way as above, one can reduce* $(**)$ *to*

$$\frac{b^2-a^2+ac}{bc}+\frac{c^2-b^2+ab}{ac}+\frac{a^2-c^2+bc}{ab}\leq 3.$$

It is easy to check that the substitution

$$x=s-a,\ y=s-b,\ z=s-c \quad\Leftrightarrow\quad a=y+z,\ b=x+z,\ c=x+y$$

turns the latter inequality into

$$3xyz\leq xy^2+yz^2+zx^2,$$

which is true by AM-GM (note that $x,y,z>0$ *by the triangle inequality).*

5087. Proposed by Tatsunori Irie.

Let $n\geq 2$ *be an integer and let* $a_1,a_2,\ldots,a_n$ *be integers with* $1\leq a_i\leq n$ $(1\leq i\leq n)$ *such that* $a_1+a_2+\cdots+a_n\equiv 0 \pmod n$*. Show that if the* a_i *are not all equal, then there exists a non-empty proper subset* $\{i_1,i_2,\ldots,i_j\}\subset\{1,2,\ldots,n\}$ *with* $1\leq j<n$ *such that* $a_{i_1}+a_{i_2}+\cdots+a_{i_j}\equiv 0 \pmod n$*.*

We received five solutions. Presented is the one by Michał Adamaszek.

Since the numbers a_i *are not all equal we may assume, rearranging the indices if necessary, that* $a_{n-1}\not\equiv a_n \pmod n$*. Now consider the following* n *numbers modulo* n*:*

$$\begin{aligned}&a_1\\&a_1+a_2\\&\cdots\\&a_1+a_2+\cdots+a_{n-2}+a_{n-1}\\&a_1+a_2+\cdots+a_{n-2}+a_n.\end{aligned}$$

The last two numbers are not equal modulo n*. If any other pair of numbers in this sequence are congruent modulo* n *then their difference equals* 0 *modulo* n *and is of the desired form.*

Otherwise these n *numbers are all different modulo* n*, so one of them is congruent to* 0 *and is therefore of the desired form.*

The assumption that $\sum a_i=0$ *does not seem necessary.*

5088. Proposed by To An Ky.

Let $ABCD$ be a parallelogram satisfying $BC = BD$. Let ω be a circle centered at the midpoint of segment CD, touching side BD at E. From A, construct segment AK tangent to ω at K ($K \notin \overleftrightarrow{AD}$). Show that $\angle BKE = 90°$.

We received 7 solutions, 5 of which were correct; we present 2 solutions.

Solution 1, by Tatsunori Irie.

Let $F_+(x) = \sqrt{2+x}$ and $F_-(x) = \sqrt{2-x}$. Set $x = 2\cos\theta$ $(0 < \theta < \frac{\pi}{2})$. Then

$$F_+(2\cos\theta) = 2\cos\frac{\theta}{2} \quad \textit{and} \quad F_-(2\cos\theta) = 2\cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right).$$

Consequently, each outward wrapping [i.e. successive radical] corresponds to one of the angle maps

$$T_+(\theta) = \frac{\theta}{2} \quad \textit{and} \quad T_-(\theta) = \frac{\pi}{2} - \frac{\theta}{2}.$$

Starting from $\theta_1 = \frac{\pi}{4}$ [i.e. $x_1 = \sqrt{2}$] an easy induction shows that after n wrappings the attainable angles are precisely

$$\theta_n = \frac{(2k-1)\pi}{2^{n+1}} \qquad (1 \le k \le 2^{n-1})$$

i.e. the odd multiples of $\pi/2^{n+1}$ in $(0, \frac{\pi}{2})$. These angles subdivide $(0, \frac{\pi}{2})$ into equal steps of size $\Delta\theta = \frac{\pi}{2^n}$. Given any target $y \in (0,2)$ [and tolerance $\varepsilon > 0$], let $\theta = \arccos(\frac{y}{2}) \in (0, \frac{\pi}{2})$. Choose n so large that $\frac{\pi}{2^n} < \varepsilon$ and take θ_n among the above grid points with $|\theta - \theta_n| \le \frac{\pi}{2^{n+1}}$. Then

$$|2\cos\theta - 2\cos\theta_n| \le 2|\theta - \theta_n| \le \frac{\pi}{2^n} < \varepsilon.$$

Because every θ_n is realizable by some choice of $\pm$ signs, there exists a sign sequence whose n^{th} truncation is within ε of y.

Solution 2, by Vivek Mehra.

From the 8^{th} Moscow Olympiad (1945) we have the following result:

> *The numbers $a_1, a_2, \ldots, a_n$ are equal to 1 or -1. Prove that*

$$2\sin\left(a_1 + \frac{a_1a_2}{2} + \ldots + \frac{a_1a_2\ldots a_n}{2^{n-1}}\right)\frac{\pi}{4} = a_1\sqrt{2 + a_2\sqrt{2 + \ldots + a_n\sqrt{2}}}.$$

By continuity of the sine function, we only need to show that

$$1 \pm \frac{1}{2} \pm \frac{1}{4} \pm \cdots$$

can, by a suitable choice of signs, attain values arbitrarily close to any prescribed number in the interval $(0,2)$. This is easy to see.

Editor's Comments. All solutions mimicked one of the two strategies above. The Lipschitz estimate follows from the trig identity

$$\cos(x) - \cos(y) + 2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) = 0$$

and the inequality $|\sin(t)| \le \min\{1, |t|\}$*. The Moscow formula follows by induction from the trig identity*

$$2 + 2\sin(z) = 4\sin^2\left(\frac{\pi}{4} + \frac{z}{2}\right).$$

Note that the infinite radical itself essentially never converges, in the sense that for almost all choices $s = (s_1, s_2, s_3, \ldots)$ *of signs* $s_k \in \{-1, +1\}$ *the limit*

$$\ldots + \sqrt{2 + s_3\sqrt{2 + s_2\sqrt{2 + s_1\sqrt{2}}}} = \lim_{k\to\infty} \sqrt{2 + s_k\sqrt{2 + \ldots + s_2\sqrt{2 + s_1\sqrt{2}}}}$$

does not exist. (If $x_k \to l$ *then because* $x_k^2 = 2 + s_k x_{k-1}$ *we must have* $l \ne 0$ *and consequently* $s_k = \frac{x_k^2 - 2}{x_{k-1}}$ *tends to a finite limit* s_∞*, which by nature of* s *must be* $+1$ *or* -1*. Then* $l^2 = 2 + s_\infty l$ *whose only positive roots are* $l = 2$ *and* $l = 1$*, corresponding to* s *being eventually constantly* $+1$ *or* -1 *respectively.) In fact, it is possible to produce a sign-sequence whose successive finite nested radicals are dense in* $[0, 2]$*. (Hint:* $F_+(F_-(F_+^n(x)) \to \sqrt{2}$ *for all* x*.)*

5089. Proposed by Tatsunori Irie.

Consider the infinite nested radical obtained by starting from the innermost term $\sqrt{2}$ *and wrapping outward choosing either* $+$ *or* $-$ *sign independently:*

$$\cdots\sqrt{2 \pm \sqrt{2 \pm \sqrt{2 \pm \sqrt{2}}}}.$$

Can one, by a suitable choice of the signs, obtain values arbitrarily close to any prescribed number in the interval $(0, 2)$*?*

We received 7 solutions, 5 of which were correct. We present two solutions.

Solution 1, by Tatsunori Irie.

Let $F_+(x) = \sqrt{2 + x}$ *and* $F_-(x) = \sqrt{2 - x}$*. Set* $x = 2\cos\theta$ $(0 < \theta < \frac{\pi}{2})$*. Then*

$$F_+(2\cos\theta) = 2\cos\frac{\theta}{2} \quad \textit{and} \quad F_-(2\cos\theta) = 2\cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right).$$

Consequently, each outward wrapping [i.e. successive radical] corresponds to one of the angle maps

$$T_+(\theta) = \frac{\theta}{2} \quad \textit{and} \quad T_-(\theta) = \frac{\pi}{2} - \frac{\theta}{2}.$$

Starting from $\theta_1 = \frac{\pi}{4}$ *[i.e.* $x_1 = \sqrt{2}$*] an easy induction shows that after* n *wrappings the attainable angles are precisely*

$$\theta_n = \frac{(2k-1)\pi}{2^{n+1}} \qquad (1 \le k \le 2^{n-1})$$

i.e. the odd multiples of $\pi/2^{n+1}$ *in* $(0, \frac{\pi}{2})$*. These angles subdivide* $(0, \frac{\pi}{2})$ *into equal steps of size* $\Delta\theta = \frac{\pi}{2^n}$*. Given any target* $y \in (0, 2)$ *[and tolerance* $\varepsilon > 0$*], let* $\theta = \arccos(\frac{y}{2}) \in (0, \frac{\pi}{2})$*. Choose* n *so large that* $\frac{\pi}{2^n} < \varepsilon$ *and take* θ_n *among the above grid points with* $|\theta - \theta_n| \le \frac{\pi}{2^{n+1}}$*. Then*

$$|2\cos\theta - 2\cos\theta_n| \le 2|\theta - \theta_n| \le \frac{\pi}{2^n} < \varepsilon.$$

Because every θ_n *is realizable by some choice of* $\pm$ *signs, there exists a sign sequence whose* n^{th} *truncation is within* ε *of* y*.*

Solution 2, by Vivek Mehra.

From the 8^{th} *Moscow Olympiad (1945) we have the following result:*

> *The numbers* $a_1, a_2, \ldots, a_n$ *are equal to 1 or* -1*. Prove that*

$$2\sin\left(a_1 + \frac{a_1a_2}{2} + \ldots + \frac{a_1a_2\ldots a_n}{2^{n-1}}\right)\frac{\pi}{4} = a_1\sqrt{2 + a_2\sqrt{2 + \ldots + a_n\sqrt{2}}}.$$

By continuity of the sine function, we only need to show that

$$1 \pm \frac{1}{2} \pm \frac{1}{4} \pm \cdots$$

can, by a suitable choice of signs, attain values arbitrarily close to any prescribed number in the interval $(0, 2)$*. This is easy to see.*

Editor's Comments. *All the solutions mimicked one of the two strategies above. The Lipschitz estimate follows from the trig identity*

$$\cos(x) - \cos(y) + 2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) = 0$$

and the inequality $|\sin(t)| \le \min\{1, |t|\}$*. The Moscow formula follows by induction from the trig identity*

$$2 + 2\sin(z) = 4\sin^2\left(\frac{\pi}{4} + \frac{z}{2}\right).$$

Note that the infinite radical itself essentially never converges, in the sense that for almost all choices $s = (s_1, s_2, s_3, \ldots)$ *of signs* $s_k \in \{-1, +1\}$ *the limit*

$$\ldots + \sqrt{2 + s_3\sqrt{2 + s_2\sqrt{2 + s_1\sqrt{2}}}} = \lim_{k\to\infty}\sqrt{2 + s_k\sqrt{2 + \ldots + s_2\sqrt{2 + s_1\sqrt{2}}}}$$

does not exist (If $x_k \to l$ *then because* $x_k^2 = 2 + s_kx_{k-1}$ *we must have* $l \ne 0$ *and consequently* $s_k = \frac{x_k^2 - 2}{x_{k-1}}$ *tends to a finite limit* s_∞*, which by nature of* s *must be* $+1$ *or* -1*. Then* $l^2 = 2 + s_\infty l$ *whose only positive roots are* $l = 2$ *and* $l = 1$*, corresponding to* s *being eventually constantly* $+1$ *or* -1 *respectively.) In fact, it is possible to produce a sign-sequence whose successive finite nested radicals are dense in* $[0, 2]$*. Hint:* $F_+(F_-(F_+^n(x)) \to \sqrt{2}$ *for all* x*.*

5090. Proposed by Ion Pătraşcu.

Let ABC be a triangle where O and I are the centres of its circumscribed and inscribed circles, and R and r are the radii of these two circles respectively, such that $R = r(1+\sqrt{2})$. Let D, E and F be the points where the inscribed circle touches BC, CA and AB. Let Ω_1, Ω_2, Ω_3 be the circumscribed circles pertaining to triangles BOC, COA and AOB, and D_1, E_1, F_1 be the points where these circles meet rays OD, OE and OF respectively. Prove that D_1, E_1, F_1 are collinear.

All of the 8 submissions we received were correct; we feature the solution jointly submitted by Shreya Mundhada, Harini Subramanian, and Shrimoyee Bera, which was one of several solutions based on an inversion.

We shall see that the converse also holds: D_1, E_1, F_1 are collinear if and only if $R = r(\sqrt{2}+1)$. Consider an inversion in the circumcircle of ΔABC. Points B and C are fixed, so line BC maps to Ω_1 (the circle passing through O, B, C). Therefore, the inversion must interchange D_1 (the point where Ω_1 meets the ray OD) with D (the point where the incircle touches BC). Similarly, E_1 is interchanged with E, and F_1 with F. Thus, D_1, E_1, F_1 being collinear is equivalent to $DEFO$ being cyclic. Consequently, our problem is reduced to determining when O lies on the incircle of ABC; that is, when $r = IO$.

By Euler's theorem for triangles, we know that $OI^2 = R(R-2r)$. Therefore, $R = r(\sqrt{2}+1)$ if and only if

$$OI^2 = r(\sqrt{2}+1)(r(\sqrt{2}+1)-2r) = r^2(\sqrt{2}+1)(\sqrt{2}-1) = r^2.$$

Thus, $OI = r$ if and only if $R = r(\sqrt{2}+1)$, if and only if O lies on the incircle, if and only if D_1, E_1, F_1 are collinear, so we are done.