Canadian Mathematical Olympiad
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P1. Let $a, b \geq 2$ be relatively prime integers. Let S consist of the points in the plane with integer coordinates lying strictly inside the triangle with vertices $(0,0), (a,0), (0,b)$. Determine, with proof,

$$\sum_{(x,y)\in S} (a-2x)(b-2y)$$

in terms of a and b.

Remark. Here the summation denotes that we sum the value $(a-2x)(b-2y)$ over all points (x,y) in S.

Solution 1

For simplicity let $f(x,y) = (a-2x)(b-2y)$. Let R be the region in the plane consisting of lattice points (x,y) with $1 \leq x \leq a-1$ and $1 \leq y \leq b-1$. Then S consists of the lattice points in R that lie below the line $x/a + y/b = 1$. Let T consist of the lattice points in R that are not in the aforementioned right angled triangle and lie strictly above the line $x/a + y/b = 1$.

Since a and b are coprime the line $x/a + y/b = 1$, which is the line $bx + ay = ab$, has no lattice points with $1 \leq x \leq a-1$ and $1 \leq y \leq b-1$. Indeed $bx + ay = ab$ implies $b|ay$, so $b|y$ by coprimality, which contradicts $1 \leq y \leq b-1$. Therefore R is the disjoint union of S and T.

Now on R we see that

$$\begin{aligned}\sum_{(x,y)\in R} f(x,y) &= \left(\sum_{x=1}^{a-1}(a-2x)\right)\left(\sum_{y=1}^{b-1}(b-2y)\right)\\ &= \left(a(a-1) - 2\cdot\frac{a(a-1)}{2}\right)\left(b(b-1) - 2\cdot\frac{b(b-1)}{2}\right)\\ &= 0\cdot 0 = 0.\end{aligned}$$

However, the map $F(x,y) = (a-x, b-y)$ is a bijection from S to T. This is because T consists of lattice points in R above the line $x/a + y/b = 1$, S consists of lattice points in R above that line, and $x/a + y/b < 1$ if and only if $(a-x)/a + (b-y)/b = 2 - (x/a + y/b) > 1$. Moreover,

$$f(F(x,y)) = (a-2(a-x))(b-2(b-y)) = (2x-a)(2y-b) = (a-2x)(b-2y) = f(x,y).$$

Altogether then

$$\sum_{(x,y)\in S} f(x,y) = \sum_{(x,y)\in T} f(x,y).$$

Together with

$$0 = \sum_{(x,y)\in R} f(x,y) = \sum_{(x,y)\in S} f(x,y) + \sum_{(x,y)\in T} f(x,y)$$

this implies

$$\sum_{(x,y)\in S} f(x,y) = 0$$

as desired.

Solution 2

Let

$$T = \sum_{(x,y)\in S} (a-2x)(b-2y).$$

We prove that $T = 0$ by splitting the big triangle into two smaller regions using the median from $(0,0)$ to the midpoint $\left(\frac{a}{2}, \frac{b}{2}\right)$ of the hypotenuse. This median is the line

$$y = \frac{b}{a}x.$$

Since a and b are relatively prime and at least 2, there are no lattice points on the open line segment joining $(0,0)$ to $\left(\frac{a}{2}, \frac{b}{2}\right)$: indeed, if an integer point (x,y) lay on that segment, then $ay = bx$, and since $\gcd(a,b) = 1$, this would force $a \mid x$ and $b \mid y$, which is impossible for a point strictly between the endpoints. Thus every lattice point of S lies strictly on one side or the other of this median.

Write $S = S_1 \sqcup S_2$, where S_1 is the set of lattice points of S below the median and S_2 is the set of lattice points of S above the median. We first consider S_1. If $(x,y) \in S_1$, then $0 < y < \frac{b}{a}x$, and because the point also lies inside the original triangle we have $x > 0$, $y > 0$, and $x < a - ay/b$. Now reflect (x,y) horizontally across the median of the horizontal slice of the triangle by sending it to $(a-x, y)$. Since

$$\frac{x}{a} + \frac{y}{b} < 1 \quad\Longleftrightarrow\quad \frac{a-x}{a} + \frac{y}{b} > \frac{y}{b},$$

and since (x,y) lies below the median, one checks that $(a-x,y)$ lies in the upper part of the triangle determined by the same horizontal level, still strictly inside the original triangle. More importantly, the two corresponding summands cancel:

$$(a - 2(a-x))(b-2y) = (-a+2x)(b-2y) = -(a-2x)(b-2y).$$

Thus points paired by $(x,y) \leftrightarrow (a-x,y)$ contribute zero in total.

Likewise, for the points in S_2, we pair vertically by sending (x,y) to $(x, b-y)$. This sends points in the left-hand region to corresponding points in the complementary region on the same vertical line, and again the summands cancel because

$$(a-2x)(b-2(b-y)) = (a-2x)(-b+2y) = -(a-2x)(b-2y).$$

The only points in S which cannot be paired are those on the lines $x = a/2$ or $y = b/2$, as they reflect to themselves. But $x = a/2$ and $y = b/2$ both imply

$(a-2x)(b-2y)=0$, so these terms do not contribute to the sum. Therefore the entire sum cancels under the above pairing:

$$\sum_{(x,y)\in S}(a-2x)(b-2y)=0.$$

Solution 3

Let

$$T=\sum_{(x,y)\in S}(a-2x)(b-2y).$$

We compute T by summing over the lattice points column by column. For each integer x with $1\le x\le a-1$, the points of S having first coordinate x are exactly $(x,1),(x,2),\ldots,(x,m_x)$, where m_x is the largest integer such that $\frac{x}{a}+\frac{y}{b}<1$. Since the hypotenuse of the triangle is the line $y=b-\frac{b}{a}x$, we have

$$m_x=\left\lfloor b-\frac{b}{a}x\right\rfloor.$$

Because $\gcd(a,b)=1$, the number $\frac{bx}{a}$ is not an integer for any $x=1,\ldots,a-1$, so

$$m_x=b-1-\left\lfloor\frac{bx}{a}\right\rfloor.$$

If we write $n_x=\left\lfloor\frac{bx}{a}\right\rfloor$, then $m_x=b-1-n_x$. The contribution from the x-th column to the entire sum is therefore

$$C_x=\sum_{y=1}^{m_x}(a-2x)(b-2y)=(a-2x)\sum_{y=1}^{m_x}(b-2y).$$

Now

$$\sum_{y=1}^{m_x}(b-2y)=m_xb-2\cdot\frac{m_x(m_x+1)}{2}=m_x(b-m_x-1),$$

and substituting $m_x=b-1-n_x$ yields

$$\sum_{y=1}^{m_x}(b-2y)=(b-1-n_x)n_x.$$

Thus

$$C_x=(a-2x)(b-1-n_x)n_x.$$

We now compare the x-th and $(a-x)$-th columns. Since

$$n_{a-x}=\left\lfloor\frac{b(a-x)}{a}\right\rfloor=\left\lfloor b-\frac{bx}{a}\right\rfloor=b-1-\left\lfloor\frac{bx}{a}\right\rfloor=b-1-n_x,$$

it follows that

$$(b-1-n_{a-x})n_{a-x} = n_x(b-1-n_x).$$

On the other hand,

$$a-2(a-x) = -(a-2x).$$

Hence

$$C_{a-x} = (a-2(a-x))(b-1-n_{a-x})n_{a-x} = -(a-2x)(b-1-n_x)n_x = -C_x.$$

So the contribution from column x cancels exactly with the contribution from column $a-x$. Therefore all columns cancel in pairs. If a is even, the middle column $x=\frac{a}{2}$ contributes 0 anyway, since then $a-2x=0$. It follows that

$$\sum_{(x,y)\in S}(a-2x)(b-2y) = 0.$$

P2. There are n types of coins in Wario's gold mine. Each coin of the ith type is worth d_i cents, where $d_1, \dots, d_n$ are distinct positive integers. A positive integer D is denoted *lucky* if the following holds: For each positive integer k, any collection of coins (containing any number of coins of each type) with a total value of exactly kD cents can be split into k groups, each worth D cents. Does a lucky number necessarily exist?

Solution 1

The answer is yes. We apply induction on k, repeatedly extracting a group of value D from the remaining coins; the case $k = 1$ is immediate. Thus the problem is equivalent to the following inductive step:

> Given positive integers $d_1, \dots, d_n$, does there exist a positive integer D such that whenever nonnegative n_i satisfy $\sum_i n_i d_i = mD$ for a positive integer $m \geq 2$, there exist $0 \leq m_i \leq n_i$ satisfying $\sum_i m_i d_i = D$?

Let d be any multiple of all of $d_1, \dots, d_n$ (e.g. $d = d_1 \dots d_n$ or $d = \operatorname{lcm}(d_1, \dots, d_n)$). We claim that $D = Md$ works for a suitable choice of M. Indeed, note that any d/d_i copies of d_i can be grouped together to a collection of numbers summing to d. There exist at least

$$\sum_i \left\lfloor \frac{n_i}{d/d_i} \right\rfloor$$

such groups, so it suffices to show that the quantity above is at least M. However we have that

$$\begin{aligned}\sum_i \left\lfloor \frac{n_i}{d/d_i} \right\rfloor &\geq \left(\sum_i \frac{n_i}{d/d_i} \right) - n \\ &= \left(\frac{1}{d} \sum_i n_i d_i \right) - n \\ &= mM - n \\ &\geq M + (M - n).\end{aligned}$$

Hence choosing $M = n$ works as desired.

Solution 2

We present a modified solution, based on the submission of Perry Dai, which proves $D = d_1 \dots d_n$ is lucky. As in Solution 1, we use induction on k where the base case $k = 1$ is trivial. Thus, suppose that $k \geq 2$ and $a_1 d_1 + \cdots + a_n d_n = kD$. WLOG let $a_1 d_1 \geq a_2 d_2 \geq \ldots \geq a_n d_n$. Then $a_1 d_1 \geq \frac{kD}{n} \geq \frac{2d_1 \dots d_n}{n}$, so $a_1 \geq \frac{2d_2 \dots d_n}{n}$.

Claim. We have $a_1 \geq d_i$ for all $2 \leq i \leq n$, except for a case where $n = 3$ and $d_i = 1$ for some $i \geq 2$.

Proof of claim. If $n \le 2$ or $n \ge 4$, then

$$a_1 \ge \frac{2d_2 \dots d_n}{n} = d_i \cdot \frac{2}{n} \cdot \prod_{2 \le j \le n, j \ne i} d_j \ge d_i \cdot \frac{2}{n} \cdot (n-2)! \ge d_i.$$

If $n = 3$ and no d_i is equal to 1, then $a_1 \ge \frac{2d_2d_3}{3} \ge \frac{4}{3}d_i \ge d_i$. $\square$

Let us ignore the $n = 3$ and some d_i equal to 1 case for now and continue with $a_1 \ge d_i$ for all $2 \le i \le n$.

Consider a sequence of nonnegative integers $x_1, \dots, x_n$ such that $x_1d_1 + \dots + x_nd_n = D$ and $x_i \le a_i$ for all $2 \le i \le n$, where we choose x_1 to be minimal under these constraints. Note that $\frac{D}{d_1}, 0, \dots, 0$ satisfies the aforementioned constraints, so at least one such sequence exists. If $x_1 \le a_1$ as well, then $x_1, \dots, x_n$ corresponds to a group of coins worth D cents. Suppose this is not the case, so $x_1 > a_1$. For any $2 \le i \le n$, note that replacing (x_1, x_i) with $(x_1 - d_i, x_i + d_1)$ does not change $x_1d_1 + \dots + x_nd_n$, keeps x_1 nonnegative because $x_1 > a_1 \ge d_i$ as per the claim above, and makes x_1 smaller. By the minimality of x_1, we must have $x_i + d_1 > a_i$ for all $2 \le i \le n$, so $x_i > a_i - d_1$. Then,

$$\begin{aligned} D = \sum_{i=1}^{n} x_id_i = x_1d_1 + \sum_{i=2}^{n} x_id_i &\ge (a_1+1)d_1 + \sum_{i=2}^{n}(a_i - d_1 + 1)d_i \\ = \sum_{i=1}^{n} a_id_i + \sum_{i=1}^{n} d_i - d_1\sum_{i=2}^{n} d_i &\ge 2D + \sum_{i=1}^{n} d_i - d_1\sum_{i=2}^{n} d_i \\ \implies d_1\sum_{i=2}^{n} d_i \ge D + \sum_{i=1}^{n} d_i &> d_1\left(1 + \prod_{i=2}^{n} d_i\right). \end{aligned}$$

We claim that $1 + \prod_{i=2}^{n} d_i \ge \sum_{i=2}^{n} d_i$ for any distinct positive integers $d_2, \dots, d_n$, which would imply a contradiction. Without loss of generality, rearrange them so that $d_2 < d_3 < \dots < d_n$, so $2 \le d_i$ for all $3 \le i \le n$. Then $(d_2 - 1)(d_3 \dots d_n - 1) \ge 0$ and $d_3 \dots d_n \ge d_3 + \dots + d_n$, so

$$1 + \prod_{i=2}^{n} d_i \ge d_2 + \prod_{i=3}^{n} d_i \ge d_2 + \sum_{i=3}^{n} d_i$$

as required. This completes the proof except for the case where $n = 3$ and some d_i is equal to 1.

In this remaining case, suppose the coins have values $d_1, d_2, 1$. If we have at least d_2 coins of value d_1, or at least d_1 coins of value d_2, then we can extract a group worth $D = d_1d_2$. Otherwise, the total value of the coins of value d_1 and d_2 is $\le d_1(d_2 - 1) + d_2(d_1 - 1) \le kD - d_1 - d_2$, so there are at least $d_1 + d_2$ coins of value 1. We can form a group of D by taking coins of value d_1 or d_2 until we exhausted all such coins or adding another one would make the total exceed D, then fill the remainder with coins of value 1.

P3. Turbo the snail plays a game on a board with $2n$ rows and $2n$ columns. There are $2n^2$ monsters who first choose to occupy $2n^2$ distinct cells, with Turbo's knowledge. After this, Turbo chooses any cell and labels it 1. Starting from this cell, Turbo then walks through all other $4n^2 - 1$ cells exactly once, labelling them in order with $2, 3, \dots, 4n^2$. Turbo only moves between cells which share an edge, and never returns to a cell. The final score is the sum of the labels of the cells with monsters. The monsters are trying to place themselves to maximize the score, while Turbo is trying to minimize the score based on the monsters' positions. Find, in terms of n, the largest score which the monsters can guarantee.

Solution 1

The largest score which the monsters can achieve is $4n^4$. This is obtained by placing themselves in a checkerboard pattern. Call a cell marked if a monster occupies it and empty otherwise. Clearly, Turbo's path will alternate between marked and empty. By starting at a marked cell, the score is

$$1 + 3 + \cdots + (4n^2 - 1) = 4n^4.$$

Now we will show that Turbo can always achieve a score at most $4n^4$, regardless of the monsters' positions. Note that for a $2n \times 2n$ board, there exists a Hamiltonian cycle through the cells. Consider an arbitrary monster and say that it is on cell 1. Now consider the two paths starting at this cell and going along the Hamiltonian cycle in each direction. For any monster other than the one at cell 1, its index in one path will be i and in the other will be $4n^2 + 2 - i$. Thus, the sum of the scores of these two paths is

$$\begin{aligned} 2 + \sum_{\text{monster at } i} \left(i + 4n^2 + 2 - i\right) &= 2 + (2n^2 - 1)(4n^2 + 2) \\ &= 8n^4. \end{aligned}$$

Thus, one of these two paths has a score at most $4n^4$.

Solution 2

As in the previous solution, the monsters can arrange themselves in a checkboard fashion to attain a lower bound of $4n^4$. Now we will show that any other configuration will allow Turbo to get a score at most $4n^4 - n^2$, which is slightly sharper than the previous solution.

As in the previous solution, there exists some Hamiltonian cycle. If the monsters are not in a checkboard pattern, there must be two adjacent cells in this cycle where both are monsters. Call these cells A and B. We will consider two paths going along the cycle. The first path begins at A, goes to B next, and continues along the cycle in that direction until ending in a cell adjacent to A. Likewise, the second path begins at B, goes to A next, and continues along until ending next to B.

We claim that the scores of the paths, denoted as $S_{A\to B}$ and $S_{B\to A}$, sum to $8n^4 - 2n^2$. Consider any of the $2n^2 - 2$ monsters not on A or B and say it is at index i of the first path. Then on the second path, this monster will be reached at index $4n^2 + 3 - i$. So we have

$$\begin{aligned} S_{A\to B} + S_{B\to A} &= 6 + \sum_{\text{monster at } i} \big(i + 4n^2 + 3 - i\big) \\ &= 6 + (2n^2 - 2)(4n^2 + 3) \\ &= 8n^4 - 2n^2. \end{aligned}$$

Thus, $\min(S_{A\to B}, S_{B\to A}) \le 4n^4 - n^2$ and so there exists a path which Turbo can take to achieve a score $\le 4n^4 - n^2$.

P4. A sphere with center I is inscribed in a tetrahedron $ABCD$. Suppose that the angle between any two faces of $ABCD$ is acute. Moreover, suppose that

$$\frac{\operatorname{vol}(IABC)}{BC} = \frac{\operatorname{vol}(IACD)}{CD} = \frac{\operatorname{vol}(IADB)}{DB}.$$

Show that AI is perpendicular to the plane BCD.

Remark. Here, $\operatorname{vol}(IABC)$ denotes the volume of tetrahedron $IABC$, and similarly for $IACD$ and $IADB$.

Solution 1

Let AI intersect the plane BCD at K. We have that

$$\frac{\operatorname{vol}(IABC)}{\operatorname{vol}(KABC)} = \frac{\operatorname{vol}(IACD)}{\operatorname{vol}(KACD)} = \frac{\operatorname{vol}(IADB)}{\operatorname{vol}(KADB)} = \frac{AI}{AK}.$$

Moreover, we have that

$$\frac{\operatorname{vol}(KABC)}{\operatorname{area}(KBC)} = \frac{\operatorname{vol}(KACD)}{\operatorname{area}(KCD)} = \frac{\operatorname{vol}(KADB)}{\operatorname{area}(KDB)} = \frac{1}{3}\operatorname{dist}(A, BCD).$$

Hence we have that

$$\frac{\operatorname{area}(KBC)}{BC} = \frac{\operatorname{area}(KCD)}{CD} = \frac{\operatorname{area}(KDB)}{DB}.$$

Thus K is equidistant from the three sides of BCD. Since K is in the interior of BCD, we find that K is the incenter of BCD.

Now, note that

$$\frac{\operatorname{vol}(IABC)}{BC} = \frac{1}{6}\operatorname{dist}(A, BC) \times \operatorname{dist}(I, ABC).$$

Hence we have that

$$\operatorname{dist}(A, BC) = \operatorname{dist}(A, CD) = \operatorname{dist}(A, DB).$$

Thus if we let L be the foot from A to BCD, then by the Pythagorean theorem L is also equidistant from the three sides of BCD. By the acute angle condition we find that L must also be the incenter of BCD. Thus $L = K = AI \cap BCD$, so $AI \perp BCD$ as desired.

Solution 2

As in Solution 1, define L as the foot from A to plane BCD, and show that L is the incenter of triangle BCD. Now observe that the plane ABL is precisely the plane containing line BL perpendicular to the plane BCD. Thus reflection across the plane ABL maps the line BC to the line BD, and hence the plane ABC to the plane ABD. Note the plane ABL also contains the line AB. Thus since the dihedral angles of tetrahedron $ABCD$ are acute, the plane ABL must be the plane through AB bisecting the interior dihedral angle between planes ABC and ABD. This coincides with thep lane ABI, so I lies on plane ABL.

Repeating this argument for planes ACL and ADL, we find that I lies on the intersection of the planes ABL, ACL, ADL, which is precisely the line AL. This finishes.

P5. For each $n \geq 1$, determine the maximum integer c_n for which there exists a polynomial f of degree n with rational coefficients, an irrational number a, and c_n distinct rational numbers $a_1, a_2, \ldots, a_{c_n}$ such that $f(a+a_i)$ is a rational number for all $1 \leq i \leq c_n$.

Solution 1

The answer is $c_n = n-1$. To construct a polynomial demonstrating that $c_n \geq n-1$, note that if $a, x \in \mathbb{Q}$ and $n \in \mathbb{Z}^{\geq 0}$, then $a(x+\sqrt{2})^n = c + d\sqrt{2}$ for some $c, d \in \mathbb{Q}$. With this motivation, consider solving the equation

$$f(x+\sqrt{2}) - g(x) = (x-\sqrt{2})(x-1)(x-2)\cdots(x-(n-1)),$$

for $f(x), g(x) \in \mathbb{Q}[x]$ with $\deg(f) = n$. Expanding the right hand side, it takes the form $a(x) + b(x)\sqrt{2}$, for some polynomials $a(x), b(x) \in \mathbb{Q}[x]$ of degree $n, n-1$ respectively.

Letting $f(x) = \sum_{i=0}^{n} f_i x^i$ with $f_i \in \mathbb{Q}$, we have

$$f(x+\sqrt{2}) = \sum_{i=0}^{n} f_i (x+\sqrt{2})^i = \sum_{i=0}^{n}\sum_{j=0}^{i} \binom{i}{j} f_i x^j (\sqrt{2})^{i-j}.$$

The $\sqrt{2}$-part picks up the terms where $i-j$ is odd, and we see that this is

$$\sum_{j=0}^{n}\left(\sum_{\substack{i=j+1 \\ i \not\equiv j \pmod 2}}^{n} \binom{i}{j} 2^{(i-j-1)/2} f_i\right) x^j$$

In particular, the coefficient of x^j is a linear combination of $f_{j+1}, f_{j+3}, \ldots$, each with a non-zero coefficient. Therefore, we can choose f_n to match the coefficient of x^{n-1} in $b(x)$, and f_{n-1} to match x^{n-2} in $b(x)$. Iterating backwards, we pick f_k for $k \geq 1$ to match the coefficient of x^{k-1} in $b(x)$, and take $f_0 = 0$. With this construction, we eliminate all of these terms, whence $a(x) + b(x)\sqrt{2} - f(x+\sqrt{2}) \in \mathbb{Q}[x]$; call this $-g(x)$. As the degree of $b(x)$ is $n-1$, $f_n \neq 0$, so $f(x)$ is indeed a polynomial of degree n.

This polynomial works with $a = \sqrt{2}$, since for integers i with $1 \leq i \leq n-1$, we have

$$f(i+\sqrt{2}) = g(i) \in \mathbb{Q}.$$

Now, we show that $c_n \geq n$ is impossible. Indeed, assume that a is irrational, and $f(a+a_i) = q_i \in \mathbb{Q}$ for some $a_i \in \mathbb{Q}$, $1 \leq i \leq n$. Lagrange interpolation produces a polynomial $g(x) \in \mathbb{Q}[x]$ of degree at most $n-1$ for which $g(a_i) = q_i$ for $1 \leq i \leq n$. Consider the polynomial $P(x) = f(x+a) - g(x)$. As $\deg(f) = n > \deg(g)$, $\deg(P) = n$. Furthermore, $P(a_i) = f(a_i + a) - g(a_i) = q_i - q_i = 0$, so the a_i's are the roots of P, i.e.

$$P(x) = b(x-a_1)(x-a_2)\cdots(x-a_n)$$

for some non-zero b. The coefficient of x^n is b, which must be the coefficient of x^n in $f(x)$, and is therefore rational, whence $P(x) \in \mathbb{Q}[x]$, and therefore $f(x+a) \in \mathbb{Q}[x]$. However, if the coefficient of x^{n-1} in $f(x)$ is $b' \in \mathbb{Q}$, then the coefficient of x^{n-1} in $f(x+a)$ is $bna + b'$, which is irrational, contradiction.

Solution 2

We present an alternate approach that directly uses algebraic number theory and linear algebra. The proof of the impossibility of $c_n \geq n$ was inspired by the solution of Perry Dai. This time, we start with that part, as it will be used in the construction. For the whole problem, take $f(x) = \sum_{i=0}^{n} f_i x^i$.

Suppose there is such a polynomial with $c_n \geq n$. By replacing $f(x)$ with $f(x+a_1) - f(a+a_1) \in \mathbb{Q}[x]$, we can assume that $f(a) = 0$, whence a is algebraic. Assume it has degree d (necessarily $\leq n$), hence $\mathbb{Q}(a)$ has a $\mathbb{Q}-$basis of $\{1, a, a^2, \ldots, a^{d-1}\}$. Since a is irrational, $d \geq 2$. A rational-coefficient linear combination of these terms is rational if and only if the coefficients of all a^i for $1 \leq i \leq d-1$ is zero.

Now, expand

$$f(a+a_k) = \sum_{i=0}^{n} f_i (a+a_k)^i = \sum_{i=0}^{n} \sum_{j=0}^{i} \binom{i}{j} f_i a_k^{i-j} a^j. \tag{1}$$

By replacing the terms $a^d, a^{d+1}, \ldots, a^n$ with rational linear combinations of $1, a, \ldots, a^{d-1}$, this expression is equal to

$$\sum_{i=0}^{d-1} g_i(a_k) a^i,$$

where the $g_i(a_k)$'s are polynomials in the variable a_k with rational coefficients. Since $f(a+a_k) \in \mathbb{Q}$, this implies that $g_i(a_k) = 0$ for all $1 \leq i \leq d-1$ and $1 \leq k \leq n$.

Consider just the polynomial g_1, i.e. the coefficient of a^1. In Equation (1), the terms with a^j where $0 \leq j \leq d-1$ only contribute to g_j. Thus, g_1 only sees contributions from $j = 1$. Of these, the largest exponent of a_k comes from the term $\binom{n}{1} f_n a_k^{n-1}$, where $\binom{n}{1} f_n$ is non-zero. Further contributions to g_1 come from the terms a^j with $j \geq d$, after replacing a^j with the appropriate rational linear combination of $1, a, \ldots, a^{d-1}$. However, this brings terms of the form $C a_k^{i-j}$, where $i - j \leq n - d \leq n - 2$, as $d \geq 2$. In particular, we see that g_1 has degree exactly $n-1$, with leading coefficient $\binom{n}{1} f_n \neq 0$. However, $g_1(a_k) = 0$ for $1 \leq k \leq n$, so g_1 has n distinct roots, contradicting its degree being exactly $n-1$. Thus, no such polynomial exists.

For the construction of $f(x)$, consider expanding $f(r+\sqrt{2})$, for $r = 1, 2, \ldots, n-1$. This will take the form $A_r + B_r\sqrt{2}$, where A_r and B_r are integral linear combinations of $f_0, f_1, \ldots, f_n$. Consider setting $B_r = 0$ for all $1 \leq r \leq n-1$: this is equivalent to the column vector $(f_0, f_1, \ldots, f_n)^T$ being in the kernel of an $(n-1) \times (n+1)$ dimensional matrix (the $n-1$ rows corresponding to the $n-1$ linear combinations in the B_r's). As there are $n-1$ rows, the rank is at most $n-1$, so by the rank-nullity theorem, the dimension of the kernel is at least 2. Thus, there exists a solution with at least one $f_i \neq 0$ for $1 \leq i \leq n$. In particular, we have a polynomial $f(x)$ of degree between 1 and n, for which $f(r+\sqrt{2}) \in \mathbb{Q}$ for $1 \leq r \leq n-1$. If $\deg(f) < n$, then we contradict $c_{\deg(f)} < \deg(f)$. Thus, $\deg(f) = n$, and we have a valid construction.

(We required the dimension of the kernel to be at least 2 for this last part. If it were only 1, we have a non-zero solution, but the solution would lead to the constant polynomial. This would not be a contradiction, as the problem statement is only valid for polynomials of positive degree.)